Groupoid Models of Irrational Rotation Algebras

by

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Abstract

We show how to construct certain groupoid models of irrational rotation algebras and show how this can be used to to find projections in the algebra.

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Introduction

Groupoid C^*-algebras first appeared in the works of Jean Renault and Alain Connes and have been studied ever since. A basic question one may ask is can a given C^*-algebra A be constructed as the C^*-algebra of some groupoid? One of the most general answers to this has been given by Xin Li who has shown in [3] that every classifiable simple stably finite C^*-algebra has a Cartan subalgebra. Paired with the work of Jean Renault in [5] where it is shown that a C^*-algebra has a Cartan subalgebra if and only if it is a C^*-algebra of a twisted étale essentially principal groupoid, this leads to the conclusion that every classifiable C^*-algebra has a groupoid model.

The Jiang-Su algebra $\mathcal{Z}$ plays a crucial role in classification theory, every classifiable C^*-algebra A is $\mathcal{Z}$-stable, meaning that to $A \otimes \mathcal{Z}$. In [8] it was shown that the Jiang-Su algebra $\mathcal{Z}$ has infinitely many non-isomorphic groupoid models. This was used in [9] to point out that if a $\mathcal{Z}$-stable C^*-algebra has a groupoid model, then it will consequently also have infinitely many non-isomorphic groupoid models.

The work in this thesis is based on a paper of George Elliott and Dickson Wong, where they give a groupoid construction of the Riffel projection of the irrational rotation algebra. We further explore their construction in order to find groupoid models of the same algebra, thus exploring the non-uniqueness phenomena explained above.

Review of topological groupoids and their C^*-algebras

We begin by giving a brief overview of topological groupoids and their C^*-algebra in order to fix the language and notation which will be used later on.

Étale groupoids

Definition 1. A *groupoid* consists of a set of arrows $\mathcal{G}$ with a distinguished subset of objects $\mathcal{G}^{(0)}$. The set of arrows is equipped with two maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ specifying the range and source of the arrow. The set of composable arrows $\mathcal{G}^{(2)}$ is defined as the set of pairs of arrows (γ_1, γ_2) such that $s(\gamma_1) = r(\gamma_2)$. These sets come with a multiplication map $(\gamma_1, \gamma_2) \to \gamma_1\gamma_2$ from $\mathcal{G}^{(2)} \to \mathcal{G}$ and inverse map $\gamma \to \gamma^{-1}$ from $\mathcal{G} \to \mathcal{G}$. These sets and maps adhere to list of axioms similar to those of a group.

To construct a C^*-algebra we need the groupoid to have a topology.

Definition 2. A *topological groupoid* is a groupoid $\mathcal{G}$ endowed with a locally compact topology. We ask that the object space $\mathcal{G}^{(0)}$ to be Hausdorff in the relative topology, the range, source, multiplication and inverse maps are asked to be continuous.

In general one more structure on the groupoid is necessary to construct a C^*-algebra. In order to define a convolution product we need a system of Haar measures. We will avoid this since the groupoids we will be considering are étale.

Definition 3. A topological groupoid is called *étale* if both the range map and source maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ are local homeomorphisms.

The simplest examples of such groupoids are transformation groupoids of discrete groups acting on locally compact Hausdorff spaces.

C^*-algebra of étale groupoids

The first algebra one associates to an étale groupoid $\mathcal{G}$ is the convolution algebra $C_c(\mathcal{G})$.

Definition 4. Given a Hausdorff étale groupoid $\mathcal{G}$ we equip the vector space of compactly supported functions $C_c(\mathcal{G})$ with a multiplication structure using the *convolution product*

$$(f * g)(\gamma) = \sum_{\alpha \in \mathcal{G}^{(\gamma)}} f(\alpha) g(\alpha^{-1}\gamma)$$

The $*$-operation is defined by $f^*(\gamma) = \overline{f(\gamma^{-1})}$.

Using these operation the vector space $C_c(\mathcal{G})$ becomes a $*$-algebra called the *convolution algebra* of the groupoid $\mathcal{G}$. We should note here that the sum in the convolution product is well defined. Since $\mathcal{G}$ is étale the set of arrows $\mathcal{G}^{r(\gamma)}$, which is the set of arrows with the same range as γ, is a discrete set. Since both the functions f, g have compact support, this implies that the sum in the convolution product is in fact finite.

The convolution algebra $C_c(\mathcal{G})$ can be completed to a C^*-algebra by considering the norm $\|a\| = sup\{\|\pi(a)\| — \pi : C_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H})\}$, this completion is called the (full) groupoid C^*-algebra $C^*(\mathcal{G})$.

The rotation algebra as a groupoid C^*-algebra

The rotation algebra A_θ is typically defined as the universal C^*-algebra generated by two unitary elements u and v satisfying the relation $vu = quv$, where $q = e^{2\pi i\theta}$, with $\theta \in [0, 1)$. We will now give a description of the rotation algebra A_θ as a groupoid algebra.

Let the group of integers $\mathbb{Z}$ act on the circle $\mathbb{T}$ by a rotation with angle θ. The transformation groupoid $\mathcal{G}_\theta$ for this action is given the product topology $\mathbb{Z} \times \mathbb{T}$, so it can be viewed as an infinite stack of circles each labeled by an integer. We will denote a point in this space by (z, n), where z is a point on the circle and n is an integer, this point is the groupoid arrow going from the point z to the point $z + n\theta$.

A function with compact support on $\mathcal{G}_\theta$ is a finite linear combination of functions on the individual circles in the stack. To analyze multiplication in this groupoid algebra it is enough to see how to multiply two functions f_n and f_m, whose supports belong to the n-th and m-th circles respectively. By definition of the convolution product we get $(f_n * f_m)(z, k) = \sum_k f_n(z, l) f_m(z + l\theta, k - l)$, from which we see that we will only get an interesting result if $l = n$ and $k = m + n$. This implies that $f_n * f_m$ is a function with support contained in the $m + n$-th circle defined by the formula $f_n * f_m(z, n + m) = f_n(z) f_m(z + n\theta)$.

Consider the functions u defined to be equal to z on the circle with label 0 and v defined to be equal to 1 on the circle with label 1. Clearly both u and v are unitaries and from the paragraph above we can see that $u * v(z) = z$ as a function on the circle with label 1 while $v * u(z) = e^{2\pi i\theta} z$, so these two unitaries satisfy the relation defining the rotation algebra A_θ. The $*$-algebra generated by u, v and u^*, v^* contains all polynomial functions supported on a finite set of circles, in particular it is dense in the algebra of all compactly supported functions on this groupoid.

It now becomes important whether or not θ is rational or irrational.

Definition 5. A topological groupoid is said to be *minimal* if for every object $x \in \mathcal{G}^{(0)}$ its orbit is dense in the object space $\mathcal{G}^{(0)}$.

For a rational rotation number it is clear that the groupoid $\mathcal{G}_\theta$ is not minimal. From ergodic theory we know that for an irrational rotation the orbit of any point on the circle is dense implying that the groupoid $\mathcal{G}_\theta$ is minimal. Minimality of the groupoid corresponds to the simplicity of the groupoid algebra, the following theorem can be found in [1].

Theorem 1. Let $\mathcal{G}$ be an amenable étale groupoid. Then $C^*(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is effective and minimal.

A groupoid is *effective* if the interior of its isotropy group is equal to the set of identity arrows. In our case case the isotropy group consists only of the identity arrows so the above theorem can be applied to $\mathcal{G}_\theta$.

Since the groupoid C^*-algebra $C^*(\mathcal{G}_\theta)$ contains the $*$-algebra generated by u, v satisfying $vu = quv$ and is simple we can conclude that it is in fact isomorphic to the universal C^*-algebra satisfying this relation A_θ. For example, restricting to the circle $y = 0$ we will get a circle with rotation number $\frac{1}{\theta}$.

Kronecker foliation

The Kronecker foliation groupoid $\mathcal{K}_\theta$ is the transformation groupoid of the real numbers $\mathbb{R}$ acting on the torus $\mathbb{T}^2$ by $t.(x,y) = (x+t, y+t\theta)$. In this case the group action is not discrete, so this groupoid is not étale, but this can be addressed by restricting to various transversals.

Given a closed subspace X of the object space of a groupoid $\mathcal{G}$ we can restrict to this subset by consider the groupoid with object space X defined by the set of all arrows γ such that both its source and range belong to the subset X.

For example we can restrict it to circle defined by $x = 0$. To get back to the circle traveling along the foliation we would have to travel for a time t equal to some integer n, from a point $(0,y)$ to the point $(0, y+n\theta)$. It is easy to see that this groupoid is precisely the one described above for the irrational rotation.

The group $SL_2(\mathbb{Z})$ acts on the torus, a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps a point (x,y) to the point $(ax+by, cx+dy)$. The circle $x = 0$ would get mapped to the circle (by, dy). The groupoid corresponding to this circle will also be a rotation groupoid with rotation number $A(\theta) = \frac{a\theta+b}{c\theta+d}$.

Two circle groupoid

We now analyze what will happen if we restrict the Kronecker foliation to a union of two circles of the form discussed in the previous section. For reasons to become apparent later we will "desingularize" this groupoid by replacing the point $(0,0)$ where the two circles intersect with two points $(0,0)_1$ and $(0,0)_2$, so that the object space is now a disjoint union of two circles. Every arrow going either to or from the origin $(0,0)$ is replaced by a two arrows, one for each of the new points. This groupoid will be denoted by $\mathcal{G}_{\theta_1,\theta_2}$ where θ_1 and θ_2 are the rotation numbers of the two chosen circles. We will now give a more detailed description of this groupoid.

It should first be noted that this two circle groupoid $\mathcal{G}_{\theta_1,\theta_2}$ contains disjoint copies of each of groupoids $\mathcal{G}_{\theta_1}$ and $\mathcal{G}_{\theta_2}$. These two copies are the arrows traveling from circle **1** to itself and circle **2** to itself. We now analyze the set of arrows going from one circle to the other.

Lemma 1. As a topological space the set of arrows going from circle **1** to circle **2** is a copy of the real line $\mathbb{R}$.

Proof. Draw the Kronecker foliation in the plane $\mathbb{R}^2$. The flow lines are lines with slope θ and the two circles correspond to the lines with slope $\frac{d_1}{b_1}$ and $\frac{d_2}{b_2}$, assuming $\theta_1 = \frac{a_1\theta+b_1}{c_1\theta+d_2}$ and $\theta_2 = \frac{a_2\theta+b_2}{c_2\theta+d_2}$. For each real number t there is precisely one point one line **1** which is transported by the flow lines to line **2** in time t. Since θ is irrational it cannot happen so that two of these arrows in fact define the same after taking the quotient by $\mathbb{Z}^2$. Thus the the time parameter t defines a homeomorphism between the space of arrows going from circle **1** to circle **2** and the real line $\mathbb{R}$. $\square$

From this description we can see that this groupoid is a locally compact Hausdorff étale principal groupoid. Moreover the following lemma becomes obvious.

Lemma 2. The two circle groupoid $\mathcal{G}_{\theta_1,\theta_2}$ is minimal.

This implies that the algebra $C^*(\mathcal{G}_{\theta_1,\theta_2})$ is simple.

A crucial upside of the object space of $\mathcal{G}_{\theta_1,\theta_2}$ being disconnected is that the two circles define two complementary projections $p_1 + p_2 = 1$ in the algebra $C^*(\mathcal{G}_{\theta_1,\theta_2})$.

To determine the isomorphism class of the C^*-algebra of this groupoid we proceed in two steps. We first establish that $C^*(\mathcal{G}_{\theta_1,\theta_2})$ is Riefell-Morita equivalent to an irrational rotation algebra and then use Riefell's classification of this Morita equivalence class to determine which algebra $C^*(\mathcal{G}_{\theta_1,\theta_2})$ is isomorphic to.

Rieffel-Morita Equivalence for groupoid C^*-algebras

Following the work of Jean Renault and Dana Williams [2] Rieffel-Morita equivalence of groupoid $C^* - algebras$ can be established by finding a *groupoid equivalence*.

Definition 6. Given locally compact Hausdorff groupoids $\mathcal{G}$ and $\mathcal{H}$ with open range and source maps, a locally compact Hausdorff space Z with a left $\mathcal{G}$-action and a right $\mathcal{H}$-action is called a $(\mathcal{G},\mathcal{H})$-equivalence if the following holds:

- The left $\mathcal{G}$-action commutes with the right $\mathcal{H}$-action.
- The two anchor maps $Z \to \mathcal{G}^{(0)}$ and $Z \to \mathcal{H}^{(0)}$ are open and induce isomorphisms $Z/\mathcal{H} \to \mathcal{H}^{(0)}$ and $\mathcal{G}\backslash Z \to \mathcal{G}^{(0)}$.

The crucial result concerning groupoid equivalences is that they imply Rieffel-Morita equivalence between the associated groupoid C^*-algebras (see [2]).

Theorem 2. Given two locally-compact Hausdorff groupoids $\mathcal{G}$ and $\mathcal{H}$ and a $(\mathcal{G},\mathcal{H})$-equivalence Z, the space $C_c(Z)$ can be naturally completed into a $C^*(\mathcal{G}) - C^*(\mathcal{H})$ imprimitivity bimodule. In particular the groupoid C^*-algebras $C^*(\mathcal{G})$ and $C^*(\mathcal{H})$ are Rieffel-Morita equivalent.

Recall that the two circle groupoid $\mathcal{G}_{\theta_1,\theta_2}$ consists of a copy of the single circle groupoid $\mathcal{G}_{\theta_1}$, a copy of the single circle groupoid $\mathcal{G}_{\theta_2}$ and arrows going in between the two circles. Let Z be the set of arrows in the two circle groupoid going from circle **1** to circle **2**. The groupoid $\mathcal{G}_{\theta_1}$ acts on Z on the left by precomposition, the groupoid $\mathcal{G}_{\theta_2}$ acts on Z on the right by composition. Associativity of the multiplication map implies that these actions commute, the two anchor maps are open since the groupoid is étale and the induced maps $Z/\mathcal{G}_\theta \to \mathcal{G}_\theta{}^{(0)}$ and $\mathcal{G}_\theta \backslash Z \to \mathcal{G}_\theta{}^{(0)}$ are clearly isomorphisms. This proves the following theorem.

Theorem 3. The space of arrows Z going from circle **1** to circle **2** with the natural left action of $\mathcal{G}_{\theta_1}$ and right action of $\mathcal{G}_{\theta_2}$ is a $(\mathcal{G}_{\theta_1} - \mathcal{G}_{\theta_2})$-equivalence.

Together with theorem 2 this reproves half of result of Marc Rieffel (see [6]) which states that two irrational rotation algebras A_{θ_1} and A_{θ_2} are Rieffel-Morita equivalent if and only if the rotation numbers θ_1 and θ_2 differ by an element of $SL_2(\mathbb{Z})$.

As mentioned in the previous section the two circles in the object space of $\mathcal{G}_{\theta_1,\theta_2}$ define complementary projections $p_1 + p_2 = 1$ such that the two corners $p_1 C^*(\mathcal{G}_{\theta_1,\theta_2})p_1$ and $p_2 C^*(\mathcal{G}_{\theta_1,\theta_2})p_2$ are the irrational rotation algebras A_{θ_1} and A_{θ_2} respectively. Thus the algebra $C^*(\mathcal{G}_{\theta_1,\theta_2})$ is none other but the linking algebra of A_{θ_1} and A_{θ_2}. In particular we have

Theorem 4. The groupoid algebra $C^*(\mathcal{G}_{\theta_1,\theta_2})$ is Rieffel-Morita equivalent to the irrational rotation algebra A_θ.

We will need the following result of Marc Rieffel from [7]

Theorem 5. For an irrational θ the only unital C^*-algebras which are Rieffel-Morita equivalent to the irrational rotation algebra A_θ are algebras of the form $M_n(A_{\theta'})$, where θ' is in the same orbit of $SL_2(\mathbb{Z})$ as θ.

This substantially narrows down the list of possibilities for the isomorphism class of $C^*(\mathcal{G}_{\theta_1,\theta_2})$. To completely determine the isomorphism class we will need to analyze the values of the trace on the projections p_1, p_2.

Traces and measures

From [6] we know that the irrational rotation algebra A_θ has a unique trace and that the the range of this trace on projections belongs to the set $(\mathbb{Z} + \theta\mathbb{Z}) \cap [0,1]$. This allows us to determine the rotation number by looking at the values of the trace on projections.

Similarly to how for a commutative C^*-algebra $C(X)$ traces correspond to suitable measures on X, for a groupoid C^*-algebra $C^*(\mathcal{G})$ traces correspond to suitable invariant measures on the object space $\mathcal{G}^{(0)}$. Recall that a groupoid is étale if both the range and source maps are local homeomorphisms.

Definition 7. A subset of arrows B in a groupoid $\mathcal{G}$ is called a *bisection* if the restriction of the range and source maps to B are local homeomorphisms.

Bisections in a groupoid form an inverse semigroup and this a suitable notion for how a groupoid acts on subsets of its object space.

Definition 8. A regular Borel measure μ on the object space of a groupoid $\mathcal{G}^{(0)}$ is said to be *invariant* if for any bisection B we have $\mu(r(B)) = \mu(s(B))$.

The following result can be found in the lectures note of Ian Putnam [4]

Theorem 6. For a locally compact Hausdorff étale groupoid $\mathcal{G}$ with compact object space, every invariant regular Borel measure μ defines a trace on the convolution algebra $C_c(\mathcal{G})$ via the formula

$$\tau(f) = \int_{\mathcal{G}^{(0)}} f(x) d\mu$$

This trace extends to the groupoid C^*-algebra $C^*(\mathcal{G})$.

For our two circle groupoid $\mathcal{G}_{\theta_1,\theta_2}$, every invariant measure μ will be the sum of an invariant measure μ_1 on the first circle and an invariant measure μ_2 on the second circle. Considering only the arrows mapping the first circle to itself we observe the measure μ_1 has to be invariant under rotations by multiples of θ_1. Since the set $\{n\theta_1\}_{n\in\mathbb{Z}}$ is dense in $\mathbb{R}$ we can conclude that the measure μ_1 must be invariant under all rotations, so it must be a multiple of the standard Lebesgue measure on the circle. The same is true for the second circle. We will normalize the measure μ so that $\mu(\mathcal{G}^{(0)}) = 1$, abusing notation let μ_1 and μ_2 be the measures of the first and second circle respectively, so that $\mu_1 + \mu_2 = 1$. The arrows going between the two circles determine the ratio of these two measures. To simplify calculations lets assume that the first circle chosen is $x = 0$,so that $\theta_1 = \theta$, the more general case is computed similarly by applying a suitable $SL(\mathbb{Z})$ transformation. By yet again carefully looking at the flow lines between the two circle we can compute that

Lemma 3. For the two circle groupoid $\mathcal{G}_{\theta,\theta'}$ the ratio $\frac{\mu_2}{\mu_1} = \theta'$.

This lemma implies that we have constructed the unique normalized trace τ on the two circle groupoid algebra $\mathcal{G}_{\theta,\theta'}$. From this construction we see that the values of this trace τ on the two projections p_1, p_2 are precisely the measures of the two circles μ_1, μ_2. These two trace values μ_1, μ_2 can be found from ratio in lemma 3 and the condition that $\mu_1 + \mu_2 = 1$ to conclude that $\mu_2 = \frac{1}{1+\theta'}$. Another way to express μ_2 would be to set $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, this is the fractional linear transformation which flips the rotation number, set $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, this is the fractional linear transformation which adds 1 to the rotation number, then if $\theta' = A(\theta)$, we have $\mu_2 = STA(\theta)$.

By using theorem 5 and this trace calculation we can conclude the following.

Theorem 7. The groupoid C^*-algebra $C^*(\mathcal{G}_{\theta,\theta'})$ is isomorphic to the irrational rotation algebra $A_{\frac{1}{1+\theta'}}$.

By choosing different values of θ' we see that we have constructed groupoid models for all possible irrational rotation algebras. As mentioned previously the two circles in the object space give a way to clearly see projections in these algebras, without having to solve any functional equations.

More circles

Instead of considering two circles cut out of the Kronecker foliation, we may instead take any finite number of different circles. We can repeat the "desingularization" process by replacing their common point of intersection $(0,0)$ with a new point for each of the circles. The resulting n-circle groupoid will be similar to the two circle groupoid, it will contain a disjoint copy of each of the chosen one circle groupoids and the space of arrows going in between any two of these circles will be isomorphic to a copy of the real line $\mathbb{R}$. The same arguments as above can be used to see that the C^*-algebra of this groupoid will be Rieffel-Morita equivalent to the irrational rotation algebra A_θ. The same arguments can be used to build an invariant measure corresponding to the unique trace, although the analog of lemma 3 becomes difficult to cleanly write down, but nevertheless the same conclusion can be made: the C^*-algebra of this n-circle groupoid will be an irrational rotation algebra. We have thus constructed groupoid models of the irrational rotation algebras whose objects spaces are arbitrary finite union of circles.

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