The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.

Good Luck!
Problem 1. Let $G$ be a finite group.

(a) Show that if $H \leq G$ is a proper subgroup, then there exists an $x \in G$ that is not contained in any subgroup conjugate to $H$.

(b) A maximal subgroup is a proper subgroup $M \leq G$ that is maximal for this property, i.e., if $M' \leq G$ is a proper subgroup of $G$ that contains $M$, then $M' = M$. Show that if all maximal subgroups of $G$ are conjugate, then $G$ is cyclic.

Problem 2. A chain of prime ideals of length $n$ in a commutative ring $R$ is an increasing sequence $P_0 \subset P_1 \subset \cdots \subset P_n$, where each $P_i$ is a prime ideal in $R$.

(a) Show that if $R$ is a PID, every chain of prime ideals has length 0 or 1.

(b) Exhibit a chain of prime ideals of length 2 in $\mathbb{Z}[x]$.

(c) Find a ring $R$ with a chain of prime ideals of length 2016.

Problem 3.

(a) State the structure theorem for modules over a PID.

(b) Suppose that $K$ is a field and $V$ a $K$-vector space of dimension 3. How many similarity classes of linear transformations $T : V \to V$ are there that satisfy $T^2(T - 1) = 0$? Among them, how many have $\dim \ker(T) = 1$? (Explain how you use part (a)!) Also, recall that linear transformations $S, T$ are called similar if there is a linear isomorphism $U : V \to V$ such that $S = UTU^{-1}$.

Problem 4.

(a) Prove or disprove that $f(x) = x^4 + 6x - 3$ is irreducible over the field $\mathbb{Q}(\sqrt[4]{5})$.

(b) Let $L$ be a finite Galois extension of a field $K$ with Galois group $\text{Gal}(L/K)$. Suppose that $F$ is a proper subfield of $L$ that contains $K$. Prove that $\cap_{\sigma \in \text{Gal}(L/K)} \sigma(F)$ is a Galois extension of $K$. Give complete statements of all results from Galois theory which are used in your solution.

Problem 5. Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of a finite group $G$. For $1 \leq j \leq r$, let $\rho_j$ be an irreducible representation of $G$ whose character equals $\chi_j$.

(a) Prove that if $x \in G$ and $x \neq e$, then there exists $j$ such that $\chi_j(x) \neq \chi_j(e)$.

(b) Let $y \in G$. Prove that if $\rho_j(y)$ is a scalar multiple of the identity operator for all $1 \leq j \leq r$, then $y$ belongs to the centre of $G$.

Problem 6. Let $R$ be a ring with 1 and let $M$ be a left $R$-module such that $M = S_1 \oplus S_2 \oplus \cdots \oplus S_n$, where each $S_i$ is a nonzero simple left $R$-submodule of $M$.

(a) Prove that any nonzero simple left $R$-submodule of $M$ is isomorphic to $S_i$ for some $i$.

(b) What additional conditions on the submodules $S_i$ guarantee that any nonzero simple left $R$-submodule of $M$ is equal to $S_i$ for some $i$? (Justify your answer.)