Weil Algebras and Double Lie Algebroids

by

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Abstract

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Contents

Chapter 1

Introduction

The theory of double structures originated with the work of Pradines [72, 73], who introduced the concept of a *double vector bundle*, defined in terms of local coordinate charts. This definition was reinterpreted by Mackenzie, who showed that double vector bundles are equivalent to manifolds equipped with a pair of compatible vector bundle structures. Grabowski-Rotkiewicz simplified the definition by showing that it was enough to consider the two scalar multiplications associated to the pair of vector bundle structures [23]. Thus in its most modern formulation, a double vector bundle is a manifold D equipped with two scalar multiplications κ^h and κ^v that commute: $\kappa^h_t \kappa^v_s = \kappa^v_s \kappa^h_t$ for all $t, s \in \mathbb{R}$. If we denote the base manifolds for these scalar multiplications by $A := \kappa^v_0(D)$ and $B := \kappa^h_0(D)$, then we can depict the double vector bundle D as a commuting diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & M. \end{tikzcd}$$

Loosely speaking, double vector bundles can be thought of as vector bundle objects in the category of vector bundles. They form the basis of second-order differential geometry, since the quintessential example of a double vector bundle is the double tangent bundle

$$\begin{tikzcd} TTM \arrow[r] \arrow[d] & TM \arrow[d] \\ TM \arrow[r] & M. \end{tikzcd}$$

More generally, the tangent bundle of a vector bundle $V \to M$ is a double vector bundle. The theory of double structures was further developed by the pioneering work of Mackenzie [5, 52, 55, 56], where the concepts of double Lie groupoids were introduced, along with their infinitesimal counterparts, double Lie algebroids. A double Lie algebroid is a double vector bundle for which all the sides are equipped with Lie algebroid structures, and with a certain compatibility condition between the horizontal and vertical Lie algebroid structures. Here the most important example is the tangent bundle $TA \to M$ of a Lie algebroid $A \to M$ (see [55, Example 4.6]), so, in particular, the double vector bundle TTM above is a double Lie algebroid. This thesis is concerned with the development of the theory of double structures, as well as their applications in Poisson geometry. Our focus lies on the construction of a bigraded bidifferential algebra associated to any double Lie algebroid called the Weil algebra that completely determines the double Lie algebroid structure. Along the way we will encounter many simplifications and additions to the theories of double vector bundles and double linear Poisson structures, and we will finish with a diverse range of applications throughout Poisson geometry that can be reinterpreted in our framework. Before describing our constructions and main results in detail, let us review some of the foundations of the theory of (ordinary) Lie algebras and Lie algebroids that will serve as motivation for our work.

1.1 The Chevalley-Eilenberg Complex

Throughout this section, let $A \to M$ be a Lie algebroid, and denote by $\rho\colon A \to TM$ its anchor map. Then any section $\sigma \in \Gamma(A)$ defines two kinds of derivations on the space

$$\Gamma(\wedge^\bullet A^*),$$

a contraction operator ι_σ of degree -1 and a Lie derivative operator $\mathcal{L}_\sigma$ of degree 0. By the usual Cartan formula, these operators give rise to the *Chevalley-Eilenberg* differential d_{CE} on $\Gamma(\wedge^\bullet A^*)$, an operator of degree 1 that satisfies

$$[\iota_\sigma, d_{\mathrm{CE}}] = \mathcal{L}_\sigma$$

for all $\sigma \in \Gamma(A)$. This identity implies that $d_{\mathrm{CE}}^2 = 0$, so for any Lie algebroid $A \to M$, there is an associated complex $(\Gamma(\wedge^\bullet A^*), d_{\mathrm{CE}})$ called the *Chevalley-Eilenberg complex* of A. Specific examples of this complex include the de Rham complex of a manifold M ($A = TM$), the Chevalley-Eilenberg complex of a Lie algebra (when M is a point), and the foliated de Rham complex of a foliated manifold (here $A \subseteq TM$ is obtained from the foliation using Frobenius's theorem).

It is a fundamental result of Vaĭntrob [76] that this construction is reversible. That is, suppose $A \to M$ is a vector bundle, and d_A is a degree 1 derivation on the graded algebra $\Gamma(\wedge^\bullet A^*)$ that satisfies $d_A^2 = 0$. Then $A \to M$ inherits a Lie algebroid structure defined by

$$\mathcal{L}_{\rho(\sigma)} f = \langle d_A f, \sigma\rangle, \quad \langle \alpha, [\sigma,\tau]\rangle = \langle d_A\langle\alpha,\tau\rangle, \sigma\rangle - \langle d_A\langle\alpha,\sigma\rangle,\tau\rangle - d_A\alpha(\sigma,\tau)$$

for all $f \in C^\infty(M)$, $\sigma, \tau \in \Gamma(A)$, and $\alpha \in \Gamma(A^*)$. This characterization of Lie algebroids often simplifies various constructions related to them. As a basic example, since maps on sections do not always correspond to bundle maps, the correct definition of a morphism Lie algebroids $A \to B$ is not immediately obvious using the usual definition of Lie algebroids. However, in terms of the Chevalley-Eilenberg complexes, such a morphism of Lie algebroids simply corresponds to a morphism of complexes $(\Gamma(\wedge^\bullet B^*), d_B) \to (\Gamma(\wedge^\bullet A^*), d_A)$.

Remark 1.1.1. If $V \to M$ is an A-module (i.e. it comes with a flat A-connection ∇), then one can define the Chevalley-Eilenberg complex with coefficients in V. The underlying algebra is $\Gamma(\wedge^\bullet A^* \otimes V)$, and one simply defines contractions, Lie derivatives, and a differential in the same way as indicated above, but with the modification that $\mathcal{L}_\sigma\vartheta = \nabla_\sigma\vartheta$ for $\vartheta \in \Gamma(\wedge^0 A^* \otimes V) = \Gamma(V)$.

1.2 The Weil Algebra of a Lie Algebra

Weil algebras have their roots in the theory of equivariant cohomology, where the Weil algebra $\mathscr{W}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is used as an algebraic model for the space of differential forms on the classifying bundle $EG \to BG$, where G is a Lie group integrating $\mathfrak{g}$. It appears in the Chern-Weil theory of characteristic classes, as well as in Cartan's equivariant de Rham theory. Here we will review only the definition of $\mathscr{W}\mathfrak{g}$, for information on its application in equivariant cohomology theory, see [30].

Let $\mathfrak{g}$ be a Lie algebra, and fix a basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$ with structure constants λ^i_{jk}. The *Weil algebra* of $\mathfrak{g}$ is the bigraded algebra $\mathscr{W}\mathfrak{g} := \wedge^\bullet \mathfrak{g}^* \otimes S^\bullet \mathfrak{g}^*$ generated by the elements

$$\theta^i = e_i \otimes 1, \quad \mu^i = 1 \otimes e_i,$$

where each θ^i has bidegree $(1, 0)$, and each μ^i has bidegree $(1, 1)$. This bigraded algebra comes equipped with two differentials. The first, denoted d_{K}, is defined on generators by

$$d_{\mathsf{K}}\theta^i = \mu^i, \quad d_{\mathsf{K}}\mu^i = 0.$$

Note that because of the way we set up the bigrading, this derivation has bidegree $(0, 1)$. We call d_{K} the *Koszul differential* since it is the differential of the Koszul complex of the vector space underlying $\mathfrak{g}$. The second differential, which we denote d_{CE}, is defined on generators by the formulas

$$d_{\mathrm{CE}}\theta^i = \frac{1}{2}\lambda^i_{jk}\theta^j\theta^k, \quad d_{\mathrm{CE}}\mu^i = -\lambda^i_{jk}\mu^j\theta^k.$$

In terms of the bigrading described above, d_{CE} has bidegree $(1, 0)$. We call d_{CE} the *Chevalley-Eilenberg differential*, since it may be thought of as the differential of the Chevalley-Eilenberg complex of the representation $S^\bullet\mathfrak{g}^*$ of $\mathfrak{g}$. These formulas allow for a direct verification that d_{K} and d_{CE} commute (in the graded sense), making the Weil algebra $\mathscr{W}\mathfrak{g}$ into a bicomplex.

Remark 1.2.1. Since the Weil algebra is in particular a Chevalley-Eilenberg complex, it comes equipped with contractions ι_x and Lie derivatives $\mathscr{L}_x$ by elements $x \in \mathfrak{g}$. In terms of the generators, these are defined by

$$\iota_x\theta^i = \langle e^i, x\rangle, \quad \iota_x\mu^i = 0, \quad \mathscr{L}_x\theta^i = -\lambda^i_{jk}\langle e^j, x\rangle\theta^k, \quad \mathscr{L}_x\mu^i = -\lambda^i_{jk}\langle e^j, x\rangle\mu^k.$$

These make $\mathscr{W}^{\bullet,\bullet}\mathfrak{g}$ into a so-called $\mathfrak{g}$-differential algebra, and it can be shown that $\mathscr{W}^{\bullet,\bullet}\mathfrak{g}$ satisfies a certain universal property among such objects.

1.3 The Weil Algebra of a Double Lie Algebroid

We are interested in generalizations of the theory discussed above to double Lie algebroids. Vaĭntrob's result that the Chevalley-Eilenberg complex on A completely determines the Lie algebroid structure was generalized to the setting of double structures by Voronov [77], who proved that double Lie algebroid structures on D were equivalent to pairs of commuting homological vector fields Q_h, Q_v, of bidegrees $(1, 0)$ and $(0, 1)$, on the bigraded supermanifold $D[1, 1]$ obtained from D by a parity shift in both vector bundle directions. Put differently, the algebra of (double-polynomial) functions on $D[1, 1]$ is a double complex, generalizing the Chevalley-Eilenberg complex of a Lie algebroid. Our main goal in this thesis is to provide a classical description of this bicomplex that simultaneously generalizes the constructions discussed in sections 1.1 and 1.2 above to the context of double structures. Specifically, to any double vector bundle D we will associate a bigraded algebra $\mathcal{W}^{\bullet,\bullet}(D)$ that we call the *Weil algebra* of D. This algebra will model the space of double polynomial functions on $D[1,1]$ (and therefore generalize the Chevalley-Eilenberg complex of a Lie algebroid) in the sense that double Lie algebroid structures on D are equivalent to pairs of differentials d_h, d_v on the Weil algebra of D that make $(\mathcal{W}^{\bullet,\bullet}(D), d_h, d_v)$ into a bicomplex. Moreover, the notion of Weil algebras has been extended from the setting of Lie algebras to the more general setting of Lie algebroids. This was first done in super-geometric terms by Mehta [61] and in more classical terms by Abad and Crainic [3]. In the case that $A \to M$ is a Lie algebroid, applying our construction to the double Lie algebroid

$$\begin{tikzcd} TA \arrow[r] \arrow[d] & TM \arrow[d] \\ A \arrow[r] & M \end{tikzcd}$$

gives the Weil algebra of A.

To explain our construction, let D be any double vector bundle with side bundles A, B. The sub-manifold on which the two scalar multiplications coincide is itself a vector bundle over M, called the *core* of D. We denote by

$$E = \mathrm{core}(D)^*$$

its dual bundle. There is a vector bundle $\widehat{E} \to M$ whose space of sections consists of the smooth functions on D that are *double-linear*, i.e., linear both horizontally and vertically. It fits into an exact sequence

$$0 \to A^* \otimes B^* \xrightarrow{i_{\widehat{E}}} \widehat{E} \to E \to 0,$$

where the map $\widehat{E} \to E$ is given by the restriction of double-linear functions to the core, while the map $i_{\widehat{E}}$ is given by the multiplication of linear functions on A, B. The above sequence is the dual of the $\mathcal{DVB}$ *sequence* of Chen-Liu-Sheng [13]. Our definition of the Weil algebra bundle is as follows:

Definition. The *Weil algebra bundle* of the double vector bundle D is the bundle (over M) of bigraded super-commutative algebras

$$W(D) = (\wedge A^* \otimes \wedge B^* \otimes \vee \widehat{E})/\sim\ ,$$

taking the quotient by the (fibrewise) ideal generated by elements of the form

$$\alpha\beta - i_{\widehat{E}}(\alpha \otimes \beta)$$

for $(\alpha, \beta) \in A^* \times_M B^*$. Here, generators $\alpha \in A^*$ have bidegree $(1,0)$, generators $\beta \in B^*$ have bidegree $(0,1)$, and generators $\widehat{e} \in \widehat{E}$ have bidegree $(1,1)$. The bigraded super-commutative algebra $\mathcal{W}(D) = \Gamma(W(D))$ will be called the *Weil algebra* of D.

Double vector bundles come in triples, with cyclic permutation of the roles of the vector bundles

A, B, E over M:

$$\begin{array}{ccc}
\begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & M \end{tikzcd} &
\begin{tikzcd} D' \arrow[r] \arrow[d] & E \arrow[d] \\ B \arrow[r] & M \end{tikzcd} &
\begin{tikzcd} D'' \arrow[r] \arrow[d] & A \arrow[d] \\ E \arrow[r] & M \end{tikzcd}
\end{array}$$

The double vector bundle D' is (essentially) obtained by taking the dual of D as a vector bundle over B and interchanging the roles of horizontal and vertical structures; similarly $D'' \cong (D')'$. (One has $(D'')' = D$.) See section 2.6 for a more precise description. Accordingly, we have three Weil algebras

$$\mathcal{W}(D), \quad \mathcal{W}(D'), \quad \mathcal{W}(D'').$$

A linear Lie algebroid structure of D as a vector bundle over A, also known as a $\mathcal{VB}$*-algebroid* structure of D over A, is equivalent to a double-linear Poisson structure on D'', and also to a $\mathcal{VB}$-algebroid structure of D' over E. (See [51].) Chapter 5 explains in detail how these structures are expressed in terms of the Weil algebras. In particular, one finds:

Theorem I. Let D be a double vector bundle. Then the following are equivalent:

1. a $\mathcal{VB}$-algebroid structure of D over A,
2. a vertical differential d_v on $\mathcal{W}(D)$,
3. a horizontal differential d'_h on $\mathcal{W}(D')$,
4. a Gerstenhaber bracket (of bidegree $(-1,-1)$) on $\mathcal{W}(D'')$.

Using cyclic permutations of D, D', D'', one has similar results when starting out with a $\mathcal{VB}$-algebroid structure of D over B or with a double-linear Poisson structure on D. Chapter 7 deals with the situation that D has any *two* of these structures; in particular we prove:

Theorem II. Let D be a double vector bundle, with $\mathcal{VB}$-algebroid structures over B as well as over A. Then the following are equivalent:

1. D is a double Lie algebroid,
2. the horizontal and vertical differentials $\mathrm{d}_h, \mathrm{d}_v$ on $\mathcal{W}(D)$ commute,
3. the horizontal differential d'_h on $\mathcal{W}(D')$ is a derivation of the Gerstenhaber bracket,
4. the vertical differential d''_v on $\mathcal{W}(D'')$ is a derivation of the Gerstenhaber bracket.

If one uses the identification of $\mathcal{W}(D)$ with functions on the supermanifold $D[1,1]$, the equivalence (a) $\Leftrightarrow$ (b) translates into Voronov's result [77] mentioned above; however, we will give a direct proof of this result, not using any super-geometry. More precisely, given vertical and horizontal $\mathcal{VB}$-algebroid structures, the proof will give an explicit relationship between their compatibility (or lack thereof) and the super-commutator of the two differentials.

If $D = T\mathfrak{g}$ is the tangent bundle of a Lie algebra, viewed as a double Lie algebroid with $A = \mathfrak{g}$, $B = 0$, $E = \mathfrak{g}^*$, then the three Weil algebras are

$$\mathcal{W}(T\mathfrak{g}) = \wedge\mathfrak{g}^* \otimes S\mathfrak{g}^*, \quad \mathcal{W}((T\mathfrak{g})') = \wedge\mathfrak{g} \otimes S\mathfrak{g}, \quad \mathcal{W}((T\mathfrak{g})'') = \wedge\mathfrak{g}^* \otimes \wedge\mathfrak{g}.$$

Here $\mathcal{W}(T\mathfrak{g})$ is the standard Weil algebra discussed in section 1.2, with d_h the Chevalley-Eilenberg differential for the $\mathfrak{g}$-module $S\mathfrak{g}^*$ and d_v the Koszul differential. The differential d'_h on $\mathcal{W}((T\mathfrak{g})')$ is the Koszul differential, and the Gerstenhaber bracket is a natural extension of the Lie bracket of the semi-direct product $\mathfrak{g} \ltimes \mathfrak{g}$ for the adjoint action. The differential d''_v on $\mathcal{W}((T\mathfrak{g})'')$ is the Chevalley-Eilenberg differential for the $\mathfrak{g}$-module $\wedge\mathfrak{g}$, and the Gerstenhaber bracket extends the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ (this is a special case of Kosmann-Schwarzbach's *big bracket* [41, 42]). More generally, as mentioned above, if $D = TA$ is the tangent bundle of a Lie algebroid A, the double complex $\mathcal{W}(TA)$ coincides with the Weil algebra of the Lie algebroid A.

We will also explore numerous relationships between the bigraded algebra $\mathcal{W}(D)$ and the existing literature in Poisson geometry. Returning to a general double vector bundle, let $\wedge_A D$ and $\wedge_B D$ be the exterior bundles of D viewed as vector bundles over A and B, respectively. By considering the homogeneity of sections in the A-direction, one can define distinguished subspaces of of *linear sections* of these bundles. In chapter 9, we show that these have descriptions in terms of the Weil algebras:

$$\Gamma_{\mathrm{lin}}(\wedge^\bullet_A D, A) = \mathcal{W}^{\bullet,1}(D''), \quad \Gamma_{\mathrm{lin}}(\wedge^\bullet_B D, B) = \mathcal{W}^{1,\bullet}(D').$$

As a consequence of Theorem I above, a double-linear Poisson structure on D determines a degree 1 differential on these spaces, while a $\mathcal{VB}$-algebroid structure on D over A (respectively over B) determines a Gerstenhaber bracket. For the cotangent and tangent bundles of a vector bundle $V \to M$ (with the $\mathcal{DVB}$ structures as in Section 2.2), some of these spaces have well-known interpretations:

$$\mathcal{W}^{1,\bullet}(T^*V) = \mathfrak{X}^\bullet_{\mathrm{lin}}(V), \quad \mathcal{W}^{1,\bullet}(TV) = \Omega^\bullet_{\mathrm{lin}}(V).$$

Here $\mathfrak{X}_{\mathrm{lin}}(V)$ are linear multi-vector fields with the Schouten bracket, while $\Omega_{\mathrm{lin}}(V)$ are linear differential forms with the de Rham differential. If V is a Lie algebroid over M, one also has horizontal differentials on $\mathcal{W}(T^*V)$ and on $\mathcal{W}(TV)$, coming from the $\mathcal{VB}$-algebroid structures of T^*V over V^* and TV over TM, respectively. In Section 9.4, we will see that

$$\mathcal{W}^{1,\bullet}(T^*V) \cap \ker(\mathrm{d}_h) = \mathfrak{X}^\bullet_{\mathrm{IM}}(V), \quad \mathcal{W}^{1,\bullet}(TV) \cap \ker(\mathrm{d}_h) = \Omega^\bullet_{\mathrm{IM}}(V)$$

the space of *infinitesimally multiplicative* multi-vector fields [36] and infinitesimally multiplicative differential forms [6], respectively. On the other hand, the Lie algebroid structure of V over M also induces a $\mathcal{VB}$-algebroid structure on T^*V^* over V^*, and the corresponding differential d_v on $\mathcal{W}^{1,\bullet}(T^*V^*) = \mathfrak{X}_{\mathrm{lin}}(V^*)$ is the Poisson differential for the resulting Poisson structure on V^*, identifying this space with the *deformation complex* of Crainic-Moerdijk [16]. (This may also be seen as a consequence of a result of Cabrera-Drummond [11] for the $\mathcal{VB}$-algebroid T^*V^*.) Further applications relate the Weil algebra to the Frölicher-Nijenhuis [22] and Nijenhuis-Richardson [70] brackets, to *matched pairs of Lie algebroids* [67], and to the notion of *representations up to homotopy* [3, 26, 29].

1.4 Outline

We begin in chapter 2 with a review of double vector bundles. Our treatment is entirely self-contained, beginning with the basic definitions and progressing through the pillars of the theory such as splittings, triality, and $\mathcal{DVB}$ sequences. Of particular importance to the rest of the thesis are the fat bundles $\widehat{A}$, $\widehat{B}$, and $\widehat{E}$ as well as the pairings between them (see section 2.8). While most of the results of this section were previously known, we emphasize a new point of view using associated bundles that simplifies the proofs of many of the main results. We follow up this discussion by introducing a particular class of examples in chapter 3: double normal bundles. Precisely, to any triple (M, N_1, N_2) of manifolds for which N_1 and N_2 are cleanly intersecting submanifolds of M (see section B.2), we associate a double vector bundle $\nu(M, N_1, N_2)$ that should be thought of as the simultaneous linearization of the directions normal to N_1 and the directions normal to N_2. We provide 3 alternative constructions of $\nu(M, N_1, N_2)$ and establish their equivalence. We finish the chapter with a doubled version of the deformation to the normal cone construction from algebraic geometry.

Chapter 4 presents our main construction, the Weil algebra $\mathcal{W}(D)$ of a $\mathcal{DVB}$. We start with a discussion of the space $\mathcal{S}(D)$ of double polynomial functions on D, and introduce $\mathcal{W}(D)$ as a supercommutative version of $\mathcal{S}(D)$. Following the lead of the classical constructions discussed in sections 1.1and 1.2, we proceed to study the space of derivations of $\mathcal{W}(D)$. Finally, we introduce alternative characterizations of the Weil algebra in terms of linear and core sequences and in terms of vector fields on the total space D.

The next step is to study the Weil algebras in the presence of additional structure. We begin this endeavour in chapter 5, where we suppose our $\mathcal{DVB}$ comes equipped with a double-linear Poisson structure. As in the case of the classical Weil algebra, the presence of Lie theoretic structures introduces a new collection of derivations on the Weil algebra $\mathcal{W}(D)$, leading us to Theorem I described above. Chapter 6 is devoted to studying the relations between these derivations to establish a Cartan calculus on $\mathcal{W}(D)$ analogous to the Chevalley-Eilenberg complex of a Lie algebroid. Finally, in chapter 7, we apply this Cartan calculus to the situation that D is a double Lie algebroid, resulting in our main result Theorem II above. We finish off the chapter with a detailed analysis of the Lie algebroid structure on the core of D.

The final chapters are devoted to connections with the literature. In chapter 8, we illustrate the constructions of the thesis by computing them for the fundamental example of a double Lie algebroid: the tangent prolongation of a Lie algebroid. We then use these results in chapter 9 to relate Weil algebroids to various notions appearing in the literature on Lie algebroids, including ta more detailed investigation of the applications described above. We conclude with chapter 10, where we describe further directions of study, with a particular emphasis on the idea of a van Est map for double Lie groupoids.

Chapter 2

Double Vector Bundles

2.1 Definitions

As mentioned in the introuction, the concept of a double vector bundle was introduced by Pradines [72, 73] in terms of local charts, and later reformulated as manifolds with ‘commuting’ vector bundle structures [51]. We shall work with an elegant approach due to Grabowski-Rotkiewicz [23], who observed that vector bundle structures on manifolds V are completely determined by their scalar multiplications $\kappa_t\colon V \to V$, $t \in \mathbb{R}$, and vector bundle morphisms $V \to V'$ are exactly the smooth maps intertwining scalar multiplications. To state these definitions precisely, we will need to make explicit what we mean by a scalar multiplication. For this, we recall the following theorem.

Theorem 2.1.1. *[23] Let V be a manifold equipped with an action of the multiplicative monoid $(\mathbb{R}, \cdot)$, with the action by $t \in \mathbb{R}$ denoted $\kappa_t\colon V \to V$, and let $M := \kappa_0(V)$. Then κ_t is the scalar multiplication for a vector bundle structure $V \to M$ if and only if the map $V \to TV|_M$ given by*

$$v \mapsto \frac{d}{dt}(\kappa_t v)|_{t=0}$$

is injective.

Remark 2.1.2. The vector bundle structure having κ_t as its scalar multiplication is unique, since the map $v \mapsto \frac{d}{dt}(\kappa_t v)|_{t=0}$ is an embedding $V \hookrightarrow TV|_M$ as a subbundle.

Any action of the monoid $(\mathbb{R}, \cdot)$ on a manifold V that satisfies the condition of theorem 2.1.1 will from now on be called a *scalar multiplication* on V. We can now state the definition of a double vector bundle.

Definition 2.1.3.

1. A *double vector bundle* ($\mathcal{DVB}$) is a smooth manifold D equipped with two scalar multiplications κ^h (the *horizontal scalar multiplication*) and κ^v (the *vertical scalar multiplication*) such that for all $s, t \in \mathbb{R}$ we have
$$\kappa^h_t \kappa^v_s = \kappa^v_s \kappa^h_t. \tag{2.1}$$

2. A *morphism of double vector bundles* ($\mathcal{DVB}$ morphism) from (D_1, A_1, B_1) to (D_2, A_2, B_2) is a smooth map $\varphi\colon D_1 \to D_2$ that satisfies $\kappa^h_{2,t} \circ \varphi = \varphi \circ \kappa^h_{1,t}$ and $\kappa^v_{2,s} \circ \varphi = \varphi \circ \kappa^v_{2,s}$ for all $s, t \in \mathbb{R}$.

For a given D, let $A = \kappa_0^v(D)$ and $B = \kappa_0^h(D)$ denote the base submanifolds for the scalar multiplications. It is easy to see that κ^h restricts to a scalar multiplication on A making it a vector bundle over $M := A \cap B$ (and similarly for B), and so we will represent the double vector bundle D as a square:

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M. \end{array} \tag{2.2}$$

One calls A, B the *side bundles*, and M the *base* of the double vector bundle. It is worth noting that A and B can also be defined as the fixed point sets of the scalar multiplications:

$$A = \{d \in D \mid \kappa_s^v(d) = d \ \forall s \in \mathbb{R}\}, \quad B = \{d \in D \mid \kappa_t^h(d) = d \ \forall t \in \mathbb{R}\}.$$

Since κ^h and κ^v commute, we get a map

$$\psi \colon D \to A \times_M B, \tag{2.3}$$

given by $\varphi(d) = (\kappa_0^v(d), \kappa_0^h(d))$. If we consider $A \times_M B$ as a double vector bundle with scalar multiplications $\kappa^h \times \mathrm{id}_B$ and $\mathrm{id}_A \times \kappa^v$ (see also example 2.2(1) below), then ψ is clearly a $\mathcal{DVB}$ morphism. It is an important fact that ψ is also a surjective submersion.

Lemma 2.1.4. *The $\mathcal{DVB}$ morphism $\psi \colon D \to A \times_M B$ is a surjective submersion.*

Proof. To begin, fix $m \in M$. Note that the map $T_m\psi$ restricts to the identity map on both T_mA and T_mB, and therefore it has maximal rank. This shows that ψ is a submersion on a neighbourhood of M in D. To see that it is a submersion everywhere, note that $\lim_{t,s\to 0} \kappa_t^h \kappa_s^v(D) = M$ and that for any $t, s \in \mathbb{R}$ we have

$$\psi(\kappa_t^h \kappa_s^v d) = (\kappa_t^h \times \kappa_s^v) \circ \psi(d), \tag{2.4}$$

since ψ is a $\mathcal{DVB}$ morphism. For any $d \in D$, take $s, t > 0$ small enough so that $\kappa_t^h \kappa_s^v d$ lies in a neighbourhood of M on which ψ is known to be a submersion. Then by (2.4) we have

$$T_{(\kappa_0^v(d), \kappa_0^h(d))}(\kappa_t^h \times \kappa_s^v) \circ T_d\psi = T_{\kappa_t^h \kappa_s^v(d)}\psi \circ T_{\kappa_s^b(d)}\kappa_t^h \circ T_d\kappa_s^v.$$

But κ_t^h, κ_s^v, and $(\kappa_t^h \times \kappa_s^v)$ are all diffeomorphisms and $T_{\kappa_s^v \kappa_t^h(d)}\psi$ is surjective, so it follows that $T_d\psi$ is surjective as well.

To see that ψ is surjective, note that its image certainly contains M. Being a submersion, ψ is an open map, and hence its image is an open neighbourhood of M. Once again using the identity (2.4) establishes that $\operatorname{im}\psi$ is in fact all of $A \times_M B$. ☐

A corollary of lemma 2.1.4 is that the preimage under ψ of any submanifold of $A \times_M B$ is a submanifold of D. Of particular importance is the preimage of M, which we call the *core* of D.

Definition 2.1.5 (Core)**.** Given a double vector bundle D, the submanifold

$$\mathrm{core}(D) = \psi^{-1}(M) \subseteq D$$

is called the *core* of D.

Note that core(D) is invariant under both κ^h and κ^v, and certainly $\kappa^h_0(\mathrm{core}(D)) = \kappa^v_0(\mathrm{core}(D)) = M$. Thus core$(D)$ inherits two vector bundle structures over M, given by the restrictions of κ^h and κ^v. In fact, these two vector bundle structures coincide.

Proposition 2.1.6. *For a double vector bundle D, the core may alternatively be characterized as the subset of D on which the horizontal and vertical scalar multiplications agree:*

$$\mathrm{core}(D) := \{d \in D |\ \kappa^h_t(d) = \kappa^v_t(d)\ \forall t \in \mathbb{R}\}. \tag{2.5}$$

Proof. First let $d \in D$ be such that $\kappa^h_t(d) = \kappa^v_t(d)$ for every $t \in \mathbb{R}$. Then note that $\kappa^h_0(d) = \kappa^v_0(d)$ lies in both A and B by definition, and since $A \cap B = M$ we conclude that

$$\{d \in D |\ \kappa^h_t(d) = \kappa^v_t(d)\ \forall t \in \mathbb{R}\} \subseteq \mathrm{core}(D).$$

For the converse, let $\mathcal{E}^h$ denote the Euler vector field for the vector bundle structure $\kappa^h|_{\mathrm{core}(D)}$, and let $\mathcal{E}^v$ denote the Euler vector field for $\kappa^v|_{\mathrm{core}(D)}$. The linear approximation of $\mathcal{E}^h$

$$\nu(\mathcal{E}^h)\colon \nu(\mathrm{core}(D), M) \to T\nu(\mathrm{core}(D), M)$$

is simply the Euler vector field for the vector bundle $T\nu(\mathrm{core}(D), M)$. Therefore, if we denote by φ the isomorphism $\nu(\mathrm{core}(D), M) \cong \mathrm{core}(D)$ induced by κ^v, we find that $\varphi^*\nu(\mathcal{E}^h) = \mathcal{E}^v$. On the other hand, since κ^h and κ^v commute, $\mathcal{E}^h$ is linear in κ^v (precisely: $(\kappa^v_s)^*\mathcal{E}^h = \mathcal{E}^h$ for all $s \in \mathbb{R}$). But then we have $\varphi^*\nu(\mathcal{E}^h) = \mathcal{E}^h$. We conclude that $\mathcal{E}^h = \mathcal{E}^v$, which completes the proof. ☐

From now on, we will reserve the notation

$$E = \mathrm{core}(D)^*$$

for the *dual bundle* of the core.

Remark 2.1.7. We would prefer the letter C, since we will make extensive use of a cyclic symmetry interchanging the bundles A, B, and core$(D)^*$; see Section 2.6 below. However, since C is commonly used to denote the core itself, this might cause confusion with the existing literature.

2.2 Examples

Here are some examples of double vector bundles:

1. If A, B, E are vector bundles over M, then $A \times_M B \times_M E^*$ is a double vector bundle

$$\begin{array}{ccc} A \times_M B \times_M E^* & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

   with core given by E^*. The horizontal and vertical scalar multiplications are given by $\kappa^h_t(a, b, \varepsilon) =$

$(ta, b, t\varepsilon)$ and $\kappa^v_t(a, b, \varepsilon) = (a, tb, t\varepsilon)$, respectively. The two additions are defined by

$$(a_1, b, \varepsilon_1) +_h (a_2, b, \varepsilon_2) = (a_1 + a_2, b, \varepsilon_1 + \varepsilon_2), \quad (a, b_1, \varepsilon_1) +_v (a, b_2, \varepsilon_2) = (a, b_1 + b_2, \varepsilon_1 + \varepsilon_2).$$

In particular, any vector bundle $V \to M$ can be regarded as a $\mathscr{DVB}$ in three ways, by playing the role of A, B or E^*. Moreover, if we take $E = 0$, then we obtain the double vector bundle $A \times_M B$ mentioned above.

2. If $V \to M$ is any vector bundle, then its tangent bundle and cotangent bundle are double vector bundles

$$\begin{tikzcd} TV \arrow[r] \arrow[d] & TM \arrow[d] \\ V \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} T^*V \arrow[r] \arrow[d] & V^* \arrow[d] \\ V \arrow[r] & M \end{tikzcd}$$

with $\mathrm{core}(TV) = V$ (thought of as the vertical bundle of $TV|_M$) and $\mathrm{core}(T^*V) = T^*M$. The $\mathscr{DVB}$ structure on TV appeared in [73]; the $\mathscr{DVB}$ structure on T^*V was first discussed in [57].

3. Suppose $V \to M$ is a subbundle of a vector bundle $W \to Q$. Then the normal and conormal bundle of V in W are double vector bundles

$$\begin{tikzcd} \nu(W,V) \arrow[r] \arrow[d] & \nu(Q,M) \arrow[d] \\ V \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} \nu^*(W,V) \arrow[r] \arrow[d] & (W|_M/V)^* \arrow[d] \\ V \arrow[r] & M \end{tikzcd}$$

with $\mathrm{core}(\nu(W,V)) = W|_M/V$ and $\mathrm{core}(\nu^*(W,V)) = \nu^*(Q,M)$.

4. Let N_1, N_2 be submanifolds of a manifold M, with clean intersection (see appendix B.2). Then there is a *double normal bundle* with base $N = N_1 \cap N_2$,

$$\begin{tikzcd} \nu(M, N_1, N_2) \arrow[r] \arrow[d] & \nu(N_1, N) \arrow[d] \\ \nu(N_2, N) \arrow[r] & N \end{tikzcd}$$

with core $TM|_N/(TN_1|_N + TN_2|_N)$. Note that the core is trivial if and only if the intersection is transverse. More details on this example are given in chapter 3.

2.3 New $\mathscr{DVB}$s from old

As with many categories, given one or more double vector bundles there are numerous ways to construct new ones. Here we briefly outline a few of these constructions, some of which we will discuss more deeply in later sections.

1. *Diagonal flips.* Given a double vector bundle D as in (2.2), the simplest way to obtain another $\mathscr{DVB}$ is to swap the horizontal and vertical scalar multiplications, which results in the diagonal

flip of D, denoted $\mathrm{flip}(D)$. Thus we have $\kappa^h_{\mathrm{flip}(D)} = \kappa^v_D$ and $\kappa^v_{\mathrm{flip}(D)} = \kappa^h_D$ resulting in a $\mathcal{DVB}$

$$\begin{tikzcd} \mathrm{flip}(D) \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & M \end{tikzcd}$$

with core E^*.

2. *Sub-$\mathcal{DVB}$s.* A sub-$\mathcal{DVB}$ of D is, as usual, a submanifold $Q \subseteq D$ that is itself a double vector bundle. By the results of [23], this happens precisely when Q is invariant under both κ^h and κ^v. The side bundles and core of Q are subbundles of the side bundles and core of D, and the total base of Q is a submanifold of the total base of D. As a specific instance of this, suppose that $\varphi\colon D_1 \to D_2$ is a $\mathcal{DVB}$ morphism. Then the kernel, $\ker\varphi := \varphi^{-1}(M)$ is a sub-$\mathcal{DVB}$ of D_1 because φ intertwines the scalar multiplications and elements of M are fixed by both κ^h_2 and κ^v_2. Similarly, the image of φ is a sub-$\mathcal{DVB}$. The diagrams for these $\mathcal{DVB}$'s are

$$\begin{tikzcd} \varphi^{-1}(M) \arrow[r] \arrow[d] & (\varphi|_B)^{-1}(M) \arrow[d] \\ (\varphi|_A)^{-1}(M) \arrow[r] & M, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{im}\,\varphi \arrow[r] \arrow[d] & \mathrm{im}(\varphi|_B) \arrow[d] \\ \mathrm{im}(\varphi|_A) \arrow[r] & M, \end{tikzcd}$$

with cores $(\varphi|_{E^*})^{-1}(M)$ and $\mathrm{im}(\varphi|_{E^*})$.

3. *Pullbacks.* Given a double vector bundle D with total base M and a smooth map $f\colon N \to M$, one can consider the pullback (as a fibre bundle) $f^*D = \{(n,d) \in N \times D \mid \kappa^h_0\kappa^v_0(d) = f(n)\}$. This space becomes a double vector bundle

$$\begin{tikzcd} f^*D \arrow[r] \arrow[d] & f^*B \arrow[d] \\ f^*A \arrow[r] & N \end{tikzcd}$$

with core f^*E^*. Here f^*A, f^*B, and f^*E^* denote the pullbacks of A, B, and E^* (as vector bundles) along f, and the scalar multiplications on f^*D are given by $\kappa^h_{f^*D} = (\mathrm{id}_N \times \kappa^h_D)|_{f^*D}$ and $\kappa^v_{f^*D} = (\mathrm{id}_N \times \kappa^v_D)|_{f^*D}$. In the special case that $f\colon N \hookrightarrow M$ is an embedding of N as a submanifold, we denote f^*D as $D|_N$ and call it the restriction of D to N.

4. *Products and direct sums.* Suppose we have two double vector bundles D_i with side bundles A_i and B_i and core E^*_i over total base M_i for $i = 1, 2$. The product $D_1 \times D_2$ becomes a $\mathcal{DVB}$ with horizontal scalar multiplication $\kappa^h_{D_1} \times \kappa^h_{D_2}$ and vertical scalar multiplication $\kappa^v_{D_1} \times \kappa^v_{D_2}$. Its diagram is

$$\begin{tikzcd} D_1 \times D_2 \arrow[r] \arrow[d] & B_1 \times B_2 \arrow[d] \\ A_1 \times A_2 \arrow[r] & M_1 \times M_2 \end{tikzcd}$$

with core $E^*_1 \times E^*_2$. In the case that $M_1 = M_2$, we also define the direct sum of D_1 and D_2 as $D_1 \oplus D_2 = (D_1 \times D_2)|_M$, where M is identified with its diagonal in $D_1 \times D_2$. This gives a $\mathcal{DVB}$

of the form

$$\begin{tikzcd} D_1 \oplus D_2 \arrow[r] \arrow[d] & B_1 \oplus B_2 \arrow[d] \\ A_1 \oplus A_2 \arrow[r] & M \end{tikzcd}$$

with core $E_1^* \oplus E_2^*$. Here the $\oplus$ notation for the side bundles and the core is used to denote the Whitney sum of vector bundles.

5. *Horizontal and vertical duals.* Recall that the total space D of a $\mathscr{DVB}$ has two vector bundle structures, $D \to B$ and $D \to A$. Dualizing each of these bundles, we obtain two new spaces D^h (the horizontal dual of D) and D^v (the vertical dual of D) respectively. As it turns out, each of these spaces defines a $\mathscr{DVB}$:

$$\begin{tikzcd} D^h \arrow[r] \arrow[d] & B \arrow[d] \\ E \arrow[r] & M, \end{tikzcd} \qquad \begin{tikzcd} D^v \arrow[r] \arrow[d] & E \arrow[d] \\ A \arrow[r] & M. \end{tikzcd}$$

Note that there are a wide range of different notations for these $\mathscr{DVB}$s in the literature, including D_B^* and D_A^* (e.g., [29]), and $D \overset{*}{\times} B$ and $D \overset{*}{\times} A$ (e.g.,[27]). The horizontal and vertical scalar multiplication on D^h are characterized by the property

$$\langle \kappa_t^h(\phi), d\rangle = t\langle \phi, d\rangle = \langle \kappa_t^v(\phi), \kappa_t^v(d)\rangle,$$

for $\phi \in D^h,\ d \in D$ in the same fibre over B. Similarly, for $\psi \in D^v,\ d \in D$ in the same fibre over A,

$$\langle \kappa_t^h(\psi), \kappa_t^h(d)\rangle = t\langle \psi, d\rangle = \langle \kappa_t^v(\psi), d\rangle.$$

One finds $\mathrm{core}(D^h) = A^*,\ \mathrm{core}(D^v) = B^*$. Clearly, $\mathrm{flip}(D^v) = \mathrm{flip}(D)^h$. Furthermore, it was discovered by Mackenzie [53, Theorem 3.1] that the vector bundles $D^h \to E$ and $D^v \to E$ are dual to one another. We will explore these structures more in section 2.6

6. *Hom Spaces.* Let D_1 and D_2 be two $\mathscr{DVB}$s as in 4 above, and let $f\colon M_1 \to M_2$ be a smooth map. Denote by $\mathrm{Hom}_f(D_1, D_2)$ the set of $\mathscr{DVB}$ morphisms whose restriction to M_1 is f. Then $\mathrm{Hom}_f(D_1, D_2)$ determines a double vector bundle, with scalar multiplications defined by $\kappa^h\varphi = \varphi \circ \kappa_{D_1}^h$ and $\kappa^v\varphi = \varphi \circ \kappa_{D_1}^v$ for all $\varphi \in \mathrm{Hom}_f(D_1, D_2)$. Its diagram is given by

$$\begin{tikzcd} \mathrm{Hom}_f(D_1, D_2) \arrow[r] \arrow[d] & \mathrm{Hom}_f(B_1, B_2) \arrow[d] \\ \mathrm{Hom}_f(A_1, A_2) \arrow[r] & M_1 \end{tikzcd} ,$$

where the side bundles are spaces of vector bundle morphisms with base map f. It is straightforward to check that the set of all maps on which the two scalar multiplications agree coincides with the set of maps whose image lies in E_2^*, so the core is given by $\mathrm{Hom}_f(D_1, E_2^*)$ (here we view E_2^* as a $\mathscr{DVB}$ with trivial side bundles).

7. *Quotients.* Contrary to the case of ordinary vector bundles, a sub-$\mathscr{DVB}$ of D is not enough on

its own to determine a quotient. Indeed, an arbitrary sub-$\mathcal{DVB}$ does not, in general, induce any natural equivalence relation on D for which the quotient is a double vector bundle. For this reason, quotients of double vector bundles do not appear in the literature for the most part (although the special case that the sub-$\mathcal{DVB}$ has the same side bundles as D is discussed in [48, Section 2.5.2]). In section 2.10, we will determine the additional structure needed for a sub-$\mathcal{DVB}$ to determine a quotient $\mathcal{DVB}$.

2.4 Splittings

Let D be a double vector bundle over M, with side bundles A, B and with $\mathrm{core}(D) = E^*$. A *splitting* (or *decomposition*) of D is a $\mathcal{DVB}$ isomorphism

$$D \to A \times_M B \times_M E^*,$$

inducing the identity on A, B, E^*. Here the $\mathcal{DVB}$ structure on the right hand side is as in Example 2.2(1).

Example 2.4.1. Let $V \to M$ be a vector bundle, and TV its tangent bundle regarded as a $\mathcal{DVB}$. A splitting of TV is equivalent to a linear connection ∇ on V. (Cf. [29, Example 2.12].)

Theorem 2.4.2. *Every double vector bundle admits a splitting.*

Remark 2.4.3. This result was stated in [29] with a reference to [23]; a detailed proof was given in the Ph.D. thesis of del Carpio-Marek [17]. (The recent paper [32] by Heuer and Jotz Lean generalizes this result to n-fold vector bundles.) Below we present a somewhat shorter argument.

Proof. Regard D and $A \times_M B$ as vector bundles over A; their restrictions to the submanifold $M \subseteq A$ are canonically $B \oplus E^*$ and B, respectively. The surjective submersion $\varphi\colon D \to A \times_M B$ from (2.3), regarded as a morphism of vector bundles over A, restricts along M to the obvious projection $B \oplus E^* \to B$. This restriction has a canonical splitting $B \to B \oplus E^*, b \mapsto (b, 0)$. Choose any extension to a splitting of vector bundles over A,

$$\psi_1\colon A \times_M B \to D.$$

Then ψ_1 intertwines the vertical scalar multiplications κ_t^v, but not necessarily the horizontal scalar multiplications. Applying the normal bundle functor, we obtain a $\mathcal{DVB}$ morphism

$$\nu(\psi_1)\colon \nu(A \times_M B, B) \to \nu(D, B).$$

But recall that for any vector bundle $V \to M$, one has a canonical isomorphism $TV|_M = V \oplus TM$, giving rise to an isomorphism of vector bundles $\nu(V, M) \cong V$. In a similar fashion, we have canonical $\mathcal{DVB}$ isomorphisms $\nu(D, B) \cong D$ and $\nu(A \times_M B, B) \cong A \times_M B$. Under these identifications, $\nu(\psi_1) =: \psi$ is the desired splitting $A \times_M B \to D$. ☐

Remark 2.4.4. As a consequence of the theorem above, any two double vector bundles with isomorphic side bundles and isomorphic cores are isomorphic as $\mathcal{DVB}$s. When we wish to emphasize a stronger notion of equivalence, we will use the term *canonical*. By a canonical $\mathcal{DVB}$ isomorphism, we mean one that does not depend on a choice of splittings.

Combining the existence of splittings with local trivializations of A, B, E^*, we see in particular that every double vector bundle D is a fibre bundle over its base manifold M, with bundle projection $q = \kappa_0^h \kappa_0^v \colon D \to M$. Its fibres $D_m = q^{-1}(m)$ are *double vector spaces* (i.e., double vector bundles over a point).

2.5 The associated principal bundle

Given non-negative integers $n_1, n_2, n_3 \in \mathbb{Z}_{\geq 0}$, put $A_0 = \mathbb{R}^{n_1},\ B_0 = \mathbb{R}^{n_2},\ E_0 = \mathbb{R}^{n_3}$, and let D_0 be the double vector space

$$D_0 = A_0 \times B_0 \times E_0^* \tag{2.6}$$

with $\kappa_t^h(a, b, \varepsilon) = (a, tb, t\varepsilon)$ and $\kappa_t^v(a, b, \varepsilon) = (ta, b, t\varepsilon)$. (Cf. Example 2.2(1).) For any vector space V_0, we denote by $\mathrm{GL}(V_0)$ its general linear group; it comes with standard representations on V_0 and on the dual space V_0^*. Thus, $\mathrm{GL}(A_0) \times \mathrm{GL}(B_0) \times \mathrm{GL}(E_0)$ has a standard action on the double vector space (2.6).

Lemma 2.5.1. *[28] The group of $\mathcal{DVB}$ automorphisms of $D_0 = A_0 \times B_0 \times E_0^*$ is a semi-direct product*

$$\mathrm{Aut}(D_0) = (\,\mathrm{GL}(A_0) \times \mathrm{GL}(B_0) \times \mathrm{GL}(E_0)) \ltimes (A_0^* \otimes B_0^* \otimes E_0^*). \tag{2.7}$$

with the standard action of $\mathrm{GL}(A_0) \times \mathrm{GL}(B_0) \times \mathrm{GL}(E_0)$, *and with* $\omega \in A_0^* \otimes B_0^* \otimes E_0^*$ *acting as*

$$(a, b, \varepsilon) \mapsto (a, b, \varepsilon + \omega(a, b)). \tag{2.8}$$

Proof. First we define the abelian group of *statomorphisms* of D_0 (this terminology comes from [28]):

$$\mathrm{Stat}(D_0) = \{\varphi \in \mathrm{Aut}(D_0) \mid \varphi|_{A_0} = \mathrm{id}_{A_0},\ \varphi|_{B_0} = \mathrm{id}_{B_0},\ \varphi|_{E_0^*} = \mathrm{id}_{E_0^*}\}.$$

Then by definition, we get an exact sequence of Lie groups

$$1 \to \mathrm{Stat}(D_0) \to \mathrm{Aut}(D_0) \to \mathrm{GL}(A_0) \times \mathrm{GL}(B_0) \times \mathrm{GL}(E_0) \to 1.$$

Since this sequence is right split (by the standard action of $\mathrm{GL}(A_0) \times \mathrm{GL}(B_0) \times \mathrm{GL}(E_0)$ on D_0), it will be enough to prove that $\mathrm{Stat}(D_0) = A_0^* \otimes B_0^* \otimes E_0^*$. First note that any $\omega \in A_0^* \otimes B_0^* \otimes E_0^*$ clearly defines a statomorphism via (2.8). Now let $\varphi \in \mathrm{Stat}(D_0)$ and write $\varphi = (\varphi_A, \varphi_B, \varphi_{E^*})$. Since φ is a $\mathcal{DVB}$ morphism, we get in particular that $\varphi_A(a, sb, s\varepsilon) = \varphi_A(a, b, \varepsilon)$ for all $s \in \mathbb{R}$. Setting $s = 0$ yields $\varphi_A(a, b, \varepsilon) = \varphi_A(a, 0, 0) = (a, 0, 0)$ since φ is a statomorphism. Similarly, $\varphi_B(a, b, \varepsilon) = (0, b, 0)$, so we are left with determining φ_{E^*}. A calculation analogous to the one above shows that $\varphi_{E^*}(a, 0, 0) = \varphi_{E^*}(0, b, 0) = 0$, and so we get

$$\varphi_{E^*}(a, b, \varepsilon) = \varphi_{E^*}(a, 0, \varepsilon) + \varphi_{E^*}(a, b, 0) = \varphi_{E^*}(0, 0, \varepsilon) + \varphi_{E^*}(a, b, 0) = \varepsilon + \varphi_{E^*}(a, b, 0),$$

where we have used the fact that φ preserves the two additions in D_0 (see example 1). But by equivariance with respect to κ^h and κ^v, $\varphi_{E^*}(ta, b, 0) = \varphi_{E^*}(0, tb, 0) = t\varphi_{E^*}(a, b, 0)$, which shows that $\varphi_{E^*}(-, -, 0) \colon A \otimes B \to E^*$ is bilinear, completing the proof. ∎

As a consequence of this result, we obtain more information about the splitting of double vector bundles, obtained by consider the space of statomorphisms from $A \times_M B \times_M E^*$ to itself.

Corollary 2.5.2. *The set of splittings of D is an affine space with $\Gamma(A^* \otimes B^* \otimes E^*)$ as its space of translations.*

Given a double vector bundle D, take n_1, n_2, n_3 to be the ranks of the bundles A, B, E. An isomorphism of double vector spaces $D_m \to D_0$ will be called a *frame of D at $m \in M$*. Clearly, any two frames are related by the action of $\mathrm{Aut}(D_0)$. We define the *frame bundle* of D to be the principal $\mathrm{Aut}(D_0)$-bundle $P \to M$ whose fibres P_m are the set of frames at m.

Remark 2.5.3. In [44], the semi-direct product (2.7) is regarded as a *double Lie group*, and a general theory of double principal bundles for double Lie groups is developed.

Many constructions with double vector bundles may be expressed in terms of bundles associated to P. In particular, D is itself an associated bundle for the action (2.8),

$$D = (P \times D_0)/\mathrm{Aut}(D_0). \tag{2.9}$$

(Pradines' original definition [72] of double vector bundles was in terms of local trivializations with $\mathrm{Aut}(D_0)$-valued transition functions.) A splitting of D amounts to a reduction of the structure group to $\mathrm{GL}(A_0) \times \mathrm{GL}(B_0) \times \mathrm{GL}(E_0) \subseteq \mathrm{Aut}(D_0)$; the fibres Q_m of the reduction $Q \subseteq P$ are all those $\mathcal{DVB}$ isomorphisms $D_m \to D_0$ that also preserve the splittings.

Let D_0^- be equal to D_0 as a double vector space, but with the new action of $\mathrm{Aut}(D_0)$, where $\omega \in A_0^* \otimes B_0^* \otimes E_0^*$ acts by $(a, b, \varepsilon) \mapsto (a, b, \varepsilon - \omega(a, b))$, while $\mathrm{GL}(A_0) \times \mathrm{GL}(B_0) \times \mathrm{GL}(E_0)$ acts in the standard way. The resulting double vector bundle

$$D^- = (P \times D_0^-)/\mathrm{Aut}(D_0).$$

will feature in some of the constructions below.

Lemma 2.5.4. *There is a canonical $\mathcal{DVB}$ isomorphism $D^- \to D$ that is the identity on the side bundles but minus the identity on the core.*

Proof. The isomorphism is induced by the $\mathrm{Aut}(D_0)$-equivariant isomorphism of double vector spaces $D_0^- \to D_0,\ (a, b, \varepsilon) \mapsto (a, b, -\varepsilon)$. □

2.6 Triality of double vector bundles

By cyclic permutation of the roles of A_0, B_0, E_0, the action (2.8) of $\mathrm{Aut}(D_0)$ on $D_0 = A_0 \times B_0 \times E_0^*$ gives rise to similar actions on $D_0' = B_0 \times E_0 \times A_0^*$ and $D_0'' = E_0 \times A_0 \times B_0^*$. The bilinear pairings

$$D_0 \times_{B_0} D_0' \to \mathbb{R},\ \ ((a, b, \varepsilon), (b, e, \alpha)) \mapsto \alpha(a) - \varepsilon(e), \tag{2.10}$$

and similar maps given by cyclic permutation, are $\mathrm{Aut}(D_0)$-equivariant. Taking associated bundles, we obtain three double vector bundles

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} D' \arrow[r] \arrow[d] & E \arrow[d] \\ B \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} D'' \arrow[r] \arrow[d] & A \arrow[d] \\ E \arrow[r] & M \end{tikzcd}$$

with bilinear pairings

$$D \times_B D' \to \mathbb{R}, \quad D' \times_E D'' \to \mathbb{R}, \quad D'' \times_A D \to \mathbb{R} \tag{2.11}$$

The bundles D', D'' are closely related to the horizontal and vertical duals:

Proposition 2.6.1. *There are canonical $\mathscr{DVB}$ isomorphisms*

$$D^h \cong \mathrm{flip}(D')^-, \quad D^v \cong \mathrm{flip}(D'')^-$$

that are the identity on the side bundles and on the core.

Proof. We give the proof for D^h (the argument for D^v is similar). It suffices to consider the double vector space D_0. Write D as an associated bundle (2.9). Then

$$D^h = (P \times D_0^h)/\mathrm{Aut}(D_0)$$

where the $\mathrm{Aut}(D_0)$-action on $D_0^h = E_0 \times B_0 \times A_0^*$ is given by the standard action of $\mathrm{GL}(A_0) \times \mathrm{GL}(B_0) \times \mathrm{GL}(E_0)$, while $\omega \in A_0^* \otimes B_0^* \otimes E_0^*$ acts as

$$(e, b, \alpha) \mapsto (e, b, \alpha - \omega(b, e)).$$

This action is dictated by invariance of the duality pairing

$$D_0 \times_{B_0} D_0^h \to \mathbb{R}, \ ((a, b, \varepsilon), (e, b, \alpha)) \mapsto \alpha(a) + \varepsilon(e).$$

The $\mathscr{DVB}$-isomorphism $D_0^h \to \mathrm{flip}(D_0')$, $(e, b, \alpha) \mapsto (e, b, -\alpha)$ is $\mathrm{Aut}(D_0)$-equivariant, and induces a $\mathscr{DVB}$-isomorphism $D^h \to \mathrm{flip}(D')$ that is the identity on the sides but minus the identity on the core. Now use Lemma 2.5.4. □

Remark 2.6.2. Using the isomorphisms from Proposition 2.6.1, the second pairing in (2.11) translates into Mackenzie's pairing $D^v \times_E D^h \to \mathbb{R}$ [53, Theorem 3.1]. We also recover the result of Mackenzie [53] and Konieczna and Urbański [38], giving a canonical $\mathscr{DVB}$ isomorphism

$$((D^h)^v)^h \cong ((D^v)^h)^v$$

that is the identity on the side bundles and on the core; indeed, by iteration of Proposition 2.6.1 we see that both are identified with D^-. As a special case, if $D = T^*V$ we have that $D^v = TV$, $(D^v)^h = TV^*$,

$((D^v)^h)^v = T^*V^*$, and we recover the canonical $\mathcal{DVB}$ isomorphism

$$T^*(V^*) \cong \mathrm{flip}(T^*V)^- \tag{2.12}$$

of Mackenzie-Xu [57].

Example 2.6.3. Consider $D = TV$ as a double vector bundle with sides $A = V$ and $B = TM$. The natural pairing between tangent and cotangent vectors identifies $D^v = T^*V$, while the tangent prolongation of the pairing $V \times_M V^* \to \mathbb{R}$ identifies $D^h = TV^*$. The three double vector bundles D, D', D'' are therefore,

$$\begin{array}{ccc} TV \longrightarrow TM & \mathrm{flip}(T(V^*))^- \longrightarrow V^* & \mathrm{flip}(T^*V)^- \longrightarrow V \\ \downarrow \qquad\quad \downarrow & \downarrow \qquad\qquad\qquad \downarrow & \downarrow \qquad\qquad \downarrow \\ V \longrightarrow M & TM \longrightarrow M & V^* \longrightarrow M \end{array}$$

Put differently, there is a canonical $\mathcal{DVB}$ isomorphism $(TV)' \to \mathrm{flip}(T(V^*))$ that is the identity on the side bundles and minus identity on the core V^*, and a canonical $\mathcal{DVB}$ isomorphism $(TV)'' \to \mathrm{flip}(T^*V)$ that is the identity on the side bundles and minus the identity on the core T^*M.

2.7 Double-linear functions

For any vector bundle $V \to M$, the fibrewise linear functions on V are identified with the sections of the dual bundle V^*. We will similarly associate to any double vector bundle D a vector bundle whose space of sections are the functions on D that are *double-linear*, i.e., linear for both scalar multiplications. Throughout this discussion, we will find it convenient to present D as an associated bundle $D = (P \times D_0)/\mathrm{Aut}(D_0)$ with $D_0 = A_0 \times_M B_0 \times_M E_0^*$. Consider the $\mathrm{Aut}(D_0)$-action on

$$\widehat{E}_0 = (A_0^* \otimes B_0^*) \oplus E_0,$$

where $\mathrm{GL}(A_0) \times \mathrm{GL}(B_0) \times \mathrm{GL}(E_0)$ acts in the standard way, while elements $\omega \in A_0^* \otimes B_0^* \otimes E_0^* \cong \mathrm{Hom}(E_0, A_0^* \otimes B_0^*)$ act as

$$(\nu, e) \mapsto (\nu - \omega(e), e).$$

The projection $\widehat{E}_0 \to E_0$ is $\mathrm{Aut}(D_0)$-equivariant, with kernel $A_0^* \otimes B_0^*$. Taking associated bundles, we obtain a vector bundle

$$\widehat{E} = (P \times \widehat{E}_0)/\mathrm{Aut}(D_0)$$

with an exact sequence of vector bundles over M,

$$0 \longrightarrow A^* \otimes B^* \xrightarrow{i_{\widehat{E}}} \widehat{E} \longrightarrow E \longrightarrow 0. \tag{2.13}$$

Proposition 2.7.1. *The space of sections of $\widehat{E}$ is canonically isomorphic to the space of double-linear functions on D. Under this identification, the quotient map to E is given by restriction of double-linear functions to* $\mathrm{core}(D) = E^*$*, and the inclusion map $i_{\widehat{E}}$ is given by the multiplication of pull-backs of linear functions on A and on B.*

Proof. It suffices to prove these claims for the double vector space D_0. Using a Taylor expansion, we see that the double-linear functions on $D_0 = A_0 \times B_0 \times E_0^*$ are $\widehat{E}_0 = (A_0^* \otimes B_0^*) \oplus E_0$, where E_0 is interpreted as linear functions on E_0^* and $A_0^* \otimes B_0^*$ as linear combinations of products of linear functions on A_0, B_0. $\square$

In terms of this interpretation through double-linear functions, the exact sequence (2.13) was discussed by Chen-Liu-Sheng [13] as the dual of their $\mathcal{DVB}$ *sequence*. Since we prefer to work with (2.13) rather than its dual, we will simply call (2.13) the $\mathcal{DVB}$ sequence of D. For clarity, we will emphasize this point by making it a definition.

Definition 2.7.2 ($\mathcal{DVB}$ sequence)**.** The sequence (2.13) (and *not its dual*) is called the $\mathcal{DVB}$ *sequence* of D.

A central result of [13] is that the double vector bundle D may be recovered from its $\mathcal{DVB}$ sequence. We will prove this result using the associated bundle construction.

Proposition 2.7.3. *[13] The double vector bundle D is the sub-double vector bundle of*

$$\widehat{D} = A \times_M B \times_M \widehat{E}^*$$

consisting of all $(a, b, \widehat{\varepsilon}) \in A \times_M B \times_M \widehat{E}^*$ such that $i_{\widehat{E}}^*(\widehat{\varepsilon}) = a \otimes b$. A splitting of D is equivalent to a splitting of the exact sequence (2.13).

Proof. We will prove the first claim for the typical fibre D_0, with the general result then being a consequence of the associated bundle construction. Define

$$\Omega(D_0) = \{(a, b, \widehat{\varepsilon}) \in A_0 \times_M B_0 \times_M \widehat{E}_0^* \mid i_{\widehat{E}}^*(\widehat{\varepsilon}) = a \otimes b\}.$$

Since the $\mathcal{DVB}$ sequence (2.13) for D_0 is canonically split, $i_{\widehat{E}_0}^* : A \otimes B \oplus E^* \to A \otimes B$ is simply projection to the first factor. This gives

$$\Omega(D_0) = \{(a, b, a \otimes b + \varepsilon) \in A_0 \times_M B_0 \times_M \widehat{E}_0^*\}.$$

Then the map $D_0 \to A_0 \times_M B_0 \times_M \widehat{E}_0^*$ defined by $(a, b, \varepsilon) \mapsto (a, b, a \otimes b + \varepsilon)$ is clearly an injective $\mathcal{DVB}$ morphism with image $\Omega(D_0)$.

For the second claim, note that any splitting of the sequence (2.13) induces an isomorphism $\widehat{E} \cong A^* \otimes B^* \oplus E$ under which $i_{\widehat{E}}^*$ is simply projection. Then by what we have proven above, we obtain an identification

$$D \cong \{(a, b, a \otimes b + \varepsilon) \mid a \in A, b \in B, \varepsilon \in E^*\}.$$

Composing this idenfication with the map $(a, b, a \otimes b + \varepsilon) \mapsto (a, b, \varepsilon)$ yields the desired $\mathcal{DVB}$ splitting $D \xrightarrow{\sim} A \times_M B \times_M E^*$. $\square$

Remark 2.7.4. A direct consequence is that every double vector bundle D comes with a map

$$D \to \widehat{E}^* \tag{2.14}$$

given by the inclusion $D \hookrightarrow \hat{D}$ followed by projection to $\widehat{E}^*$. This map is a $\mathcal{DVB}$-morphism if the vector bundle $\widehat{E}^*$ is regarded as a double vector bundle (with zero sides). In terms of the associated bundle construction, this is induced by the map

$$D_0 = A_0 \times B_0 \times E_0^* \to \widehat{E}_0^* = (A_0 \otimes B_0) \oplus E_0^*, \quad (a, b, \varepsilon) \mapsto a \otimes b + \varepsilon.$$

Remark 2.7.5. The inclusion $D \hookrightarrow \widehat{D}$ dualizes to a surjective $\mathcal{DVB}$ morphism

$$\widehat{D}' = B \times_M \widehat{E} \times_M A^* \to D'.$$

Replacing D' with D (thus D with D''), this shows that every double vector bundle also arises as a *quotient* of a split double vector bundle.

Using a similar construction for the bundles $D',\ D''$, we obtain bundles $\widehat{A}$ and $\widehat{B}$, with inclusion maps

$$i_{\widehat{E}}\colon A^* \otimes B^* \to \widehat{E}, \quad i_{\widehat{A}}\colon B^* \otimes E^* \to \widehat{A}, \quad i_{\widehat{B}}\colon E^* \otimes A^* \to \widehat{B}, \tag{2.15}$$

and exact sequences obtained from (2.13) by cyclic permutation of A, B, E. In Section 2.9 below we will identify the bundles $\widehat{A}, \widehat{B}$ with those introduced by Gracia-Saz and Mehta [29]; the corresponding exact sequences appear as Equation (26) in that reference.

Remark 2.7.6. Since a splitting of D is equivalent to a splitting of D', D'', we see that a splitting of D is equivalent to a splitting of any one of the three vector bundle maps $\widehat{A} \to A,\ \widehat{B} \to B$ or $\widehat{E} \to E$.

2.8 The three pairings

In what follows, we will denote elements of the bundles $\widehat{A}, \widehat{B}, \widehat{E}$ by $\widehat{a}$, $\widehat{b}$, $\widehat{e}$, and their images in A, B, E by a, b, e.

Proposition 2.8.1. *There are canonical bilinear pairings*

$$\begin{aligned} \langle\cdot,\cdot\rangle_{E^*}&\colon \widehat{B} \times_M \widehat{A} \to E^*,\\ \langle\cdot,\cdot\rangle_{A^*}&\colon \widehat{E} \times_M \widehat{B} \to A^*,\\ \langle\cdot,\cdot\rangle_{B^*}&\colon \widehat{A} \times_M \widehat{E} \to B^*, \end{aligned} \tag{2.16}$$

with the properties

$$\langle \widehat{b},\ i_{\widehat{A}}(\mu)\rangle_{E^*} = \mu(b), \quad \mu \in B^* \otimes E^*,\ \widehat{b} \in \widehat{B}, \tag{2.17}$$

$$\langle i_{\widehat{B}}(\nu),\ \widehat{a}\rangle_{E^*} = -\nu(a), \quad \nu \in E^* \otimes A^*,\ \widehat{a} \in \widehat{A}, \tag{2.18}$$

and similar properties obtained by cyclic permutations of A, B, E. The pairings are related by the identity

$$\langle \widehat{b}, \widehat{a}\rangle_{E^*}(e) + \langle \widehat{e}, \widehat{b}\rangle_{A^*}(a) + \langle \widehat{a}, \widehat{e}\rangle_{B^*}(b) = 0. \tag{2.19}$$

Proof. Using the associated bundle construction, it suffices to define the corresponding pairings for the double vector space D_0, and check that they are $\mathrm{Aut}(D_0)$-equivariant. We have

$$\widehat{A}_0 = (B_0^* \otimes E_0^*) \oplus A_0, \quad \widehat{B}_0 = (E_0^* \otimes A_0^*) \oplus B_0, \quad \widehat{E}_0 = (A_0^* \otimes B_0^*) \oplus E_0.$$

Put

$$\langle\cdot,\cdot\rangle_{E_0^*}\colon \widehat{B}_0\times\widehat{A}_0\to E_0^*,\quad \langle(\nu,b),\,(\mu,a)\rangle_{E_0^*}=\mu(b)-\nu(a); \tag{2.20}$$

This is clearly equivariant for the actions of $\mathrm{GL}(A_0)\times\mathrm{GL}(B_0)\times\mathrm{GL}(E_0)$. For the action of $\omega\in A_0^*\otimes B_0^*\otimes E_0^*$, observe that in the pairing between

$$\omega.(\mu,a)=(\mu-\omega(a),a),\quad \omega.(\nu,b)=(\nu-\omega(b),b),$$

the terms involving ω cancel. The properties (2.17) and (2.18) hold by definition. Furthermore, given $\widehat{a}=(\mu,a)\in\widehat{A}_0$, $\widehat{b}=(\nu,b)\in\widehat{B}_0$, $\widehat{e}=(\rho,e)\in\widehat{E}_0$ the three terms in (2.19) are $\mu(e,a)-\nu(b,e)$, $\rho(a,b)-\mu(e,a)$ and $\nu(b,e)-\rho(a,b)$, hence their sum is zero. □

Replacing D with $\mathrm{flip}(D)$ reverses the role of A and B. Hence, the three inclusion maps (2.15) are unchanged, but the three pairings (2.16) all change sign. On the other hand, for D^- we have:

Proposition 2.8.2. *The bundles $\widehat{A},\widehat{B},\widehat{C}$ for D are canonically isomorphic to those for D^-. Under this identification, replacing D with D^- changes the signs of the inclusion maps (2.15) and also of the pairings (2.16).*

Proof. Let $\widehat{A}^-,\widehat{B}^-,\widehat{E}^-$ be the corresponding bundles for D^-. The $\mathrm{Aut}(D_0)$-equivariant isomorphism

$$\widehat{E}_0^-\to\widehat{E}_0,\ (\nu,c)\mapsto(-\nu,c)$$

gives the desired isomorphism $\widehat{E}^-\to\widehat{E}$, and similar for $\widehat{A}^-$, $\widehat{B}^-$. One readily checks that these isomorphisms give sign changes for the inclusions and pairings. □

2.9 Geometric interpretations

The bundles $\widehat{A},\widehat{B},\widehat{E}$ and the pairings between them have various geometric interpretations, in terms of functions and vector fields on D.

We begin by recalling analogous interpretations for vector bundles $V\to M$. The space $\mathfrak{X}(V)_{[r]}$ of vector fields on V that are homogeneous of degree r for the scalar multiplication (i.e., $\kappa_t^*X=t^rX$ for $t\neq 0$) is trivial if $r<-1$, while the *core* and *linear* vector fields

$$\mathfrak{X}(V)_{[-1]}:=\mathfrak{X}_{\mathrm{core}}(V),\quad \mathfrak{X}(V)_{[0]}=:\mathfrak{X}_{\mathrm{lin}}(V) \tag{2.21}$$

are identified with sections of V (via the *vertical lift*, taking a section $\sigma\in\Gamma(V)$ to the corresponding fibrewise constant vector field $\sigma^\sharp$), and infinitesimal automorphisms of V, respectively. On the other hand, $C^\infty(V)_{[0]}=C^\infty(M)$ and $C^\infty(V)_{[1]}=\Gamma(V^*)$.

The pairing $V\times_M V^*\to\mathbb{R}$ is realized as the map $\mathfrak{X}(V)_{[-1]}\otimes C^\infty(V)_{[1]}\to C^\infty(V)_{[0]}$ given by Lie derivative, $X\otimes f\mapsto \mathcal{L}_Xf$. We can also take a dual viewpoint ('Fourier transform'), using the identifications $C^\infty(V^*)_{[0]}=C^\infty(M)$, $C^\infty(V^*)_{[1]}=\Gamma(V)$, $\mathfrak{X}(V^*)_{[-1]}=\Gamma(V^*)$. Here, we realize the pairing $V\times_M V^*\to\mathbb{R}$ as *minus* the Lie derivative, $h\otimes Z\mapsto -\mathcal{L}_Zh$. (Working with multi-vector fields, it is convenient to think of these pairings as Schouten brackets (see section A.1) between 1-vector fields and 0-vector fields.)

For a double vector bundle, let $\mathfrak{X}(D)_{[k,l]}$ be the space of vector fields on D that are homogeneous of degree k horizontally and of degree l vertically. Similar notation will be used for smooth functions, differential forms, and soon. Let $\Gamma(D,A)$ be the sections of D as a vector bundle over A, and $\Gamma_{\mathrm{lin}}(D,A)$ the subspace of sections that are homogeneous of degree 0 horizontally, i.e., such that the corresponding map $A \to D$ is κ^h_t-equivariant. Linear sections of D over B are defined similarly. We have

$$\Gamma_{\mathrm{lin}}(D,A) \cong \mathfrak{X}(D)_{[0,-1]}, \quad \Gamma_{\mathrm{lin}}(D,B) \cong \mathfrak{X}(D)_{[-1,0]}, \quad \Gamma(E^*) \cong \mathfrak{X}(D)_{[-1,-1]}. \tag{2.22}$$

To verify (2.22), note that vector fields $X \in \mathfrak{X}(D)_{[k,l]}$ with $k=-1$ or $l=-1$ annihilate $C^\infty(D)_{[0,0]} = C^\infty(M)$, and are thus vertical for the bundle projection $D \to M$. Hence, it suffices to check for the double vector space $D_0 = A_0 \times B_0 \times E_0^*$. But

$$\mathfrak{X}(D_0)_{[0,-1]} \cong B_0 \oplus (A_0^* \otimes E_0^*), \quad \mathfrak{X}(D_0)_{[-1,0]} = A_0 \oplus (B_0^* \otimes E_0^*), \quad \mathfrak{X}(D_0)_{[-1,-1]} \cong E_0^*,$$

where elements of a vector space are seen as constant vector fields on the vector space, and elements of the dual space as linear functions. Indeed, with this interpretation the elements of A_0, B_0, E_0^* have homogeneity bidegrees $(-1,0), (0,-1), (-1,-1)$ respectively, while elements of A_0^*, B_0^*, E_0 have homogeneity bidegrees $(1,0), (0,1), (1,1)$.

Proposition 2.9.1. *The space of sections of $\widehat{E}$ is canonically isomorphic to*

1. *the space $C^\infty(D)_{[1,1]}$ of double-linear functions on D,*
2. *the space $\Gamma_{\mathrm{lin}}(D',B)$ of linear sections of D' over B,*
3. *the space $\Gamma_{\mathrm{lin}}(D'',A)$ of linear sections of D'' over A,*
4. *the space $\mathfrak{X}(D')_{[0,-1]}$ of vector fields on D' of homogeneity $(0,-1)$,*
5. *the space $\mathfrak{X}(D'')_{[-1,0]}$ of vector fields on D'' of homogeneity $(-1,0)$.*

Similar descriptions hold for sections of $\widehat{A}, \widehat{B}$.

Proof. It suffices to prove these descriptions for the double vector spaces $D_0 = A_0 \times B_0 \times E_0^*$. We have already remarked that $\widehat{E}_0$ is the space of double-linear functions on D_0. For (b), note that sections of $D_0' = B_0 \times E_0 \times A_0^*$ over B_0 are smooth functions $B_0 \to E_0 \times A_0^*$; such a function defines a linear section if and only if its first component is a constant map $B_0 \to E_0$, while its second component is a linear map $B_0 \to A_0^*$. Hence, we obtain $\Gamma_{\mathrm{lin}}(D_0', B_0) = (A_0^* \otimes B_0^*) \oplus E_0 = \widehat{E}_0$. The proof of (c) is similar, and by the counterparts of (2.22) for D_0', D_0'' the properties (b),(c) are equivalent to (d),(e). ☐

Remark 2.9.2. In the work of Gracia-Saz and Mehta [29, Section 2.4], the isomorphism $\Gamma(\widehat{A}) \cong \Gamma_{\mathrm{lin}}(D,B)$ is used as the definition of $\widehat{A}$.

Using these geometric interpretations, the three inclusion maps (2.15) are realized as the bilinear maps

$$\begin{aligned}
i_{\widehat{E}}\colon\ & C^\infty(D)_{[1,0]} \times C^\infty(D)_{[0,1]} \to C^\infty(D)_{[1,1]}, && (f,g) \mapsto fg\\
i_{\widehat{A}}\colon\ & C^\infty(D)_{[0,1]} \times \mathfrak{X}(D)_{[-1,-1]} \to \mathfrak{X}(D)_{[-1,0]}, && (g,Z) \mapsto gZ\\
i_{\widehat{B}}\colon\ & \mathfrak{X}(D)_{[-1,-1]} \times C^\infty(D)_{[1,0]} \to \mathfrak{X}(D)_{[0,-1]}, && (Z,f) \mapsto fZ
\end{aligned} \tag{2.23}$$

while the three pairings (2.16) are

$$\begin{aligned}
&\langle\cdot,\cdot\rangle_{E^*} \quad \mathfrak{X}(D)_{[0,-1]} \times \mathfrak{X}(D)_{[-1,0]} \to \mathfrak{X}(D)_{[-1,-1]}, \quad (X,Y) \mapsto [X,Y] \\
&\langle\cdot,\cdot\rangle_{A^*} \quad C^\infty(D)_{[1,1]} \times \mathfrak{X}(D)_{[0,-1]} \to C^\infty(D)_{[1,0]}, \quad (h,X) \mapsto -\mathcal{L}_X h \\
&\langle\cdot,\cdot\rangle_{B^*} \quad \mathfrak{X}(D)_{[-1,0]} \times C^\infty(D)_{[1,1]} \to C^\infty(D)_{[0,1]}, \quad (Y,h) \mapsto \mathcal{L}_Y h.
\end{aligned} \tag{2.24}$$

The identity (2.19) just amounts to $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$. To verify (2.23) and (2.24), it is enough to consider the double vector space D_0, but there it follows by a routine check from the definitions.

2.10 Quotients of double vector bundles

In [63], a theory of quotients is developed in the category of *graded multi-bundles*. The key insight is that quotients are determined not by an action of the sub-object itself, but rather by the action of a suitable Lie algebra bundle. Double vector bundles are a special case of graded multi-bundles, and so in this section we review Meinrenken's quotient construction for $\mathcal{DVB}$s (theorem 2.10.1), and we apply some of the machinery discussed earlier in this chapter to the study of such quotients. Suppose Q is a sub-$\mathcal{DVB}$ of D with the same total base M. As mentioned in section 2.3, Q alone is not enough to determine a quotient $\mathcal{DVB}$. To understand the additional structure necessary, for any arbitrary double vector bundle D, consider the bigraded vector bundle

$$\mathfrak{g} = \widehat{A} \oplus \widehat{B} \oplus E^*.$$

Using the first pairing described in (2.16), we can give $\mathfrak{g}$ the structure of a bundle of two-step nilpotent Lie algebras. Specifically, the bracket is defined by

$$[(\widehat{a}_1, \widehat{b}_1, \varepsilon_1), (\widehat{a}_2, \widehat{b}_2, \varepsilon_2)] = (0, 0, \langle \widehat{b}_1, \widehat{a}_2\rangle_{E^*} - \langle \widehat{b}_2, \widehat{a}_1\rangle_{E^*}).$$

On the other hand, by proposition 2.9.1 we have the isomorphisms $\Gamma(\widehat{A}) \cong \mathfrak{X}_{[-1,0]}(D)$, $\Gamma(\widehat{B}) \cong \mathfrak{X}_{[0,-1]}(D)$, and $\Gamma(E^*) \cong \mathfrak{X}_{[-1,-1]}(D)$. These identifications combine to define an action of the bigraded Lie algebra bundle $\mathfrak{g}$ on D. To describe the action explicitly, we will use the associated bundle construction. As usual, let D_0 denote a double vector space that will play the role of the generic fibre of D. Then the vector bundle actions of E_0^*, $\widehat{A}_0$, and $\widehat{B}_0$ on D_0 are described below. For any $(a, b, \varepsilon) \in D_0$:

1. the formula
$$\varepsilon_1 \cdot (a, b, \varepsilon) = (a, b, \varepsilon + \varepsilon_1), \quad \varepsilon_2 \in E_0^*$$
defines the *core action* of E_0^* on D_0.

2. The formula
$$(\mu, a_1) \cdot (a, b, \varepsilon) = (a + a_1, b, \varepsilon - \mu(b)), \quad (\mu, a_1) \in \widehat{A}_0 = \mathrm{Hom}(B_0, E_0^*) \oplus A_0$$
defines the *fat bundle action* of $\widehat{A}_0$ on D_0.

3. The formula

$$(\nu, b_1) \cdot (a, b, \varepsilon) = (a, b + b_1, \varepsilon - \nu(a)), \quad (\nu, b_1) \in \widehat{B}_0 = \mathrm{Hom}(A_0, E_0^*) \oplus B_0$$

defines the *fat bundle action* of $\widehat{B}_0$ on D_0.

By taking associated bundles, we obtain the core action and fat bundle actions of E^*, $\widehat{A}$, and $\widehat{B}$ on D. Let us verify that these formulas lead to the correct commutator action. For $(\mu, a_1) \in \widehat{A}_0$ and $(\nu, b_1) \in \widehat{B}_0$, we compute

$$\begin{aligned}
(\mu, a_1)(\nu, b_1)(-\mu, -a_1)(-\nu, -b_1) \cdot (a, b, \varepsilon) &= (\mu, a_1)(\nu, b_1)(-\mu, -a_1) \cdot (a, b - b_1, \varepsilon + \nu(a)) \\
&= (\mu, a_1)(\nu, b_1) \cdot (a - a_1, b - b_1, \varepsilon + \nu(a) + \mu(b) - \mu(b_1)) \\
&= (\mu, a_1) \cdot (a - a_1, b, \varepsilon + \mu(b) - \mu(b_1) + \nu(a_1)) \\
&= (a, b, \varepsilon + \nu(a_1) - \mu(b_1)) \\
&= -\langle (\nu, b_1), (\mu, a_1) \rangle_{E^*} \cdot (a, b, \varepsilon) \\
&= [(\mu, a_1), (\nu, b_1)] \cdot (a, b, \varepsilon),
\end{aligned}$$

as required. Note that this calculation also explains our sign choice in the definition of the fat bundle actions.

Now let us return to the situation above, with $Q \subseteq D$ a sub-$\mathcal{DVB}$, whose diagram is given by

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] & V \arrow[d] \\
H \arrow[r] & M
\end{tikzcd} \tag{2.25}$$

with core K^*, and let $\mathfrak{g}$ denote the Lie algebra bundle described above for D. Like any $\mathcal{DVB}$, Q comes equipped with a core action by the vector bundle K^*. In addition to Q, the extra data that determines a quotient $\mathcal{DVB}$ amounts to a bigraded Lie subalgebra bundle $\mathfrak{h}$ of $\mathfrak{g}$ that preserves the core action of K^*. More specifically, let $\mathfrak{g}_{E^*}$ denote the bundle of abelian ideals that preserve the core E^* of D:

$$\mathfrak{g}_{E^*} = \mathrm{Hom}(B, E^*) \oplus \mathrm{Hom}(A, E^*) \oplus E^*.$$

Then since $K \subseteq E^*$, we can consider the subbundle

$$\mathfrak{g}_{K^*} = \mathrm{Hom}(B, K^*) \oplus \mathrm{Hom}(A, K^*) \oplus K^*.$$

The condition on $\mathfrak{h} \subseteq \mathfrak{g}$ is that

$$\mathfrak{h} \cap \mathfrak{g}_{E^*} = \mathfrak{g}_{K^*}.$$

We make these statements precise in the theorem below.

Theorem 2.10.1 ($\mathcal{DVB}$ quotients)**.** *[63] Let Q be a sub-$\mathcal{DVB}$ of D with diagram (2.25). Then surjective $\mathcal{DVB}$ morphisms with domain D and kernel Q are in bijective correspondence with pairs of subbundles $\widetilde{H} \subseteq \widehat{A}$ and $\widetilde{V} \subseteq \widehat{B}$ such that:*

1. *the map $\widehat{A} \to A$ restricts to a surjection $\widetilde{H} \to H$ with kernel* $\mathrm{Hom}(B, K^*)$*, and*

2. *the map* $\widehat{B} \to B$ *restricts to a surjection* $\widetilde{V} \to V$ *with kernel* $\mathrm{Hom}(A, K^*)$.

The image of such a $\mathcal{DVB}$ *morphism is given by the quotient of* D *by the action of the Lie subalgebra bundle*

$$\mathfrak{h} = \widetilde{H} \oplus \widetilde{V} \oplus K.$$

Proof. First choose any two splittings of D and Q. In terms of these splittings, the inclusion map $i_Q: Q \hookrightarrow D$ takes the form $(h, v, k) \mapsto (h, v, k + \omega(h, v))$ for some $\omega \in \mathrm{Hom}(A \otimes B, C)$. Using corollary 2.5.2, we may arrange that $\omega = 0$. We will therefore assume that $D = A \times_M B \times_M E^*$ and that $Q = H \times_M V \times_M K^*$. If we suppose that such bundles $\widetilde{H}$ and $\widetilde{V}$ exist, then the quotient of $A \times_M B \times_M E^*$ by the action of $\mathfrak{h}$ is simply

$$D/\mathfrak{h} = A/H \times_M B/V \times_M E^*/K^*,$$

which is a decomposed $\mathcal{DVB}$ and comes along with an obvious quotient map.

Now suppose there exists a double vector bundle D/Q and a surjective $\mathcal{DVB}$ morphism $\pi: D \to D/Q$ with kernel Q (the notation D/Q is meant to be merely suggestive). We define $\widetilde{H} \subseteq \widehat{A}$ and $\widetilde{V} \subseteq \widehat{B}$ to be the subbundles whose action on D preserves the fibres of π. Analogous to above, a suitable choice of splittings ensures that π takes the form $\pi(a, b, \varepsilon) = (\pi_1(a), \pi_2(b), \pi_3(\varepsilon))$, and the subgroup of $\widehat{A} = \mathrm{Hom}(B, E^*) \oplus A$ that preserves the fibres of π consists of those (h, μ) for which

$$(\pi_1(a) + \pi_1(h), \pi_2(b), \pi_3(\varepsilon) - \pi_3(\mu(b))) = \pi(a + h, b, \varepsilon - \mu(b)) = \pi(a, b, \varepsilon) = (\pi_1(a), \pi_2(b), \pi_3(\varepsilon))$$

for all $a \in A$, $b \in B$, $\varepsilon \in E^*$. It follows that $\pi(h) = 0$ and $\pi_3(\mu(b)) = 0$, hence we find that $h \in H$ and $\mu(b) \in K^*$ for all $b \in B$. Thus $\widetilde{H} = \mathrm{Hom}(B, K^*) \oplus H$, and a similar argument confirms that $\widetilde{V} = \mathrm{Hom}(A, K^*) \oplus V$. To see that D/Q is indeed completely recovered as $D/\mathfrak{h}$, note that if

$$\pi(a_1, b_1, c_1) = \pi(a_2, b_2, c_2)$$

then $a_1 - a_2 \in H$, $b_1 - b_2 \in V$, and $\varepsilon_1 - \varepsilon_2 \in K^*$ so that the elements $(a_1, b_1, \varepsilon_1)$ and $(a_2, b_2, \varepsilon_2)$ are related by the action of $\mathfrak{h}$. $\square$

Note that the two scalar multiplications on $D/\mathfrak{h}$, which we denote by $\kappa^{h,\mathfrak{h}}$ and $\kappa^{v,\mathfrak{h}}$, are defined by

$$\kappa_t^{h,\mathfrak{h}}([d]) = [\kappa_t^h d], \quad \kappa_s^{v,\mathfrak{h}}([d]) = [\kappa_s^v d],$$

which are easily seen to be well-defined using the definition of the core and fat bundle actions above. Let us now take a look at a few special cases of this construction.

Example 2.10.2. (i) Suppose that Q has a full core, in other words that $K^* = E^*$. Then the subbundles $\widetilde{H}$ and $\widetilde{V}$ must necessarily be the preimages of H and V inside $\widehat{A}$ and $\widehat{B}$ respectively, and so in this case there is a canonical quotient

$$\begin{tikzcd} D/Q \arrow[r] \arrow[d] & B/V \arrow[d] \\ A/H \arrow[r] & M \end{tikzcd}$$

with trivial core.

(ii) On the other side of things, suppose that $H = A$ and $V = B$. Then the quotient by $\mathfrak{h}$ eliminates the side bundles and takes the quotient of E^* by K^*, producing the *total quotient* [48, Section 2.5.2]

$$\begin{tikzcd} E^*/K^* \arrow[r] \arrow[d] & M \arrow[d] \\ M \arrow[r] & M. \end{tikzcd}$$

We will encounter a more specific example of $\mathcal{DVB}$ quotients in section 3.3.

Remark 2.10.3. Suppose now that Q has only one side bundle in common with D, say $V = B$. Then $Q \to B$ is a vector subbundle of $D \to B$, and one can take the quotient as vector bundles over B. The kernel of this quotient map (considered as a surjective $\mathcal{DVB}$ morphism) would be $H \times_M K^*$, giving $\widetilde{V} = \mathrm{Hom}(H, K^*)$. Thus taking the quotient by this map would result in a $\mathcal{DVB}$ with vertical side bundle equal to B. Note that this differs from any quotient of D by Q as described above, since any such $\mathcal{DVB}$ quotient would have the zero bundle as its vertical side bundle.

There are a few equivalent ways to describe the additional data defining a $\mathcal{DVB}$ quotient. First, observe that the subbundle $\widetilde{H} \subseteq \widehat{A}$ fits into an exact sequence

$$0 \to B^* \otimes K^* \to \widetilde{H} \to H \to 0,$$

and the subbundle $\widetilde{V}$ fits into a similar exact sequence. These sequences are examples of $\mathcal{DVB}$ sequences, and so by the results of [13], they correspond to double vector bundles

$$\begin{tikzcd} \Omega(H) \arrow[r] \arrow[d] & V \arrow[d] \\ K \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} \Omega(V) \arrow[r] \arrow[d] & K \arrow[d] \\ H \arrow[r] & M, \end{tikzcd}$$

with cores H^* and V^* respectively. In order to see how these $\mathcal{DVB}$s are related to D and Q, start by choosing an extension of the subbundle $Q \to V$ of $D \to B$ to the whole base B. Then the quotient $(D/Q)_B$ (taken as vector bundles over B) becomes a double vector bundle

$$\begin{tikzcd} D/Q_B \arrow[r] \arrow[d] & B \arrow[d] \\ A/H \arrow[r] & M \end{tikzcd}$$

with core E^*/K^*. Cyling once through the triality of this $\mathcal{DVB}$ gives the double vector bundle (in the notation of section 2.6 this would be called $(D/Q_B)'$)

$$\begin{tikzcd} \mathrm{Ann}(Q)_B \arrow[r] \arrow[d] & \mathrm{Ann}(K^*) \arrow[d] \\ B \arrow[r] & M \end{tikzcd}$$

with core $\mathrm{Ann}(H)$. We then recover $\Omega(H)$ by taking the quotient (as vector bundles over B) of D' by $\mathrm{Ann}(Q)_B$. Similarly, one can produce a double vector bundle $\mathrm{Ann}(Q)_A$ and recover $\Omega(V)$ as the quotient $D''/\mathrm{Ann}(Q)_A$ taken as vector bundles over A. Thus we see that a $\mathcal{DVB}$ quotient is equivalent to a vector bundle quotient of E^*, together with extensions to a pair of quotients of D' (as a vector bundle over B) and D'' (as a vector bundle over A). We observe that this alternate viewpoint yields a natural explanation for the case $H = A$ and $V = B$ having a canonical quotient, since no choices of extensions are required in this case.

For yet another viewpoint, the extensions Q_B and Q_A of the subbundles $Q \to V$ and $Q \to H$ that define the quotients discussed above actually determine $\mathcal{DVB}$s themselves. Of course, we have the inclusions $Q \subseteq Q_B, Q_A \subseteq D$, so we see that $\mathcal{DVB}$ quotients are equivalent to a choice of intermediate $\mathcal{DVB}$s

$$\begin{tikzcd} Q_B \arrow[r] \arrow[d] & B \arrow[d] \\ H \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} Q_A \arrow[r] \arrow[d] & V \arrow[d] \\ A \arrow[r] & M, \end{tikzcd} \tag{2.26}$$

that lie between Q and D. Let us relate these double vector bundles back to the data of theorem 2.10.1. If we think of $\Gamma(\widehat{A})$ as consisting of vector fields of bidegree $(-1, 0)$ on D (see proposition 2.9.1), then $\Gamma(\widetilde{H})$ consists of those vector fields on D that are tangent to Q_B. Similarly, $\Gamma(\widetilde{V})$ consists of vector fields that are tangent to Q_A.

In section 2.7 we stressed the importance of $\mathcal{DVB}$ sequences (see definition 2.7.2), so let us end our discussion of quotients with a description of the corresponding sequence.

Proposition 2.10.4. *Let $Q \subseteq D$ be a sub-$\mathcal{DVB}$, and let $\mathfrak{h}$ be a Lie subalgebra bundle as in theorem 2.10.1. Then the space of double-linear functions on the quotient $D/\mathfrak{h}$ can be identified with the collection of all double-linear functions on D that are $\mathfrak{h}$-invariant:*

$$C^\infty(D/\mathfrak{h})_{[1,1]} = (C^\infty(D)_{[1,1]})^{\mathfrak{h}}.$$

Moreover, it fits into a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(\mathrm{Ann}(H)) \otimes \Gamma(\mathrm{Ann}(V)) \arrow[r] \arrow[d, hook] & C^\infty(D/\mathfrak{h})_{[1,1]} \arrow[r] \arrow[d, hook] & \Gamma(\mathrm{Ann}(K^*)) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \Gamma(A^*) \otimes \Gamma(B^*) \arrow[r] & C^\infty(D)_{[1,1]} \arrow[r] & \Gamma(E) \arrow[r] & 0. \end{tikzcd}$$

Proof. As in the proof of theorem 2.10.1, we will assume that $D = A \times_M B \times_M E^*$ and $Q = H \times_M V \times_M K^*$ so that

$$D/\mathfrak{h} = A/H \times_M B/V \times_M E^*/K^*.$$

As usual, we identify smooth functions on $D/\mathfrak{h}$ with smooth functions on D that are $\mathfrak{h}$-invariant. This identification is given by $C^\infty(D)^{\mathfrak{h}} \to C^\infty(D/\mathfrak{h})$, $f \mapsto \bar{f}$, where

$$\bar{f}([a], [b], [\varepsilon]) = f(a, b, \varepsilon).$$

The inverse assignment is given by $C^\infty(D/\mathfrak{h}) \to C^\infty(D)^{\mathfrak{h}}$, $f \mapsto f \circ \pi$, where $\pi\colon D \to D/\mathfrak{h}$ is the quotient map. Both of these assignments preserve the property of being double-linear. Indeed, if f is a double-linear function on D, then

$$(\bar{f} \circ \kappa_t^{h,\mathfrak{h}})([a], [b], [\varepsilon]) = f([ta], [b], [t\varepsilon]) = f(ta, b, t\varepsilon) = tf(a, b, \varepsilon) = t\bar{f}([a], [b], [\varepsilon])$$

which shows that $\bar{f}$ is horizontally linear. To show that it is vertically linear is similar. On the other hand, if f is a double-linear function on $D/\mathfrak{h}$, then

$$f \circ \pi \circ \kappa_t^h = f \circ \kappa_t^{h,\mathfrak{h}} \circ \pi = (tf) \circ \pi = t(f \circ \pi)$$

showing that $f \circ \pi$ is horizontally linear, and seeing that it is vertically linear is identical. This proves that $C^\infty(D/\mathfrak{h})_{[1,1]} = (C^\infty(D)_{[1,1]})^{\mathfrak{h}}$. Now consider the restriction of the map $C^\infty(D)_{[1,1]} \to \Gamma(E)$ to $(C^\infty(D)_{[1,1]})^{\mathfrak{h}}$. We claim that the image of this restriction lies in $\Gamma(\mathrm{Ann}(K^*))$. To see this, note that any $\mathfrak{h}$-invariant function is in particular invariant under the core action of K^*, so for any $\not{k} \in K^*$ we have

$$f(0, 0, \not{k}) = f(0, 0, 0) = 0$$

by invariance and double-linearity. Thus $f|_{E^*} \in \Gamma(\mathrm{Ann}(K^*))$, as claimed. The corresponding map $\operatorname{res} E^* \colon (C^\infty(D)_{[1,1]})^{\mathfrak{h}} \to \Gamma(\mathrm{Ann}(K^*))$ is surjective, since for any $e \in \Gamma(\mathrm{Ann}(K^*))$ the function $(a, b, \varepsilon) \mapsto (0, 0, \varepsilon(e))$ is double-linear and $\mathfrak{h}$-invariant. This gives a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \ker(\operatorname{res} E^*) \arrow[r] \arrow[d, hook] & C^\infty(D/\mathfrak{h})_{[1,1]} \arrow[r] \arrow[d, hook] & \Gamma(\mathrm{Ann}(K^*)) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \Gamma(A^*) \otimes \Gamma(B^*) \arrow[r] & C^\infty(D)_{[1,1]} \arrow[r] & \Gamma(E) \arrow[r] & 0,
\end{tikzcd}$$

so we are left only with determining the kernel $\ker(\operatorname{res} E^*)$. This kernel consists of all maps of the form $(a, b, \varepsilon) \mapsto \alpha(a)\beta(b)$ for some $\alpha \in \Gamma(A^*)$, $\beta \in \Gamma(B^*)$ that are $\mathfrak{h}$-invariant. But if a nonzero such map is $\widetilde{H}$-invariant, then for any $h \in H$ we have

$$0 = \alpha(a + h)\beta(b) = \alpha(a)\beta(b) + \alpha(h)\beta(b)$$

for all $a \in A$, $b \in B$, from which we conclude that $\alpha(h) = 0$. Similarly, one finds that $\beta(v) = 0$ for all $v \in V$, so we conclude that $\alpha \in \Gamma(\mathrm{Ann}(H))$ and $\beta \in \Gamma(\mathrm{Ann}(V))$. Conversely, any such function is clearly $\mathfrak{h}$-invariant, so the proof of the proposition is complete. $\square$

Chapter 3

Example: The Double Normal Bundle

In this section we will study the double normal functor. That is, let (M, N_1, N_2) be any triple of manifolds for which N_1 and N_2 are cleanly intersecting submanifolds of M (see section B.2). Such a triple will be called a *manifold triple*, and the double normal functor will associate to (M, N_1, N_2) a double vector bundle $\nu(M, N_1, N_2)$ that simultaneously linearizes the directions normal to N_1 and the directions normal to N_2. We will present three different constructions of $\nu(M, N_1, N_2)$, mirroring the case of the classical normal functor.

3.1 First Construction: As the Spectrum of an Algebra

Our main construction of the double normal functor will be analogous to the construction of $\nu(M, N)$ as the spectrum of an algebra described in section B.1. Before presenting this construction, it will be helpful to review some preliminaries on smooth manifolds that arise as the spectra of algebras. Our discussion of this material follows [31, Section 2], which the reader can consult for more information. Let $\mathcal{A}$ be a commutative algebra over the real numbers, and consider the *spectrum* of $\mathcal{A}$, defined as the space of algebra homomorphisms to $\mathbb{R}$:

$$\operatorname{Spec}\mathcal{A} = \operatorname{Hom}_{\mathrm{Alg}}(\mathcal{A}, \mathbb{R}).$$

We first topologize this space by giving it the topology with the fewest open sets for which the evaluation map $\operatorname{ev}(a)\colon \operatorname{Spec}(\mathcal{A}) \to \mathbb{R}$, $\operatorname{ev}(a)(\varphi) = \varphi(a)$ is continuous for every $a \in \mathcal{A}$. Next we define the sheaf $\mathcal{S}_{\mathcal{A}}$ to be the smallest subsheaf of the sheaf of continuous real-valued functions on $\operatorname{Spec}\mathcal{A}$ that includes all functions of the form

$$f = g(\operatorname{ev}(a_1), \ldots, \operatorname{ev}(a_k))$$

for $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \mathcal{A}$, and $g \in C^\infty(\mathbb{R}^k)$. The main result we will need describes when the space $\operatorname{Spec}\mathcal{A}$ with the topology above can be made into a smooth manifold with sheaf of smooth functions given by $\mathcal{S}_{\mathcal{A}}$.

Lemma 3.1.1. *[31, Lemma 2.4] Let $\mathcal{A}$ be a commutative algebra over the real numbers. Then* $\operatorname{Spec}\mathcal{A}$

is a smooth manifold of dimension n with sheaf of smooth functions given by $\mathcal{S}_{\mathcal{A}}$ if and only if for every point in $\operatorname{Spec}\mathcal{A}$ *there exists an open neighbourhood Ω of that point and elements $a_1, \ldots, a_n \in \mathcal{A}$ such that:*

1. *for every $f \in \mathcal{S}_{\mathcal{A}}(\Omega)$, there exists some $g \in C^\infty(\mathbb{R}^n)$ with*

$$f = g(\mathrm{ev}(a_1), \ldots, \mathrm{ev}(a_n)).$$

2. *The map*

$$(\mathrm{ev}(a_1), \ldots, \mathrm{ev}(a_n)) \colon \Omega \to \mathbb{R}^n$$

is a homemomorphism onto an open subset of $\mathbb{R}^n$.

Of couse, the main point is that if the elements $a_1, \ldots, a_n \in \mathcal{A}$ are as described in the lemma, then the functions $\mathrm{ev}(a_1), \ldots, \mathrm{ev}(a_n)$ act as coordinates for the manifold $\operatorname{Spec}\mathcal{A}$.

Now recall that we say a function $f \in C^\infty(M)$ vanishes to order at least p on a submanifold N if it is a sum of products of p or more functions on M that all vanish on N. For a manifold triple (M, N_1, N_2), consider the bifiltration of the algebra of smooth functions on M:

$$C^\infty(M) = \bigcup_{p,q \geq 0} I^{p,q}(M, N_1, N_2), \tag{3.1}$$

where $I^{p,q}(M, N_1, N_2)$ denotes the set of smooth functions on M that vanish to order at least p on N_1 and vanish to order at least q on N_2. By convention we say that every function on M vanishes to order at least 0 on any submanifold, so that $I^{p,0}(M, N_1, N_2)$ consists of all functions on M that vanish to order at least p on N_1, $I^{0,q}(M, N_1, N_2)$ consists of all functions on M that vanish to order at least q on N_2, and $I^{0,0}(M, N_1, N_2) = C^\infty(M)$. By definition, the associated bigraded algebra of this bifiltration is

$$\mathcal{A}(M, N_1, N_2) = \bigoplus_{p,q \geq 0} \mathcal{A}^{p,q}(M, N_1, N_2), \tag{3.2}$$

where $\mathcal{A}^{p,q}(M, N_1, N_2)$ is defined to be

$$\mathcal{A}^{p,q}(M, N_1, N_2) = I^{p,q}(M, N_1, N_2)/I^{p+1,q}(M, N_1, N_2) + I^{p,q+1}(M, N_1, N_2).$$

From now on we will simply write $\mathcal{A}^{p,q}$ and $I^{p,q}$, leaving the dependence on a manifold triple understood. Our first definition of the double normal bundle will be as the spectrum of the algebra $\mathcal{A}$. Before making this definition official, let us prove that the spectrum of $\mathcal{A}$ is in fact a smooth manifold. For this, it will be helpful to examine the degree $(0,0)$ component of $\mathcal{A}$, which is by definition $\mathcal{A}^{0,0} = C^\infty(M)/(I^{1,0} + I^{0,1})$. If we observe that the ideal $I^{1,0} + I^{0,1}$ consists precisely of functions that vanish on the intersection $N_1 \cap N_2$, we see that $\mathcal{A}^{0,0}$ is given by $\operatorname{Spec}(C^\infty(N)) = N$, where we have introduced the notation $N := N_1 \cap N_2$.

Lemma 3.1.2. *For a manifold triple (M, N_1, N_2), the spectrum of the algebra $\mathcal{A}(M, N_1, N_2)$ is a smooth manifold with dimension equal to the dimension of M.*

Proof. We will use lemma 3.1.1, so we begin by fixing a point $\varphi \in \operatorname{Spec}\mathcal{A}$. By restricting φ to $\mathcal{A}^{0,0} = C^\infty(N)$, we obtain a point $p \in N$ for which $\varphi|_{\mathcal{A}^{0,0}}$ is given by evluation at p. Choose a neighbourhood

U of p and smooth functions $x_1, \ldots, x_m$ on M that are local coordinates on U and are adapted to the manifold triple (M, N_1, N_2), as in proposition B.2.2. If we define $\Omega \subseteq \operatorname{Spec} \mathscr{A}$ to be the open subset of $\operatorname{Spec} \mathscr{A}$ consisting of all characters whose restriction to $\mathscr{A}^{0,0}$ is given by evaluation at a point $u \in U$, then it follows from Taylor's theorem that the elements $[x_1], \ldots, [x_m]$ satisfy property (1) from lemma 3.1.1, where by $[x_j]$ we mean the class of x_j in the appropriate bihomogeneous component of $\mathscr{A}$:

$$[x_j] \in \begin{cases} \mathscr{A}^{1,1} \text{ for } j = 1, \ldots, e, \\ \mathscr{A}^{1,0} \text{ for } j = e+1, \ldots, k, \\ \mathscr{A}^{0,1} \text{ for } j = k+1, \ldots, \ell, \\ \mathscr{A}^{0,0} \text{ for } j = \ell+1, \ldots, m. \end{cases}$$

Furthermore, the map

$$(\operatorname{ev}([x_1]), \ldots, \operatorname{ev}([x_m])) \colon \operatorname{Spec} \mathscr{A} \to \mathbb{R}^{\ell} \times \mathbb{R}^{m-\ell}$$

maps Ω homeomorphically onto $\mathbb{R}^{\ell} \times W$, where $W \subseteq R^{m-\ell}$ is the range of the coordinates $x_{\ell+1}, \ldots, x_m$ on $N \cap U$. □

Definition 3.1.3 (Double Normal Bundle)**.** Let (M, N_1, N_2) be a manifold triple. Then the *double normal bundle* of (M, N_1, N_2) is $\nu(M, N_1, N_2) := \operatorname{Spec}(\mathscr{A})$.

When dealing with coordinates on $\nu(M, N_1, N_2)$, we will omit the square brackets from our notation, and simply use

$$\operatorname{ev}(x_1), \ldots, \operatorname{ev}(x_m) \tag{3.3}$$

from now on. Our next claim is that the double normal bundle $\nu(M, N_1, N_2)$ is in fact a double vector bundle, whose space of double polynomial functions coincides with $\mathscr{A}$. We split the proof of these facts across the next two lemmas.

Proposition 3.1.4. *Let (M, N_1, N_2) be a manifold triple. Then the double normal bundle $\nu(M, N_1, N_2)$ is the total space of a double vector bundle.*

Proof. To show that $\operatorname{Spec}(\mathscr{A})$ is in fact a double vector bundle, it is enough to produce two commuting scalar multiplications. For any $t \in \mathbb{R}$, consider the map $a \mapsto t \cdot_h a$ defined on homogeneous elements by

$$t \cdot_h a = t^p a, \quad a \in \mathscr{A}^{p,q}.$$

This map is an algebra endomorphism for any t, so it induces a map κ_t^h on the spectrum by dualizing:

$$\kappa_t^h \colon \operatorname{Spec}(\mathscr{A}) \to \operatorname{Spec}(\mathscr{A}), \quad \kappa_t^h \varphi(a) = \varphi(t \cdot_h a).$$

To see that this map is smooth, let us examine its coordinate representation in the coordinates 3.3. If $\kappa_t(\varphi)$ lies in a neighbourhood on which these coordinates are defined, then for any $i = 1, \ldots, m$, we have

$$\operatorname{ev}(x_i) \circ \kappa_t^h(\varphi) = \begin{cases} t \operatorname{ev}(x_i)(\varphi) & i = 1, \ldots, e \\ \operatorname{ev}(x_i)(\varphi) & i = e+1, \ldots, m. \end{cases}$$

Therefore the coordinate representation of κ_t^h simply rescales the first e entries by t, which is certainly smooth. Note further that when $t \neq 0$, the map κ_t^h is in fact a diffeomorphism. Since $\kappa_{t\cdot s}^h = \kappa_t^h \circ \kappa_s^h$, this defines an action of $\mathbb{R}_{>0}$ on $\operatorname{Spec}(\mathscr{A})$. The base $\kappa_0(\operatorname{Spec}\mathscr{A})$ consists of characters that vanish on all elements with $p > 0$, and hence it is idenified with $\operatorname{Spec}\mathscr{A}^{0,\bullet}$. The map $\operatorname{Spec}\mathscr{A} \to T\operatorname{Spec}\mathscr{A}|_{\operatorname{Spec}\mathscr{A}^{0,\bullet}}$, $\varphi \mapsto \frac{d}{dt}(\kappa_t\varphi)|_{t=0}$ is given in terms of the coordinates $\operatorname{ev}(x_1), \ldots, \operatorname{ev}(x_m)$ by

$$\begin{pmatrix} \operatorname{ev}(x_1)(\varphi) \\ \vdots \\ \operatorname{ev}(x_m)(\varphi) \end{pmatrix} \mapsto \left(\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \operatorname{ev}(x_{e+1})(\varphi) \\ \vdots \\ \operatorname{ev}(x_m)(\varphi) \end{pmatrix}, \begin{pmatrix} \operatorname{ev}(x_1)(\varphi) \\ \vdots \\ \operatorname{ev}(x_e)(\varphi) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)$$

which is clearly injective. Thus κ_t^h is a scalar multiplication on $\operatorname{Spec}\mathscr{A}$ with base $\operatorname{Spec}\mathscr{A}^{0,\bullet}$. Using the map $a \mapsto s \cdot_v a$ defined by

$$s \cdot_v a = s^q a, \quad a \in \mathscr{A}^{p,q}$$

for any $s \in \mathbb{R}$, one similarly obtains a scalar multiplication κ_s^v on $\operatorname{Spec}\mathscr{A}$ with base $\operatorname{Spec}(\mathscr{A}^{\bullet,0})$. Since κ_t^h and κ_s^v clearly commute, we conclude that $\operatorname{Spec}(\mathscr{A})$ is a double vector bundle. □

The proof of proposition 3.1.4 shows that the side bundles of $\nu(M, N_1, N_2)$ are given by $\operatorname{Spec}(\mathscr{A}^{\bullet,0})$ and $\operatorname{Spec}(\mathscr{A}^{0,\bullet})$. Our first task will be to describe these side bundles in more conventional terms. First consider the horizontal side bundle $\operatorname{Spec}(\mathscr{A}^{0,\bullet})$, with scalar multiplication given by restriction of the map κ_t^h defined above. The base of this bundle is $\operatorname{Spec}(\mathscr{A}^{0,0}) = N$. We claim that the only linear functions on $\operatorname{Spec}(\mathscr{A}^{0,\bullet})$ are the maps $\operatorname{ev}(a)$ for $a \in \mathscr{A}^{0,1}$. Before we state this claim precisely, note that each $I^{i,j}$ is a module over $I^{0,0} = C^\infty(M)$, from which it follows that $\mathscr{A}^{0,1}$ is a $C^\infty(M)$-module. But $\mathscr{A}^{0,1}$ is annihilated by $I^{1,0} + I^{0,1}$, and hence $\mathscr{A}^{0,1}$ is a module over $\mathscr{A}^{0,0} = C^\infty(N)$.

Lemma 3.1.5. *The map $\mathscr{A}^{0,1} \to C^\infty_{[1]}(\operatorname{Spec}(\mathscr{A}^{0,\bullet}))$ given by $a \mapsto \operatorname{ev}(a)$ is an isomorphism of $C^\infty(N)$-modules.*

Proof. It is clear that the map is an injective $C^\infty(N)$-module homomorphism, so we only need to show that it is surjective. First note that by lemma 3.1.2, it is enough to prove surjectivity for functions $f \in C^\infty(\operatorname{Spec}(\mathscr{A}^{0,\bullet}))$ of the form $f = g(\operatorname{ev}(x_{k+1}), \ldots, \operatorname{ev}(x_m))$ for some $g \in C^\infty(\mathbb{R}^{m-k})$, where $x_{k+1}, \ldots, x_m$ are the coordinates as in proposition B.2.2 that do not necessarily vanish on N_1. Then linearity of f means that for all $y_{k+1}, \ldots, y_m \in \mathbb{R}$ we have

$$g(ty_{k+1}, \ldots, ty_\ell, y_{\ell+1}, \ldots, y_m) = tg(y_{k+1}, \ldots, y_m). \tag{3.4}$$

Taking the derivative with respect to t gives

$$\sum_{i=k+1}^{\ell} y_i \partial_i g(ty_{k+1}, \ldots, ty_\ell, y_{\ell+1}, \ldots, y_m) = g(y_{k+1}, \ldots, y_m).$$

By setting $t = 0$ in the equation above, we see that g is a combination of linear monomials:

$$g(y_{k+1}, \dots, y_m) = \sum_{i=k+1}^{\ell} y_i \partial_i g(0, \dots, 0, y_{\ell+1}, \dots y_m).$$

Since $x_{\ell+1}, \dots, x_m$ all have degree 0, the functions $f_i := \partial_i g(0, \dots, 0, \mathrm{ev}(x_{\ell+1}), \dots, \mathrm{ev}(x_m))$ are all smooth functions on $\mathscr{A}^{0,0} = N$. Moreover, the equation above shows that $f = \mathrm{ev}(a)$, where $a = \sum_{i=k+1}^{\ell} f_i x_i$, completing the proof. $\square$

Hence $\mathrm{Spec}(\mathscr{A}^{0,\bullet})$ is dual to the bundle whose space of sections is given by $\mathscr{A}^{0,1}$. Note that $I^{1,0}$ is the vanishing ideal of N_1, $I^{0,1}$ is the vanishing ideal of N_2, and $I^{1,1}$ is their intersection. If we let $J \subseteq C^\infty(N_2)$ denote the ideal consisting of functions on N_1 that vanish on N, then since N_1 and N_2 intersect cleanly, we a have surjective map $I^{0,1} \to J$ given by restriction (see corollary B.2.3). The kernel of this map is the intersection $I^{1,1}$, and thus we have

$$\mathscr{A}^{0,1} = I^{0,1}/(I^{0,2} + I^{1,1}) \cong \left(I^{0,1}/I^{1,1}\right)/\left(I^{0,1}/I^{1,1}\right)^2 \cong J/J^2. \tag{3.5}$$

This shows that the space of sections of $\nu(N_1, N)^*$ may be identified with $\mathscr{A}^{0,1}$ (see section B), and we conclude that $\mathrm{Spec}(\mathscr{A}^{0,\bullet}) = \nu(N_1, N)$. A similar discussion shows that the other side bundle $\mathrm{Spec}(\mathscr{A}^{\bullet,0})$ is isomorphic to the vector bundle $\nu(N_2, N) \to N$. Putting everything together, we obtain the following theorem.

Theorem 3.1.6. *Let (M, N_1, N_2) be a manifold triple. Then the double normal bundle of (M, N_1, N_2) is a double vector bundle*

$$\begin{tikzcd} \nu(M, N_1, N_2) \arrow[r] \arrow[d] & \nu(N_2, N) \arrow[d] \\ \nu(N_1, N) \arrow[r] & N. \end{tikzcd} \tag{3.6}$$

Our next goal is to compute the $\mathscr{DVB}$ sequence of the double normal bundle. As mentioned in section 2.7, the $\mathscr{DVB}$ sequence of $\nu(M, N_1, N_2)$ encodes the structure of $\nu(M, N_1, N_2)$ completely, and computing it will allow us to connect our construction with the other constructions to be presented below. To this end, we compute the spaces $C^\infty_{[1,1]}(\mathrm{Spec}\,\mathscr{A})$ and $\Gamma(E)$, where as usual E denotes the dual bundle to $\mathrm{core}(\mathrm{Spec}(\mathscr{A}))$. By comparing with lemma 3.1.5, one can identify $C^\infty_{[1,1]}(\mathrm{Spec}\,\mathscr{A})$.

Lemma 3.1.7. *The map $\mathscr{A}^{1,1} \to C^\infty_{[1,1]}(\mathrm{Spec}\,\mathscr{A})$ given by $a \mapsto \mathrm{ev}(a)$ is an isomorphism of $C^\infty(N)$-modules.*

Proof. As above, we will show that the map is surjective, and we will take $f \in C^\infty_{[1,1]}(\mathrm{Spec}\,\mathscr{A})$ to be of the form $f = g(\mathrm{ev}(x_1), \dots, \mathrm{ev}(x_m))$ for some $g \in C^\infty(\mathbb{R}^k)$, and with $x_1, \dots, x_m$ adapted corodinates as in proposition B.2.2. By the argument of lemma 3.1.5, since f is linear in the "p-grading", g takes the form

$$g(y_1, \dots, y_m) = \sum_{j=1}^{k} y_j \partial_j g(0, \dots, 0, y_{k+1}, \dots, y_m).$$

Since f is also linear in the "q-grading", we obtain the following condition on g:

$$t\sum_{j=1}^{k} y_j\partial_j g(0,\dots,0,y_{k+1},\dots,y_m) = \sum_{j=1}^{e} ty_j\partial_j g(0,\dots,0,ty_{k+1},\dots,ty_\ell,y_{\ell+1},\dots,y_m) + \sum_{j=e+1}^{k} y_j\partial_j g(0,\dots,0,ty_{k+1},\dots,ty_\ell,y_{\ell+1},\dots,y_m)$$

for all t. By differentiating this equation with respect to t, we get

$$\sum_{j=1}^{k} y_j\partial_j g(0,\dots,0,y_{k+1},\dots,y_m) = \sum_{j=1}^{e} y_j\partial_j g(0,\dots,0,ty_{k+1},\dots,ty_\ell,y_{\ell+1},\dots,y_m) + \sum_{j=1}^{e}\sum_{i=k+1}^{\ell} ty_jy_i\partial_i\partial_j g(0,\dots,0,ty_{k+1},\dots,ty_\ell,y_{\ell+1},\dots,y_m).$$

Since this equation must hold for all $y_1,\dots,y_m$, the quadratic terms on the right hand side must vanish. So for each $i = k+1,\dots,\ell$ we have $\partial_i\partial_j g(0,\dots,0,y_{k+1},\dots,y_m) = 0$ for each j. But then the terms $\partial_j g(0,\dots,0,y_{k+1},\dots,y_m)$ do not depend on y_i, and so if we denote by

$$f_j = \partial_j g(0,\dots,0,\mathrm{ev}(x_{\ell+1}),\dots,\mathrm{ev}(x_m)) \in C^\infty(N)$$

we see that $f = \mathrm{ev}(a)$, where $a = \sum_{j=1}^{e} f_jx_j \in \mathcal{A}^{1,1}$. $\square$

We now turn our attention to the linear functions on the core. Since the core of a double vector bundle can be thought of as the submanifold on which the two scalar multiplications agree, we have

$$\mathrm{core}(\operatorname{Spec}\mathcal{A}) = \{\varphi \in \operatorname{Spec}\mathcal{A} \mid \varphi(a) = 0 \text{ for all } a \in \mathcal{A}^{p,q} \text{ with } p \neq q\}.$$

By the argument of lemma 3.1.5, we see that the map $\mathcal{A}^{1,1} \to C^\infty_{[1]}(\mathrm{core}(\operatorname{Spec}\mathcal{A}))$ given by $a \mapsto \mathrm{ev}(a)$ is a surjective homomorphism of $C^\infty(N)$-modules. However, this time the map is not injective, since $\mathrm{ev}(a) = 0$ if $a = a_1a_2$ for $a_1 \in \mathcal{A}^{1,0}$ and $a_2 \in \mathcal{A}^{0,1}$. Since $\mathcal{A}$ is generated by $\mathcal{A}^{0,0}, \mathcal{A}^{1,0}, \mathcal{A}^{0,1}$, and $\mathcal{A}^{1,1}$, it follows that this is exactly the kernel so we have

$$\Gamma(E) \cong \mathcal{A}^{1,1}/\mathcal{A}^{1,0}\mathcal{A}^{0,1}.$$

We have now arrived at the following result.

Theorem 3.1.8. *On the level of sections, the $\mathcal{DVB}$ sequence of $\nu(M,N_1,N_2)$ is*

$$0 \to \mathcal{A}^{1,0}\otimes\mathcal{A}^{0,1} \to \mathcal{A}^{1,1} \to \mathcal{A}^{1,1}/\mathcal{A}^{1,0}\mathcal{A}^{0,1} \to 0. \tag{3.7}$$

Before moving on to alternative constructions of $\nu(M,N_1,N_2)$, we will conclude this section with a simple example. Recall that for a vector bundle $V \to M$, there is a canonical isomorphism $\nu(V,M) \cong V$. We claim that the situation of applying the double normal functor to a double vector bundle is analogous. In order to make sense of this statement, we require the following lemma.

Lemma 3.1.9. *The submanifolds A, B of a double vector bundle D intersect cleanly (see B.2 for details on clean intersections). That is, for any $m \in M$ we have*

$$T_mM = T_mA \cap T_mB.$$

Proof. For any $m \in M$, we have a decomposition $T_mA = T_mM \oplus A_m$. On the first component, the scalar multiplication $T_m\kappa^h$ acts trivially, while it acts simply as κ^h on the second component. Hence the fixed point set of $T_m\kappa^h$ is precisely T_mM. But B is the fixed point set of κ^h, and therefore T_mB is the fixed point set of $T_m\kappa^h$ (inside T_mD, so we conclude that $T_mA \cap T_mB = T_mM$, as claimed. □

Example 3.1.10 (Double Vector Bundles). Let (D, A, B) be a double vector bundle with core E^*. By lemma 3.1.9, we may consider the double normal bundle $\nu(D, A, B)$. We claim that there is a canonical $\mathcal{DVB}$ isomorphism $\nu(D, A, B) \cong D$. To see this, introduce coordinates as in proposition B.2.2. Then by considering Taylor expansions in these coordinates, one obtains isomorphisms

$$\mathcal{A}^{1,0} \cong B^* \quad \mathcal{A}^{0,1} \cong A^* \quad \mathcal{A}^{1,1}/\mathcal{A}^{1,0}\mathcal{A}^{0,1} \cong E.$$

Similar reasoning leads to the identification $\mathcal{A}^{1,1} \cong (A^* \otimes B^*) \oplus E$, which shows that the $\mathcal{DVB}$ sequences (2.13) and (3.7) agree.

3.2 Second Construction: Iterating the Normal Functor

Our second construction of $\nu(M, N_1, N_2)$ will proceed by applying the classical normal functor twice. First, suppose $V \to M$ is any vector bundle with scalar multiplication by $t \in \mathbb{R}_{\geq 0}$ denoted by κ_t, and let $W \to N$ be a subbundle along a submanifold N. This means precisely that W is invariant under κ_t for all t, with $\kappa_0(W) = N$ (see [23]). Thus each κ_t induces a map $\nu(\kappa_t)\colon \nu(V, W) \to \nu(V, W)$, which we claim defines a scalar multiplication for the pair $(\nu(V, W), \nu(M, N))$. Indeed, recall that the tangent lift $T\kappa_t$ defines a scalar multiplication on (TM, TN), and so injectivity of the map $[X] \mapsto \frac{d}{dt}(\nu(\kappa_t X))|_{t=0}$ follows from the injectivity of the map $X \mapsto \frac{d}{dt}(T\kappa_t X)|_{t=0}$ and the fact that $\kappa_t^{-1}(W) = W$. We therefore obtain the following diagram of vector bundles

$$\begin{tikzcd} \nu(V,W) \arrow[r] \arrow[d] & W \arrow[d] \\ \nu(M,N) \arrow[r] & N. \end{tikzcd} \tag{3.8}$$

Since $\nu(\kappa_t)$ is linear in the structure $\nu(V, W) \to W$ for all t, it follows that the square (3.8) is a double vector bundle. Viewing $\nu(V, W)$ as a bundle over W, the restriction to the submanifold N is given by

$$\nu(V, W)|_N = TV|_N/TW|_N = (V|_N \oplus TM|_N)/(W \oplus TN) = V|_N/W \oplus \nu(M, N).$$

This shows that on the restriction $\nu(V, W)|_N$, the map $\nu(\kappa_0)$ is simply projection to the second factor, and that the core of the double vector bundle (3.8) is given by $E^* = V|_N/W$.

Now let us return to our manifold triple (M, N_1, N_2), and consider the vector bundle $\nu(M, N_1) \to N_1$. Since N_1 and N_2 intersect cleanly, the bundle $\nu(N_2, N) \to N$ is actually a subbundle of $\nu(M, N_1) \to N_1$.

While the proof of this fact is straightforward, we include it here to highlight the use of the clean intersection assumption.

Lemma 3.2.1. *Let M be a manifold and let N_1 and N_2 be cleanly intersecting submanifolds of M. Then $\nu(N_2, N)$ is a vector subbundle of $\nu(M, N_1)$.*

Proof. Consider the map of pairs $\iota\colon (N_2, N) \hookrightarrow (M, N_1)$ given by inclusion. Applying the normal functor, we obtain a map

$$\nu(\iota) : \nu(N_2, N) \to \nu(M, N_1).$$

We claim that this map is injective if and only if N_1 and N_2 intersect cleanly. Indeed, if $v \in TN_2|_N$, then $v + TN$ lies in the kernel of $\nu(\iota)$ precisely when $T\iota(v) \in TN_1$. In other words, we have $\ker \nu(\iota) = TN_1 \cap TN_2|_N/TN$, which is zero exactly if N_1 and N_2 intersect cleanly. □

As a result of lemma 3.2.1 and the discussion above we obtain a double vector bundle

$$\begin{tikzcd} \nu(\nu(M, N_1), \nu(N_2, N)) \arrow[r] \arrow[d] & \nu(N_2, N) \arrow[d] \\ \nu(N_1, N) \arrow[r] & N \end{tikzcd} \tag{3.9}$$

with core given by

$$\nu(M, N_1)|_N/\nu(N_2, N) \cong TM|_N/(TN_1|_N{+}TN_2|_N).$$

In section 3.4 below, we prove that there is an isomorphism of double vector bundles between (3.6) and (3.9), and so this process of iterating the classical normal functor provides an alternate description of the double normal bundle.

3.3 Third Construction: As a Quotient $\mathscr{DVB}$

The most straightforward definition of the classical normal bundle is as a quotient of the usual tangent bundle. Analogously, we can obtain the double normal bundle by starting with the double tangent bundle TTM obtained by taking $V = TM$ in example 2.22:

$$\begin{tikzcd} TTM \arrow[r] \arrow[d] & TM \arrow[d] \\ TM \arrow[r] & M. \end{tikzcd} \tag{3.10}$$

Just like with the usual tangent bundle (see A.2), there are various methods for lifting a smooth function $f \in C^\infty(M)$ to a smooth function on the double tangent bundle. To describe these lifting processes, let us first recall an algebraic description of TTM. First, consider the algebra

$$\mathbb{A}_1 = \mathbb{R}[\varepsilon]/\varepsilon^2.$$

The (ordinary) tangent bundle TM can be defined as the space of algebra homomorphisms from $C^\infty(M)$ to $\mathbb{A}_1$:

$$TM = \mathrm{Hom}_{\mathrm{Alg}}(C^\infty(M), \mathbb{A}_1).$$

Here the scalar multiplication κ_t is defined by $\kappa_t(v) = m_t \circ v$, where $m_t : \mathbb{A}_1 \to \mathbb{A}_1$ is the map $a_0 + a_1\varepsilon \mapsto a_0 + ta_1\varepsilon$. Taking this one step further, we obtain the double tangent bundle TTM as a space of algebra homomorphisms:

$$TTM = \mathrm{Hom}_{\mathrm{Alg}}(C^\infty(M), \mathbb{A}_1 \otimes \mathbb{A}_1).$$

If we write an arbitrary element of $\mathbb{A}_1 \otimes \mathbb{A}_2$ as $\sum_{0\le i,j\le 1} a_{i,j}\varepsilon_1^i\varepsilon_2^j$, then the two scalar multiplications are given by $\kappa_t^h(v) = v \circ m_t^h$ and $\kappa_s^v(v) = v \circ m_s^v$, where m_t^h and m_s^v are the maps

$$\begin{aligned} m_t^h &: \sum_{0\le i,j\le 1} a_{i,j}\varepsilon_1^i\varepsilon_2^j \mapsto \sum_{0\le i,j\le 1} t^i a_{i,j}\varepsilon_1^i\varepsilon_2^j, \\ m_s^v &: \sum_{0\le i,j\le 1} a_{i,j}\varepsilon_1^i\varepsilon_2^j \mapsto \sum_{0\le i,j\le 1} s^j a_{i,j}\varepsilon_1^i\varepsilon_2^j. \end{aligned}$$

Now suppose $f \in C^\infty(M)$. Then the action of $v \in TTM$ on f can be resolved into component functions $f^{(i,j)} : TTM \to \mathbb{R}$ for $0 \le i, j \le 1$ as follows:

$$X(f) = f^{(0,0)}(v) + f^{(1,0)}(v)\varepsilon_1 + f^{(0,1)}(v)\varepsilon_2 + f^{(1,1)}(v)\varepsilon_1\varepsilon_2.$$

Furthermore, the function $f^{(i,j)}$ is bihomogeneous of bidegree (i, j) since by construction we have

$$f^{(i,j)}(\kappa_t^h v) = t^i f^{(i,j)}(v), \quad f^{(i,j)}(\kappa_s^v v) = s^j f^{(i,j)}(v)$$

for all $t, s \in \mathbb{R}$. In this way we determine four different procedures for lifting a function $f \in C^\infty(M)$ to a bihomogeneous function $f^{(i,j)} \in C^\infty(TTM)_{[i,j]}$ on the double tangent bundle TTM. Furthermore, using the definition of $f^{(i,j)}$, we can obtain various results that establish how these lifts behave with respect to the algebra structure on $C^\infty(M)$. For example, we have

$$(fg)^{(1,1)} = f^{(0,0)}g^{(1,1)} + f^{(1,0)}g^{(0,1)} + f^{(0,1)}g^{(1,0)} + f^{(1,1)}g^{(0,0)} \tag{3.11}$$

For more information on the algebraic approach to tangent functors touched on above, see [37, Chapter VIII].

Remark 3.3.1. 1. If $x_1, \ldots, x_m$ are smooth coordinates on the manifold M, then the functions

$$x_1^{(0,0)}, \ldots, x_m^{(0,0)}, x_1^{(1,0)}, \ldots, x_m^{(1,0)}, x_1^{(0,1)}, \ldots, x_m^{(0,1)}, x_1^{(1,1)}, \ldots, x_m^{(1,1)} \tag{3.12}$$

constitute coordinates for the double tangent bundle TTM. Here the $(0,0)$ lifts correspond to coordinates on the total base M, the $(1,0)$ lifts correspond to fibre coordinates on the vertical base (similarly the $(0,1)$ lifts are for the horizontal base), and the $(1,1)$ lifts correspond to fibre coordinates on the core.

2. The lifting processes described above can also be obtained by iterating the vertical and tangent lift operations of section A.2. Specifically, any function $f \in C^\infty(M)$ induces (a priori) eight functions on TTM: first one lifts f to TM by using either a vertical lift ($f^\sharp$) or a tangent lift (f_T) then each of these two functions on TM can be lifted to TTM either along the vertical projection or the horizontal one (in each case, one has again the choice of vertical lift or tangent lift). It turns out that this only produces four unique functions, and these four functions coincide with the lifts

described above. For example, $f^{(1,0)}$ can be obtained by first taking the tangent lift $f_T \in C^\infty(M)$, and then taking the vertical lift along the horizontal projection (or vice versa).

Having established lifting procedures for smooth functions, one can then define the corresponding lifts $X \mapsto X^{(k,\ell)}$ for $-1 \leq k, \ell \leq 0$ of a vector field $X \in \mathfrak{X}(M)_{[k,\ell]}$ to a vector field $X^{(k,\ell)} \in \mathfrak{X}(TTM)$ by their actions on the generators of $\mathcal{S}(TTM)$:

$$\mathcal{L}_{X^{(k,\ell)}} f^{(i,j)} = (\mathcal{L}_X f)^{k+i,\ell+j}.$$

Alternatively, as with the function lifts, one could define them by iterating the vertical and tangent lift operations on vector fields. By the results of section 2.9, the space of sections of the bundle $\widehat{A} = \widehat{TM}$ is given by $\mathfrak{X}_{[-1,0]}(D)$. It is therefore $C^\infty(M)$-generated by vector fields of the forms $X^{(-1,0)}$ and $f^{(0,1)}X^{(-1,-1)}$ for $X \in \mathfrak{X}(M)$, with the exact sequence

$$0 \to \Gamma(T^*M \otimes TM) \to \mathfrak{X}_{[-1,0]}(D) \to \Gamma(TM) \to 0$$

being determined by the prescriptions $f^{(0,1)} \otimes X \to f^{(0,1)}X^{(-1,-1)}$ and $X^{(-1,0)} \mapsto X$.

As with any double vector bundle (see lemma 2.1.4), there is a surjective submersion $\psi: TTM \to TM \times_M TM$. The preimage of the sub-double vector bundle $TN_1 \times_N TN_2$ under this map yields a sub-$\mathcal{DVB}$ of TTM since ψ is a surjective submersion as well as a $\mathcal{DVB}$ morphism. We will denote this preimage by $TT(M, N_1, N_2)$, giving a $\mathcal{DVB}$

$$\begin{tikzcd} TT(M,N_1,N_2) \arrow[r] \arrow[d] & TN_2|_N \arrow[d] \\ TN_1|_N \arrow[r] & N. \end{tikzcd} \tag{3.13}$$

The core of this $\mathcal{DVB}$ is simply $\psi^{-1}(N)$, and is therefore given by $TM|_N$. Let us briefly describe a few alternate descriptions of $TT(M, N_1, N_2)$. In terms of coordinates, if $x_1, \ldots, x_m$ are coordinates for the manifold triple (M, N_1, N_2) as in proposition B.2.2, then $TT(M, N_1, N_2)$ can be described by the vanishing of some of the associated lifted coordinates 3.12:

$$x_1^{(0,0)} = \cdots = x_\ell^{(0,0)} = x_1^{(1,0)} = \cdots = x_k^{(1,0)} = x_1^{(0,1)} = \cdots = x_e^{(0,1)} = x_{k+1}^{(0,1)} = \cdots = x_\ell^{(0,1)} = 0.$$

Extending this idea, we may describe $TT(M, N_1, N_2)$ by its vanishing ideal. If a function of the form $f^{(0,0)}$ vanishes on $TT(M, N_1, N_2)$, then f must vanish everywhere along N so that $f \in I^{1,0} + I^{0,1}$. Meanwhile if a function $f^{(1,0)}$ vanishes on $TT(M, N_1, N_2)$, then its tangent lift f_T must vanish on $TN_1|_N$, which happens in particular if $f \in I^{1,0}$ (and similarly for functions $f^{(0,1)}$). Using the coordinate description above, we see that these functions are sufficient to generate all functions that vanish on $TT(M, N_1, N_2)$, so the vanishing ideal is generated by the set

$$\{f^{(0,0)} \mid f \in I^{1,0} + I^{0,1}\} \cup \{f^{(1,0)} \mid f \in I^{1,0}\} \cup \{f^{(0,1)} \mid f \in I^{0,1}\}.$$

Our goal is to obtain $\nu(M, N_1, N_2)$ as a quotient of $TT(M, N_1, N_2)$. To do this, we should find a suitable Lie algebra of vector fields acting on $TT(M, N_1, N_2)$, and quotient by the action of this Lie subalgebra (see section 2.10). We will start by considering the vector fields $X \in \mathfrak{X}_{[-1,0]}(TTM)$ that are tangent to $TT(M, N_1, N_2)$. Note that all vector fields of the form $X^{(-1,-1)}$ are tangent to $TT(M, N_1, N_2)$, and for the vector fields of the form $X^{(-1,0)}$ we have the following lemma.

Lemma 3.3.2. *A vector field on TTM of the form $X^{(-1,0)}$ for $X \in \mathfrak{X}(M)$ is tangent to $TT(M, N_1, N_2)$ if and only if $X|_N \in \Gamma(TN_1|_N)$.*

Proof. A vector field on TTM is tangent to $TT(M, N_1, N_2)$ precisely when it preserves the vanishing ideal of $TT(M, N_1, N_2)$. For degree reasons, $X^{(-1,0)}$ always annihilates functions of the form $f^{(0,0)}$ and $f^{(0,1)}$. Thus $X^{(-1,0)}$ is tangent to $TT(M, N_1, N_2)$ if and only if $\mathcal{L}_{X^{(-1,0)}} f^{(1,0)}|_{TT(M,N_1,N_2)} = 0$ for all $f \in I^{1,0}$. But we have $\mathcal{L}_{X^{(-1,0)}} f^{(1,0)} = (\mathcal{L}_X f)^{(0,0)}$, so this is equivalent to the condition $\mathcal{L}_X f|_N = 0$ for all $f \in I^{1,0}$, which is in turn equivalent to $X|_N \in \Gamma(TN_1|_N)$. $\square$

Using this lemma, we see that the fat bundle $\widehat{TN_1|_N}$ of $TT(M, N_1, N_2)$ is generated by vector fields of the form $f^{(1,0)} X^{(-1,-1)}$ with no conditions on f, X, as well as vector fields of the form $X^{(-1,0)}$ for $X|_N \in \Gamma(TN_1|_N)$, with corresponding exact sequence

$$0 \to T^*N_2|_N \otimes TM|_N \to \widehat{TN_1|_N} \to TN_1|_N \to 0.$$

One has a similar description for the fat bundle $\widehat{TN_2|_N}$, leading to the Lie algebra bundle

$$\mathfrak{g} = \widehat{TN_1|_N} \times \widehat{TN_2|_N} \times \Gamma(TM|_N).$$

that acts on $TT(M, N_1, N_2)$. To determine the quotient, we first take the subbundle $\widetilde{H} \subseteq \widehat{TN_1|_N}$ defined as

$$\widetilde{H} = \langle f^{(1,0)} X^{(-1,-1)}, Y^{(-1,0)} \mid X|_N \in \Gamma(TN_1|_N + TN_2|_N), Y|_N \in \Gamma(TN) \rangle,$$

where the angled brackets enclose the generators as a $C^\infty(N)$-module. This bundle $\widetilde{H}$ fits into an exact sequence

$$0 \to T^*N_2 \otimes (TN_1|_N + TN_2|_N) \to \widetilde{H} \to TN \to 0.$$

By choosing $\widetilde{V} \subseteq \widehat{TN_2|_N}$ similarly, by theorem 2.10.1 we obtain the $\mathcal{DVB}$ quotient

$$\begin{tikzcd} TT(M, N_1, N_2)/\mathfrak{h} \arrow[r] \arrow[d] & \nu(N_2, N) \arrow[d] \\ \nu(N_1, N) \arrow[r] & N. \end{tikzcd} \tag{3.14}$$

with core $TM|_N/(TN_1|_N + TN_2|_N)$, where $\mathfrak{h} = \widetilde{H} \times \widetilde{V} \times (TN_1|_N + TN_2|_N)$. This gives our third and final description of the double normal bundle.

Remark 3.3.3. In section 2.10, we saw that one could also define $\mathcal{DVB}$ quotients using a sub double vector bundle Q and extensions of Q to intermediate double vector bundles Q_1 and Q_2. In this context, Q is the sub $\mathcal{DVB}$

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & TN \arrow[d] \\ TN \arrow[r] & N. \end{tikzcd}$$

of $TT(M, N_1, N_2)$ whose vanishing ideal is generated by

$$\{f^{(0,0)}, f^{(1,0)}, f^{(0,1)} \mid f \in I^{1,0} + I^{0,1}\} \cup \{f^{(1,1)} \mid f \in I^{1,1}\}.$$

We can then obtain Q_1 from Q by removing the functions of the form $f^{(0,1)}$ for $f \in I^{1,0}$ from our generators. In other words, Q_1 is the sub $\mathscr{DVB}$

$$\begin{tikzcd} Q_1 \arrow[r] \arrow[d] & TN_2|_N \arrow[d] \\ TN \arrow[r] & N. \end{tikzcd}$$

of $TT(M, N_1, N_2)$ whose vanishing ideal has

$$\{f^{(0,0}, f^{(1,0)} \mid f \in I^{1,0} + I^{0,1}\} \cup \{f^{(0,1)} \mid f \in I^{0,1}\} \cup \{f^{(1,1)} \mid f \in I^{1,1}\}$$

as a set of generators. Using an argument similar to the proof of lemma 3.3.2, one can prove that $\widetilde{H}$ as defined above consists of those vector fields in $\mathfrak{X}(TTM)_{[-1,0]}$ that are tangent to Q_1. The description of Q_2 and $\widetilde{V}$ is then similar.

3.4 Equivalence of Constructions

We will now prove that the three different constructions described above all yield canonically isomorphic structures. Our strategy for this will to compute the $\mathscr{DVB}$ sequences associated to the constructions of sections 3.2 and 3.3, and prove that they are both canonically isomorphic (as $\mathscr{DVB}$ sequences) to (3.7).

As mentioned above, our goal in this section is to establish the equivalence of the constructions in sections 3.1 and 3.2. To avoid overly cumbersome notation, we will denote $\nu(\nu(M, N_1), \nu(N_2, N))$ simply by $\nu \circ \nu$. We begin by stating the result precisely.

Theorem 3.4.1. *Let (M, N_1, N_2) be a manifold triple. Then there exists a canonical map*

$$\nu(M, N_1, N_2) \to \nu(\nu(M, N_1), \nu(N_2, N))$$

that is a double vector bundle isomorphism between (3.6) and (3.9).

By the results of [13], it is enough to construct the isomorphism on the level of $\mathscr{DVB}$ sequences. As a first step, note that we have already shown that $\nu(N_1, N)^* \cong \mathscr{A}^{0,1}$ and $\nu(N_2, N)^* \cong \mathscr{A}^{1,0}$ in section 3.1 (see (3.5)). Thus we start by considering the space $C^\infty_{[1,1]}(\nu \circ \nu)$ of double linear functions on $\nu \circ \nu$.

Lemma 3.4.2. *There is an isomorphism of $C^\infty(N)$-modules*

$$\mathscr{A}^{1,1} \to C^\infty_{[1,1]}(\nu \circ \nu).$$

Proof. Any function in $C^\infty_{[1,1]}(\nu \circ \nu)$ is in particular a linear function for the horizontal structure, and is therefore of the form $\nu(\varphi)$ for some function $\varphi \in C^\infty(\nu(M, N_1))$ that vanishes on $\nu(N_2, N)$. Since $\nu(\varphi)$ is homogeneous of degree one in the vertical structure as well, it follows that φ is homogeneous of degree 1 by functoriality of ν. But then φ itself is of the form $\nu(\psi)$ for some $\psi \in I^{1,0}$ (see (3.1)). Since $\nu(\psi)$

vanishes on $\nu(N_2, N)$, ψ itself must vanish on N_2 as well. In other words, we have a surjective map

$$\begin{aligned} \nu^2\colon I^{1,1} &\to C^\infty_{[1,1]}(\nu\circ\nu) \\ \psi &\mapsto \nu(\nu(\psi)). \end{aligned}$$

There are two ways for ψ to lie in the kernel of this map: either $\nu(\psi) = 0$ or $\nu(\psi) \neq 0$ but $\nu(\nu(\psi)) = 0$. In the former case, the derivatives of ψ in the directions tangent to N_1 vanish, in which case $\psi \in I^{2,0}$. In the latter case, the derivatives of $\nu(\psi)$ in the directions tangent to $\nu(N_2, N)$ vanish, which implies that the derivatives of ψ vanish in directions tangent to N_2 so that $\psi \in I^{0,2}$. This shows that $\ker \nu^2 = I^{2,1} + I^{1,2}$ (the reverse inclusion being clear), which completes the proof. □

Before proceeding to the proof of the theorem, it remains only to understand the space of linear functions on the core. If, as usual, we denote the core of the double vector bundle $\nu\circ\nu$ by E^*, then the lemma below represents the last piece of the $\mathscr{DVB}$ sequence for $\nu\circ\nu$.

Lemma 3.4.3. *There is an isomorphism of $C^\infty(N)$-modules*

$$I^{1,1}/I^{1,0}I^{0,1} \to C^\infty_{[1]}(E^*).$$

Proof. Using standard linear algebra:

$$\begin{aligned} C^\infty_{[1]}(TM|_N/(TN_1|_N + TN_2|_N)) &= \Gamma(\mathrm{Ann}(TN_1|_N + TN_2|_N)) \\ &= \Gamma(\mathrm{Ann}(TN_1|_N)) \cap \Gamma(\mathrm{Ann}(TN_2|_N)), \end{aligned}$$

where Ann denotes an annihilator bundle inside $T^*M|_N$. Exterior differentiation gives a natural surjection

$$I^{1,1} \to \Gamma(\mathrm{Ann}(TN_1|_N)) \cap \Gamma(\mathrm{Ann}(TN_2|_N)).$$

The kernel of this map is $I^{1,0} \cap I^{0,1} \cap J^2$, where $J \subseteq C^\infty(M)$ denotes the ideal of functions that vanish on N. Now since N_1 and N_2 intersect cleanly, it follows that $J = I^{1,0} + I^{0,1}$ (simply dualize the definition of clean intersection). Thus the kernel of the map above is

$$I^{1,1} \cap (I^{2,0} + I^{1,0}I^{0,1} + I^{0,2}).$$

Since $I^{2,0} \subseteq I^{1,0}$ and $I^{0,2} \subseteq I^{0,1}$, this is just equal to $I^{2,1} + I^{1,0}I^{0,1} + I^{1,2}$. Moreover, the ideals $I^{2,1}$ and $I^{1,2}$ are contained in $I^{1,0}I^{0,1}$ (one way to see this is to check the generators of each ideal in the coordinates of proposition B.2.2), and so in the end the kernel is simply $I^{1,0}I^{0,1}$. □

Proof of theorem 3.4.1. By the results of the two lemmas above, the $\mathscr{DVB}$ sequence of (3.8) is determined by the sequence of $C^\infty(N)$ modules below. We note that the injection is given by $[f]\otimes[g] \mapsto [fg]$, and the surjection is the standard one induced by the inclusion $I^{2,1} + I^{1,2} \subseteq I^{1,0}I^{0,1}$.

$$0 \to \mathscr{A}^{1,0} \otimes \mathscr{A}^{0,1} \to \mathscr{A}^{1,1} \to I^{1,1}/I^{1,0}I^{0,1} \to 0 \tag{3.15}$$

which we claim agrees with (3.7). To see this, consider the map obtained by the composition of quotient maps $I^{1,1} \to \mathscr{A}^{1,1} \to \mathscr{A}^{1,1}/\mathscr{A}^{1,0}\mathscr{A}^{0,1}$. Its kernel consists of all functions $f \in I^{1,1}$ that lie in $I^{1,2} + I^{2,1}$, as well as those functions that lie in $I^{1,0}I^{0,1}$. In other words, the kernel is $I^{2,1}+I^{0,1}I^{1,0}+I^{1,2}$. However, as observed above, there is an inclusion of ideals $I^{1,2}+I^{2,1} \subseteq I^{1,0}I^{0,1}$, and so we get an isomorphism

$$I^{1,1}/I^{1,0}I^{0,1} \xrightarrow{\sim} \mathscr{A}^{1,1}/\mathscr{A}^{1,0}\mathscr{A}^{0,1}.$$

Since this isomorphism is induced by a composition of quotient maps, it forms a canonical isomorphism of $\mathscr{DVB}$ sequences from (3.15) to (3.7) when the left and middle morphisms are taken to be the identity maps. □

Next we will use a similar strategy to prove that $\nu(M, N_1, N_2)$ is canonically isomorphic to the quotient $TT(M, N_1, N_2)/\mathfrak{h}$ described in section 3.3. Fortunately, most of the work done above for the previous equivalence can be reused here, so we can dive right in to the proof.

Theorem 3.4.4. *Let (M, N_1, N_2) be a manifold triple. Then there exists a canonical map*

$$\nu(M, N_1, N_2) \to TT(M, N_1, N_2)/\mathfrak{h}$$

that is a double vector bundle isomorphism between (3.6) *and* (3.14)*.*

Proof. By proposition 2.10.4, on the level of sections the $\mathscr{DVB}$ sequence of $TT(M, N_1, N_2)/\mathfrak{h}$ is

$$0 \to \mathscr{A}^{1,0} \otimes \mathscr{A}^{0,1} \to C^\infty(TT(M, N_1, N_2))^{\mathfrak{h}}_{[1,1]} \to \mathscr{A}^{1,1}/\mathscr{A}^{1,0}\mathscr{A}^{0,1} \to 0,$$

where we have used the facts that $\mathscr{A}^{1,0} \cong \Gamma(\mathrm{Ann}(TN)) \subseteq \Gamma(T^*N_1|_N)$ and $\mathscr{A}^{0,1} \cong \Gamma(\mathrm{Ann}(TN)) \subseteq \Gamma(T^*N_2|_N)$ (see equation 3.5), and that $\Gamma(\mathrm{Ann}(TN_1|_N + TN_2|_N) \cong \mathscr{A}^{1,1}/\mathscr{A}^{1,0}\mathscr{A}^{0,1}$ (see the proof of theorem 3.4.1 above). Let us compute the middle term of the sequence above. By the definition of Q_1, a function of the form $f^{(1,1)}$ for $f \in C^\infty(M)$ is $\widetilde{H}$-invariant precisely when $f \in I^{1,1}$. The same holds true if we replace $\widetilde{H}$ by $\widetilde{V}$ or by $TN_1|_N + TN_2|_N$, so we get a map

$$\begin{aligned} I^{1,1} &\to C^\infty(TT(M, N_1, N_2)_{[1,1]})^{\mathfrak{h}} \\ f &\mapsto f^{(1,1)}. \end{aligned}$$

This map is surjective, since if $g^{(1,0)}h^{(0,1)}$ is $\mathfrak{h}$-invariant, then $g \in I^{1,0}$ since it is $\widetilde{H}$-invariant, $h \in I^{0,1}$ since it is $\widetilde{V}$-invariant, and it follows that $g^{(1,0)}$ and $h^{(0,1)}$ vanish on $TT(M, N_1, N_2)$, so the double-linear functions on $TT(M, N_1, N_2)$ that are $\mathfrak{h}$-invariant are all of the form $f^{(1,1)}$. The kernel of this mapconsists of those functions of the form $f^{(1,1)}$ that lie in the vanishing ideal of $TT(M, N_1, N_2)$. Consulting the generators of this ideal and using equation 3.11, we conclude that the kernel is generated by products of smooth functions on M for which the terms on the right hand side of 3.11 all vanish. Now if $fg \in I^{1,1}$, there are four possibilities:

$$f \in I^{1,1}, \quad g \in I^{1,1}, \quad f \in I^{1,0} \text{ and } g \in I^{0,1}, \quad f \in I^{0,1} \text{ and } g \in I^{1,0}.$$

In the first case, the only term on the right hand side of equation 3.11 that does not necessarily vanish on $TT(M, N_1, N_2)$ is $f^{(1,1)}g^{(0,0)}$. In order for this term to be zero, we need $g^{(0,0)}|_{TT(M,N_1,N_2)} = 0$, which means that $g \in I^{1,0} + I^{0,1}$ and thus $fg \in I^{1,2} + I^{2,1}$. The second case is similar. For the last case, the only term that may be nonzero is $f^{(1,0)}g^{(0,1)}$. For this term to vanish, we need either $f \in I^{1,1}$ or

$g \in I^{1,1}$, landing us in one of the first two cases. The second to last case is handled similarly, and so we conclude that the kernel is given by $I^{1,2} + I^{2,1}$, completing the proof. $\square$

3.5 Lifting Processes for $\nu(M, N_1, N_2)$

In section 3.3 we discussed various processes for lifting functions and vector fields on M to the total space of the double tangent bundle TTM. In this section, we will adapt these lifts to the setting of the double normal bundle $\nu(M, N_1, N_2)$ of a manifold triple (M, N_1, N_2). First suppose we have a function $f \in I^{p,q}$, and let $[f]_{p,q}$ denote its equivalence class in $I^{p,q}/(I^{p+1,q} + I^{p,q+1})$. This class determines a function $f^{(p,q)}$ on $\nu(M, N_1, N_2) = \operatorname{Spec} \mathcal{A}$ by evaluation:

$$\begin{aligned} f^{(p,q)} \colon \operatorname{Spec} \mathcal{A} &\to \mathbb{R} \\ \varphi &\mapsto \varphi([f]_{p,q}). \end{aligned}$$

Note that by construction, the function $f^{(p,q)}$ is homogeneous of bidegree (p, q). To lift vector fields, observe that the space $\mathfrak{X}(M)$ inherits a bilfiltration from the one on $C^\infty(M)$ that starts in degree $(-1, -1)$:

$$\mathfrak{X}(M) = \bigcup_{i,j \geq -1} \mathfrak{X}_{(i,j)}(M, N_1, N_2),$$

where $X \in \mathfrak{X}_{(i,j)}(M, N_1, N_2)$ if and only if the associated Lie derivative $\mathcal{L}_X$ takes $I^{p,q}$ to $I^{p+i,q+j}$. As in the case of $C^\infty(M)$, we will usually omit from our notation the dependence of the bifiltration on the manifold triple (M, N_1, N_2) and simply write $\mathfrak{X}_{(i,j)}(M)$. As special cases, note that $\mathfrak{X}_{(0,-1)}(M)$ consists of vector fields tangent to N_1, and similarly $\mathfrak{X}_{(-1,0)}(M)$ consists of vector fields tangent to N_2.

Lemma 3.5.1. *Any vector field $X \in \mathfrak{X}_{(i,j)}(M)$ induces a vector field $X_{(i,j)}$ on $\nu(M, N_1, N_2)$ that is bihomogeneous of degree (i, j). That is,*

$$(\kappa^h_t)^* X^{(i,j)} = t^i X^{(i,j)}, \quad (\kappa^v_s)^* X^{(i,j)} = s^j X^{(i,j)}, \quad \forall t, s \in \mathbb{R}.$$

Moreover, the assignment $X \mapsto X^{(i,j)}$ is compatible with the Lie bracket of vector fields. Specifically, for $X \in \mathfrak{X}_{(i_1,j_1)}(M)$ and $Y \in \mathfrak{X}_{(i_2,j_2)}(M)$ we have

$$[X^{(i_1,j_1)}, Y^{(i_2,j_2)}] = [X, Y]^{(i_1+i_2, j_1+j_2)}.$$

Proof. Any double vector bundle D is completely determined by the bigraded algebra bundle $\mathcal{S}^{\bullet,\bullet}(D)$ consisting of double polynomial functions on D. In the case of $\nu(M, N_1, N_2)$, theorem 3.1.8 implies that $\mathcal{S}^{\bullet,\bullet}(\nu(M, N_1, N_2)) \cong \mathcal{A}$, so to define a vector field on $\nu(M, N_1, N_2)$ it is enough to define a derivation on $\mathcal{A}$. Given $X \in \mathfrak{X}_{(i,j)}(M)$, define $X_{(i,j)} \in \mathfrak{X}(\nu(M, N_1, N_2)$ by

$$\mathcal{L}_{X_{(i,j)}}([f]_{p,q}) = [\mathcal{L}_X(f)] \quad f \in I^{p,q}.$$

This is well-defined by definition of $\mathfrak{X}_{(i,j)}(M)$, and the fact that it is a derivation follows from the fact that X is. The compatibility of the assignment $X \mapsto X^{(i,j)}$ with Lie brackets is due to the equation $\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$. $\square$

The story above transports to the setting of multi-vector fields in a straightforward way. That is, the spaces $\mathfrak{X}^k(M)$ come equipped with a bifiltration

$$\mathfrak{X}^k(M) = \bigcup_{n,m \geq -k} \mathfrak{X}^k_{(n,m)}(M),$$

and any $\pi \in \mathfrak{X}^k_{(n,m)}(M)$ defines a multi-vector field $\pi_{(n,m)}$ on $\nu(M, N_1, N_2)$ that satisfies

$$(\kappa^h_t)^* \pi^{(n,m)} = t^n \pi^{(n,m)}, \quad (\kappa^v_s)^* \pi^{(n,m)} = s^m \pi^{(n,m)}$$

for all $t, s \in \mathbb{R}$. Here $\mathfrak{X}^k_{(n,m)}(M)$ is spanned by those $X_1 \wedge \ldots \wedge X_k$ with $X_\ell \in \mathfrak{X}_{(i_\ell, j_\ell)}(M)$ satisfying $i_1 + \cdots + i_k = n$ and $j_1 + \cdots + j_k = m$. The existence of $\pi^{(n,m)}$ then follows from the lemma above and the fact that the wedge product of multi-vector fields on a double vector bundle is compatible with the bihomogeneous grading. Moreover, the assignment $\pi \mapsto \pi^{(n,m)}$ is compatible with the Schouten bracket of multi-vector fields.

Remark 3.5.2.

1. Given a function $f \in I^{p,q}$ for $0 \leq p, q \leq 1$, we now have two different lifts: one to the double tangent bundle TTM, and another to the double normal bundle $\nu(M, N_1, N_2)$. Observe that we have chosen to use the same notation $f^{(p,q)}$ for both of these operations. This can be justified by the fact that the lift to the double tangent bundle induces the corresponding lift to the double normal bundle through the quotient construction described in section 3.3. In any case, we will rely on context to make it clear which lift we are using.

2. Dual to the discussion of multi-vector fields above, there is a similar theory for differential forms on M. In particular, $\Omega(M)$ inherits a bifiltration

$$\Omega(M) = \bigcup_{i,j \geq 0} \Omega_{(i,j)}(M),$$

where $\Omega_{(i,j)}(M)$ is spanned by forms $df_1 \wedge \ldots \wedge df_n$ with $f_k \in I^{p_k, q_k}$ and such that $p_1 + \cdots + p_n = i$ and $q_1 + \cdots + q_n = j$. Then any form $\alpha \in \Omega_{(i,j)}(M)$ induces a form $\alpha^{(i,j)}$ on $\nu(M, N_1, N_2)$ that is bihomogeneous of degree (i, j). By construction, the bifiltrations on $C^\infty(M)$, $\mathfrak{X}(M)$ and $\Omega(M)$ are compatible with the usual operators of calculus: the exterior derivative, contractions, and Lie derivatives.

3.6 The Double Deformation Space

Recall that for a manifold pair (M, N), the *deformation to the normal cone* is a manifold $\mathcal{D}(M, N)$ fibred over $\mathbb{R}$ that can be thought of as stretching out the directions normal to N as $t \to 0$. Specifically, as a set it is given by

$$\mathcal{D}(M, N) = \nu(M, N) \sqcup (M \times \mathbb{R}^*),$$

while its smooth structure is determined by the following properties:

1. The map $\pi\colon \mathcal{D}(M, N) \to \mathbb{R}$ that is the zero map on $\nu(M, N)$ and acts as $(m, t) \mapsto t$ on $M \times \mathbb{R}^*$ is a surjective submersion.

2. The map $\gamma\colon \mathcal{D}(M,N) \to M$ defined to be the composition $\nu(M,N) \to N \hookrightarrow M$ for $t=0$ and by $(m,t) \mapsto m$ on $M \times \mathbb{R}^*$ is smooth.

3. For any $f \in C^\infty(M)$ such that $f|_N = 0$, the map $\tilde{f}$ given by $\nu(f)$ on $\nu(M,N)$ and by $f(m,t) = t^{-1}f(m)$ on $M \times \mathbb{R}^*$ is smooth.

Put another way, the smooth structure is such that $C^\infty(\mathcal{D}(M,N))$ is generated by π as well as functions of the form

$$f^{(p)}(x,t) = \begin{cases} f^{(p)}(x), & t = 0, \\ \frac{1}{t^p} f(x), & t \neq 0, \end{cases}$$

where $f \in C^\infty(M)$ is a function that vanishes on N to order p. Here we are using the lifts (B.1) to the normal bundle, so note that using the language above, we have $f^{(0)} = \kappa^* f$ and $f^{(1)} = \tilde{f}$. For more information on this approach to deformation spaces, see [4] or [33]. As with the normal bundle, the deformation space $\mathcal{D}(M,N)$ may also be described as the spectrum of a suitable algebra. Specifically, let $\mathfrak{L}(M,N)$ denote the algebra consisting of Laurent polynomials

$$\sum_{n \in \mathbb{Z}} a_n t^{-n}$$

whose coefficients are smooth functions $a_n \in C^\infty(M)$ that vanish to order n on N, where by convention this puts no restrictions on a_n for $n \leq 0$. Then $\mathrm{Spec}(\mathfrak{A}(M,N))$ is a manifold, and moreover it is equal to $\mathcal{D}(M,N)$ (see [31, Section 3]).

We will present an analogous construction in the setting of double structures. Given a manifold triple (M,N_1,N_2), the *double deformation space* $\mathcal{D}(M,N_1,N_2)$ will be a manifold fibred over $\mathbb{R}^2$, where the fibre over $(0,0)$ is $\nu(M,N_1,N_2)$, the fibres over $0 \times \mathbb{R}^*$ are isomorphic to $\nu(M,N_1)$, the fibres over $0 \times \mathbb{R}^*$ are isomorphic to $\nu(M,N_2)$, and the fibres everywhere else are diffeomorphic to M itself. Thus as a set it is given by

$$\mathcal{D}(M,N_1,N_2) = \nu(M,N_1,N_2) \sqcup (\nu(M,N_1) \times \mathbb{R}^*) \sqcup (\mathbb{R}^* \times \nu(M,N_2)) \sqcup (M \times \mathbb{R}^* \times \mathbb{R}^*),$$

and we will denote the fibration $\mathcal{D}(M,N_1,N_2) \to \mathbb{R}^2$ by (s,t). To understand the smooth structure on this space, let us extend some of the lifting processes discussed in section 3.5 above to the set $\mathcal{D}(M,N_1,N_2)$. Once again we will continue to use the same notation for these lifts even though we have changed the domain of the lift, and rely on context to distinguish which one we mean. For any smooth function $f \in I^{p,q}$, we define the function $f^{(p,q)}$ on $\mathcal{D}(M,N_1,N_2)$ by the formulas

$$f^{(p,q)}(s,t,x) = \begin{cases} f^{(p,q)}(x) & s = t = 0, \\ \frac{1}{t^q} f^{(p)}(x) & s = 0, t \neq 0, \\ \frac{1}{s^p} f^{(q)}(x) & s \neq 0, t = 0, \\ \frac{1}{s^p t^q} f(x) & s \neq 0, t \neq 0. \end{cases}$$

Here we have used to lifting processes related to the usual normal bundle explained in section B.1, for convenience we recall that $f^{(0)}$ is the pullback along the map $\nu(M,N_i) \to N_i \hookrightarrow M$, while $f^{(1)}$ denotes $\nu(f) \in C^\infty(\nu(M,N_i))$, which is defined whenever $f|_{N_i} = 0$. We also note that $f^{(0,0)}$ may alternatively be defined as the pullback of f along the map $\gamma\colon \mathcal{D}(M, N_1, N_2) \to M$ that is built out of the natural maps

$$\gamma_{s,t}\colon M \to M, \quad \gamma_{0,t}\colon \nu(M, N_1) \to N_1 \hookrightarrow M, \quad \gamma_{s,0}\colon \nu(M, N_2) \to N_2 \hookrightarrow M, \quad \gamma_{0,0}\colon \nu(M, N_1, N_2) \to N \hookrightarrow M,$$

for $s, t \neq 0$. With all of this machinery in place, we now topologize $\mathcal{D}(M, N_1, N_2)$ by giving it the topology with the fewest sets for which the functions $f^{(p,q)}$ for $f \in I^{p,q}$ as well as the map $\gamma\colon \mathcal{D}(M, N_1, N_2) \to M$ and the fibration (s, t) are all continuous. Our first claim is that this topology leads to a smooth structure on $\mathcal{D}(M, N_1, N_2)$.

Theorem 3.6.1. *Given a manifold triple (M, N_1, N_2), the space $\mathcal{D}(M, N_1, N_2)$ is a smooth manifold.*

Proof. Choose coordinates $x_1, \ldots, x_m$ on M such that (see proposition B.2.2)

$$\begin{aligned} N_1 &= \{x_1 = \cdots = x_e = 0,\ x_{e+1} = \cdots = x_k = 0\}, \\ N_2 &= \{x_1 = \cdots = x_e = 0,\ x_{k+1} = \cdots = x_\ell = 0\}. \end{aligned}$$

and let $U \subseteq M$ be the open subset on which these coordinates are defined. Then the subset $\mathcal{D}(U, N_1 \cap U, N_2 \cap U) = \gamma^{-1}(U)$ is an open subset of $\mathcal{D}(M, N_1, N_2)$ since γ is continuous, and the functions

$$x_1^{(1,1)}, \ldots, x_e^{(1,1)}, x_{e+1}^{(1,0)}, \ldots, x_k^{(1,0)}, x_{k+1}^{(0,1)}, \ldots, x_\ell^{(0,1)}, x_{\ell+1}^{(0,0)}, \ldots, x_m^{(0,0)}, s, t \tag{3.16}$$

are all defined on $\gamma^{-1}(U)$. These functions define a map

$$\gamma^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^* \times \mathbb{R}^*$$

that is continuous, injective, and has open image. Let us use the coordinates x_m to identify U with a subset of $\mathbb{R}^m = \mathbb{R}^{m-\ell-k-e} \times \mathbb{R}^e \times \mathbb{R}^{k-e} \times \mathbb{R}^{\ell-k-e}$, and to identify $\nu(U, N_1 \cap U, N_2 \cap U)$ with $N \cap U \times \mathbb{R}^e \times \mathbb{R}^{k-e} \times \mathbb{R}^{\ell-k-e}$, as well as similar identifications for $\nu(M, N_1)$ and $\nu(M, N_2)$. Then we can give an explicit formula for the inverse of the map above:

$$(u, x, y, z, s, t) \mapsto \begin{cases} (u, x, y, z, 0, 0) & s = 0, t = 0, \\ (u, sx, y, sz, s, 0), & s \neq 0, t = 0, \\ (u, x, ty, tz, 0, t), & s = 0, t \neq 0, \\ (u, sx, ty, stz, s, t) & s \neq 0, t \neq 0. \end{cases}$$

To see that this map is continuous, it is enough to check that it's composition with the fibration (s, t), the map γ, and the maps $f^{(p,q)}$ for $f \in I^{p,q}$ (one can even restrict to $0 \leq p, q \leq 1$) are all continuous, which can be verified directly. Thus the functions (3.16) give $\mathcal{D}(M, N_1, N_2)$ the structure of a topological manifold. In fact, they define a smooth structure on $\mathcal{D}(M, N_1, N_2)$, which follows from the fact that $x_1, \ldots, x_m$ forms a smooth coordinate system since s, t are global coordinates. $\square$

It is worth noting that it follows from the proof of the theorem above that the algebra of smooth functions $C^\infty(\mathcal{D}(M, N_1, N_2))$ is generated by functions of the form $f^{(p,q)}$ for $f \in I^{p,q}$ along with the maps s, t. Having confirmed that $\mathcal{D}(M, N_1, N_2)$ is smooth, we now give it a name.

Definition 3.6.2 (Double Deformation Space)**.** Let (M, N_1, N_2) be a manifold triple. Then the *double deformation space* of (M, N_1, N_2) is the smooth manifold $\mathcal{D}(M, N_1, N_2)$.

In complete analogy with the double normal bundle, the double deformation space admits two alternative characterizations, one in terms of iterated functors and another in terms of algebra spectra. First let us understand the iterated functor approach. Using lemma 3.2.1, we may identify the deformation space $\mathcal{D}(N_2, N)$ as a submanifold of the space $\mathcal{D}(M, N_1)$. We claim that by taking another deformation space, we end up with the double deformation space. To start with, we observe that the underlying sets of $\mathcal{D}(M, N_1, N_2)$ and $\mathcal{D}(\mathcal{D}(M, N_1), \mathcal{D}(N_2, N))$ are equal, which can be seen easily through the following diagram

$$\begin{tikzcd}
 & & & \mathcal{D}(\mathcal{D}(M,N_1),\mathcal{D}(N_2,N)) \arrow[dll, dash, "t=0"'] \arrow[drr, dash, "t\neq 0"] & & & \\
 & \nu(\mathcal{D}(M,N_1),\mathcal{D}(N_2,N)) \arrow[dl, dash, "s=0"'] \arrow[dr, dash, "s\neq 0"] & & & & \mathcal{D}(M,N_1) \arrow[dl, dash, "s=0"'] \arrow[dr, dash, "s\neq 0"] & \\
\nu(\nu(M,N_1),\nu(N_2,N)) & & \nu(M,N_2) & & \nu(M,N_1) & & M
\end{tikzcd}$$

This now gives two smooth structures on the underlying set of $\mathcal{D}(M, N_1, N_2)$: the one described above and another by thinking of it as a (usual) deformation space. In fact, these two structures are the same.

Proposition 3.6.3. *The smooth structure on $\mathcal{D}(M, N_1, N_2)$ described in theorem 3.6.1 agrees with the smooth structure on $\mathcal{D}(\mathcal{D}(M, N_1), \mathcal{D}(N_2, N))$ coming from the usual deformation to the normal cone construction.*

Proof. The smooth functions on $\mathcal{D}(M, N_1, N_2)$ are generated by functions of the form $f^{(p,q)}$ for $f \in I^{p,q}$. On the other hand, smooth functions on $\mathcal{D}(\mathcal{D}(M, N_1), \mathcal{D}(N_2, N))$ are generated by functions of the form $f^{(q)}$ for $f \in J^q$, where J denotes the ideal of smooth functions on $\mathcal{D}(M, N_1)$ that vanish on $\mathcal{D}(N_2, N)$. In turn, smooth functions on $\mathcal{D}(M, N_1)$ are generated by functions of the form $f^{(p)}$ for $f \in I^{p,0}$. But we have $f^{(p)} \in J$ precisely when $f \in I^{p,1}$, and so in order to prove the proposition, it suffices to show that the functions $f^{(p,q)}$ and $(f^{(p)})^{(q)}$ for $f \in I^{p,q}$ generate the same collection of smooth functions. At all points x away from the $(0, 0)$-fibre we have equality $f^{(p,q)}(x) = (f^{(p)})^{(q)}(x)$, so we need only consider the analogous question for the lifts to $\nu(M, N_1, N_2)$ and $\nu(\nu(M, N_1), \nu(N_2, N)$. But functions of the form $f^{(p,q)}$ generate the smooth functions on $\nu(M, N_1, N_2)$ while functions of the form $(f^{(p)})^{(q)}$ generate the smooth functions on $\nu(\nu(M, N_1), \nu(N_2, N))$, and so the result follows from theorem 3.4.1. $\square$

Remark 3.6.4. For some time now, deformation spaces have been a tool used in index theory, see for example [33]. Recently, in [66], iterated deformations spaces were introduced as a means of generalizing various index theorems, relating in particular to hypoelliptic pseudo-differential operators. The construction outlined there begins with a manifold M, a submanifold $N \subseteq M$, and another submanifold $V \subseteq \mathcal{D}(M, N)$. The iterated deformation space is then defined to be $\mathcal{D}(\mathcal{D}(M, N), V)$. The discussion above shows that our double deformation spaces $\mathcal{D}(M, N_1, N_2)$ provide an alternate description of these iterated deformation spaces in the setting where the submanifold V can itself be identified with a deformation space.

Each fibre of the double deformation space $\mathscr{D}(M, N_1, N_2)$ is a smooth manifold that can be defined as the spectrum of an appropriate algebra. Therefore it is natural to ask whether $\mathscr{D}(M, N_1, N_2)$ itself can be described as a spectrum. This is indeed the case, which can be seen using the so-called Rees algebra of the bifiltration 3.1. More specifically, let $\mathfrak{DL}(M, N_1, N_2)$ denote the algebra consisting of Laurent polynomials in two variables (or "double Laurent polynomials") of the form

$$\sum_{p,q} a_{p,q} s^{-p} t^{-q},$$

where $a_{p,q}$ are smooth functions on M such that $a_{p,q} \in I^{p,q}$ whenever $p, q > 0$. As usual, we will omit the dependence on the manifold triple whenever possible, writing simply $\mathfrak{DL}$. The spectrum of this algebra comes with a fibration to $\mathbb{R}^2$, given by the map

$$\begin{aligned} \operatorname{Spec} \mathfrak{DL} &\to \mathbb{R}^2 \\ \varphi &\mapsto (\varphi(s), \varphi(t)). \end{aligned}$$

Note that the fibre over (λ, μ) is given by the spectrum of the algebra $\mathfrak{DL}/\langle t - \lambda, s - \mu\rangle$. Let us describe these fibres in a more tangible way.

Lemma 3.6.5. *Let (M, N_1, N_2) be a manifold triple. As a set, we have $\operatorname{Spec} \mathfrak{DL} = \mathscr{D}(M, N_1, N_2)$, with the fibration over $\mathbb{R}^2$ being given by the map $\varphi \mapsto (\varphi(s), \varphi(t))$.*

Proof. First we consider the case $\lambda = \mu = 0$ by examining the map

$$\begin{aligned} \mathfrak{DL} &\to \mathscr{A} \\ \sum_{p,q} a_{p,q} s^{-p} t^{-q} &\mapsto \sum_{p,q \geq 0} [a_{p,q}]_{p,q}, \end{aligned}$$

where $\mathscr{A}$ denotes the algebra (3.2), and $[a_{p,q}]_{p,q}$ denotes the class of $a_{p,q}$ in $I^{p,q}/(I^{p+1,q} + I^{p,q+1})$. The map is clearly surjective, and its kernel consists of all Laurent polynomials spanned by monomials of the form $a_{p,q} s^{-p} t^{-q}$ with $p, q < 0$ as well as those monomials of the form $a_{p,q} s^{-p} t^{-q}$ for which we have $a_{p,q} \in I^{p',q'}$ with either $p' > p$ or $q' > q$ (or both). Monomials of the former kind clearly lie in $\langle s, t\rangle$, as do monomials of the latter kind since then $a_{p,q} s^{-p} t^{-q} = (a_{p,q} s^{-p'} t^{-q'}) \cdot s^{p'-p} t^{q'-q}$. Moreover, these monomials span $\langle s, t\rangle$, so we have shown that $\mathfrak{DL}/\langle s, t\rangle \cong \mathscr{A}$, which in turn means that the fibre over $(0, 0)$ is given by $\operatorname{Spec} \mathscr{A} = \nu(M, N_1, N_2)$.

Now suppose that $\lambda \neq 0$ but $\mu = 0$, and consider the map

$$\begin{aligned} \mathfrak{DL} &\to \mathfrak{L}(M, N_1) \\ \sum_{p,q} a_{p,q} s^{-p} t^{-q} &\mapsto \sum_{q} \left(\sum_{p} \lambda^{-p} a_{p,q} \right) t^{-q} \end{aligned}$$

given by evaluation at $s = \lambda$. Once again, this map is clearly surjective, and we claim its kernel is $\langle s - \lambda\rangle$. To see this, we may use "long division" to write any Laurent polynomial as

$$\sum_{p,q} a_{p,q} s^{-p} t^{-q} = (s - \lambda) \sum_{p,q} \left(\sum_{j \geq 0} a_{p-j,q} \lambda^{j} \right) s^{-p} t^{-q},$$

as can be verified by expanding the right hand side. If this Laurent polynomial satisfies $\sum_{p,q} a_{p,q}\lambda^{-p}t^{-q} = 0$, then by multiplying this equation through by an approrpiate power of λ and reindexing if necessary, we see that only finitely many terms of the form $\sum_{j\geq 0} a_{p-j,q}\lambda^j$ can be nonzero, so the right hand side of the equation above lies in $\langle s-\lambda\rangle$. Since the reverse inclusion is obvious, we conclude that $\mathfrak{DL}/\langle s-\lambda\rangle \cong \mathfrak{L}(M,N_1)$. Composing with evaluation at $t=0$ then tells us that the fibre over $(\lambda,0)$ is given by the 0-fibre of $\mathscr{D}(M,N_1)$, in other words it is $\nu(M,N_1)$. If $\lambda=0$ but $\mu\neq 0$, one similarly finds that the fibre over (λ,μ) is given by $\nu(M,N_2)$.

Finally, suppose that neither λ nor μ are equal to zero. Then the isomorphism $\mathfrak{DL}/\langle s-\lambda\rangle \cong \mathfrak{L}(M,N_1)$ described above still stands, and so the fibre over (λ,μ) is equal to the μ-fibre of $\mathscr{D}(M,N_1)$, in other words, it is M. □

Recall that the topology on $\operatorname{Spec}\mathfrak{DL}$ is the topology with the fewest open sets for which the evaluation maps are all continuous. Note that in this context, the map $\gamma\colon \operatorname{Spec}\mathfrak{DL}\to\operatorname{Spec}C^\infty(M)$ is induced by the inclusion $C^\infty(M)\hookrightarrow\mathfrak{DL}$, which is continuous since composing it with any evaluation map on $\operatorname{Spec}C^\infty(M)$ yields an evaluation map on $\operatorname{Spec}\mathfrak{DL}$. Observe that under the fibration described in lemma 3.6.5, the functions $f^{(p,q)}$ on $\mathscr{D}(M,N_1,N_2)$ for $f\in I^{p,q}$ are identified with the evaluation maps $\operatorname{ev}(fs^{-p}t^{-q})$. It follows that the two topologies on $\mathscr{D}(M,N_1,N_2)$ implied by the lemma above coincide. Moreover, it is straightforwad to prove that $\operatorname{Spec}\mathfrak{DL}$ is in fact a smooth manifold. Away from the $(0,0)$-fibre, this follows from the fact that $\operatorname{Spec}\mathfrak{L}(M,N_1)$, $\operatorname{Spec}\mathfrak{L}(M,N_2)$, and $\operatorname{Spec}C^\infty(M)$ are all smooth. Near the $(0,0)$-fibre, one can simply replicate the proof of theorem 3.6.1, replacing the coordinates by the evaluation functions

$$\begin{gathered}\operatorname{ev}(x_1s^{-1}t^{-1}),\dots,\operatorname{ev}(x_es^{-1}t^{-1}),\operatorname{ev}(x_{e+1}s^{-1}),\dots,\operatorname{ev}(x_ks^{-1}),\\ \operatorname{ev}(x_{k+1}t^{-1}),\dots,\operatorname{ev}(x_\ell t^{-1}),\operatorname{ev}(x_{\ell+1}),\dots,\operatorname{ev}(x_m),s,t,\end{gathered}$$

which as before are defined on the open set $\gamma^{-1}(U)$, where $U\subseteq M$ is an open set on which the coordinates $x_1,\dots,x_m$ are defined. As noted above, these coordinates are precisely the same as the coordinates (3.16), and so we observe the following result.

Proposition 3.6.6. *The smooth structure on $\mathscr{D}(M,N_1,N_2)$ described in theorem 3.6.1 agrees with the smooth structure on* $\operatorname{Spec}\mathfrak{DL}$ *coming from the usual manifold structure on spectrums of suitable algebras.*

Chapter 4

Weil Algebras

Recall that any vector bundle $V \to M$ may be recovered from its graded algebra $\mathcal{S}^\bullet(V) = C^\infty(V)_{[\bullet]} \cong \Gamma(\wedge^\bullet V^*)$ of fiberwise polynomial functions as the set of algebra morphisms (spectrum) $V \cong \mathrm{Hom}_{\mathrm{alg}}(\mathcal{S}(V), \mathbb{R})$; the algebra endomorphism given as multiplication by t^k on $\mathcal{S}^k(V)$ corresponds to the scalar multiplication on V, its infinitesimal generator is the Euler vector field. In super-geometry, the super-commutative graded algebra $\Gamma(\wedge V^*)$ is regarded as the algebra of functions on the graded manifold $V[1]$, where the [1] signifies a degree shift.

In a similar way, double vector bundles may be recovered for their bigraded algebra $\mathcal{S}^{\bullet,\bullet}(D)$ of double-polynomial functions, as $D = \mathrm{Hom}_{\mathrm{alg}}(\mathcal{S}(D), \mathbb{R})$. The two scalar multiplications correspond to the two gradings on $\mathcal{S}(D)$. In this section we give a geometric model $\mathcal{W}(D)$ for the bigraded algebra of double-polynomial functions on the supermanifold $D[1,1]$, obtained by parity change of the two vector bundle directions. (See [77] for details.)

As we shall see, both $\mathcal{S}(D)$ and $\mathcal{W}(D)$ are generated by double-polynomial functions of bidegree (r, s) with $r, s \leq 1$, modulo a quadratic relation. As pointed out to us by a referee, the existence of this type of presentation is an instance of a very general construction due to Grabowski-Jóźwikowski-Rotkievicz [25].

Throughout this section, we consider a fixed double vector bundle D over M, with side bundles A, B and with $E = \mathrm{core}(D)^*$. It will be convenient to regard D as an associated bundle $(P \times D_0)/\mathrm{Aut}(D_0)$ with $D_0 = A_0 \times B_0 \times E_0^*$.

4.1 Double-Polynomial Functions

A smooth function on a double vector bundle D will be called a (homogeneous) *double-polynomial of bidegree* (k, ℓ) if it is homogeneous of degree k for the horizontal scalar multiplication, and of degree ℓ for the vertical scalar multiplication. The space of such functions is denoted $\mathcal{S}^{k,\ell}(D) = C^\infty(D)_{[k,\ell]}$; their direct sum over all $k, \ell \geq 0$ is denoted $\mathcal{S}(D)$. In order to avoid confusion in this section, we will denote the symmetric tensor powers of a vector space by $\vee$ rather than by S.

Lemma 4.1.1. *The space $\mathcal{S}^{k,\ell}(D)$ of double-polynomial functions on D of bidegree (k, ℓ) is the space of sections of a vector bundle*

$$S^{k,\ell}(D) \to M.$$

Proof. We will first construct the space $S^{k,\ell}(D_0)$, and appeal to the associated bundle construction to obtain $S^{k,\ell}(D)$. A function $f \in C^\infty(D_0)$ lies in $\mathcal{S}^{k,\ell}(D_0)$ when

$$f(ta, b, t\varepsilon) = t^k f(a, b, \varepsilon), \quad f(a, sb, s\varepsilon) = s^\ell f(a, b, \varepsilon), \tag{4.1}$$

for all $t, s \in \mathbb{R}$. Choosing bases of A_0, B_0, and E_0^*, we see that $\mathcal{S}^{k,\ell}(D_0)$ is spanned by those monomials of total degree $k+\ell$ that are of degree no more than k in the A_0 and E_0^* coordinates and of degree no more than ℓ in the B_0 and E_0^* coordinates. In other words, we have

$$S^{k,\ell}(D_0) = \bigoplus_{i+n=k, j+n=\ell} \vee^i A_0^* \otimes \vee^j B_0^* \otimes \vee^n E_0. \tag{4.2}$$

The action of $\mathrm{Aut}(D_0)$ on D_0 (see lemma 2.5.1) induces a linear representation of $\mathrm{Aut}(D_0)$ on $C^\infty(D_0)$ via $\varphi.f = f \circ \varphi$, for which the defining equations (4.1) of $\mathcal{S}^{k,\ell}(D)$ are clearly equivariant. The vector bundle $S^{k,\ell}(D)$ is then given by $S^{k,\ell}(D) = (P \times S^{k,\ell}(D_0))/\mathrm{Aut}(D_0)$. $\square$

Summing over all bidegrees, we obtain a bundle of bigraded commutative algebras

$$S(D) = \bigoplus_{k,\ell} S^{k,\ell}(D).$$

Moreover, the construction of $S(D)$ is functorial. That is, given a $\mathcal{DVB}$ morphism $\varphi \colon D_1 \to D_2$ with base map $\Phi \colon M_1 \to M_2$, we get algebra morphisms $S(\varphi)_m \colon S(D_2)_{F(m)} \to S(D_1)_m$ for every $m \in M$, which induces a comorphism of bigraded algebra bundles $S(\varphi) \colon S(D_1) \dashrightarrow S(D_2)$. On the level of sections, the map $\mathcal{S}(D_2) \to \mathcal{S}(D_1)$ is simply given by the pullback of double polynomial functions along φ. In the case that $M_1 = M_2$ with $F = \mathrm{id}$, the comorphism $S(\varphi)$ can be seen as an ordinary morphism of algebra bundles: $S(\varphi) \colon S(D_2) \to S(D_1)$. Applying this functoriality to the $\mathcal{DVB}$ morphism $D \hookrightarrow \widehat{D}$ described in proposition 2.7.3, we expect to be able to recover the bundle $S(D)$ as a quotient of the bundle $S(\widehat{D})$, which the next proposition confirms.

Proposition 4.1.2. *The algebra bundle $S(D)$ is the bundle of bigraded commutative algebras*

$$S(D) = (\vee A^* \otimes \vee B^* \otimes \vee \widehat{E})/\sim$$

where the generators $\alpha \in A^*$, $\beta \in B^*$, $\widehat{e} \in \widehat{E}$ have bidegrees $(1,0)$, $(0,1)$, $(1,1)$, respectively. Here the kernel of the quotient map is the ideal generated by elements of the form

$$\alpha\beta - i_{\widehat{E}}(\alpha \otimes \beta)$$

for $(\alpha, \beta) \in A^* \times_M B^*$.

Proof. We consider the double vector space D_0 and we let

$$\bar{S}(D_0) = \vee A_0^* \otimes \vee B_0^* \otimes \vee((A_0^* \otimes B_0^*) \oplus E_0)/\sim$$

denote the proposed construction for $S(D_0)$. Since the $\mathcal{DVB}$ sequence for D_0 is canonically split, the quotient relation $\sim$ allows us to write the generators $\hat{e} = (\alpha \otimes \beta, e)$ as $\hat{e} = \alpha \otimes \beta \otimes (0, e)$. Hence $\bar{S}(D)$ is generated by A_0^* in bidegree $(1,0)$, by B_0^* in bidegree $(0,1)$, and by $E_0 \subseteq \widehat{E}_0$ in bidegree $(1,1)$, which agrees with (4.2). This shows that $\bar{S}(D_0) = S(D_0)$, and the general result follows by taking associated bundles. ☐

By applying a similar construction to the double vector bundles D' and D'', we also have the bigraded algebra bundles $S(D') \to M$ and $S(D'') \to M$. In the next section, we will anti-symmetrize the construction of $S(D)$ described in proposition 4.1.2 to obtain Weil algebra bundles $W(D)$, $W(D')$, and $W(D'')$. Before proceeding to the Weil algebra bundles, however, we record one more fact about double-polynomial functions. Recall that a vector bundle is completely determined by its space of polynomial functions. We observe that the situation for double vector bundles is analogous.

Lemma 4.1.3. *The double vector bundle D may be recovered from $\mathcal{S}(D)$ via*

$$D \cong \mathrm{Spec}(\mathcal{S}(D)).$$

Proof. This follows from example 3.1.10, where it is show that the $\mathcal{DVB}$ sequences for D and $\nu(D, A, B)$ coincide. Since $\mathcal{S}(D)$ is generated by $\Gamma(A^*)$, $\Gamma(B^*)$, and $\Gamma(\widehat{E})$ with relations defined by the $\mathcal{DVB}$ sequence of D, it follows that $\mathcal{S}(D) = \mathcal{A}(D, A, B)$ (see equation 3.2). This gives

$$\mathrm{Spec}(\mathcal{S}(D)) = \mathrm{Spec}(\mathcal{A}(D, A, B)) = \nu(D, A, B) \cong D.$$

☐

4.2 Definition of $W(D)$

The Weil algebra bundle is obtained from the description of $S(D)$, given in Proposition 4.1.2, by replacing commutativity with super-commutativity:

Definition 4.2.1. The *Weil algebra bundle* $W(D)$ is the bundle of bigraded super-commutative algebras given as

$$W(D) = (\wedge A^* \otimes \wedge B^* \otimes \vee \widehat{E})/\sim,$$

where the generators $\alpha \in A^*,\ \beta \in B^*,\ \widehat{e} \in \widehat{E}$ have bidegrees $(1,0),\ (0,1),\ (1,1)$ respectively. Here the kernel of the quotient map is the ideal generated by elements of the form

$$\alpha\beta - i_{\widehat{E}}(\alpha \otimes \beta)$$

with $(\alpha, \beta) \in A^* \times_M B^*$.

In the definition above, $\otimes$ denotes the usual tensor product of superalgebras. For $x \in W^{p,q}(D)$ we write

$$|x| = p + q$$

for the total degree; thus super-commutativity means $x_1x_2 = (-1)^{|x_1||x_2|}x_2x_1$. The bigraded algebra of sections $\mathcal{W}(D) = \Gamma(W(D))$ is called the *Weil algebra* of D. In super-geometric terms, it is the algebra of smooth functions on the supermanifold $D[1,1]$. Similar to the construction of $S(D)$, we claim that the association $D \mapsto \mathcal{W}(D)$ is functorial.

Lemma 4.2.2. *Given a morphism of double vector bundles $\varphi\colon D_1 \to D_2$, there exists a morphism of bigraded super-commutative algebras*

$$\mathcal{W}(\varphi)\colon \mathcal{W}(D_2) \to \mathcal{W}(D_1)$$

making the association $D \mapsto \mathcal{W}(D)$ a contravariant functor.

Proof. By the results of [13], any $\mathcal{DVB}$ morphism $\varphi\colon D_1 \to D_2$ induces vector bundle morphisms $A_2^* \to A_1^*$, $B_2^* \to B_1^*$, and $\widehat{E}_2 \to \widehat{E}_1$ that intertwine the $\mathcal{DVB}$ sequences of D_2 and D_1. Applying the exterior and symmetric algebra functors appropriately to these maps and passing to the quotient yields a bigraded algebra morphism $\mathcal{W}(\varphi)\colon \mathcal{W}(D_2) \to \mathcal{W}(D_1)$. Every step in this process is functorial ($\mathcal{DVB}$ sequence, exterior and symmetric algebras, quotient), hence the Weil algebra assignment is as well. $\square$

In terms of bundles, we get a comorphism $W(D_1) \dashrightarrow W(D_2)$ that becomes an honest morphism of bundles of super-commutative algebras $W(\varphi)\colon W(D_2) \to W(D_1)$ in the case that the total bases of D_1 and D_2 are the same and φ restricts to the identity there.

The definition gives a number of straightforward properties of $W(D)$:

1. In degree $p \leq 1$, $q \leq 1$, $W^{p,q}(D)$ coincides with $S^{p,q}(D)$:

$$W^{0,0,}(D) = M, \quad W^{1,0}(D) = A^*, \quad W^{0,1}(D) = B^*, \quad W^{1,1}(D) = \widehat{E}.$$

2. An argument identical to the proof of proposition 4.1.2 shows that the decomposed $\mathcal{DVB}$ given by $A \times_M B \times_M E^*$ has Weil algebra $\wedge A^* \otimes \wedge B^* \otimes \vee E$. Therefore, by functoriality, a choice of splitting $D \cong A \times_M B \times_M E^*$ gives an algebra bundle isomorphism

$$W(D) \cong \wedge A^* \otimes \wedge B^* \otimes \vee E.$$

3. As a particular case of the observation above, we have $W(D_0) = \wedge A_0^* \otimes \wedge B_0^* \otimes \vee E_0$. This allows for an alternative definition of the Weil algebra bundle as $W(D) = (P \times W(D_0))/\mathrm{Aut}(D_0)$.

Replacing D with D' and D'', we have three bigraded algebra bundles $W(D), W(D'), W(D'')$ over M, where the roles of A, B, and E are cyclically permuted. In particular,

$$W^{1,1}(D) = \widehat{E}, \quad W^{1,1}(D') = \widehat{A}, \quad W^{1,1}(D'') = \widehat{B}.$$

The pairings (2.16) between these bundles extend to

$$\begin{aligned} \langle \cdot, \cdot \rangle_{E^*}&\colon W^{p,1}(D'') \times_M W^{1,q}(D') \to \wedge^{p+q-1} E^*, \\ \langle \cdot, \cdot \rangle_{A^*}&\colon W^{p,1}(D) \times_M W^{1,q}(D'') \to \wedge^{p+q-1} A^*, \\ \langle \cdot, \cdot \rangle_{B^*}&\colon W^{p,1}(D') \times_M W^{1,q}(D) \to \wedge^{p+q-1} B^*. \end{aligned} \tag{4.3}$$

Here $\langle \cdot, \cdot \rangle_{E^*}$ is the unique extension of the given pairing such that

$$\langle \alpha, \widehat{a} \rangle_{E^*} = -\alpha(a), \quad \langle \widehat{b}, \beta \rangle_{E^*} = \beta(b)$$

for the cases $p = 0, q = 1$ and $p = 1, q = 0$, and such that the following bilinearity property holds:

$$\langle \lambda x,\, y\rangle_{E^*} = \lambda\langle x, y\rangle_{E^*}, \quad \langle x,\, y\lambda\rangle_{E^*} = \langle x, y\rangle_{E^*}\lambda$$

for $\lambda \in \wedge E^*,\ x \in W^{\bullet,1}(D''),\ y \in W^{1,\bullet}(D')$ (with the same base points). The discussion for the pairings $\langle\cdot,\cdot\rangle_{A^*},\ \langle\cdot,\cdot\rangle_{B^*}$ is similar. In Section 4.5, we will give geometric interpretations of these pairings.

4.3 Properties of $W(D)$

In this section we will discuss how the functor $D \mapsto \mathcal{W}(D)$ behaves with respect to some of the operations between double vector bundles described in chapter 2. All of the following properties are verified by tracing through the sequence of operations that make up the Weil algebra functor.

1. *Core-negation.* The description of the Weil algebra bundle for D^- is obtained from that for D by replacing the sign of the inclusion map $i_{\widehat{E}}$. That is, $W(D^-)$ has the same generators, but the defining relation becomes $\alpha\beta = -i_{\widehat{E}}(\alpha\otimes\beta)$. The map on generators $\alpha\mapsto\alpha,\ \beta\mapsto\beta,\ \widehat{e}\mapsto -\widehat{e}$ extends to an isomorphism of algebra bundles $W(D^-)\to W(D)$.

2. *Diagonal flip.* It is clear from the definitions that we have

$$\mathcal{W}^{p,q}(D) = \mathcal{W}^{q,p}(\mathrm{flip}(D))$$

for any double vector bundle D.

3. *Horizontal and vertical duals.* By combining the previous two observations and proposition 2.6.1, we obtain a description of the Weil algebras for the horizontal and vertical duals D^h and D^v of D. Specifically we see that

$$\mathcal{W}^{p,q}(D^h) = \mathcal{W}^{q,p}(D'), \quad \mathcal{W}^{p,q}(D^v) = \mathcal{W}^{q,p}(D'')$$

but with the generators in bidegree $(1,1)$ appearing with opposite sign.

4. *Direct sums.* Given two double vector bundles D_1 and D_2 with total bases M_1 and M_2, the bigraded algebra bundle for the product $D_1\times D_2$ is given by the (graded) tensor product of the two Weil algebras, thought of as a bundle over $M_1\times M_2$:

$$\mathcal{W}(D_1\times D_2)_{(m_1,m_2)} = \mathcal{W}(D_1)_{m_1}\otimes\mathcal{W}(D_2)_{m_2}.$$

Similarly, in the case that $M_1 = M_2$ the Weil algebra of the direct sum $D_1\oplus D_2$ is also described as the graded tensor product of $\mathcal{W}(D_1)$ and $\mathcal{W}(D_2)$, this time thought of as a bundle over M:

$$\mathcal{W}(D_1\oplus D_2)_m = \mathcal{W}(D_1)_m\otimes\mathcal{W}(D_2)_m$$

5. *Sub-$\mathcal{DVB}$s and quotients.* By functoriality, any sub-$\mathcal{DVB}$, $Q\subseteq D$ determines a surjective map $\mathcal{W}(D)\to\mathcal{W}(Q)$. Thus we may recover the Weil algebra of Q as a quotient of the Weil algebra of D. Specifically, if Q takes the form (2.25), then $\mathcal{W}(Q) = \mathcal{W}(D)/I$, where I is the ideal generated by $\Gamma(\mathrm{Ann}(H))\subseteq\Gamma(A^*)$, $\Gamma(\mathrm{Ann}(V))\subseteq\Gamma(B^*)$, and $\Gamma(\mathrm{Ann}(\widehat{K}))\subseteq\Gamma(\widehat{E})$. On the other side of the

coin, a $\mathcal{DVB}$ quotient $D/\mathfrak{h}$ determines an injection $\mathscr{W}(D/\mathfrak{h}) \hookrightarrow \mathscr{W}(D)$ that identifies $\mathscr{W}(D/\mathfrak{h})$ with the subalgebra of $\mathscr{W}(D)$ generated by $\Gamma(\mathrm{Ann}(H))$, $\Gamma(\mathrm{Ann}(V))$, and those elements of $\Gamma(\widehat{E})$ that are invariant under the action of $\mathfrak{h}$ (see proposition 2.10.4).

4.4 Derivations of $W(D)$

For a vector bundle $V \to M$, the graded bundle $\mathrm{Der}(\wedge V^*)$ of *fiberwise* superderivations of $\wedge V^*$ is the free $\wedge V^*$-module generated by contractions. Thus

$$\mathrm{Der}(\wedge V^*) = \wedge V^* \otimes V$$

as a bundle of graded super-Lie algebras, where the elements $1 \otimes v$ have degree -1.

Given a double vector bundle D, we are interested in the structure of the bigraded bundle

$$\mathrm{Der}(W(D)) = \bigoplus_{r,s} \mathrm{Der}^{r,s}(W(D)).$$

Here $\mathrm{Der}^{r,s}(W(D)) \to M$ is the bundle of fiberwise superderivations of bidegree (r, s) of the algebra bundle $W(D) \to M$: its space of sections consists of bundle maps $\delta\colon W(D) \to W(D)$ of bidegree (r, s) with the superderivation property

$$\delta(xy) = \delta(x)y + (-1)^{|\delta||x|}x\delta(y)$$

for homogeneous elements x, y, where $|\delta| = r + s$ and $|x|$ are the total degrees of δ and x. The following result describes the structure of $\mathrm{Der}(W(D))$ as a $W(D)$-module and as a bundle of graded Lie algebras.

Theorem 4.4.1. *Let $m \in M$ and $\widehat{a} \in \widehat{A}_m$, $\widehat{b} \in \widehat{B}_m$, and $\varepsilon \in E^*_m$. There are unique* contraction operators

$$\iota_h(\widehat{a}) \in \mathrm{Der}^{-1,0}(W(D))_m, \quad \iota_v(\widehat{b}) \in \mathrm{Der}^{0,-1}(W(D))_m, \quad \iota(\varepsilon) \in \mathrm{Der}^{-1,-1}(W(D))_m$$

such that

$$\iota_h(\widehat{a})v = \langle \widehat{a}, v\rangle_{B^*}, \quad \iota_v(\widehat{b})u = (-1)^{|u|}\langle u, \widehat{b}\rangle_{A^*}, \quad \iota(\varepsilon)e = -\varepsilon(e) \tag{4.4}$$

for all $u \in W^{1,\bullet}(D)_m$, $v \in W^{\bullet,1}(D)_m$, $e \in W^{1,1}(D)_m = \widehat{E}_m$. The contraction operators satisfy the commutation relations

$$[\iota_v(\widehat{b}),\, \iota_h(\widehat{a})] = \iota(\langle \widehat{b}, \widehat{a}\rangle_{E^*}), \quad \widehat{a} \in \widehat{A},\ \widehat{b} \in \widehat{B}, \tag{4.5}$$

while all other commutations of contractions are zero. The $W(D)_m$-module $\mathrm{Der}(W(D))_m$ is generated by the three types of contraction operators, subject to the relations

$$\iota_h(i_{\widehat{A}}(\beta \otimes \varepsilon)) = m_L(\beta) \circ \iota(\varepsilon), \qquad \iota_v(i_{\widehat{B}}(\varepsilon \otimes \alpha)) = -m_R(\alpha) \circ \iota(\varepsilon), \qquad \alpha \in A^*_m,\ \beta \in B^*_m,\ \varepsilon \in E^*_m, \tag{4.6}$$

where $m_L(\beta)$ denotes left multiplication by β, and $m_R(\alpha)$ denotes right multiplication by α.

Proof. For degree reasons, the proposed expressions for the contractions determine the formulas on generators of $W(D)_m$. Specifically, $\iota_h(\widehat{a})$ is given on generators $\alpha \in A^*_m$, $\beta \in B^*_m$, $\widehat{e} \in \widehat{E}_m$ by

$$\alpha \mapsto \alpha(a), \quad \beta \mapsto 0, \quad \widehat{e} \mapsto \langle \widehat{a}, \widehat{e}\rangle_{B^*},$$

while $\iota_v(\widehat{b})$ is given by

$$\alpha \mapsto 0,\ \beta \mapsto \beta(b),\ \widehat{e} \mapsto \langle \widehat{e}, \widehat{b}\rangle_{A^*},$$

and $\iota(\varepsilon)$ is given by

$$\alpha \mapsto 0,\ \beta \mapsto 0,\ \widehat{e} \mapsto -\varepsilon(e).$$

To see that these formulas hence extend to derivations on all of $W(D)_m$, we need to verify that they are compatible with the defining relations of the Weil algebra. For $\iota_h(\widehat{a})$ we have

$$\iota_h(\widehat{a})(i_{\widehat{E}}(\alpha \otimes \beta)) = \langle \widehat{a}, i_{\widehat{E}}(\alpha \otimes \beta)\rangle_{B^*} = \alpha(a)\beta = \iota_h(\widehat{a})(\alpha)\beta - \alpha\iota_h(\widehat{a})(\beta).$$

The calculation for $\iota_v(\widehat{b})$ is similar:

$$\iota_v(\widehat{b})(i_{\widehat{E}}(\alpha \otimes \beta)) = \langle i_{\widehat{E}}(\alpha \otimes \beta), \widehat{b}\rangle_{A^*} = -\alpha\beta(b) = \iota_v(\widehat{b})(\alpha)\beta - \alpha\iota_v(\widehat{b})(\beta).$$

For the last alleged derivation, note that if $\widehat{e} \in \mathrm{im}(i_{\widehat{E}})$ then $e = 0$ and so both sides of the desired relation for $\iota(\varepsilon)$ are zero. Next we prove the relations (4.5). Since we have shown that $\iota_h(\widehat{a}), \iota_v(\widehat{b})$, and $\iota(\varepsilon)$ extend to derivations, it is enough to check the formula on generators. Note that on the generators α, β, both sides of the equation vanish for degree reasons, while for generators $\widehat{e}$, we have

$$[\iota_v(\widehat{b}), \iota_h(\widehat{a})](\widehat{e}) = \iota_v(\widehat{b})\iota_h(\widehat{a})(\widehat{e}) + \iota_h(\widehat{a})\iota_v(\widehat{b})(\widehat{e}) = \langle \widehat{a}, \widehat{e}\rangle_{B^*}(b) + \langle \widehat{e}, \widehat{b}\rangle_{A^*}(a) = -\langle \widehat{b}, \widehat{a}\rangle_{E^*}(e) = \iota(\langle \widehat{b}, \widehat{a}\rangle_{E^*})(\widehat{e}),$$

where we have used the identity (2.19). The other two possible commutations result in a derivation of total degree -3, and therefore vanish identically (see below).

For the final claim, note that the three types of contraction operators define a $W(D)_m$-module morphism

$$W(D)_m \otimes (\widehat{A}_m \oplus \widehat{B}_m \oplus E^*_m) \to \mathrm{Der}(W(D))_m \tag{4.7}$$

whose kernel contains elements of the form

$$1 \otimes i_{\widehat{A}}(\beta \otimes \varepsilon) - \beta \otimes \varepsilon, \qquad 1 \otimes i_{\widehat{B}}(\varepsilon \otimes \alpha) + \alpha \otimes \varepsilon \tag{4.8}$$

with $\alpha \in A^*_m,\ \beta \in B^*_m,\ \varepsilon \in E^*_m$. We have to show that (4.7) is surjective, with kernel the submodule generated by elements of the form (4.8).

It suffices to prove this for the double vector space $D_0 = A_0 \times B_0 \times E^*_0$. Here $W(D_0)$ is simply a tensor product $\wedge A^*_0 \otimes \wedge B^*_0 \otimes \vee E_0$, and hence $\mathrm{Der}(W(D_0)) = W(D_0) \otimes (A_0 \oplus B_0 \oplus E^*_0)$. Since $\widehat{A}_0 = A_0 \oplus (B^*_0 \otimes E^*_0)$ and $\widehat{B}_0 = B_0 \oplus (E^*_0 \otimes A^*_0)$, it is immediate that the module map (4.7) (with D replaced by D_0) is surjective. Its kernel contains elements of the form (4.8); hence it also contains the $W(D_0)$-submodule generated by elements of this form. But this submodule is a complement to the submodule $W(D_0) \otimes (A_0 \oplus B_0 \oplus E^*_0)$, and is therefore the entire kernel of (4.7). □

In particular, we see that the bundle $\mathrm{Der}^{r,s}(W(D))$ is zero if $r < -1$ or $s < -1$, while

$$\mathrm{Der}^{-1,0}(W(D)) = \widehat{A}, \quad \mathrm{Der}^{0,-1}(W(D)) = \widehat{B}, \quad \mathrm{Der}^{-1,-1}(W(D)) = E^*. \tag{4.9}$$

Proposition 4.4.2. *The horizontal contractions extend to an isomorphism of left $\wedge B^*$-modules*

$$\iota_h : W^{\bullet,1}(D') \to \mathrm{Der}^{-1,-1+\bullet}(W(D)),\ x \mapsto \iota_h(x) \tag{4.10}$$

such that

$$\iota_h(x)z = \langle x, z\rangle_{B^*}, \quad x \in W^{\bullet,1}(D'),\ z \in W^{1,\bullet}(D). \tag{4.11}$$

The vertical contractions extend to an isomorphism of left $\wedge A^*$-modules

$$\iota_v : W^{1,\bullet}(D'') \to \mathrm{Der}^{-1+\bullet,-1}(W(D)),\ y \mapsto \iota_v(y) \tag{4.12}$$

given by

$$\iota_v(y)z = -(-1)^{(|y|+1)(|z|+1)}\,\langle z, y\rangle_{A^*}, \quad y \in W^{1,\bullet}(D''), \quad z \in W^{\bullet,1}(D) \tag{4.13}$$

Proof. The proposed formulas determine $\iota_h(x),\ \iota_v(y)$ on generators. To show that the formula (4.13) for $\iota_v(y)$ gives a well-defined $\wedge A^*$-module homomorphism, it suffices to check that the right hand side is linear in the argument y for the left $\wedge A^*$-module structure and linear in the argument z for the right $\wedge A^*$-module structure. Indeed, replacing y with αy changes the right hand side to

$$(-1)^{|y|(|z|+1)}\langle z,\, \alpha y\rangle_{A^*} = (-1)^{|y|\,|z|}\langle z,\, y\rangle_{A^*}\alpha = (-1)^{(|y|+1)(|z|+1)}\alpha\langle z,\, y\rangle_{A^*}.$$

Similarly, replacing z with $z\alpha$ for $\alpha \in A^*$ changes the right hand side to

$$(-1)^{(|y|+1)|z|}\langle z\alpha, y\rangle_{A^*} = (-1)^{|y|\,|z|}\alpha\langle z,\, y\rangle_{A^*} = (-1)^{(|y|+1)(|z|+1)}\langle z,\, y\rangle_{A^*}\alpha$$

as required. The argument for $\iota_h(x)$ is similar. □

Remark 4.4.3. The sign in (4.13) comes from the fact that we are using the left $\wedge A^*$-module structures, whereas the pairing is bilinear for the right $\wedge A^*$-module structure in the second argument. It is also worth noting that $\iota_h(\varepsilon) = \iota_v(\varepsilon) = \iota(\varepsilon)$ for $\varepsilon \in E^*$.

4.5 Linear and core sections of $\wedge_A D$

We have already encountered the linear and core sections of a double vector bundle D over its side bundles. We shall now consider the generalization to the exterior algebra bundles, and relate it to the Weil algebra bundles. Throughout, D will denote a double vector bundle with sides A, B and with $\mathrm{core}(D) = E^*$. Given a double vector bundle D, we denote by

$$\wedge^n_A D \to A$$

its exterior powers as a vector bundle over A. The horizontal scalar multiplications $\kappa^h_t : D \to D$ are vector bundle endomorphisms of $D \to A$, hence they extend to algebra bundle endomorphisms $\wedge^\bullet \kappa^h_t$ of $\wedge^\bullet_A D \to A$. A section $\sigma : A \to \wedge^n_A D$ is *homogeneous of degree* k if it satisfies

$$\sigma(\kappa^h_t(a)) = t^k\ (\wedge^n \kappa^h_t)(\sigma(a))$$

for all $t \in \mathbb{R}$; the space of such sections is denoted $\Gamma(\wedge^n_A D, A)_{[k]}$.

Definition 4.5.1. The spaces of *core sections* and *linear sections* of $\wedge^n_A D$ over A are defined as follows:

$$\Gamma_{\mathrm{core}}(\wedge^n_A D, A) = \Gamma(\wedge^n_A D, A)_{[-n]},$$
$$\Gamma_{\mathrm{lin}}(\wedge^n_A D, A) = \Gamma(\wedge^n_A D, A)_{[-n+1]}.$$

The spaces $\Gamma_{\mathrm{core}}(\wedge^n_B D, B)$ and $\Gamma_{\mathrm{lin}}(\wedge^n_B D, B)$ are defined similarly.

The core sections $\Gamma_{\mathrm{core}}(\wedge^\bullet_A D, A)$ are a super-commutative graded algebra under the wedge product, and $\Gamma_{\mathrm{lin}}(\wedge^\bullet_A D, A)$ is a graded module over this algebra. The significance of these spaces is clarified by the following result.

Proposition 4.5.2. *The space $\Gamma(\wedge^n_A D, A)_{[k]}$ is zero if $k < -n$, and for $k = -n$ is given by*

$$\Gamma_{\mathrm{core}}(\wedge^n_A D, A) \cong \Gamma(\wedge^n E^*). \tag{4.14}$$

The space of linear sections fits into a short exact sequence

$$0 \to \Gamma(\wedge^n E^* \otimes A^*) \to \Gamma_{\mathrm{lin}}(\wedge^n_A D, A) \to \Gamma(\wedge^{n-1} E^* \otimes B) \to 0. \tag{4.15}$$

Proof. Recall that when D is viewed as a vector bundle over A, its restriction to the submanifold M is the direct sum $D = E^* \oplus B$. Hence, the restriction of sections to $M \subseteq A$ gives a map

$$\Gamma(\wedge_A D, A) \to \Gamma(\wedge(E^* \oplus B)) = \Gamma(\wedge E^* \otimes \wedge B).$$

We claim that the restriction of core sections gives the isomorphism (4.14), while restriction of linear sections gives a map from $\Gamma_{\mathrm{lin}}(\wedge^n_A D, A)$ onto $\Gamma(\wedge^{n-1} E^* \otimes B)$, with kernel $\Gamma(\wedge^n E^* \otimes A^*)$ spanned by products of core sections with linear functions on the base A. Using the associated bundle construction, it suffices to prove these claims for the double vector space $D_0 = A_0 \times B_0 \times E_0^*$. We have

$$\Gamma(\wedge^n_{A_0} D_0, A_0) = C^\infty(A_0, \bigoplus_{i+j=n} \wedge^j E_0^* \otimes \wedge^i B_0).$$

The elements of $\wedge^j E_0^* \otimes \wedge^{n-j} B_0 \to M$ (regarded as constant sections of $\wedge^n_{A_0} D_0$) are homogeneous of degree $-j$. To obtain a section that is homogeneous of degree k, we must multiply by a polynomial on A_0 of degree $k + j$. Thus,

$$\Gamma(\wedge^n_{A_0} D_0, A_0)_{[k]} = \bigoplus_j \wedge^j E_0^* \otimes \wedge^{n-j} B_0 \otimes \vee^{k+j} A_0^*$$

where the sum is over all j with $0 \le j \le n$ and $k + j \ge 0$. In particular, this space is zero if $k < -n$, and is equal to $\wedge^n E_0^*$ for $k = -n$. Specializing to $k = -n + 1$ this shows

$$\Gamma_{\mathrm{lin}}(\wedge^n_{A_0} D_0, A_0) = (\wedge^n E_0^* \otimes A_0^*) \oplus (\wedge^{n-1} E_0^* \otimes B_0). \tag{4.16}$$

Hence, the map $\Gamma_{\mathrm{lin}}(\wedge^n_{A_0} D_0, A_0) \to \wedge^{n-1} E_0^* \otimes B_0$ is surjective, with kernel $\wedge^n E_0^* \otimes A_0^*$. $\square$

The linear and core sections of $\wedge_A D \to A$ are graded subspaces of Weil algebras, as follows.

Proposition 4.5.3. *There is a canonical isomorphism* $\Gamma_{\mathrm{core}}(\wedge^\bullet_A D, A) \cong \mathcal{W}^{\bullet,0}(D'') = \Gamma(\wedge^\bullet E^*)$ *as graded algebras, and an isomorphism of graded left modules over this algebra,*

$$\Gamma_{\mathrm{lin}}(\wedge^\bullet_A D, A) \cong \mathcal{W}^{\bullet,1}(D'').$$

Similarly, there is a canonical isomorphism of graded algebras, $\Gamma_{\mathrm{core}}(\wedge^\bullet_B D, B) \cong \mathcal{W}^{0,\bullet}(D') = \Gamma(\wedge^\bullet E^*)$ *and an isomorphism of right modules over this algebra,*

$$\Gamma_{\mathrm{lin}}(\wedge^\bullet_B D, B) \cong \mathcal{W}^{1,\bullet}(D').$$

Proof. It suffices to prove the claim for the double vector space $D_0 = A_0 \times B_0 \times E_0^*$. But $W^{\bullet,0}(D_0'') = \wedge^\bullet E_0^*$, $W^{0,\bullet}(D_0') = \wedge^\bullet E_0^*$ as graded algebras. Furthermore, the isomorphism of graded left $\wedge E_0^*$-modules

$$W^{\bullet,1}(D_0'') = (\wedge^\bullet E_0^* \otimes A_0^*) \oplus (\wedge^{\bullet-1} E_0^* \otimes B_0)$$

is exactly the description of linear sections of $\wedge^n_{A_0} D_0$, see (4.16). Similarly for

$$W^{1,\bullet}(D_0') = (B_0^* \otimes \wedge^\bullet E_0^*) \oplus (A_0 \otimes \wedge^{\bullet-1} E_0^*).$$

as graded right $\wedge E_0^*$-modules. □

With these identifications, the pairing $\langle\cdot,\cdot\rangle_{E^*}\colon W^{p,1}(D'') \times_M W^{1,q}(D') \to \wedge^{p+q-1}E^*$ (cf. (4.3)) translates into a $\Gamma(\wedge E^*)$-bilinear pairing

$$\langle\cdot,\cdot\rangle_{E^*}\colon \Gamma_{\mathrm{lin}}(\wedge^p_A D, A) \times \Gamma_{\mathrm{lin}}(\wedge^q_B D, B) \to \Gamma(\wedge^{p+q-1}E^*). \tag{4.17}$$

4.6 Multi-vector Fields on D

The linear and core sections of $\wedge_A D \to A$ and $\wedge_B D \to B$, and their pairings, have a simple interpretation in terms of the space $\mathfrak{X}^\bullet(D)$ of multi-vector fields on D. The sections of $\wedge^n_A D$ are identified with n-vector fields on D that are homogeneous of degree $-n$ with respect to the vertical vector bundle structure (see section 8.2 for a more detailed discussion of vector fields on the total space of a vector bundle). A similar description applies to sections of $\wedge^n_B D$. As before, we let $\mathfrak{X}^n(D)_{[k,l]}$ denote the space of n-vector fields that are homogeneous of degree k horizontally and of degree l vertically. This space is trivial if $k < -n$ or $l < -n$, while $\mathfrak{X}^n(D)_{[-n,-n]} \cong \Gamma(\wedge^n E^*)$ is identified with $\Gamma_{\mathrm{core}}(\wedge^n_A D, A)$ and also with $\Gamma_{\mathrm{core}}(\wedge^n_B D, B)$. Furthermore, we have canonical isomorphisms

$$\Gamma_{\mathrm{lin}}(\wedge^p_A D, A) \cong \mathfrak{X}^p(D)_{[1-p,-p]}, \qquad \Gamma_{\mathrm{lin}}(\wedge^q_B D, B) \cong \mathfrak{X}^q(D)_{[-q,1-q]},$$

obtained by extending the isomorphisms $\mathfrak{X}(D)_{[-1,-1]} \cong \Gamma_{\mathrm{core}}(D,A) \cong \Gamma_{\mathrm{core}}(D,B)$, $\mathfrak{X}(D)_{[-1,0]} \cong \Gamma_{\mathrm{lin}}(D,B)$, and $\mathfrak{X}(D)_{[0,-1]} \cong \Gamma_{\mathrm{lin}}(D,A)$ described in section 2.9. The first isomorphism is compatible with the left module structure over $\Gamma(\wedge E^*)$, the second isomorphism with the right module structure, realized as wedge product of the corresponding multivector fields from the left or right, respectively.

Proposition 4.6.1. *With the above identifications, the pairing* (4.17) *is given by the Schouten bracket*

$$\langle x, y\rangle_{E^*} = [\![x, y]\!]$$

for all $x \in \mathfrak{X}^p(D)_{[1-p,-p]}$ *and* $y \in \mathfrak{X}^q(D)_{[-q,1-q]}$.

Proof. The Schouten bracket of elements $\lambda \in \mathfrak{X}^n(D)_{[-n,-n]}$ with $x \in \mathfrak{X}^p(D)_{[1-p,-p]}$ or with $y \in \mathfrak{X}^q(D)_{[-q,1-q]}$ is zero, for degree reasons. Hence, the derivation property of the Schouten bracket shows that $[\![x, y]\!]$ is $\Gamma(\wedge E^*)$-bilinear, for the left module structure on $\mathfrak{X}^p(D)_{[1-p,-p]}$ and the right module structure on $\mathfrak{X}^q(D)_{[-q,1-q]}$. The pairing $\langle x, y\rangle_{E^*}$ has the same bilinearity property. It therefore suffices to prove the formula for $p, q \leq 1$. If $p = q = 1$, we are dealing with the pairing of vector fields $X \in \mathfrak{X}(D)_{[0,-1]} = \Gamma(\widehat{B})$ and $Y \in \mathfrak{X}(D)_{[-1,0]} = \Gamma(\widehat{A})$, and the claim was already noted in Section 2.9. If $p = 0$, $q = 1$ we have $x = \alpha \in \Gamma(A^*)$, $y = \widehat{a} \in \Gamma(\widehat{A})$ with the pairing $\langle \alpha, \widehat{a}\rangle_{E^*} = -\alpha(a)$. After identification of $\widehat{a}$ with a vector field $Y \in \mathfrak{X}^1(D)_{[-1,0]}$ and α with a function $f \in C^\infty(D)_{[1,0]}$, this coincides with $\mathcal{L}_Y f = -[\![f, Y]\!]$, as required. Similarly, for $p = 1$, $q = 0$ we have $x = \widehat{b} \in \Gamma(\widehat{B})$ and $y = \beta \in \Gamma(B^*)$, with pairing $\langle \widehat{b}, \beta\rangle_{A^*} = \beta(b)$, which, after identification of $\widehat{b}$ with a vector field $X \in \mathfrak{X}^1(D)_{[0,-1]}$ and β with a function $g \in C^\infty(D)_{[0,1]}$, coincides with $-\mathcal{L}_X g = -[\![X, g]\!]$. $\square$

Chapter 5

Poisson Double Vector Bundles

5.1 Reminder on Poisson vector bundles

Given a vector bundle $p: V \to M$, one knows that the following structures are equivalent:

(i) a linear Poisson structure π on $V \to M$,

(ii) a degree -1 Poisson bracket $\{\cdot,\cdot\}$ on the algebra of polynomial functions on V,

(iii) a Lie algebroid structure on the dual bundle, $V^* \Rightarrow M$,

(iv) a degree -1 Gerstenhaber bracket (*Schouten bracket*) on $\Gamma(\wedge V^*)$,

(v) a degree 1 differential d_{CE} on $\Gamma(\wedge V)$.

Here (and from now on) we write $A \Rightarrow M$ to indicate a Lie algebroid over M; the notation (which we learned from [7]) suggests the differentiation of a Lie groupoid $G \rightrightarrows M$ when source and target become 'infinitesimally close'. Let us briefly recall how these equivalences come about. Given a linear Poisson tensor π on V, the corresponding Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(V)$ restricts to a bracket on the space of polynomial functions on V, and is uniquely determined by this restriction. The Poisson bivector π being linear is equivalent to the bracket of linear functions being again linear, thus to $\{\cdot,\cdot\}$ having degree -1. Hence, (i)$\Leftrightarrow$ (ii). The Poisson bracket is in fact already determined by its restriction to linear and basic functions on V. Using the identification of linear functions with sections $\sigma \in \Gamma(V^*)$, this gives the equivalence (ii)$\Leftrightarrow$(iii), where the Lie bracket and anchor are expressed by

$$[\![\sigma_1, \sigma_2]\!] = \{\sigma_1, \sigma_2\}, \quad p^*(\mathcal{L}_{\mathsf{a}(\sigma)} f) = \{\sigma, p^*(f)\}. \tag{5.1}$$

This Lie algebroid bracket extends to a Schouten bracket on the algebra $\Gamma(\wedge V^*)$, with $[\![\sigma, p^* f]\!] = p^*(\mathcal{L}_{\mathsf{a}(\sigma)} f)$ as the bracket between generators of degrees 1 and 0, hence (iii)$\Leftrightarrow$(iv). The Chevalley-Eilenberg differential d_{CE} on $\Gamma(\wedge V)$ is the unique degree 1 derivation such that

$$\iota(\sigma)\mathrm{d}_{CE} f = \mathcal{L}_{\mathsf{a}(\sigma)}(f) \tag{5.2}$$

for $f \in C^\infty(M)$ and $\sigma \in \Gamma(V^*)$, and such that

$$[\iota(\sigma_1), [\iota(\sigma_2), \mathrm{d}_{CE}]] = \iota([\![\sigma_1, \sigma_2]\!]) \tag{5.3}$$

for all $\sigma_1, \sigma_2 \in \Gamma(V^*)$; one can recover the bracket and anchor from these identities, giving the equivalence (iii)⇔(v).

5.2 Results for Poisson double vector bundles

We are interested in the counterparts of these correspondences for double vector bundles. A Poisson bivector field $\pi \in \mathfrak{X}^2(D)$ on a double vector bundle is called *double-linear* if it is linear for both vector bundle structures, i.e., homogeneous of bidegree $(-1,-1)$. Following Mackenzie [56], a double vector bundle with a double-linear Poisson bivector field π is called a *Poisson double vector bundle.*

Theorem 5.2.1. *Let D be a double vector bundle with sides A, B and with $\mathrm{core}(D) = E^*$. The following are equivalent.*

(i) a double-linear Poisson structure π on $D \to M$,

(ii) a bidegree $(-1,-1)$ Poisson bracket $\{\cdot,\cdot\}$ on the algebra $\mathcal{S}(D)$ of double polynomials,

(iii) a $\mathcal{VB}$-algebroid structure on D' over B,

(iv) a $\mathcal{VB}$-algebroid structure on D'' over A,

(v) a Lie algebroid structure on $\widehat{E}$, together with representations on A^* and B^*, and an invariant bilinear pairing $A^* \times_M B^* \to \mathbb{R}$, with certain compatibility conditions (cf. Theorem 5.4.3 below),

(vi) a bidegree $(-1,-1)$ Gerstenhaber bracket on the Weil algebra $\mathcal{W}(D)$,

(vii) a bidegree $(0,1)$ differential d'_v on the Weil algebra $\mathcal{W}(D')$,

(viii) a bidegree $(1,0)$ differential d''_h on the Weil algebra $\mathcal{W}(D'')$.

Some of these equivalences are already known: Given π, the corresponding Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(D)$ restricts to the subalgebra $\mathcal{S}(D)$ of double-polynomial functions, and is uniquely determined by this restriction (since the differentials of functions in this subalgebra span the cotangent bundle everywhere). The bivector π being double-linear means precisely that this Poisson bracket has bidegree $(-1,-1)$, hence (i)⇔(ii). The equivalence with (iii), (iv) is due to Mackenzie [56] (see also [7]): Regarding D as a vector bundle over B, and using the nondegenerate pairing $D \times_B D' \to \mathbb{R}$ from (2.11), the Poisson structure π determines a Lie algebroid structure $D' \Rightarrow B$. The bivector field π being linear in the vertical direction $D \to A$ implies that the horizontal scalar multiplication on D' is by Lie algebroid morphisms, which shows that D' is a $\mathcal{VB}$-algebroid. Similarly, from the pairing $D'' \times_A D \to \mathbb{R}$ we obtain a $\mathcal{VB}$-algebroid structure on $D'' \Rightarrow A$. We depict these $\mathcal{VB}$-algebroid structures on D', D'' by

$$\begin{array}{ccc} D' & \longrightarrow & E \\ \Downarrow & & \Downarrow \\ B & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} D'' & \Longrightarrow & A \\ \downarrow & & \downarrow \\ E & \Longrightarrow & M \end{array} \tag{5.4}$$

where the double arrow indicates Lie algebroid directions. In particular, we see that the bundle E becomes a Lie algebroid $E \Rightarrow M$. (The Lie algebroid structures on E coming from the $\mathcal{VB}$-algebroid structures on D', D'' coincide; indeed, we will see below that both are induced from a $\mathcal{VB}$-algebroid structure $\widehat{E} \to M$.) The characterizations (v), (vi), (vii) will be consequences of theorems 5.4.3, 5.5.1, and proposition 5.6.1 below, while (viii) is obtained by applying (vii) to the flip of D.

5.3 Examples of Poisson double vector bundles

As a preparation for the general situation, let us consider some special cases.

Example 5.3.1. Any Poisson vector bundle $V \to M$ can be seen as a Poisson double vector bundle with zero side bundles, thus $A = B = M,\ E = V^*$. In this case,

$$\begin{tikzcd} D = V \arrow[r] \arrow[d] & M \arrow[d] \\ M \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} D' = V^* \arrow[r] \arrow[d, Rightarrow] & V^* \arrow[d, Rightarrow] \\ M \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} D'' = V^* \arrow[r, Rightarrow] \arrow[d] & M \arrow[d] \\ V^* \arrow[r, Rightarrow] & M \end{tikzcd}$$

Example 5.3.2. Suppose D is a Poisson double vector bundle for which the side bundle A is zero. Then

$$\begin{tikzcd} D = B \times_M E^* \arrow[r] \arrow[d] & B \arrow[d] \\ M \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} D' = B \times_M E \arrow[r] \arrow[d, Rightarrow] & E \arrow[d, Rightarrow] \\ B \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} D'' = E \times_M B^* \arrow[r, Rightarrow] \arrow[d] & M \arrow[d] \\ E \arrow[r, Rightarrow] & M \end{tikzcd}$$

Here $D' \Rightarrow B$ is the *action Lie algebroid* for a representation of $E \Rightarrow M$ on B, and the second diagram describes $D'' \Rightarrow M$ as the *semi-direct product Lie algebroid* for the dual E-representation on B^*.

Example 5.3.3. Suppose D is a *vacant* Poisson double vector bundle, that is, with a zero core:

$$\begin{tikzcd} D = A \times_M B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} D' = A^* \times_M B \arrow[r] \arrow[d, Rightarrow] & M \arrow[d, Rightarrow] \\ B \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} D'' = A \times_M B^* \arrow[r, Rightarrow] \arrow[d] & A \arrow[d] \\ M \arrow[r, Rightarrow] & M \end{tikzcd}$$

We claim that double-linear Poisson structures π on $D = A \times_M B$ are equivalent to bilinear pairings $(\cdot,\cdot)\colon A^* \times_M B^* \to \mathbb{R}$. To see this, note that the bigraded algebra $\mathcal{S}(D)$ is generated by $\mathcal{S}^{0,0}(D) = C^\infty(M)$, $\mathcal{S}^{1,0}(D) = \Gamma(A^*)$, and $\mathcal{S}^{0,1}(D) = \Gamma(B^*)$. Given π, it follows for degree reasons that the only non-trivial Poisson bracket of generators are between $\alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$; the resulting pairing $(\alpha, \beta) = \{\alpha, \beta\}$ is $C^\infty(M)$-linear by the derivation property. Conversely, given the pairing, we define a bi-derivation by letting $\{\alpha, \beta\} = (\alpha, \beta)$, and setting all other brackets between generators equal to zero. This bi-derivation satisfies the Jacobi identity, since triple brackets between generators are always zero. This proves the claim. The Lie algebroid structure on D' is that of an action Lie algebroid for the translation action of A^* on B, given by the map $\Gamma(A^*) \to \Gamma(B) = \mathfrak{X}(B)_{[-1]},\ \alpha \mapsto (\alpha, \cdot)$, and similarly for D''. Since $E = 0$, we have $\widehat{E} = A^* \otimes B^*$. The sections of this bundle have a Lie bracket, coming from its identification with double-linear functions on D:

$$[\alpha_1 \otimes \beta_1,\, \alpha_2 \otimes \beta_2] \equiv \{\alpha_1\beta_1,\, \alpha_2\beta_2\} = (\alpha_1, \beta_2)\, \alpha_2\beta_1 - (\alpha_2, \beta_1)\alpha_1\beta_2;$$

thus $\widehat{E}$ becomes a Lie algebroid with zero anchor. Likewise, the Poisson bracket of such functions with $\alpha \in \Gamma(A^*)$ or $\beta \in \Gamma(B^*)$ defines representations of this Lie algebroid on A^*, B^*, respectively. Explicitly, these actions are determined by the formulas

$$\nabla_{\alpha_1 \otimes \beta_1} \alpha_2 = -(\alpha_2, \beta_1)\alpha_1, \quad \nabla_{\alpha_1 \otimes \beta_1} \beta_2 = (\alpha_1, \beta_2)\beta_1. \tag{5.5}$$

Example 5.3.4. Suppose that $E \Rightarrow M$ is a Lie algebroid, together with representations $\nabla^{A^*}, \nabla^{B^*}$ on A^*, B^*, and that $(\cdot,\cdot)\colon A^* \times_M B^* \to \mathbb{R}$ is a bilinear pairing that is E-invariant in the sense that

$$(\nabla_e^{A^*}\alpha, \beta) + (\alpha, \nabla_e^{B^*}\beta) = \mathcal{L}_{\mathsf{a}(e)}(\alpha, \beta)$$

for $\alpha \in \Gamma(A^*),\ \beta \in \Gamma(B^*),\ e \in \Gamma(E)$. Then $D = A \times_M B \times_M E^*$ becomes a Poisson double vector bundle, with the non-zero brackets on generators given as

$$\{\alpha, \beta\} = (\alpha, \beta), \quad \{e, \alpha\} = \nabla_e^{A^*}\alpha, \quad \{e, \beta\} = \nabla_e^{B^*}\beta, \quad \{e_1, e_2\} = [\![e_1, e_2]\!], \quad \{e, f\} = \mathcal{L}_{\mathsf{a}(e)}f,$$

for $f \in C^\infty(M) = \mathcal{S}^{0,0}(D)$, $\alpha \in \Gamma(A^*) = \mathcal{S}^{1,0}(D)$, $\beta \in \Gamma(B^*) = \mathcal{S}^{0,1}(D)$, and $e, e_1, e_2 \in \Gamma(E) \subseteq \mathcal{S}^{1,1}(D)$. The Jacobi identity and the biderivation property of $\{\cdot,\cdot\}$ follow from the definition of Lie algebroids and their representations, together with the invariance of the pairing $(\cdot,\cdot)$.

5.4 The Lie algebroid structure on $\widehat{E}$

In this section, we will concentrate on the characterization (v) of Poisson double vector bundles from Theorem 5.2.1. Suppose D is a Poisson double vector bundle, with corresponding Poisson bracket $\{\cdot,\cdot\}$. Recall that the algebra $\mathcal{S}(D) = \bigoplus \mathcal{S}^{r,s}(D)$ is generated by

$$\mathcal{S}^{1,1}(D) = \Gamma(\widehat{E}), \quad \mathcal{S}^{1,0}(D) = \Gamma(A^*), \quad \mathcal{S}^{0,1}(D) = \Gamma(B^*), \quad \mathcal{S}^{0,0}(D) = C^\infty(M);$$

we will use these identifications without further comment, and for example think of $\alpha \in \Gamma(A^*)$ as a function on D. The Poisson bracket gives bilinear maps $\mathcal{S}^{r,s}(D) \times \mathcal{S}^{r',s'}(D) \to \mathcal{S}^{r+r'-1,s+s'-1}(D)$, and is uniquely determined by the resulting maps on generators,

$$[\![\cdot,\cdot]\!]\colon\ \mathcal{S}^{1,1}(D) \times \mathcal{S}^{1,1}(D) \to \mathcal{S}^{1,1}(D), \quad [\![\widehat{e}_1, \widehat{e}_2]\!] = \{\widehat{e}_1, \widehat{e}_2\}, \tag{5.6}$$

$$\mathsf{a}\colon\ \mathcal{S}^{1,1}(D) \times \mathcal{S}^{0,0}(D) \to \mathcal{S}^{0,0}(D), \quad \mathsf{a}(\widehat{e}, f) = \{\widehat{e}, f\}, \tag{5.7}$$

$$\nabla^{A^*}\colon\ \mathcal{S}^{1,1}(D) \times \mathcal{S}^{1,0}(D) \to \mathcal{S}^{1,0}(D), \quad \nabla_{\widehat{e}}^{A^*}\alpha = \{\widehat{e}, \alpha\}, \tag{5.8}$$

$$\nabla^{B^*}\colon\ \mathcal{S}^{1,1}(D) \times \mathcal{S}^{0,1}(D) \to \mathcal{S}^{0,1}(D), \quad \nabla_{\widehat{e}}^{B^*}\beta = \{\widehat{e}, \beta\}, \tag{5.9}$$

$$(\cdot,\cdot)\colon\ \mathcal{S}^{1,0}(D) \times \mathcal{S}^{0,1}(D) \to \mathcal{S}^{0,0}(D), \quad (\alpha, \beta) = \{\alpha, \beta\} \tag{5.10}$$

(the other brackets between generators are zero, for degree reasons).

Lemma 5.4.1. *The formulas (5.6)–(5.10) define a Lie algebroid structure on $\widehat{E}$, with representations on A^*, B^*, and with an invariant bilinear pairing $(\cdot,\cdot)\colon A^* \times_M B^* \to \mathbb{R}$.*

Proof. The derivation property of the Poisson bracket shows that (5.7) is $C^\infty(M)$-linear in the first argument and satisfies a Leibniz rule in the second argument; hence that $\mathsf{a}(\widehat{e}) = \mathsf{a}(\widehat{e}, \cdot)$ comes from a bundle map $\mathsf{a}\colon \widehat{E} \to TM$. The Jacobi identity for $\{\cdot,\cdot\}$ implies that (5.6) is a Lie bracket on $\Gamma(\widehat{E})$, and the derivation property for $\{\cdot,\cdot\}$ shows that $[\![\cdot,\cdot]\!]$ satisfies the Leibniz rule for the anchor map a, hence that $\widehat{E}$ is a Lie algebroid. Further applications of the Jacobi identity and derivation property of $\{\cdot,\cdot\}$ show that $\nabla^{A^*}, \nabla^{B^*}$ are representations of the Lie algebroid $\widehat{E}$ on A^*, B^*, and that the pairing $(\cdot,\cdot)$ is $\widehat{E}$-invariant. $\square$

Remark 5.4.2. The Lie algebroid structure $\widehat{E} \Rightarrow M$, and its action on A^*, B^* were first observed by Gracia-Saz and Mehta, [29, Section 4.3] in terms of the $\mathcal{VB}$-algebroid $D' \Rightarrow B$; the pairing $(\cdot,\cdot)$ corresponds to the 'core-anchor' map defined in [29]. The relevant formulas are recovered from the correspondence between Poisson vector bundles and Lie algebroid structures, as reviewed in Section 5.1: Using the identifications $\Gamma_{\text{lin}}(D', B) = \Gamma(\widehat{E})$ and $\Gamma_{\text{core}}(D', B) = \Gamma(A^*)$, the Lie bracket $[\cdot,\cdot]_{D'}$ is given by the formulas

$$[\widehat{e}_1, \widehat{e}_2]_{D'} = \{\widehat{e}_1, \widehat{e}_2\} = [\widehat{e}_1, \widehat{e}_2], \quad [\widehat{e}, \alpha]_{D'} = \{\widehat{e}, \alpha\} = \nabla^{A^*}_{\widehat{e}} \alpha, \quad [\alpha_1, \alpha_2]_{D'} = \{\alpha_1, \alpha_2\} = 0$$

while the anchor map $\mathsf{a}_{D'}\colon D' \to TB$ is described by

$$\mathcal{L}_{\mathsf{a}_{D'}(\widehat{e})} f = \{\widehat{e}, f\} = \mathcal{L}_{\mathsf{a}(\widehat{e})} f, \quad \mathcal{L}_{\mathsf{a}_{D'}(\widehat{e})} \beta = \{\widehat{e}, \beta\} = \nabla^{B^*}_{\widehat{e}} \beta,$$

$$\mathcal{L}_{\mathsf{a}_{D'}(\alpha)} f = \{\alpha, f\} = 0, \quad \mathcal{L}_{\mathsf{a}_{D'}(\alpha)} \beta = \{\alpha, \beta\} = (\alpha, \beta).$$

(Here $f \in C^\infty(M)$ is viewed as a function on B, via pullback, and $\beta \in \Gamma(B^*)$ is viewed as a linear function on B.) In particular, we see that the $\mathcal{VB}$-algebroid $D' \Rightarrow B$ directly determines the data (5.6)–(5.10), and Lemma 5.4.1 recovers the 'side and core representations' of the 'fat Lie algebroid $\widehat{E}$', in the terminology of [29, Section 4.3]. Similarly, we can describe the data (5.6)–(5.10) in terms of the $\mathcal{VB}$-algebroid structure $D'' \Rightarrow A$.

The Lie algebroid representations of $\widehat{E}$ on A^*, B^* and the bilinear form satisfy certain compatibility conditions. Recall from Example 5.3.3 that the pairing $(\cdot,\cdot)\colon A^* \times_M B^* \to \mathbb{R}$ defines a Lie algebroid structure on $A^* \otimes B^*$, with zero anchor, and that this Lie algebroid comes with natural representations on A^*, B^*. The data for $\widehat{E}$ must 'extend' these data for its subbundle $i_{\widehat{E}}(A^* \otimes B^*)$:

Theorem 5.4.3. *Let D be a double vector bundle. A Lie algebroid structure on the bundle $\widehat{E} \to M$, together with Lie algebroid representations on A^* and B^* and an invariant bilinear pairing $(\cdot,\cdot)\colon A^* \times_M B^* \to \mathbb{R}$, defines a double-linear Poisson structure on D if and only if the following compatibility conditions are satisfied:*

(i) The image of $i_{\widehat{E}}\colon \Gamma(A^* \otimes B^*) \hookrightarrow \Gamma(\widehat{E})$ is a Lie algebra ideal (in particular, $i_{\widehat{E}}(A^* \otimes B^*)$ is a Lie subalgebroid of $\widehat{E}$),

(ii) the $\widehat{E}$-representations on A^*, B^* extend those of its Lie subalgebroid $i_{\widehat{E}}(A^* \otimes B^*)$,

(iii) the $\widehat{E}$-representation on $i_{\widehat{E}}(A^* \otimes B^*)$ is the tensor product of those on A^*, B^*.

Condition (i) determines a Lie algebroid structure on E, in such a way that

$$0 \to A^* \otimes B^* \xrightarrow{i_{\widehat{E}}} \widehat{E} \to E \to 0$$

is an exact sequence of Lie algebroids.

Proof. Throughout, we denote by α, α_1 sections of A^*, by β, β_1 sections of B^*, and by $\widehat{e}, \widehat{e}_1$ sections of $\widehat{E}$. Suppose first that a double-linear Poisson structure on D is given, determining the Lie algebroid structure on $\widehat{E}$, representations on A^*, B^*, and a pairing $(\cdot,\cdot)$. On the level of sections, the inclusion $i_{\widehat{E}}$ is just the multiplication map $\alpha \otimes \beta \mapsto \alpha\beta$. Thus the compatibility conditions (i)-(iii) follow from the derivation property of the Poisson bracket $\{\cdot,\cdot\}$. Indeed, for condition (i) we have

$$[\![\widehat{e}, i_{\widehat{E}}(\alpha\otimes\beta)]\!] = \{\widehat{e},\alpha\beta\} = \{\widehat{e},\alpha\}\beta + \alpha\{\widehat{e},\beta\} = i_{\widehat{E}}(\{\widehat{e},\alpha\}\otimes\beta + \alpha\otimes\{\widehat{e},\beta\}),$$

showing that $\operatorname{im} i_{\widehat{E}}$ is an ideal. To establish condition (ii), note that

$$\nabla_{i_{\widehat{E}}(\alpha\otimes\beta)}\alpha_1 = \{\alpha\beta,\alpha_1\} = -(\alpha_1,\beta)\alpha, \quad \nabla_{i_{\widehat{E}}(\alpha\otimes\beta)}\beta_1 = \{\alpha\beta,\beta_1\} = (\alpha,\beta_1)\beta,$$

which shows that the representations of $\widehat{E}$ on A^* and B^* extend the formulas (5.5). Finally,

$$[\![\widehat{e}, i_{\widehat{E}}(\alpha\otimes\beta)]\!] = \{\widehat{e},\alpha\beta\} = \{\widehat{e},\alpha\}\beta + \alpha\{\widehat{e},\beta\} = \nabla_{\widehat{e}}\alpha\cdot\beta + \alpha\cdot\nabla_{\widehat{e}}\beta,$$

so the representation of $\widehat{E}$ on $i_{\widehat{E}}(A^*\otimes B^*)$ is the tensor product of the representations on A^* and B^*.

Conversely, suppose (i),(ii),(iii) are satisfied. Recall from Proposition 2.7.3 that D is a sub-double vector bundle of $\widehat{D} = A\times_M B\times_M \widehat{E}^*$. The formulas of Example 5.3.4 (with E replaced by $\widehat{E}$) define a double-linear Poisson structure on $\widehat{D}$. We will show that D is a Poisson submanifold of $\widehat{D}$, and hence is a Poisson double vector bundle. The ideal $\mathscr{I}\subseteq\mathcal{S}(\widehat{D})$ of double-polynomial functions vanishing on D is generated by functions of the form

$$\alpha\beta - i_{\widehat{E}}(\alpha\otimes\beta) \tag{5.11}$$

with $\alpha\in\Gamma(A^*),\beta\in\Gamma(B^*)$. To show that $\mathscr{I}$ is an ideal for the Poisson bracket, it suffices to show that the Poisson bracket of functions (5.11) with any of the generators lies in the ideal. For $\widehat{e}\in\Gamma(\widehat{E})$ we have

$$\{\widehat{e},\ \alpha\beta - i_{\widehat{E}}(\alpha\otimes\beta)\} = (\nabla^{A^*}_{\widehat{e}}\alpha)\beta - i_{\widehat{E}}(\nabla^{A^*}_{\widehat{e}}\alpha\otimes\beta) + \alpha\nabla^{B^*}_{\widehat{e}}\beta - i_{\widehat{E}}(\alpha\otimes\nabla^{B^*}_{\widehat{e}}\beta)\in\mathscr{I}, \tag{5.12}$$

where we used (i) and (iii). For $\alpha_1\in\Gamma(A^*)$ we compute

$$\{\alpha_1, \alpha\beta - i_{\widehat{E}}(\alpha\otimes\beta)\} = \alpha(\alpha_1,\beta) + \nabla^{A^*}_{i_{\widehat{E}}(\alpha\otimes\beta)}\alpha_1 = 0, \tag{5.13}$$

where we used (ii). A similar argument applies to generators $\beta_1\in\Gamma(B^*)$. Finally, for $f\in C^\infty(M)$

$$\{f, \alpha\beta - i_{\widehat{E}}(\alpha\otimes\beta)\} = \mathscr{L}_{\mathsf{a}(i_{\widehat{E}}(\alpha\otimes\beta))}(f) = 0 \tag{5.14}$$

since $\mathsf{a}\circ i_{\widehat{E}} = 0$ by (i). □

Remark 5.4.4. Luca Vitagliano has pointed out to us that Theorem 5.4.3 can also be obtained as a consequence of [20, Theorem 2.33].

Let us note the following consequences of the discussion above:

Proposition 5.4.5. *Let D be a Poisson double vector bundle, with side bundles A, B and with $E = \operatorname{core}(D)^*$. Then*

1. *$A\times_M B$ inherits a double-linear Poisson structure, with $\varphi\colon D\to A\times_M B$ a Poisson map.*
2. *The subbundle $\operatorname{core}(D)$ is a Poisson-Dirac submanifold of D: every smooth function on the core extends to a smooth function on D with Hamiltonian vector field tangent to the core.*
3. *$\widehat{D} = A\times_M B\times_M\widehat{E}^*$ acquires a double-linear Poisson structure, such that $D\subseteq\widehat{D}$ is a Poisson submanifold.*

Proof. For (a), observe that the image of the pullback map $\varphi^*\colon \mathcal{S}(A\times_M B)\to\mathcal{S}(D)$ is the subalgebra generated by $\Gamma(A^*),\Gamma(B^*)$; by the bracket relations (5.6)–(5.10) it is a Poisson subalgebra. Part (c) is contained in the proof of Theorem 5.4.3. For (b), note that it is enough to prove the analogous statement for $\widehat{D}$, since D is a Poisson submanifold and $\mathrm{core}(D)=D\cap\mathrm{core}(\widehat{D})$. But functions on $\widehat{E}^*$ extend canonically to functions on $\widehat{D}$, by taking the pullback under $\widehat{D}\to\widehat{E}^*$. The vanishing ideal of $\widehat{E}^*$ is generated by $\Gamma(A^*),\Gamma(B^*)\subseteq C^\infty(\widehat{D})$, and is preserved under Poisson brackets with pullbacks of functions on $\widehat{E}^*$. This means that the Hamiltonian vector fields of the latter are tangent to $\widehat{E}^*$. $\square$

5.5 Gerstenhaber brackets

Our next aim is to interpret double-linear Poisson structures on D in terms of a 'Gerstenhaber' bracket on the Weil algebra $\mathcal{W}(D)$, as in item (vi) of Theorem 5.2.1. We make the following definitions. Let $\mathcal{A}$ be a bigraded commutative superalgebra. A *bidegree* $(-1,-1)$ *Gerstenhaber bracket* on $\mathcal{A}$ is a bilinear map $[\![\cdot,\cdot]\!]\colon\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ of bidegree $(-1,-1)$, such that $\mathcal{A}[1,1]$ (i.e., the space $\mathcal{A}$ with bidegrees shifted down by $(1,1)$) becomes a bigraded super-Lie algebra, and for $x\in\mathcal{A}^{p,q}$ the map $[\![x,\cdot]\!]$ is a superderivation of bidegree $(p-1,q-1)$ of the algebra structure on $\mathcal{A}$: In particular

$$[\![x,y]\!]=-(-1)^{|x||y|}[\![y,x]\!],\quad [\![x,yz]\!]=[\![x,y]\!]z+(-1)^{|x||y|}y[\![x,z]\!].$$

From now on, we we will omit the explicit mention of the bidegree $(-1,-1)$, taking the degree shifts to be understood. (This deviates from the work of Huebschmann [35], where the degree shift is taken to be $(1,0)$.) Note also that we will reserve the symbol $[\![\cdot,\cdot]\!]$ for Gerstenhaber brackets on bigraded superalgebras, to avoid confusion with various other Lie brackets and commutators.

Theorem 5.5.1. *A double-linear Poisson structure π on a double vector bundle D is equivalent to a Gerstenhaber bracket $[\![\cdot,\cdot]\!]$ on the Weil algebra $\mathcal{W}(D)$.*

Proof. First we observe that for $r,s\le 1$, the spaces $\mathcal{W}^{r,s}(D)$ coincide with $\mathcal{S}^{r,s}(D)$:

$$\mathcal{W}^{1,1}(D)=\Gamma(\widehat{E}),\quad \mathcal{W}^{1,0}(D)=\Gamma(A^*),\quad \mathcal{W}^{0,1}(D)=\Gamma(B^*),\quad \mathcal{W}^{0,0}(D)=C^\infty(M).$$

A Gerstenhaber bracket on $\mathcal{W}(D)$ gives bilinear maps $\mathcal{W}^{r,s}(D)\times\mathcal{W}^{r',s'}(D)\to\mathcal{W}^{r+r'-1,s+s'-1}(D)$. The following formulas define a Lie algebroid structure on $\widehat{E}$, together with representations of this Lie algebroid on A^*,B^*, and a bilinear pairing $(\cdot,\cdot)$ between A^* and B^*:

$$[\widehat{e}_1,\widehat{e}_2]=[\![\widehat{e}_1,\widehat{e}_2]\!],\quad L_{\mathsf{a}(\widehat{e})}(f)=[\![\widehat{e},f]\!],\tag{5.15}$$

$$\nabla^{A^*}_{\widehat{e}}\alpha=[\![\widehat{e},\alpha]\!],\qquad \nabla^{B^*}_{\widehat{e}}\beta=[\![\widehat{e},\beta]\!],\qquad (\alpha,\beta)=-[\![\alpha,\beta]\!].\tag{5.16}$$

To see that this makes D into a Poisson $\mathcal{DVB}$, we need to show that these structures satisfy properties (i)-(iii) of theorem 5.4.3. Note that the formulas above can be obtained from (5.6)–(5.10) by replacing Poisson brackets with Gerstenhaber brackets, except for an extra minus sign in the last formula. Since Gerstenhaber brackets and Poisson brackets share similar properties, one can simply reproduce the argument for the "only if"' part of theorem 5.4.3, taking care to account for signs. As a reminder, we note that the compatibility conditions (i)-(iii) follow from the derivation property and the Jacobi identity for $[\![\cdot,\cdot]\!]$.

Conversely, given a double-linear Poisson structure π and the associated data from Theorem 5.4.3, we obtain a Gerstenhaber bracket as follows. Consider the Poisson double vector bundle $\widehat{D} = A \times_M B \times_M \widehat{E}^*$ (cf. Example 5.3.4). On the super-commutative algebra $\mathscr{W}(\widehat{D})$, we define a super-biderivation $[\![\cdot,\cdot]\!]$ of bidegree $(-1,-1)$, by taking (5.15) and (5.16) to be the nontrivial brackets between generators. This super-biderivation satisfies the super-Jacobi identity, as one checks on generators. Finally, by essentially the same argument as for the Poisson bracket on $\mathcal{S}(D)$, this Gerstenhaber bracket descends to $\mathscr{W}(D) = \mathscr{W}(\widehat{D})/\sim$. In repeating the calculations (5.12) – (5.14), the second equation encounters a minus sign since $[\![\alpha_1, \alpha\beta]\!] = -(\alpha_1, \beta)\,\alpha$ in contrast to $\{\alpha_1, \alpha\beta\} = (\alpha_1, \beta)\,\alpha$; this is compensated by the extra sign in the last equation of (5.16). □

5.6 Differentials

Suppose D is a Poisson double vector bundle. The corresponding $\mathcal{VB}$-algebroid structure $D' \Rightarrow B$ (dual to the Poisson vector bundle $D \to B$) gives a Chevalley-Eilenberg differential $\mathrm{d}_{D'}$ on $\Gamma(\wedge_B^\bullet D, B)$. Since $\mathrm{d}_{D'}$ commutes with the scalar multiplication κ_t^v on D and on B, it restricts to a differential on core and linear sections of $\wedge_B D$ over B. Since $\mathrm{d}_{D'}$ is a derivation with respect to the wedge product, we see that the core sections become a differential graded algebra, and the linear sections a differential graded module over this differential graded algebra.

On the other hand, recall from Proposition 4.5.3 that the linear and core sections of $\wedge_B D$ over B are identified with $\mathscr{W}^{1,\bullet}(D')$ and $\mathscr{W}^{0,\bullet}(D')$, respectively. Hence, a bidegree $(0,1)$ differential on $\mathscr{W}(D')$ again restricts to differentials on the core and linear sections, making them into differential graded algebras and differential graded modules, respectively. To prove the characterization of double-linear Poisson structures on D in terms of differentials on $\mathscr{W}(D')$, then, it suffices to show that we can reverse these constructions.

Proposition 5.6.1. *Suppose $\Gamma_{\mathrm{core}}(\wedge_B^\bullet D, B)$ and $\Gamma_{\mathrm{lin}}(\wedge_B^\bullet D, B)$ are equipped with differentials d, for which they are a differential graded algebra and differential graded module respectively. Then there are unique extensions of d to*

1. *a bidegree $(0,1)$ differential d_v' on the algebra $\mathscr{W}(D')$,*
2. *a degree 1 differential $\mathrm{d}_{D'}$ on the algebra $\Gamma(\wedge_B D, B)$,*

as superderivations for the algebra structures.

Proof. 1. By definition, $\mathscr{W}(D)$ is generated by elements in bidegree (i,j) with $0 \le i,j \le 1$. Put $\mathrm{d}_v' x = \mathrm{d}x$ whenever x is one of these generators. To show that this definition extends as a superderivation , we have to verify that it is compatible with the relations between generators. The defining relation of $\mathscr{W}(D')$ (aside from super-commutativity and $C^\infty(M)$-linearity) is that for $\beta \in \Gamma(B^*) = \Gamma_{\mathrm{lin}}(\wedge_B^0 D, B)$ and $\varepsilon \in E^* = \Gamma_{\mathrm{core}}(\wedge^1 BD, B)$, the linear section $i_{\widehat{A}}(\beta \otimes \varepsilon)$ of $\widehat{A} = \Gamma_{\mathrm{lin}}(\wedge_B^1 D, B)$ coincides with the product $\beta\varepsilon$. Thus, we need that

$$\mathrm{d}(i_{\widehat{A}}(\beta \otimes \varepsilon)) = (\mathrm{d}\beta)\,\varepsilon - \beta\,(\mathrm{d}\varepsilon).$$

But the linear section $i_{\widehat{A}}(\beta \otimes \varepsilon)$ is simply the product of the linear function β with the core section ε; hence the desired identity follows from the assumption that linear sections are a differential graded module over the core sections.

2. The algebra of sections of $\wedge_B D$ over B has a subalgebra

$$\Gamma_{\mathrm{pol}}(\wedge_B D, B) = \bigoplus_{m,n\geq 0} \Gamma(\wedge_B^n D, B)_{[-n+m]}$$

of *polynomial sections*, in the notation of Section 4.5. It is a super-commutative bigraded algebra, with bigrading given by m, n, and with the for the $\mathbb{Z}_2$-grading given as the mod 2 reduction of n. It is generated by its components in degree $m, n \leq 1$, which coincide with those for $\mathscr{W}(D')$, and the relations between these generators are the same as for $\mathscr{W}(D')$, with the exception that the relation $\beta_1\beta_2 = -\beta_2\beta_1$ for $\beta_1, \beta_2 \in \Gamma(B^*)$ gets replaced with $\beta_1\beta_2 = \beta_2\beta_1$. The same argument as for $\mathscr{W}(D')$ shows that d extends to a superderivation $\mathrm{d}_{D'}$ of $\Gamma_{\mathrm{pol}}(\wedge_B D, B)$. By [29, Theorem 3.15], the latter determines a Lie algebroid structure on D' over B, which, in turn, extends the differential to all sections of $\wedge_B D$ over B.

$\square$

Remark 5.6.2. In [29], the double complex $\Gamma_{\mathrm{pol}}(\wedge_B D, B)$ is denoted $\Omega^{\bullet,\bullet}(D')$. As explained there, it may be regarded as the space of double-polynomial functions on the supermanifold $D'[1,0]$, using a parity change only in the vertical vector bundle direction. Note also that in the notation of Cabrera-Drummond [11], $\Gamma_{\mathrm{pol}}(\wedge_B^\bullet D, B)_{[k-\bullet]}$ is the space of k-homogeneous cochains for the $\mathcal{VB}$-algebroid $D' \Rightarrow B$.

5.7 Poisson Structures on Manifold Triples

Recall from chapter 3 that for two cleanly intersecting submanifolds N_1, N_2 of M, we have a double vector bundle $\nu(M, N_1, N_2)$. Moreover, the space of multivector fields $\mathfrak{X}^\bullet(M)$ inherits a bifiltration for which the bihomogeneous elements lift to multivector fields on $\nu(M, N_1, N_2)$ that are bihomogeneous with respect to the scalar multiplications (see section 3.5). This allows us to furnish examples Poisson double vector bundles by first starting with a Poisson structure on M that is compatible (in a certain sense) with the submanifolds N_1, N_2, which we now show.

Proposition 5.7.1. *Let (M, π) be a Poisson manifold, and let N_1 and N_2 be cleanly intersecting coisotropic submanifolds of M. Then π induces a double linear Poisson structure $\pi^{(0,0)}$ on the double vector bundle $\nu(M, N_1, N_2)$.*

Proof. Recall that a submanifold N is said to be *coisotropic* if the vector field $X_f := \pi(f, \cdot)$ is tangent to N whenever $f \in C^\infty(M)$ vanishes on N. Hence the fact that N_1 and N_2 are both coisotropic means precisely that $\pi \in \mathfrak{X}^2_{(0,0)}$. By lemma 3.5.1, it induces a double linear bivector $\pi^{(0,0)}$ on $\nu(M, N_1, N_2)$. To see that $\pi^{(0,0)}$ is also Poisson, note that we have

$$[\pi^{(0,0)}, \pi^{(0,0)}] = [\pi, \pi]^{(0,0)} = 0,$$

where we have used lemma 3.5.1 (here [,] denotes the Schouten bracket of multi-vector fields). $\square$

Double linear Poissons structures are objects that are dual (in a certain sense) to $\mathcal{VB}$-algebroids, higher analogs of Lie algebroids that have been given much attention in the literature in recent years (see for example, [29]). Thus in much the same way that Lie algebroids appear in the study of Poisson manifolds, one may use the theory of $\mathcal{V}B$-algebroids developed in this chapter to study situations in which multiple coisotropic submanifolds appear. One example of a situation where this arises is in the coisotropic calculus of Weinstein [78], where the goal is to view coisotropic submanifolds of a product of Poisson manifolds as generalized morphisms, and determine under what conditions one can take the composition of two such coisotropic submanifolds.

Chapter 6

Derivations of the Weil Algebras of Poisson $\mathcal{DVB}$s

In the preceeding chapters, we have described numerous derivations on the Weil algebras for Poisson $\mathcal{DVB}$s and $\mathcal{VB}$-algebroids, including contraction operators, differentials, and Gerstenhaber brackets. In this chapter, we investigate how all of these operators interact with each other and with the pairings between the Weil algebras. Throughout, D will denote a Poisson double vector bundle, with side bundles A, B and with $\mathrm{core}(D)^* = E$. Equivalently, the double vector bundles D', D'' are $\mathcal{VB}$-algebroids $D' \Rightarrow B$, $D'' \Rightarrow A$, respectively. In the last chapter, we saw how these structures are equivalent to a Gerstenhaber bracket $[\![\cdot,\cdot]\!]$ on $\mathcal{W}(D)$, a vertical differential d'_v on $\mathcal{W}(D')$, and a horizontal differential d''_h on $\mathcal{W}(D'')$.

6.1 The Differential and Contractions on $\mathcal{W}(D')$

Recall from section 4.4 that any $\widehat{e} \in \Gamma(\widehat{E})$ defines a contraction operator $\iota'_v(\widehat{e})$ of bidegree $(0,-1)$ on $\mathcal{W}(D')$. We would like to understand what the relationship is between these contraction operators and the differential d'_h. However, we know that the isomorphisms $\mathcal{W}^{0,\bullet}(D') \cong \Gamma_{\mathrm{core}}(\wedge^\bullet_B D, B)$ and $\mathcal{W}^{1,\bullet}(D') \cong \Gamma_{\mathrm{lin}}(\wedge^\bullet_B D, B)$ take d'_h to the Chevalley-Eilenberg differential d_{CE} of the Lie algebroid $D' \Rightarrow B$ (see section 5.6). Moreover, any $\sigma \in \Gamma(D', B)$ determines a contraction operator $\iota_{D'}(\sigma)$ on $\Gamma(\wedge_B D, B)$ in the usual way. Since we know how the operators d_{CE} and $\iota_{D'}(\sigma)$ interact, it will be enough to understand the relationship between the $\iota'_v(\widehat{e})$ contractions and the $\iota_{D'}(\sigma)$ contractions. The following lemma can be established with a simple check on generators.

Lemma 6.1.1. 1. *The isomorphism $\Gamma(\widehat{E}) \cong \Gamma_{\mathrm{lin}}(D', B)$ identifies the contraction operators*

$$\iota'_v(\widehat{e})\colon \mathcal{W}^{p,\bullet}(D') \to \mathcal{W}^{p,\bullet}(D')$$

for $p = 0, 1$ with the operator $\iota_{D'}(\widehat{e})$ on $\Gamma_{\mathrm{core}}(\wedge_B D, B)$ $(p = 0)$ and on $\Gamma_{\mathrm{lin}}(\wedge_B D, B)$ $(p = 1)$.

2. *The isomorphism $\Gamma(A^*) \cong \Gamma_{\mathrm{core}}(D', B)$ identifies the contraction operator*

$$\iota'_v(\alpha)\colon \mathcal{W}^{1,\bullet}(D') \to \mathcal{W}^{0,\bullet}(D')$$

with the operator $\iota_{D'}(\alpha)\colon \Gamma_{\mathrm{lin}}(\wedge_B D, B) \to \Gamma_{\mathrm{core}}(\wedge_B D, B)$.

Combining this lemma with the classical results for the Chevalley-Eilenberg differential of a Lie algebroid, we obtain the following explicit relationships.

Proposition 6.1.2. *The derivation* d'_v *satisfies*

$$[\iota'_v(\widehat{e}_1), [\iota'_v(\widehat{e}_2), \mathrm{d}'_v]] = \iota'_v([\![\widehat{e}_1, \widehat{e}_2]\!]), \tag{6.1}$$

$$[\iota'_v(\widehat{e}), [\iota'_v(\alpha), \mathrm{d}'_v]] = -\iota'_v(\nabla^{A^*}_{\widehat{e}}\alpha), \tag{6.2}$$

$$[\iota'_v(\alpha_1), [\iota'_v(\alpha_2), \mathrm{d}'_v]] = 0, \tag{6.3}$$

for $\widehat{e}, \widehat{e}_1, \widehat{e}_2 \in \Gamma(\widehat{E}),\ \alpha, \alpha_1, \alpha_2 \in \Gamma(A^*)$*. Furthermore,*

$$\iota'_v(\widehat{e})\mathrm{d}'_v f = \mathcal{L}_{\mathsf{a}(\widehat{e})} f, \qquad \iota'_v(\widehat{e})\mathrm{d}'_v\beta = \nabla^{B^*}_{\widehat{e}}\beta, \tag{6.4}$$

$$\iota'_v(\alpha)\mathrm{d}'_v f = 0, \qquad \iota'_v(\alpha)\mathrm{d}'_v\beta = -(\alpha, \beta), \tag{6.5}$$

for all $\widehat{e} \in \Gamma(\widehat{E}),\ \alpha \in \Gamma(A^*),\ f \in C^\infty(M),\ \beta \in \Gamma(B^*)$*.*

Proof. Since (6.1)–(6.3) are equalities of derivations, it suffices to check on $\mathcal{W}^{0,\bullet}(D'), \mathcal{W}^{1,\bullet}(D')$. Since the identifications of these spaces with core and linear sections of $\wedge_B D$ takes ι'_v, d'_v to the contractions and Lie algebroid differential of $\Gamma(\wedge_B D, B)$, and since the right hand sides can be expressed in terms of Lie algebroid brackets (see Remark 5.4.2), the three equalities (6.1)–(6.3) correspond to the formula (5.3) for the bracket of Lie algebroids. Similarly, (6.4), (6.5) correspond to the formula (5.2) for the anchor of a Lie algebroid. $\square$

6.2 The Gerstenhaber Bracket and the Differentials

The Gerstenhaber bracket on $\mathcal{W}(D)$ restricts to a bracket on $\mathcal{W}^{1,1+\bullet}(D)$, making the latter into a graded Lie algebra with a representation $x \mapsto [\![x, \cdot]\!]$ on $\mathcal{W}^{0,\bullet}(D) = \Gamma(\wedge B^*)$. Likewise, $\mathcal{W}^{1+\bullet,1}(D)$ is a graded Lie algebra with a representation on $\mathcal{W}^{\bullet,0}(D) = \Gamma(\wedge A^*)$ by graded superderivations $x \mapsto [\![x, \cdot]\!]$. The next proposition gives a way of obtaining these representations through the contraction operators and the differentials.

Proposition 6.2.1. *For* $x \in \mathcal{W}^{1,\bullet}(D)$ *and* $y \in \mathcal{W}^{0,\bullet}(D) = \Gamma(\wedge B^*) = \mathcal{W}^{\bullet,0}(D')$,

$$\iota'_v(x)\mathrm{d}'_v y = [\![x, y]\!]. \tag{6.6}$$

Similarly, for $x \in \mathcal{W}^{1,\bullet}(D)$ *and* $y \in \mathcal{W}^{0,\bullet}(D) = \Gamma(\wedge B^*) = \mathcal{W}^{\bullet,0}(D')$,

$$\iota''_h(x)\mathrm{d}''_h y = [\![x, y]\!]. \tag{6.7}$$

Proof. For $x \in \mathcal{W}^{1,q}(D)$, both $[\![x, \cdot]\!]$ and $[\iota'_v(x), \mathrm{d}'_v]$ define superderivations of degree $q-1$ on $\Gamma(\wedge B^*)$. Since $\iota'_v(x)$ vanishes on $\mathcal{W}^{\bullet,0}(D')$, Equation (6.6) amounts to the equality of these two superderivations. It suffices to verify this on generators y of $\Gamma(\wedge B^*)$, given by $f \in C^\infty(M)$ or $\beta \in \Gamma(B^*)$. Furthermore, since ι'_v is a left $\Gamma(\wedge B^*)$-module morphism, we only need to consider the cases that x is a generator of

$\mathscr{W}^{1,\bullet}(D)$, given by $\alpha \in \Gamma(A^*)$ or $\widehat{e} \in \Gamma(\widehat{E})$. These verifications are as follows, using (6.5):

$$\begin{aligned} [\![\alpha, f]\!] &= 0 = \iota'_v(\alpha)\mathrm{d}'_v f, \qquad [\![\alpha, \beta]\!] = -(\alpha, \beta) = \iota'_v(\alpha)\mathrm{d}'_v \beta, \\ [\![\widehat{e}, f]\!] &= \mathscr{L}_{\mathsf{a}(\widehat{e})} f = \iota'_v(\widehat{e})\mathrm{d}'_v f, \quad [\![\widehat{e}, \beta]\!] = \nabla^{B^*}_{\widehat{e}} \beta = \iota'_v(\widehat{e})\mathrm{d}'_v \beta. \end{aligned}$$

Equation (6.7) may be proved along similar lines, or obtained from (6.6) by using the flip operation. □

Note that (6.6) can be written in various other ways:

$$[\![x, y]\!] = -(-1)^{(|x|+1)|y|}\langle \mathrm{d}'_v y,\, x\rangle_{B^*} = -(-1)^{(|x|+1)|y|}\iota_h(\mathrm{d}'_v y)x. \tag{6.8}$$

Another natural thing to ask is what happens when you contract by the bracket of two elements. This question is answered below.

Proposition 6.2.2. *For $x_1, x_2 \in \mathscr{W}^{1,\bullet}(D)$,*

$$[\iota'_v(x_1), [\iota'_v(x_2), \mathrm{d}'_v]] = (-1)^{|x_2|}\iota'_v([\![x_1, x_2]\!]) \tag{6.9}$$

For $x_1, x_2 \in \mathscr{W}^{\bullet,1}(D)$,

$$[\iota''_h(x_1), [\iota''_h(x_2), \mathrm{d}''_h]] = (-1)^{|x_2|}\iota''_h([\![x_1, x_2]\!]). \tag{6.10}$$

Proof. Equation (6.9) holds for $x_i = \widehat{e}_i \in \Gamma(\widehat{E})$ by (6.1) and for $x_1 = \widehat{e} \in \Gamma(\widehat{E}),\ x_2 = \alpha \in \Gamma(A^*)$ by (6.2). Since both sides change sign by $(-1)^{(|x_1|+1)(|x_2|+1)}$ under interchange of x_1, x_2, this establishes the identity for generators. The general case follows by induction: the statement for x_1, x_2 implies that for $x_1, \beta x_2$ with $\beta \in \Gamma(B^*)$, as follows:

$$\begin{aligned} [\iota'_v(x_1), [\iota'_v(\beta\, x_2), \mathrm{d}'_v]] &= [\iota'_v(x_1), \beta[\iota'_v(x_2), \mathrm{d}'_v] + (-1)^{|x_2|+1}(\mathrm{d}'_v\beta)\, \iota'_v(x_2)] \\ &= (-1)^{|x_1|+1}\beta[\iota'_v(x_1), [\iota'_v(\, x_2), \mathrm{d}'_v]] + (-1)^{|x_2|+1}(\iota'_v(x_1)\mathrm{d}'_v\beta)\ \iota'_v(x_2) \\ &= (-1)^{|x_1|+|x_2|+1}\iota'_v(\beta[\![x_1, x_2]\!]) + (-1)^{|x_2|+1}\iota'_v([\![x_1, \beta]\!]x_2) \\ &= (-1)^{|x_2|+1}\iota'_v[\![x_1,\, \beta x_2]\!]. \end{aligned}$$

The arguments for $\mathrm{d}''_h,\ \iota''_h(x)$ are analogous; alternatively, one can use the flip operation. □

For later reference, observe the following consequence of Proposition 6.2.2.

Corollary 6.2.3. *For $\phi \in \mathscr{W}^{1,\bullet}(D'')$, and $x_i \in \mathscr{W}^{\bullet,1}(D)$,*

$$(-1)^{|x_2|}\iota''_h(x_1)\iota''_h(x_2)\mathrm{d}''_h\phi = \iota''_h([\![x_1, x_2]\!])\phi - [\![x_1, \iota''_h(x_2)\phi]\!] + (-1)^{|x_1||x_2|}[\![x_2, \iota_h(x_1)\phi]\!].$$

Proof. The left hand side $(-1)^{|x_2|}\iota''_h(x_1)\iota''_h(x_2)\mathrm{d}''_h\phi$ may be written as a sum of three terms,

$$(-1)^{|x_2|}[\iota''_h(x_1), [\iota''_h(x_2), \mathrm{d}''_h]]\phi - \iota''_h(x_1)\, \mathrm{d}''_h\, \iota''_h(x_2)\phi + (-1)^{|x_1|\,|x_2|}\iota''_h(x_2)\, \mathrm{d}''_h\, \iota''_h(x_1)\phi.$$

By Proposition 6.2.2, the first term is $\iota''_h([\![x_1, x_2]\!])\phi$. For the second term, note that $\iota''_h(x_2)\phi \in \mathscr{W}^{0,\bullet}(D'') = \Gamma(\wedge A^*)$. But on sections of $\wedge A^*$, the composition $\iota''_h(x_1) \circ \mathrm{d}''_h$ acts as $[\![x_1, \cdot]\!]$, again using Proposition 6.2.2. Hence, the second term is $-[\![x_1, \iota''_h(x_2)\phi]\!]$. The third term is dealt with similarly. □

6.3 Interaction Between the Differentials

The differential d'_v on $\mathscr{W}(D')$ restricts to a differential on

$$\mathscr{W}^{1,\bullet}(D') = \Gamma_{\mathrm{lin}}(\wedge_B^\bullet D, B) \equiv \Gamma_{\mathrm{lin}}(\wedge_B^\bullet (D')^v, B).$$

On the other hand, we also have the horizontal differential on

$$\mathscr{W}^{\bullet,1}(D'') = \Gamma_{\mathrm{lin}}(\wedge_A D, A) \equiv \Gamma_{\mathrm{lin}}(\wedge_A^\bullet (D'')^h, A)$$

coming from the horizontal $\mathcal{VB}$-algebroid structure $D'' \Rightarrow A$ under the identification of D with the flip of the horizontal dual $(D'')^h$, it is again the restriction of the Chevalley-Eilenberg differential. In other words, it is the space $\mathsf{CE}^\bullet_{\mathcal{VB}}(D'')$ in the notation of [?]. The Lie algebroid differential d on $\Gamma(\wedge E^*)$ may be interpreted as d'_v on $\mathscr{W}^{0,\bullet}(D')$ or as d''_h on $\mathscr{W}^{\bullet,0}(D'')$, and both $\mathscr{W}^{1,\bullet}(D')$ and $\mathscr{W}^{\bullet,1}(D'')$ are differential graded modules over this algebra. Understanding the relationship between these various differentials will pave the way for our study of double Lie algebroids in the next chapter.

Proposition 6.3.1. *The $\wedge E^*$-valued pairing $\langle\cdot,\cdot\rangle_{E^*}$ (cf. (4.3)) satisfies*

$$\mathrm{d}\langle x, y\rangle_{E^*} = \langle \mathrm{d}''_h x, y\rangle_{E^*} + (-1)^{|x|+1}\langle x, \mathrm{d}'_v y\rangle_{E^*},$$

for all $x \in \mathscr{W}^{p,1}(D'')$ and $y \in \mathscr{W}^{1,q}(D')$. Here $|x| = p+1$.

Before proving the proposition, recall that for any Poisson manifold (Q,π), the Schouten bracket defines a degree 1 differential on the graded algebra $\mathfrak{X}^\bullet(Q)$ of multi-vector fields on Q:

$$[\pi,\cdot]\colon \mathfrak{X}^p(Q) \to \mathfrak{X}^{p+1}(Q).$$

If $Q = V$ is a Poisson vector bundle, thus π is homogeneous of degree -1, then this differential preserves the graded subalgebra $\mathfrak{X}^\bullet_{\mathrm{core}}(V)$ of core multi-vector fields, as well as the module $\mathfrak{X}^\bullet_{\mathrm{lin}}(V)$ of linear multi-vector fields. It is well-known that the identification $\mathfrak{X}^\bullet_{\mathrm{core}}(V) = \Gamma(\wedge^\bullet V)$ intertwines the differential $[\pi,\cdot]$ with the Lie algebroid differential for $V^* \Rightarrow M$.

Proof of Proposition 6.3.1. As explained in Section 4.6,

$$\mathfrak{X}^q(D)_{[-q,1-q]} \cong \mathscr{W}^{1,q}(D'),\quad \mathfrak{X}^p(D)_{[1-p,-p]} \cong \mathscr{W}^{p,1}(D''),\quad \mathfrak{X}^n(D)_{[-n,-n]} \cong \Gamma(\wedge^n E^*).$$

By a check on generators, one finds that the differential d'_v is realized as $-[\pi,\cdot]$, while d''_h, d are realized as $[\pi,\cdot]$. Furthermore, according to Proposition 4.6.1 the pairing between $x \in \mathfrak{X}^p(D)_{[1-p,-p]}$ and $y \in \mathfrak{X}^q(D)_{[-q,1-q]}$ is expressed in terms of the Schouten bracket as $\langle x, y\rangle_{E^*} = [x,y]$. The proposition thus translates into the Jacobi identity

$$[\pi,[x,y]] = [[\pi,x],y] + (-1)^{|x|}[x,[\pi,y]]. \qquad \square$$

A consequence of Proposition 6.3.1 is the following result about contraction operators.

Proposition 6.3.2. *For $x \in \mathcal{W}^{p,1}(D'')$, we have the following equality of superderivations of $\mathcal{W}(D')$,*

$$[\mathrm{d}'_v, \iota'_h(x)] = \iota'_h(\mathrm{d}''_h x). \tag{6.11}$$

Similarly, for $y \in \mathcal{W}^{1,q}(D')$ we have the equality of superderivations of $\mathcal{W}(D'')$,

$$[\mathrm{d}''_h, \iota''_v(y)] = \iota''_v(\mathrm{d}'_v y). \tag{6.12}$$

Proof. Both sides of (6.11) are superderivations of bidegree $(-1, p)$. Hence, they both vanish on $W^{0,\bullet}(D')$. On sections $y \in \mathcal{W}^{1,q}(D')$, the identity becomes

$$\mathrm{d}'_v \iota'_h(x)y + (-1)^{|x|}\iota'_h(x)\mathrm{d}'_v y = \iota'_h(\mathrm{d}''_h x)y.$$

After expressing the horizontal contractions in terms of the pairing $\langle \cdot, \cdot \rangle_{E^*}$, this identity reads as

$$\mathrm{d}\langle x, y\rangle_{E^*} + (-1)^{|x|}\langle x,\, \mathrm{d}'_v y\rangle_{E^*} = \langle \mathrm{d}''_h x, y\rangle_{E^*}.$$

which is just the statement of Proposition 6.3.1. Similarly, the two sides of (6.12) are superderivations of bidegree $(q, -1)$. Applying both sides to $x \in \mathcal{W}^{p,1}(D'')$, the identity becomes

$$\mathrm{d}''_h \iota''_v(y)x + (-1)^{|y|}\iota''_v(y)\mathrm{d}''_h x = \iota''_v(\mathrm{d}'_v y)x,$$

which may be written

$$(-1)^{(|x|+1)(|y|+1)}\mathrm{d}\langle x, y\rangle_{E^*} + (-1)^{|y|}(-1)^{|x|(|y|+1)}\langle \mathrm{d}''_h x, y\rangle_{E^*} = (-1)^{(|x|+1)|y|}\langle x,\, \mathrm{d}'_v y\rangle_{E^*}.$$

After multiplying by the sign $(-1)^{(|x|+1)(|y|+1)}$, this becomes

$$\mathrm{d}\langle x, y\rangle_{E^*} - \langle \mathrm{d}''_h x, y\rangle_{E^*} = (-1)^{|x|+1}\langle x,\, \mathrm{d}'_v y\rangle_{E^*}$$

which again is a reformulation of Proposition 6.3.1. $\square$

Chapter 7

Double Lie Algebroids

7.1 Definition and Basic Properties

The concept of a *double Lie algebroid* was introduced by Mackenzie in [55, 51, 54], as the infinitesimal counterpart to double Lie groupoids. It is given by a double vector bundle with *compatible* horizontal and vertical $\mathcal{VB}$-algebroid structures

$$\begin{tikzcd} D \arrow[r, Rightarrow] \arrow[d, Rightarrow] & B \arrow[d, Rightarrow] \\ A \arrow[r, Rightarrow] & M \end{tikzcd} \tag{7.1}$$

To formulate the compatibility condition, recall that a vertical $\mathcal{VB}$-algebroid structure makes D'' into a Poisson double vector bundle; hence D' becomes a $\mathcal{VB}$-algebroid horizontally,

$$\begin{tikzcd} D' \arrow[r, Rightarrow] \arrow[d] & E \arrow[d] \\ B \arrow[r, Rightarrow] & M \end{tikzcd}$$

Similarly, a horizontal $\mathcal{VB}$-algebroid structure on D makes D' into a double Poisson vector bundle, and hence D'' becomes a $\mathcal{VB}$-algebroid vertically,

$$\begin{tikzcd} D'' \arrow[r] \arrow[d, Rightarrow] & A \arrow[d, Rightarrow] \\ E \arrow[r] & M \end{tikzcd}$$

The compatibility condition is that the two Lie algebroids $D' \Rightarrow E$ and $D'' \Rightarrow E$, with their natural duality pairing, form a *Lie bialgebroid*, as defined by Mackenzie-Xu [57]. In the formulation of Kosmann-Schwarzbach [39], this means that the Chevalley-Eilenberg differential d_{CE} on $\Gamma(\wedge_E D', E)$, defined by the identification of $D' \to E$ with the dual of the Lie algebroid $D'' \Rightarrow E$, is a derivation of the Schouten bracket $[\cdot,\cdot]$ for the Lie algebroid $D' \Rightarrow E$:

$$\mathrm{d}_{CE}[\lambda_1, \lambda_2] = [\mathrm{d}_{CE}\lambda_1, \lambda_2] + (-1)^{n_1-1}[\lambda_1, \mathrm{d}_{CE}\lambda_2], \tag{7.2}$$

for $\lambda_i \in \Gamma(\wedge_E^{n_i} D', E),\ i = 1, 2.$

Examples 7.1.1. 1. Mackenzie arrived at the definition of a double Lie algebroid by applying the Lie functor to an $\mathcal{LA}$-groupoid. For instance, any Poisson Lie groupoid $G \rightrightarrows M$ [57], with Lie algebroid $A \Rightarrow M$, gives rise to a double Lie algebroid structure on T^*A, by applying the Lie functor to its cotangent Lie algebroid $T^*G \rightrightarrows A^*$. Similarly, for a *double Lie groupoid* [5, 51, 54], applying the Lie functor twice produces a double Lie algebroid.

2. The tangent bundle of a Lie algebroid $V \Rightarrow M$ is a double Lie algebroid [55, Example 4.6]

$$\begin{tikzcd} TV \arrow[r, Rightarrow] \arrow[d, Rightarrow] & TM \arrow[d, Rightarrow] \\ V \arrow[r, Rightarrow] & M \end{tikzcd}$$

One may regard TV as being obtained by applying the Lie functor to the $\mathcal{LA}$-groupoid $V \times V \rightrightarrows V$ (the pair groupoid of V).

3. *Matched pairs of Lie algebroids*, due to Lu [49] and Mokri [67], are a generalization of a matched pair of Lie algebras [60] (also known as double Lie algebras [50] or twilled extensions [40]). Two Lie algebroids $A \Rightarrow M,\ B \Rightarrow M$, with actions of A on B and of B on A, are a matched pair if the brackets and actions define a Lie algebroid structure on the direct sum $A \oplus B \Rightarrow M$. Mackenzie [55] proved that matched pairs of Lie algebroids are equivalent to *vacant* double Lie algebroids

$$\begin{tikzcd} D = A \times_M B \arrow[r, Rightarrow] \arrow[d, Rightarrow] & B \arrow[d, Rightarrow] \\ A \arrow[r, Rightarrow] & M \end{tikzcd}$$

i.e. such that $\mathrm{core}(D) = 0$.

7.2 Weil Algebra of a Double Lie Algebroid

The following theorem gives equivalent formulations of Mackenzie's definition of a double Lie algebroid in terms of the Weil algebras of the three double vector bundles D, D', D''.

Theorem 7.2.1. *Let D be a double vector bundle. The following are equivalent:*

1. *A double Lie algebroid structure on D;*

2. *a Gerstenhaber bracket on the bigraded superalgebra $\mathcal{W}(D')$, together with a differential d'_h of bidegree $(1, 0)$ that is a derivation of the Gerstenhaber bracket;*

3. *a Gerstenhaber bracket on the bigraded superalgebra $\mathcal{W}(D'')$, together with a differential d''_v of bidegree $(0, 1)$ that is a derivation of the Gerstenhaber bracket;*

4. *commuting differentials $\mathrm{d}_h, \mathrm{d}_v$ on $\mathcal{W}(D)$, of bidegrees $(1, 0)$ and $(0, 1)$, respectively,*

We will break up the proof into several steps. Consider first equivalence (a) $\Leftrightarrow$ (c).

Lemma 7.2.2. *A double Lie algebroid structure on D is equivalent to a Gerstenhaber bracket on $\mathcal{W}(D'')$, together with a differential d_v'' of bidegree $(0,1)$ that is a derivation of the Gerstenhaber bracket.*

Proof. As discussed in Section 4.5, there are canonical identifications

$$\mathcal{W}^{0,\bullet}(D'') \cong \Gamma_{\mathrm{core}}(\wedge_E^\bullet D', E), \quad \mathcal{W}^{1,\bullet}(D'') \cong \Gamma_{\mathrm{lin}}(\wedge_E^\bullet D', E).$$

These spaces generate $\mathcal{W}(D'')$ as an algebra, and also $\Gamma(\wedge_E D', E)$ as a module over $C^\infty(M)$. Furthermore, by definition, the restriction of the Gerstenhaber bracket on $\mathcal{W}(D'')$ to these spaces agrees with the Lie algebroid bracket for $D' \Rightarrow E$, while the differential d_v'' coincides with the Chevalley-Eilenberg differential for $D'' \Rightarrow E$. Hence, d_v'' being a derivation of the Gerstenhaber bracket amounts to the defining compatibility condition of a double Lie algebroid. □

Theorem 5.2.1 shows that for a horizontal $\mathcal{VB}$-algebroid structure $D \Rightarrow B$ on a double vector bundle D is equivalent to a vertical differential d_v on $\mathcal{W}(D)$, and also to a horizontal differential d_h' on $\mathcal{W}(D')$. By Proposition 6.3.2, these are related by

$$\mathrm{d}_v y = \mathrm{d}_h' y, \qquad [\mathrm{d}_v, \iota_h(x)] = \iota_h(\mathrm{d}_h' x), \tag{7.3}$$

for all $y \in \Gamma(\wedge^\bullet B^*)$ and all $x \in \mathcal{W}^{\bullet,1}(D')$. (The first identity uses that both $\mathcal{W}^{0,\bullet}(D)$ and $W^{\bullet,0}(D')$ are identified with sections of $\wedge B^*$.) On the other hand, using Theorem 5.2.1 again, a vertical $\mathcal{VB}$-algebroid structure $D \Rightarrow A$ is equivalent to a horizontal differential d_h on $\mathcal{W}(D)$, and also to a Gerstenhaber bracket $[\![\cdot,\cdot]\!]$ on $\mathcal{W}(D')$. According to Propositions 6.2.1 and 6.2.2, these are related by

$$\iota_h(x)\,\mathrm{d}_h y = [\![x, y]\!], \quad [\iota_h(x_1), [\iota_h(x_2), \mathrm{d}_h]] = (-1)^{|x_2|}\iota_h([\![x_1, x_2]\!]), \tag{7.4}$$

for all $y \in \Gamma(\wedge^\bullet B^*)$ and all $x, x_1, x_2 \in \mathcal{W}^{\bullet,1}(D')$. Consider now the situation that both a horizontal and a vertical $\mathcal{VB}$-algebroid structure are given.

Lemma 7.2.3. *Given a horizontal $\mathcal{VB}$-algebroid structure $D \Rightarrow B$ and a vertical $\mathcal{VB}$-algebroid structure $D \Rightarrow A$, the super-commutator $[\mathrm{d}_h, \mathrm{d}_v]$ satisfies*

$$\iota_h(x)[\mathrm{d}_h,\, \mathrm{d}_v]y = (-1)^{|x|+1}(\mathrm{d}_h'[\![x, y]\!] - [\![\mathrm{d}_h' x, y]\!] - (-1)^{|x|}[\![x, \mathrm{d}_h' y]\!]) \tag{7.5}$$

for $x \in \mathcal{W}^{\bullet,1}(D')$ and $y \in \Gamma(\wedge^\bullet B^*)$, as well as

$$[\iota_h(x_1), [\iota_h(x_2), [\mathrm{d}_h, \mathrm{d}_v]]] = (-1)^{|x_1|}\iota_h(\mathrm{d}_h'[\![x_1, x_2]\!] - [\![\mathrm{d}_h' x_1, x_2]\!] - (-1)^{|x_1|}[\![x_1, \mathrm{d}_h' x_2]\!]) \tag{7.6}$$

for $x_1, x_2 \in \mathcal{W}^{\bullet,1}(D')$.

Proof. The two identities follow from the calculations, using (7.3) and (7.4),

$$\begin{aligned}
\iota_h(x)[\mathrm{d}_h, \mathrm{d}_v]y &= \iota_h(x)\mathrm{d}_h\mathrm{d}_v\, y + \iota_h(x)\mathrm{d}_v\mathrm{d}_h y \\
&= [\![x, \mathrm{d}_v y]\!] + (-1)^{|x|}[\mathrm{d}_v, \iota_h(x)]\mathrm{d}_h y - (-1)^{|x|}\mathrm{d}_v\iota_h(x)\mathrm{d}_h y \\
&= [\![x, \mathrm{d}_h' y]\!] + (-1)^{|x|}\iota_h(\mathrm{d}_h' x)\mathrm{d}_h y - (-1)^{|x|}\mathrm{d}_h'[\![x, y]\!] \\
&= (-1)^{|x|+1}(\mathrm{d}_h'[\![x, y]\!] - [\![\mathrm{d}_h' x, y]\!] - (-1)^{|x|}[\![x, \mathrm{d}_h' y]\!])
\end{aligned}$$

and

$$\begin{aligned}
\iota_h(\mathrm{d}'_h[\![x_1,x_2]\!]) &= [\mathrm{d}_v, \iota_h([\![x_1,x_2]\!])] \\
&= (-1)^{|x_2|}[\mathrm{d}_v, [\iota_h(x_1), [\iota_h(x_2), \mathrm{d}_h]]] \\
&= (-1)^{|x_2|}[[\mathrm{d}_v, \iota_h(x_1)], [\iota_h(x_2), \mathrm{d}_h]] - (-1)^{|x_1|+|x_2|}[\iota_h(x_1), [[\mathrm{d}_v, \iota_h(x_2)], \mathrm{d}_h]] \\
&\quad + (-1)^{|x_1|}[\iota_h(x_1), [\iota_h(x_2), [\mathrm{d}_v, \mathrm{d}_h]]] \\
&= (-1)^{|x_2|}[\iota_h(\mathrm{d}'_h x_1), [\iota_h(x_2), \mathrm{d}_h]] - (-1)^{|x_1|+|x_2|}[\iota_h(x_1), [\iota_h(\mathrm{d}'_h x_2), \mathrm{d}_h]] \\
&\quad + (-1)^{|x_1|}[\iota_h(x_1), [\iota_h(x_2), [\mathrm{d}_v, \mathrm{d}_h]]] \\
&= \iota_h([\![\mathrm{d}'_h x_1, x_2]\!]) + (-1)^{|x_1|}\iota_h([\![x_1, \mathrm{d}'_h x_2]\!]) + (-1)^{|x_1|}[\iota_h(x_1), [\iota_h(x_2), [\mathrm{d}_v, \mathrm{d}_h]]].
\end{aligned}$$

□

We now have all the tools we need to establish Theorem 7.2.1:

Proof of Theorem 7.2.1. The equivalence (a) ⇔ (c) was already established in Lemma 7.2.3. Consider now the implication (b) ⇒ (d). Using (7.5) and (7.6), we see that if d'_h is a derivation of the Gerstenhaber bracket, then

$$\iota_h(x)[\mathrm{d}_h, \mathrm{d}_v]y = 0, \quad [\iota_h(x_1), [\iota_h(x_2), [\mathrm{d}_h, \mathrm{d}_v]]] = 0$$

for all $x, x_1, x_2 \in \mathcal{W}^{\bullet,1}(D')$ and $y \in \Gamma(\wedge^\bullet B^*) = \mathcal{W}^{\bullet,0}(D')$. The first equation shows that $[\mathrm{d}_h, \mathrm{d}_v]y = 0$, so that $[\mathrm{d}_h, \mathrm{d}_v]$ vanishes on $\mathcal{W}^{\bullet,0}(D')$. Using the second equation, and induction on q, it then follows that $[\mathrm{d}_h, \mathrm{d}_v]$ vanishes on all $\mathcal{W}^{\bullet,q}(D')$, hence that $\mathrm{d}_h, \mathrm{d}_v$ super-commute. For the reverse implication (d) ⇒ (b), suppose $[\mathrm{d}_h, \mathrm{d}_v] = 0$. Equations (7.5) and (7.6) tell us that

$$\mathrm{d}'_h[\![x,y]\!] - [\![\mathrm{d}'_h x, y]\!] - (-1)^{|x|}[\![x, \mathrm{d}'_h y]\!] = 0, \quad \mathrm{d}'_h[\![x_1,x_2]\!] - [\![\mathrm{d}'_h x_1, x_2]\!] - (-1)^{|x_1|}[\![x_1, \mathrm{d}'_h x_2]\!] = 0$$

for all $x, x_1, x_2 \in \mathcal{W}^{\bullet,1}(D')$ and $y \in \mathcal{W}^{\bullet,0}(D')$. This means that d'_h acts as a derivation of the Gerstenhaber bracket on generators, and hence in general. We have thus shown (b) ⇔ (d). The equivalence (c) ⇔ (d) follows by applying a flip, which interchanges the horizontal and vertical structures. □

7.3 The Core of a Double Lie Algebroid

It was pointed out in [55, Section 4] that for any double Lie algebroid D, the core E^* acquires the structure of a Lie algebroid. This fact may be seen as a consequence of the fact that the base of any Lie bialgebroid is Poisson [57, Proposition 3.6]. It may also be obtained using the Weil algebras, as follows. Recall that $\mathcal{W}(D')$ has a vertical differential and $\mathcal{W}(D'')$ a horizontal differential, which are derivations of the Gerstenhaber brackets on these algebras.

Proposition 7.3.1. *The core E^* of a double Lie algebroid D has a Lie algebroid structure, with bracket given in terms of the identification $\Gamma(\wedge E^*) = \mathcal{W}^{\bullet,0}(D'')$ by*

$$[\varepsilon_1, \varepsilon_2] = [\![\varepsilon_1, \mathrm{d}''_v \varepsilon_2]\!], \qquad \mathcal{L}_{\mathsf{a}(\varepsilon)}(f) = [\![\varepsilon, \mathrm{d}''_v f]\!],$$

or in terms of the identification $\Gamma(\wedge E^*) = \mathcal{W}^{0,\bullet}(D')$ *by*

$$[\varepsilon_1, \varepsilon_2] = -[\![\varepsilon_1, \mathrm{d}'_h\varepsilon_2]\!], \qquad \mathcal{L}_{\mathsf{a}(\varepsilon)}(f) = -[\![\varepsilon, \mathrm{d}'_h f]\!].$$

Proof. Note that $[\![\varepsilon_1, \mathrm{d}''_v\varepsilon_2]\!] = [\![\mathrm{d}''_v\varepsilon_1, \varepsilon_2]\!]$, which implies skew-symmetry of the bracket $[\cdot,\cdot]$. The Jacobi identity for $[\cdot,\cdot]$ follows from that for the Gerstenhaber bracket:

$$\begin{aligned}
[\varepsilon_1, [\varepsilon_2, \varepsilon_3]] &= [\![\varepsilon_1, \mathrm{d}''_v[\![\varepsilon_2, \mathrm{d}''_v\varepsilon_3]\!]]\!] = [\![\varepsilon_1, [\![\mathrm{d}''_v\varepsilon_2, \mathrm{d}''_v\varepsilon_3]\!]]\!] \\
&= [\![[\![\varepsilon_1, \mathrm{d}''_v\varepsilon_2]\!], \mathrm{d}''_v\varepsilon_3]\!] + [\![\mathrm{d}''_v\varepsilon_2, [\![\varepsilon_1, \mathrm{d}''_v\varepsilon_3]\!]]\!] = [[\varepsilon_1, \varepsilon_2], \varepsilon_3] + [\varepsilon_2, [\varepsilon_1, \varepsilon_3]].
\end{aligned}$$

Furthermore, if $f \in C^\infty(M)$,

$$[\![\varepsilon_1, \mathrm{d}''_v(f\varepsilon_2)]\!] = f[\![\varepsilon_1, \mathrm{d}''_v\varepsilon_2]\!] + [\![\varepsilon_1, \mathrm{d}''_v f]\!]\varepsilon_2,$$

so that $\mathcal{L}_{\mathsf{a}(\varepsilon)}(f) = [\![\varepsilon, \mathrm{d}''_v f]\!]$ defines an anchor map for this bracket. The expression of the Lie algebroid structure in terms of D' follows from

$$[\![\mathrm{d}''_v\varepsilon_1, \varepsilon_2]\!] = \iota'_h(\mathrm{d}''_v\varepsilon_1)\mathrm{d}'_h\varepsilon_2 = \langle \mathrm{d}''_v\varepsilon_1, \mathrm{d}'_h\varepsilon_2\rangle_{E^*} = \iota''_v(\mathrm{d}'_h\varepsilon_2)\mathrm{d}''_v\varepsilon_1 = [\![\mathrm{d}'_h\varepsilon_2, \varepsilon_1]\!] = -[\![\varepsilon_1, \mathrm{d}'_h\varepsilon_2]\!]. \qquad \square$$

The Lie algebroid bracket $[\varepsilon_1, \varepsilon_2] = [\![\varepsilon_1, \mathrm{d}''_v\varepsilon_2]\!]$ on $\Gamma(E^*)$ extends to a Schouten bracket on $\Gamma(\wedge E^*)$, by

$$[\lambda_1, \lambda_2] = [\![\lambda_1, \mathrm{d}''_v\lambda_2]\!].$$

Chapter 8

Example: Tangent Prolongation of a Vector Bundle

We will now illustrate the constructions presented in this thesis by taking a close look at the main example of a double Lie algebroid, the tangent bundle of a Lie algebroid. Since the earlier chapters require only a double vector bundle (and not a double Lie algebroid), we will start with a vector bundle $V \to M$. The two $\mathscr{DVB}$s we will be considering in this chapter are those of example 2.22:

$$\begin{array}{ccc} TV & \longrightarrow & TM \\ \downarrow & & \downarrow \\ V & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} T^*V & \longrightarrow & V^* \\ \downarrow & & \downarrow \\ V & \longrightarrow & M. \end{array}$$

In this context, we will encounter the *jet bundle* $J^1(V)$ and the *Atiyah algebroid* $\mathrm{At}(V)$ (often denoted by $\mathfrak{D}(V)$, or similar). For $\sigma \in \Gamma(V)$, we denote by $j^1(\sigma) \in \Gamma(J^1(V))$ its jet prolongation. As a reminder, the fibers $J^1(V)_m$ of the jet bundle consist of equivalence classes of sections of V, where two sections are equivalent if their values at m as well as their derivatives at m coincide. Given a section σ, we denote by $j^1(\sigma)_m$ its equivalence class, and one obtains the jet prolongation $j^1(\sigma) \in \Gamma(J^1(V))$ by allowing m to vary. The jet bundle comes with a quotient map $J^1(V) \to V$ taking sections of the form $fj^1(\sigma)$ to $f\sigma$; this defines a short exact sequence

$$0 \to T^*M \otimes V \xrightarrow{i_{J^1(V)}} J^1(V) \to V \to 0 \tag{8.1}$$

with $i_{J^1(V)}(\mathrm{d}f \otimes \sigma) = j^1(f\sigma) - fj_1(\sigma)$. On the other hand, the Atiyah algebroid comes with a short exact sequence

$$0 \to V \otimes V^* \xrightarrow{i_{\mathrm{At}(V)}} \mathrm{At}(V) \xrightarrow{\mathsf{a}} TM \to 0 \tag{8.2}$$

where a is the anchor. We shall find it convenient to use the identification $\Gamma(\mathrm{At}(V)) \cong \mathfrak{X}_{\mathrm{lin}}(V)$ (cf. (2.21)) to interpret sections δ of the Atiyah algebroid in terms of the corresponding linear vector field $\widetilde{\mathsf{a}}(\delta)$ on V; its restriction to the zero section is $\mathsf{a}(\delta)$. From this perspective, $\widetilde{\mathsf{a}}(i_{\mathrm{At}(V)}(\sigma \otimes \tau)) = \phi_\tau \sigma^\sharp$, where $\phi_\tau \in C^\infty(V)$ is the linear function defined by $\tau \in \Gamma(V^*)$, and $\sigma^\sharp \in \mathfrak{X}(V)_{[-1]}$ denotes the vertical lift of $\sigma \in \Gamma(V)$ (see A.2). The representation of $\mathrm{At}(V)$ on V is given by the Lie bracket, $(\nabla_\delta \sigma)^\sharp = [\widetilde{\mathsf{a}}(\delta), \sigma^\sharp]$, and the dual representation ∇^* on V^*, defined as

$$\mathscr{L}_{\mathsf{a}(\delta)}\langle\tau,\sigma\rangle = \langle\tau,\nabla_\delta\sigma\rangle + \langle\nabla^*_\delta\tau,\,\sigma\rangle, \tag{8.3}$$

is realized by the Lie derivative of $\widetilde{\mathsf{a}}(\delta)$ on linear functions, $\phi_\tau \mapsto \mathscr{L}_{\widetilde{\mathsf{a}}(\delta)}\phi_\tau$.

Throughout this chapter we will use several operations that lift geometric objects on V to objects on TV and T^*V, namely the vertical, tangent, and cotangent lifts of functions, vector fields, and sections. We have included a review of all of these processes in section A.2. As a quick notation reference, the superscript $\sharp$ denotes a vertical lift, the subscript T denotes a tangent lift, and the subscript T^* denotes a cotangent lift.

8.1 $\mathscr{DVB}$ Sequences of TV and T^*V

For $D = TV$ we have $A = V,\ B = TM,\ E = V^*$. One finds that

$$\widehat{A} = J^1(V),\quad \widehat{B} = \mathrm{At}(V),\quad \widehat{E} = J^1(V^*). \tag{8.4}$$

In terms of $\Gamma(\widehat{A}) \cong \mathfrak{X}(D)_{[-1,0]},\ \Gamma(\widehat{B}) \cong \mathfrak{X}(D)_{[0,-1]},\ \Gamma(\widehat{E}) \cong C^\infty(D)_{[1,1]}$, these identifications are given by

$$j^1(\sigma) \mapsto (\sigma^\sharp)_T,\quad \delta \mapsto \widetilde{\mathsf{a}}(\delta)^\sharp,\quad j^1(\tau) \mapsto (\phi_\tau)_T.$$

Using (2.23) and (2.24), we obtain the three inclusions

$$i_{\widehat{E}}(\tau\otimes \mathrm{d}f) = i_{J^1(V^*)}(\mathrm{d}f\otimes\tau),\quad i_{\widehat{A}}(\mathrm{d}f\otimes\sigma) = i_{J^1(V)}(\mathrm{d}f\otimes\sigma),\quad i_{\widehat{B}}(\sigma\otimes\tau) = i_{\mathrm{At}(V)}(\sigma\otimes\tau)$$

and the three pairings

$$\langle\delta,\,j^1(\sigma)\rangle_V = \nabla_\delta\sigma,\quad \langle j^1(\tau),\,\delta\rangle_{V^*} = -\nabla^*_\delta\tau,\quad \langle j^1(\sigma),\ j^1(\tau)\rangle_{T^*M} = \mathrm{d}\langle\tau,\sigma\rangle$$

for $\sigma\in\Gamma(V),\ \tau\in\Gamma(V^*),\ \delta\in\Gamma(\mathrm{At}(V))$. As a sample computation, note that (2.23) gives

$$i_{\widehat{E}}(\tau\otimes\mathrm{d}f) = (\phi_\tau)^\sharp f_T = (f\phi_\tau)_T - f^\sharp(\phi_\tau)_T,$$

which lies in $C^\infty(TV)_{[1,1]}$. This coincides with the image of $i_{J^1(V^*)}(\mathrm{d}f\otimes\tau) = j^1(f\tau) - fj^1(\tau)$. The exact sequences (2.13) are just the standard exact sequences for the jet bundles and the Atiyah algebroid.

Remark 8.1.1. The $E^* = V$-valued pairing between $\widehat{A} = J^1(V)$ and $\widehat{B} = \mathrm{At}(V)$ was observed by Chen-Liu in [12, Section 2].

Let us also note that by Remark 2.7.6, a splitting of $D = TV$ is equivalent to a splitting of any one of the exact sequences for $J^1(V^*),\ J^1(V)$ or $\mathrm{At}(V)$; in turn, these are equivalent to a linear connection on the vector bundle V.

Now let us turn to the cotangent bundle T^*V of V. We stress again that we will be using the notation described in secion A.2, in particular, $\{\ ,\ \}$ denotes the canonical Poisson bracket on the cotangent bundle. For $D = T^*V$, we have that $A = V,\ B = V^*,\ E = TM$ with

$$\widehat{A} = J^1(V),\quad \widehat{B} = J^1(V^*),\quad \widehat{E} = \mathrm{At}(V).$$

In terms of the identifications of their spaces of sections with $\mathfrak{X}(D)_{[-1,0]}$, $\mathfrak{X}(D)_{[0,-1]}$, $C^\infty(D)_{[1,1]}$, these isomorphisms are given by

$$j^1(\sigma) \mapsto \{\phi_{\sigma^\sharp}, \cdot\}, \quad j^1(\tau) \mapsto \{(\phi_\tau)^\sharp, \cdot\}, \quad \delta \mapsto \phi_\delta.$$

Using (2.23) and (2.24), we find that the three pairings are the same as for TV (with the order of the two entries interchanged), while each of the three inclusion maps changes sign. This is consistent with $T^*V = \text{flip}(TV)^-$, see Proposition 2.8.2. The three exact sequences (2.13) are the standard exact sequences for the jet bundles and the Atiyah algebroid, up to a sign change of the three inclusion maps.

8.2 The Weil Algebras $\mathscr{W}(TV)$ and $\mathscr{W}(T^*V)$

Using the $\mathscr{DVB}$ sequences computed above, we now describe the Weil algebras of the tangent and cotangent prolongations. First consider $D = TV$, so that $A = V,\ B = TM,\ C = V^*$. The Weil algebra $\mathscr{W}(TV)$ is generated by functions $f \in C^\infty(M)$ (bidegree $(0,0)$) and their de Rham differentials $\mathrm{d}f$ (bidegree $(0,1)$), together with sections $\tau \in \Gamma(V^*)$ (bidegree $(1,0)$) and their 1-jets $j^1(\tau) \in \Gamma(J^1(V^*))$ (bidegree $(1,1)$), subject to relations of $C^\infty(M)$-linearity and the relation that

$$\tau\ \mathrm{d}f = j^1(f\tau) - fj^1(\tau).$$

Here we used that $i_{\widehat{E}}(\tau \otimes \mathrm{d}f) = i_{J^1(V^*)}(\tau \otimes \mathrm{d}f)$. The contraction operators are computed from the pairings (see the proof of theorem 4.4.1):

$$\begin{array}{llll}
\iota_h(j^1(\sigma)): & \mathrm{d}f \mapsto 0 & \tau \mapsto \langle \tau, \sigma\rangle, & j^1(\tau) \mapsto -\mathrm{d}\langle \tau, \sigma\rangle, \\
\iota_v(\delta): & \mathrm{d}f \mapsto \mathcal{L}_{\mathsf{a}(\delta)} f, & \tau \mapsto 0, & j^1(\tau) \mapsto \nabla^*_\delta \tau, \\
\iota(\sigma): & \mathrm{d}f \mapsto 0, & \tau \mapsto 0, & j^1(\tau) \mapsto -\langle \tau, \sigma\rangle.
\end{array}$$

As per section 4.5, $\mathscr{W}(TV)$ may also be described in terms of linear and core multivector fields on V. The *core n-vector fields* $\mathfrak{X}^n_{\text{core}}(V) \equiv \Gamma_{\text{core}}(\wedge^n_V TV, V)$ are the sections of $\wedge^n V$, regarded as vertical 'fiberwise constant' multi-vector fields on V:

$$\mathfrak{X}^n_{\text{core}}(V) = \Gamma(\wedge^n V).$$

The *linear n-vector fields* $\mathfrak{X}^n_{\text{lin}}(V) = \Gamma_{\text{lin}}(\wedge^n_V TV, V)$ may be defined by their property that the evaluation on linear 1-forms on V is a linear function on V [24]. The short exact sequence (4.15) specializes to

$$0 \to \Gamma(\wedge^n V \otimes V^*) \to \mathfrak{X}^n_{\text{lin}}(V) \to \Gamma(\wedge^{n-1} V \otimes TM) \to 0; \tag{8.5}$$

here the inclusion of $\Gamma(\wedge^n V \otimes V^*)$ is as the subspace of linear n-vector fields on V that are tangent to the fibers of $V \to M$. In local vector bundle coordinates, with x_i the coordinates on the base and y_j the coordinates on the fiber, the linear n-vector fields on V are of the form

$$\sum a^j_{j_1\cdots j_n}(x)\, y_j \frac{\partial}{\partial y_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial y_{j_n}} + \sum b_{i\, j_1\cdots j_{n-1}}(x) \frac{\partial}{\partial y_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial y_{j_{n-1}}} \wedge \frac{\partial}{\partial x_i}.$$

The Schouten bracket of multi-vector fields defines a graded Lie algebra structure on $\mathfrak{X}^\bullet_{\mathrm{lin}}(V)[1]$, with a representation on $\mathfrak{X}^\bullet_{\mathrm{core}}(V) = \Gamma(\wedge^\bullet V)$. These are the *multi-differentials* in the work of Iglesias-Ponte, Laurent-Gengoux, and Xu [36].

For the case $D = T^*V$, so that $A = V,\ B = V^*,\ C = TM$. the Weil algebra $\mathscr{W}(T^*V)$ is generated by functions $f \in C^\infty(M)$ (bidegree $(0,0)$), sections $\sigma \in \Gamma(V)$ (bidegree $(0,1)$), sections $\tau \in \Gamma(V^*)$ (bidegree $(1,0)$), and infinitesimal automorphisms $\delta \in \Gamma(\mathrm{At}(V))$ (bidegree $(1,1)$), subject to relations of $C^\infty(M)$-linearity and the relation that the product $\tau\sigma$ in the Weil algebra, for $\tau \in \Gamma(V^*)$ and $\sigma \in \Gamma(V)$, equals the section $i_{\widehat{E}}(\tau \otimes \sigma) = -i_{\mathrm{At}(V)}(\sigma \otimes \tau) = -\phi_\tau\, \sigma^\sharp \in \Gamma(\mathrm{At}(V))$. Again, the various contraction operators may be computed from the pairings.

$$\begin{array}{llll}
\iota_h(j^1(\sigma)): & \sigma \mapsto 0 & \tau \mapsto \langle \tau, \sigma\rangle, & \delta \mapsto \nabla_\delta \sigma,\\
\iota_v(j^1\tau): & \sigma \mapsto -\langle \tau, \sigma\rangle, & \tau \mapsto 0, & \delta \mapsto -\nabla^*_\delta \tau,\\
\iota(df): & \sigma \mapsto 0, & \tau \mapsto 0, & \delta \mapsto -\mathscr{L}_{\mathsf{a}(\delta)} f.
\end{array}$$

We now relate the Weil algebra $\mathscr{W}(T^*V)$ to linear and core differential forms on V. The space $\Omega^n_{\mathrm{core}}(V) = \Gamma_{\mathrm{core}}(\wedge^n_V T^*V, V)$ is just $\Omega^n(M)$, viewed as the space of basic n-forms on V via pullback. The space

$$\Gamma_{\mathrm{lin}}(\wedge^n_V T^*V, V) = \Omega^n_{\mathrm{lin}}(V)$$

of *linear n-forms* on V consists of n-forms α with $\kappa_t^*\alpha = t\alpha$ where κ_t is scalar multiplication by t on V. (Note that the homogeneity of n-forms on V relative to pullback κ_t^* is not the same as homogeneity as sections of $\wedge^n_V T^*V$ over V.) In local bundle bundle coordinates, with x_i the coordinates on the base and y_j the coordinates on the fiber, the 1-forms $\mathrm{d}x_i$, seen as local sections of $\wedge^\bullet_V T^*V$, have homogeneity degree -1 while the $\mathrm{d}y_j$ have homogeneity 0. A general linear n-form is locally given by an expression

$$\sum a_{j\, i_1\cdots i_{n-1}}(x)\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{n-1}} \wedge \mathrm{d}y_j + \sum b_{j\, i_1\cdots i_n}(x) y_j \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n}.$$

The short exact sequence (4.15) becomes

$$0 \to \Gamma(\wedge^n T^*M \otimes V^*) \to \Omega^n_{\mathrm{lin}}(V) \to \Gamma(\wedge^{n-1} T^*M \otimes V^*) \to 0; \tag{8.6}$$

here the inclusion of $\Gamma(\wedge^n T^*M \otimes V^*)$ is as the space of linear n-forms on V that are horizontal for the projection to M, while the projection to $\Gamma(\wedge^{n-1}T^*M \otimes V^*)$ is given by contraction with sections of V (regarded as the space $\mathfrak{X}_{\mathrm{core}}(V)$ of fiberwise constant vector fields on V). The exact sequence (8.6) has a canonical splitting [6]: every element of $\Omega^n_{\mathrm{lin}}(V)$ decomposes uniquely as $\nu + \mathrm{d}\mu$ where $\nu \in \Gamma(\wedge^n T^*M \otimes V^*)$ and $\mu \in \Gamma(\wedge^{n-1}T^*M \otimes V^*)$. Using the Mackenzie-Xu isomorphism (2.12) we obtain a similar interpretation

$$\Gamma_{\mathrm{lin}}(\wedge^q_{V^*} T^*V, V^*) = \Omega^q_{\mathrm{lin}}(V^*).$$

Equation (4.17) defines an $\Omega(M)$-bilinear pairing

$$\langle\cdot,\cdot\rangle_{T^*M} \colon \Omega^p_{\mathrm{lin}}(V) \times \Omega^q_{\mathrm{lin}}(V^*) \to \Omega^{p+q-1}(M), \tag{8.7}$$

(using the right $\Omega(M)$-module structure in the second argument), given in low degrees by

$$\langle\phi_\tau,\phi_\sigma\rangle_{T^*M}=0,\quad \langle\phi_\tau,\mathrm{d}\phi_\sigma\rangle_{T^*M}=-\langle\mathrm{d}\phi_\tau,\phi_\sigma\rangle_{T^*M}=\langle\tau,\sigma\rangle,\quad \langle\mathrm{d}\phi_\tau,\mathrm{d}\phi_\sigma\rangle_{T^*M}=\mathrm{d}\langle\tau,\sigma\rangle$$

for $\tau\in\Gamma(V^*)$, $\sigma\in\Gamma(V)$ (with ϕ_τ,ϕ_σ the corresponding linear functions).

8.3 The Double-Linear Poisson Structure on T^*V

The canonical Poisson structure on T^*V is compatible with the vector bundle structure on V in the sense that it determines a double linear Poisson structure $^*V$ as a double vector bundle. This in turn determines a $\mathcal{VB}$-algebroid structure $TV\Rightarrow V$, and in particular a vertical differential d_v on the Weil algebra $\mathscr{W}(TV)$. This differential is given on the generators by

$$\mathrm{d}_v(f)=\mathrm{d}f,\quad \mathrm{d}_v(\mathrm{d}f)=0,\quad \mathrm{d}_v(\tau)=-j^1(\tau),\quad \mathrm{d}_v(j^1(\tau))=0.$$

On $\mathscr{W}^{0,\bullet}(TV)=\Gamma(\wedge T^*M)$, it agrees with the de Rham differential; on $\mathscr{W}^{1,\bullet}(TV)=\Gamma_{\mathrm{lin}}(\wedge_V T^*V,V)$ it is the restriction of the de Rham differential to linear forms. For example, we may verify that

$$\iota_v(\delta)\mathrm{d}_v\tau=-\nabla^*_\delta\tau=-\langle j^1(\tau),\delta\rangle_{V^*}=-\iota_v(\delta)j^1(\tau)$$

as required; similarly $\iota_v(\delta)\mathrm{d}_v f=\mathcal{L}_{\mathsf{a}(\delta)}f=-\langle\mathrm{d}f,\delta\rangle_{V^*}=\iota_v(\delta)\mathrm{d}f$.

On the other hand, the Gerstenhaber bracket on $\mathscr{W}(T^*V)$ is easily obtained from the Poisson bracket (see the proof of theorem 5.5.1):

$$[\![\tau,\sigma]\!]=\langle\tau,\sigma\rangle,\quad [\![\delta,\sigma]\!]=\nabla_\delta\sigma,\quad [\![\delta,\tau]\!]=\nabla^*_\delta\tau,\quad [\![\delta,f]\!]=\mathcal{L}_{\mathsf{a}(\delta)}f,\quad [\![\delta_1,\delta_2]\!]=[\delta_1,\delta_2].$$

The first equality follows from $\{\tau,\phi_{\sigma^\sharp}\}=-\mathcal{L}_{\sigma^\sharp}\tau=-\langle\tau,\sigma\rangle$, the second from $\{\phi_{\widetilde{\mathsf{a}}(\delta)},\phi_{\sigma^\sharp}\}=\phi_{[\widetilde{\mathsf{a}}(\delta),\sigma^\sharp]}=\phi_{(\nabla_\delta\sigma)^\sharp}$, and so on.

8.4 Application to Lie algebroids

Now suppose that the vector bundle $V\to M$ carries the structure of a Lie algebroid, $V\Rightarrow M$. Then we obtain $\mathcal{VB}$-algebroid structures $TV\Rightarrow TM$ and $T^*V\Rightarrow V^*$, as well as a Poisson structure on TV^*, to be described below. Let us discuss the resulting structures on the Weil algebras.

The $\mathcal{VB}$-algebroid $TV\Rightarrow TM$ is the *tangent prolongation* of the Lie algebroid $V\Rightarrow M$ [14, 57]: its anchor is the tangent map to the anchor of V composed with the canonical involution of TTM, and the Lie bracket is such that the tangent lift $\Gamma(V,M)\to\Gamma(TV,TM)$, $\sigma\mapsto T\sigma$ is bracket preserving. The resulting Lie algebroid structure on $\widehat{A}=J^1(V)$ is the *jet prolongation* of the Lie algebroid; the bracket is uniquely characterized by $[j^1(\sigma_1),j^1(\sigma_2)]=j^1([\sigma_1,\sigma_2])$, and its representations on $B=TM$ and on $E^*=V$ are given by

$$\nabla^{TM}_{j^1(\sigma)}\mu=\mathcal{L}_{\mathsf{a}(\sigma)}\mu,\qquad \nabla^V_{j^1(\sigma_1)}\sigma_2=[\sigma_1,\sigma_2]$$

for $\mu\in\Omega^1(M)$, $\sigma,\sigma_1,\sigma_2\in\Gamma(V)$. See [15] for a detailed discussion. The invariant bilinear pairing $(\cdot,\cdot)\colon B^*\times_M E^*\to\mathbb{R}$ is given by the anchor $\mathsf{a}\colon V\to TM$. The resulting horizontal differential d_h on the Weil algebra $\mathscr{W}(TV)$ is uniquely determined by its properties that d_h agrees with the Lie algebroid differential d_{CE} on $\mathscr{W}^{\bullet,0}(TV) = \Gamma(\wedge V^*)$ and satisfies $[\mathrm{d}_h, \mathrm{d}_v] = 0$, where d_v was described above.

Now consider the Weil algebra of $(TV)' = \mathrm{flip}((TV^*)^-)$, which by the results above is $C^\infty(M)$-linearly generated by differentials $\mathrm{d}f \in \Gamma(T^*M)$ (identified with the tangent lifts f_T), sections $\sigma \in \Gamma(V)$ (identified with vertical lifts σ^v) and their jets $j^1(\sigma) \in \Gamma(J^1(V))$ (identified with σ_T), subject to the relation $j^1(f\sigma) - fj^1(\sigma) = \mathrm{d}f\ \sigma$. It has a horizontal differential, characterized by $\mathrm{d}'_h(f) = \mathrm{d}f$, $\mathrm{d}'_h(\sigma) = j^1(\sigma)$, that is a derivation of the Gerstenhaber bracket. To describe the latter, note that the Lie algebroid structure on V defines a linear Poisson structure (5.1) on V^*; its tangent lift is a double-linear Poisson structure on TV^*. By definition of the tangent lift of Poisson structures,

$$\{(\sigma_1)_T, (\sigma_2)_T\} = [\sigma_1, \sigma_2]_T, \quad \{(\sigma_1)_T, (\sigma_2)_v\} = [\sigma_1, \sigma_2]_v,$$

$$\{\sigma_T, f_v\} = (\mathscr{L}_{\mathsf{a}(\sigma)} f)_v, \quad \{\sigma_T, f_T\} = (\mathscr{L}_{\mathsf{a}(\sigma)} f)_T, \quad \{\sigma_v, f_T\} = (\mathscr{L}_{\mathsf{a}(\sigma)} f)_v.$$

We read off the Gerstenhaber brackets as

$$[\![j^1(\sigma_1), j^1(\sigma_2)]\!] = j^1([\sigma_1, \sigma_2]), \quad [\![j^1(\sigma_1), \sigma_2]\!] = [\sigma_1, \sigma_2],$$

$$[\![j^1(\sigma), f]\!] = \mathscr{L}_{\mathsf{a}(\sigma)} f, \quad [\![j^1(\sigma), \mathrm{d}f]\!] = \mathrm{d}\mathscr{L}_{\mathsf{a}(\sigma)} f, \quad [\![\sigma, \mathrm{d}f]\!] = -\mathscr{L}_{\mathsf{a}(\sigma)} f.$$

Note that d'_h is a derivation of the Gerstenhaber bracket, as required.

Finally, the Weil algebra of $(TV)'' = \mathrm{flip}(T^*V)^- \cong T^*V^*$ (using the Mackenzie-Xu isomorphism (2.12)) is $C^\infty(M)$-linearly generated by sections of V, V^*, $\mathrm{At}(V)$, subject to the relation that for $\sigma \in \Gamma(V)$, $\tau \in \Gamma(V^*)$ the product $\sigma\tau$ equals $i_{\mathrm{At}(V)}(\sigma \otimes \tau)$. From the Poisson bracket relations between the corresponding functions $\phi_{\sigma^\sharp}$, τ, ϕ_δ on T^*V, we read off the Gerstenhaber brackets

$$[\![\delta_1, \delta_2]\!] = [\delta_1, \delta_2], \quad [\![\delta, \sigma]\!] = \nabla_\delta \sigma, \quad [\![\delta, \tau]\!] = \nabla^*_\delta \tau, \quad [\![\delta, f]\!] = \mathscr{L}_{\mathsf{a}(\delta} f, \quad [\![\sigma, \tau]\!] = -\langle \tau, \sigma\rangle.$$

On the other hand, the Lie algebroid structure on V determines a $\mathscr{VB}$-algebroid structure $T^*V^* \Rightarrow V^*$, and hence vertical differential d''_v. The latter agrees with the Chevalley-Eilenberg differential on $\mathscr{W}^{0,\bullet}(T^*V^*) = \Gamma(\wedge^\bullet V^*)$, while

$$\mathrm{d}''_v \sigma = -\delta(\sigma) \in \Gamma(\mathrm{At}(V))$$

where $\delta(\sigma) \in \Gamma(\mathrm{At}(V))$ is the infinitesimal automorphism given in terms of its representation on V by $\nabla_{\delta(\sigma_1)}(\sigma_2) = [\sigma_1, \sigma_2]$. This follows from the formulas of Theorem 6.1.2:

$$\iota''_v(j^1(\sigma_1))\mathrm{d}''_v \sigma_2 = \nabla^V_{j^1(\sigma_1)} \sigma_2 = [\sigma_1, \sigma_2] = -\nabla_{\delta(\sigma_2)} \sigma_1 = -\iota''_v(j^1(\sigma_1))\delta(\sigma_2).$$

Finally, for $\delta \in \Gamma(\mathrm{At}(V))$ the differential $\mathrm{d}''_v \delta \in \mathscr{W}^{1,2}((TV)'')$ is described by the formula

$$\iota''_v(j^1(\sigma_1))\iota''_v(j^1(\sigma_2))\mathrm{d}'_v \delta = \nabla_\delta[\sigma_1, \sigma_2] - [\sigma_1, \nabla_\delta \sigma_2] + [\nabla_\delta \sigma_1, \sigma_2],$$

which may be deduced from Corollary 6.2.3. One finds that the differential on $\mathscr{W}^{1,\bullet}(T^*V^*) \cong \Gamma_{\mathrm{lin}}(\wedge^\bullet_{TM}(TV^*), TM)$ coincides with the restriction of the Chevalley-Eilenberg differential of the tangent prolongation $TV \Rightarrow TM$. (Recall that the dual of the tangent prolongation is the bundle $TV^* \to TM$.)

Chapter 9

Applications, Connections with Other Work

In this section, we indicate connections between the results and constructions presented above and various ideas appearing in the literature.

9.1 Matched Pairs of Lie Algebroids

Consider a matched pair of Lie algebroids A, B, corresponding to a vacant double Lie algebroid $D = A \times_M B$, as in Example 7.1.1(3). Thus

$$\begin{tikzcd} D = A \times_M B \arrow[r, Rightarrow] \arrow[d, Rightarrow] & B \arrow[d, Rightarrow] \\ A \arrow[r, Rightarrow] & M \end{tikzcd} \qquad \begin{tikzcd} D' = B \times_M A^* \arrow[r, Rightarrow] \arrow[d] & M \arrow[d] \\ B \arrow[r, Rightarrow] & M \end{tikzcd} \qquad \begin{tikzcd} D'' = A \times_M B^* \arrow[r] \arrow[d, Rightarrow] & A \arrow[d, Rightarrow] \\ M \arrow[r] & M \end{tikzcd}$$

with corresponding Weil algebra bundles

$$W(D) = \wedge A^* \otimes \wedge B^*, \qquad W(D') = \wedge B^* \otimes \vee A, \qquad W(D'') = \wedge A^* \otimes \vee B.$$

The double Lie algebroid structure defines commuting differentials $\mathrm{d}_h,\ \mathrm{d}_v$ on $\mathscr{W}(D)$. This double complex was described in the work of Laurent-Gengoux, Stienon and Xu [46, Section 4.2]. Identifying $W(D) = \wedge(A \oplus B)^*$ (with the total grading), the sum $\mathrm{d}_h + \mathrm{d}_v$ is a degree 1 differential, in such a way that the bundle maps to $\wedge A^*, \wedge B^*$ give cochain maps on sections. We hence see that $A \oplus B$ becomes a Lie algebroid, with A, B as Lie subalgebroids. The Weil algebra $\mathscr{W}(D')$ has a Gerstenhaber bracket $[\![\cdot,\cdot]\!]$ and a compatible horizontal differential d'_h. The restriction of the differential to $\mathscr{W}^{\bullet,0}(D') = \Gamma(\wedge^\bullet B^*)$ gives the Lie algebroid structure on B, and the restriction to $\mathscr{W}^{\bullet,1}(D') = \Gamma(\wedge^{\bullet-1} B^* \otimes A)$ gives the action of this Lie algebroid on A. On the other hand, the restriction of the Gerstenhaber bracket to $\mathscr{W}^{1,1}(D') = \Gamma(A)$ recovers the Lie algebroid bracket of A, and the bracket with elements of $\mathscr{W}^{1,0}(D') = \Gamma(B^*)$ recovers the A-action on B^*. The fact that the differential d'_h on $\Gamma(\wedge B^* \otimes A)$ is a derivation of the (Gerstenhaber) bracket on this space is thus an equivalent formulation of the compatibility condition. A similar discussion applies to D''.

9.2 Multi-derivations

In [16], motivated by the study of deformations of Lie algebroids, Crainic and Moerdijk associate to any vector bundle $V \to M$ a graded vector space $\mathrm{Der}^\bullet(V)$ of *multi-derivations of* V, equipped with a Gerstenhaber bracket. Its simplest description is in terms of the isomorphism (see [16, Section 4.9])

$$\mathrm{Der}^\bullet(V) \cong \mathfrak{X}^{1+\bullet}_{\mathrm{lin}}(V^*),$$

with bracket the usual Schouten bracket of multivector fields. A Lie algebroid structure on V defines a compatible degree 1 differential on this space, given by Schouten bracket $[\![\pi, \cdot]\!]$ with the Poisson bivector field $\pi \in \mathfrak{X}^2_{\mathrm{lin}}(V^*)$ dual to the Lie algebroid structure. As shown in [16], the Maurer-Cartan elements of this *deformation complex* describe the deformations of the Lie algebroid structure. See also the work of Esposito-Tortorella-Vitagliano [20], where the deformation complex is generalized further to the setting of $\mathcal{VB}$-algebroids. For any vector bundle $V \to M$, the cotangent bundle T^*V is a Poisson double vector bundle, hence $\mathscr{W}(T^*V)$ inherits a Gerstenhaber bracket. By Proposition 4.5.3 and the discussion in section 8.2, the isomorphism

$$\mathscr{W}^{\bullet,1}(T^*V) \cong \Gamma_{\mathrm{lin}}(\wedge^\bullet_{V^*} TV^*, V^*) \cong \mathfrak{X}^\bullet_{\mathrm{lin}}(V^*)$$

takes this to the Schouten bracket of multivector fields. Given a Lie algebroid structure on V, the resulting horizontal differential on $\mathscr{W}^{\bullet,1}(T^*V)$ is $\mathrm{d}_h = [\![\pi, \cdot]\!]$, which is the differential on the deformation complex. In conclusion, the deformation complex is identified with $\mathscr{W}^{1+\bullet,1}(T^*V)$. Alternatively, this follows from the fact (Remark 5.6.2) that $\mathscr{W}^{\bullet,1}(T^*V)$ for $T^*V \Rightarrow V^*$ is the linear Chevalley-Eilenberg complex $\mathsf{CE}^\bullet_{\mathcal{VB}}(T^*V)$ of Cabrera-Drummond [11]; the isomorphism of the latter with the deformation complex was observed in [11, page 312].

9.3 Abad-Crainic's Weil algebra of a vector bundle V

Given a vector bundle $V \to M$, Abad and Crainic [3] define a bigraded *Weil algebra* $\mathfrak{W}^{\bullet,\bullet}(V)$ as follows. An element of $\mathfrak{W}^{p,q}(V)$ is given by a sequence of $\mathbb{R}$-multilinear skew-symmetric maps

$$c_i \colon \underbrace{\Gamma(V) \times \cdots \times \Gamma(V)}_{p-i \quad \text{times}} \to \Omega^{q-i}(M, \vee^i V^*).$$

Here c_0 is considered the 'leading term', c_1 measures the failure of c_0 to be multi-linear, c_2 measures the failure of c_1 to be multi-linear, and so on. (See [3] for details.) We claim that every $w \in \mathscr{W}^{p,q}(TV)$ gives rise to such a sequence, thereby identifying $\mathfrak{W}(V) \cong \mathscr{W}(TV)$.

For any double vector bundle D as in (2.2), there is a canonical surjective morphism of bigraded algebra bundles $\Pi \colon W(D) \to W(B \times_M E^*) = \wedge B^* \otimes \vee E$, induced by the $\mathcal{DVB}$ morphism $B \times_M E^* \hookrightarrow D$. Explicitly, the maps $\Pi \colon W^{p,q}(D) \to \wedge^{q-p} B^* \otimes \vee^p E$ are given by p-fold contractions with elements $\varepsilon \in E^*$. In the case of $D = TV$, with $A = V,\ B = TM,\ E = V^*$, we obtain projection maps

$$\Pi \colon W^{p,q}(TV) \to \wedge^{q-p} T^*M \otimes \vee^p V.$$

On the other hand, for $\sigma \in \Gamma(V)$, its jet prolongation $j^1(\sigma) \in \Gamma(J^1(V)) = \Gamma(\widehat{A})$ defines a contraction operator $\iota(j^1(\sigma))$ of bidegree $(-1,0)$. The map c_i corresponding to $w \in \mathscr{W}^{p,q}(TV)$ is given by

$$c_i(\sigma_{i+1},\ldots,\sigma_p) = \Pi\Big(\iota(j^1(\sigma_p))\cdots\iota(j^1(\sigma_i))\,w\Big) \in \Omega^{q-i}(M, \vee^i V^*).$$

The relation $j^1(f\sigma) - fj^1(\sigma) = i_{J^1(V)}(\mathrm{d}f\otimes\sigma) = \sigma\mathrm{d}f$ implies that $\iota(j^1(f\sigma)) - f\ \iota(j^1(\sigma)) = -\mathrm{d}f\ \iota(\sigma)$, where $\iota(\sigma)$ is the contraction operator of bidegree $(-1,-1)$ defined by σ (regarded as a section of E^*). Consequently, the failure of $C^\infty(M)$-linearity of c_i is expressed in terms of c_{i+1}, leading to the conditions in [3].

9.4 IM-forms and IM-multivector fields

Let $V \to M$ be a vector bundle. In Section 8.2, we discussed the spaces $\mathfrak{X}^\bullet_{\mathrm{lin}}(V)$ of linear multivector fields and $\Omega^\bullet_{\mathrm{lin}}(V)$ of linear differential forms on V. The Schouten bracket of multivector fields and the de Rham differential of forms restrict to these linear subspaces.

Given a Lie algebroid structure $V \Rightarrow M$, there are notions of *infinitesimally multiplicative* (IM) multi-vector fields and differential forms,

$$\mathfrak{X}^\bullet_{\mathrm{IM}}(V) \subseteq \mathfrak{X}^\bullet_{\mathrm{lin}}(V), \qquad \Omega^\bullet_{\mathrm{IM}}(V) \subseteq \Omega^\bullet_{\mathrm{lin}}(V).$$

These are designed to be the infinitesimal versions of multiplicative multivector fields or forms on Lie groupoids.

IM-multivector fields were introduced by Iglesias-Ponte, Laurent-Gengoux and Xu [36] under the name of *k-differentials*. To define them, note that for any vector bundle V, the graded Lie algebra $\mathfrak{X}^{1+\bullet}(V)$ acts on $\Gamma(\wedge V)$ by derivations. Using the identification $\Gamma(\wedge V) \cong \mathfrak{X}^\bullet_{\mathrm{core}}(V)$, this action is just the Schouten bracket of multi-vector fields. In particular, for $\delta \in \mathfrak{X}^k_{\mathrm{lin}}(V)$ and $\sigma \in \Gamma(\wedge^l V)$ we have that $\delta.\sigma \in \Gamma(\wedge^{k+l-1}V)$. If V is a Lie algebroid, then the bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(V)$ extends to the exterior algebra. The element δ is called an *IM-multivector field* if it is a derivation of this Lie bracket:

$$\delta.[\![\sigma_1,\sigma_2]\!] = [\![\delta.\sigma_1,\sigma_2]\!] + (-1)^{|\delta|(|\sigma_1|+1)}[\![\sigma_1,\delta.\sigma_2]\!] \tag{9.1}$$

for all $\sigma_i \in \Gamma(\wedge^{l_i}V)$, $i = 1,2$. Here $|\delta|= k+1$. Using the derivation property with respect to wedge product, it is actually enough to have this condition for $l_1 = l_2 = 1$. The *universal lifting theorem* [36], generalizing earlier results of Mackenzie-Xu [58, 59], integrates any such δ to a multiplicative vector field on the corresponding (local) Lie groupoid.

To describe the IM condition for forms, recall that any $\phi \in \Omega^k_{\mathrm{lin}}(V)$ can be uniquely written as $\phi = \nu + \mathrm{d}_{\mathrm{Rh}}\mu$ where $\nu \in \Gamma(V^*\otimes\wedge^k T^*M)$ and $\mu \in \Gamma(V^*\otimes\wedge^{k-1}T^*M)$ (viewed as linear differential forms), and d_{Rh} denotes the de Rham differential on linear forms. If V is a Lie algebroid, then $\phi = \nu + \mathrm{d}_{\mathrm{Rh}}\mu$ is called an *IM form* if the following three conditions are satisfied

$$\iota_{\mathsf{a}(\sigma_1)}\mu(\sigma_2) = -\iota_{\mathsf{a}(\sigma_2)}\mu(\sigma_1), \tag{9.2}$$

$$\mu([\![\sigma_1,\sigma_2]\!]) = \mathcal{L}_{\mathsf{a}(\sigma_1)}\mu(\sigma_2) - \iota_{\mathsf{a}(\sigma_2)}\mathrm{d}_{\mathrm{Rh}}\ \mu(\sigma_1) - \iota_{\mathsf{a}(\sigma_2)}\nu(\sigma_1), \tag{9.3}$$

$$\nu([\![\sigma_1,\sigma_2]\!]) = \mathcal{L}_{\mathsf{a}(\sigma_1)}\nu(\sigma_2) - \iota_{\mathsf{a}(\sigma_2)}\mathrm{d}_{\mathrm{Rh}}\ \nu(\sigma_1), \tag{9.4}$$

for all $\sigma_1, \sigma_2 \in \Gamma(V)$. These conditions are due to Bursztyn and Cabrera [6] (see [8, 9] for the case of

2-forms); as shown in [6], these are exactly the conditions ensuring that ϕ integrates to a multiplicative form on the associated (local) Lie groupoid.

We will now give interpretations of IM multivector fields and IM forms in terms of the Weil algebras. Recall from section 8.2 that for any vector bundle $V \to M$,

$$\mathfrak{X}^\bullet_{\mathrm{lin}}(V) = \mathcal{W}^{1,\bullet}(T^*V), \quad \Omega^\bullet_{\mathrm{lin}}(V) = \mathcal{W}^{1,\bullet}(TV).$$

The first isomorphism is compatible with the Gerstenhaber bracket $[\![\cdot,\cdot]\!]$ on $\mathcal{W}(T^*V)$ defined by the Poisson structure on T^*V, the second isomorphism with the vertical differential d_v on $\mathcal{W}(TV)$ defined by the $\mathcal{VB}$-algebroid structure $TV \Rightarrow V$. A Lie algebroid structure $V \Rightarrow M$ gives $\mathcal{VB}$-algebroid structures $T^*V \Rightarrow V^*$ and $TV \Rightarrow TM$, resulting in horizontal differentials d_h on both $\mathcal{W}(T^*V)$ and $\mathcal{W}(TV)$. The second part of the following result is due to Bursztyn and Cabrera [6]; for a more restrictive notion of IM forms it was observed in [3, Section 6].

Theorem 9.4.1. *For any Lie algebroid $V \Rightarrow M$,*

$$\mathfrak{X}^\bullet_{\mathrm{IM}}(V) = \mathcal{W}^{1,\bullet}(T^*V) \cap \ker(\mathrm{d}_h), \quad \Omega^\bullet_{\mathrm{IM}}(V) = \mathcal{W}^{1,\bullet}(TV) \cap \ker(\mathrm{d}_h).$$

Proof. Consider a $\mathcal{VB}$-algebroid $D \Rightarrow B$ over $A \Rightarrow M$, so that the Weil algebra $\mathcal{W}(D)$ carries a horizontal differential d_h. Then $\mathcal{W}(D')$ has a Gerstenhaber bracket. By Corollary 6.2.3, an element $\phi \in \mathcal{W}^{1,\bullet}(D)$ is d_h-closed if and only if

$$\iota_h([\![x_1, x_2]\!])\phi = [\![x_1, \iota_h(x_2)\phi]\!] + (-1)^{|x_1||x_2|+1}[\![x_2, \iota_h(x_1)\phi]\!] \tag{9.5}$$

for all $x_i \in \mathcal{W}^{p_i,1}(D')$. (It suffices to verify this on generators.)

Suppose now that D also carries a double-linear Poisson structure; thus D'' is a double Lie algebroid. In particular $B^* = \mathrm{core}(D'')$ is a Lie algebroid, with the bracket

$$[\sigma_1, \sigma_2] := [\![\sigma_1, \mathrm{d}'_v\sigma_2]\!]$$

for $\sigma_1, \sigma_2 \in \Gamma(\wedge B^*)$. (We have in mind the case $D = T^*V$; the Lie algebroid structure on $B^* = V$ being the standard one.) The space $\mathcal{W}^{1,\bullet}(D)[1]$ is a graded Lie algebra for the Gerstenhaber bracket, with a representation on $\Gamma(\wedge B^*)$ given by (cf. (6.8))

$$\phi.\sigma := [\![\phi, \sigma]\!] = -(-1)^{(|\phi|+1)\,|\sigma|}\iota_h(\mathrm{d}'_v\sigma)\phi$$

for $\sigma \in \Gamma(\wedge B^*)$. Let $x_i = \mathrm{d}'_v\sigma_i$ with $\sigma_i \in \Gamma(\wedge B^*)$. Then $[\![x_1, x_2]\!] = \mathrm{d}'_v[\sigma_1, \sigma_2]$, and therefore

$$\phi.[\sigma_1, \sigma_2] = -(-1)^{(|\phi|+1)(|\sigma_1|+|\sigma_2|+1)}\iota_h([\![x_1, x_2]\!])\phi. \tag{9.6}$$

On the other hand,

$$\begin{aligned}
&(-1)^{|\phi|(|\sigma_1|+1)}[\sigma_1, \phi.\sigma_2] + [\phi.\sigma_1, \sigma_2] \\
&= -(-1)^{(|\phi|+1)(|\sigma_1|+|\sigma_2|+1)}\Big([\![x_1, \iota_h(x_2)\phi]\!] + (-1)^{(|\phi|+1)(|\sigma_2|+1)}[\![\iota_h(x_1)\phi, x_2]\!]\Big) \\
&= -(-1)^{(|\phi|+1)(|\sigma_1|+|\sigma_2|+1)}\Big([\![x_1, \iota_h(x_2)\phi]\!] - (-1)^{(|\sigma_1|+1)(|\sigma_2|+1)}[\![x_2, \iota_h(x_1)\phi]\!]\Big).
\end{aligned}$$

By (9.5), if ϕ is d_h-closed then this expression coincides with (9.6), proving that $\phi.\sigma = [\phi.\sigma_1, \sigma_2] + (-1)^{|\phi|(|\sigma_1|+1)}[\sigma_1, \phi.\sigma_2]$. For $D = T^*V$, the converse is true, because in that case the space $\mathscr{W}^{\bullet,1}(D')$ is spanned, as a $C^\infty(M)$-module, by $\mathrm{d}_v'\mathscr{W}^{\bullet,0}(D')$. The case of IM-differential forms can be discussed similarly; in terms of the Abad-Crainic approach to the Weil algebra $\mathscr{W}(TV)$ this is done in [6]. □

9.5 Frölicher-Nijenhuis and Nijenhuis-Richardson brackets

Suppose $V \to M$ is a Lie algebroid, so that V^* has a linear Poisson structure. The double-linear Poisson structure on TV^* defines a Gerstenhaber bracket on $\mathscr{W}(TV^*)$, compatible with the vertical differential d^v. Hence,

$$\mathscr{W}^{1,1+\bullet}(TV^*) \cong \Omega^{\bullet+1}_{\mathrm{lin}}(V^*) \cong \Omega^{\bullet+1}(M,V) \oplus \Omega^{\bullet}(M,V) \tag{9.7}$$

becomes a differential graded Lie algebra. It comes with a morphism of graded Lie algebras (also $\Omega(M)$-module morphism)

$$\Omega^{\bullet+1}(M,V) \oplus \Omega^{\bullet}(M,V) \to \mathrm{Der}^{\bullet}(\Omega(M)) \tag{9.8}$$

given by Gerstenhaber bracket with elements of $\mathscr{W}^{0,\bullet}(TV^*) \cong \Omega(M)$. One verifies that the first summand in (9.8) acts as contractions via the anchor $V \to TM$, the second as Lie derivatives.

For the Lie algebroid $V = TM$, the map (9.8) is an isomorphism, hence we recover the bracket on $\Omega^{\bullet+1}(M,TM) \oplus \Omega^{\bullet}(M,TM)$ given by the Frölicher-Nijenhuis bracket on the first summand, the Nijenhuis-Richardson bracket on the second summand, and a cross term. See [37, Chapter II.8] for a detailed discussion; see also [24, 69] for related brackets and generalizations to Lie algebroids.

9.6 Representations up to homotopy

Representations up to homotopy were introduced by Evens-Lu-Weinstein [21] and Abad and Crainic [1] as generalizations of the usual concept of representations of a Lie algebroid. Among other things, they give a notion of the adjoint action of a Lie algebroid on itself, which is generally not possible using ordinary representations. The essential idea is to represent Lie algebroids on complexes of vector bundles rather than just single vector bundles. We will adopt the definition from [1].

Definition 9.6.1. Let $A \to M$ be a Lie algebroid. A *representation up to homotopy* of A is a $\mathbb{Z}$-graded vector bundle $\mathsf{U}_\bullet$ over M along with a degree 1 differential δ on sections of $\wedge A^* \otimes \mathsf{U}$ (using the graded tensor product) satisfying

$$\delta(\omega\eta) = d_A(\omega)\eta + (-1)^k\omega\delta(\eta)$$

for all $\omega \in \Gamma(\wedge^k A^*)$, $\eta \in \Gamma(\wedge A^* \otimes \mathsf{U})$.

Given a Lie algebroid $A \to M$, Gracia-Saz and Mehta [29] showed how to construct a 2-step representation up to homotopy of A from a horizontal $\mathcal{VB}$-algebroid

$$\begin{tikzcd} D \arrow[r, Rightarrow] \arrow[d] & B \arrow[d] \\ A \arrow[r, Rightarrow] & M \end{tikzcd}$$

having A as its horizontal side bundle. The construction depends on the choice of a splitting of D, and the resulting graded vector bundle is

$$\mathsf{U} = E^*[1] \oplus B;$$

that is, $\mathsf{U}_{-1} = E^*$ and $\mathsf{U}_0 = B$. We briefly review their construction, making use of some of our observations in chapter 5. By Lemma 5.4.1, the double-linear Poisson structure on D' gives the following data:

- a Lie algebroid structure on $\widehat{A}$,
- $\widehat{A}$-representations $\widehat{\nabla}^{B^*}$, $\widehat{\nabla}^{E^*}$ on B^* and on E^*,
- an invariant pairing $(\cdot,\cdot)\colon B^* \times E^* \to \mathbb{R}$.

A choice of splitting of the double vector bundle D is equivalent to a choice of splitting $s\colon A \to \widehat{A}$. In general, s need not preserve Lie brackets, and so we can consider its curvature tensor $\Omega \in \Gamma(\wedge^2 A^* \otimes (B^* \otimes E^*))$ defined by

$$\Omega(a_1, a_2) = s([\![a_1, a_2]\!]) - [s(a_1), s(a_2)], \quad a_1, a_2 \in \Gamma(A).$$

The U_0–U_{-1}-component of δ is the linear map $\Gamma(\wedge^\bullet A^* \otimes B) \to \Gamma(\wedge^{\bullet+2} A^* \otimes E^*)$ given by Ω (with the identification $B^* \otimes E^* \cong \mathrm{Hom}(B, E^*)$). The U_{-1}–U_0-component of δ is the linear map $\Gamma(\wedge^\bullet A^* \otimes E^*) \to \Gamma(\wedge^\bullet A^* \otimes B)$ defined by the pairing $(\cdot,\cdot)$ viewed as a bundle map $E^* \to B$. The connection $\widehat{\nabla}^{E^*}$ pulls back under s to a non-flat A-connection ∇^{E^*} on E^*; its extension to a map $\nabla^{E^*}\colon \Gamma(\wedge^\bullet A^* \otimes E^*) \to \Gamma(\wedge^{\bullet+1} A^* \otimes E^*)$ is the U_{-1}–U_{-1}-component of δ. Similarly, the flat $\widehat{A}$-connection on B pulls back to a non-flat A-connection, and the resulting map on sections gives the U_0–U_0-component. See [29, Theorem 4.10]. This establishes a one-to-one correspondence:

Theorem 9.6.2. *[29, Theorem 4.11 (2)] Let D be a double vector bundle with side bundles A, B and core E^*, such that $A \Rightarrow M$ is a Lie algebroid. After choosing a splitting $s\colon A \to \hat{A}$, there is a one-to-one correspondence between horizontal $\mathcal{VB}$-algebroid structures $D \Rightarrow B$ extending $A \Rightarrow M$, and representations up to homotopy of A on $\mathsf{U} = E^*[1] \oplus B$.*

This correspondence has a simple interpretation in terms of the Weil algebras. Recall from chapter 5 that horizontal $\mathcal{VB}$-algebroid structures $D \Rightarrow B$ are in one-to-one correspondence with vertical differentials d_v on $\mathcal{W}(D'')$. This restricts to a differential d_v on $\mathcal{W}^{1,\bullet}(D'') \cong \Gamma_{\mathrm{lin}}(\wedge^\bullet_E D', E)$. Once we choose a splitting of D, or equivalently a vector bundle splitting $s\colon A \to \widehat{A}$, we obtain a decomposition (see Proposition 4.5.2)

$$\mathcal{W}^{1,\bullet}(D'') \cong \Gamma(\wedge^\bullet A^* \otimes B) \oplus \Gamma(\wedge^{\bullet+1} A^* \otimes E^*).$$

With $\mathsf{U}_{-1} = E^*$ and $\mathsf{U}_0 = A^*$, the differential d defines a representation up to homotopy of A on U. To see that this correspondence is bijective, we note that a vertical differential on the bigraded algebra $\mathcal{W}(D'')$ is uniquely determined by its restrictions to $\mathcal{W}^{1,\bullet}(D'')$ and $\mathcal{W}^{0,\bullet}(D'') = \Gamma(\wedge^\bullet A^*)$ thanks to the Leibniz rule.

Remark 9.6.3. Horizontal $\mathcal{VB}$-algebroid structures on D are also equivalent to differentials d_h of bidegree $(1,0)$ on $\mathcal{W}(D)$. After a choice of splitting, this induces a degree 1 operator on $\mathcal{W}^{\bullet,1}(D) \cong \Gamma(\wedge^\bullet A^* \otimes E) \oplus \Gamma(\wedge^{\bullet+1} A^* \otimes B^*)$, giving a representation up to homotopy of A on the bundle $B^*[1] \oplus E$. This representation up to homotopy is dual to the one on $E^*[1] \oplus B$, as discussed in [29, Section 4.5].

Let us turn now to the case where D has both a horizontal and a vertical $\mathcal{VB}$-algebroid structure. Then both D' and D'' are Poisson double vector bundles, which is equivalent to the following structures:

- Lie algebroid structures on $\widehat{A}$ and $\widehat{B}$,
- $\widehat{A}$-representations on B^*, E^* and $\widehat{B}$-representations on A^*, E^*,
- an $\widehat{A}$-invariant pairing $B^* \times E^* \to \mathbb{R}$ and a $\widehat{B}$-invariant pairing $A^* \times E^* \to \mathbb{R}$

subject to certain compatibility conditions (Theorem 5.4.3). It is natural to ask what additional compatibility conditions on these data ensure that D is a double Lie algebroid. This question was answered in the work of Gracia-Saz, Jotz Lean, Mackenzie, and Mehta [26, Theorem 3.4] using a splitting and a notion of *matched pair* for representations up to homotopy.

The representations up to homotopy of Lie algebroids discussed above have a corresponding global object, namely representations up to homotopy of Lie groupoids. In [18], del Hoyo and Davide reinterpret this concept global concept by developing a theory of Lie 2-groupoids. Specifically, they introduce the general linear Lie 2-groupoid $\mathrm{GL}(\mathsf{U})$ of a two-term graded vector bundle U, and show that two term representations up to homotopy of a Lie groupoid $G \rightrightarrows M$ on U are equivalent to pseudo-functors $G \dashrightarrow \mathrm{GL}(\mathsf{U})$. The discussion above therefore implies a relationship between Weil algebras and these general linear 2-groupoids that would be interesting to uncover further.

Chapter 10

Outlook: Further Directions

10.1 The Double Lie Functor

Recall that for (ordinary) Lie groupoids, one has the Lie functor

$$\mathrm{Lie}\colon \text{Lie Groupoids} \to \text{Lie Algebroids}$$

that "differentiates" a Lie groupoid $G \rightrightarrows M$, producing it's associated Lie algebroid $A \to M$. The vector bundle structure underlying A is given by the normal bundle of the space of units inside G: $A = \nu(G, M)$. To describe the anchor map $A \to TM$, we observe that the source and target maps s, t agree on M so that the difference $Tt - Ts\colon TG \to TM$ vanishes on TM, and therefore descends to a map $\nu(G, M) \to TM$. Finally, let $S_t \subseteq G$, $t \in \mathbb{R}$, be a one-parameter family of bisections of G with $S_0 = M$. Taking the differential at $t = 0$, we obtain a smooth section $\sigma \in \Gamma(A)$, and so we may regard $\Gamma(A)$ as the Lie algebra of the (infinite dimensional) Lie group $\Gamma(G)$, which gives the bracket:

$$\Gamma(A) = \mathrm{Lie}(\Gamma(G)).$$

The Lie functor has been extended to the setting of double Lie structures in the work of Mackenzie [55, 51, 54], associating to any double Lie groupoid a corresponding double Lie algebroid though differentiation processes (in fact, as mentioned previously, differentiating double Lie groupoids was how the notion of double Lie algebroids came to be). Before describing Mackenzie's approach to the double Lie functor, let us recall the basics of double Lie groupoids. Consider a diagram of the form

$$\begin{tikzcd} \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{V} \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{H} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \tag{10.1}$$

in which all four sides are Lie groupoids. Such a diagram is called a *double Lie groupoid* [5] (see also [62]) if the structure maps (source, target, multiplication, inversion) for either groupoid structure on $\mathcal{G}$ are Lie groupoid morphisms with respect to the other structure. The various structure maps for the horizontal Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{V}$ will be distinguished by the subscript 'h', while the structure maps for the vertical Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{H}$ will be distinguished by the subscript 'v'. In contrast, the structure maps for the Lie groupoids $\mathscr{H} \rightrightarrows M$ and $\mathscr{K} \rightrightarrows M$ will not be given subscripts, and we will rely on context to clarify which one is meant.

Example 10.1.1. Suppose $G \rightrightarrows M$ is a (ordinary) Lie groupoid, with source and target maps denoted by s_G and t_G respectively. Then the diagram

$$\begin{tikzcd} \mathrm{Pair}(G) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Pair}(M) \arrow[d, shift left] \arrow[d, shift right] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

is a double Lie groupoid. Any pair (g', g) of elements of G defines a unique element in $\mathrm{Pair}(G)$, with the properties

$$s_v(g', g) = g, \quad t_v(g', g) = g', \quad s_h(g', g) = (s_G g', s_G g), \quad t_h(g', g) = (t_G g', t_G g).$$

The vertical multiplication is defined by

$$(g', g) = (g_1', g_1) \circ_v (g_2', g_2) \iff g' = g_1',\ g_1 = g_2',\ g = g_2.$$

On the other hand, the horizontal multiplication is

$$(g_1', g_1) \circ_h (g_2', g_2) = (g_1' g_2', g_1 g_2),$$

where juxtaposition denotes multiplication in G. Thus the vertical units are of the form (g, g) for $g \in G$, while the horizontal units are of the form (m', m) for $m', m \in M \subseteq G$.

Mackenzie's approach to the double Lie functor was to proceed in stages. By first applying the Lie functor to the Lie groupoid $\mathscr{G} \rightrightarrows \mathscr{H}$, one obtains an intermediate object that is a Lie algebroid in one direction, and a Lie groupoid in the other. Such objects were called $\mathscr{LA}$ groupoids, and by applying the Lie functor once more (to the Lie groupoid part), one obtains a double Lie algebroid. Mackenzie showed that proceeding in the other order (that is, starting by applying the Lie functor to $\mathscr{G} \rightrightarrows \mathscr{V}$ instead) produces the same result, and so the double Lie functor factors through the category of $\mathscr{LA}$-groupoids as follows:

$$\begin{tikzcd} & \mathscr{LA}\text{ Groupoids} \arrow[dr] & \\ \text{Double Lie Groupoids} \arrow[ur] \arrow[dr] & & \text{Double Lie Algebroids} \\ & \mathscr{LA}\text{ Groupoids} \arrow[ur] & \end{tikzcd}$$

As an example, let $G \rightrightarrows M$ be a Lie groupoid with $\mathrm{Lie}(G) = A$. Then the three diagrams

$$\begin{tikzcd} TG \arrow[r, shift left] \arrow[r, shift right] \arrow[d, Rightarrow] & TM \arrow[d, Rightarrow] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Pair}(A) \arrow[r, Rightarrow] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Pair}(M) \arrow[d, shift left] \arrow[d, shift right] \\ A \arrow[r, Rightarrow] & M \end{tikzcd} \qquad \begin{tikzcd} TA \arrow[r, Rightarrow] \arrow[d, Rightarrow] & TM \arrow[d, Rightarrow] \\ A \arrow[r, Rightarrow] & M \end{tikzcd}$$

comprise all of the differential objects associated to the double Lie groupoid $\mathrm{Pair}(\mathscr{G})$ of example 10.1.1.

Using the double normal bundle construction of chapter 3, one could develop a theory of the double Lie functor that passes directly from double Lie groupoids to double Lie algebroids, without factoring through the category of $\mathcal{LA}$ groupoids. More specifically, if $\mathcal{G}$ is a double Lie groupoid, then the double vector bundle underlying its associated double Lie algebroid is given by $\nu(\mathcal{G}, \mathcal{H}, \mathcal{V})$ (the proof that $\mathcal{H}$ and $\mathcal{V}$ intersect cleanly is a straightforward generalization of lemma 3.1.9). One could then adapt the techniques for defining the usual Lie functor to the case where we have two families of bisections of $\mathcal{G}$ (one for $\mathcal{G} \to \mathcal{H}$ and one for $\mathcal{G} \to \mathcal{V}$) that are compatible in a suitable sense.

Finally, using deformation spaces (see section 3.6), one could unite the two approaches to the double Lie functor into one cohesive theory. To see how deformation spaces enter the picture, consider an ordinary Lie groupoid $G \rightrightarrows M$, with associated Lie algebroid $A \to M$. Observe that the deformation space $\mathcal{D}(G, M)$ has A lying in the 0-fibre, while all other fibres are copies of G. Moreover, $\mathcal{D}(G, M)$ is itself a Lie groupoid (see, for example CITE), and a bisection of $\mathcal{D}(M, N)$ is made up of a one-parameter family of bisections of G away from $t = 0$, along with the section of A corresponding to the derivative of this family sitting in the $t = 0$ fibre. By analogy, we expect that one could fully develop the theory of the double Lie functor using the double deformation space $\mathcal{D}(\mathcal{G}, \mathcal{H}, \mathcal{V})$. This deformation space has the advantage of containing all of the infinitesimal objects associated to $\mathcal{G}$ simultaneously. Indeed, the double Lie algebroid $\nu(\mathcal{G}, \mathcal{H}, \mathsf{K})$ lies in the $s = t = 0$ fibre, while the $\mathcal{LA}$ groupoids $\nu(\mathcal{G}, \mathcal{H})$ and $\nu(\mathcal{G}, \mathcal{V})$ lie in the $s = 0, t \neq 0$ and $s \neq 0, t = 0$ fibres respectively. We believe that the development of the constructions briefly discussed above would lead to a deeper understanding of the differentiation processes associated to double Lie structures, which in turn would be helpful in tackling the difficult problems surrounding the integration of double Lie structures (see [7]).

10.2 van Est maps

Recall that the classical van Est map [75] is a morphism from the cochain complex of a Lie group G to the Chevalley-Eilenberg cochain complex of its Lie algebra $\mathfrak{g}$. This map was extended by Weinstein and Xu [79] to the case of Lie groupoids G and their Lie algebroids A, thus obtaining a morphism of cochain complexes

$$\mathrm{VE}\colon \widetilde{C}^{\infty}(B_\bullet G) \to \Gamma(\wedge^\bullet A^*). \tag{10.2}$$

Here B_pG is the space of p-arrows in G, and the tilde signifies the normalized complex for the simplicial manifold $B_\bullet G$. In [61, Chapter 6], Mehta generalized (10.2) to a van Est map from 'Q-groupoids' into the double complex of the corresponding 'Q-algebroid'; in particular this gives a version of (10.2) for differential forms. We note here that Q-groupoids are a supergeometric analog of $\mathcal{LA}$ groupoids. A construction in more classical terms was given by Abad and Crainic [3], in terms of the Weil algebra $\mathfrak{W}(A) = \mathcal{W}(TA)$. The van Est map (10.2) described in [3] is a morphism of double complexes

$$\mathrm{VE}\colon \widetilde{\Omega}^\bullet(B_\bullet G) \to \mathcal{W}^{\bullet,\bullet}(TA). \tag{10.3}$$

A geometric construction of this map was provided in [47]. This map has since been generalized in a few different directions, for example the work of Cabrera and Drummond in [11] adapts it to the setting of $\mathcal{VB}$ groupoids, and the recent thesis of Angulo [2] develops a van Est map in the context of Lie 2-algebras and Lie 2-groups. The next step in this story would be to describe a van Est map for double Lie groupoids $\mathcal{G}$. As mentioned above, by applying the Lie functor to one of the groupoid structures of $\mathcal{G}$ results in an $\mathcal{LA}$ groupoid, which we will briefly denote by Ω, and applying the Lie functor once more gives a double Lie algebroid D [55, 51, 54]. So a van Est theory for double Lie groupoids would take the form

$$\begin{tikzcd} \widetilde{C}^\infty(B_{\bullet,\bullet}\mathcal{G}) \arrow[r] \arrow[rr, bend left] & \mathsf{C}^{\bullet,\bullet}(\Omega) \arrow[r] & \mathcal{W}^{\bullet,\bullet}(D) \end{tikzcd}$$

for certain double complex $\widetilde{C}^\infty(B_{\bullet,\bullet}\mathcal{G})$, where $\mathsf{C}^{\bullet,\bullet}(\Omega)$ is Mehta's double complex of an $\mathcal{LA}$ groupoid. In other words, it is a map of double complexes $\widetilde{C}^\infty(B_{\bullet,\bullet}\mathcal{G}) \to \mathcal{W}^{\bullet,\bullet}(D)$ that factors through (a classical analog of) the van Est map of Mehta. Let us briefly describe the double complex $\widetilde{C}^\infty(B_{\bullet,\bullet}\mathcal{G})$. Much like the case for Lie groupoids, to any double Lie groupoid we can associate a bisimplicial manifold $B_{\bullet,\bullet}(\mathcal{G})$. In the language of simplicial sets this is called the *nerve* of $\mathcal{G}$, and is therefore sometimes denoted $\mathcal{N}_{\bullet,\bullet}(\mathcal{G})$ (see for example [62]). Explicitly, for $p,q \geq 0$ we have

$$B_{p,q}(\mathcal{G}) = \left\{ \begin{pmatrix} g_{11} & \cdots & g_{1q} \\ \vdots & \ddots & \vdots \\ g_{p1} & \cdots & g_{pq} \end{pmatrix} \middle| s_h g_{ij} = t_h g_{i(j+1)}, s_v g_{ij} = t_v g_{(i+1)j},\ 0 \leq i < p, 0 \leq j < q \right\}$$

with the conventions that $B_{0,0}(\mathcal{G}) = M$, $B_{p,0}(\mathcal{G}) = B_p\mathcal{H}$, and $B_{0,q}(\mathcal{G}) = B_q\mathcal{V}$. To understand the bisimplicial structure of the collection $B_{p,q}(\mathcal{G})$, first consider the special cases where one (or both) of p,q are zero or one. For $p = q = 0$, the horizontal degeneracy map is the inclusion of units $M \hookrightarrow \mathcal{H}$, while the vertical degeneracy map is given by $M \hookrightarrow \mathcal{V}$. If $p \geq 1$, then for $q = 0$, the horizontal face and degeneracy maps are defined by the simplicial structure on $B_p(\mathcal{H})$, while the vertical face and degeneracy maps are trivial. For $p \geq 1$, $q = 1$, then we remark that $B_{p,1}(\mathcal{G}) = B_p(\mathcal{G} \rightrightarrows \mathcal{H})$, which defines the horizontal simplicial structures, while the vertical face maps are given by the p-fold vertical source and target maps: $\partial_0^v = s_v^p$ and $\partial_1^v = t_v^p$. The cases $p = 0$ and $p = 1$ (with $q \geq 1$) are handled similarly.

Now assume that p and q are both at least 2. Then for every $0 \leq i \leq p$, we have a horizontal face map $\partial_i^h \colon B_{p,q}(\mathcal{G}) \to B_{p-1,q}(\mathcal{G})$ that collapses the i^{th} row. That is, ∂_i^h is given by

$$\partial_i^h \begin{pmatrix} g_{11} & \cdots & g_{1q} \\ \vdots & \ddots & \vdots \\ g_{p1} & \cdots & g_{pq} \end{pmatrix} = \begin{pmatrix} g_{11} & \cdots & g_{1q} \\ \vdots & & \vdots \\ g_{i1} \circ_v g_{(i+1)1} & \cdots & g_{iq} \circ_v g_{(i+1)q} \\ \vdots & & \vdots \\ g_{p1} & \cdots & g_{pq} \end{pmatrix}$$

for $0 < i < p$ (here $\circ_v$ denotes multiplication in the groupoid $\mathcal{G} \rightrightarrows \mathcal{H}$), and

$$\partial_0^h \begin{pmatrix} g_{11} & \cdots & g_{1q} \\ \vdots & \ddots & \vdots \\ g_{p1} & \cdots & g_{pq} \end{pmatrix} = \begin{pmatrix} g_{21} & \cdots & g_{2q} \\ \vdots & \ddots & \vdots \\ g_{p1} & \cdots & g_{pq} \end{pmatrix}, \quad \partial_p^h \begin{pmatrix} g_{11} & \cdots & g_{1q} \\ \vdots & \ddots & \vdots \\ g_{p1} & \cdots & g_{pq} \end{pmatrix} = \begin{pmatrix} g_{11} & \cdots & g_{1q} \\ \vdots & \ddots & \vdots \\ g_{(p-1)1} & \cdots & g_{(p-1)q} \end{pmatrix}.$$

Similarly, for any $0 \leq j \leq q$ we have a vertical face map ∂_j^v that collapses the j^{th} column.

To describe the degeneracy maps, suppose that $p, q \geq 1$. Then for every $0 \leq 1 \leq p$, the horizontal degeneracy map ϵ_i^h is given by injecting a trivial row in the $i+1$ position:

$$\epsilon_i^h\begin{pmatrix} g_{11} & \cdots & g_{1q} \\ \vdots & \ddots & \vdots \\ g_{p1} & \cdots & g_{pq}\end{pmatrix} = \begin{pmatrix} g_{11} & \cdots & g_{1q} \\ \vdots & & \vdots \\ g_{i1} & \cdots & g_{iq} \\ s_v g_{i1} & \cdots & s_v g_{iq} \\ g_{(i+1)1} & \cdots & g_{(i+1)q} \\ \vdots & & \vdots \\ g_{p1} & \cdots & g_{pq}\end{pmatrix}$$

(Note that ϵ_0^h adds $(t_v g_{11}, \ldots, t_v g_{1q})$ as the top row.) Similarly, for every $0 \le j \le q$, we get a vertical degeneracy map ϵ_j^v that injects a trivial column in the $(j+1)$ position.

Theorem 10.2.1. *The collection of spaces $B_{p,q}(\mathcal{G})$, along with the face and degeneracy maps described above, is a bisimplicial manifold.*

Proof. See [62, Proposition 3.10]. □

The *normalized double complex* associated to the double Lie groupoid $\mathcal{G}$ is given by $(\widetilde{C}^\infty_{\bullet,\bullet}(\mathcal{G}), \delta^h, \delta^v)$, where $\widetilde{C}^\infty_{\bullet,\bullet}(\mathcal{G})$ is the bigraded algebra

$$\widetilde{C}^\infty_{p,q}(\mathcal{G}) = \{f \in C^\infty(B_{p,q}(\mathcal{G})) \mid (\epsilon_i^h)^* f = 0, (\epsilon_j^v)^* f = 0,\ 0 \le i \le p, 0 \le j \le q\},$$

and the differentials $\delta^h\colon \widetilde{C}^\infty_{p,q}(\mathcal{G}) \to \widetilde{C}^\infty_{p+1,q}(\mathcal{G})$ and $\delta^v\colon \widetilde{C}^\infty_{p,q}(\mathcal{G}) \to \widetilde{C}^\infty_{p,q+1}(\mathcal{G})$ are defined by:

$$\delta^h(f) = \sum_{i=0}^{p+1}(-1)^i(\partial_i^h)^* f, \quad \delta^v(f) = \sum_{j=0}^{q+1}(-1)^j(\partial_j^v)^* f.$$

Remark 10.2.2. For a full theory, one should also consider the *localized* version of this double complex, defined as the quotient of $\widetilde{C}^\infty_{p,q}(\mathcal{G})$ by the subcomplex of functions vanishing on a neighbourhood of $M \subseteq B_{p,q}(\mathcal{G})$.

The key step to describing a van Est theory for double Lie groupoids is to develop an appropriate theory of double Lie groupoid actions. Specifically, one should define what it means for a double Lie groupoid $\mathcal{G}$ to act on a commutative diagram of manifolds

$$\begin{tikzcd} \mathcal{Q} \arrow[r] \arrow[d] & Q_1 \arrow[d] \\ Q_2 \arrow[r] & M. \end{tikzcd} \tag{10.4}$$

The space

$$E_{p,q}(\mathcal{G}) = \left\{ \begin{pmatrix} g_{00} & \cdots & g_{0q} \\ \vdots & \ddots & \vdots \\ g_{p0} & \cdots & g_{pq}\end{pmatrix} \middle| s_h g_{ij} = s_h g_{i(j+1)}, s_v g_{ij} = s_v g_{(i+1)j},\ 0 \le i < p, 0 \le j < q \right\}$$

fits into a pair of diagrams that, given the right definition of double Lie groupoid actions, should consitute a (double) principal bundle over base $B_{p,q}(\mathcal{G})$:

$$\begin{tikzcd} E_{p,q}(\mathcal{G}) \arrow[r] \arrow[d] & F^h_{p,q}(\mathcal{G}) \arrow[d] \\ F^v_{p,q}(\mathcal{G}) \arrow[r] & B_{p,q}(\mathcal{G}), \end{tikzcd} \qquad \begin{tikzcd} E_{p,q}(\mathcal{G}) \arrow[r] \arrow[d] & E_p(\mathcal{V}) \arrow[d] \\ E_q(\mathcal{H}) \arrow[r] & M. \end{tikzcd}$$

Here the side bundles are defined as

$$F^v_{p,q}(\mathcal{G}) = \left\{ \begin{pmatrix} g_{00} & \cdots & g_{0q} \\ \vdots & \ddots & \vdots \\ g_{p0} & \cdots & g_{pq} \end{pmatrix} \middle| s_h g_{ij} = s_h g_{i(j+1)}, s_v g_{ij} = t_v g_{(i+1)j},\ 0 \le i < p, 0 \le j < q \right\},$$

and similarly for $F^h_{p,q}(\mathcal{G})$. To develop a van Est theory, one would then follow the geometric approach initiated in [47] (and later refined in [64]). That is, for each p, q, the principal bundle described above defines a pair of fibrations that are compatible in the sense that they form a double Lie algebroid $\mathcal{F}_{p,q}$. The quadruple complex $\mathcal{W}^{\bullet,\bullet}(\mathcal{F}_{\bullet,\bullet})$ then augments both $\widetilde{C}^\infty_{\bullet,\bullet}(\mathcal{G})$ and $\mathcal{W}^{\bullet,\bullet}(D)$, and the van Est map VE: $\widetilde{C}^\infty_{\bullet,\bullet}(\mathcal{G}) \to \mathcal{W}^{\bullet,\bullet}(D)$ can then be constructed systematically out of homotopy operators on $\mathcal{W}^{\bullet,\bullet}(\mathcal{F}_{\bullet,\bullet})$ using homological algebra.

Appendix A

Operations on Vector Fields and Differential Forms

The fundamental example of a double Lie algebroid is the tangent space TA of a Lie algebroid $A \to M$. For this reason, chapter 8 is devoted to explicit computations of the constructions presented in this thesis for this specific example. Throughout the course of these computations we make use of numerous operations in differential geometry, which we review here. All of the material in this appendix is known, we include it as a means of fixing our terminology and conventions as well as to provide a convenient reference for the reader. Some excellent sources for this material include the book of Kolář, Michor, and Slovák [37] and the work of Grabowski and Urbański [24].

A.1 Graded Lie Brackets

For a given smooth manifold M, there are several graded Lie algebras related to the geometry of M that are referenced in the chapters above. The most foundational one is the algebra $\mathfrak{X}^\bullet(M)$ of multivector fields on M, whose degree k component is

$$\mathfrak{X}^k(M) = \Gamma(\wedge^k TM, M),$$

with the convention that $\mathfrak{X}^0(M) = C^\infty(M)$. Contrary to the usual convention for graded algebras, the notation $\mathfrak{X}(M)$ (without the $\bullet$) does not denote the direct sum of the components, but rather its degree one part: $\mathfrak{X}(M) = \mathfrak{X}^1(M)$. We can extend the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{X}(M)$ to a graded Lie bracket $[\![\cdot, \cdot]\!]$ on $\mathfrak{X}^\bullet(M)$ by defining it on generators by the formulas

$$[\![X, f]\!] = \mathcal{L}_X f, \quad [\![X, Y]\!] = [X, Y]$$

and insisting that $[\![X, \cdot]\!]$ is a graded derivation of the wedge product. The bracket $[\![\cdot, \cdot]\!]$ is called the *Schouten bracket*. Using the definition in terms of generators, one can obtain the following more explicit form of the Schouten bracket

$$[\![f\cdot X_1\wedge\ldots\wedge X_k, g\cdot Y_1\wedge\ldots\wedge Y_\ell]\!]=\sum_{i,j}(-1)^{i+j}[X_i,Y_j]\wedge X_1\wedge\ldots\wedge\widehat{X}_i\wedge\ldots\wedge X_k\wedge Y_1\wedge\ldots\wedge\widehat{Y}_j\wedge\ldots\wedge Y_\ell$$
$$+\sum_j(-1)^{j+1}\mathcal{L}_{Y_j}f\cdot X_1\wedge\ldots\wedge X_k\wedge Y_1\wedge\ldots\wedge\widehat{Y}_j\wedge\ldots\wedge Y_\ell$$
$$+\sum_i(-1)^{i+1}\mathcal{L}_{X_i}g\cdot X_1\wedge\ldots\wedge\widehat{X}_i\wedge\ldots\wedge X_k\wedge Y_1\wedge\ldots\wedge Y_\ell,$$

where as usual the hat denotes omission from the product. If we let $\mathfrak{X}^\bullet(D)[1]$ denote the graded algebra with $\mathfrak{X}^k(M)[1]=k+1(M)$, then the Schouten bracket makes $\mathfrak{X}^\bullet(M)[1]$ into a graded Lie algebra.

Remark A.1.1. In the literature, the opposite sign convention is sometimes used to define the Schouten bracket. For example, our convention agrees with the book by Dufour and Zung [19], which, as they point out, differs from the lectures of Vaisman [74].

The importance of the Schouten bracket in Poisson geometry is that it can be used to define a cohomology theory associated to any Poisson manifold. Any Poisson bracket $\{\cdot,\cdot\}$ on a manifold M determines a multivector field $\pi\in\mathfrak{X}^2(M)$ defined by $\langle df\wedge dg,\pi\rangle=\{f,g\}$. The Schouten bracket allows us to determine which elements of $\mathfrak{X}^2(M)$ arise in this fashion, as explained by the following lemma.

Lemma A.1.2. *A multivector field $\pi\in\mathfrak{X}^2(M)$ determines a Poisson structure on M if and only if*

$$[\![\pi,\pi]\!]=0.$$

Proof. This lemma is well known and a proof can be found in a number of places. For example, it is in [19, Theorem 1.8.5]. □

Taking things one step further, we can observe that the operator $[\![\pi,\cdot]\!]$ defines a differential on $\mathfrak{X}^\bullet(M)$. Indeed, the graded Jacobi identity for $[\![\cdot,\cdot]\!]$ implies that the formula

$$(-1)^{|X|-1}[\![\pi,[\![\pi,X]\!]]\!]-[\![\pi,[\![X,\pi]\!]]\!]+(-1)^{|X|-1}[\![X,[\![\pi,\pi]\!]]\!]=0$$

holds for any X. Using the lemma as well as the skew symmetry $[\![\pi,X]\!]=(-1)^{|X|}[\![X,\pi]\!]$ shows that $[\![\pi,[\![\pi,X]\!]]\!]=0$. The cohomology of the complex

$$(\mathfrak{X}^\bullet(M),[\![\pi,\cdot]\!])$$

is called the *Poisson cohomology* (or *Lichnerowicz -Poisson cohomology*) of M. For more information on Schouten brackets and Poisson cohomology, we recommend the reader consult the book of Laurent-Gengoux, Pichereau, and Vanhaecke [45], or the aforementioned book of Dufour and Zung [19].

Let us move on to considering the space $\Omega^\bullet(M,TM)$ of TM-valued differential forms on M, whose degree k component consists of multilinear maps

$$\Upsilon\colon\underbrace{\mathfrak{X}(M)\otimes\ldots\otimes\mathfrak{X}(M)}_{k\text{ copies}}\to\mathfrak{X}(M)$$

that are skew symmetric. Alternatively, we have $\Omega^k(M,TM)=\Gamma(\wedge^kT^*M\otimes TM)$. Note that in particular, the degree 0 component consists of vector fields on M: $\Omega^0(M,TM)=\Gamma(TM)$. Associated to any $\Upsilon \in \Omega^k(M,TM)$, we obtain a derivation ι_Υ of degree $k-1$ on the space $\Omega^\bullet(M)$ of differential forms on M via the formula

$$\iota_\Upsilon\omega(X_1,\ldots,X_{k+\ell-1}) = \frac{1}{k!\,(\ell-1)!}\sum_{\sigma\in S_{k+\ell-1}} \mathrm{sgn}(\sigma)\omega(\Upsilon(X_{\sigma(1)},\ldots,X_{\sigma_k}),X_{\sigma(k+1)},\ldots,X_{\sigma(k+\ell-1)})$$

for $\omega \in \Omega^\ell(M)$. Note that any derivation D of degree $k-1$ of $\Omega^\bullet(M)$ is determined by it's restriction to one forms $D|_{\Omega^1(M)}\colon \Omega^1(M) \to \Omega^k(M)$, which may be viewed as an element of $\Gamma(\wedge^k T^*(M)\otimes TM)$. The operator ι_Υ is merely the extension of Υ to all of $\Omega(M)$, in particular, the assignment $\Upsilon \mapsto \iota_\Upsilon$ is injective. By the usual Cartan formula, one then obtains the operator $\mathcal{L}_\Upsilon$ on $\Omega^\bullet(M)$ that takes a form to its Lie derivative along Υ,

$$\mathcal{L}_\Upsilon = \iota_\Upsilon \circ d + (-1)^{|\Upsilon|-1} d\circ\iota_\Upsilon,$$

where d denotes the de Rham differential. When the degree of Υ is zero, the operators ι_Υ and $\mathcal{L}_\Upsilon$ are just the usual contraction and Lie derivative of a vector field. These operators suggest two natural ways to define a Lie bracket on $\Omega^\bullet(M,TM)$: one can either insist that the contraction operators preserve the bracket, or that the Lie derivatives do. The former choice leads to the *Nijenhuis-Richardson*(NR) bracket [70], while the latter gives the *Frölicher-Nijenhuis*(FN) bracket [22]. In other words, we have two Lie brackets $[\cdot,\cdot]_{\mathrm{NR}}$ and $[\cdot,\cdot]_{\mathrm{FN}}$ on $\Omega^\bullet(M,TM)$ that are determined by

$$\iota_{[\Upsilon_1,\Upsilon_2]_{\mathrm{NR}}} = [\iota_{\Upsilon_1},\iota_{\Upsilon_2}], \quad \mathcal{L}_{[\Upsilon_1,\Upsilon_2]_{\mathrm{FN}}} = [\mathcal{L}_{\Upsilon_1},\mathcal{L}_{\Upsilon_2}] \tag{A.1}$$

for all $\Upsilon_1,\Upsilon_2 \in \Omega^\bullet(M,TM)$. Here the brackets on the right hand sides of the two equations denote the (super) commutator of two derivations, and so the Nijenhuis-Richardson bracket is of degree -1, while the Frölicher-Nijenhuis bracket is of degree 0. Note also that the FN bracket extends the usual Lie bracket of vector fields. It is instructive to see why these formulas do indeed give unique and well-defined Lie brackets on $\Omega^\bullet(M,TM)$. For the NR bracket, one can give a simple formula by extending the contraction operators ι_Υ to $\Omega^\bullet(M,TM)$ by the definition $\iota_\Upsilon(\omega\otimes X) = \iota_\Upsilon\omega\otimes X$, giving

$$[\Upsilon_1,\Upsilon_2]_{\mathrm{NR}} = \iota_{\Upsilon_1}\Upsilon_2 + (-1)^{|\Upsilon_1|+1}\iota_{\Upsilon_2}\Upsilon_1.$$

Indeed, a check on one-forms shows that contraction by the right hand side agrees with $[\iota_{\Upsilon_1},\iota_{\Upsilon_2}]$, and it therefore must give the NR bracket by injectivity of $\Upsilon\mapsto\iota_\Upsilon$. To understand the FN bracket, we use the following lemma.

Lemma A.1.3. *Any degree k derivation D of $\Omega(M)$ can be written in the form*

$$D = \mathcal{L}_{\Upsilon_1} + \iota_{\Upsilon_2}$$

for unique $\Upsilon_1 \in \Omega^k(M,TM)$ and $\Upsilon_2 \in \Omega^{k+1}(M,TM)$. Moreover, $\Upsilon_2 = 0$ if and only if $[D,d]=0$.

Proof. This lemma is well-known, we include a short proof here for illustration purposes. Note that given $X_1,\ldots,X_k \in \mathfrak{X}(M)$, the assignment $f\mapsto Df(X_1,\ldots,X_k)$ is a derivation of $C^\infty(M)$. Therefore there is a unique vector field $\Upsilon(X_1,\ldots,X_k)$ such that $D-\mathcal{L}_{\Upsilon_1(X_1,\ldots,X_k)}$ vanishes along $C^\infty(M)$. But this means that $D-\mathcal{L}_{\Upsilon_1(X_1,\ldots,X_n)}$ is given by a contraction operator, and the first claim follows by injectivity of $\iota\colon\Omega^\bullet(M,TM)\to \mathrm{Der}^{\bullet-1}(\Omega(M))$. Moreover, the Jacobi identity gives $[\mathcal{L}_{\Upsilon_1},d] = \frac{1}{2}[\iota_{\Upsilon_1},[d,d]] = 0$, and so we get $[D,d] = \mathcal{L}_{\Upsilon_2}$. The second claim then follows from the injectivity of the assignment $\Upsilon\mapsto\mathcal{L}_\Upsilon$. $\square$

It follows from this lemma that the FN bracket is unique and well-defined, since we certainly have $[[\mathcal{L}_{\Upsilon_1}, \mathcal{L}_{\Upsilon_2}], d] = 0$ for any $\Upsilon_1, \Upsilon_2 \in \Omega^\bullet(M, TM)$ by the Jacobi identity and the observation in the proof of the lemma that the commutator of a Lie derivative with the exterior derivative vanishes. Both the Nijenhuis-Richardson bracket and the Frölicher-Nijenhuis bracket have numerous applications throughout geometry. For example, the Nijenhuis-Richardson bracket can be used to study deformations of Lie algebra structures [71], since a bilinear form Q on a vector space is a Lie bracket if and only if $[Q,Q]_{\mathrm{NR}} = 0$. Meanwhile the Frölicher-Nijenhuis bracket can be used to study complex manifolds, since an almost complex structure $J\colon TM \to TM$ is a complex structure precisely if $[J,J]_{\mathrm{FN}} = 0$ (this is the *Newlander-Nirenberg* theorem [68]). For more information on the NR and FN brackets, the reader can consult the books of Kolář-Michor-Slovák [37, Section 30] and of Michor [65, Chapter IV].

Remark A.1.4. All of the brackets described in this section can be generalized from the tangent bundle TM to arbitary Lie algebroids A. That is, the space $\Gamma(\wedge^\bullet A)$ can be endowed with a Gerstenhaber bracket that agrees with the Schouten bracket when $A = TM$ (see e.g [43]), and the space $\Gamma(\wedge^\bullet A^* \otimes A)$ inherits brackets that generalize the NR and FN brackets [69]. These generalizations are all explained in [24, Section 1].

A.2 Tangent and Cotangent Lifts

Given a vector bundle $V \to M$, there are various ways of lifting geometric objects acting on M and E (functions, vector fields, sections) to objects acting on the tangent and cotangent bundles, TM, TE, T^*M, and T^*E. These lifting processes are used extensively in the computations in chapter 8, so we will review them here.

To start off, any smooth function on the base $f \in C^\infty(M)$ defines two smooth functions on TM, the *vertical lift* $f^\sharp$ and the *tangent lift* f_T. Here $f^\sharp$ is given by the pullback of f along the tangent bundle projection, while f_T is the one-form df viewed as a linear function on TM. Explicitly:

$$f^\sharp(X_m) = f(m), \quad f_T(X_m) = \langle df_m, X_m \rangle, \tag{A.2}$$

for $X_m \in T_mM$. Clearly any function that is homogeneous of degree 0 on TM is of the form $f^\sharp$ for some $f \in C^\infty(M)$. Similarly, since the space of one-forms is spanned by elements of the form df for $f \in C^\infty(M)$, it follows that the tangent lifts f_T span $C^\infty(TM)_{[1]}$. But TM is a vector bundle, which allows us to define a vector field on TM by its action on the generators of $\mathcal{S}(TM)$ (the polynomial functions on TM), namely $f^\sharp$ and f_T for $f \in C^\infty(M)$. So for $X \in \mathfrak{X}(M)$, we define the vertical lift $X^\sharp$ and the tangent lift X_T by the conditions

$$\mathcal{L}_{X^\sharp}(f^\sharp) = 0, \quad \mathcal{L}_{X^\sharp}(f_T) = (\mathcal{L}_X f)^\sharp = \mathcal{L}_{X_T}(f^\sharp), \quad \mathcal{L}_{X_T}(f_T) = (\mathcal{L}_X f)_T. \tag{A.3}$$

Note that $X^\sharp$ is homogeneous of degree -1, while X_T is homogeneous of degree 0. Our next goal will be to extend these operations from vector fields to multivector fields. First, we extend the assignment $X \mapsto X^\sharp$ as an algebra homomorphism $\mathfrak{X}^\bullet(M) \to \mathfrak{X}^\bullet(TM)$, i.e $(X \wedge Y)^\sharp = X^\sharp \wedge Y^\sharp$ for all $X, Y \in \mathfrak{X}^\bullet(M)$. Having done this, we extend the assignment $X \mapsto X_T$ by the rule $(X \wedge Y)_T = X_T \wedge Y^\sharp + X^\sharp \wedge Y_T$, which ensures that the tangent lift of a multi-vector field of degree k is always homogeneous of degree $1-k$ (in other words, the tangent lift of a multivector field is always *linear*). That these are the right extension rules will be made apparent by the next lemma, which tells us that these operations on multivector fields behave with the Schouten bracket in the way we would hope.

Lemma A.2.1. *For any pair of multivector fields $\xi, \zeta \in \mathfrak{X}(M)$, we have*

$$[\xi^\sharp, \zeta^\sharp] = 0, \quad [\xi_T, \zeta^\sharp] = [\xi, \zeta]^\sharp = [\xi^\sharp, \zeta_T], \quad [\xi_T, \zeta_T] = [\xi, \zeta]_T.$$

Proof. When X, Y are vector fields, these formulas follow from the identity $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$, as we can check on the generators $f^\sharp, f_T$ for $f \in C^\infty(M)$. For the first bracket, simply note that $[\mathcal{L}_{X^\sharp}, \mathcal{L}_{Y^\sharp}]$ is of degree -2 and so is automatically zero. For the middle bracket, we have

$$\mathcal{L}_{[X_T, Y^\sharp]} f^\sharp = \mathcal{L}_{X_T} \mathcal{L}_{Y^\sharp} f^\sharp - \mathcal{L}_{Y^\sharp} \mathcal{L}_{X_T} f^\sharp = -\mathcal{L}_{Y^\sharp} (\mathcal{L}_X f)^\sharp = 0$$
$$\mathcal{L}_{[X_T, Y^\sharp]} f_T = (\mathcal{L}_{X_T} \mathcal{L}_{Y^\sharp} - \mathcal{L}_{Y^\sharp} \mathcal{L}_{X_T}) f_T = \mathcal{L}_{X_T} (\mathcal{L}_Y f)^\sharp - \mathcal{L}_{Y^\sharp} (\mathcal{L}_X f)_T = (\mathcal{L}_X \mathcal{L}_Y f)^\sharp - (\mathcal{L}_Y \mathcal{L}_X f)^\sharp = \left(\mathcal{L}_{[X,Y]} f\right)^\sharp,$$

from which it follows that $[X_T, Y^\sharp] = [X, Y]^\sharp$. The identites $[X^\sharp, Y_T] = [X, Y]^\sharp$ and $[X_T, Y_T] = [X, Y]_T$ are established similarly. The general result then follows from induction using the derivation property of the Schouten bracket with respect to the wedge product. For example, if $\xi \in \mathfrak{X}^k(M)$, $\zeta \in \mathfrak{X}^{\tilde{k}}(M)$, and $X \in \mathfrak{X}(M)$, we compute

$$\begin{aligned}
[\xi_T, (\zeta \wedge X)_T] &= [\xi_T, \zeta_T \wedge X^\sharp + \zeta^\sharp \wedge X_T] \\
&= [\xi_T, \zeta_T] \wedge X^\sharp + (-1)^k \zeta_T \wedge [\xi_T, X^\sharp] + [\xi_T, \zeta^\sharp] \wedge X_T + (-1)^k \zeta^\sharp \wedge [\xi_T, X_T] \\
&= [\xi, \zeta]_T \wedge X^\sharp + (-1)^k \zeta_T \wedge [\xi, X]^\sharp + [\xi, \zeta]^\sharp \wedge X_T + (-1)^k \zeta^\sharp \wedge [\xi, X]_T \\
&= ([\xi, \zeta] \wedge X)_T + (-1)^k (\zeta \wedge [\xi, X])_T \\
&= [\xi, \zeta \wedge X]_T.
\end{aligned}$$

□

The situation for the cotangent bundle T^*M is slightly different. Rather than having two ways of lifting functions to functions and vector fields to vector fields, instead we have some “mixing”. That is, we can lift a function to a function (vertical lift), a function to a vector field (Hamiltonian), a vector field to a function (duality), or a vector field to a vector field (cotangent lift). First note that we can define the vertical lift of a function in the same way as for the tangent bundle, simply by pulling back along the projection. To distinguish it from the vertical lift to the tangent bundle, we will denote this operation by $f \mapsto f^\flat$. Next, any vector field $X \in \mathfrak{X}(M)$ defines a linear function on T^*M by duality, which we denote by ϕ_X. Moreover, being dual to a Lie algebroid, T^*M comes with a canonical Poisson structure $\{\cdot, \cdot\}$, which can be defined by the property

$$\{\phi_X, \phi_Y\} = \phi_{[X,Y]}.$$

As usual, any function $f \in C^\infty(T^*M)$ determines a Hamiltonian vector field $X_f := \{f, \cdot\}$ on T^*M, so given a vector field $X \in \mathfrak{X}(M)$, we can define its *cotangent lift* X_{T^*} to be the Hamiltonian vector field of the induced linear function ϕ_X. Symbolically, X_{T^*} is defined by

$$\mathcal{L}_{X_{T^*}} f = \{\phi_X, f\}.$$

With these notations, the Leibniz rule for the canonical Poisson bracket becomes the relation

$$(fX)_{T^*} = fX_{T^*} + \phi_X X_f.$$

Remark A.2.2. In some of the literature, the terminology differs from above. The most common difference is to use the name *complete lifts* for both the tangent and cotangent lifts, allowing context to clarify the bundle being lifted to (see for example [24]). Additionally, in some works related to the cotangent bundle, the term "vertical lift" refers to a process of lifting tensor fields to the cotangent bundle that does not preserve the bidegree of tensor fields [80]. On functions and vector fields, this vertical lift is defined in our notation by $f \mapsto f^\flat$, $X \mapsto \phi_X$.

Finally, consider the case when $V \xrightarrow{p} M$ is a vector bundle. A section of the dual bundle, $\tau \in \Gamma(V^*)$, defines a linear function on V in the usual way. Since we will be mixing this identification with the lifting processes described above, we will denote this linear function by φ_τ, singling out the duality in the tangent bundle with reserved notation. We can then define the *vertical lift* of a section $\sigma \in \Gamma(V)$ to be the unique vector field $\sigma^\sharp \in \Gamma(TV) = \mathfrak{X}(V)$ satisfying the conditions

$$\mathcal{L}_{\sigma^\sharp}(p^* f) = 0, \quad \mathcal{L}_{\sigma^\sharp}\varphi_\tau = p^*\langle \tau, \sigma \rangle$$

for all $f \in C^\infty(M)$, $\tau \in \Gamma(V^*)$.

Remark A.2.3. As with the brackets described in the previous section, the constructions presented in this section admit generalizations from the tangent bundle TM of a manifold to an arbitrary Lie algebroid $A \to M$. A detailed account of these generalizations can be found in [24].

Appendix B

Normal Bundles and Clean Intersections

One new construction appearing in this thesis is the means of obtaining a double normal space $\nu(M, N_1, N_2)$ out of a manifold M and two submanifolds N_1, N_2 (see chapter 3). If the submanifolds N_1 and N_2 have a "nice enough" intersection, then the space $\nu(M, N_1, N_2)$ is a double vector bundle. The precise condition on N_1 and N_2 turns out to be that their intersection is *clean*, so we review this notion here, as well as some preliminaries on normal bundles that generalize to the setting of the double normal bundle.

B.1 Normal Bundles

In this section we recall the definition of normal bundles and point out some of their uses that we will need later on. It's convenient to work with the category of *manifold pairs*. The objects in this category are pairs (M, N), where M is a smooth manifold and $N \subseteq M$ is a submanifold, and the morphisms between two such pairs (M, N) and (M', N') are smooth maps $\Phi\colon M \to M'$ such that $\Phi(N) \subseteq N'$. The normal bundle may be thought of as a functor ν that associates to any manifold pair (M, N) the vector bundle

$$\nu(M, N) = TM|_N/TN$$

over N. To any morphism $\Phi\colon (M, N) \to (M', N')$, ν associates the vector bundle morphism $\nu(\Phi) = T\Phi|_N$, which passes to the quotient since $\Phi(N) \subseteq \Phi(N')$. The normal bundle $\nu(M, N)$ admits alternate characterizations that are more algebraic in nature. To describe them, let $I \subseteq C^\infty(M)$ be the ideal of functions that vanish along N. Then we obtain a filtration of the space of smooth functions on M given by

$$C^\infty(M) \supseteq I \supseteq I^2 \supseteq \dots$$

The associated graded algebra A of this filtration can be identified with the space of polynomial functions on the normal bundle $\nu(M, N)$. Indeed, for any $p \in N$ let A_p denote the quotient of A by the vanishing ideal of p, then the identification is determined by the map

$$\begin{aligned}(A_p)_1 &\to \mathcal{S}(T_pM/T_pN)\\ [\mathrm{pr}_1 f] &\mapsto ([X_p] \mapsto X_p(f)),\end{aligned}$$

where $\mathrm{pr}_1 \colon A \to A_1$ denotes the projection to the first component. Thus we obtain the normal bundle as the character spectrum of A:

$$\nu(M, N) = \operatorname{Spec} A := \operatorname{Hom}_{\mathrm{Alg}}(A, \mathbb{R}).$$

Furthermore, we see that the space of linear functions on $\nu(M, N)$ is precisely A_1. This gives a third description of the normal bundle: it is dual to the bundle having I/I^2 as its space of sections.

A natural question to ask is how this normal functor interacts with the tangent functor. It turns out that the two functors commute, in the sense that the two double vector bundles

$$\begin{tikzcd} \nu(TM, TN) \arrow[r] \arrow[d] & TN \arrow[d] \\ \nu(M, N) \arrow[r] & N \end{tikzcd} \qquad \begin{tikzcd} T\nu(M, N) \arrow[r] \arrow[d] & TN \arrow[d] \\ \nu(M, N) \arrow[r] & N \end{tikzcd}$$

are isomorphic (see [10, Appendix A]). This compatibility allows us to use the normal bundle to linearly approximate vector fields tangent to N and to produce neighbourhoods of N that are linear in the directions normal to N. Specifically, any vector field $X \in \mathfrak{X}(M)$ that is tangent to M can be thought of as a map of manifold pairs $X \colon (M, N) \to (TM, TN)$. Applying the normal functor and then the isomorphism of double vector bundles above, we get a vector field

$$\nu(X) \colon \nu(M, N) \to T\nu(M, N)$$

that we call the *linear approximation* of X. As a final note about normal bundles, we observe that the lifting processes described in section A.2 above can be adapted from the tangent bundle to the normal bundle. Indeed, for any function $f \in I^n$, let $[f]_n$ denote it's equivalence class in $A_n = I^n/I^{n+1}$. This class determines a function $f^{(n)}$ on $\nu(M, N)$ by the formula

$$f^{(n)} = \mathrm{ev}_{[f]_n} \colon \operatorname{Spec} A \to \mathbb{R} \tag{B.1}$$

$$\varphi \mapsto \varphi([f]_n). \tag{B.2}$$

Note that in the particular cases that $n = 0$, $f^{(0)}$ is the pullback of $f|_N$ along the projection , while for the case $n = 1$, $f^{(1)} = \nu(f)$. We also observe that by construction, $f^{(n)}$ is homogeneous of degree n for the vector bundle structure on $\nu(M, N)$. As with the tangent bundle, one can then also lift vector fields: the filtration on $C^\infty(M)$ induces a filtration on $\mathfrak{X}(M)$, and then the lift $X_(n)$ of a vector field $X \in \mathfrak{X}(M)_{(n)}$ is defined to satisfy the obvious relations with the lfits of functions defined above.

B.2 Clean Intersections

As mentioned above, our goal in this section is to describe a compatibility condition between two submanifolds $N_1, N_2 \subseteq M$ that guarantees that the space $\nu(M, N_1, N_2)$ constructed in chapter 3 is a double vector bundle. A natural starting point is the condition of transverse intersection. Recall that we say N_1, N_2 *intersect transversally* if

$$T_pM = T_pN_1 + T_pN_2$$

for all $p \in N_1 \cap N_2$. Note that this restriction is rather strong, for example if $\dim N_1 + \dim N_2 < \dim M$ then N_1 and N_2 intersect transversally only if their intersection is empty. In fact, it is too strong for our purposes, since if N_1 and N_2 have transverse intersection then not only will $\nu(M, N_1, N_2)$ be a double vector bundle, it will be a *vacant* double vector bundle. This suggests that we can relax the condition of transverse intersection and still obtain a $\mathscr{DVB}$, with the cost of introducing a nontrivial core. The correct way to relax the notion of transverse intersections turns out to be that of *clean intersection*, the definition of which we now recall.

Definition B.2.1 (Cleanly intersecting submanifolds)**.** Let M be a manifold, and let N_1, N_2 be submanifolds. Then N_1 and N_2 are said to *intersect cleanly* if $N_1 \cap N_2$ is a submanifold of M such that

$$T_p(N_1 \cap N_2) = T_pN_1 \cap T_pN_2$$

for all $p \in N_1 \cap N_2$.

The main result about cleanly intersecting submanifolds is that locally N_1 and N_2 look like coordinate subspaces of M. A proof of this result, stated more precisely below, can be found in [34, Proposition C.3.2].

Proposition B.2.2. *Let N_1 and N_2 be cleanly intersecting submanifolds of M. Then there exist coordinates $\{x_1, \dots, x_m\}$ on M such that*

$$\begin{aligned} N_1 &= \{x_1 = \cdots = x_e = 0,\ x_{e+1} = \cdots = x_k = 0\}, \\ N_2 &= \{x_1 = \cdots = x_e = 0,\ x_{k+1} = \cdots = x_\ell = 0\}. \end{aligned}$$

The number $e = \operatorname{codim} N_1 + \operatorname{codim} N_2 - \operatorname{codim}(N_1 \cap N_2)$ appearing above is called the *excess* of the intersection. Note that a clean intersection is transverse precisely when $e = 0$. The final result on clean intersections that we need is the following consequence of proposition B.2.2, which was useful in determining the $\mathscr{DVB}$ sequence of the double normal bundle in chapter 3.

Corollary B.2.3. *Let N_1 and N_2 be cleanly intersecting submanifolds of M, and suppose $f \in C^\infty(N_1)$ is a function that vanishes on $N_1 \cap N_2$. Then f is the restriction of a smooth function on M that vanishes on N_2.*

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