Department of Mathematics
University of Toronto

Topology Comprehensive Exam

September 8, 2017

Topology I

Problem 1. Let K and L be embedded (i.e. regular) submanifolds of a smooth manifold M.

a) What does it mean for K and L to intersect transversally? State a precise definition.
b) Prove that if K and L intersect transversally, then $K \cap L$ is an embedded submanifold, and compute its dimension in terms of $\dim K, \dim L, \dim M$.
c) Let $S^2 \subset \mathbb{R}^3$ be given by $\{(x_1, x_2, x_3) \ : \ x_1^2 + x_2^2 + x_3^2 = 1\}$. Write the tangent bundle of S^2 as a subset of $T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$ and prove that it is an embedded submanifold.

Problem 2. Let M be a smooth, connected manifold and fix any pair of points $p, q \in M$. Prove that there exists a diffeomorphism $\varphi : M \to M$ taking p to q, i.e. $\varphi(p) = q$.

Problem 3. A 2-form $\omega \in \Omega^2(M)$ on the smooth $2n$-dimensional manifold M is called *symplectic* when it is closed, i.e. $d\omega = 0$, and its n^{th} exterior power $\omega^n = \omega \wedge \cdots \wedge \omega$ does not vanish at any point. Prove the following:

a) The 2-sphere S^2 has a symplectic form.
b) For all n, the torus T^{2n} has a symplectic form.
c) For $n \geq 2$, the sphere S^{2n} does not have any symplectic forms.

Topology II

Problem 4. Give an example of closed manifold M such that its fundamental group is isomorphic to the cyclic group of order 3 (that is, to $\mathbb{Z}_3$). Prove your answer.

Problem 5. Find all homology groups of the complement $S^2 \times S^2 \setminus \Delta(S^2)$, where $\Delta(S^2)$ denotes the diagonal $\{(x, x), x \in S^2\} \subset \{(x, y), x \in S^2, y \in S^2\} = S^2 \times S^2$.

Problem 6. Show that if a closed orientable manifold M of dimenion $2k$ has $H_{k-1}(M; \mathbb{Z})$ torsion-free, then $H_k(M; \mathbb{Z})$ is also torsion-free.