# A NON-ARCHIMEDEAN DEFINABLE CHOW THEOREM

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# ABSTRACT

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O-minimality has had some striking applications to number theory. The utility of o-minimal structures originates from the remarkably tame topological properties satisfied by sets definable in such structures. Despite the rigidity that it imposes, the theory is sufficiently flexible to allow for a range of analytic constructions. An illustration of this 'tame' property is the following surprising generalization of Chow's theorem proved by Peterzil and Starchenko - A closed analytic subset of a complex algebraic variety that is also definable in an o-minimal structure, is in fact algebraic. While the o-minimal machinery aims to capture the archimedean order topology of the real line, it is natural to wonder if such a machinery can be set up over non-archimedean fields. In this thesis, we explore a non-archimedean analogue of an o-minimal structure and prove a version of the definable Chow theorem in this context.

*To my first math teacher,*
*my grandfather.*

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The proof of the non-archimedean definable Chow theorem is based on an alternate proof of Peterzil and Starchenko's definable Chow theorem. I learned of this alternate proof at a lecture series given by Benjamin Bakker. I am very grateful to him for the inspiring set of lectures that went a long way in helping me understand the work of Peterzil and Starchenko.

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# CONTENTS

# 1 INTRODUCTION

The interaction between algebraic and analytic geometry has been an immensely fruitful one. Over the complex numbers $\mathbb{C}$, analytic geometry has been the subject of study since Riemann. While a complex algebraic variety is locally described as the zero locus of finitely many polynomial equations, a complex analytic space is locally described as the vanishing locus of finitely many *holomorphic* functions in some domain in affine space $\mathbb{C}^n$. Similarly, while morphisms between algebraic varieties are described in local coordinates by polynomial equations, morphisms between complex analytic varieties are given locally by holomorphic functions. To any complex algebraic variety one may functorially associate an analytic variety to obtain the so-called analytification functor:

$$\begin{aligned} \text{an} : \text{Alg.Var.}/\mathbb{C} &\to \mathbb{C}\text{-An.Var.} \\ X &\mapsto X^{\text{an}} \end{aligned}$$

from the category Alg.Var./$\mathbb{C}$ of algebraic varieties over $\mathbb{C}$ to the category of complex analytic spaces $\mathbb{C}$-An.Var.

Given a complex algebraic variety $X$, an important question in geometry is how much of the algebraic structure is retained in the analytification $X^{\text{an}}$? Theorems that lead to comparisons between the algebraic and analytic structures have been a cornerstone of algebraic geometry. There is indeed a long history attached to such comparison questions. Two fundamental results in this direction are the theorem of Chow and the celebrated generalization in Serre's GAGA paper [Ser56], which in turn has paved the way for modern algebraic geometry.

## 1.1 THE THEOREM OF CHOW AND SERRE'S GAGA

In 1949, Wei-Liang Chow proved the following remarkable result:

**Theorem** (Chow, [Cho49])**.** *For a projective complex algebraic variety $X$, every closed analytic subvariety of $X^{\text{an}}$ is in fact algebraic.*

While Chow's original proof was a careful analysis of the analytic simplices in analytic subvarieties of projective space, an alternate proof of Chow's theorem was provided by Remmert and Stein in 1953 [RS53].

Remmert and Stein reproved Chow's theorem as a simple consequence of their result on analytic continuations of analytic subvarieties. Questions concerning the extensions of analytic functions, or analytic sets have been fundamental in studying the behaviour of singularities of analytic spaces. A classical theorem in this regard includes the famous Riemann extension theorem.

**Theorem 1.1** (Riemann extension theorem)**.** *Let $(X, \mathcal{O}_X)$ be a normal complex analytic space and let $Y \subseteq X$ be a closed analytic subvariety of $X$ that is of positive codimension everywhere. Suppose $f \in \mathcal{O}(X \setminus Y)$ be an analytic function that is locally bounded at every point of $Y$. Then $f$ extends to a global holomorphic function $f^* \in \mathcal{O}(X)$.*

The analytic continuation result of Remmert–Stein is concerned with the analytic continuation of an analytic set along the complement of another analytic set of lower dimension. The Remmert–Stein theorem is important from the point of view of complex geometry. For instance, it is a crucial input to Remmert's proper mapping theorem.

**Theorem 1.2** (Remmert–Stein, [RS53])**.** *Let $X$ be a complex analytic space and $T \subseteq X$ a closed analytic subset of dimension $d$. Suppose $Y \subseteq (X \setminus T)$ is a closed analytic subset of $X \setminus T$. If every irreducible component of $Y$ has dimension $> d$ then the closure $\overline{Y}$ of $Y$ in $X$ is a closed analytic subset of $X$.*

### SERRE'S GAGA THEOREM

We also briefly mention the remarkable generalization of Chow's theorem due to Serre in his celebrated GAGA paper [Ser56]. For a complex algebraic variety $X$, we denote by $\mathrm{Coh}(X)$ (resp. $\mathrm{Coh}(X^{\mathrm{an}})$) the abelian category of coherent sheaves on $X$ (resp. $X^{\mathrm{an}}$). The analytification map $i_X : X^{\mathrm{an}} \to X$ gives rise to a natural analytification functor on coherent sheaves:

$$\begin{aligned} \mathrm{An} : \mathrm{Coh}(X) &\longrightarrow \mathrm{Coh}(X^{\mathrm{an}}) \\ \mathcal{F} &\longmapsto \mathcal{F}^{\mathrm{an}} := i_X^*(\mathcal{F}) \end{aligned}$$

Serre's GAGA proves that when $X$ is projective, the analytification functor An is an equivalence of categories and furthermore, the algebraic and analytic cohomology groups of such a coherent sheaf also agree. In particular, for a projective variety $X$, the correspondence between coherent ideal sheaves of $\mathcal{O}_X$ and those of $\mathcal{O}_{X^{\mathrm{an}}}$, enables one to reprove the theorem of Chow.

## 1.2 O-MINIMALITY IN COMPLEX GEOMETRY

A natural question one may ask is what happens in the setting of non-proper or non-projective algebraic varieties. Evidently, the conclusion of Chow's theorem no longer holds. For instance, infinite discrete subsets of $\mathbb{C}$ that accumulate at $\infty$, such as the set of integers $\mathbb{Z} \subseteq \mathbb{C}$, are analytic subvarieties of the complex affine line that are not algebraic. A more illuminating example, is provided by sets with essential singularities at infinity. Consider the graph of the complex exponential function $\Gamma := \{(z, e^z) : z \in \mathbb{C}\}$. Then $\Gamma$ is an analytic subvariety of the complex plane $\mathbb{C}^2$ that is not algebraic.

The remarkable insight of the work of Peterzil and Starchenko is to use some ideas from model theory (in particular from the theory of o-minimality) in order to systematically rule out essential singularities at infinity. In this manner they are able to prove stronger versions of a number of classical results in complex geometry, including a version of Chow's theorem that holds in the non-proper case.

### *O-minimal structures*

The theory of o-minimality has its origins in model theory, and was initially developed alongside real analytic geometry. But in recent years, the theory has had a number of striking applications to diverse fields such as Diophantine geometry and Hodge theory. The utility of o-minimal structures originates from the remarkably tame topological properties satisfied by sets definable in such structures.

Grothendieck, in his famous 'Esquisse d'un Programme' [Gro97], proposed that the axioms of a topological space are far too general for the purposes of geometry. The existence of space-filling curves and nowhere differentiable continuous functions should really be wild pathologies. He posited the existence of axioms of a 'topologie modérée' which should be better suited for the purposes of geometry and goes on to describe suitable topological properties one would like from such a recasting of the foundations of topology. The example he has in mind is that of Hironaka's real semi-analytic sets. In many regards, o-minimality is widely considered as a possible candidate of Grothendieck's conjectured 'tame topology'.

Roughly speaking, a structure on the set of real numbers $\mathbb{R}$ (or more generally on any set) is the data of a Boolean algebra of subsets of $\mathbb{R}^n$ for every $n \geq 0$ that are closed under Cartesian products and coordinate projections. The subsets in the Boolean algebra are called definable sets. A structure is said to be o-minimal if the collection of definable subsets of $\mathbb{R}$ is precisely the collection of finite unions of intervals and points in $\mathbb{R}$.

**Definition 1.3.** A *structure* on $\mathbb{R}$ is a collection $(\mathfrak{S}_m)_{m\geq 0}$ where each $\mathfrak{S}_m$ is a collection of subsets of $\mathbb{R}^m$ with the following properties:

(i) $\mathfrak{S}_m$ is a Boolean algebra of subsets of $\mathbb{R}^m$;
(ii) If $\mathcal{S} \in \mathfrak{S}_m$ then $\mathbb{R} \times \mathcal{S} \in \mathfrak{S}_{m+1}$ and $\mathcal{S} \times \mathbb{R} \in \mathfrak{S}_{m+1}$.
(iii) The diagonal $\{(x, x) : x \in \mathbb{R}\} \in \mathfrak{S}_2$.
(iv) If $\mathcal{S} \in \mathfrak{S}_m$ then $\mathrm{pr}(\mathcal{S}) \in \mathfrak{S}_{m-1}$, where $\mathrm{pr} : \mathbb{R}^m \to \mathbb{R}^{m-1}$ denotes the projection omitting the last factor.
(v) $+, \cdot : \mathbb{R}^2 \to \mathbb{R}$ are definable i.e. their graphs are in $\mathfrak{S}_3$.

We call the subsets of $\mathbb{R}^m$ that are in our structure $\mathfrak{S}_m$ as *definable sets* and similarly a function $f : A \to B$ (where $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$) is said to be *definable* if the graph of $f$ is a definable subset of $\mathbb{R}^{m+n}$.

**Definition 1.4** (An o-minimal structure on $\mathbb{R}$)**.** We say that a structure $(\mathfrak{S}_m)_{m\geq 0}$ on $\mathbb{R}$ is *o-minimal* if $\mathfrak{S}_1$ is the collection of finite unions of (open) intervals and points of $\mathbb{R}$

We now provide some examples of o-minimal structures. Note that real algebraic sets (i.e. subsets of $\mathbb{R}^m$ described as the zero locus of polynomials with real coefficients) are always definable in any structure. However, the collection of real algebraic sets does not form a structure since coordinate projections of real algebraic sets may not remain algebraic. This is seen for instance when projecting the graph of $y = x^2$ onto the $y$-axis.

As a consequence, we see that the order relation on $\mathbb{R}$ is also always definable. Indeed, the graph of the order relation is the projection of a real algebraic set:

$$\{(x, y) \in \mathbb{R}^2 : x \leq y\} = \mathrm{pr}_{(x,y)}(\{(x, y, z) \in \mathbb{R}^3 : y = x + z^2\})$$

and is therefore definable. Consequently, sets that are cut out by polynomial inequalities are also definable in any structure. Sets that are defined via polynomial inequalities are called the semialgebraic sets. It is a classical theorem of Tarski and Seidenberg that the collection of semialgebraic sets is in fact a structure, and furthermore it is o-minimal.

**Example 1.5** (The structure $\mathbb{R}_{\mathrm{alg}}$ of real semialgebraic sets)**.** A subset $S \subseteq \mathbb{R}^m$ is said to be a semialgebraic set if it is a finite union of sets of the form

$$\left\{ \underline{x} \in \mathbb{R}^m : \bigwedge_{i=1}^{r} \left(f_i(\underline{x}) = 0\right) \wedge \bigwedge_{j=1}^{s} \left(g_j(\underline{x}) > 0\right) \right\}$$

where $f_i(\underline{x})$ and $g_j(\underline{x})$ are polynomials in $\mathbb{R}[x_1, \ldots, x_m]$.

We set for every $m \geq 0, \mathbb{R}_{\mathrm{alg},m} := \{S \subseteq \mathbb{R}^m : S \text{ is semialgebraic}\}$. Then $\mathbb{R}_{\mathrm{alg}} := (\mathbb{R}_{\mathrm{alg},m})_{m\geq 0}$ is an o-minimal structure on $\mathbb{R}$ and is called the *structure of real semialgebraic sets*.

Another important example of an o-minimal structure is furnished by the collection of real subanalytic sets. Similar to the semialgebraic sets, one could consider subsets of $\mathbb{R}^m$ cut out by inequalities among *real analytic* functions. However, to obtain an o-minimal structure, one cannot expect all real analytic functions to be definable. For instance, the set of zeroes of the real sine function, $\sin : \mathbb{R} \to \mathbb{R}$ is an infinite discrete subset of $\mathbb{R}$ and is therefore not definable in any o-minimal structure. The solution is to consider 'overconvergent' analytic functions, or more precisely, real analytic functions on $[0,1]^m$ that actually converge in an open neighbourhood of $[0,1]^m$.

**Example 1.6** (The structure $\mathbb{R}_{\text{an}}$ of real subanalytic sets)**.** We define the structure $\mathbb{R}_{\text{an}}$ to be the smallest structure on $\mathbb{R}$ in which all the semialgebraic subsets are definable, and in which the graphs of all the restricted analytic functions $f : [0,1]^m \to \mathbb{R}$ converging in an open neighbourhood of $[0,1]^m$ are definable.

It turns out that a subset $X \subseteq \mathbb{R}^m$ is definable in $\mathbb{R}_{\text{an}}$ if and only if $X \subseteq \mathbb{P}^m(\mathbb{R})$ is *subanalytic* i.e. is locally in $\mathbb{P}^m(\mathbb{R})$ cut out by Boolean combinations of inequalities between real analytic functions. It is a well-known result due to Gabrielov and Łojasiewicz that $\mathbb{R}_{\text{an}}$ is in fact an o-minimal structure.

**Example 1.7** (The structure $\mathbb{R}_{\text{exp}}$)**.** Let $\mathbb{R}_{\text{exp}}$ denote the smallest structure on $\mathbb{R}$ in which the graph of the real exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is definable.

Due to a remarkable quantifier-simplification result of Wilkie [Wil96, 'Second Main Theorem'], it turns out that a subset $S \subseteq \mathbb{R}^m$ is definable in $\mathbb{R}_{\text{exp}}$ if and only if it is the image under a projection $\mathbb{R}^{m+k} \to \mathbb{R}^m$ of a set of the form $\{(\underline{x}, \underline{y}) \in \mathbb{R}^{m+k} : P(\underline{x}, \underline{y}, e^{\underline{x}}, e^{\underline{y}}) = 0\}$, where $P$ is a real polynomial in $2m + 2k$ variables, and $\underline{x} = (x_1, \ldots, x_m), \underline{y} = (y_1, \ldots, y_k)$ and $e^{\underline{x}} = (e^{x_1}, \ldots, e^{x_m}), e^{\underline{y}} = (e^{y_1}, \ldots, e^{y_m})$. In conjunction with the famous 'fewnomials' result of Khovanskii [Kho91], one proves that in fact $\mathbb{R}_{\text{exp}}$ is indeed an o-minimal structure.

**Example 1.8** (The structure $\mathbb{R}_{\text{an,exp}}$)**.** The structure $\mathbb{R}_{\text{an,exp}}$ is defined to be the smallest structure on $\mathbb{R}$ for which the sets definable in the structure $\mathbb{R}_{\text{an}}$ are definable in $\mathbb{R}_{\text{an,exp}}$ and for which the graph of the real exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is also definable.

A deep theorem of van den Dries and Miller [vdDM94] then proves that $\mathbb{R}_{\text{an,exp}}$ is in fact o-minimal. For applications to Hodge theory and number theory it turns out that the structure $\mathbb{R}_{\text{an,exp}}$ is the one that is used the most.

*Tame topological properties of o-minimally definable sets*

Sets definable in o-minimal structures satisfy strong finiteness properties and have rather pleasant topological properties. To name a few: sets definable in o-minimal structures have only finitely many connected components; quasi-finite definable maps are uniformly quasi-finite; a definable function $f : \mathbb{R} \to \mathbb{R}$ is a piecewise monotone and continuous function except at finitely many points; there is a nice dimension theory of definable sets; definable sets admit a stratification: for every $k \geq 0$ a decomposition into finitely many disjoint $C^k$-manifolds. We refer the reader to the book of van den Dries [vdD98] for a fantastic introduction to the theory of o-minimality.

By identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ one can now talk about definable subsets of $\mathbb{C}^n$. One can go further and globalize the notion of definability to subsets of complex algebraic varieties via affine charts. Since the 'transition maps' on an intersection of two affine charts of an algebraic variety are given by polynomial functions (which are always definable) the notion of definability is indeed well-defined. Furthermore, it is possible to now construct a notion of 'definable complex manifolds', or more generally 'definable complex analytic spaces', as being built from local definable charts glued via definable holomorphic transition maps. This set of ideas has been carried out in a series of fantastic works, [PS08], [PS08], [PS10] where Peterzil and Starchenko develop a theory of complex analysis and complex analytic geometry in the o-minimal definable category. Owing to the strong finiteness and tame topological properties possessed by definable sets in the o-minimal category they are able to prove strong definable versions of a number of classical results in complex geometry including a definable Riemann extension, Remmert–Stein and finally a definable version of Chow's algebraization theorem.

**Theorem 1.9.** (The definable Chow theorem of Peterzil–Starchenko, [PS08, Theorem 5.1]). *Let $X$ be a reduced, complex algebraic variety and $Y \subseteq X^{\mathrm{an}}$ a closed analytic subvariety of the complex analytic variety $X^{\mathrm{an}}$ associated to $X$. Suppose $Y$ as a subset of $X(\mathbb{C})$ is definable in an o-minimal structure. Then $Y$ is algebraic.*

In recent groundbreaking work of Bakker, Brunebarbe and Tsimerman [BBT18], the work of Peterzil–Starchenko is extended even further. The authors develop the theory of coherent sheaves on definable complex analytic spaces, and much like Serre's GAGA theorem, prove a correspondence between categories of definable coherent sheaves and coherent algebraic sheaves. The authors go further by applying their generalization to prove a conjecture of Griffiths on the algebraicity of images of period maps.

## 1.3 NON-ARCHIMEDEAN ANALOGUES

In recent years, there has been an immense amount of progress in non-archimedean geometry. Consequently, non-archimedean techniques are becoming increasingly important in number theory and Diophantine geometry, as evinced by the recent $p$-adic proof by Lawrence and Venkatesh of Faltings' theorem or by the development of the Chabauty–Coleman–Kim method. A natural question therefore is the pursuit of analogues of the above o-minimal machinery to the setting of non-archimedean analytic spaces.

Over non-archimedean fields the existence of a nice theory of analytic geometry is in itself rather surprising. The fact that the underlying metric topology of such fields is totally disconnected leads to significant difficulties in creating a meaningful theory of analytic geometry. Analytic continuation in the usual sense does not hold any longer. In the 1960s, Marc Krasner [Kra66] attempts to solve these difficulties and develops a notion of analytic functions on certain open subsets $U$ of a non-archimedean algebraically closed field $k$. However, Krasner already realizes the need to restrict the class of allowable open subsets $U$ and thus in this theory one is only allowed to work with subsets that satisfy a certain property of analytic continuation. In the celebrated paper [Tat71], John Tate generalizes the ideas of Krasner and is able to lay the foundations for a more robust analytic geometry over non-archimedean fields. The subject of rigid analytic geometry introduced by Tate, today has been further generalized in several directions and forms the building block for modern non-archimedean analytic geometry.

Consider an algebraically closed field $K$ complete with respect to a non-trivial, non-archimedean absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$. Let us denote its ring of integers by $K^\circ$ and the unique maximal ideal of $K^\circ$ by $K^{\circ\circ}$. The basic object in rigid geometry is the $n$-dimensional unit disk $\mathbb{B}^n := (K^\circ)^n$. On $\mathbb{B}^n$, we consider the ring of convergent power series (also referred to as the Tate algebra):

$$T_n(K) := \{ \sum_{I=(i_1,\ldots,i_n)\in\mathbb{N}^n} a_I t_1^{i_1} \ldots t_n^{i_n} : \lim_{\sum_j i_j \to \infty} |a_I| = 0 \}.$$

There is a bijection between the set of maximal ideals $\mathrm{Max}(T_n(K))$ and the unit disk $(K^\circ)^n$. The local model for a rigid space is, analogous to the complex case, the zero locus of finitely many analytic functions converging on the unit polydisk. However, due to the totally disconnected nature of the metric topologies, the topology underlying rigid spaces is no longer an actual topology but rather a Grothendieck toplogy, that dictates which

sets must be declared open and which coverings are admissible. We refer the reader to the book [BGR84] for an introduction to the theory of rigid geometry.

A number of results that hold in the setting of complex analytic spaces continue to hold for rigid analytic spaces. As in complex geometry, there is a natural analytfication functor from the category Alg.Var./$\mathbb{C}_p$ of algebraic varieties over $\mathbb{C}_p$ (or more generally any non-trivially valued non-archimedean field) to the category $\mathbb{C}_p$-Rig.An. of rigid analytic varieties over $\mathbb{C}_p$:

$$\begin{aligned} \text{an} : \text{Alg.Var.}/\mathbb{C}_p &\to \mathbb{C}_p\text{-Rig.An.} \\ X &\mapsto X^{\text{an}}. \end{aligned}$$

Furthermore, the analogues of Riemann extension, Remmert–Stein, Chow's theorem (see [Ł74]) and Serre's GAGA theorems hold in the setting of rigid analytic spaces as well.

While there isn't a satisfactory analogue of o-minimality in the non-archimedean setting, there is nevertheless, a rather pleasant and deep theory of subanalytic geometry over non-archimedean algebraically closed fields (see [Lip93], [LRoob]).

*Subanalytic sets in the non-archimedean setting*

As in the setting of real subanalytic geometry one would like to define subsets to be semianalytic when they are locally Boolean combination of sets cut out by inequalities among local analytic functions and similarly subanalytic sets as sets obtained via coordinate projections of semianalytic sets. For an algebraically closed field $K$, complete with respect to a non-trivial *non-archimedean* absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$, let us identify the unit polydisk $(K^\circ)^n$ with the underlying set of the affinoid space $\text{Sp}(T_n(K))$ associated to the $n$-dimensional Tate algebra $T_n(K)$. One may thus start by making the following natural definition: Say that a subset $X \subseteq (K^\circ)^n$ is semianalytic if there exist finitely many affinoid subdomains $U_i$ of $(K^\circ)^n$ such that $X \subseteq \cup_{i=1}^s U_i$ and for each $i$, $X \cap U_i$ is a finite union of sets of the form:

$$\{x \in U_i : |f_{ij}(x)| \leq |g_{ij}(x)|, |f'_{ij}(x)| < |g'_{ij}(x)| \text{ for } j = 1, \ldots, r_i\}$$

where $f_{ij}, g_{ij}, f'_{ij}, g'_{ij} \in \mathcal{O}(U_i)$, for $j = 1, \ldots, r_i, i = 1, \ldots, s$.

Considering coordinate projections of semianalytic sets, we may now define a notion of subanalytic sets. The question then is with these definitions, are rigid subanalytic sets closed under basic operations such as taking

complements, closures, and furthermore, do they satisfy tame topological properties analogous to those satisfied by real subanalytic sets, and more generally by sets definable in o-minimal structures. Such properties have indeed been proved by Leonard Lipshitz and Zachary Robinson in a series of remarkable works [Lip93], [Lip88], [LRoob], [LRooa]. In fact, it turns out that the authors construct a larger class of subsets than those described above and are able to prove a remarkable number of analogues of classical results of real subanalytic geometry for this larger class of subsets.

To briefly motivate their definition, consider the $n$-dimensional *open* polydisk $(K^{\circ\circ})^n$, which we may consider as an admissible open subset of the closed disk $\mathrm{Sp}(T_n(K))$. One would like to consider subsets $X \subseteq (K^{\circ\circ})^n$ that are Boolean combinations of sets of the form $\{f, g \in \mathcal{O}((K^{\circ\circ})^n) : |f| \leq |g|\}$. But it turns out that for arbitrary rigid analytic sections $f, g \in \mathcal{O}((K^{\circ\circ})^n)$ may behave rather badly as one approaches the 'boundary' $\max_i |t_i(x)| = 1$. For instance, consider the one-dimensional open disk $U := K^{\circ\circ}$ and consider any power series $f(t) := \sum_{n\geq 0} a_n t^n \in (K^{\circ})[\![t]\!]$, with $\{|a_n|\}_{n\geq 0}$ a strictly increasing sequence converging to 1. Then considering the Newton polygon of $f$, or by using the Weierstrass preparation theorem for Tate algebras, it is easy to see that $f(t) \in \mathcal{O}(U)$ has infinitely many zeroes in $K^{\circ}$. Thus, the zero set $V(f)$ is an infinite discrete subset of $K^{\circ\circ}$. Evidently, if one aims to build a theory of subanalytic sets with strong finiteness properties, one would like to exclude such sets and such functions $f(t)$ from consideration. The rather clever solution offered by Lipshitz is to consider a class of *restricted* power series, $f(\underline{t}) = \sum_{\nu\in\mathbb{N}^n} a_\nu \underline{t}^\nu \in K^{\circ}[\![t]\!]$ where the coefficients $a_\nu$ all lie in a 'quasi-Noetherian B-subring' of $K^{\circ}$. Such rings in particular have the property that the set of absolute values attained by non-zero elements of such rings is discrete in $\mathbb{R}_{>0}$ and thus ruling out the kinds of power series that were mentioned above. For every $m, n \in \mathbb{N}$, Lipshitz and Robinson define a subring $S_{m,n} \subseteq K\{x_1, \ldots, x_m\}[\![\rho_1, \ldots, \rho_n]\!]$ (called the ring of separated power series) such that every element $f \in S_{m,n}$ defines an analytic function on $(K^{\circ})^m \times (K^{\circ\circ})^n$. In [LRooc], Lipshitz and Robinson develop the commutative algebraic properties of $S_{m,n}$; the rings $S_{m,n}$ are Noetherian, Jasobson, UFDs; furthermore, versions of Weierstrass preparation theorems are proved for the rings $S_{m,n}$. The theory of semianalytic and subanalytic subsets is then developed based on the restricted analytic functions of $S_{m,n}$. We refer the reader to Chapter 2 for a brief overview of the theory of rigid subanalytic sets and for a summary of the various finiteness and topological properties that such sets satisfy.

*Results of this thesis*

Having the theory of rigid subanalytic subsets at one's disposal, one could now globalize the notion of subanalytic sets to talk about rigid subanalytic subsets of $X(\mathbb{C}_p)$, for a general algebraic variety $X$ over $\mathbb{C}_p$. A natural question that one may ask is whether the analogues of the algebraization results of Peterzil and Starchenko hold in this setting? The main result of this thesis provides an affirmative answer to this question.

As a first step, we prove a strong version of the Riemann extension theorem.

**Theorem** (A rigid subanalytic Riemann extension theorem, Theorem 2.38)**.** *Let $X = \mathrm{Sp}(A)$ be a reduced, affinoid space over $\mathbb{C}_p$. Let $Y \subseteq X$ be a closed analytic subvariety of $X$ that is everywhere of positive codimension. Then any analytic function $f \in \mathcal{O}_X(X \setminus Y)$ whose graph is a rigid subanalytic subset of $X(\mathbb{C}_p) \times \mathbb{C}_p$ extends to a meromorphic function on all of $X$, i.e. $f \in \mathcal{M}(X)$.*

We also prove an analogue of the Definable Chow theorem in the rigid subanalytic setting. In fact, we prove this in the setting of what we refer to as 'tame structures'. The definition of a tame structure follows very closely the definition of an o-minimal structure. In [LR96], Lipshitz–Robinson prove that rigid subanalytic subsets of the one-dimensional unit disk $K^\circ$ are none other than the subsets that are Boolean combinations of disks. Thus, it is natural to consider arbitrary structures on $K^\circ$ such that the definable subsets of the closed one-dimensional unit disk $K^\circ$ are the Boolean combinations of (open or closed) disks. This is (in an imprecise sense) what we refer to as a 'tame structure'. We then prove some basic results in the dimension theory of tame structures that are needed for the proof of the Definable Chow theorem. The two key results are the invariance of dimension under definable bijections and the Theorem of the Boundary, both of which have been previously proved for rigid subanalytic subsets.

**Proposition** (Invariance of dimension under definable bijections, Proposition 3.16)**.** *Let $X \subseteq (\mathcal{O}_{\mathbb{C}_p})^m$ and $Y \subseteq (\mathcal{O}_{\mathbb{C}_p})^n$ be definable sets (in a fixed tame structure) and $f : X \to Y$ a definable bijection. Then* $\dim(X) = \dim(Y)$.

**Theorem** (Theorem of the Boundary, Theorem 3.18)**.** *Let $X \subseteq (\mathcal{O}_{\mathbb{C}_p})^m$ be a definable set. Then* $\dim(\mathrm{Fr}(X)) < \dim(X)$, *where* $\mathrm{Fr}(X)$ *denotes the frontier of $X$ in $(\mathcal{O}_{\mathbb{C}_p})^m$, that is* $\mathrm{Fr}(X) = \mathrm{cl}_{(\mathcal{O}_{\mathbb{C}_p})^m}(X) \setminus X$.

Next we prove the following theorem which may be viewed as a definable version of a classical theorem of Liouville in complex geometry.

**Proposition** (A non-archimedean definable Liouville's theorem, Theorem 4.5)**.** *Let $X$ be a reduced scheme of finite type over $\mathbb{C}_p$ and denote by $X^{an}$ the rigid analytification of $X$. Let $f \in H^0(X^{an}, \mathcal{O}_{X^{an}})$ be a global rigid analytic function on $X^{an}$ such that the graph of $f$ viewed as a subset of $X(\mathbb{C}_p) \times \mathbb{C}_p$ is definable. Then $f \in H^0(X, \mathcal{O}_X)$.*

Finally, we prove the non-archimedean version of the definable Chow theorem.

**Theorem** (The non-archimedean definable Chow theorem, Corollary 4.13)**.** *Let $V$ be a reduced algebraic variety over $\mathbb{C}_p$, and let $X \subseteq V^{\mathrm{an}}$ be a closed analytic subvariety of the rigid analytic variety $V^{\mathrm{an}}$ associated to $V$, such that $X \subseteq V(\mathbb{C}_p)$ is definable in a tame structure on $\mathbb{C}_p$. Then $X$ is algebraic.*

## 1.4 OVERVIEW OF THE THESIS

In Chapter 2 we provide the reader with some background on the theory of rigid subanalytic sets as developed by Lipshitz and Robinson. This chapter is mostly expository in nature, and the reader may find most of the results presented in the chapter to be proved in the fantastic papers [Lip88], [Lip93], [LR96], [LR00c], [LR00b] and [LR00a]. The only original contribution of this chapter is Section 2.4, where we prove a strong version of the Riemann extension theorem in the rigid subanalytic category.

In Chapter 3 we introduce the notion of tame structures, and proceed to develop some preliminary dimension theory in this context. The theorem of the boundary and the invariance of dimensions under definable bijections are proved here. In Section 3.3, we collect some results on the general dimension theory of rigid analytic varieties that shall be used in the proof of the definable Chow theorem. Most of the results in this section should be well-known to experts, nonetheless complete proofs are provided for lack of a coherent reference.

In Chapter 4, we start by proving the so-called non-archimedean definable Liouville theorem, and then finally proceed to the proof of the non-archimedean definable Chow theorem.

# 2

# BACKGROUND

*Outline of this Chapter*

In this chapter we provide a brief overview of subanalytic geometry in the non-archimedean setting. Analogous to the real case, one would like to define subanalytic sets as sets that are locally described by Boolean combinations of sets of the form $\{\underline{x} : |f(\underline{x})| \leq |g(x)|\}$ where $f, g$ are analytic functions. However, just as in the real case, for such sets to define a reasonable 'tame topology' one must restrict the class of analytic functions. Indeed, analytic functions on the on the open polydisk may behave poorly as one approaches the 'boundary'. In the non-archimedean setting, such a theory has been developed in a series of works by Leonard Lipshitz and Zachary Robinson. In the first part of this Chapter, we summarize some of the main results of their works.

Finally in Section 2.4 we prove a strong version of the Riemann extension theorem for rigid subanalytic sets.

## 2.1 RINGS OF SEPARATED POWER SERIES

Suppose $K$ is an algebraically closed field complete with respect to a non-trivial, non-archimedean absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$. We denote by $K^\circ$ the valuation ring consisting of power bounded elements of $K$, and $K^{\circ\circ}$ denotes the maximal ideal of $K^\circ$ consisting of the topologically nilpotent elements of $K$. We denote by $\widetilde{K} := K^\circ / K^{\circ\circ}$ the residue field of $K$ and $\widetilde{\ }: K^\circ \to \widetilde{K}$ shall denote the reduction map.

**Definition 2.1.** A valued subring $B \subseteq K^\circ$ is called a *B-ring* if every $x \in B$ with $|x| = 1$ is a unit in $B$.

*Remark* 2.2. Every $B$-ring is a local ring with $B \cap K^{\circ\circ}$ being its unique maximal ideal.

**Definition 2.3.** A $B$-ring $B \subseteq K^\circ$ is said to be *quasi-Noetherian* if every ideal $\mathfrak{a} \subseteq B$ has a 'quasi-finite generating set' i.e. a zero-sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq \mathfrak{a}$ such that any element $a \in \mathfrak{a}$ can be written in the form $a = \sum_{i \geq 0} b_i x_i$ for some $b_i \in B$. We note that we are not insisting that every infinite sum of the form $\sum_{i \geq 0} b_i x_i$ also lies in $\mathfrak{a}$.

**Proposition 2.4** (Properties of quasi-Noetherian rings)**.** *We have the following properties of quasi-Noetherian rings:*

1. *A Noetherian B-subring of $K^\circ$ is quasi-Noetherian.*
2. *If $B$ is quasi-Noetherian and $\{a_i\}_{i\in\mathbb{N}} \subseteq K^\circ$ is a zero-sequence then*

$$B[a_0, a_1, \ldots]_{\{a\in B[a_0,a_1,\ldots]:|a|=1\}}$$

   *is also quasi-Noetherian.*
3. *The completion of a quasi-Noetherian subring $B \subseteq K^\circ$ (with respect to the restriction of the absolute value $|\cdot|$ to $B$) is also a quasi-Noetherian subring of $K^\circ$.*
4. *The value semi-group $|B \setminus \{0\}| \subseteq \mathbb{R}_{>0}$ is a discrete subset of $\mathbb{R}_{>0}$.*

**Definition 2.5.** (a) If $R$ is a complete, Hausdorff topological ring whose topology is defined by a system of ideals $\{\mathfrak{a}_i\}_{i\in I}$ we define the *ring of convergent power series* $R\{x_1, \ldots, x_n\}$ with coefficients in $R$ as:

$$R\{x_1, \ldots, x_n\} := \{ \sum_{\nu=(\nu_1,\ldots,\nu_n)\in\mathbb{N}^n} a_\nu x_1^{\nu_1} \ldots x_n^{\nu_n} \in R[\![x_1, \ldots, x_n]\!] : \lim_{\nu_1+\ldots+\nu_n\to\infty} a_\nu = 0\}.$$

   The topology on $R\{x_1, \ldots, x_n\}$ is defined by declaring $\{\mathfrak{a}_i \cdot R\{\underline{x}\}\}_{i\in\mathbb{N}}$ to be a fundamental system of neighbourhoods of 0. With this topology $R\{x_1, \ldots, x_n\}$ is also a complete, Hausdorff topological ring.

(b) The *Tate algebra* $T_m(K)$ *in m-variables over* $K$ is defined as $T_m(K) := K \otimes_{K^\circ} K^\circ\{x_1, \ldots, x_m\}$. We equip $T_m(K)$ with the *Gauss norm* which is defined as follows: $\left\|\sum_{i\geq 0} a_i x^i\right\|_{\text{Gauss}} := \max_i\{|a_i|\}$. The Guass norm is a multiplicative norm on $T_m(K)$ that makes $T_m(K)$ a Banach $K$-algebra.

**Definition 2.6** (Rings of separated power series)**.** We fix a complete, quasi-Noetherian subring $E \subseteq K^\circ$. Denote by $\mathfrak{B}$ the following family of complete, quasi-Noetherian subrings of $K^\circ$:

$$\mathfrak{B} := \{E[a_0, a_1, \ldots]^\wedge_{\{x\in E[a_0,\ldots]:|x|=1\}} : \text{where } \{a_i\}_{i\geq 0} \subseteq K^\circ \text{ satisfies } \lim|a_i| = 0\}.$$

Define:

$$S_{m,n}(E,K)^\circ := \varinjlim_{B\in\mathfrak{B}} B\{x_1,\dots,x_m\}[\![\rho_1,\dots,\rho_n]\!],$$
$$S_{m,n}(E,K) := S_{m,n}(E,K)^\circ \otimes_{K^\circ} K.$$

For an $f \in S_{m,n}(E,K)$ we define its Gauss norm in the usual way; writing $f = \sum_{\mu\in\mathbb{N}^m,\nu\in\mathbb{N}^n} b_{\mu,\nu} x_1^{\mu_1}\cdots x_m^{\mu_n}\rho_1^{\nu_1}\cdots\rho_n^{\nu_n}$ we set $\|f\|_{\text{Gauss}} := \sup_{\mu,\nu}|b_{\mu,\nu}| = \max_{\mu,\nu}|b_{\mu,\nu}|$.

*Remark* 2.7. (a) We call $S_{m,n}(E,K)$ the ring of separated power series over $K$. When $K = \mathbb{C}_p$ for instance, we may choose $E$ to be the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$. We shall often suppress the reference to $E$ and $K$ in the notation for convenience and often refer to $S_{m,n}(E,K)$ as simply $S_{m,n}$.

(b) Note that $S_{m,0} = T_m(K)$ and that $S_{m,n} \supseteq T_{m+n}(K)$.

**Definition 2.8.** A *quasi-affinoid* algebra $A$ is a quotient of the ring of separated power series $A = S_{m,n}/I$. We equip $A$ with the residue norm inherited from the Guass norm on $S_{m,n}$. The residue norms on $A$ arising from distinct presentations of $A$ as quotients of the rings of separated power series are all equivalent to one another (see [LRooc, Corollary 5.2.4]).

**Theorem 2.9** (Lipshitz–Robinson [LRooc])**.** *The rings $S_{m,n}$ have the following properties:*

1. *$S_{m,n}$ is Noetherian, a UFD, and a Jacobson ring of Krull dimension $m+n$.*
2. *For every maximal ideal $\mathfrak{m}$ of $S_{m,n}$ the quotient ring $S_{m,n}/\mathfrak{m}$ is an algebraic extension of $K$. Furthermore, there is a bijection*

$$\{\mathfrak{n}\in \operatorname{Max}(K[\underline{x},\underline{\rho}]) : |x_i(\mathfrak{n})|\le 1, |\rho_j(\mathfrak{n})|<1\} \longleftrightarrow \operatorname{Max}(S_{m,n})$$
$$\mathfrak{n} \longmapsto \mathfrak{n}\cdot S_{m,n}.$$

**Definition 2.10** (Rings of fractions of quasi-affinoid algebras: [LRooc], [LRoob])**.**

(a) Suppose $A$ is a quasi-affinoid algebra and let $f_1,\dots,f_M,g_1,\dots,g_N,h \in A$. Choose a presentation of $A$ as a quotient of a ring of separated power series, say $A = S_{m,n}/I$. Choose lifts $F_1,\dots,F_m,G_1,\dots,G_N,H \in S_{m,n}$ of $f_1,\dots,f_M,g_1,\dots,g_N,h \in A$ respectively so that $f_i = F_i + I, g_j = G_j + I$ and $h = H + I$. We define the *ring of fractions* $A\{\underline{f}/h\}[\![\underline{g}/h]\!]_s$ associated with the elements $f_1,\dots,f_M,g_1,\dots,g_N,h$ of $A$ to be the quotient ring:

$$A\{\underline{f}/h\}[\![\underline{g}/h]\!]_s := S_{m+M,n+N}/J$$

where $J$ is the ideal generated by the image of $I$ and the elements $(H \cdot x_{m+i} - F_i)$ and $(H \cdot \rho_{n+j} - G_j)$ for $i \in \{1, \ldots, M\}, j \in \{1, \ldots, N\}$. We remark that the ring $A\{\underline{f}/h\}[\![\underline{g}/h]\!]_s$ only depends on the quasi-affinoid algebra $A$ and the elements $f_i, g_j, h \in A$ and is independent of the chosen presentation of $A$ as a quotient of an $S_{m,n}$ (see [LRooc, Definition 5.3.1]).

(b) We inductively define the *generalized ring of fractions over* $T_m$ as follows: the identity map $T_m \to T_m$ is a generalized ring of fractions over $T_m$; If $T_m \to A$ is a generalized ring of fractions over $T_m$ then for any $f, g \in A$ we declare $T_m \to A\{f/g\}$ and $T_m \to A[\![f/g]\!]_s$ to also be generlized rings of fractions over $T_m$.

*Remark* 2.11. The natural inclusion $T_{m+n} \hookrightarrow S_{m,n}$ is a generalized ring of fractions over $T_{m+n}$. Indeed, $S_{m,n} = T_{m+n}[\![x_{m+1}, \ldots, x_{m+n}]\!]_s$.

**Definition 2.12** (Quasi-rational subdomains)**.** Let $A$ be a quasi-affinoid algebra and set $X := \mathrm{Max}(A)$. Given elements $f_1, \ldots, f_M, g_1, \ldots, g_N, h \in A$ that generate the unit ideal in $A$ we define the *quasi-rational subdomain* associated to this set of elements to be the subset $U \subseteq \mathrm{Max}(A)$ defined by

$$U := \{\mathfrak{m} \in \mathrm{Max}(A) : |f_i(\mathfrak{m})| \leq |h(\mathfrak{m})|, |g_j(\mathfrak{m})| < |h(\mathfrak{m})|, 1 \leq i \leq M, \\ 1 \leq j \leq N\}.$$

In fact, it turns out that the induced map $\mathrm{Max}(A\{\underline{f}/h\}[\![\underline{g}/h]\!]_s) \to \mathrm{Max}(A)$ is a bijection onto $U = \mathrm{Max}(A\{\underline{f}/h\}[\![\underline{g}/h]\!]_s)$ in $\mathrm{Max}(A)$.

*Remark* 2.13.

(a) Quasi-rational subdomains satisfy a universal property much like the affinoid subdomains of rigid analytic geometry. Namely, suppose $U$ is a quasi-rational subdomain of $\mathrm{Max}(A)$ described as in the above definition. Suppose also that one is given a map $\varphi : A \to B$ of quasi-affinoid $K$-algebras such that the induced map on maximal ideals, $\Phi : \mathrm{Max}(B) \to \mathrm{Max}(A)$ has its image $\Phi(\mathrm{Max}(B))$ contained in $U$. Then there exists a unique map of quasi-affinoid $K$-algebras $\psi : A\{\underline{f}/h\}[\![\underline{g}/h]\!]_s \to B$ such that $\varphi : A \to B$ is the composition of the natural map $i : A \to A\{\underline{f}/h\}[\![\underline{g}/h]\!]_s$ with $\psi$.

(b) From the above remark, we thus see that the quasi-affinoid algebra $A\{\underline{f}/h\}[\![\underline{g}/h]\!]_s$ is uniquely associated to underlying subset $U$. Thus if $U \subseteq \mathrm{Max}(A)$ is a quasi-rational subdomain described as in the above definition we denote the $A$-algebra $A\{\underline{f}/h\}[\![\underline{g}/h]\!]_s$ by $\mathcal{O}(U)$.

(c) Quasi-rational subdomains are clopen with respect to the metric topology on $\mathrm{Max}(A)$.

(d) Finite intersections of quasi-rational subdomains are again quasi-rational.

(e) The complement of a quasi-rational subdomain is a finite union of quasi-rational subdomains.

(f) It is not true that a quasi-rational subdomain $V \subseteq U$ of a quasi-rational subdomain $U \subseteq X$ is quasi-rational in $X$. Nonetheless, such subsets do form an important part ofthe theory and they do satisfy a universal property analogous to that mentioned above for quasi-rational subdomains. This brings us to the definition of an $R$-subdomain.

**Definition 2.14** ($R$-subdomains)**.** For a quasi-affinoid $K$-algebra $A$, we define below when a subset $U \subseteq \mathrm{Max}(A)$ is said to be an $R$-subdomain and simultaneously define its 'ring of functions', $\varphi_U : A \to \mathcal{O}(U)$ where $\varphi$ is a map of quasi-affinoid $K$-algebras such that the induced map on maximal spectra $\Phi_U : \mathrm{Max}(\mathcal{O}(U)) \to \mathrm{Max}(A)$ is a bijection onto $U$. We define these notions simultaneously in an inductive manner. Quasi-rational subdomains of $\mathrm{Max}(A)$ are declared to be $R$-subdomains with their ring of functions $A \to \mathcal{O}(U)$ defined as in Remark 2.13(a). If $U \subseteq \mathrm{Max}(A)$ is an $R$-subdomain of $\mathrm{Max}(A)$ with ring of functions $A \to \mathcal{O}(U)$, and if $V \subseteq U = \mathrm{Max}(\mathcal{O}(U))$ is a quasi-rational subdomain of the quasi-affinoid $\mathrm{Max}(\mathcal{O}(U))$ then we declare $V$ to be an $R$-subdomain of $\mathrm{Max}(A)$ with its ring of functions to be the composition $A \to \mathcal{O}(U) \to \mathcal{O}(V)$.

**Definition 2.15** (Domain of a ring of generalised fraction over a Tate algebra)**.** Suppose $\varphi : T_m \to A$ is a generalised ring of fractions over $A$. Denote the the induced map on maximal spectra by $\Phi : \mathrm{Max}(A) \to \mathrm{Max}(T_m)$. Then we define the *domain of* $\varphi : T_m \to A$, denoted simply $\mathrm{Dom}(A)$, to be the subset of points $x \in \mathrm{Max}(T_m)$ such that there exists a quasi-rational subdomain $U \subseteq \mathrm{Max}(T_m)$ containing $x$ such that $\Phi : \Phi^{-1}(U) \to U$ is a bijection.

*Remark* 2.16. (a) Alternatively, one may define $\mathrm{Dom}(A)$ in an inductive manner using the fact that if $\varphi : T_m \to A$ is a generalised ring of fractions over $T_m$ and $f, g, h \in A$ then

$$\mathrm{Dom}(A\{f/h\}) = \{x \in \mathrm{Dom}(A) : |f(x)| \leq |h(x)| \neq 0\}$$
$$\mathrm{Dom}(A[\![g/h]\!]_s) = \{x \in \mathrm{Dom}(A) : |g(x)| < |h(x)|\}.$$

(b) For a generalized ring of fractions $\varphi : T_m \to A$ over $T_m$ and any complete valued field extension $F$ of $K$ we may interpret $\mathrm{Dom}(A)(F)$ as a subset of $(F^\circ_{\mathrm{alg}})^m$ (see [LRoob, p. 122]). Furthermore, any element $f \in A$, defines a locally analytic function $\overline{f} : \mathrm{Dom}(A)(F) \to F_{\mathrm{alg}}$ described by a locally convergent power series with coefficients in $K$. We also sometimes extend this locally analytic function by zero to its complement in $(F^\circ_{\mathrm{alg}})^m$.

**Definition 2.17.** For a generalised ring of fractions $\varphi : T_m \to A$ over $T_m$ and an element $f \in S$ we denote by $\Delta(f)$ the set of all its partial derivatives $\frac{\partial^{|\nu|} f}{\partial x_1^{\nu_1} \cdots \partial x_m^{\nu_m}}$ for all $\nu \in \mathbb{N}^m$.

## 2.2 RIGID SUBANALYTIC SETS

**Definition 2.18** (The language $L$ of mutiplicatively valued rings)**.** Denote by $L = (+, \cdot, |\cdot|, 0, 1; \tilde{\cdot}, \tilde{<}, \tilde{0}, \tilde{1})$ the language of multiplicatively valued rings. Note that $L$ is a two-sorted language, the operations $+, -, \cdot$ and elements $0, 1$ refer to corresponding operations and elements of the underlying ring and $\tilde{\cdot}, \tilde{0}, \tilde{1}$ are the underlying operations and elements on the value group $\cup\{\tilde{0}\}$.

We set $S := \cup_{m,n\in\mathbb{N}} S_{m,n}(E, K)$ and $T := \cup_{m\geq 0} T_m$. Consider any subset $\mathcal{H} \subseteq S$ such that $\Delta(\mathcal{H}) \subseteq \mathcal{H}$. The two main examples of such $\mathcal{H}$ are provided by $\mathcal{H} = S$ or $\mathcal{H} = T$.

We define now the language $L_{\mathcal{H}}$ introduced by Lipshitz–Robinson [LRoob] which are used to define subanalytic sets. $L_{\mathcal{H}}$ is a three-sorted language; the first sort for the closed unit disk $K^\circ$, the second sort for $K^{\circ\circ}$ the open unit disk and the last sort for the totally ordered value group$\cup\{\tilde{0}\}$. The sort structure is merely a bookkeeping device; the first sort helps us to keep track of non-strict inequalities of the form $|f| \leq |g|$ whereas the second sort helps us to keep track of strict inequalities.

**Definition 2.19** (The language $L_{\mathcal{H}}$)**.** The language $L_{\mathcal{H}}$ is the language obtained by augmenting to the language $L$ defined above, symbols for every function in $\mathcal{H}$; i.e. for every $f \in \mathcal{H}$, if $f \in S_{m,n}$ we add a function symbol to $L_{\mathcal{H}}$ with arity $m$ for the first sort and $n$ for the second sort. Thus,

$$L_{\mathcal{H}} := (+, \cdot, |\cdot|, 0, 1, \{f\}_{f\in\mathcal{H}}; \tilde{\cdot}, \tilde{<}, \tilde{0}, \tilde{1}).$$

**Definition 2.20** (Globally $\mathcal{H}$-semianalytic, locally semianalytic, and $\mathcal{H}$-subanalytic sets)**.**

(a) For a complete, valued field $F$ over $K$, a subset $X \subseteq (F^\circ_{\text{alg}})^m$ is said to be *globally $\mathcal{H}$-semianalytic* (resp. *$\mathcal{H}$-subanalytic*) if $X$ is definable by a quantifier-free (resp. existential) $L_{\mathcal{H}}$-formula, i.e. if there exists a quantifier-free (resp. existential) first-order formula $\phi(x_1, \ldots, x_m)$ such that $(a_1, \ldots, a_m) \in X$ if and only if $F_{\text{alg}} \models \phi(a_1, \ldots, a_m)$.

(b) In the special case that $\mathcal{H} = S$, the $\mathcal{H}$-semianalytic (resp. subanalytic) sets are referred to as the *globally quasi-affinoid semianalytic* (resp. *quasi-affinoid subanalytic*) sets. Similarly, in the case that $\mathcal{H} = T$, the $\mathcal{H}$-semianalytic (resp. subanalytic) sets are referred to as *affinoid semianalytic* (resp. *affinoid subanalytic*) sets.

(c) In the special case that $\mathcal{H} = S$, we denote the language $L_{\mathcal{H}}$ by $L_{\text{an}}$. Furthermore, in this case we also define the language $L^*_{\text{an}}$ as follows. $L^*_{\text{an}}$ is the language where we augment to $L_{\text{an}}$ function symbols for every function $f : \text{Max}(T_m(K)) \to K_{\text{alg}}$ such that there exists a finite cover of $\text{Max}(T_m(K))$ by $R$-subdomains $\text{Max}(T_m(K)) = \cup_{i=1}^{l} U_l$ and functions $f_i \in \mathcal{O}(U_i)$ such that for every $i$, $f|_{U_i}$ agrees with the function represented by $f_i$ on $U_i$. A subset $X \subseteq (F^\circ_{\text{alg}})^m$ is said to be *locally semianalytic* if $X$ is defined by a quantifier-free $L^*_{\text{an}}$-formula.

The globally $\mathcal{H}$-semianalytic sets are in other words Boolean combinations of sets defined by inequalities among the analytic functions in $\mathcal{H}$. Similarly, the $\mathcal{H}$-subanalytic sets, being defined by *existential* formulas are precisely the sets obtained by coordinate projections of $\mathcal{H}$-semianalytic sets from higher dimensions.

Just as in the real subanalytic setting, one would now ask whether subanalytic sets satisfy basic closure properties. For instance, are they closed under taking complements, closures? It turns out that they are. Lipshitz-Robinson [LRoob] prove a remarkable quantifier-simplification theorem for the language $L_{\mathcal{H}}$ (recalled below), which would imply that any *arbitrary* $L_{\mathcal{H}}$-definable set is also $\mathcal{H}$-subanalytic. Since complements and closures are all first-order definable in $\mathcal{L}_{\mathcal{H}}$, the required closure properties would then follow.

Lipshitz and Robinson's proof of the quantifier simplification theorem for $L_{\mathcal{H}}$, is actually obtained as a consequence of a striking quantifier-*elimination* theorem in a slightly expanded language $\mathcal{L}_{\mathcal{E}(\mathcal{H})}$ which we introduce below.

The expanded language $\mathcal{L}_{\mathcal{E}(\mathcal{H})}$, roughly speaking contains function symbols for every function that is *existentially* definable from functions in $\mathcal{H}$ (the precise definitions are given below).

The need to expand our language to include such functions is reflected in the fact that for an $f \in \mathcal{H}$, the Weierstrass data outputted by the Weierstrass division theorems in the context of the algebras $S_{m,n}$ are only existentially definable over $\mathcal{H}$.

We also note that for a generalized ring of fractions $\varphi : T_m \to A$ over $T_m$ and for an element $f \in A$, the induced analytic function $f : \text{Dom}(A)(F) \to F_{\text{alg}}$ might not necessarily be in $\mathcal{H}$ but is nevertheless existentially definable over $\mathcal{H}$.

**Definition 2.21** (Existentially definable analytic functions)**.** [LRoob, Definition 2.6]. Given a complete valued field extension $F$ of $K$, a subset $X \subseteq (F^\circ_{\text{alg}})^m$, and a function $f : X \to F_{\text{alg}}$, we say that $f$ is *existentially*

*definable from the functions* $g_1, \dots, g_l$ if there exists a quantifier-free formula $\phi$ in the language $L$ of multiplicatively valued rings, such that

$$y = f(x) \iff \exists \underline{z},\ \phi(x, y, \underline{z}, g_1(x, y, \underline{z}), \dots, g_l(x, y, \underline{z})).$$

**Definition 2.22** (The expanded language $L_{\mathcal{E}(\mathcal{H})}$)**.**

(a) We set $\mathcal{E}(\mathcal{H})$ to consist of all functions $f : \mathrm{Dom}(A)(F) \to F_{\mathrm{alg}}$ for a generalized ring of fractions $\varphi : T_m \to A$ over $T_m$ and $f \in A$ such that all of its partial derivatives, i.e. all the functions in $\Delta(f)$ are existentially definable from functions in $\mathcal{H}$.

(b) The language $L_{\mathcal{E}(\mathcal{H})}$ is the three-sorted language obtained by augmenting $L_{\mathcal{H}}$ with function symbols for every $f \in \mathcal{E}(\mathcal{H})$.

**Theorem 2.23** (The uniform quantifier elimination theorem of Lipshitz and Robinson [LRoob])**.** *Fix a subset* $\mathcal{H} \subseteq S$ *such that* $\Delta(\mathcal{H}) = \mathcal{H}$*. Let* $\varphi(\underline{x})$ *be an* $L_{\mathcal{E}(\mathcal{H})}$*-formula. Then there exists a quantifier-free* $L_{\mathcal{E}(\mathcal{H})}$*-formula* $\psi(\underline{x})$ *such that for every complete valued field extension* $F$ *of* $K$ *we have that*

$$F_{\mathrm{alg}} \models \left(\forall \underline{x}, \varphi(\underline{x}) \iff \psi(\underline{x})\right).$$

**Corollary 2.24** (Quantifier simplification for $L_{\mathcal{H}}$)**.** *For every* $L_{\mathcal{H}}$*-formula* $\varphi(\underline{x})$*, there exists an* existential $L_{\mathcal{H}}$*-formula* $\psi(\underline{x})$ *such that for every complete valued field* $F$ *extending* $K$ *we have that*

$$F_{\mathrm{alg}} \models \forall \underline{x},\ \varphi(\underline{x}) \iff \psi(\underline{x}).$$

*In other words, every* $L_{\mathcal{H}}$*-definable subset is in fact* $\mathcal{H}$*-subanalytic. In particular, the closures and complements of* $\mathcal{H}$*-subanalytic sets are again* $\mathcal{H}$*-subanalytic.*

## 2.3 TAME PROPERTIES OF SUBANALYTIC SETS

*Dimension theory of subanalytic sets*

In [LRooa] Lipshitz–Robinson develop the dimension theory results for rigid subanalytic sets. Dimension of a subanalytic set is defined in a manner analogous to that in the theory of o-minimality.

**Definition 2.25** (Dimension and local dimensions of subanalytic sets)**.**

(a) The dimension of a non-empty $\mathcal{H}$-subanalytic subset $X \subseteq (F_{\mathrm{alg}}^{\circ})^m$ is the largest integer $n \in \{0, 1, \dots, m\}$ such that there exists some subset $J \subseteq \{0, 1, \dots, m\}$ of size $n$ with the property that if $\pi_J : (F_{\mathrm{alg}}^{\circ})^m \to (F_{\mathrm{alg}}^{\circ})^n$ denotes the projection onto the $n$-coordinates in $J$, then $\pi_J(X)$ has non-empty interior. If $X = \varnothing$, we set $\dim(X) := -\infty$.

(b) For an $\mathcal{H}$-subanalytic subset $X \subseteq (F^\circ_{\text{alg}})^m$ and a point $x \in (F^\circ_{\text{alg}})^m$, we define the local dimension $\dim_x(X)$ of $X$ at $x$ by:

$$\dim_x(X) := \min\{\dim(X \cap U) : U \text{ is an open polydisc containing } x\}$$

*Remark* 2.26. We note that the statement that the dimension of a subanalytic set is $d$, can be expressed by a first-order sentence in $L_{\mathcal{H}}$.Thanks to the uniform quantifier elimination theorem Theorem 2.23 it now follows that the dimension of a subanalytic set only depends on the $L_{\mathcal{H}}$-formula defining it and does not depend on the complete valued field $F$ extending $K$.

**Theorem 2.27** (Results in the dimension theory of rigid subanalytic sets)**.**

*(a)* [LRooa, Lemma 2.3]. *The dimension of an $\mathcal{H}$-subanalytic set $X \subseteq (F^\circ_{\text{alg}})^m$ is the maximum of the local dimensions of $X$ at its points; that is,*

$$\dim(X) = \max_{x \in X} \dim_x(X).$$

*(b)* [LRooa, Lemma 4.2]. *The Krull dimension of the ring $S_{m,n}/I$ is equal to the dimension of the quasi-affinoid subanalytic set $V(I) \subseteq (F^\circ_{\text{alg}})^m$.*

*(c)* [LRooa, Theorem 4.3, 'Theorem of the Boundary']. *For an $\mathcal{H}$-subanalytic subset $X \subseteq (F^\circ_{\text{alg}})^m$ we have that*

$$\dim(\text{cl}(X) \setminus X) < \dim(X)$$

*where $\text{cl}(X)$ denotes the metric closure of $X$ in $(F^\circ_{\text{alg}})^m$.*

*(d)* [LRooa, Theorem 4.4, The Smooth stratification theorem]. *Given an $\mathcal{H}$-subanalytic subset $X \subseteq (F^\circ_{\text{alg}})^m$, there is a partition of $X$ into finitely many $\mathcal{H}$-subanalytic subsets $X = X_1 \cup \ldots \cup X_l$ such that each $X_j$ is an $F_{\text{alg}}$-analytic submanifold of $(F^\circ_{\text{alg}})^m$.*

*Characterisation of subanalytic sets in low dimensions*

**Theorem 2.28.**

*(a)* [LR99, Theorem 1.1]. *Let $V \subseteq (F^\circ_{\text{alg}})^m$ be locally semianalytic such that $\dim(V) \leq 2$ and let $W \subseteq V$ be subanalytic. Then $W$ is locally semianalytic.*

*(b)* [LR99, Theorem 4.6]. *Let $V \subseteq (F^\circ_{\text{alg}})^m$ be semialgebraic, with $\dim(V) \leq 1$, and let $W \subseteq V$ be subanalytic. Then $W$ is semialgebraic.*

*(c)* [LR96, Theorem 1.1] *Let $W \subseteq F^\circ_{\text{alg}}$ be $\mathcal{H}$-subanalytic. Then $W$ is a Boolean combination of open or closed disks.*

*A Łojasiewicz inequality*

We suppose for this subsection that $K$ is algebraically closed as well.

**Theorem 2.29.** [Lip93, Theorem 5.6] *Suppose $F_i(\underline{x}, \underline{\rho}) \in S_{m,n}(E, K)^\circ$, for $i = 1, \ldots, L$ is a finite collection of separated power series over $K$. Let $Z \subseteq (K^\circ)^m \times (K^{\circ\circ})^n$ be the quasi-affinoid subanalytic set given by the vanishing locus of the $F_i(\underline{x}, \underline{\rho})$. Let us also denote by $d((\underline{x}, \underline{\rho}), Z)$ the distance of a point $(\underline{x}, \underline{\rho}) \in (K^\circ)^m \times (K^{\circ\circ})^n$ to the closed subset $Z$.*

*Then there exist $b, c \in \mathbb{R}_{>0}$ such that for every $\alpha \in \mathbb{R}_{>0}$ the following holds: if for all $i = 1, \ldots, L$, $|F_i(\underline{x}, \underline{\rho})| < b\alpha^c$, then $d((\underline{x}, \underline{\rho}), Z) < \alpha$.*

**Corollary 2.30.**

*(a)* [Lip93, Corollary 5.9]. *Suppose $X \subseteq (K^\circ)^m \times (K^{\circ\circ})^n$ is an $\mathcal{H}$-subanalytic subset that is closed in the metric topology and let $f : X \to \mathbb{R}_{\geq 0}$ be a continuous, $\mathcal{H}$-subanalytic function. Suppose that*

$$\inf\{f(\underline{x}, \underline{\rho}) : (\underline{x}, \underline{\rho}) \in X\} = 0.$$

*Then there exists a point $(\underline{y}, \underline{\lambda}) \in X$ with $f(\underline{y}, \underline{\lambda}) = 0$.*

*(b)* [Lip93, Theorem 5.11]. *Suppose $X \subseteq (K^\circ)^m \times (K^{\circ\circ})^n$ is an $\mathcal{H}$-subanalytic subset that is closed in the metric topology and let $f, g : X \to \mathbb{R}_{\geq 0}$ be continuous, $\mathcal{H}$-subanalytic functions. Suppose that $g^{-1}(0) \subseteq f^{-1}(0)$. Then there exist $\alpha > 0, c > 0$ such that for all $x \in X$*

$$|f(x)|^\alpha \leq c|g(x)|.$$

*(c)* [Lip93, Theorem 5.13]. *Let $A, B$ be two closed, $\mathcal{H}$-subanalytic subsets of $(K^\circ)^m \times (K^{\circ\circ})^n$ and suppose that $A \cap B \neq \varnothing$. Then there exists an $\alpha > 0, c > 0$ such that for all $(\underline{x}, \underline{\rho}) \in (K^\circ)^m \times (K^{\circ\circ})^n$*

$$d((\underline{x}, \underline{\rho}), A \cap B)^\alpha < c \max\left(d((\underline{x}, \underline{\rho}), A), d((\underline{x}, \underline{\rho}), B)\right).$$

*Uniform boundedness of fibers*

**Theorem 2.31.** [Lip88, Theorem 2] *Suppose $X \subseteq (K^\circ)^{m+n}$ is an affinoid subanalytic subset. For $\underline{\lambda} \in (K^\circ)^m$ let $X_{\underline{\lambda}} := \{\underline{x} \in (K^\circ)^n : (\underline{\lambda}, \underline{x}) \in X\}$. Then there is a bound $\beta$ depending on $X$ such that for every $\underline{\lambda} \in (K^\circ)^m$, $X_{\underline{\lambda}}$ has at most $\beta$ isolated points. In particular, a quasi-finite subanalytic family of subanalytic sets, is uniformly quasi-finite.*

## 2.4 A RIGID SUBANALYTIC RIEMANN EXTENSION THEOREM

In this section we prove a version of the Riemann extension theorem in the setting of rigid subanalytic sets. We consider this as a starting point for the non-archimedean definable Chow theorem.

Throughout this section and in the sequel, by subanalytic (without further qualification) we shall simply mean quasi-affinoid subanalytic, i.e. $\mathcal{H}$-subanalytic with $\mathcal{H} = S = \cup_{m,n} S_{m,n}(E, K)$. We also assume in this section that $K$ is algebraically closed. We shall denote by $\mathbb{B}^d$ the $d$-dimensional rigid analytic closed unit disk over $K$, that is $\mathbb{B}^d = \mathrm{Sp}(T_d(K))$.

It is convenient to extend the notion of subanalytic sets to subsets of $K^n$. We make the following definition:

**Definition 2.32.** A subset $\mathcal{S} \subseteq K^n$ is said to be a subanalytic subset of $K^n$ if the following equivalent conditions are satisfied:

(i) $\pi_n^{-1}(\mathcal{S}) \subseteq (K^\circ)^{n+1}$ is subanalytic, where

$$\pi_n : (K^\circ)^{n+1} \setminus \{0\} \to \mathbb{P}^n(K^\circ) = \mathbb{P}^n(K)$$

is the map sending $(z_0, z_1, \ldots, z_n) \mapsto [z_0 : z_1 : \ldots : z_n]$.

We view $K^n \subseteq \mathbb{P}^n(K)$ via the map $(z_0, z_1, \ldots, z_{n-1}) \mapsto [z_0 : z_1 : \ldots : z_{n-1} : 1]$.

(ii) For every map $\epsilon : \{1, 2, \ldots, n\} \to \{\pm 1\}$ the set

$$\mathcal{T}_\epsilon := \{(\alpha_1, \ldots, \alpha_n) \in (K^\circ)^n : \text{ if } \epsilon(r) = -1, \alpha_r \neq 0, \text{ and } (\alpha_1^{\epsilon(1)}, \ldots, \alpha_i^{\epsilon(i)}, \ldots, \alpha_n^{\epsilon(n)}) \in \mathcal{S}\}$$

is a subanalytic subset of $(K^\circ)^n$.

It follows that the collection of subanalytic subsets of $K^n$ forms a Boolean algebra of subsets, closed under projections, and moreover forms a structure on $K$ in the sense of [vdD98, Ch 1, (2.1)].

**Definition 2.33.** Let $X$ be a separated rigid analytic variety over $K$ and let $S \subseteq X$ be a subset. Then we say that $S$ is *locally subanalytic* in $X$ if there exists an admissible cover by admissible affinoid opens $X = \cup_i X_i$ and closed immersions $\beta_i : X_i \hookrightarrow \mathbb{B}^{d_i}$ such that for all $i$, $\beta_i(S \cap X_i)$ is subanalytic in $(K^\circ)^{d_i}$.

It is easy to see that if it is true for one admissible affinoid cover and some choice of embeddings $\beta_i$, then it's true for any other such cover and embeddings.

**Definition 2.34.** Let $V/K$ be a finite-type reduced scheme over $K$. We say that a subset $S \subseteq V(K)$ is subanalytic if there exists a finite affine open cover $V = \cup_i U_i = \cup_i \mathrm{Spec}(A_i)$ and closed embeddings $U_i(K) \overset{\beta_i}{\hookrightarrow} K^{n_i}$ (arising from a presentation of $A_i$ as a quotient of $K[t_1, \ldots, t_{n_i}]$) such that for all $i$, $\beta_i(S \cap U_i(K))$ is subanalytic.

*Remark* 2.35. We note that if $S \subseteq V(K)$, is subanalytic, then for *every* finite affine open cover $U_i$ of $V$ and for any choice of presentations $\beta_i : K[t_1, \ldots, t_{n_i}] \twoheadrightarrow \mathcal{O}(U_i)$, we have that $\beta_i(S \cap U_i(K)) \subseteq K^{n_i}$ is subanalytic.

*Remark* 2.36. Suppose $V$ is a separated finite type scheme over $K$ and $V^{\mathrm{an}}$ is the associated rigid analytic variety, with analytification map $a_V : V^{\mathrm{an}} \to V$, then we note that the map $a_V$ need not necessarily take a locally subanalytic set on $V^{\mathrm{an}}$ to a subanalytic set of $V(K)$ in the sense of Definition 2.34. Indeed, if we consider the affine line $\mathbb{A}^1_{\mathbb{C}_p}$, and the subset $S := \cup_{n \geq 0}\{z \in \mathbb{C}_p : |p^{-2n}| \leq |z| \leq |p^{-(2n+1)}|\}$. Then $S$ is not a rigid subanalytic subset of the algebraic affine line $\mathbb{A}^1(\mathbb{C}_p)$ nevertheless it is a locally subanalytic subset of the analytification $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$.

But if $V$ is proper then locally subanalytic sets of $V^{\mathrm{an}}$ are indeed subanalytic in $V(K)$.

**Lemma 2.37.** *Suppose $V$ is a variety over $K$. Let $V^{\mathrm{an}}$ denote the associated rigid analytic space over $K$, with analytification map $a_V : V^{\mathrm{an}} \to V$. Then, if $S \subseteq V(K)$ is subanalytic as in Definition 2.34 then $a_V^{-1}(S) \subseteq V^{\mathrm{an}}$ is locally subanalytic as in Definition 2.33.*

*Moreover, if $V$ is* proper *over $K$ then the converse holds, i.e. $S \subseteq V(K)$ is subanalytic $\iff$ $a_V^{-1}(S) \subseteq V^{\mathrm{an}}$ is locally subanalytic.*

*Proof.* This follows from the fact that proper rigid spaces are quasicompact, and in particular, when $V$ is proper, $V^{\mathrm{an}}$ has an admissible covering by *finitely many* affinoids. □

### THE RIGID SUBANALYTIC RIEMANN EXTENSION THEOREM

We now turn to the proof of the following version of the Riemann extension theorem.

**Theorem 2.38** (The rigid subanalytic Riemann extension theorem)**.** *Suppose $X$ is a separated and reduced rigid analytic space over the algebraically closed field $K$. Let $Y \subseteq X$ be a closed analytic subvariety of $X$ that is everywhere of positive codimension. Then any analytic function $f \in \mathcal{O}_X(X \setminus Y)$ whose graph is a locally subanalytic subset of $X(K) \times K$ extends to a meromorphic function on all of $X$, i.e. $f \in \mathcal{M}(X)$.*

*Outline of the proof*

The proof is inspired by Lütkebohmert's proof of the usual non-archimedean Riemann extension theorem [L̈74]. We make a series of reductions in the course of the proof. We summarize the main reduction steps below.

STEP 1: The question of extending $f$ meromorphically along $X$ is local for the $G$-topology of $X$ and thus we may assume that $X = \mathrm{Sp}(A)$ is a reduced affinoid. Further, working over irreducible components of $X$, we also assume that $X = \mathrm{Sp}(A)$ is irreducible and thus that $A$ is an integral domain.

STEP 2: Choose a Noether normalization $\pi : X \to \mathbb{B}^d$. We show in Lemma 2.43 that if we prove our theorem for $\mathbb{B}^d$ and the analytic subset $\pi(Y) \subseteq \mathbb{B}^d$, we can conclude the theorem for $X$. Thus, we may assume $X = \mathbb{B}^d$ is the $d$-dimensional rigid unit disk over $K$.

STEP 3: Since $\mathrm{Sing}(Y)$ is of codimension at least 2 in $X$, by the non-archimedean Levi extension theorem [L̈74, Theorem 4.1], it suffices to extend $f$ meromorphically to an $f^* \in \mathcal{M}(X \setminus \mathrm{Sing}(Y))$. Replace $X, Y$ by $X \setminus \mathrm{Sing}(Y), Y \setminus \mathrm{Sing}(Y)$ respectively. Once more using Step 1, we reduce to the case where $Y$ is regular/smooth and $X$ is an affinoid subdomain of $\mathbb{B}^d$.

STEP 4: Since $X$ and $Y$ are smooth over the algebraically closed field $K$, we are now in a position to use a result of Kiehl (recalled below, Theorem 2.41) which tells us that *locally* $Y \subseteq X$ looks like $Z \times \{0\} \subseteq Z \times \mathbb{B}^n$ for a smooth affinoid space $Z$. We may even assume that $n = 1$ since if $Y$ is codimension at least 2, the result we seek is a special case of the non-archimedean Levi extension theorem. In all we are down to the case where $X = Z \times \mathbb{B}^1$ and $Y = Z \times \{0\}$ for a smooth, reduced affinoid space $Z$ over $K$.

STEP 5: This final case is proved separately in Lemma 2.44.

We first recall Kiehl's tubular neighbourhod result. We need the following definition.

**Definition 2.39.** ([Kie67, Definition 1.11].) We say that an affinoid algebra $A$ over the non-trivially valued non-archimedean field $k$ is *absolutely regular at a maximal ideal* $x$ of $A$ if for every complete valued field $K$ extending $k$ and for every maximal ideal $y$ of $A\widehat{\otimes}_k K$ above $x$, the localization $(A\widehat{\otimes}_k K)_y$ is a regular local ring. If the affinoid algebra $A$ over $k$ is absolutely regular at every one of its maximal ideals we say that $A$ is absolutely regular.

*Remark* 2.40. For a maximal ideal $x$ of an affinoid algebra $A$ over an *algebraically closed* (or more generally perfect) non-archimedean field $k$, $A$ is absolutely regular at $x$ if and only if the localization $A_x$ is a regular local ring.

**Theorem 2.41.** (Kiehl's tubular neighbourhood theorem, [Kie67, Theorem 1.18]). *Suppose $A$ is an affinoid algebra over a non-trivially valued non-archimedean field $k$ and let $\mathfrak{a}$ be an ideal of $A$ generated by $f_1, \ldots, f_l \in A$. Suppose that the quotient affinoid algebra $A/\mathfrak{a}$ is absolutely regular and that $A$ is absolutely regular at every point of $V(\mathfrak{a})$. Then there exists an $\epsilon \in k^\times$ such that the '$\epsilon$-tube' around $V(\mathfrak{a})$,*

$$\mathrm{Sp}(B) := \{x \in \mathrm{Sp}(A) : |f_j(x)| \leq |\epsilon|, \forall j = 1, \ldots, l\}$$

*has an admissible affinoid covering $(\mathrm{Sp}(B_i) \to \mathrm{Sp}(B)), i = 1, \ldots, r$ along with isomorphisms*

$$\phi_i : (B_i/\mathfrak{a}B_i)\{x_1, \ldots, x_{n_i}\} \xrightarrow{\cong} B_i$$

*from the free affinoid algebra over $B_i/\mathfrak{a}B_i$ in the variables $x_1, \ldots, x_{n_i}$, such that the elements $\phi_i(X_1), \ldots, \phi_i(x_{n_i})$ generate the ideal $\mathfrak{a}B_i$.*

We recall a result on the number of zeroes of a convergent power series in one variable that shall be used in our proof and then prove the Lemma that allows us to make the reduction mentioned in Step 2.

**Lemma 2.42.** *Suppose $f(t) \in K\{t\}$ is an element of the one-dimensional Tate algebra over $K$. Let $\epsilon(f) := \max\{i \geq 0 : |a_i| = \|f\|_{Gauss}\}$. Then the number of zeroes of $f$ (counting multiplicities) in the closed unit disk $K^\circ$ is at least $\epsilon(f)$.*

*Proof.* Note that $f$ is "$t$-distinguished" of degree $\epsilon(f)$ (see [BGR84, §5.2.1, Definition 1]). By the Weierstrass Preparation Theorem for Tate algebras ([BGR84, §5.2.2, Theorem 1]) we may write $f = e \cdot \omega$ where $e \in K\{t\}^\times$ and $\omega \in K[t]$ is a polynomial of degree $\epsilon(f)$, and $\omega$ has $\epsilon(f)$ zeroes (counting multiplicities) in $K^\circ$. □

**Lemma 2.43.** *Let $\pi : X \to S$ be a finite morphism of reduced, irreducible affinoids over $K$ of the same dimension. Suppose $Y \subseteq X$ is a closed analytic subvariety of $X$. Let $T := \pi(Y)$. Suppose that every analytic function $g \in \mathcal{O}_S(S \setminus T)$ extends uniquely to a meromorphic function $g^* \in \mathcal{M}(S)$ on $S$. Then every analytic function $f \in \mathcal{O}_X(X \setminus Y)$ extends to a meromorphic function on $X$.*

*Proof.* Following the proof of [Lütkebohmert74, Satz 1.7], we see that for any $f \in \mathcal{O}_X(X \setminus Y)$ there is a meromorphic $f^* \in \mathcal{M}(X)$ such that $f^*|_{X \setminus \pi^{-1}(T)} = f|_{X \setminus \pi^{-1}(T)}$. However, two meromorphic functions that agree on the complement of a positive codimensional analytic subvariety must agree everywhere (see [Lütkebohmert74, Lemma 1.1]). It thus follows that $f^*|_{X \setminus Y} = f$. □

We are now equipped to fully prove Theorem 2.38.

PROOF OF THE RIGID SUBANALYTIC RIEMANN EXTENSION THEOREM, THEOREM 2.38.

*Proof.* We first reduce to the case where $X$ is a reduced affinoid space over $K$. Indeed, to make this reduction consider an admissible covering of $X$ by affinoid subdomains $X = \cup_{i\in I} U_i$. For each $i \in I$, $U_i \cap Y$ is an analytic subvariety of $U_i$ of positive codimension at every point of $U_i$ and furthermore $f|_{U_i\setminus Y}$ has a subanalytic graph in $U_i(K) \times K$. Suppose that we were able to find for every $i$, meromorphic functions $f_i^* \in \mathcal{M}(U_i)$ such that $f_i^*|_{U_i\setminus Y} = f|_{U_i\setminus Y}$. Then we note that for any $i, j \in I$ the meromorphic functions $f_i^*|_{U_i\cap U_j}$ and $f_j^*|_{U_i\cap U_j}$ agree on the complement of the positive codimensional subvariety $Y \cap U_i \cap U_j$ and hence by [Lü74, Lemma 1.1], agree on $U_i \cap U_j$. Thus, the $\{f_i^*\}_{i\in I}$ glue to a global meromorphic function $f^*$ on $X$ extending $f$. We may thus assume henceforth that $X = \mathrm{Sp}(A)$ is a reduced affinoid space over $K$.

By working on the irreducible components of $X$ we may assume that $X$ is irreducible, and hence that $A$ is an integral domain. Choose a Noether normalization for $X$, i.e. a finite surjective morphism $\pi : X \to \mathbb{B}^d$ where $d = \dim(X)$, and with the help of Lemma 2.43 we further assume that $X = \mathbb{B}^d$ is the $d$-dimensional unit disk over $K$.

Let $f \in \mathcal{O}(\mathbb{B}^d \setminus Y)$ be an analytic function such that its graph is subanalytic. In order to show that $f$ extends meromorphically to $X$, it suffices to show that $f$ extends to an $f^* \in \mathcal{M}(\mathbb{B}^d \setminus \mathrm{Sing}(Y))$ such that $f^*|_{\mathbb{B}^d\setminus Y} = f$. Indeed, since $\mathrm{Sing}(Y)$ is an analytic subset of codimension at least 2 in $\mathbb{B}^d$, we have an isomorphism $\mathcal{M}(\mathbb{B}^d) \xrightarrow{\cong} \mathcal{M}(\mathbb{B}^d \setminus \mathrm{Sing}(Y))$ by the non-archimedean Levi extension theorem [Lü74, Theorem 4.1]. Consider an admissible affinoid covering $\mathbb{B}^d \setminus \mathrm{Sing}(Y) = \cup_i U_i$. We remark that affinoid subdomains being finite unions of rational subdomains are indeed rigid subanalytic sets and hence $f|_{U_i\setminus Y}$ has a subanalytic graph. Using [Lü74, Lemma 1.1], we may work individually over each $U_i$, i.e. we are reduced to proving the theorem in the situation where $X = \mathrm{Sp}(A)$ is an affinoid subdomain of $\mathbb{B}^d$ and $Y \subseteq X$ is a *regular* analytic subvariety of $X$ of positive codimension everywhere.

Applying the 'tubular neighbourhood' result of Kiehl [Kie67, Theorem 1.18], we obtain an admissible covering $(\mathrm{Sp}(B_i) \to \mathrm{Sp}(B)), i = 1, \ldots, l$ of some '$\epsilon$-tube' $\mathrm{Sp}(B)$ around $Y$ in $X = \mathrm{Sp}(A)$. It suffices now to prove that for every $i = 1, \ldots, l$, $f|_{\mathrm{Sp}(B_i)\setminus Y}$ extends to a meromorphic function $f_i^* \in \mathcal{M}(\mathrm{Sp}(B_i))$. Indeed, the $f_i^*$ must necessarily glue to a meromorphic function $f^* \in \mathcal{M}(\mathrm{Sp}(B))$ (using [Lü74, Lemma 1.1]) such that $f^*|_{\mathrm{Sp}(B)\setminus Y} =$

$f|_{\mathrm{Sp}(B)\setminus Y}$. Since the functions $f^* \in \mathcal{M}(\mathrm{Sp}(B))$ and $f \in \mathcal{O}(X \setminus Y)$ agree on the intersection $\mathrm{Sp}(B) \setminus Y$ and noting that $\mathrm{Sp}(B) \cup (X \setminus Y) = X$ is an *admissible* open cover of $X = \mathrm{Sp}(A)$, the sections $f^*$ and $f$ glue to a global meromorphic function on $X$.

We are thus reduced to proving the theorem in the case that $X = \mathrm{Sp}(B_i/\mathfrak{a}B_i) \times \mathbb{B}^{n_i}$ and $Y = \mathrm{Sp}(B_i/\mathfrak{a}B_i) \times \{\underline{0}\}$. If $n_i \geq 2$, then the codimension of $Y$ is at least 2, and in this case the theorem follows as a special case of the non-archimedean Levi extension theorem [Lütkebohmert74, Theorem 4.1]. Thus we may even assume that $n_i = 1$. In all, we are reduced to proving the following special case of the theorem in Lemma 2.44. □

**Lemma 2.44.** *Suppose $Z = \mathrm{Sp}(A)$ is a reduced, irreducible affinoid space, $X = Z \times \mathbb{B}^1$, and $Y = Z \times \{0\} \subseteq X$. Then every analytic function $f \in \mathcal{O}(X \setminus Y)$ whose graph is a subanalytic subset of $X \times K$ extends to a meromorphic function $f^* \in \mathcal{M}(X)$.*

*Proof.* We denote the coordinate on $\mathbb{B}^1$ as $t$. Let $|\cdot|$ represent the supremum norm on the reduced affinoid $A$. Note that $A$ is a Banach algebra over $K$ when endowed with its supremum norm and the supremum norm is equivalent to any residue norm on $A$ (see [BGR84, §6.2.4, Theorem 1]).

We may expand $f(t) = \sum_{i\geq 0} a_i t^i + \sum_{j>0} b_j t^{-j}$ with $a_i, b_j \in A$, such that $\lim_i |a_i| = 0$ and for every $R > 0, \lim_j |b_j| R^j = 0$. Since $\sum_i a_i t^i \in A\{t\}$, we have that $\sum_i a_i t^i$ is a rigid subanalytic function on $Z \times \mathbb{B}^1$, and thus the function $g := \sum_{j>0} b_j t^{-j}$ defined on $X \setminus Y$ has a graph that is a subanalytic subset of $X \times K$. In particular, for each $z \in Z$, the function $g(z,t) = \sum_{j>0} b_j(z) t^{-j}$ on the punctured disc $\mathbb{B}^1 \setminus \{0\}$ is also subanalytic. Since discrete subanalytic sets must be finite, we get that for each fixed $z$ either $g(z,t)$ is identically zero on $\mathbb{B}^1 \setminus \{0\}$ or $g(z,t)$ has finitely many zeroes in $\mathbb{B}^1 \setminus \{0\}$.

Consider $h_z(y) := g(z, y^{-1}) = \sum_{j>0} b_j(z) y^j$. The growth hypothesis $\forall R \in K^\times, \lim_{j\to 0} |b_j||R|^j = 0$ on the $b_j$ implies that $h_z(y) \in K\{R^{-1}y\}$ for every $R \in K^\times$. The number of zeroes of $g(z,t)$ on the annulus $|R|^{-1} \leq |t| \leq 1$ is the number of zeroes of $h_z(y)$ on $1 \leq |y| \leq |R|$. For each $R \in K^\times$, we set $h_{z,R}(y) := \sum_{j>0}(b_j R^{-j}) y^j$ so that by Lemma 2.42 the number of zeroes of $h_z(y)$ on the closed disk $|y| \leq R$, is given by $\epsilon(h_{z,R}(y))$.

Now for $i < j$ if $b_i(z), b_j(z) \neq 0$, then for $R$ large enough $|b_i(z)|R^i \leq |b_j(z)|R^j$ and thus $\epsilon(h_{z,R}) \geq j$. Thus, if $b_j(z) \neq 0$ for infinitely many $j$, $h_z(y)$ has infinitely many zeroes going off to $\infty$ and therefore also $g(z,t)$ has an infinite discrete zero set in $\mathbb{B}^1 \setminus \{0\}$ which as noted above is not possible. Thus, for each $z \in Z$, $b_j(z)$ is eventually 0. In other words, $Z = \cup_{m\geq 0} \cap_{j>m} V(b_j)$. If the set $\cap_{j>m} V(b_j)$ is not equal to $Z$ then it is a nowhere dense closed subset of $Z$. By the Baire category theorem, $Z$ cannot be a countable union of nowhere dense closed subsets and therefore for

large enough $m$, $\cap_{j>m} V(b_j) = V(\sum_{j>m}(b_j))$ must be equal to $Z$. Since $Z$ is reduced, this means that the $b_j \in A$ are eventually zero. Thus $f$ has a finite order pole along $Y$ and hence extends meromorphically. This completes the proof of the Lemma and thus also of Theorem 2.38. □

# 3

# TAME STRUCTURES

*Outline of this chapter*

In this chapter we introduce the notion of a tame structure. The definition of a tame structure closely follows the definition of an o-minimal structure on $\mathbb{R}$ and is suitably adapted as a generalization of the non-archimedean rigid subanalytic sets discussed in the previous chapter. Let $K$ be a non-trivially valued non-archimedean field with valuation ring $R$ and totally ordered value group $(\Gamma, <)$. A 'structure' on $R$, is going to be a collection of subsets of $R^n$ for every $n \geq 0$. In fact, it turns out to be convenient to keep track of definable subsets of the value group $\Gamma$ as well. Thus, in this setting a 'structure' on $R$, is actually a collection of subsets of $R^m \times \Gamma^n$ for $m, n \geq 0$ that are closed under the natural first-order operations (see Definition 3.1 for the precise conditions). A 'tame structure' is then defined to be one where the definable subsets of $R$ are precisely the Boolean combinations of disks of $R$. In Section 3.1, we provide these preliminary definitions and prove some elementary properties of sets definable in tame structures.

In Section 3.2, we develop the basic dimension theory of sets definable in tame structures. As in the o-minimal setting, the dimension of a non-empty definable set $X \subseteq R^m$ is defined as the largest $d \leq m$ such that for some coordinate projection $\pi : R^m \to R^d$ we have that the interior of $\pi(X)$ in $R^d$ is non-empty. The two key results we prove in this section are:

- the invariance of dimension under definable bijections (Proposition 3.16) and
- the Theorem of the Boundary, Theorem 3.18 which states that for a definable set $X \subseteq R^m$, $\dim(\mathrm{cl}(X) \setminus X) < \dim(X)$.

The purpose of Section 3.3 is to collect together some results in the dimension theory of rigid geometry that are needed for the sequel. Most importantly, we connect the usual notion of dimension in the rigid analytic setting with the concept of definable dimension of the previous section (Lemma 3.22). We also prove in Lemma 3.24, a result on the dimensions of local rings of equidimensional rigid varieties. This lemma is used in the course of the proof of the definable Chow theorem.

## 3.1 PRELIMINARIES

*Notations and conventions for this section*

For a subset $X$ of a topological space $Y$ endowed with the subspace topology, the interior, closure, and frontier of $X$ inside $Y$ are denoted by $\mathrm{int}_Y(X), \mathrm{cl}_Y(X)$ and $\mathrm{Fr}_Y(X)$ respectively. We often omit writing the subscript $Y$ when the ambient topological space is clear from the context. We recall that the frontier of $X$ in $Y$ is defined as $\mathrm{Fr}_Y(X) := \mathrm{cl}_Y(X) \setminus X$.

$K$ denotes a field complete with respect to a non-trivial non-archimedean absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$. $R$ denotes the valuation ring of $K$, $\Gamma^\times := |K|$ the value group of $K$, and $\Gamma := \Gamma^\times \cup \{0\}$. We choose a pseudo-uniformizer $\varpi \in K^\times$, i.e. a non-zero element $\varpi \in R$ with $|\varpi| < 1$.

For an element $\underline{x} = (x_1, \ldots, x_n) \in K^n$, we set $\|\underline{x}\| := \max_{1 \leq i \leq n} |x_i|$. For $\underline{x} = (x_1, \ldots, x_n) \in K^n$ and $\underline{r} = (r_1, \ldots, r_n) \in \Gamma^n$, denote by $\mathbb{D}(\underline{x}; \underline{r}) := \{\underline{y} = (y_1, \ldots, y_n) \in K^n : |x_i - y_i| < r_i \text{ for all } i\}$ and let $\overline{\mathbb{D}}(\underline{x}; \underline{r}) := \{\underline{y} \in K^n : |x_i - y_i| \leq r_i \text{ for all } i\}$. The set $\mathbb{D}(\underline{x}, \underline{r})$ is referred to as an *open polydisk (or simply open disk) of poyradius* $\underline{r}$ and $\overline{\mathbb{D}}(\underline{x}, \underline{r})$ as the *closed polydisk/disk of polyradius* $\underline{r}$.

*We shall assume from now on that $K$ is second countable, i.e. that $K$ has a countable dense subset.* However, when working with the collection of $\mathcal{H}$-subanalytic sets, the hypothesis of second countability can be eliminated from most statements(see Remark 3.19).

*Tame structures*

**Definition 3.1.** A *structure* on $(R, \Gamma)$ is a collection $(\mathfrak{S}_{m,n})_{m,n \geq 0}$ where each $\mathfrak{S}_{m,n}$ is a collection of subsets of $R^m \times \Gamma^n$ with the following properties:

(i) $\mathfrak{S}_{m,n}$ is a Boolean algebra of subsets of $R^m \times \Gamma^n$

(ii) If $\mathcal{S} \in \mathfrak{S}_{m,n}$ then $R \times \mathcal{S} \in \mathfrak{S}_{m+1,n}$ and $\mathcal{S} \times \Gamma \in \mathfrak{S}_{m,n+1}$.

(iii) The diagonal $\{(x, x) : x \in R\} \in \mathfrak{S}_{2,0}$, and similarly $\{(\alpha, \alpha) \in \Gamma^2 : \alpha \in \Gamma\} \in \mathfrak{S}_{0,2}$.

(iv) If $\mathcal{S} \in \mathfrak{S}_{m,n}$ then $\mathrm{pr}(\mathcal{S}) \in \mathfrak{S}_{m-1,n}$ and $\mathrm{pr}'(\mathcal{S}) \in \mathfrak{S}_{m,n-1}$, where $\mathrm{pr} : R^m \times \Gamma^n \to R^{m-1} \times \Gamma^n$ denotes the projection forgetting the last $R$ factor and similarly $\mathrm{pr}' : R^m \times \Gamma^n \to R^m \times \Gamma^{n-1}$ denotes the projection omitting the last $\Gamma$ factor.

**Definition 3.2.** We say that a structure $(\mathfrak{S}_{m,n})_{m,n \geq 0}$ on $(R, \Gamma)$ is *tame* if

- $+, \cdot : R^2 \to R$ are definable i.e. their graphs are in $\mathfrak{S}_{3,0}$.
- $|\cdot| : R \to \Gamma$ is definable i.e. its graph $\{(x, |x|) : x \in R\} \subseteq R \times \Gamma$ is in $\mathfrak{S}_{1,1}$
- $\mathfrak{S}_{0,1}$ is the collection of finite unions of (open) intervals and points in the totally ordered abelian group $\Gamma$

- $\mathfrak{S}_{1,0}$ is the collection of subsets of $R$ consisting of the Boolean combination of disks (open or closed).

*Remark* 3.3. It follows from the axioms that in a tame structure $(R, \Gamma)$, the ordering on $\Gamma$ is also definable, i.e. the set $\{(\lambda, \mu) \in \Gamma^2 : \lambda < \mu\}$ is in $\mathfrak{S}_{0,2}$.

For the remainder of this section, we fix a tame structure on $(R, \Gamma)$, and definability of sets and maps will be in reference to this fixed structure.

**Example 3.4** (Rigid subanalytic sets)**.** Suppose $K$ is algebraically closed. For such a $K$, the central example of a tame structure shall be those of the rigid subanalytic subsets of Lipshitz [LRoob] and the $\mathcal{H}$-subanalytic sets defined in [LRoob]. Indeed, it is proved in [LR96], that the subanalytic subsets of $R$ are exactly the Boolean combinations of disks.

For the sequel it shall also be convenient to talk about definable subsets of $K^n$. We make the following definition:

**Definition 3.5.** We say that a subset $\mathcal{S} \subseteq K^n$ is a definable subset of $K^n$ if the following equivalent conditions are satisfied:

(i) $\pi_n^{-1}(\mathcal{S}) \subseteq R^{n+1}$ is definable, where

$$\pi_n : R^{n+1} \setminus \{0\} \to \mathbb{P}^n(R) = \mathbb{P}^n(K)$$

is the map sending $(z_0, z_1, \ldots, z_n) \mapsto [z_0 : z_1 : \ldots : z_n]$.

We view $K^n \subseteq \mathbb{P}^n(K)$ via the map $(z_0, z_1, \ldots, z_{n-1}) \mapsto [z_0 : z_1 : \ldots : z_{n-1} : 1]$.

(ii) For every map $\epsilon : \{1, 2, \ldots, n\} \to \{\pm 1\}$ the set

$$\begin{aligned}\mathcal{T}_\epsilon := \{(\alpha_1, \ldots, \alpha_n) \in R^n : \text{ if } \epsilon(r) = -1, \alpha_r \neq 0,\\ \text{and } (\alpha_1^{\epsilon(1)}, \ldots, \alpha_i^{\epsilon(i)}, \ldots, \alpha_n^{\epsilon(n)}) \in \mathcal{S}\}\end{aligned}$$

is a definable subset of $R^n$.

It follows that the collection of definable subsets of $K^n$ form a Boolean algebra of subsets, closed under projections, and moreover forms a structure on $K$ in the sense of [vdD98, Ch 1, (2.1)].

**Lemma 3.6** (Basic Properties of definable sets and functions)**.** *(i) A polynomial map $\phi : K^n \to K^m$ is definable (i.e. its graph is a definable subset of $K^{n+m}$). In particular, zero sets of polynomials with K-coefficients are definable subsets of $K^n$.*

*(ii) For definable functions $f, g : K^n \to K$, the set $\{\underline{\mathbf{z}} \in K^n : |f(\underline{\mathbf{z}})| \leq |g(\underline{\mathbf{z}})|\}$ is a definable subset of $K^n$.*

(iii) *For a definable function $f : S \to K$ on a definable subset $S \subseteq K^m$, we have that $|f(S)| \subseteq \Gamma$ is a finite union of open intervals and points.*

(iv) *Suppose $f : K^n \to K$ is a definable function then the partial derivative $\frac{\partial f}{\partial z_i}$ exists on a definable subset $\Delta_i \subseteq K^n$ and defines a definable function $\Delta_i \to K$.*

*Proof.* All of these facts follow from the definition of a tame structure. We note in particular that $+, \cdot : K^2 \to K$ and $|\cdot| : K \to \Gamma$ are definable, and that subsets defined by a first-order formula involving definable sets and definable functions must themselves be definable. □

**Definition 3.7.** Let $V/K$ be a finite-type reduced scheme over $K$. We say that a subset $S \subseteq V(K)$ is definable if there exists a finite affine open cover $V = \cup_i U_i = \cup_i \mathrm{Spec}(A_i)$ and closed embeddings $U_i(K) \overset{\beta_i}{\hookrightarrow} K^{n_i}$ (arising from a presentation of $A_i$ as a quotient of $K[t_1, \ldots, t_{n_i}]$) such that for all $i$, $\beta_i(S \cap U_i(K))$ is definable.

*Remark* 3.8. We note that if $S \subseteq V(K)$, is definable, then for *every* finite affine open cover $U_i$ of $V$ and for any choice of presentations $\beta_i : K[t_1, \ldots, t_{n_i}] \twoheadrightarrow \mathcal{O}(U_i)$, we have that $\beta_i(S \cap U_i(K)) \subseteq K^{n_i}$ is definable.

## 3.2 DIMENSION THEORY OF TAME STRUCTURES

Parallel to the notion of definable dimension in o-minimality, in this section, we shall develop the basic dimension theory in the context of tame structures. In particular, we prove the so-called 'Theorem of the Boundary' (Theorem 3.18), which shall be an important input in the proof of the Definable Chow theorem.

For this section, we shall retain the Notations and Conventions introduced in the previous section. We note that our field $K$ is assumed to be second-countable. Throughout this section, we fix a tame structure on $(R, \Gamma)$ and definability will be with regards to the fixed structure.

We recall the following definition from [LRooa, Definition 2.1]:

**Definition 3.9.** (a) For any subset $X \subseteq K^m$, we define its dimension, denoted $\dim(X)$ as the largest non-negative integer $d \leq m$ such that there exists a collection of $d$ coordinates $I \subseteq \{1, \ldots, m\}$ (with $|I| = d$) such that if $\mathrm{pr}_I : K^m \to K^d$ denotes the projection to these coordinates, the image $\mathrm{pr}_I(X)$ of $X$ is a subset of $K^d$ with non-empty interior.

(b) For a subset $X \subseteq K^m$ and a point $x \in X$, the local dimension of $X$ at $x$, denoted $\dim_x(X)$ is defined by:

$$\dim_x(X) := \min\{\dim(U \cap X) : U \subseteq K^m \text{ is an open containing } x\}$$

**Lemma 3.10.** *If $X \subseteq R^m$ is definable, then one of $X$ or its complement $X^c$ contains a non-empty open disk of $R^m$.*

*Proof.* Induct on $m$. For $m = 0, 1$ this is clear. Let $m \geq 2$ and suppose $X \subseteq R^m$ is definable. Consider the projection to the first coordinate $\mathrm{pr} : R^m \to R$. For a point $s \in R$, and for a set $Y \subseteq R^m$, we denote by $Y_s \subseteq R^{m-1}$ the set $\mathrm{pr}^{-1}(s) \cap Y = \left(\{s\} \times R^{m-1}\right) \cap Y$ viewed as a subset of $R^{m-1}$. Consider the following two sets:

$$S_1 := \{s \in R : X_s \text{ contains a non-empty disk of } R^{m-1}\}$$
$$S_2 := \{s \in R : (X^c)_s \text{ contains a non-empty disk of } R^{m-1}\}.$$

$S_1, S_2$ are definable. Also, for every fixed $s \in R$, $R^{m-1} = X_s \cup (X^c)_s$. Thus, by the inductive hypothesis, for every $s$ one of $X_s$ or $X_s^c$ must contain a non-empty disk of $R^{m-1}$, i.e. $R = S_1 \cup S_2$. By the $m = 1$ case, one of $S_1$ or $S_2$ contains a non-empty disk. Replacing $X$ by $X^c$ if necessary, we may assume without loss of generality, that $S_1$ contains a non-empty open 1-dimensional disk $D \subseteq S_1 \subseteq R$. Recall that $R$ is assumed to be second countable. Let $\{D_i \subseteq R^{m-1} : i \geq 1\}$ be a countable collection of non-empty open disks in $R^{m-1}$, forming a basis of the metric topology of $R^{m-1}$. For each $i$ define

$$T_i := \{s \in D : X_s \supseteq D_i\}$$

We have that $\cup_i T_i = D \subseteq R$. Since definable subsets of $R$ are Boolean combinations of disks, either $T_i$ is finite or $T_i$ has non-empty interior. $K$ being complete, is uncountable and hence, there is some $i$ such that $T_i$ contains a non-empty open disk of $R$. Say that $T_1$ contains a non-empty open disk $D' \subseteq T_1$. Then $D' \times D_1 \subseteq X$, i.e. $X$ contains an $m$-dimensional disk. □

**Corollary 3.11.** *For a definable set $X \subseteq R^m$, we have*

$$\mathrm{int}(X) = \varnothing \iff \mathrm{int}(\mathrm{cl}(X)) = \varnothing.$$

*Proof.* Suppose $\mathrm{int}(X) = \varnothing$, however the closure $\mathrm{cl}(X)$ has non-empty interior. Let $\overline{D} \subseteq \mathrm{cl}(X)$ be a closed disk of $R^m$, with positive radius. Then $\overline{D}$ is definably homeomorphic to $R^m$ (by scaling the coordinates). So we may apply the above Lemma 3.10 to definable subsets of $\overline{D}$. In particular, since $X \cap \overline{D}$, has empty interior, by the Lemma $X^c \cap \overline{D}$ contains a non-empty open disk, i.e. $\mathrm{cl}(X) \setminus X$ has non-empty interior, which is impossible. □

**Corollary 3.12.** *1. Suppose* $\bigcup_{i=1}^{\infty} X_i = R^m$ *for a countable collection of definable subsets* $X_i$*. Then, there is some* $i \geq 1$ *such that* $\text{int}(X_i) \neq \varnothing$.

*2. For a countable collection* $\{X_i\}_{i\geq 1}$ *of definable subsets of* $R^m$*, we have*

$$\dim(\cup_i X_i) = \max(\dim(X_i)).$$

*Proof.* Follows from the Baire Category theorem and Corollary 3.11. □

**Corollary 3.13.** *For a definable set* $X \subseteq R^m$*, we have* $\dim(X) = \dim(\text{cl}(X))$.

*Proof.* Follows from Corollary 3.11. □

**Lemma 3.14.** *Let* $f : R^m \hookrightarrow R^n$ *be an injective definable map. Then*

$$\dim(f(R^m)) \geq m.$$

*Proof.* Induct on $m$.

If $m = 1$, then since $f(R^1)$ is infinite, there must necessarily be some coordinate projection $\text{pr} : R^n \to R$ such that the image $\text{pr}(f(R^1))$ is infinite. By the tameness axiom, infinite definable sets of $R$ contain non-empty disks. So $\dim(f(R)) \geq 1$.

Now suppose $m \geq 2$. For any $y \in R$, denote by $L_y$ the $(m-1)$-dimensional line, $L_y := R^{m-1} \times \{y\}$. By the inductive hypothesis, we have that $\dim(f(L_y)) \geq (m-1)$. So there exists a choice of $(m-1)$-coordinates (depending on $y$) of $R^n$, such that the projection of $f(L_y)$ to those corresponding coordinates has non-empty interior. For each choice of $I = (i(1), \ldots, i(m-1))$ with $1 \leq i(1) < \ldots < i(m-1) \leq n$, let $\pi_I : R^n \to R^{m-1}$ denote the corresponding projection. Let $T_I := \{y \in R : \pi_I(f(L_y))$ contains a non-empty disk $\}$. Then $\cup_I T_I = R$. Hence, there is a choice of $I$ such that $T_I$ contains a closed disk of positive radius say $D$. Replacing $R^m$ by $R^{m-1} \times D$, (and rearranging coordinates if necessary) we may assume that for all $y \in R$, $\pi(f(L_y))$ contains a non-empty open disk of $R^{m-1}$ where $\pi : R^n \to R^{m-1}$ is the projection to the first $(m-1)$-coordinates. Enumerate a countable basis $\{B_i : i \geq 1\}$ of non-empty open disks of $R^{m-1}$. Let $\Lambda_i := \{y \in R : \pi(f(L_y)) \supseteq B_i\}$. Then $\Lambda_i$ is definable in $R$ and $\cup_{i\geq 1}\Lambda_i = R$. So there is some $i$ such that $\Lambda_i$ contains a closed disk, say $D'$, of positive radius. Again replacing $R^m$ by $R^{m-1} \times D'$ we may assume that for all $y \in R$, $\pi(f(L_y)) \supseteq B_0$ where $B_0 \subseteq R^{m-1}$ is some fixed non-empty open disk of $R^{m-1}$. Let $X := f(R^m) \subseteq R^n$. For every $b \in B_0$, and for every $y \in R$, we have that $X_b \cap f(L_y) \neq \varnothing$. Thus for every $b \in B_0$, the set $X_b$ is infinite and so some projection of $X_b$ to the remaining $(n-(m-1))$-coordinates must be infinite and hence contains a non-empty open 1-dimensional disk. For each of these remaining coordinates, $j \in \{m, \ldots, n\}$

let $S_j := \{b \in B_0 : \mathrm{pr}_j(X_b)$ contains a non-empty open disk$\}$. Since $\cup_j S_j = B_0$, by Corollary 3.12 some $S_j$ contains a non-empty open disk. Shrinking $B_0$ further to this smaller disk and rearranging the coordinates if necessary, we may assume that for all $b \in B_0, \mathrm{pr}_m(X_b)$ contains a non-empty open disk of $R$. Enumerate the disks of $R$, i.e. let $\{C_i : i \geq 1\}$ be a countable basis of non-empty open disks of $R$. Let $\Gamma_i := \{b \in B_0 : \mathrm{pr}_m(X_b) \supseteq C_i\}$. Then $\Gamma_i$ are definable and $\cup_i \Gamma_i = B_0$. By Corollary 3.12, we have an $i$ such that $\Gamma_i$ contains a non-empty open disk say $B' \subseteq \Gamma_i$. Then $\mathrm{pr}_{(1,\ldots,m)}(X) \supseteq B' \times C_i$, and therefore $\dim(f(R^m)) = \dim(X) \geq m$. □

**Lemma 3.15.** *Let $X \subseteq R^m$ be definable. Let $d \leq m$, and let $\mathrm{pr}_{(1,\ldots,d)} : R^m \to R^d$ denote the projection to the first $d$-coordinates. Suppose that $\mathrm{pr}_{(1,\ldots,d)}(X) = B$ is a closed polydisc in $R^d$ of positive polyradius, such that the restriction of the projection to $X$, $\mathrm{pr}_{(1,\ldots,d)} : X \to R^d$ is a quasi-finite surjection, with all the fibers having the same size of say $N$ elements. Then there exists a smaller closed polydisk of positive polyradius $B' \subseteq B$ and $N$ definable maps $s_i : B' \to R^{m-d}, 1 \leq i \leq N$ such that $X \cap (B' \times R^{m-d})$ is the disjoint union of the graphs of the $s_i, 1 \leq i \leq N$.*

*Proof.* Induct on $N$. If $N = 1$, then the projection $\mathrm{pr}_{(1,\ldots,d)} : X \to B$ is a definable bijection and the Lemma is clear in this case, since $X$ is evidently the graph of the definable inverse of this bijection, composed with the projection to the last $(m-d)$-coordinates.

Suppose $N \geq 2$. For every $m \geq 1$, define

$$D_m := \{b \in B : \text{ for all } x_1 \neq x_2 \in X_b, \|x_1 - x_2\| \geq |\varpi^m|\}.$$

Note that $\cup_{m \geq 1} D_m = R^d$, and thus by Corollary 3.12 for an $m_0, D_{m_0}$ has non-empty interior. Replacing $B$ with a smaller disk in this interior, and shrinking $X$ too, we assume that for all $b \in B$, and for all $x_1 \neq x_2 \in X_b, \|x_1 - x_2\| \geq |\varpi^{m_0}|$. Now, cover $R^{m-d}$ by countably many non-empty open disks $\{\Delta_j\}_{j \geq 1}$ of polyradius strictly less than $|\varpi^{m_0}|$. Since $\bigcup_{j \geq 1} \mathrm{pr}_{(1,\ldots,d)}\left((B \times \Delta_j) \cap X\right) = B$, by Corollary 3.12, for some $j \geq 1, \mathrm{pr}_{(1,\ldots,d)}\left((B \times \Delta_j) \cap X\right)$ has non-empty interior in $R^d$. We replace $B$ by a smaller closed disk of positive radius contained in this interior. Thus, we now have that $\mathrm{pr}_{(1,\ldots,d)} : (B \times \Delta_j) \cap X \to B$ is a *bijection* (since distinct points in the fiber of this projection are at least $|\varpi^{m_0}|$ apart in some coordinate, however the polydisc $\Delta_j$ has polyradius $< |\varpi^{m_0}|$ by choice.) Thus the inverse of this bijection, provides a definable section $s : B \hookrightarrow X$. And letting $s_1 := \mathrm{pr}_{(d+1,\ldots,m)} \circ s$, we see that the graph of $s_1$ is exactly $(B \times \Delta_j) \cap X$. Let $Y := X \cap (B \times (R^{m-d} \setminus \Delta_j))$. Then $\mathrm{pr}_{(1,\ldots,d)} : Y \to B$ is a quasi-finite surjection with fibers of constant cardinality $N - 1$. We now

apply the induction hypothesis to $\mathrm{pr}_{(1,\ldots,d)} : Y \to B$ to finish the proof of the Lemma. □

**Proposition 3.16** (Invariance of dimensions under definable bijections)**.** *Let $X \subseteq R^m$ and $Y \subseteq R^n$ be definable sets and $f : X \to Y$ a definable bijection. Then* $\dim(X) = \dim(Y)$.

*Proof.* It suffices to prove that $\dim(X) \leq \dim(Y)$, since the inverse $f^{-1} : Y \to X$ is also definable. Let $d = \dim(X)$. Suppose the projection of $X$ to the first $d$-coordinates contains a $d$-dimensional non-empty open disk $B$, i.e. $\mathrm{pr}_{(1,\ldots,d)}(X) \supseteq B$. Replacing $X$ by $X \cap (B \times R^{m-d})$ we may assume that $\mathrm{pr}_{(1,\ldots,d)}(X) = B$.

**Claim.** *There is a non-empty open disk $B' \subseteq B$ such that the projection map $\mathrm{pr}_{(1,\ldots,d)} : X \to B$ is quasi-finite over $B'$ with constant fiber cardinality of size $N$.*

For each $j \in \{d+1, \ldots, m\}$, let

$$T_j := \{b \in B : \mathrm{pr}_j(X_b) \text{ contains a non-empty open disk}\}$$

and for each natural number $k \geq 1$, let $F_k := \{b \in B : |X_b| = k\}$. Then $B = \bigcup_{k=1}^{\infty} F_k \cup \bigcup_{d+1 \leq j \leq m} T_j$. If for any $k \geq 1, F_k$ has non-empty interior, the Claim would be proved. So suppose each $F_k$ has empty interior; then by Corollary 3.12 there is some $j$ such that $T_j$ has non-empty interior. Suppose w.l.o.g that $T_{d+1}$ contains a non-empty open disk. Replacing $B$ by this smaller disk (and modifying $X$ appropriately), we may assume that $T_{d+1} = B$. Let $\{B_i : i \geq 1\}$ be an enumeration of a countable basis of non-empty open disks of $R$, and let $\mathcal{K}_i := \{b \in B : \mathrm{pr}_{d+1}(X_b) \supseteq B_i\}$. Then $\cup_i \mathcal{K}_i = B$ and so by Corollary 3.12 there is an $i_o$ such that $\mathcal{K}_{i_0}$ contains a non-empty open disk $D$. Replacing $B$ by $D$ we assume that for all $b \in B, \mathrm{pr}_{d+1}(X_b) \supseteq B_{i_0}$. But then $\mathrm{pr}_{(1,\ldots,d+1)}(X) \supseteq B \times B_{i_0}$ contradicting that $\dim(X) = d$, and thus proving the claim.

Replacing $B$ with $B'$ obtained from the above claim, and replacing $X$ by $X \cap (B' \times R^{m-d})$ we assume $\mathrm{pr}_{(1,\ldots,d)} : X \to B$ is quasi-finite with constant fiber cardinality of size $N \geq 1$. By Lemma 3.15, (after possibly shrinking $B$ further) we can find a definable section $s : B \hookrightarrow X$. By Lemma 3.14 now, $\dim(f(s(B))) \geq d$ and as $Y \supseteq f(s(B))$ we get that $\dim(Y) \geq d$ as was needed. □

**Lemma 3.17.** *Let $D \subseteq R^d$ be a closed polydisk of positive polyradius. Let $s : D \to R$ be a definable function. Then given any $\epsilon > 0$, there exists a smaller closed polydisk $D' \subseteq D$ or positive polyradius such that $s(D')$ is contained in a disk of diameter $< \epsilon$, i.e. for all $x, y \in D', |s(x) - s(y)| < \epsilon$.*

*Proof.* Cover $R$ by countably many non-empty open disks $\{B_i\}_{i\geq 1}$ each of diameter $< \epsilon$. Then, $D = \cup_i s^{-1}(B_i)$. By Corollary 3.12, there is an $i \geq 1$, such that $s^{-1}(B_i)$ has non-empty interior in $R^d$. For such an $i$, take $D' \subseteq s^{-1}(B_i)$ to be a closed polydisk of positive polyradius. □

**Theorem 3.18** (Theorem of the Boundary)**.** *Let $X \subseteq R^m$ be a definable set. Then* $\dim(\mathrm{Fr}(X)) < \dim(X)$.

*Proof.* Let $d = \dim(X)$. By Corollary 3.13, we first note that $\dim(\mathrm{Fr}(X)) \leq \dim(\mathrm{cl}(X)) = \dim(X) = d$. Suppose for the sake of contradiction, that $\dim(\mathrm{Fr}(X)) = d$, and that the projection of $\mathrm{Fr}(X)$ to the first $d$-coordinates has non-empty interior. Thus if $\pi : R^m \to R^d$, denotes the projection to the first $d$-coordinates, there exists a closed polydisk $D$ of positive polyradius in $R^d$, such that $\pi(\mathrm{Fr}(X)) \supseteq D$.

So we have $D \subseteq \pi(\mathrm{Fr}(X)) \subseteq \pi(\mathrm{cl}(X)) \subseteq \mathrm{cl}(\pi(X))$, and hence in particular $\mathrm{cl}_D(\pi(X) \cap D) = D$. By Corollary 3.13, $\pi(X) \cap D$ contains a smaller closed disk of positive radius say $D' \subseteq \pi(X) \cap D \subseteq D$. Replacing $D$ with $D'$ and $X$ with $X \cap (D' \times R^{m-d})$, we may assume that $D = \pi(X) = \pi(\mathrm{Fr}(X))$.

We note that in the argument that follows, we shall often replace $D$ with a smaller disk. This is justified since, if $D' \subseteq D$ is a smaller closed disk of positive radius, replacing $D$ by $D'$ and $X$ by $X \cap (D' \times R^{m-d})$ does not change the property that $D = \pi(X) = \pi(\mathrm{Fr}(X))$.

Continuing our proof, for each $j \in \{d+1, \ldots, m\}$ we let $\Lambda_j := \{b \in D : \pi_j(X_b)$ has non-empty interior$\}$. If for some $j$, $\Lambda_j$ has non-empty interior, then again using the same trick of enumerating a countable basis of disks in $R$, and following the same line of argument we would conclude that $\pi_{(1,\ldots,d,j)}(X)$ contains a $(d+1)$-dimensional disk. This is not possible since $\dim(X) = d$. Therefore, for each $j \in \{d+1, \ldots, m\}$ we must have that $\mathrm{int}(\Lambda_j) = \varnothing$. Therefore, by Corollary 3.12, $D \setminus \cup_j \Lambda_j$ contains a closed disk of positive polyradius. Replacing $D$ with this smaller closed disk, we may assume then that $\pi : X \to D$ is a quasi-finite surjection. Moreover, using the argument that we made in the Proof of the claim in the proof of Proposition 3.16, we may assume the fibers of $\pi : X \to D$ have constant finite cardinality of size $N$. Running the exact same argument for $\mathrm{Fr}(X)$ instead of $X$ and shrinking $D$ if necessary, we also assume that $\pi : \mathrm{Fr}(X) \to D$ is quasi-finite surjection with fibers of constant size say $M$.

Further shrinking $D$ to a smaller closed disk, we may assume by Lemma 3.15, that $X$ is the disjoint union of graphs of $N$ definable functions $s_i : D \to R^{m-d}$. If we let $T_i$ denote the graph of $s_i$, then since $X = \cup_{1\leq i\leq N} T_i$, we have $\mathrm{Fr}(X) \subseteq \cup_i \mathrm{Fr}(T_i)$. Furthermore, since $D = \pi(\mathrm{Fr}(X)) = \cup_i \pi(\mathrm{Fr}(T_i))$, for some $i$ we must have that $\pi(\mathrm{Fr}(T_i))$ has

non-empty interior. We may then replace $D$ by a smaller disk in this interior, and $X$ by $T_i$ and assume thus that $X$ is the graph of a definable function $s : D \to R^{m-d}$. Furthermore, running the argument in the above paragraphs again, we may ensure that the property that $\pi : \mathrm{Fr}(X) \to D$ is a quasi-finite surjection of constant fiber cardinality still holds. In all, we have therefore reduced to the following situation:

*$X$ is the graph of a definable function $s : D \to R^{m-d}$ such that $\pi(\mathrm{Fr}(X)) = D = \pi(X)$, and such that $\pi : \mathrm{Fr}(X) \to D$ is a quasi-finite surjection with constant fiber cardinality of size $M$.*

Applying Lemma 3.15 to $\mathrm{Fr}(X)$, we assume that $\mathrm{Fr}(X)$ is the disjoint union of graphs of $M$ definable functions $g_j : D \to R^{m-d}, 1 \leq j \leq M$. Let $Y_j$ denote the graph of $g_j$, so that $\mathrm{Fr}(X) = \cup_{j=1}^{M} Y_j$. We note that $X \cap Y_j = \varnothing$ for each $j$, or in other words for each $j$ and for every $b \in D, \left\| s(b) - g_j(b) \right\| \neq 0$. For every $m \geq 1$, define $E_m := \{b \in D : \left\| s(b) - g_j(b) \right\| > |\varpi^m| \text{ for each } j\}$. We have $\cup_{m \geq 1} E_m = D$, and thus by Corollary 3.12 some $E_m$ has non-empty interior. Replacing $D$ by a smaller closed disk contained in this interior, we may assume that there is some $m_0$ large enough, such that for all $1 \leq j \leq M$ and for all $b \in D, \left\| s(b) - g_j(b) \right\| > |\varpi^{m_0}|$. Applying Lemma 3.17, and shrinking $D$ to a smaller disk, we may assume that for all $x, y \in D, \left\| s(x) - s(y) \right\| < |\varpi^{m_0}|$.

Now choose a $b \in D$, and consider $y = (b, g_1(b)) \in Y_1$. Since $y \in \mathrm{Fr}(X)$, in particular $y$ is in the closure of $X$, and thus there must exist some $b' \in D$ such that $\left\| s(b') - g_1(b) \right\| < |\varpi^{m_0}|$. However, by our choice of $m_0$, $\left\| s(b) - g_1(b) \right\| > |\varpi^{m_0}|$. By the non-archimedean triangle inequality, we therefore get that $\left\| s(b') - s(b) \right\| > |\varpi^{m_0}|$, contradicting the conclusion of the previous paragraph. □

*Remark* 3.19. Note that in all our proofs we have extensively made use of the standing assumption that $K$ is second countable. However, when the tame structure under consideration is that of the $\mathcal{H}$-subanalytic sets this assumption can be removed, exploiting the model completeness and *uniform* quantifier elimination results of [LRoob]. See for example the argument used in the Proof of [LRooa, Lemma 2.3]. Running the same argument given there, with appropriate modifications, enables us to reduce the proof of the Theorem of the Boundary for $\mathcal{H}$-subanalytic sets to the case where $K$ is second countable.

## 3.3 RECOLLECTIONS ON THE DIMENSION THEORY OF RIGID ANALYTIC VARIETIES

*Summary of this section*

In this section we collect a few auxiliary results relating to the dimension theory of rigid analytic spaces. These results shall be used in the sequel. We prove in Lemma 3.22 that the usual notion of dimension in rigid geometry defined via Krull dimensions of associated rings of analytic functions agrees with the notion of definable dimension defined above via coordinate projections. In Lemma 3.24, we show that for any point $x$ of a reduced, equidimensional rigid variety $X$, every minimal prime ideal of the local ring $\mathcal{O}_{X,x}$ has the same coheight. This result is used in the sequel in the course of proving the definability of the étale locus of a certain finite map. While the results of this section should be fairly standard, we provide their proofs for completeness. We suggest that the reader return to this section as and when the lemmas here are referenced.

**Definition 3.20.** If $X$ is a rigid variety over $K$, we define its dimension, denoted $\dim(X)$, by

$$\dim(X) := \max_{x \in X}\{\dim(\mathcal{O}_{X,x})\}.$$

**Lemma 3.21.** *Let $Y = \mathrm{Sp}(A)$ be a K-affinoid space. Let $\{Y_i\}_{1 \leq i \leq m}$ denote the finitely many irreducible components of $Y$. Then,*

1. $\dim(Y) = \dim(A)$ *and*
2. *For any point $y \in Y$, $\dim(\mathcal{O}_{Y,y}) = \max\{\dim(Y_j) : y \in Y_j\}$.*

*Proof.* These facts are rather standard. Due to lack of an explicit reference, we provide a proof nevertheless.

*Proof of (1):* For a point $y \in Y$, corresponding to $\mathfrak{m} \in \mathrm{MaxSpec}(A)$, we have $\widehat{A}_{\mathfrak{m}} = \widehat{\mathcal{O}}_{Y,y}$ [BGR84, §7.3.2 Proposition 3]. Since the Krull dimension of a Noetherian local ring is preserved under completion (see [Sta20, Tag 07NV]) we get,

$$\begin{aligned}\dim(A) &= \max_{\mathfrak{m} \in \mathrm{MaxSpec}(A)}\{\dim(A_{\mathfrak{m}})\} = \max_{\mathfrak{m} \in \mathrm{MaxSpec}(A)}\{\dim(\widehat{A}_{\mathfrak{m}})\} \\ &= \max_{y \in Y}\{\dim(\widehat{\mathcal{O}}_{Y,y})\} = \max_{y \in Y}\{\dim(\mathcal{O}_{Y,y})\} = \dim(Y).\end{aligned}$$

*Proof of (2):* From the argument above, if $\mathfrak{m} \in \mathrm{MaxSpec}(A)$, corresponds to $y$, we know that $\dim(\mathcal{O}_{Y,y}) = \dim(A_{\mathfrak{m}})$. If the irreducible component $Y_i$ corresponds to the minimal prime $\mathfrak{p}_i \subset A$, then we note that $\dim(A_{\mathfrak{m}}) =$

$\max\{\dim(A_{\mathfrak{m}}/\mathfrak{p}_j A_{\mathfrak{m}}) : \mathfrak{p}_j \subseteq \mathfrak{m}\} = \max\{\dim\left((A/\mathfrak{p}_j)_{\mathfrak{m}}\right) : \mathfrak{p}_j \subseteq \mathfrak{m}\}$. Now, $A/\mathfrak{p}_j$ is an affinoid algebra that is an integral domain, and this implies that $\dim(A/\mathfrak{p}_j)_{\mathfrak{m}} = \dim(A/\mathfrak{p}_j)$ - see Lemma 3.23-1. below. Thus, $\dim(\mathcal{O}_{Y,y}) = \dim(A_{\mathfrak{m}}) = \max\{\dim(A/\mathfrak{p}_j) : \mathfrak{p}_j \subseteq \mathfrak{m}\} = \max\{\dim(Y_j) : y \in Y_j\}$. □

**Lemma 3.22.** *Suppose $Y = \mathrm{Sp}(A)$ is a K-affinoid space. Suppose $\pi : T_n(K) \twoheadrightarrow A$ is a surjective homomorphism of K-algebras. Via $\pi$ we may view $i : Y \hookrightarrow R^n$ as a subset of the n-dimensional unit ball $R^n$. Then,*

1. *The dimension of $i(Y)$ as a subset of $R^n$ (in the sense of Definition 3.9) is the same as the dimension of $Y$ as a rigid analytic space.*
2. *For a point $y \in Y$, the local dimension $\dim_{i(y)} i(Y)$ (in the sense of Definition 3.9) is equal to $\dim(\mathcal{O}_{Y,y})$.*
3. *Suppose $X$ is a rigid space over $K$ and $i : X \hookrightarrow \mathbb{A}_K^{n,\mathrm{an}}$ a closed immersion. Then $\dim(X)$ equals the dimension of $i(X)$ viewed as a subset of $K^n$ (as in Definition 3.9). For a point $x \in X$, the local dimension $\dim_{i(x)} i(X)$ (as in Definition 3.9) is equal to $\dim(\mathcal{O}_{X,x})$.*

*Proof of (1):* This is a special case of [LRooa, Lemma 4.2]. Alternatively, we may reduce to the case that $K$ is second-countable (see Remark 3.19). Then, using Noether's normalization for affinoid algebras, if $d = \dim(A)$, we have a quasi-finite subanalytic surjection $i(Y) \twoheadrightarrow R^d$. And then we may use an argument very similar to that of Proposition 3.16. We omit the details for the alternate argument.

*Proof of (2)*: By definition, we have that

$$\dim_{i(y)} i(Y) = \min\{\dim(U \cap i(Y)) : U \subseteq R^n \text{ is an open set containing } i(y)\}.$$

We may take this minimum instead over all closed polydisks $\overline{\mathbb{D}}$ of $R^n$ of positive polyradius containing $i(y)$, i.e.

$$\dim_{i(y)} i(Y) = \min\left\{\dim\left(\overline{\mathbb{D}}(i(y), \underline{r}) \cap i(Y)\right) : \underline{r} > \underline{0}\right\}.$$

Since $i^{-1}(\overline{\mathbb{D}}(i(y), \underline{r}))$ is an affinoid subdomain of $Y$, from the first part (1), we have that

$$\dim\left(\overline{\mathbb{D}}(i(y), \underline{r}) \cap i(Y)\right) = \dim(i^{-1}(\overline{\mathbb{D}}(i(y), \underline{r})))$$

where the dimension on the right side is the dimension of the affinoid subdomain $i^{-1}(\overline{\mathbb{D}}(i(y), \underline{r}))$ as an analytic space. Furthermore, note that the affinoid subdomains of the form $i^{-1}(\overline{\mathbb{D}}(i(y), \underline{r}))$ are cofinal in the

collection of all affinoid subdomains of $Y$ containing $y$ (use for example [Con99, Lemma 1.1.4] to see this). Therefore,

$$\dim_{i(y)} i(Y) = \min\{\dim(W) : W \subseteq Y \text{ is an affinoid subdomain containing } y\}.$$

The right-hand-side is indeed equal to $\dim(\mathcal{O}_{Y,y})$ (follows from [Duco7, 1.17] and Lemma 3.21-(2)).

*Proof of (3):* Follows immediately from (1) and (2). □

**Lemma 3.23.** 1. *Suppose $A$ is a $K$-affinoid algebra that is an integral domain. Then every maximal ideal of $A$ has the same height.*

2. *Suppose $Y$ is an irreducible rigid analytic variety. Then $Y$ is equidimensional, i.e. for all $y \in Y$, $\dim(\mathcal{O}_{Y,y}) = \dim(Y)$.*

*Proof.* For (1), use Noether normalization for affinoid algebras, the Going Down theorem ([Sta20, Tag 00H8]) and [BGR84, Chapter 2, Proposition 17].

For (2), we refer the reader to the paragraph preceding [Con99, Lemma 2.2.3]. □

**Lemma 3.24.** *Let $X$ be a reduced equidimensional rigid space over $K$, i.e. $\dim(\mathcal{O}_{X,x}) = \dim(X)$ for all $x \in X$. Then for every $x \in X$ and for every minimal prime ideal $\mathfrak{q}$ of $\mathcal{O}_{X,x}$ we have $\dim(\mathcal{O}_{X,x}/\mathfrak{q}) = \dim(X)$.*

*Proof.* Evidently for every $x \in X$, $\dim(\mathcal{O}_{X,x}/\mathfrak{q}) \leq \dim(\mathcal{O}_{X,x}) = \dim(X)$. Suppose the Lemma was false. Then for some $x$, we would have

$$\dim(\mathcal{O}_{X,x}/\mathfrak{q}) < \dim(X).$$

Since $\mathcal{O}_{X,x}$ is Noetherian ([BGR84, §7.3.2 Proposition 7]), $\mathfrak{q}$ is finitely generated; say $\mathfrak{q} = (h_1, \ldots, h_m)$, for elements $h_i \in \mathcal{O}_{X,x}$. We may choose an open affinoid domain $\mathrm{Sp}(B)$ in $X$ containing $x$ such that the $h_i$ are (images of elements) in $B$. Let $\mathfrak{n} \in \mathrm{MaxSpec}(B)$ be the maximal ideal corresponding to the point $x$, and let $J := (h_1, \ldots, h_m)B$ be the ideal in $B$ generated by the $h_i$.

We claim first that $JB_{\mathfrak{n}}$ is a minimal prime ideal of $B_{\mathfrak{n}}$. To see this, note that since $B_{\mathfrak{n}} \hookrightarrow \mathcal{O}_{X,x}$ is a faithfully flat map, (as these local rings have the same completions), $JB_{\mathfrak{n}}$ is the contraction of $J\mathcal{O}_{X,x} = \mathfrak{q}$ (see [Sta20, Tag 05CK]) and is therefore a prime ideal. Moreover, since $B_{\mathfrak{n}} \hookrightarrow \mathcal{O}_{X,x}$ is faithfully flat, it has the Going-Down property ([Sta20, Tag 00HS]). Therefore, as $\mathfrak{q}$ is a minimal prime of $\mathcal{O}_{X,x}$, its contraction $JB_{\mathfrak{n}}$, must also be minimal.

We have $(B_{\mathfrak{n}}/JB_{\mathfrak{n}})\widehat{\ } = \widehat{B_{\mathfrak{n}}}/J\widehat{B_{\mathfrak{n}}} = \widehat{\mathcal{O}}_{X,x}/\mathfrak{q}\widehat{\mathcal{O}}_{X,x} = (\mathcal{O}_{X,x}/\mathfrak{q})\widehat{\ }$. Hence, $\dim(B_{\mathfrak{n}}/JB_{\mathfrak{n}}) = \dim(\mathcal{O}_{X,x}/\mathfrak{q})$. Now let $\mathfrak{p} \subseteq B$ denote the contraction of $JB_{\mathfrak{n}}$ to $B$, so then $\mathfrak{p}$ is a minimal prime of $B$ contained in $\mathfrak{n}$. Then $\dim(B_{\mathfrak{n}}/JB_{\mathfrak{n}}) = \dim\left((B/\mathfrak{p}B)_{\mathfrak{n}}\right) = \dim(B/\mathfrak{p}B)$, where the last equality follows from the fact that $B/\mathfrak{p}$ being an affinoid algebra that is an integral domain, all its maximal ideals have the same height (see Lemma 3.23).

Therefore, in all we have shown that $\dim(B/\mathfrak{p}) = \dim(\mathcal{O}_{X,x}/\mathfrak{q})$, for a minimal prime $\mathfrak{p}$ of $B$. And since we are assuming that $\dim(\mathcal{O}_{X,x}/\mathfrak{q}) < \dim(X)$, this means that $\dim(B/\mathfrak{p}) < \dim(X)$. However, now find a closed point $\mathfrak{n}' \in \mathrm{MaxSpec}(B)$ containing $\mathfrak{p}$ but not containing any other minimal prime of $B$ (this is possible since $B$ is Jacobson and so closed points are dense). If $\mathfrak{n}'$ corresponds to the point $x' \in X$, we have $\dim(X) > \dim(B/\mathfrak{p}) \geq \dim(B_{\mathfrak{n}'}) = \dim(\mathcal{O}_{X,x'})$. This contradicts the equidimensionality of $X$. □

# 4

# THE NON-ARCHIMEDEAN DEFINABLE CHOW THEOREM

*Outline of this chapter*

The goal of this section is to prove a version of the Definable Chow theorem in the non-archimedean setting. More precisely, we prove the following result:

**Theorem 4.1.** *Let $X$ be a closed analytic subset of $\mathbb{A}^{n,an}_{\mathbb{C}_p}$. Suppose that for some tame structure on $\mathbb{C}_p$, $X$ is definable as a subset of $\mathbb{A}^n(\mathbb{C}_p) = \mathbb{C}_p^n$. Then $X$ is algebraic i.e. $X$ is the vanishing locus of a finite collection of polynomials in $\mathbb{C}_p[t_1, \ldots, t_n]$.*

We outline the major steps of the proof below:

STEP 0: Our first step is to show that for a reduced variety $X$ over $\mathbb{C}_p$, a global analytic function $f \in H^0(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}})$ whose graph is definable, must be algebraic. This is the content of Theorem 4.5. The Proposition may be seen as a non-archimedean definable analogue of Liouville's theorem from complex analysis. The proof proceeds by a devissage argument:

- First, when $X = \mathbb{A}^n_{\mathbb{C}_p}$ - Lemma 4.2.
- Second, when $X$ is a smooth affine variety over $\mathbb{C}_p$ - Lemma 4.3, using Noether normalization to reduce to the first case.
- And lastly, for a general reduced variety $X$ ( Theorem 4.5), using Lemma 4.4 to reduce to the smooth case.

STEP 1: Now suppose $X \subseteq \mathbb{A}^{n,\text{an}}_{\mathbb{C}_p}$ is as in the statement of Theorem 4.1. We shall induct on $\dim(X) + n$. By the Theorem of the Boundary, $\dim(\text{Fr}(X)) < \dim(X)$ and so we can find a point $q \in \mathbb{P}^n(\mathbb{C}_p) \setminus \text{cl}(X)$.

STEP 2: The projection from $q$ onto a hyperplane $\mathbb{H} \subseteq \mathbb{C}_p^n$ not containing it, $\pi|_X : X \to \mathbb{H}$ is *finite*. The image $Y = \pi(X)$ is an analytic subset of $\mathbb{A}^{n-1,\text{an}}_{\mathbb{C}_p}$, and by induction therefore algebraic.

STEP 3: The étale locus $U \subseteq Y$ of $\pi|_X : X \to Y$ is definable (thanks to Lemma 4.9), and of smaller dimension, therefore algebraic.

STEP 4: The characteristic polynomial of the finite étale map

$$\pi : \pi^{-1}(U^{\mathrm{an}}) \to U^{\mathrm{an}},$$

has coefficients in $H^0(U^{\mathrm{an}})$ that are *definable*. By Step 0, we shall then conclude $\pi^{-1}(U^{\mathrm{an}}) \subseteq X$ is algebraic. The complement in $X$ is of smaller dimension, and by induction thus algebraic.

## 4.1 A NON-ARCHIMEDEAN DEFINABLE LIOUVILLE THEOREM

**Lemma 4.2.** *Let $(X, \mathcal{O}_X) = \mathbb{A}^{n,an}_{\mathbb{C}_p}$ be the rigid $n$-dimensional affine plane over $\mathbb{C}_p$ and let $f \in H^0(X, \mathcal{O}_X)$ be a global analytic function. Suppose $f$ viewed as a function $f : \mathbb{C}_p^n \to \mathbb{C}_p$ is definable. Then $f$ is a polynomial function.*

*Proof.* We prove this by induction on $n$.

*Case of $n = 1$:* A function $f \in H^0(\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}, \mathcal{O}_{\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}})$ is given by a globally convergent power series $f(z) = \sum_{i\geq 0} a_i z^i$. Thus, $\lim_{i\to\infty}(p^{ir} \cdot |a_i|) = 0$ for every $r \geq 0$. For a given $r \geq 0$, the number of zeroes of $f(z)$ on the disk $\{z \in \mathbb{C}_p : |z| \leq p^r\}$ is the number of zeroes (with the same multiplicities) of $g_r(t) := f(p^{-r}t) = \sum_{\geq 0} a_i p^{-ir} t^i$ in the unit disk $|t| \leq 1$, which by Lemma 2.42 is at least $\epsilon(g_r)$. Now given any $i < j$ with $a_i, a_j \neq 0$, we note that for $r$ large enough $p^{rj}|a_j| \geq p^{ri}|a_i|$ and thus $\epsilon(g_r) \geq j$. Thus, if $a_i \neq 0$ for infinitely many $i$, $f$ must have infinitely many zeroes. However, as $f$ is definable, $f^{-1}(0)$ is a definable subset of $\mathbb{C}_p$ that is discrete, and must therefore be necessarily finite. Hence, it cannot be the case that $a_i \neq 0$ for infinitely many $i$, i.e. $f$ is a polynomial.

*Proof for general $n \geq 1$:* The global analytic function $f \in H^0(\mathbb{A}^{n,an}_{\mathbb{C}_p})$ is again given by a globally convergent power series on $\mathbb{C}_p^n$. Thus, we write $f(z_1, \ldots, z_n) = \sum_{i\geq 0} a_i(z_1, \ldots, z_{n-1}) z_n^i$, where $a_i \in H^0(\mathbb{A}^{n-1,an}_{\mathbb{C}_p})$. Moreover, each $a_i(z_1, \ldots, z_{n-1})$ is also definable viewed as a function on $\mathbb{C}_p^{n-1}$ (by Lemma 3.6.item (iv)). By induction, we have that the $a_i(z_1, \ldots, z_{n-1})$ are polynomials in $\mathbb{C}_p[z_1, \ldots, z_{n-1}]$. From the $n = 1$ case, it must be that for every $\underline{\lambda} \in \mathbb{C}_p^{n-1}$, the sequence $a_i(\underline{\lambda})$ is eventually 0. In other words, $\mathbb{C}_p^{n-1} = \bigcup_{j\geq 0} \cap_{i\geq j} V(a_i)$, a countable union of closed subsets of $\mathbb{C}_p^{n-1}$. By the Baire Category Theorem, this is only possible if for some $j \geq 0, \mathbb{C}_p^{n-1} = \bigcap_{i\geq j} V(a_i)$, i.e. $a_i = 0$ for all $i \geq j$ and hence $f$ is a polynomial. □

**Lemma 4.3.** *Let $X$ be an integral, locally Cohen-Macaulay scheme of finite type over $\mathbb{C}_p$ and denote by $X^{an}$ the rigid analytification of $X$. Let $f \in H^0(X^{an}, \mathcal{O}_{X^{an}})$*

*be a global rigid analytic function on $X^{an}$ such that the graph of $f$ viewed as a subset of $X(\mathbb{C}_p) \times \mathbb{C}_p$ is definable. Then $f \in H^0(X, \mathcal{O}_X)$.*

*Proof.* By passing to a *finite* affine cover of $X$ we may assume $X = \mathrm{Spec}(A)$ for a domain $A$ that is Cohen-Macaulay and a $\mathbb{C}_p$-algebra of finite type. Choose a Noether normalization of $A$, i.e. a finite inclusion $i : \mathbb{C}_p[t_1, \ldots, , t_d] \hookrightarrow A$. Since $A$ is Cohen-Macaulay, $i$ is locally free by Hironaka's Miracle Flatness criterion [Sta20, Tag 00R4]. There is a finite set of polynomials $p_i(\underline{t}), 1 \leq i \leq m$ generating the unit ideal in $\mathbb{C}_p[\underline{t}]$ such that $A[p_i^{-1}]$ is free over $\mathbb{C}_p[\underline{t}][p_i^{-1}]$ for each $i$. Moreover, it is easy to see that $\mathbb{C}_p[\underline{t}][p_i^{-1}]$ is finite free over another pure polynomial subring in $d$ variables (just need to change variables). Thus, by replacing $A$ with $A[p_i^{-1}]$ and modifying the Noether normalization map as above, we are in the case where $A$ is finite free over the polynomial ring $\mathbb{C}_p[t_1, \ldots, t_d]$, say of rank $r$.

Let $a_1, \ldots, a_r \in A$ be a module-basis over $\mathbb{C}_p[t_1, \ldots, t_d]$. It follows that $H^0(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ is a free module over $H^0\left(\mathbb{A}^{d,\mathrm{an}}\right)$ again with basis $a_1, \ldots, a_r$. Thus, $f$ can be written uniquely as $f = \sum_{1 \leq k \leq r} a_k \cdot g_k(\underline{t})$ with $g_k(\underline{t}) \in H^0\left(\mathbb{A}^{d,\mathrm{an}}\right)$. To finish the proof, it suffices to show that the $g_k(\underline{t})$ have definable graphs in $\mathbb{C}_p^{d+1}$, since then we may appeal to the previous Lemma 4.2 to conclude that the $g_k$ are polynomials. By continuity, it in fact suffices to show that $g_k(\underline{t})|_U$ has a definable graph for some Zariski dense open subset $U$ of $\mathbb{C}_p^d$. Since the Noether normalization map $i : \mathrm{Spec}(A) \to \mathbb{A}^d_{\mathbb{C}_p}$ is generically étale, we may take $U \subseteq \mathbb{A}^d_{\mathbb{C}_p}$ to be the locus over which it is étale. For any point $\underline{u} \in U$, letting $i^{-1}(\underline{u}) = \{P_1, \ldots, P_r\}$, we have $r$ linear equations in $r$ variables:

$$f(P_j) = \sum_{1 \leq k \leq r} a_k(P_j) g_k(\underline{u})$$

for each $j \in \{1, \cdots, r\}$. Over the étale locus the matrix $\left(a_k(P_j)\right)_{1 \leq k,j \leq r}$ is invertible and thus we may write for each $k$, the function $g_k(u)$ as an explicit linear combination of $\{f(P_j) : 1 \leq j \leq r\}$, with coefficients being rational function in $a_k(P_j)$. Note that permuting the ordering of the $P_j$, leaves the specific linear combination invariant. Thus, the graph of the function $g_k : U \to \mathbb{C}_p$ can be expressed as a first-order formula with all its terms using definable functions and sets– indeed, we note that $f$ is definable by assumption, and that $U, X(\mathbb{C}_p), i, a_j$ being algebraic are also definable. We thus obtain that the $g_k(\underline{u})$ are definable over $U$, concluding the proof.

□

**Lemma 4.4.** *Let $A$ be a reduced finite-type $\mathbb{C}_p$-algebra and let $X = \mathrm{Spec}(A)$. Let $\{X_i\}$ denote the set of irreducible components of $X$, given their reduced induced structures. Suppose $f \in H^0(X^{an}, \mathcal{O}_{X^{an}})$ is a global rigid analytic function such that for every $i$, there is a dense open subset $U_i \subseteq X_i$ such that $f|_{U_i^{an}} \in H^0(U_i, \mathcal{O}_{U_i})$. Then $f \in H^0(X, \mathcal{O}_X)$.*

*Proof.* By our assumptions on $f$, we may view $f$ as an element of the total ring of fractions $Q(A)$ of $A$. To show that $f \in A$, it suffices to show that for every maximal ideal $\mathfrak{m}$ of $A$, the image of $f$ in $Q(A)_\mathfrak{m}$ (the localization of $Q(A)$ at the multiplicative set $A \setminus \mathfrak{m}$) is also in $A_\mathfrak{m}$. Indeed, writing $f = a/s$ with $a, s \in A$ and $s$ a non-zerodivisor, if $f \notin A$, then $a \notin sA$. So we may choose a maximal ideal $\mathfrak{m}$ containing $(sA : a) = \{b \in A : ba \in sA\}$. However, for this choice of $\mathfrak{m}$, $f \notin A_\mathfrak{m}$.

So let us now fix a maximal ideal $\mathfrak{m}$ of $A$. We note that $Q(A)_\mathfrak{m}$ is in fact the total ring of fractions $Q(A_\mathfrak{m})$ of $A_\mathfrak{m}$. We also note that since $f \in H^0(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$, in particular, $f \in \widehat{A_\mathfrak{m}}$, since $\widehat{A_\mathfrak{m}} = \widehat{\mathcal{O}}_{X^{\mathrm{an}},\mathfrak{m}}$ ([Con99, Lemma 5.1.2 (2)]). For notational simplicity let $B := A_\mathfrak{m}, K := Q(B)$ and $\widehat{K} := K \otimes_B \widehat{B}$. We have *inclusions* $K \subseteq \widehat{K}$ and $\widehat{B} \subseteq \widehat{K}$ and $f \in K \cap \widehat{B}$. We must show that $f \in B$. Since $B \subseteq \widehat{B}$ is faithfully flat it suffices to show that $f \otimes 1 = 1 \otimes f$ in $\widehat{B} \otimes_B \widehat{B}$ [Sta20, Tag 023M]. Since $f \in K$, the equality $f \otimes 1 = 1 \otimes f$ evidently holds in $\widehat{K} \otimes_K \widehat{K}$ and we further note that $\widehat{B} \otimes_B \widehat{B}$ *injects* into $(\widehat{B} \otimes_B \widehat{B}) \otimes_B K = \widehat{K} \otimes_K \widehat{K}$ since $B$ injects into $K$ and $\widehat{B} \otimes_B \widehat{B}$ is $B$-flat. Hence $f \otimes 1 = 1 \otimes f$ holds in $\widehat{B} \otimes_B \widehat{B}$, as was to be shown. □

**Theorem 4.5** (A non-archimedean definable Liouville theorem)**.** *Let $X$ be a reduced scheme of finite type over $\mathbb{C}_p$ and denote by $X^{an}$ the rigid analytification of $X$. Let $f \in H^0(X^{an}, \mathcal{O}_{X^{an}})$ be a global rigid analytic function on $X^{an}$ such that the graph of $f$ viewed as a subset of $X(\mathbb{C}_p) \times \mathbb{C}_p$ is definable. Then $f \in H^0(X, \mathcal{O}_X)$.*

*Proof.* Again, by passing to a finite affine open cover we may assume that $X$ is affine. For each irreducible component $X_i$ of $X$, let $U_i \subseteq X_i$ be a dense open subset of $X_i$ that is *smooth* over $\mathbb{C}_p$ (in particular, locally Cohen-Macaulay). The restriction $f|_{U_i^{\mathrm{an}}} \in H^0({U_i}^{\mathrm{an}}, \mathcal{O}_{{U_i}^{\mathrm{an}}})$ is definable and hence by Lemma 4.3 we have that $f|_{U_i^{\mathrm{an}}} \in H^0(U_i, \mathcal{O}_{U_i})$. From Lemma 4.4 we conclude that $f \in H^0(X, \mathcal{O}_X)$. □

*Remark* 4.6. It is clear that reducedness of the underlying variety $X$ is necessary in the above Theorem 4.5 since the graph of a function on the underlying $\mathbb{C}_p$-points doesn't record the nilpotent structure. For example, take $X = \mathbb{A}^1_{\mathbb{C}_p}[\epsilon] = \mathrm{Spec}(\mathbb{C}_p[t, \epsilon]/(\epsilon^2))$. Choose any function $g \in H^0(X^{\mathrm{an}})$ which is not in $H^0(X)$ and take $f = \epsilon \cdot g$.

## 4.2 PROOF OF THE NON-ARCHIMEDEAN DEFINABLE CHOW THEOREM

We now turn towards proving our main Theorem 4.1:

**Theorem 4.7.** *Let $X$ be a closed analytic subset of $\mathbb{A}^{n,an}_{\mathbb{C}_p}$ that is also definable as a subset of $\mathbb{A}^n_{\mathbb{C}_p} = \mathbb{C}_p^n$. Then $X$ is an algebraic subset i.e. $X$ is defined as the vanishing locus of a finite collection of polynomials in $\mathbb{C}_p[t_1, \ldots, t_n]$.*

*Remark* 4.8. Recall that if $\mathscr{A}$ is a rigid analytic space over $\mathbb{C}_p$, then by a *closed analytic subset* $X \subseteq \mathscr{A}$ we mean that there is a closed immersion of rigid spaces $\mathscr{X} \overset{i}{\hookrightarrow} \mathscr{A}$ such that $i(\mathscr{X}) = X$. Equivalently, $X$ is cut out by the vanishing locus of a coherent $\mathcal{O}_{\mathscr{A}}$-ideal, or more concretely, there is an admissible affinoid covering $\mathscr{A} = \cup_{i\in I} U_i$, and for each $i \in I$, finitely many functions $f_1^{(i)}, \ldots, f_{n(i)}^{(i)}$ in $\mathcal{O}_{\mathscr{A}}(U_i)$ such that $X \cap U_i$ is the vanishing locus of $\{f_1^{(i)}, \ldots, f_{n(i)}^{(i)}\}$. Moreover, we note that given a closed analytic subset $X \subseteq \mathscr{A}$ as above, there is a canonical structure of a reduced rigid analytic space that can be put on $X$, with a canonical closed immersion $X \hookrightarrow \mathscr{A}$ (see [BGR84, §9.5.3, Proposition 4]). We shall refer to this reduced structure as the *reduced induced structure* on $X$.

As was outlined earlier, the proof of the theorem shall proceed by inducting on the dimension of the definable set $X \subseteq \mathbb{C}_p^n$, (which agrees with the dimension of $X$ as an analytic space - Lemma 3.22). First, we prove a preparatory Lemma concerning the étale locus of a finite morphism of rigid varieties that shall be used in the proof.

**Lemma 4.9.** *Suppose $\pi : X \to Y$ is a finite surjective morphism of reduced rigid analytic varieties over $\mathbb{C}_p$. Suppose $X$ is equidimensional at every point (i.e. for all $x \in X$, $\dim(\mathcal{O}_{X,x}) = \dim(X)$) and suppose $Y$ is irreducible and normal (i.e. for all $y \in Y, \mathcal{O}_{Y,y}$ is a normal domain). Let $N$ be the generic fiber cardinality of $\pi$ (i.e. $N = \operatorname{rank}_{\mathcal{O}_Y}(\pi_*\mathcal{O}_X)$). Then for $y \in Y$, $\pi$ is étale at every point in the fiber of $y$ if and only if $|\pi^{-1}(y)| = N$.*

*Remark* 4.10.

(a). We recall that a morphism of rigid spaces $\pi : X \to Y$ is said to be étale at a point $x \in X$ iff the induced map on local rings $\mathcal{O}_{Y,\pi(x)} \to \mathcal{O}_{X,x}$ is flat and unramified (see [dJvdP96, §3]).

(b). When we say the generic fiber cardinality is $N$ we mean that for every $y \in Y$, we have $N = \dim_{Q(\mathcal{O}_{Y,y})} \left( (\pi_*\mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} Q(\mathcal{O}_{Y,y}) \right)$. Here $Q(\mathcal{O}_{Y,y})$ denotes the fraction field of the domain $\mathcal{O}_{Y,y}$. Since $Y$ is connected, the dimension on the right hand side is indeed independent of the point $y$. To see this, it suffices to work over a connected affinoid open $\operatorname{Sp}(A)$ of $Y$. Then $A$ must be a domain, and since $\pi$ is a finite map, $\pi^{-1}(\operatorname{Sp}(A))$ is an affinoid open $\operatorname{Sp}(B)$ of $X$ with

the induced map $A \to B$ making $B$ a finite $A$-module. For a point $y \in \mathrm{Sp}(A)$ corresponding to the maximal ideal $\mathfrak{m}$ of $A$, we have that $\dim_{Q(\mathcal{O}_{Y,y})}\left((\pi_*\mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} Q(\mathcal{O}_{Y,y})\right) = \dim_{Q(\mathcal{O}_{Y,y})}\left(B \otimes_A Q(\mathcal{O}_{Y,y})\right) = \dim_{Q(A)} B \otimes_A Q(A)$.

*Proof of Lemma 4.9.* By working locally over connected affinoid opens of $Y$, we may assume that $Y = \mathrm{Sp}(A)$ is affinoid. Since $\pi$ is a finite morphism, $X$ is also affinoid, and $X = \mathrm{Sp}(B)$ with the induced map $A \to B$ making $B$ a finite $A$-module. The assumptions on $Y$ imply that $A$ is a normal integral domain. Let $K$ denote its fraction field and let $N = \dim_K(B \otimes_A K)$ be the generic fiber cardinality of $\pi$.

For a point $x \in X$, if we denote the maximal ideals corresponding to $x, \pi(x)$ by $\mathfrak{n} \subseteq B, \mathfrak{m} \subseteq A$ respectively, then we note that since $\widehat{A}_{\mathfrak{m}} = \widehat{\mathcal{O}}_{Y,\pi(x)}$ and $\widehat{B}_{\mathfrak{n}} = \widehat{\mathcal{O}}_{X,x}$ the map $\mathcal{O}_{Y,\pi(x)} \to \mathcal{O}_{X,x}$ is flat and unramified if and only if the same holds for the map $A_{\mathfrak{m}} \to B_{\mathfrak{n}}$ (the fact that both maps are unramified simultaneously is easy to see, whereas for flatness one may use the local flatness criterion [Mat89, Theorems 22.1 and 22.4]).

Suppose now $y \in Y$ is a point corresponding to the maximal ideal $\mathfrak{m}$ of $A$ such that $\pi$ is étale at every point of $\pi^{-1}(y)$. Then from the above $B/\mathfrak{m}B$ must be unramified over $A/\mathfrak{m}$ and thus $|\pi^{-1}(y)| = \dim_{A/\mathfrak{m}} B/\mathfrak{m}B$. Similarly, it folows that $B \otimes_A A_{\mathfrak{m}}$ is finite flat(hence free) over $A_{\mathfrak{m}}$ and hence $\mathrm{rank}_{A_{\mathfrak{m}}}(B \otimes_A A_{\mathfrak{m}}) = \dim_{A/\mathfrak{m}} B/\mathfrak{m}B = \dim_K(B \otimes_A K) = N$. Therefore, we see that $|\pi^{-1}(y)| = N$.

Before turning to prove the converse direction, we first show that $\dim(X) = \dim(Y)$. By Lemma 3.21, $\dim(X) = \dim(B)$ and $\dim(Y) = \dim(A)$. Since $\pi : \mathrm{MaxSpec}(B) \to \mathrm{MaxSpec}(A)$ is surjective, the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ contains all the closed points of $\mathrm{Spec}(A)$. If $I$ denotes the kernel of $A \to B$, then $A/I \hookrightarrow B$ is a finite *inclusion* of rings, and so by the Going Up Theorem, the image $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is the vanishing locus $V(I)$. Thus, we must have $V(I) \supseteq \mathrm{MaxSpec}(A)$. However since $A$ is Jacobson (by [BGR84, §5.2.6, Theorem 3]), this implies $V(I) = \mathrm{Spec}(A)$ and hence $I = 0$. Thus, $A \hookrightarrow B$ is a finite inclusion of rings and therefore $\dim(A) = \dim(B)$.

Now suppose $y \in Y$ is a point such that $|\pi^{-1}(y)| = N$. Let $\pi^{-1}(y) = \{x_1, \ldots, x_N\}$. We would like to show that $\pi$ is étale at each $x_i$. We have a canonical isomorphism $B \otimes_A \mathcal{O}_{Y,y} = (\pi_*\mathcal{O}_X)_y \cong \prod_{i=1}^N \mathcal{O}_{X,x_i}$ (see [Con99, pp. 481-482]). If $L$ denotes the fraction field of $\mathcal{O}_{Y,y}$, we have that $B \otimes_A L \cong \prod_{i=1}^N \mathcal{O}_{X,x_i} \otimes_{\mathcal{O}_{Y,y}} L$.

*Subclaim: For each $i$, the natural map $\mathcal{O}_{X,x_i} \to \mathcal{O}_{X,x_i} \otimes_{\mathcal{O}_{Y,y}} L$ is injective.*

*Proof of Subclaim:* Note that $\mathcal{O}_{X,x_i} \otimes_{\mathcal{O}_{Y,y}} L$ is the localisation of $\mathcal{O}_{X,x_i}$ at the (image inside $\mathcal{O}_{X,x_i}$ of the) multiplicative set $\mathcal{O}_{Y,y} \setminus \{0\}$. Thus, the

claim is equivalent to showing that $\mathcal{O}_{X,x_i}$ is a torsion-free $\mathcal{O}_{Y,y}$-module. Equivalently, we must show that the image of $\mathcal{O}_{Y,y} \setminus \{0\}$ inside $\mathcal{O}_{X,x_i}$ does not contain any zero-divisors of the ring $\mathcal{O}_{X,x_i}$. Since $\mathcal{O}_{X,x_i}$ is a reduced ring, the set of zero-divisors of $\mathcal{O}_{X,x_i}$ is the union of the minimal prime ideals of $\mathcal{O}_{X,x_i}$ ([Sta20, Tag 00EW]). Therefore, it suffices to prove that every minimal prime ideal $\mathfrak{q}$ of $\mathcal{O}_{X,x_i}$ contracts to the $(0)$-ideal of $\mathcal{O}_{Y,y}$. If we set $\mathfrak{q} \cap \mathcal{O}_{Y,y} =: \mathfrak{p}$, then $\mathcal{O}_{Y,y}/\mathfrak{p} \hookrightarrow \mathcal{O}_{X,x_i}/\mathfrak{q}$ a finite *inclusion* of domains and hence, $\dim(\mathcal{O}_{Y,y}/\mathfrak{p}) = \dim(\mathcal{O}_{X,x_i}/\mathfrak{q})$. We now have the chain of equalities:

$$\dim(\mathcal{O}_{Y,y}/\mathfrak{p}) = \dim(\mathcal{O}_{X,x_i}/\mathfrak{q}) = \dim(X) = \dim(Y) = \dim(\mathcal{O}_{Y,y})$$

where the second equality is from Lemma 3.24, the third from the previous paragraphs and the last from Lemma 3.23. But since $\mathcal{O}_{Y,y}$ is an integral domain the equality $\dim(\mathcal{O}_{Y,y}/\mathfrak{p}) = \dim(\mathcal{O}_{Y,y})$ is only possible if $\mathfrak{p} = (0)$. This completes the proof of the subclaim. □

The subclaim shows in particular, that $\mathcal{O}_{X,x_i} \otimes_{\mathcal{O}_{Y,y}} L$ must be non-zero for each $i$. But since, $\dim_L(B \otimes_A L) = N$, this is only possible if $L = \mathcal{O}_{X,x_i} \otimes_{\mathcal{O}_{Y,y}} L$ for each $i$. In particular, $\mathcal{O}_{X,x_i} \subseteq L$. However, since $\mathcal{O}_{Y,y}$ is a normal domain, and since $\mathcal{O}_{X,x_i}$ is finite over $\mathcal{O}_{Y,y}$, we get that $\mathcal{O}_{Y,y} = \mathcal{O}_{X,x_i}$, and so $\pi$ is evidently étale at $x_i$. □

*Proof of Theorem 4.1.* We induct on $d + n$ where $d := \dim(X)$. If $d = 0$, then $X$ is finite, hence algebraic. And if $d = n$, then $X = \mathbb{C}_p^n$ and we're done.

Suppose that $n > d \geq 1$. It follows from Lemma 3.22 that the set $S = \{x \in X : \dim(\mathcal{O}_{X,x}) < d\}$ is a *definable* subset of $\mathbb{C}_p^n$. Moreover, the closure of $S$ in $\mathbb{C}_p^n$, $\mathrm{cl}(S)$ is the union of the irreducible components of $X$ of dimension $< d$, and so by the induction hypothesis, $\mathrm{cl}(S)$ is an algebraic subset of $\mathbb{C}_p^n$. It suffices to then show that $X \setminus S$ is an algebraic subset of $\mathbb{C}_p^n$. Note that $X \setminus S$ is the union of the irreducible components of dimension $d$ and therefore $X \setminus S$ is a closed, analytic subset of $\mathbb{C}_p^n$. Thus, replacing $X$ by $X \setminus S$ we may assume that $X$ is *equidimensional* of dimension $d$.

*Finding a point $q \in \mathbb{P}^n(\mathbb{C}_p) \setminus \mathrm{cl}(X)$:* Embed $\mathbb{C}_p^n \subseteq \mathbb{P}^n(\mathbb{C}_p)$ inside projective $n$-space and denote the homogeneous coordinates of $\mathbb{P}^n(\mathbb{C}_p)$ by $Z_1, \ldots, Z_{n+1}$. Let $\mu$ denote the point $[1 : 0 : \ldots : 0] \in \mathbb{P}^n(\mathbb{C}_p) \setminus \mathbb{C}_p^n$, and consider the neighbourhood $\Delta := \{|Z_1| \geq |Z_2|, \ldots, |Z_1| \geq |Z_{n+1}|\} \subseteq \mathbb{P}^n(\mathbb{C}_p)$ of the point $\mu$. The neighbourhood $\Delta$ is naturally homeomorphic to the closed unit $n$-dimensional disk, $\mathcal{O}_{\mathbb{C}_p}^n$, via the map $\varphi : \Delta \to \mathcal{O}_{\mathbb{C}_p}^n$ that sends $[Z_1 : \ldots : Z_{n+1}] \mapsto \left(\frac{Z_2}{Z_1}, \ldots, \frac{Z_{n+1}}{Z_1}\right)$ and $S := \varphi(X \cap \Delta)$ is a definable subset of $\mathcal{O}_{\mathbb{C}_p}^n$ of dimension $\leq d$ contained in $\mathcal{O}_{\mathbb{C}_p}^{n-1} \times \mathcal{O}_{\mathbb{C}_p} \setminus \{0\}$. We note that since $\mathrm{cl}(S) \cap \mathcal{O}_{\mathbb{C}_p}^{n-1} \times \{0\} \subseteq \mathrm{Fr}(S)$, and from the Theorem of the Boundary (Theorem 3.18), since $\dim(\mathrm{Fr}(S)) < d \leq (n-1)$ we can find

a point $q \in \mathcal{O}_{\mathbb{C}_p}^{n-1} \times \{0\}$ such that $q \notin \mathrm{cl}(S)$, and pulling back via $\varphi$ to $\Delta$, we find a point $q \in \mathbb{P}^n(\mathbb{C}_p) \setminus \mathbb{C}_p^n$ such that $q \notin \mathrm{cl}_{\mathbb{P}^n(\mathbb{C}_p)}(X)$. The point $q \in \mathbb{P}^n(\mathbb{C}_p) \setminus \mathbb{C}_p^n = \mathbb{P}^{n-1}(\mathbb{C}_p)$ defines a line in $\mathbb{C}_p^n$.

Consider any $(n-1)$-dimensional linear subspace $\mathbb{H} \subseteq \mathbb{C}_p^n$, not containing the line defined by $q$, and let $\pi : \mathbb{C}_p^n \to \mathbb{H}$ denote the projection onto $\mathbb{H}$ with kernel being the line defined by $q$. We are free to make a linear change of coordinates on $\mathbb{C}_p^n$, and so we may even assume for simplicity that $q = [0 : \ldots : 0 : 1] \in \mathbb{P}^{n-1}(\mathbb{C}_p)$ and that $\pi : \mathbb{C}_p^n \to \mathbb{C}_p^{n-1}$ is the projection to the first $(n-1)$-coordinates.

**Lemma 4.11.** *The projection $\pi|_X : X \to \mathbb{C}_p^{n-1}$ is a finite morphism of rigid analytic spaces (endowing $X$ with the reduced induced structure).*

*Proof.* *$\pi|_X$ is quasi-finite:* Indeed, for $\underline{z} \in \mathbb{C}_p^{n-1}$, $\pi^{-1}(\underline{z}) \cap X$ is a closed analytic subset of the 1-dimensional line $\pi^{-1}(\underline{z})$ and is in addition definable. If $\dim(\pi^{-1}(\underline{z}) \cap X) = 1$, then $\pi^{-1}(\underline{z}) \subseteq X$, which would imply that $q = [0 : \ldots : 0 : 1] \in \mathrm{cl}_{\mathbb{P}^n(\mathbb{C}_p)}(X)$ contradicting our choice of $q$. Thus, $\dim(\pi^{-1}(\underline{z}) \cap X) = 0$, i.e. $\pi^{-1}(\underline{z}) \cap X$ is finite.

To show that $\pi|_X$ is a finite morphism, it thus remains to show that $\pi|_X$ is a proper morphism of rigid spaces ([BGR84, §9.6.3, Corollary 6]). In order to prove this, we consider the map $\pi|_X$ on the level of the associated Berkovich spaces. Note that $X$ being a closed analytic subvariety of rigid affine $n$-space, is a quasi-separated rigid space and has an admissible affinoid covering of finite type, and moroeover its associated Berkovich analytic space is 'good' in the sense of [Ber93, Remark 1.2.16 & §1.5]. Recall that the morphism $\pi|_X : X^{\mathrm{Berk}} \to \mathbb{A}_{\mathbb{C}_p}^{n-1,\mathrm{Berk}}$ of good $\mathbb{C}_p$-analytic spaces is proper if it is topologically proper and boundaryless (or 'compact and closed' in the terminology of [Ber90, pp. 50]).

*$\pi|_X$ is separated and topologically proper:* $\pi|_X$ is indeed separated. If $E(0, \underline{r})$ denotes the closed polydisc of polyradius $\underline{r}$ in $\mathbb{A}_{\mathbb{C}_p}^{n-1,\mathrm{Berk}}$, i.e. $E(0, \underline{r}) = \mathcal{M}(\mathbb{C}_p\{r_1^{-1}T_1, \ldots, r_{n-1}^{-1}T_{n-1}\})$, then we claim that $\pi^{-1}(E(0, \underline{r})) \cap X^{\mathrm{Berk}}$ is bounded in $\mathbb{A}_{\mathbb{C}_p}^{n,\mathrm{Berk}}$. If it weren't, there would be a sequence of points $x_i \in \pi^{-1}(E(0, \underline{r})) \cap X^{\mathrm{Berk}}$ with $|T_n(x_i)| \to \infty$ as $i \to \infty$. We may even find a sequence $x_i \in X$ since by [Ber90, Proposition 2.1.15], the set of rigid points is everywhere dense. But this would again imply that $q = [0 : \ldots : 0 : 1] \in \mathrm{cl}_{\mathbb{P}^n(\mathbb{C}_p)}(X)$ contradicting the choice of $q$. Since every compact subset of $\mathbb{A}_{\mathbb{C}_p}^{n-1,\mathrm{Berk}}$ is contained in some $E(0, \underline{r})$, it follows that the inverse image of compact sets under the map $\pi|_X : X^{\mathrm{Berk}} \to \mathbb{A}_{\mathbb{C}_p}^{n-1,\mathrm{Berk}}$ are compact. Thus, $\pi|_X$ is topologically proper.

$\pi|_X$ is boundaryless: Since $X^{\mathrm{Berk}} \hookrightarrow \mathbb{A}_{\mathbb{C}_p}^{n,\mathrm{Berk}}$ is a closed immersion, $\mathrm{Int}(X^{\mathrm{Berk}}/\mathbb{A}_{\mathbb{C}_p}^{n,\mathrm{Berk}}) = X^{\mathrm{Berk}}$. By [Ber93, Proposition 3.1.3 (ii)] it suffices to

note that $\mathrm{Int}(\mathbb{A}^{n,\mathrm{Berk}}_{\mathbb{C}_p}/\mathbb{A}^{n-1,\mathrm{Berk}}_{\mathbb{C}_p}) = \mathbb{A}^{n,\mathrm{Berk}}_{\mathbb{C}_p}$. To see this last equality, for any $x \in \mathbb{A}^{n,\mathrm{Berk}}_{\mathbb{C}_p}$ let $y = \pi(x)$, and choose an affinoid neighbourhood $E(0,\underline{r}) \subseteq \mathbb{A}^{n-1,\mathrm{Berk}}_{\mathbb{C}_p}$ containing $y$ in its interior. Choosing an $R \in |\mathbb{C}_p^\times|$ with $R > |T_n(x)|$, we see that $\chi_x : \mathbb{C}_p\{r_1^{-1}T_1, \ldots, r_{n-1}^{-1}T_{n-1}, R^{-1}T_n\} \to \mathcal{H}(x)$ is inner over $\mathbb{C}_p\{r_1^{-1}T_1, \ldots, r_{n-1}^{-1}T_{n-1}\}$, i.e. $x \in \mathrm{Int}(\mathbb{A}^{n,\mathrm{Berk}}_{\mathbb{C}_p}/\mathbb{A}^{n-1,\mathrm{Berk}}_{\mathbb{C}_p})$.

Therefore, the map $\pi|_X : X^{\mathrm{Berk}} \to \mathbb{A}^{n-1,\mathrm{Berk}}_{\mathbb{C}_p}$ is proper and hence by [Ber90, Proposition 3.3.2], so is $\pi|_X : X \to \mathbb{A}^{n-1,\mathrm{an}}_{\mathbb{C}_p}$. □

Since $\pi|_X : X \to \mathbb{A}^{n-1,\mathrm{an}}_{\mathbb{C}_p}$ is finite, the image $Y := \pi(X)$ is a closed analytic subvariety of $\mathbb{C}_p^{n-1}$, by [BGR84, §9.6.3, Proposition 3]. In addition, as $Y$ is a definable subset, by the induction hypothesis $Y$ is an algebraic subset of $\mathbb{C}_p^{n-1}$. Endowing $Y$ with its structure as a reduced closed affine algebraic subvariety of $\mathbb{A}^{n-1}_{\mathbb{C}_p}$, the morphism $\pi|_X$ gives rise to a finite, surjective morphism of rigid analytic spaces $\pi|_X : X \to Y^{\mathrm{an}}$.

**Lemma 4.12.** *There is a Zariski dense open $U \subseteq Y$, such that $\pi|_X^{-1}(U^{\mathrm{an}}) \to U^{\mathrm{an}}$ is a finite, étale surjection of rigid varieties.*

*Proof.* Let $\{Y_i\}_{1\leq i\leq r}$ denote the finitely many irreducible components of $Y$, thus $\{Y_i^{\mathrm{an}}\}_{1\leq i\leq r}$ being those of $Y^{\mathrm{an}}$. Let $U_i \subseteq Y_i \setminus \bigcup_{j\neq i} Y_j$, be a non-empty, *principal* open subset of $Y$ so that each $U_i$ is an integral (reduced and irreducible) open subvariety of $Y$, (hence $U_i^{\mathrm{an}}$ is a reduced and irreducible admissible open of $Y^{\mathrm{an}}$ [Con99, Theorem 5.1.3 (2)]). $U_i$ being a principal open subset of $Y \subseteq \mathbb{C}_p^{n-1}$, may be viewed as a closed affine subvariety of $\mathbb{C}_p^n$. By Lemma 4.9, the étale locus $E_i \subseteq U_i^{\mathrm{an}}$ of $\pi : \pi^{-1}(U_i^{\mathrm{an}}) \to U_i^{\mathrm{an}}$ is definable as it may be defined using a first-order formula expressing $E_i$ as the subset of points in $U_i^{\mathrm{an}}$ whose fiber under $\pi$ has cardinality equal to the generic fiber cardinality over $U_i^{\mathrm{an}}$. Moreover, the complement $U_i^{\mathrm{an}} \setminus E_i$ is a closed analytic subvariety of $U_i^{\mathrm{an}} \subseteq \mathbb{C}_p^n$ of dimension $< \dim(U_i) = d$. By the induction hypothesis, $U_i^{\mathrm{an}} \setminus E_i$ is a Zariski closed algebraic subset of $U_i$ and hence $E_i$ is a Zariski dense open of $U_i$. Now setting $U = \cup_i E_i$, completes the proof of the Lemma. □

Let $U$ be as in Lemma 4.12 above, let $\{U_j\}$ be the finitely many open connected components of $U$ and let $V_j := \pi|_X^{-1}(U_j)$. Suppose the fiber cardinality of $\pi|_{V_j} : \pi|_X^{-1}(U_j) \to U_j$ is $N_j$. The characteristic polynomial of $T_n|_{V_j}$ over $U_j$ (here $T_n$ being the last coordinate function of $\mathbb{C}_p^n$) is a polynomial of degree $N_j$ with coefficients in $\mathcal{O}_{Y^{\mathrm{an}}}(U_j^{\mathrm{an}})$. Moreover, since the $U_j$ are Zariski opens and since $X$ is definable it follows that the coefficients are also definable since they may be defined as symmetric polynomials in the fibers of $\pi|_X$. Hence, by Theorem 4.5 the characteristic

polynomial in fact has coefficients in $\mathcal{O}_Y(U_j)$. If $W \subseteq \mathbb{C}_p^{n-1}$ is a Zariski open subset such that $W \cap Y = U$, then it follows from the above that $X \cap (W \times \mathbb{C}_p)$ is a closed algebraic subset of $W \times \mathbb{C}_p$.

If we let $Z$ denote the Zariski closure of $X \cap (W \times \mathbb{C}_p)$ in $\mathbb{C}_p^n$, then $Z$ is also the closure of $X \cap (W \times \mathbb{C}_p)$ in the metric topology of $\mathbb{C}_p^n$, ([Con99, Theorem 5.1.3 (2)]) and hence $Z \subseteq X$. Moreover, $X \setminus (W \times \mathbb{C}_p) = \pi|_X^{-1}(Y \setminus U)$ and so $\dim(X \setminus (W \times \mathbb{C}_p)) < \dim(Y) \leq d$. By the induction hypothesis $X \setminus (W \times \mathbb{C}_p)$ is thus a closed algebraic subset of $\mathbb{C}_p^n$ and since $X = Z \cup (X \setminus (W \times \mathbb{C}_p))$, we get that $X$ is algebraic, finishing the Proof of Theorem 4.1. □

We obtain as an immediate Corollary:

**Corollary 4.13.** *Let $V$ be a reduced algebraic variety over $\mathbb{C}_p$, and let $X \subseteq V^{\mathrm{an}}$ be a closed analytic subvariety of the rigid analytic variety $V^{\mathrm{an}}$ associated to $V$, such that $X \subseteq V(\mathbb{C}_p)$ is definable in a tame structure on $\mathbb{C}_p$. Then $X$ is algebraic.*

For a *proper* rigid variety, every closed analytic subvariety is definable in the tame structure of the rigid subanalytic sets. Thus the familiar version of Chow's theorem for proper varieties follows from Theorem 4.1:

**Corollary 4.14** (Chow's theorem for proper varieties)**.** *Every closed analytic subset of the rigid analytic variety associated to a proper algebraic variety over $\mathbb{C}_p$ is algebraic.*

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