PSEUDO-ANOSOV HOMEOMORPHISMS CONSTRUCTED USING POSITIVE DEHN TWISTS

by

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ABSTRACT

Pseudo-Anosov homeomorphisms constructed using positive Dehn twists

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The mapping class group is the group orientation preserving homeomorphisms of a surface up to isotopy. The mapping class group encodes information about the symmetries of a surface. We focus on studying the pseudo-Anosov mapping classes, which are the elements of the group that mix the underlying surface in a complex way. These maps have applications in physics, notably in fluid dynamics, since we can stir a disk of fluid to create topological chaos [BAS00], and in the study of magnetic fields since pseudo-Anosov maps create odd magnetic fields [Gil93]. Pseudo-Anosov maps also appear in industrial applications such as food engineering and polymer processing [GTF06].

We introduce a construction of pseudo-Anosov homeomorphisms on n-times punctured spheres and surfaces with higher genus using only sufficiently many positive half-twists [Ver19]. These constructions can produce explicit examples of pseudo-Anosov maps with various number-theoretic properties associated to the stretch factors, including examples where the trace field is not totally real and the Galois conjugates of the stretch factor are on the unit circle.

We construct explicit examples of geodesics in the mapping class group and show that the shadow of a geodesic in mapping class group to the curve graph does not have to be a quasi-geodesic [RV]. We also show that the quasi-axis of a pseudo-Anosov element of the mapping class group may not have the strong contractibility property. Specifically, we show that, after choosing a generating set carefully, one can find a pseudo-Anosov homeomorphism f, a sequence of points w_k and a sequence of radii r_k so that the ball $B(w_k, r_k)$ is disjoint from a quasi-axis a of f, but for any projection map from mapping class group to a, the diameter of the image of $B(w_k, r_k)$ grows like $\log(r_k)$.

To my parents and brother,
thank-you for your endless support.

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PUBLICATIONS

Parts of this thesis have been submitted for publication and are currently available on the arXiv.

The content of Chapter 2 is presented in [Ver19] (submitted for publication and available on the arXiv).

The content of Chapter 3 is presented in [RV] (submitted for publication and available on the arXiv).

CONTENTS

1

MAPPING CLASS GROUPS

In this thesis, we will study the geometry and topology of the mapping class group. The mapping class group of a surface is defined to be the isotopy classes of orientation preserving homeomorphisms of a surface. The study of the mapping class group first dates back to Dehn in the 1920s, and has become a central topic in mathematics since the work of Birman, Hatcher and Thurston. The mapping class group has many connections to other branches of mathematics which includes algebraic geometry, dynamics, geometric group theory, low-dimensional topology, number theory, and Teichmüller theory.

In this section, we will provide a brief overview of mapping class groups, providing the reader with the background required for this thesis. While an effort has been made to include all of the relevant background information required to understand this thesis, we recommend reading *A Primer on Mapping Class Groups* by Benson Farb and Dan Margalit [FM12] for a thorough treatment of the topic.

1.1 DEFINING THE MAPPING CLASS GROUP

In this thesis, we will be studying the mapping class groups of surfaces. A *surface* is defined to be a 2-dimensional manifold. Möbius proved the following result which provides us with a classification of surfaces. For a proof of this theorem, one could see, for example [Ish96].

Theorem 1.1 (Möbius)**.** *Any closed, connected, orientable surface is homeomorphic to the connect sum of a 2-dimensional sphere with $g \geq 0$ tori. Any compact, connected, orientable surface is obtained from a closed surface by removing $b \geq 0$ open disks with disjoint closures. The set of homeomorphism types of compact surfaces is in bijective correspondence with the set $\{(g, b) : g, b \geq 0\}$.*

The g in Theorem 1.1 is the *genus* of the surface, and b is the number of *boundary components*. By removing n points from the interior of S, we obtain a noncompact surface. When we remove these points, we say that the resulting surface has n *punctures*.

When we use the word "surface" in this thesis, we mean a compact, connected, oriented surface which is possibly punctured. We use the triple (g, b, n) to specify our surface, the notation $S = S_{g,n,b}$ for a surface of genus

g with n punctures and b boundary components, and the notation $S_{g,n}$ to denote a surface of genus g, with n punctures and empty boundary. The *Euler characteristic* of a surface S is a topological invariant which is used to describe the shape or structure of the surface, regardless of how the surface is bent or stretched. The Euler characteristic is defined to be

$$\chi(S) = 2 - 2g - (b + n).$$

Since $\chi(S)$ is an invariant of the homeomorphism class of S, we can determine S, up to homeomorphism, by any three of the numbers g, b, n, and $\chi(S)$.

The curvature of the surface lets us determine whether a surface admits a *hyperbolic metric*. Intuitively, the curvature is how much our surface deviates from being a plane. A surface S is said to admit a hyperbolic metric if there exists a complete, finite-area Riemannian metric of S with constant curvature -1 where the boundary of S (if nonempty) is totally geodesic. The following theorem tells us that we can use the Euler characteristic to determine whether or not a surface admits a hyperbolic metric.

Theorem 1.2. *Let S be any surface (perhaps with punctures or boundary). If $\chi(S) < 0$ then S admits a hyperbolic metric. If $\chi(S) = 0$ then S admits a Euclidean metric.*

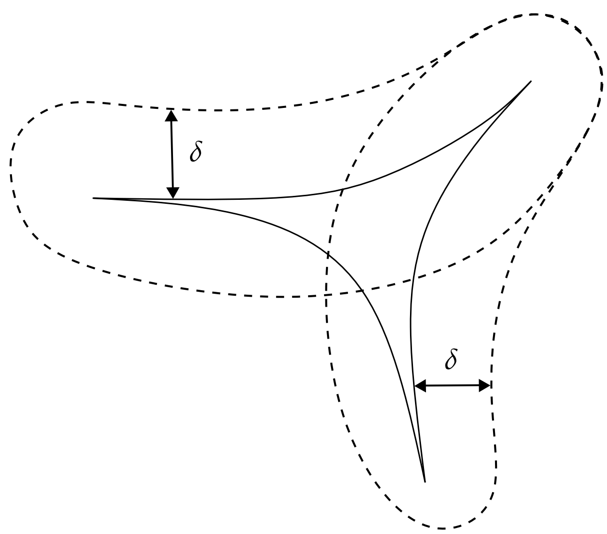


Figure 1.1: A δ-thin triangle.

We say that a geodesic metric space X is *Gromov hyperbolic* (or *δ-hyperbolic*) if it satisfies the *thin triangles condition*. We say that a geodesic metric space X satisfies the thin traingles condition if there exists some $\delta \geq 0$ such that, for any x, y and $z \in X$, the geodesic $[xz]$ is contained in a

δ-neighborhood of $[xy] \cup [yz]$, see Figure 1.1. We note that this is only one of several equivalent conditions for X to be Gromov hyperbolic.

A *closed curve* in the surface is a continuous map $S^1 \to S$. A closed curve is said to be *simple* if it is embedded and is said to be *essential* if it is not homotopic to a point, a puncture, or a boundary component. A *multicurve* in a surface is the union of a finite collection of disjoint simple closed curves in the surface.

One way to count the number of intersections between homotopy classes of closed curves is by counting the minimal number of unsigned intersections. The *geometric intersection number* between free homotopy classes a and b of simple closed curves in a surface is defined to be the minimal number of intersection points between a representative curve in the class a and a representative curve in the class b:

$$i(a, b) = \min\{|\alpha \cap \beta| : \alpha \in a, \beta \in b\}.$$

Notice that the geometric intersection number is symmetric. If $\alpha \in a$ and $\beta \in b$ are such that $i(a, b) = |\alpha \cap \beta|$, then α and β are considered to be in *minimal position*.

Let $\mathrm{Homeo}^+(S, \partial S)$ denote the group of orientation-preserving homeomorphisms of S that restrict to the identity on ∂S. Endow this group with the compact-open topology. We define the *mapping class group* of S, denoted $\mathrm{Map}(S)$, to be the group

$$\mathrm{Map}(S) \ = \pi_0(\mathrm{Homeo}^+(S, \partial S)).$$

In other words, $\mathrm{Map}(S)$ is the group of isotopy classes of elements of $\mathrm{Homeo}^+(S, \partial S)$ where isotopies are required to fix the boundary pointwise. The elements of $\mathrm{Map}(S)$ are called *mapping classes* and we apply elements of the mapping class group from right to left. The *pure mapping class group*, denoted $\mathrm{PMap}(S)$, is the subgroup of the mapping class group which fix the punctures pointwise.

An intuitive way to think of the mapping class group is as follows: Let S be a surface and let $f, g \in \mathrm{Homeo}^+(S, \partial S)$. Consider the resulting surfaces $f(S)$ and $g(S)$. If we can bend and stretch the surface $f(S)$ obtain the surface $g(S)$, then f and g belong in the same isotopy class. If we need to cut the surface $f(S)$ at any point in order to obtain $g(S)$, then f and g are in different isotopy classes.

Example 1.3 (Examples of mapping classes)**.** In Figure 1.2 we indicate a homeomorphism of order 6 for $g = 6$. This example is able to be generalized to show that for a surface S_g there exists a homeomorphism of order g.

The mapping class represented by ϕ also has order g. This can be seen by considering a simple closed curve α in S_g and noticing that $\alpha, \phi(\alpha), \ldots \phi^{g-1}(\alpha)$ are pairwise nonisotopic, and $\phi^g(\alpha) = \alpha$.

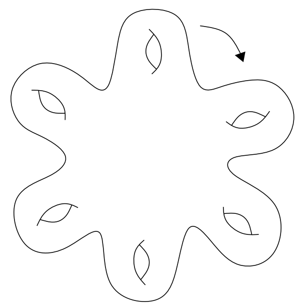


Figure 1.2: By rotating the surface of genus 6, we obtain an order 6 element of $\mathrm{Map}(S_6)$.

To see another nice example, take a $(4g+2)$-gon and glue together the opposite sides together to obtain a surface of genus g, S_g. We can get elements of $\mathrm{Map}(S_g)$ by rotating the $(4g+2)$-gon by any number of "clicks".

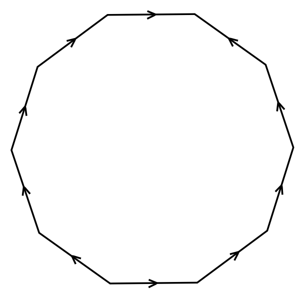


Figure 1.3: A rotation of $2\pi/10$ will give an order 10 element of S_2.

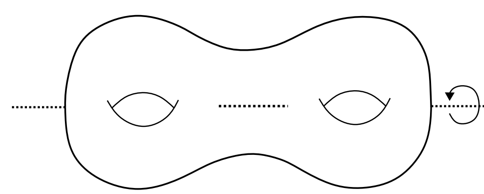


Figure 1.4: Hyperelliptic involution of a surface of genus 2.

If we rotate the $(4g+2)$-gon by $(2g+1)$ clicks, we obtain a "hyperelliptic involution". To visualize the hyperelliptic involution, we consider the surface S_g in $\mathbb{R}^3$ and rotate the surface by π around the axis indicated in Figure 1.4.

Computing the simplest mapping class group

Working directly from the definitions, we will describe the mapping class group of the closed disk, D^2. This computation is frequently used to help compute the mapping class group of other surfaces.

Lemma 1.4 (Alexander lemma)**.** *The group* $\mathrm{Map}(D^2)$ *is trivial.*

Lemma 1.4 tells us that if we have a homeomorphism ϕ of D^2 which is the identity on the boundary ∂D^2, then it is isotopic to the identity.

Proof. Identify D^2 with the closed unit disk in $\mathbb{R}^2$. Let $\phi\colon D^2 \to D^2$ be a homeomorphism which is the identity on the boundary. Define

$$F(x,t) = \begin{cases} (1-t)\phi\left(\frac{x}{1-t}\right) & 0 \le |x| < 1-t \\ x & 1-t \le |x| \le 1 \end{cases}$$

for $0 \le t < 1$, and we define $F(x,1)$ to be the identity map of D^2. F is an isotopy from ϕ to the identity. □

The Alexander method

In Section 1.1, we computed the mapping class group of a disk. To compute the mapping class group of other surfaces, we often use the following procedure: find a collection of curves and/or arcs which cut up the surface into a collection of disks and apply Lemma 1.4. This will show that the action of the mapping class group of the surface is determined by the action of the isotopy class onto the collection of these curves and/or arcs. In other words, we can understand an element of the mapping class group by looking at the action of this element on simple closed curves in the surface. Therefore, we can convert the problem of computing a mapping class group into a combinatorial problem. This is stated precisely in the following theorem.

Theorem 1.5 (Alexander method)**.** *Let S be a compact surface, possibly with marked points, and let $\phi \in \mathrm{Homeo}^+(S, \partial S)$. Let $\gamma_1, \ldots, \gamma_n$ be a collection of essential simple closed curves and simple proper arcs in S which have the following properties.*

- *The γ_i are pairwise in minimal position.*
- *The γ_i are pairwise nonisotopic.*
- *For distinct i, j, k, at least one of $\gamma_i \cap \gamma_j$, $\gamma_j \cap \gamma_k$, or $\gamma_i \cap \gamma_k$ is empty.*

(1) If there is a permutation σ of $\{1,\ldots,n\}$ so that $\phi(\gamma_i)$ is isotopic to $\gamma_{\sigma(i)}$ relative to ∂S for each i, then $\phi(\cup\gamma_i)$ is isotopic to $\cup\gamma_i$ relative to ∂S.

If we regard $\cup\gamma_i$ as a (possibly disconnected) graph Γ in S, with vertices at the intersection points and at the endpoints to arcs, then the composition of ϕ with this isotopy gives and automorphism ϕ_* of Γ.

(2) Suppose now that $\{\gamma_i\}$ fills S. If ϕ_* fixes each vertex and each edge of Γ, with orientations, then ϕ is isotopic to the identity. Otherwise, ϕ has a nontrivial power that is isotopic to the identity.

1.2 DEHN TWISTS

Dehn twists are elements of infinite order in the mapping class group. They are the “simplest" mapping classes in the sense that they have representatives with the “smallest" possible supports. To define Dehn twists, we start by defining the twist map.

Definition 1.6 (Twist map)**.** Consider the annulus $A = S^1 \times [0,1]$. Orient A by embedding it in the plane via the map $(\theta,t) \mapsto (\theta, t+1)$ and take the orientation induced by the standard orientation of the plane. Let $T\colon\ A \to A$ be the *twist map* of A given by the formula

$$T(\theta,t) = (\theta - 2\pi t, t).$$

Notice that T is an orientation preserving homeomorphism that fixes ∂A point-wise. Also notice that instead of using $\theta - 2\pi t$ we could have used $\theta + 2\pi t$. Our choice is the “right twist" while the other is a “left twist".

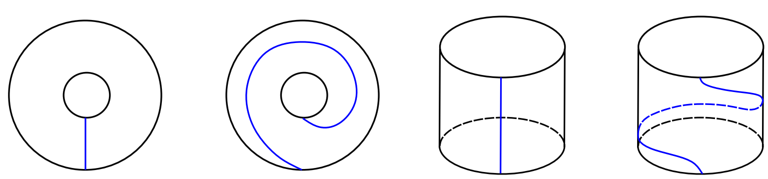


Figure 1.5: Two views of the map T.

Figure 1.5 shows two ways to visualize the twist map, T. We now use the map T to define a Dehn twist on a surface S.

Definition 1.7 (Dehn twist)**.** Let S be an arbitrary (oriented) surface and let α be a simple closed curve in S. Let N be a regular neighborhood of α, and choose an orientation preserving homeomorphism $\phi\colon\ A \to N$.

We obtain a homeomorphism $D_\alpha\colon S \to S$, called a *Dehn twist about* α, as follows:

$$D_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{if } x \in N \\ x & \text{if } x \in S \setminus N \end{cases}$$

In other words, to complete a Dehn twist we fix every point outside of the annulus N while performing the twist map on the annulus N.

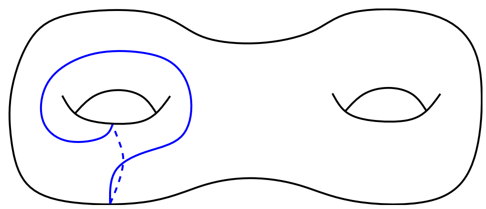


Figure 1.6: Dehn twist on a surface of genus 2.

In Figure 1.6 we illustrate a Dehn twist on a surface of genus 2.

Dehn twists via cutting and gluing

Another way to think of a Dehn twist is as an operation which involves cutting the surface, twisting a boundary component, and gluing the surface back together.

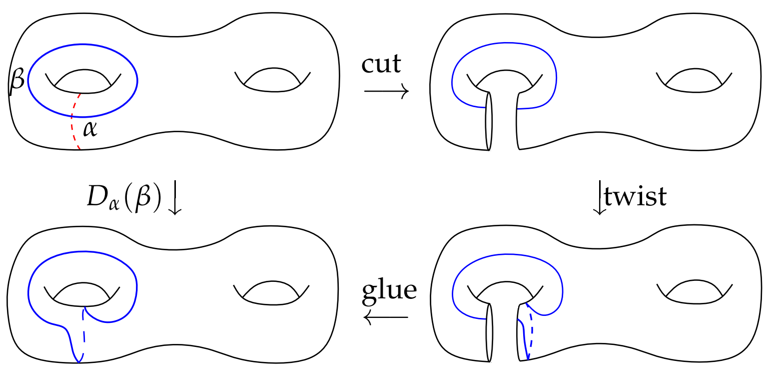


Figure 1.7: Viewing the Dehn twist as a cutting and gluing operation.

To see how this works, start by cutting the surface S along the curve α. Take a neighborhood of one of the boundary components and twist this neighborhood by an angle of 2π. After twisting, reglue the two boundary components back together. See Figure 1.7 to see a visual of this procedure. By cutting, twisting and regluing, we obtain a well-defined homeomorphism of the surface which is equivalent to a Dehn twist around the curve α. It is important to twist only a neighborhood of one of the boundary components. To see why this is important, consider a separating curve α, cut along α, twist one of the subsurfaces by 2π and reglue. This results in the identity homemorphism which is not what we set out to do!

Dehn twists on the torus

Earlier, we calculated the mapping class group of the disk. Another important mapping class group to compute is the mapping class group of the torus, $\text{Map}(T^2)$. We often use this mapping class group to give us some intuition when we're studying the mapping class group of other surfaces. One of the reasons we like to use this mapping class group is because it's easy for us to understand since $\text{Map}(T^2)$ is isomorphic to the group of integral matrices, a fact which was proven by Dehn [Deh87].

Theorem 1.8. *The homomorphism*

$$\sigma\colon\ \text{Map}(T^2) \to SL(2,\mathbb{Z}) \tag{1.1}$$

given by the action on $H_1(T;\mathbb{Z}) \cong \mathbb{Z}^2$ is an isomorphism.

We won't give an explicit proof of this theorem, but we'll give a brief outline so one can see why this is true. Consider the curves α and β as shown on the torus in Figure 1.8. It has been shown that $\text{Map}(T^2)$ is generated by Dehn twists around the curves α and β, which we will denote D_α and D_β.

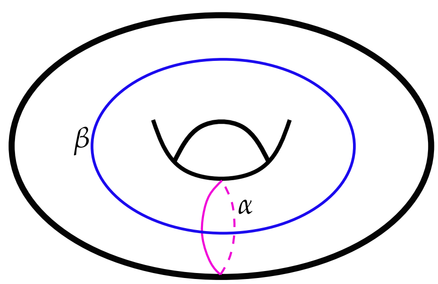


Figure 1.8: The curves α and β on the torus. Dehn twists around these two curves generate $\text{Map}(T^2)$.

We are also able to construct a torus by gluing together the opposite sides of a 4-gon. We consider this 4-gon to be the fundamental domain of the torus, and the universal cover is an integer lattice in $\mathbb{R}^2$. We can think of the curve α as the line $[0,1]$x$\{0\}$ on the lattice, and the curve β as the line $\{1\}$x$[0,1]$ on the lattice.

It follows that the homeomorphisms D_α and D_β on T^2 are able to be induced by the following linear transformations on $\mathbb{R}^2$:

$$D_\alpha = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and

$$D_\beta = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Since $SL(2, \mathbb{Z})$ is generated by these two matrices, we find that there is an isomorphism between $\mathrm{Map}(T^2)$ and $SL(2, \mathbb{Z})$.

The inclusion homomorphism

We are now going to describe a homomorphism which is used to help simplify many proofs. In this thesis, we will use this map, often called the *inclusion homomorphism*, to simplify the proof of the *lantern relation* in Chapter 3.

Let S' be a closed subsurface of S. We define a homomorphism

$$\eta\colon\ \mathrm{Map}(S') \to \mathrm{Map}(S).$$

Let $f \in \mathrm{Map}(S')$, and let $\phi \in \mathrm{Homeo}^+(S', \partial S')$ be a homeomorphism which is a representative of f. If $\hat{\phi}$ is the element of $\mathrm{Homeo}^+(S, \partial S)$ such that $\hat{\phi}$ agrees with ϕ on S' and is the identity outside of S', then define $\eta(f)$ to be the isotopy class of $\hat{\phi}$. The map η must be well-defined since any homotopy between two elements of $\phi \in \mathrm{Homeo}^+(S', \partial S')$ will give a homotopy between the corresponding elements of $\mathrm{Homeo}^+(S, \partial S)$.

The following theorem describes the kernel of the homomorphism η.

Theorem 1.9 (The inclusion homomorphism)**.** *Let S' be a closed subsurface of the surface S. We assume that S' is not homeomorphic to a closed annulus and that no component of $S\setminus$ is an open disk. Let $\eta\colon\ \mathrm{Map}(S') \to \mathrm{Map}(S)$ be the induced inclusion map, described above. Let $\alpha_1, \ldots, \alpha_m$ denote the boundary components of S which bound once-punctured disks in $S \setminus S'$ and let $\{\beta_1, \gamma_1\}, \ldots, \{\beta_n, \gamma_n\}$ denote the pairs of boundary components of S' that bound annuli in $S \setminus S'$. Then the kernel of η is the free abelian group*

$$\ker(\eta) = \langle T_{\alpha_1}, \ldots, T_{\alpha_m}, T_{\beta_1} T_{\gamma_1}^{-1}, \ldots, T_{\beta_n} T_{\gamma_n}^{-1} \rangle$$

In particular, if no connected component of $S \setminus S'$ is an open annulus, an open disk, or an open once-marked disk, then η is injective.

1.3 GENERATING THE MAPPING CLASS GROUP

Earlier in this chapter, we mentioned that the mapping class group of a torus is generated by Dehn twists about two simple closed curves. A natural question to ask if we can generalize this for all surfaces of genus g. In particular, can we write any element of the mapping class group as a composition of Dehn twists about easy to understand simple closed curves? This question can be answered in the affirmative, and this result has been a key ingredient to prove many results about mapping class groups.

The first answer in the affirmative was from Dehn in the 1920's. Dehn proved that for a surface of genus g, $\mathrm{Map}(S_g)$ is generated by $2g(g-1)$ Dehn twists [Deh87]. In 1967, Mumford [Mum67] showed that only Dehn twists around nonseparating curves were needed. In 1964, Lickorish [Lic64] gave an independent proof that $\mathrm{Map}(S_g)$ is generated by the Dehn twists about $3g-1$ nonseparating curves which are known as the *Lickorish generators*.

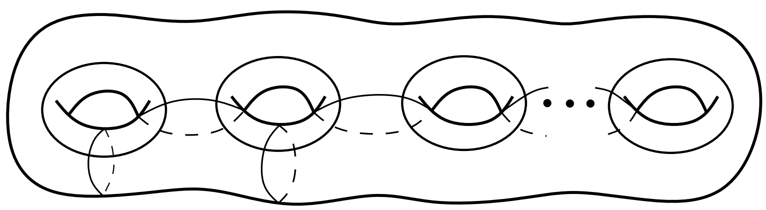


Figure 1.9: The Humphries generators for the mapping class group of a surface of genus g.

In 1976, Humphries [Hum79] was able to decrease the number of generators in the mapping class group to be the twists about the $2g+1$ curves, pictured in Figure 1.9. These $2g+1$ generators are often referred to as the *Humphries generators*. Humphries also showed that any Dehn twist generating set for the mapping class group of a surface of genus g must contain at least $2g+1$ elements.

This work can be generalized further to show that there is a finite set of generators for the mapping class group of any surface $S_{g,n}$ of genus $g \geq 0$ with $n \geq 0$ punctures.

The $2g+n$ twists about the simple closed curves in Figure 1.10 give a generating set for $\mathrm{PMap}(S_{g,n})$. By taking this generating set together with a set of elements of $\mathrm{Map}(S_{g,n})$, we can obtain a finite set of generators of $\mathrm{Map}(S_{g,n})$.

Theorem 1.10. *For any $g, n \geq 0$, the group $\mathrm{Map}(S_{g,n})$ is generated by a finite number of Dehn twists and half-twists.*

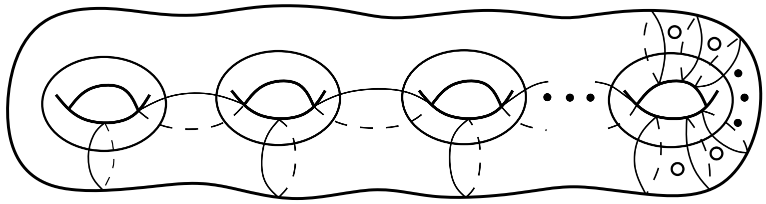


Figure 1.10: The Humphries generators for the pure mapping class group of a surface of genus g with n punctures.

The proof that the mapping class group is finitely generated is done through an induction argument on genus. The key ingredient to prove the inductive step is to show that the *complex of curves*, $\mathcal{C}(S_g)$, is connected when $g \geq 2$. The complex of curves, defined by Harvey [Har81], is a combinatorial object which encodes the intersection patterns of simple closed curves in S_g.

Definition 1.11 (Complex of curves, $\mathcal{C}(S_g)$)**.** The *complex of curves* is an abstract simplicial complex associated to a surface S. Its 1-skeleton is given by the following data:
Vertices: There is one vertex of $\mathcal{C}(S)$ for each isotopy class of essential simple closed curves in S.
Edges: There is an edge between any two vertices of $\mathcal{C}(S)$ corresponding to isotopy classes a and b with $i(a,b) = 0$, ie. a and b are disjoint.

Notice that $\mathcal{C}(S)$ is a *flag complex*, which means that $k+1$ vertices span a k-simplex of $\mathcal{C}(S)$ if and only if they are pairwise connected by edges. In this thesis, we will only make use of the 1-skeleton of the complex of curves, and we will denote the 1-skeleton by $\mathcal{C}(S)$. We note that in the complex of curves, a puncture has the same effect as a boundary component, so in this thesis we will consider our surfaces to be punctured.

1.4 THE NIELSEN-THURSTON CLASSIFICATION

The Nielsen-Thurston classification of elements of $\text{Map}(S)$ is one of the central theorems in the study of mapping class groups. The Nielsen-Thurston classification of elements tells us that every element $f \in \text{Map}(S)$ is of one of three special types: periodic, reducible, or pseudo-Anosov [Nie27, Nie29, Nie32, Nie44, Thu88].

The first type of mapping class is a periodic mapping class.

Definition 1.12 (Periodic mapping class)**.** An element $f \in \text{Map}(S)$ is *periodic* if it is of finite order.

Another way to think of a periodic mapping class is to think of them as the class of homeomorphisms which eventually map any curve on the surface back to itself. The following theorem proves that every periodic mapping class has a representative diffeomorphism which has finite order.

Theorem 1.13 (Fenchel-Neilsen). *Suppose $\chi(S) < 0$. If $f \in \mathrm{Map}(S)$ is an element of finite order k, then there is a representative homeomorphism $\phi \in \mathrm{Homeo}^+(S)$ so that ϕ has order k. Further, ϕ can be chosen to be an isometry of some hyperbolic metric on S.*

This theorem was generalized by Kerckhoff from finite cyclic groups to arbitrary finite groups [Ker83].

The next type of mapping class is a reducible mapping class.

Definition 1.14 (Reducible mapping class). An element $f \in \mathrm{Map}(S)$ is *reducible* if there is a nonempty set $\{c_1, \ldots, c_n\}$ of isotopy classes of essential simple closed curves in S so that $i(c_i, c_j) = 0$ for all i and j and so that $\{f(c_i)\} = \{c_i\}$. The collection is called a *reduction system* for f. In this case, we can understand f via the following procedure:

- Choose a representative $\{\gamma_i\}$ of the $\{c_i\}$ with $\gamma_i \cap \gamma_j = \emptyset$ for $i \neq j$.
- Choose a representative ϕ of f with $\{\phi(\gamma_i)\} = \{\gamma_i\}$.
- Consider the homeomorphism of the noncompact, possibly disconnected surface $S - \cup\gamma_i$ induced by ϕ.

In particular, we break f into irreducible pieces.

Another way to think of a reducible mapping class is to think of them as the class of homeomorphisms which will eventually map a subset of curves on the surface back to themselves. This means that all periodic mapping classes are reducible, but it is not the case that a reducible mapping class must be periodic.

Finally, we have the pseudo-Anosov mapping classes, the mapping classes which will be the focus of our thesis.

Definition 1.15 (Pseudo-Anosov mapping class). An element $f \in \mathrm{Map}(S)$ is *pseudo-Anosov* if it is neither periodic nor reducible.

Another way to think of pseudo-Anosov mapping classes is to think of them as the class of homeomorphisms which never map any curve on the surface back to itself. The following theorem by Thurston is able to give an equivalent definition of pseudo-Anosov mapping classes and will be the definition we will make use of most often [Thu88].

Theorem 1.16 (Thurston). *An element $f \in \mathrm{Map}(S)$ is called pseudo-Anosov if there is a pair of transverse measured foliations $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$ on S, a number $\lambda > 1$, and a representative homeomorphism ϕ so that*

$$\phi \cdot (\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda\mu_u) \text{ and } \phi \cdot (\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda^{-1}\mu_s)$$

The measured foliations are called the unstable folitation and stable foliation, respectively, and the number λ is called the stretch factor of ϕ (or of f). The map ϕ is called a pseudo-Anosov homeomorphism.

1.5 TRAIN-TRACKS

In the 1970's, Thurston introduced the notion of train-tracks. A train-track is a combinatorial tool which we use to study simple closed curves on surfaces. When we are studying how pseudo-Anosov homeomorphisms act on a surface, we can use train-tracks to determine information attached to the homeomorphism, such as the stretch factor.

To illustrate how we construct train-tracks, we will work through a key example which Thurston studied. Let $S_{0,4}$ denote the sphere with four punctures. Consider one of the punctures of the four-times punctured sphere to lie at infinity, which allows us to regard $S_{0,4}$ as the three-times punctured plane. Define $f \in \text{Map}(S_{0,4})$ to be $f = \sigma_1^{-1}\sigma_2$, where σ_2 and σ_1 are the half twists pictured in 1.11. We will show that this map defines a pseudo-Anosov homeomorphism by constructing and anlyzing the train-track associated to this map.

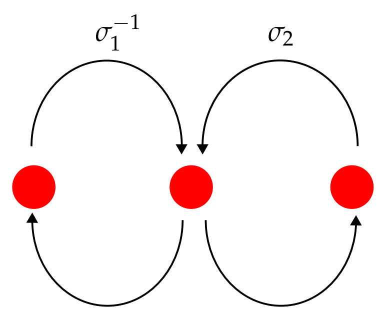


Figure 1.11: The half twists used to define the map $f = \sigma_1^{-1}\sigma_2$.

Thurston's idea is that we can understand homeomorphisms by iterating them on simple closed curves. We do this for f, while keeping track of what happens to a chosen isotopy class c of simple closed curves in $S_{0,4}$ as f is iterated. Such as isotopy class is shown in Figure1.12, along with the image of c under the first two iterations of f.

As we iterate f the number of strands grows quickly, and it quickly becomes difficult for us to keep track of the strands on the train-track. Indeed, the number of horizontal strands after the fourth iteration in Figure 1.12 is 10, and we were to continue iterating we would find that the number of horizontal strands for $f^5(c)$ is 188, for $f^{10}(c)$ is 21892, and for $f^{100}(c)$ we obtain an integer with 42 digits! Since the number of strands grows so quickly, this is an efficient map to use when we would want

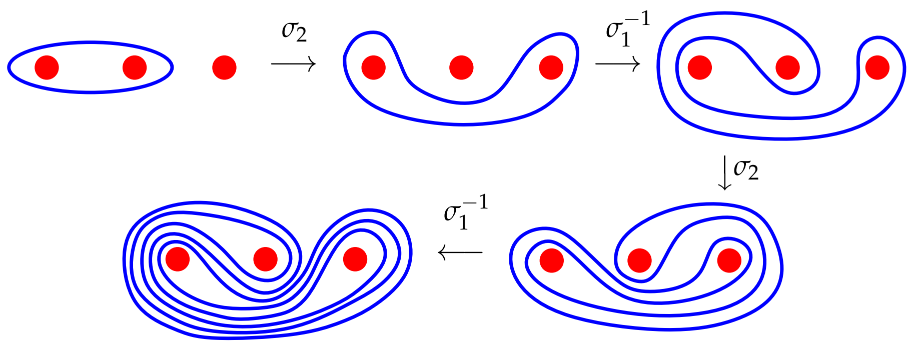


Figure 1.12: The image of the curve under the first four iterations of f.

something to mix efficiently, for instance, this map is often used to pull taffy.

One way to measure the complexity of the curve under iterations of f is to draw two horizontal rays, one drawn outwards from the leftmost puncture and the second drawn outwards from the rightmost puncture, and count the number of times $f^n(c)$ intersects each of the rays. By keeping track of the stands in this way, we can see that the Fibonacci sequence appears as we continue iterating the sequence. Indeed, we find that $f^n(c)$ intersects the left and right hand rays F_{2n+1} and F_{2n} times, respectively, where $F_0 = 0$, $F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ is the ith Fibonacci number. Further, this implies that the number of strands grows exponentially.

One of the downsides to this technique is that we are only looking at one of the isotopy classes of our curve, c. What we would like is to understand how our map acts on all the isotopy classes of simple closed curves in $S_{0,4}$.

Observe that the isotopy class $f^2(c)$, shown in Figure 1.12, can be represented by the data in Figure 1.13 as follows. Replace n "parallel strands" of $f^2(c)$ by a single strand labelled n. We think of this taking n parallel strands and pinching (or homotoping) the strands together to create one strand. Once we have pinched the n strands together, we label the new strand with the number of strands which we pinched together. This process is sometimes called *zipping* strands together. This process is done in a continuous manner so we are not able to cross the punctures during this procedure. This means that at the four punctures, the pinched together strands must split into two strands, as shown in Figure 1.13. We make note that this process is not canonical since there are different ways to draw and interpret the action of f on the curve c. By choosing to pinch the strands in different ways, we can obtain many different train-tracks which represent the same map.

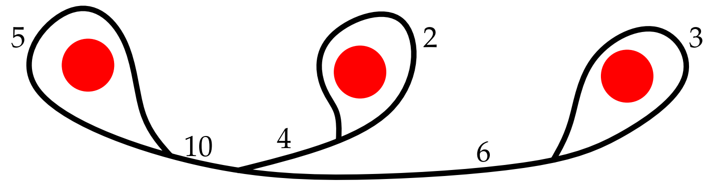


Figure 1.13: A train-track for the map f.

After completing this process, we have obtained a finite graph τ embedded in $S_{0,4}$ which we call a *train-track*. A train-track on a surface S is an embedded 1-complex τ satisfying the following three properties.

- Each edge (called a *branch*) is a smooth path with well-defined tangent vectors at the endpoints, and at any vertex (called a *switch*) the incident edges are mutually tangent. The tangent vector at the switch pointing toward the edge can have two possible directions which divides the ends of edges at the switch into two sets. The end of a branch of τ which is incident on a switch is called "*incoming*" if the one sided tangent vector of the branch agrees with the direction at the switch and "*outgoing*" otherwise.

- Neither the set of "incoming" nor the set of "outgoing" branches are permitted to be empty. The valence of each switch is at least 3, except for possibly one bivalent switch in a closed curve component.

- Finally, we require the components of $S \backslash \tau$ to have negative generalized Euler characteristic: for a surface R whose boundary consists of smooth arcs meeting at cusps, define $\chi'(R)$ to be the Euler characteristic $\chi(R)$ minus $1/2$ for every outward-pointing cusp (internal angle 0), or plus $1/2$ for each inward pointing cusp (internal angle 2π).

A *train route* is a non–degenerate smooth path in τ. A train route traverses a switch only by passing from an incoming to an outgoing edge (or vice-versa). A *transverse measure* on τ is a function μ which assigns a non-negative real number, $\mu(b)$, to each branch which satisfies the *switch condition*: For any switch, the sums of μ over incoming and outgoing branches are equal. A train-track is *recurrent* if there is a transverse measure which is positive on every branch. Equivalently a train-track is reurrent if each branch is contained in a closed train route.

Fix a reference hyperbolic metric on S. A *geodesic lamination* in S is a closed set foliated by geodesics [CB88]. A geodesic lamination is *measured* if it supports a measure on arcs transverse to its leaves, which is invariant under isotopies preserving the leaves.

If σ is a train-track which is a subset of τ we write $\sigma < \tau$ and say σ is a *subtrack* of τ; we may also say that τ is an *extention* of σ. If there is a homotopy of S such that every train route on σ is taken to a train route on τ we say σ is *carried* on τ and write $\sigma \prec \tau$.

We now illustrate how train-tracks make it easy for us to keep track of the isotopy class of any simple closed curve carried by τ under any number of iterations of f. We go back to the example of a train-track we constructed on $S_{0,4}$, τ, endowed with a measure ν. The measure on τ is able to be given by two weights, x and y, as shown in Figure 1.14. Every other weight on τ is able to be determined from these weights by using the switch condition, defined earlier. In our previous example, it's easy to check that the curve c is given by the coordinates $(0, 2)$, and the curve $f^2(c)$ is given by the coordinates $(6, 4)$.

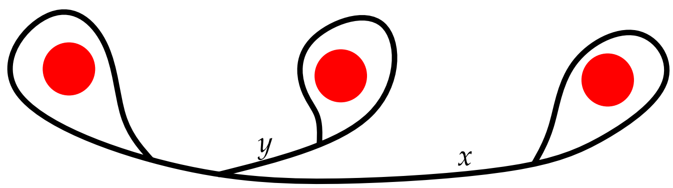


Figure 1.14: A train-track for the map f.

We now apply the map f to the train-track (τ, ν) directly. After we have applied out map to the train-track, we will perform the pinching (or zippering) process as decribed above to $f(\tau, \nu)$: we homotope the parallel edges of $f(\tau, \nu)$ together while keeping track of the weights of the edges which are homotoped to a single edge. To see this process illustrated, see Figure 1.15. The result of this process is another train-track which has the same underlying train-track τ, but with different weights. Since the resulting train-track $f(\tau, \nu)$ has the same underlying train-track (τ, ν), notice that, by the definition above, $f(\tau, \nu)$ is *carried* by (τ, ν). Given a pseudo-Anosov mapping class f, it is always possible to find a train-track which is carried by f.

Theorem 1.17. *Given a pseudo-Anosov mapping class f on a surface S we can find a train-track τ such that $f(\tau)$ is carried by τ.*

In Figure 1.15, we see that the weights of the train-track transform as follows: the edge of (τ, ν) with the weight x has the weight $2x + y$ in $f(\tau, \nu)$, while the edge of (τ, ν) with weight y has the weight $x + y$ in $f(\tau, \nu)$. Since these weights are able to determine the weights of the other branches, we see that f acts on the original measured train-track (τ, ν) by changing the weights in a linear way. Therefore, we can describe the action through the *train-track matrix* for f:

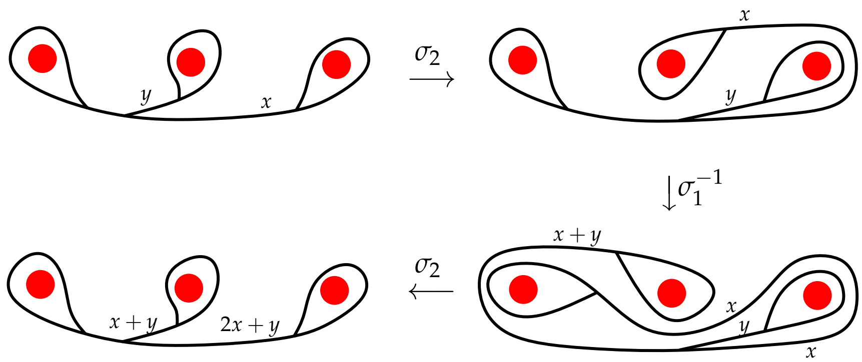


Figure 1.15: Applying the map f to the train-track τ.

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

This matrix gives us a way to encode the action of iterates of f on any isotopy class of simple closed curves carried by τ, not only the action on the curve c. To see this, let b be an isotopy class which corresponds to the train-track τ with starting weights (x_0, y_0). Then $f^n(b)$ is the isotopy class of simple closed curves corresponding to τ with weights (x_n, y_n) given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

The weights (x_n, y_n) determine a measured train-track (τ, ν_n) given by the switch conditions on τ. From (τ, ν_n) we can build the simple closed curve $f^n(b)$ as we did above. In fact, notice that our example from above is the sepcial case where $(x_0, y_0) = (0, 2)$.

As with any matrix, we can determine the eigenvalues and eigenvectors of M. The matrix M has eigenvalues

$$\lambda = \frac{3+\sqrt{5}}{2} \text{ and } \lambda^{-1} = \frac{3-\sqrt{5}}{2}$$

with eigenvectors

$$v_\lambda = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 2 \end{pmatrix} \text{ and } v_{\lambda^{-1}} = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}.$$

Since the eigenvector $v_{\lambda_{-1}}$ contains a negative value, it does not correspond to a measured train-track. The values we care about are the eigenvalue $\lambda > 1$ with its corresponding eigenvector v_λ since it tells us that the norm of the vector (x_n, y_n) grows like λ^n as $n \longrightarrow \infty$. Another important consequence is that the value λ is the stretch factor of f, which describes the stretching and contraction along the unstable and stable foliations, respectively.

For each pseudo-Anosov mapping class, the transition matrix M is Perron-Frobenius and the positive eigenvector determines and invariant measure.

Definition 1.18 (Perron-Frobenius matrix)**.** A *Perron-Frobenious matrix* is one with entries $a_{i,j} \geq 0$ such that some power of the matrix has strictly positive entries.

It is also known that the eigenvalue of this eigenvector will correspond to the stretch factor of the pseudo-Anosov map [PP87].

Theorem 1.19. *Given a pseudo-Anosov mapping class, f, there exists a train-track τ invariant under the action of f such that the matrix M which determines the action on the transverse measures is Perron-Frobenious. The positive eigenvector determines an invariant measure corresponding to the invariant foliation and the eigenvalue is the stretch factor.*

2 A NEW CONSTRUCTION OF PSEUDO-ANOSOV HOMEOMORPHISMS USING POSITIVE DEHN TWISTS

2.1 BACKGROUND AND MOTIVATION

We recall that the Nielsen-Thurston classification of elements states that each element in $\mathrm{Map}(S_{g,n})$ is either periodic, reducible, or pseudo-Anosov [Thu88]. An element $f \in \mathrm{Map}(S_{g,n})$ is pseudo-Anosov if there is a representative homeomorphism ϕ, a real number $\lambda > 1$ and a pair of transverse measured foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ such that $\phi(\mathcal{F}^u) = \lambda\mathcal{F}^u$ and $\phi(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s$. λ is called the stretch factor (or dilatation) of f, $\mathcal{F}^u$ and $\mathcal{F}^s$ are the unstable foliation and stable foliation, respectively, and the map ϕ is a pseudo-Anosov homeomorphism.

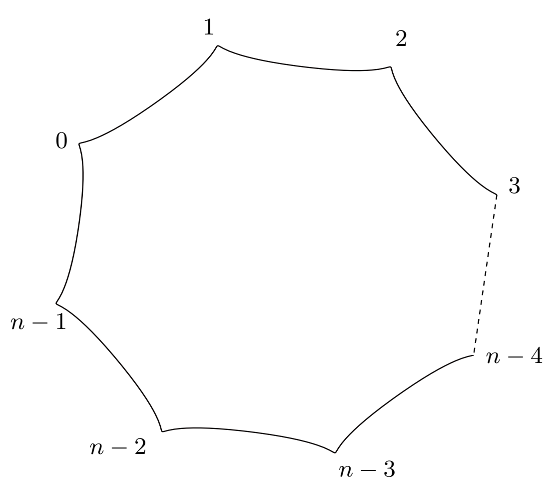


Figure 2.1: Labelling of the punctures on the n-times punctured sphere.

In this chapter, we will provide a new construction of pseudo-Anosov mapping classes on n-times punctured spheres. In particular, consider the n-times punctured sphere, $S_{0,n}$, with a clockwise labelling of the punctures, as depicted in Figure 2.1.

In this paper, we will consider the following curves on the n-times punctured sphere. Consider the plane with n punctures, where the punctures are located at the vertices of a regular n-gon. Then for any subset of the punctures, there is a unique isotopy class of curves that surrounds those punctures and is convex in the Euclidean metric. This curve in the plane projects to a curve on the sphere with n punctures via stereographic projection. This is well defined up to change of coordinates. Using these curves, it is possible to associate a half-twist to each puncture on $S_{0,n}$.

Definition 2.1. Consider a simple closed curve on the n-times punctured sphere obtained as described above. We say that such a curve, call it γ, separates the punctures k and l, if γ bounds a twice punctured disc containing k and l. Denote the curve separating puncture j and $j-1$ mod n by α_j. Define the half-twist associated to puncture j, denoted D_j, as the half-twist around α_j. The square of a half twist is called a Dehn twist.

In this paper, our choice of a positive half-twist will be a right half-twist.

Consider a set of punctures $p = \{p_1, \ldots, p_i\}$ such that $|p_k - p_j| \geq 2$ mod n for each $j, k \in \{1, \ldots, i\}$. Then every curve associated to a puncture from this set is disjoint from any other curve associated to a puncture from this set. As the curves in p are disjoint, we are able to perform the half-twists associated to the curves in p simultaneously as a multi-twist. We denote this multi-twist by $D_p = D_{p_i} \ldots D_{p_1}$.

Additionally, we define the following map:

Definition 2.2. Define the map ρ as follows:

$$\begin{aligned} \rho \colon \mathbb{Z}_n &\to \mathbb{Z}_n \\ j &\mapsto j+1 \mod n. \end{aligned}$$

This is the map which permutes the numbers $1, \ldots, n$ cyclically.

Using the above notation, we are able to define pseudo-Anosov maps on $S_{0,n}$ by considering the different partitions of the punctures of the sphere. We recall that a partition of a set is a grouping of the set's elements into non-empty subsets in such a way that every element is included in one and only one of the subsets. We define a partition of the punctures, $\mu = \{\mu_1, \ldots, \mu_k\}$, where each $\mu_i \subset \{1, \ldots, n\}$, to be evenly spaced if $\rho(\mu_i) = \mu_{i+1}$ for $1 \leq i \leq k$. Notice that if a partition is evenly spaced, then $|\mu_i| = |\mu_j|$ for all $\mu_i, \mu_j \in \mu$.

Example 2.3. Consider the 6-times punctured sphere.

We label the punctures in a clockwise fashion as in Figure 2.2. Up to spherical symmetry, we notice that we may partition the punctures evenly

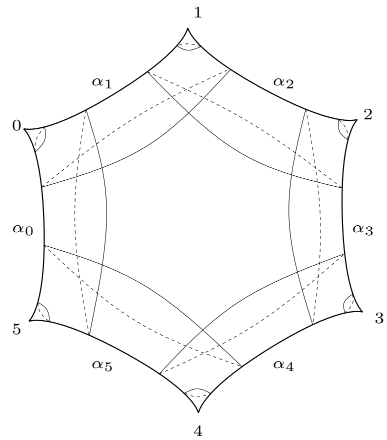


Figure 2.2: The 6-times punctured sphere.

in two different ways. First, we could place every third puncture in the same set to get

$$\mu = \{\{0,3\},\{1,4\},\{2,5\}\} = \{\mu_1,\mu_2,\mu_3\},$$

and secondly we could place every other puncture in the same set to get

$$\bar{\mu} = \{\{0,2,4\},\{1,3,5\}\} = \{\bar{\mu}_1,\bar{\mu}_2\}.$$

To each evenly spaced partition of the punctures of the sphere $S_{0,n}$, we can define an associated pseudo-Anosov mapping class on $S_{0,n}$.

Theorem 2.4. *Consider the surface $S_{0,n}$. Let $\{\mu_i\}_{i=1}^k$, for $1 < k < n$, be an evenly spaced partition of the punctures of $S_{0,n}$. Then*

$$\phi = \prod_{i=1}^{k} D_{\mu_i}^{q_i} = D_{\mu_k}^{q_k} \dots D_{\mu_2}^{q_2} D_{\mu_1}^{q_1},$$

where $q_j = \{q_{j_1}, \dots q_{j_l}\}$ are tuples of integers greater than one, is a pseudo-Anosov homeomorphism of $S_{0,n}$.

Example 2.5. Returning to the partitions from Example 2.3, we notice that Theorem 2.4 tells us that the maps

$$\phi = D_5^2 D_2^2 D_4^2 D_1^2 D_3^2 D_0^2 = D_{\mu_3}^2 D_{\mu_2}^2 D_{\mu_1}^2,$$

and

$$\bar{\phi} = D_5^2 D_3^2 D_1^2 D_4^2 D_2^2 D_0^2 = D_{\bar{\mu}_2}^2 D_{\bar{\mu}_1}^2$$

are pseudo-Anosov mapping homeomorphisms. We prove this example in detail in Section 2.3.

Consider an even partition, μ, of the n-times punctured sphere as in Theorem 2.4. One can modify the partition μ to obtain a partition of $n+1$ punctures, μ'. To the modified partition μ', one can definite a pseudo-Anosov mapping class on $S_{0,n+1}$

Theorem 2.6. *Consider the surface $S_{0,n}$, and consider a partition of the n punctures into $1 < k < n$ sets $\{\mu_i\}_{i=1}^k$ such that the partition is evenly spaced, as in Theorem 2.4. For each subset μ_i, we create the subset μ_i' as follows: For each puncture $j \in \mu_i$ such that $j \leq k$, include puncture j in μ_i'. If puncture $j \in \mu_i'$ is such that $j \geq k+1$, include puncture $j+1$ in μ_i'. Let μ_{k+1}' be the subset containing only puncture $k+1$. Then $\{\mu_i'\}_{i=1}^{k+1}$ is a partition of the punctures of $S_{0,n+1}$ and*

$$\phi' = \prod_{i=1}^{k} D_{\mu_i'}^{q_i'} = D_{\mu_{k+1}'}^{q_{k+1}'} D_{\mu_k'}^{q_k'} \dots D_{\mu_2'}^{q_2'} D_{\mu_1'}^{q_1'},$$

where $q_j' = \{q_{j_1}', \dots q_{j_l}'\}$ are tuples of integers greater than one, is a pseudo-Anosov homeomorphism of $S_{0,n+1}$.

Example 2.7. We consider the evenly spaced partition

$$\mu = \{\{0,3\},\{1,4\},\{2,5\}\}$$

found in Examples 2.3 and 2.5. We apply Theorem 2.6 to obtain the partition

$$\mu' = \{\{0,4\},\{1,5\},\{2,6\},\{3\}\} = \{\mu_1', \mu_2', \mu_3', \mu_4'\},$$

which implies that the map

$$\phi' = D_3^2 D_6 D_2^2 D_5^2 D_1^2 D_4^2 D_0^2 = D_{\mu_4}^2 D_{\mu_3}^2 D_{\mu_2}^2 D_{\mu_1}^2$$

is a pseudo-Anosov homeomorphism.

Using applying Theorem 2.6 more than once to an evenly spaced partition, it is possible to construct a pseudo-Anosov homeomorphisms which is unable to be constructed using either the Penner or Thurston constructions.

Theorem 2.8. *There exists a pseudo-Anosov mapping class obtained from the construction of Theorem 2.6 which cannot be obtained from either the Thurston or the Penner constructions.*

Remark 2.9 (The Braid Group). There is a well defined map from the braid group on n strands, B_n, to the mapping class group of a disk with n punctures, $\text{Map}(D_n)$ (see Section 9.1.3 of [FM12] for a detailed description). In fact, the map $B_n \to \text{Map}(D_n)$ is an isomorphism. By the Alexander trick, any homeomorphism of the disk which fixes the boundary is isotopic to the identity. Therefore, there is a canonical embedding $B_n \hookrightarrow \mathrm{Map}(S_{n+1})$, where one of the punctures of $\mathrm{Map}(S_{0,n+1})$ is a marked point [?]. Additionally, there is the forgetful map from $\mathrm{Map}(S_{0,n+1}) \to \mathrm{Map}(S_{0,n})$, where we "fill in" the marked point in $S_{0,n+1}$. This is the same forgetful map as in Birman's exact sequence. We are able to compose these two maps to obtain a map from the braid group to the n-times punctured sphere. It follows that if a map in $\mathrm{Map}(S_n)$ is pseudo-Anosov, the corresponding map in B_n is pseudo-Anosov.

One elementary construction of pseudo-Anosov homeomorphisms was provided by Thurston [Thu88]. We recall that parabolic isometries correspond to those non-identity elements of $\mathrm{PSL}(2,\mathbb{R})$ with trace ± 2. We recall that A is a multi-curve if A is the union of a finite collection of disjoint simple closed curves in S.

Theorem 2.10 (Thurston). *Suppose that*

$$A = \{\alpha_1, \dots, \alpha_n\}$$

and

$$B = \{\beta_1, \dots, \beta_m\}$$

are multicurves in S so that A and B are filling, that is, A and B are in minimal position and the complement of $A \cup B$ is a union of disks and once punctured disks. There is a real number $\mu = \mu(A, B)$, and a representation $\rho\colon \langle D_A, D_B \rangle \to$ $\mathrm{PSL}(2,\mathbb{R})$ given by

$$D_\alpha \mapsto \begin{pmatrix} 1 & \mu^{1/2} \\ 0 & 1 \end{pmatrix}$$

and

$$D_\beta \mapsto \begin{pmatrix} 1 & 0 \\ -\mu^{1/2} & 1 \end{pmatrix}.$$

with the following properties:

1. *An element $f \in \langle D_A, D_B \rangle$ is periodic, reducible, or pseudo-Anosov according to whether $\rho(f)$ is elliptic, parabolic, or hyperbolic.*

2. *When $\rho(f)$ is parabolic f is a multitwist.*

3. *When $\rho(f)$ is hyperbolic the pseudo-Anosov homeomorphism f has stretch factor equal to the larger of the two eigenvalues of $\rho(f)$.*

After the work of Thurston, Penner gave the following very general construction of pseudo-Anosov homeomorphisms [Pen88]:

Theorem 2.11 (Penner). *Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ be a pair of multicurves on a surface S, and suppose that A and B are filling. Then any product of positive Dehn twists about a_j and negative Dehn twists about b_k is pseudo-Anosov provided that all $n + m$ Dehn twists appear in the product at least once.*

There has been a considerable amount of research regarding the number-theoretic properties of the stretch factors of pseudo-Anosov maps. Hubert and Lanneau proved that for Thurston's construction, the field $\mathbb{Q}(\lambda + 1/\lambda)$ is always totally real [HL06], and Shin and Strenner proved that for Penner's construction the Galois conjugates of the stretch factor are never on the unit circle [SS15].

For the new pseudo-Anosov mapping classes constructed in this paper, it is shown that there is a large variety of number-theoretic properties associated to the stretch factors. In contrast to the above results for the maps from Penner and Thurston's constructions, it is possible to find explicit examples of pseudo-Anosov mapping classes using this new construction so that $\mathbb{Q}(\lambda + 1/\lambda)$ is not totally real and the Galois conjugates of the stretch factors are on the unit circle. However, these results are not able to be generalized as it is also possible to find explicit examples where $\mathbb{Q}(\lambda + 1/\lambda)$ is totally real and the Galois conjugates of the stretch factors are not on the unit circle, where $\mathbb{Q}(\lambda + 1/\lambda)$ is totally real and the Galois conjugates of the stretch factors are on the unit circle, and where $\mathbb{Q}(\lambda + 1/\lambda)$ is not totally real and the Galois conjugates of the stretch factors are not on the unit circle.

2.2 MAIN CONSTRUCTION ON n-TIMES PUNCTURED SPHERES

We begin by recalling some of the basics of train-tracks. Recall that in Section 1.5, we began by defining train-tracks and worked through a fundamental example. Here we include the other relevant definitions related to train-tracks which we will require. For a thorough treatment of the topic, the author recommends *Combinatorics of Train Tracks* by Penner and Harer [PH92].

Let α be a simple closed curve which intersects τ. We say α intersects τ *efficiently* if $\alpha \cup \tau$ has no bigon complementary regions. A track τ is *transversely recurrent* if every branch of τ is crossed by some simple curve α which intersects τ transversely and efficiently. A track is called *birecurrent* if it is both recurrent and transversely recurrent.

A train-track is called τ *large* if every component of $S \backslash \tau$ is a polygon or a once-punctured polygon, and τ is called *generic* if every switch is trivalent.

If τ is a recurrent train-track, we let $P(\tau)$ denote the *polyhedron of measures* supported on τ. We make note that $P(\tau)$ is preserved by scaling. By $int(P(\tau))$ we will denote the set of weights on τ where each branch has a positive weight. We say that σ *fills* τ if $\sigma \prec \tau$ and $int(P(\sigma)) \subseteq int(P(\tau))$. Similarly, a curve α fills τ if $\alpha \prec \tau$ and α traverses every branch of τ.

If σ is a large train-track, then *diagonal extension* of σ is a track κ such that $\sigma < \kappa$ and every branch of $\kappa \backslash \sigma$ is a *diagonal* of σ ie. its endpoints terminate in the corner of a complementary region of σ. We let $E(\sigma)$ denote the set of all recurrent diagonal extensions of σ, and we note that this set is finite. Let $PE(\sigma)$ denote $\bigcup_{\kappa \in E(\sigma)} P(\kappa)$, and $int(PE(\sigma))$ denotes the set of measures $\mu \in PE(\sigma)$ where each branch of σ has a positive weight.

The following lemma, which is similar to the Nesting Lemma from [MM99], will allow us to determine whether a map ϕ is pseudo-Anosov by analyzing the train-track τ associated to ϕ, and the matrix which represents the induced action on the space of weights on τ. This lemma is one of the key steps to proving that the homeomorphisms constructed in this paper are pseudo-Anosov.

Lemma 2.12. *Let τ be a large, generic, birecurrent train-track. Let $\phi\colon S \to S$ be a map which preserves τ. If the matrix associated to τ is a Perron-Frobenius matrix, then ϕ is a pseudo-Anosov map.*

Before we are able to prove this lemma, we must first state a few lemmas from Masur and Minsky [MM99].

Let μ be a measured lamination on a surface S. The following lemma provides a sufficient condition for when μ is contained in $int(PE(\sigma))$.

Lemma 2.13 (Masur-Minsky)**.** *There exists $\delta > 0$ (depending only on S) for which the following holds. Let $\sigma < \tau$ where σ is a large track. If $\mu \in P(\tau)$ and, for every branch b of $\tau \backslash \sigma$ and b' of σ, $\mu(b) < \delta\mu(b')$, then σ is recurrent and $\mu \in int(PE(\sigma))$.*

Let σ and τ be two large recurrent tracks such that $\sigma \prec \tau$. In this case we say that the train-tracks σ and τ are *nested*. The following two lemms tell us that when we have nested train tracks that their diagonal extentions are also nested in a suitable sense. Additionally, these lemmas tell us that the way in which the diagonal branches cover each other is controlled.

Lemma 2.14 (Masur-Minsky)**.** *Let σ and τ be large recurrent tracks, and suppose $\sigma \prec \tau$. If σ fills τ, then $PE(\sigma) \subseteq PE(\tau)$. Even if σ does not fill τ, we have $PN(\sigma) \subseteq PC(\tau)$.*

Lemma 2.15 (Masur-Minsky)**.** *Let $\sigma \prec \tau$ where σ is a large recurrent track, and let $\sigma' \in E(\sigma)$, $\tau' \in E(\tau)$ such that $\sigma' \prec \tau'$. Then any branch b of $\tau' \backslash \tau$ is traversed by branches of σ' with degree at most m_0, a number depending only on S.*

The final lemma we will need from Masur-Minsky is the following lemma which gives us a relation between nesting train tracks and their distance in the complex of curves.

Lemma 2.16 (Masur-Minsky). *If σ is a large birecurrent train-track and $\alpha \in int(PE(\sigma))$ then*

$$d_{\mathcal{C}(S)}(\alpha, \beta) \leq 1 \implies \beta \in PE(\sigma).$$

In other words,

$$\mathcal{N}_1(int(PE(\sigma))) \subset PE(\sigma),$$

where $\mathcal{N}_1$ denotes a radius 1 neighborhood in $\mathcal{C}(S)$.

With the following lemmas in hand, we may now prove Lemma 2.12

Proof of Lemma 2.12. Consider $\mu \in int(P(\phi(\tau)))$. Therefore, μ is a measure which is positive on every branch of $\phi(\tau)$. Since $\phi(\tau)$ preserves τ, this implies that μ is also a measure which is positive on any branch of τ. Therefore, we have that $int(P(\phi(\tau))) \subseteq int(P(\tau))$, which implies that $\phi(\tau)$ fills τ.

We now follow the argument found in [MM99] for Theorem 4.6 from which shows that if $\phi(\tau)$ fills τ, then there exists some $k \in \mathbb{N}$ such that $\phi^k(PE(\tau)) \subset int(PE(\tau))$.

Suppose that $\tau' \in E(\tau)$ is a diagonal extension. By Lemma 2.14 $\phi(\tau')$ is carried by some $\tilde{\tau} \in E(\tau)$. There exists a constant $c_0(S)$ such that, for some $c \leq c_0$ the power $\phi' = \phi^c$ takes τ' to a train-track carried by τ' because the number of train-tracks in $E(\tau_0)$ is bounded in terms of the topology of S. Let $\mathcal{B}$ represent the branch set of τ', and $\mathcal{B}_\tau \subset \mathcal{B}$ represent the branch set of τ. In the coordinates of $\mathbb{R}^{\mathcal{B}}$ we may represent ϕ' as an integer matrix M, with a submatrix M_τ whichs gives us the restriction to $\mathbb{R}^{\mathcal{B}_\tau}$. Penner shows in [Pen88] that M_τ^n has all positive entries where n is the dimension $|\mathcal{B}_\tau|$. In fact, Penner shows that $|M_\tau^n(x_\tau)| \geq 2|x_\tau|$ for any vector x_τ which represents a measure on τ. Indeed, M_τ has a unique eigenspace in the positive cone of $\mathbb{R}^{\mathcal{B}_\tau}$, which corresponds to $[\mu]$. On the other hand, for a diagonal branch $b \in \mathcal{B} \setminus \mathcal{B}_\tau$, Lemma MM1Lemma4.3 shows that $|M^i(x)| \leq m_\tau |x|$ for all $x \in \mathbb{R}^{\mathcal{B}}$ and all powers $i > 0$. Since τ is generic, we have that any transverse measure x on τ' must put a positive measure on a branch of $\mathcal{B}_\tau$. This implies that given $\delta > 0$ there exists k_1, depending only on δ and S, such that for some $k \leq k_1$ we have $\max_{b \in \mathcal{B} \setminus \mathcal{B}_\tau} \phi^k(x)(b) \leq \delta \min_{b \in \mathcal{B}_\tau} h^k(x)(b)$, for any $x \in P(\tau')$. We apply this to each $\tau' \in E(\tau)$, and by applying Lemma 2.13, we see that, for an appropriate choice of δ,

$$\phi^k(PE(\tau)) \subset int(PE(\tau)) \tag{2.1}$$

Using the above argument, we find that $\phi^{jk}(\tau)$ fills $\phi^{(j-1)k}(\tau)$, from which it follows that

$$PE(\phi^{jk}(\tau)) \subset int(PE(\phi^{(j-1)k}(\tau))) \tag{2.2}$$

for any j.

By way of contradiction, suppose that ϕ is not a pseudo-Anosov map. Then there exists a curve α on the surface S disjoint from τ such that there exists an $m \in \mathbb{N}$ such that $\phi^m(\alpha) = \alpha$.

Since there are elements of $PE(\tau)$ which are not in $\phi^k(PE(\tau))$, it is possible to find $\gamma \in \mathcal{C}(S)$ such that $\gamma \notin PE(\tau)$ and $\phi^k(\gamma) \in PE(\tau)$. Then $\phi^{2k}(\gamma) \in int(PE(\tau))$, which implies that $d_{\mathcal{C}(S)}(\gamma, \phi^{2k}(\gamma)) \geq 1$ by Lemma 2.16.

Since $\phi^{jk}(\gamma) \in PE(\phi^{(j-1)k}(\tau))$ for $j \geq 1$, we use Equation 2.2 to find

$$\begin{aligned}
&\phi^{3k}(\gamma) \in PE(\phi^{2k}(\tau)) \subset int(PE(\phi^k(\tau))) \\
&\phi^{3k}(\gamma) \in PE(\phi^k(\tau)) \subset int(PE(\tau)) \\
&\phi^{3k}(\gamma) \in PE(\tau).
\end{aligned}$$

By Lemma 2.16, we have that for any k,

$$\mathcal{N}_1(int(PE(\phi^k(\tau)))) \subset PE(\phi^k(\tau)).$$

Therefore, we find that

$$\begin{aligned}
\phi^{3k}(\gamma) \in PE(\phi^{2k}(\tau)) \subset \mathcal{N}_1(int(PE(\phi^k(\tau)))) \subset PE(\phi^k(\tau)) \\
\subset \mathcal{N}_1(int(PE(\tau))) \subset PE(\tau).
\end{aligned}$$

Therefore, since $\gamma \notin PE(\tau)$, we have that $d_{C_0(S)}(\gamma, \phi^{3k}(\gamma)) \geq 2$.

We continue inductively to show that $d_{C_0(S)}(\gamma, \phi^{jk}(\gamma)) \geq j-1$, which implies that as $j \to \infty$, $d_{C_0(S)}(\gamma, \phi^{jk}(\gamma)) \to \infty$. Since

$$\begin{aligned}
d_{C_0(S)}(\alpha, \phi^{jk}(\alpha)) &\geq d_{C_0(S)}(\gamma, \phi^{jk}(\gamma)) - d_{C_0(S)}(\alpha, \gamma) - d_{C_0(S)}(h^{jk}(\alpha), h^{jk}(\gamma)) \\
&= d_{C_0(S)}(\gamma, \phi^{jk}(\gamma)) - 2d_{C_0(S)}(\alpha, \gamma)
\end{aligned} \tag{2.3}$$

we have $d_{C_0(S)}(\alpha, \phi^{jk}(\alpha)) \to \infty$, which contradicts that there exists some m such that $\phi^m(\alpha) = \alpha$. Therefore ϕ is a pseudo-Anosov map. □

Now that we have established Lemma 2.12, we are able to prove Theorem 2.4.

Proof of Theorem 2.4. Fix some value of $n \in \mathbb{N}$, and consider the surface $S_{0,n}$. Fix $k > 1$, $k \in \mathbb{N}$, and fix a partition $\mu = \{\mu_1, \ldots, \mu_k\}$ of the n punctures of $S_{0,n}$ such that $\rho(\mu_{i-1}) = \rho(\mu_{(i \mod k)})$. We prove that

$$\phi = \prod_{i=1}^{k} D_{\mu_i}^{q_i} = D_{\mu_k}^{q_k} \ldots D_{\mu_2}^{q_2} D_{\mu_1}^{q_1}$$

is a pseudo-Anosov mapping class.

We first construct the train-track τ so that $\phi(\tau)$ is carried by τ. Consider the partition $\mu = \{\mu_1, \ldots, \mu_k\}$. Construct a k-valent spine by having a branch loop around the highest labelled puncture in each of the sets μ_i, with each of these branches meeting in the center where they are smoothly connected by a k-gon, such as in Figures 2.3 (a) and 2.5 (a). For the remaining labelled punctures in μ_i, loop a branch around each puncture and have this branch will turn left towards the k-valent spine meeting the branch of the spine whose label is next in the ordering, such as in Figures 2.3 (b) and 2.5 (b).

For each k, $D_{\mu_i}^{q_i}$ acts locally the same. In particular, each half-twist in $D_{\mu_i}^{q_i}$ involves a branch located around puncture b on the k-valent spine and the branch located around puncture b' which is directly next to puncture b in the clockwise direction. As we consider a right half-twist to be positive, we notice that the branch at puncture b will begin to turn into the branch at puncture b', see Figures 2.4 and 2.6 for examples. Therefore, after the twist, the branch around puncture b' is now on the k-valent spine, and the branch around puncture b is directly next to the branch at puncture b' in the counter clockwise direction. Branches which are neither on the k-valent spine nor directly clockwise to the k-valent spine are unaffected by $D_{\mu_i}^{q_i}$. Thus, after each application of $D_{\mu_i}^{q_i}$, the train-track rotates clockwise by $\frac{2\pi}{n}$. Since τ has a rotational symmetry of order k, we notice that $\phi(\tau)$ is carried by τ.

Let M_τ denote the matrix representing the induced action of the space of weights on τ. To prove that M_τ is Perron-Frobenius, fix an initial weight on each branch. For each application of $D_{\mu_i}^{q_i}$, the labels on and directly next in the clockwise direction to the k-valent spine will become a linear combination of the labels associated to these two branches. In particular, let w be the weight of a branch on the k-valent spine, and let w' be the weight of the branch directly next in the clockwise direction to the branch on the k-valent spine. After applying l half-twists, we see that the weight of branch w is $lw' + (l-1)w$ and the weight of branch w' is $(l+1)w' + lw$. Since τ rotates clockwise by $\frac{2\pi}{n}$ after each application of $D_{\mu_i}^{q_i}$, we know that after k applications of ϕ each branch will be a linear combination of the initial weights from each of the branches where the constants of this linear combination will be strictly positive integers. Equivalently, this implies that each entry in M_τ^k is a strictly positive integer value. This implies that the matrix M_τ is Perron-Frobenius.

To finish the proof, note that each of the train-tracks which were constructed above are large, generic, and birecurrent. Therefore, we apply Lemma 2.12 which completes the proof that ϕ is a pseudo-Anosov mapping class. □

We now provide a proof for Theorem 2.6.

Proof of Theorem 2.6. Fix some value of $n \in \mathbb{N}$, some $k > 1$, $k \in \mathbb{N}$, and a partition $\mu = \{\mu_1, \ldots, \mu_k\}$ of the n punctures of $S_{0,n}$ such that $\rho(\mu_{i-1}) = \rho(\mu_{(i \mod k)})$. We perform the modification outlined in the statement of the theorem to obtain a partition, $\mu' = \{\mu'_1, \ldots, \mu'_k, \mu'_{k+1}\}$, on the $(n+1)$-times punctured sphere, $S_{0,n+1}$, which defines the map

$$\phi' = \prod_{i=1}^{k+1} D_{\mu_i}^{q'_i} = D_{\mu'_{k+1}}^{q'_{k+1}} D_{\mu'_k}^{q'_k} \ldots D_{\mu'_2}^{q'_2} D_{\mu'_1}^{q'_1}.$$

We prove that ϕ' is a pseudo-Anosov mapping class.

We first construct the train-track τ' so that $\phi'(\tau')$ is carried by τ'. We begin by considering the train-track τ associated to the map $\phi = \prod_{i=1}^{k} D_{\mu_i}^{q_i}$ defined by the partition μ. We then add in a new puncture onto the sphere between punctures k and $k+1$, and relabel the punctures, see Figures 2.7 (a) and 2.9 (a) for examples. Add a branch from puncture $k+1$ so that it turns tangentially into the k-valent spine meeting the same branch on the spine as the branches associated to punctures $1, \ldots, j$, see Figures 2.7 (b) and 2.9 (b) for examples. We denote this modified train-track by τ'.

To show that $\phi'(\tau')$ is carried by τ', we notice that by the same reasoning in the proof of Theorem 2.4 that for each $1 \le i < k+1$, the application of $D_{\mu'_i}^{q'_i}$ will rotate the train-track clockwise by $\frac{2\pi}{n}$. After the first k applications of $D_{\mu'_i}^{q'_i}$, we have rotated the train-track by $\frac{2\pi k}{n}$, which is not quite τ'. By applying the final twist $D_{\mu'_{k+1}}^{q'_{k+1}}$, we find $\phi'(\tau') = \tau'$ and thus $\phi'(\tau')$ is carried by τ'. See Figures 2.8 and 2.10 for examples.

For the same reasoning as in the proof of Theorem 2.4, the matrix representing the induced action on the space of weights on τ' will be a Perron-Frobenius. To finish the proof, we note that each of the train-tracks that we have constructed are large, generic, and birecurrent. Therefore, we are able to apply Lemma 2.12 which completes the proof that the map is a pseudo-Anosov mapping class. □

2.3 INTRODUCTORY EXAMPLES

In this section, we present two detailed examples on how to use Theorems 2.4 and 2.6. Consider the six-times punctured sphere. By following Theorem2.4, we construct two pseudo-Anosov maps on $S_{0,6}$.

Example 2.17. Consider the six-times punctured sphere and label the punctures of the sphere as introduced in Figure 2.1. Up to spherical symmetry, there are two unique partitions of the six punctures so that the labels of the punctures are evenly spaced, namely

$$\mu = \{\{0,3\},\{1,4\},\{2,5\}\} = \{\mu_1,\mu_2,\mu_3\}$$

and

$$\bar{\mu} = \{\{0,2,4\},\{1,3,5\}\} = \{\bar{\mu}_1,\bar{\mu}_2\}.$$

Recall that we define the half-twist associated to puncture j as the half-twist around the curve separating punctures j and $j-1$. Therefore, these partitions can define the two maps,

$$\phi = D_5^2 D_2^2 D_4^2 D_1^2 D_3^2 D_0^2 = D_{\mu_3}^2 D_{\mu_2}^2 D_{\mu_1}^2,$$

and

$$\bar{\phi} = D_5^2 D_3^2 D_1^2 D_4^2 D_2^2 D_0^2 = D_{\bar{\mu}_2}^2 D_{\bar{\mu}_1}^2,$$

respectively. We prove that both maps are pseudo-Anosov.

We first prove that $\phi = D_{\mu_3}^2 D_{\mu_2}^2 D_{\mu_1}^2$ is a pseudo-Anosov map on $S_{0,6}$. To prove that ϕ is a pseudo-Anosov map, we find a train-track τ on $S_{0,6}$ so that $\phi(\tau)$ is carried by τ and show that the matrix presentation of ϕ in the coordinates given by τ is a Perron-Frobenius matrix.

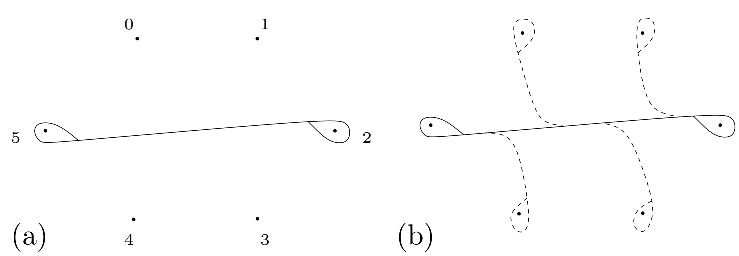


Figure 2.3: Constructing the train-track for the map ϕ.

First, we describe how we construct the train-track for the map ϕ based on the partition μ. Notice that μ has three subsets containing two punctures. As the punctures in each partition are so that $|i-j| \geq 2 \mod 6$, the twists associated to the punctures in each subset are disjoint. Since there are two twists in each subset and the partition is evenly spaced, the train-track has rotational symmetry of order two. Therefore, we construct a two-valent spine around the punctures labelled 2 and 5, pictured in 2.3 (a). Since there are three subsets, there are two branches turning tangentially into each of the two nodes on the two-valent spine, where these branches will turn left towards the spine, pictured in 2.3 (b).

The series of images in Figure 2.4 depict the train-track τ and its images under successive applications of the Dehn twists associated to ϕ. These images prove that $\phi(\tau)$ is carried by τ, and for every application of $D^2_{\mu_i}$, the train-track τ rotates clockwise by $\frac{2\pi}{6}$. By keeping track of the weights on τ, we calculate that the induced action on the space of weights on τ is given by the following matrix:

$$A = \begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 2 \\ 6 & 3 & 2 & 4 & 0 & 4 \\ 12 & 6 & 3 & 6 & 0 & 8 \\ 0 & 0 & 2 & 3 & 2 & 0 \\ 4 & 0 & 4 & 6 & 3 & 2 \\ 6 & 0 & 8 & 12 & 6 & 3 \end{pmatrix}$$

Note that the space of admissible weights on τ is the subset of $\mathbb{R}^6$ given by positive real numbers a, b, c, d, e and f such that $a + b + f = c + d + e$. The linear map described above preserves this subset. The square of the matrix A is strictly positive, which implies that the matrix is a Perron-Frobenius matrix. In fact, the top eigenvalue is $9 + 4\sqrt{5}$ which is associated to a unique irrational measured lamination F carried by τ that is fixed by ϕ. Lastly, since the train-track τ is large, generic, and birecurrent, we are able to apply Lemma 2.12 from Section 2.2 which finishes the proof that this map is a pseudo-Anosov.

Notice that we can perform each of the half twists to any power and still have the exact same train-track constructed above. However, the labels associated to the branches will subsequently increase or decrease in value according to how many twists are applied to each curve. Since all the twists are positive, we still have that all values in the resulting matrix will be positive and will be a Perron-Frobenius. An application of Lemma 2.12 from Section 2.2 will give our desired result.

We will now show that $\bar{\phi} = D^2_{\bar{\mu}_2} D^2_{\bar{\mu}_1}$ is a pseudo-Anosov on $S_{0,6}$, which follows a similar argument as above.

We again analyze the partition $\bar{\mu}$ as it determines the construction of our train-track $\bar{\tau}$. Notice that $\bar{\mu}$ has two subsets containing three twists each. Since there are three punctures in each subset and the partition is evenly spaced, the train-track has rotational symmetry of order three. Therefore, we will construct a three-valent spine around the punctures

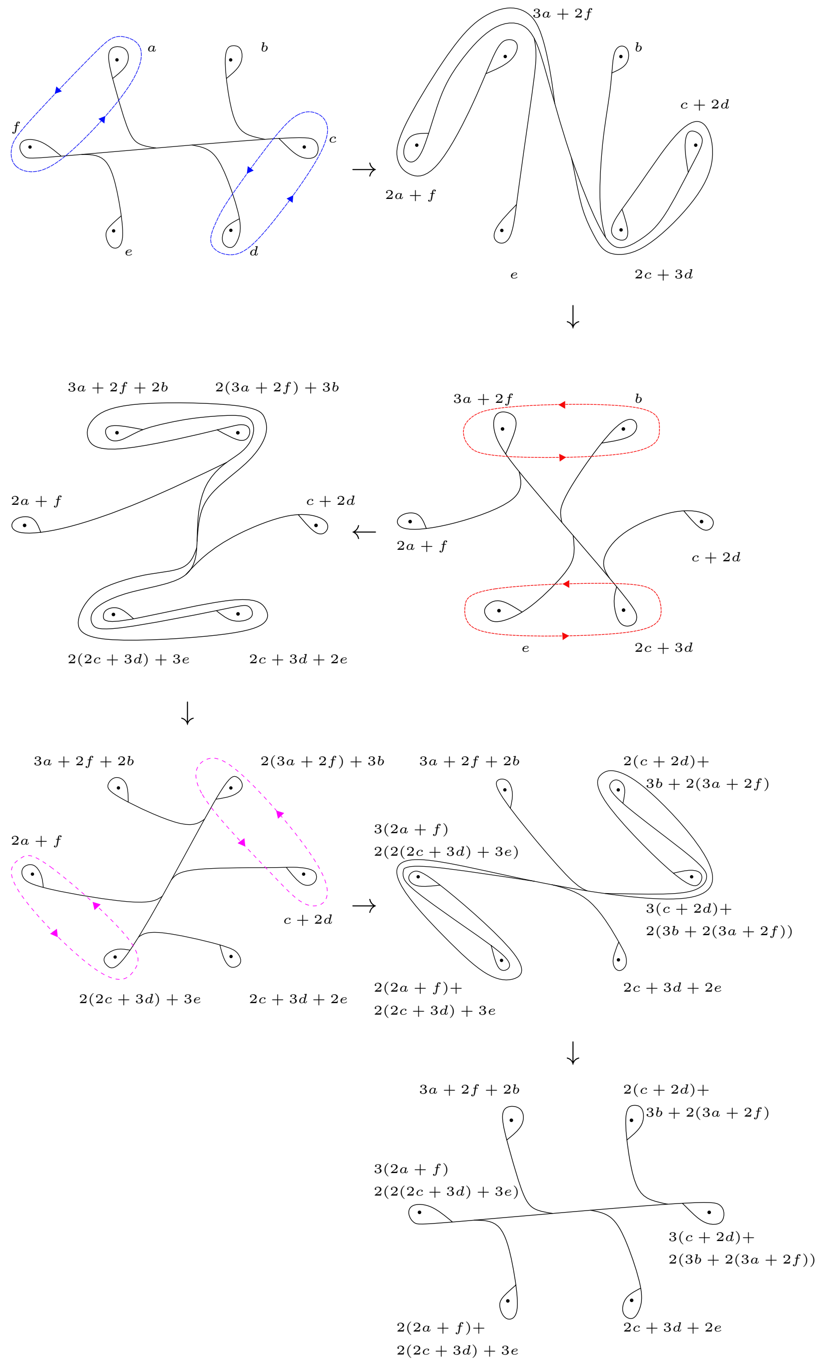


Figure 2.4: The train-track $\phi(\tau)$ is carried by τ.

labelled 1, 3 and 5, pictured in 2.5 (a). Since there are two subsets, there is one branch turning tangentially towards each of the three nodes on the three-valent spine, where these branches will be turning left towards the spine, pictured in 2.5 (b).

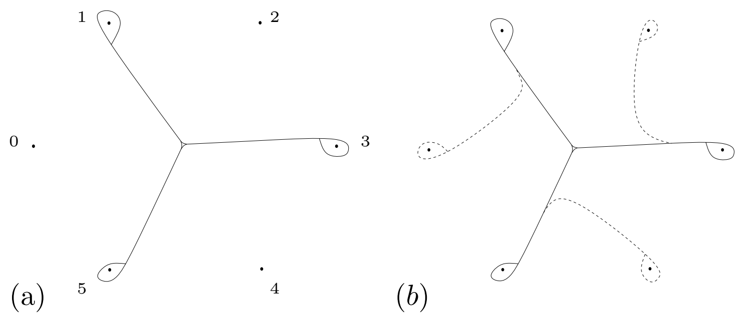


Figure 2.5: Constructing the train-track for the map $\bar{\phi}$.

The series of images in Figure 2.6 depict the train-track $\bar{\tau}$ and its images under successive applications of the Dehn twists associated to $\bar{\phi}$. These images prove that $\bar{\phi}(\bar{\tau})$ is indeed carried by $\bar{\tau}$. We again notice that for every application of $D^2_{\bar{\mu}_i}$, the train-track $\bar{\tau}$ rotates clockwise by $\frac{2\pi}{6}$. By keeping track of the weights on $\bar{\tau}$, we calculate that the induced action on the space of weights on τ is given by the following matrix:

$$B = \begin{pmatrix} 3 & 2 & 4 & 0 & 0 & 2 \\ 6 & 3 & 6 & 0 & 0 & 4 \\ 0 & 2 & 3 & 2 & 4 & 0 \\ 0 & 4 & 6 & 3 & 6 & 0 \\ 4 & 0 & 0 & 2 & 3 & 2 \\ 6 & 0 & 0 & 4 & 6 & 3 \end{pmatrix}$$

The space of admissible weights on $\bar{\tau}$ is the subset of $\mathbb{R}^6$ given by positive real numbers a, b, c, d, e and f such that $b - a$, $d - c$, and $f - e$ are all positive and satisfy the triangle inequalities. The linear map described above preserves this subset. The square of the matrix B is strictly positive, which implies that the matrix is a Perron-Frobenius matrix. The top eigenvalue is $7 + 4\sqrt{3}$, which is associated to a unique irrational measured lamination F carried by $\bar{\tau}$ that is fixed by $\bar{\phi}$. As the train-track is large, generic, and birecurrent, we may apply Lemma 2.12 to finish the proof that this map is pseudo-Anosov.

We now modify the pseudo-Anosov maps from Example 2.17 to find two pseudo-Anosov maps on the seven-times punctured sphere. To achieve this, we apply Theorem 2.6 once to each of the maps found in 2.17. For each of these maps, we note that we can apply the modification more than once to obtain additional pseudo-Anosov maps defined on spheres with more punctures.

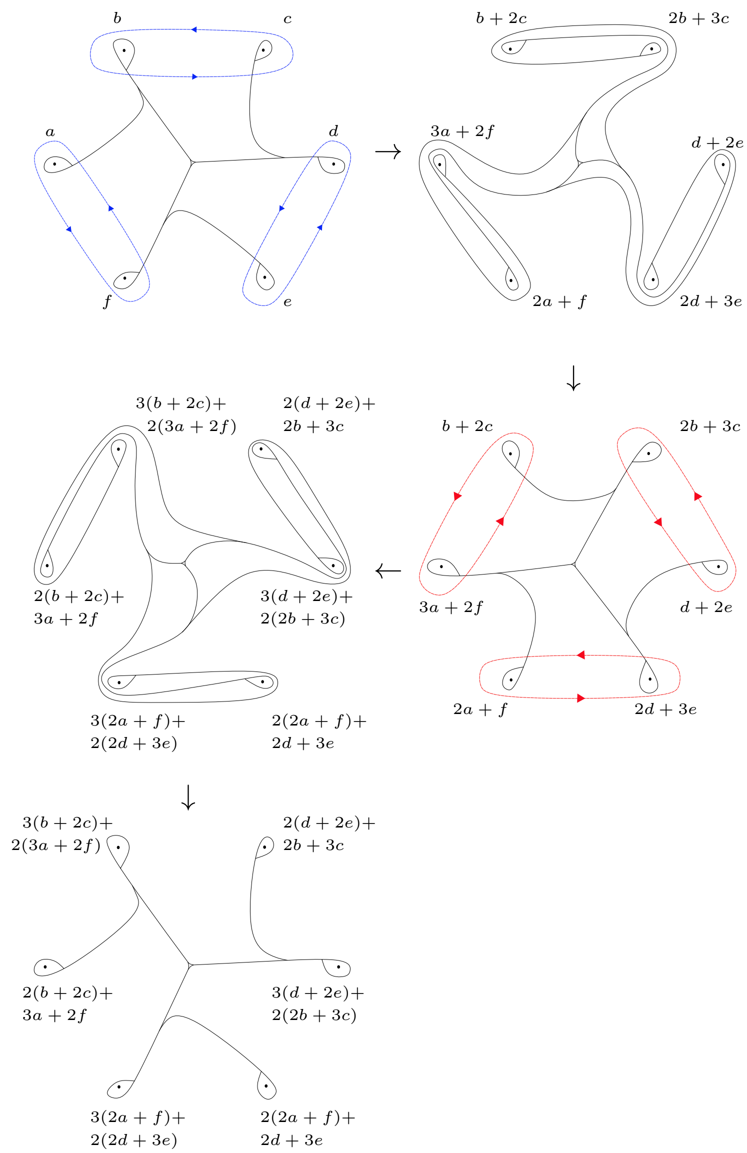


Figure 2.6: The train-track $\bar{\phi}(\bar{\tau})$ is carried by $\bar{\tau}$.

Example 2.18. We consider the seven-times punctured sphere with the labelling as introduced in Theorem 2.4. After applying Theorem 2.6 we obtain two partitions

$$\mu' = \{\{0,4\},\{1,5\},\{2,6\},\{3\}\} = \{\mu'_1, \mu'_2, \mu'_3, \mu'_4\},$$

and

$$\bar{\mu}' = \{\{0,3,5\},\{1,4,6\},\{2\}\} = \{\bar{\mu}'_1, \bar{\mu}'_2, \bar{\mu_3}'\}.$$

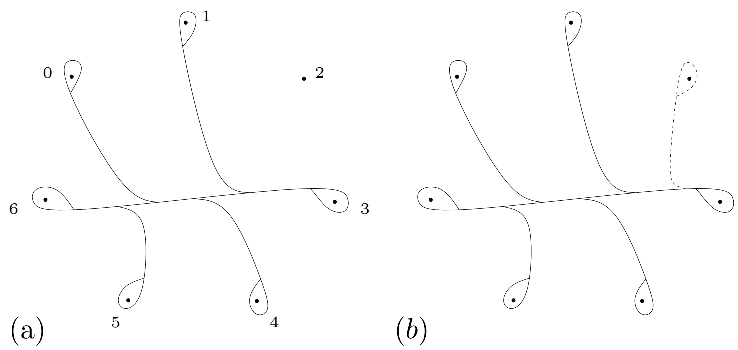


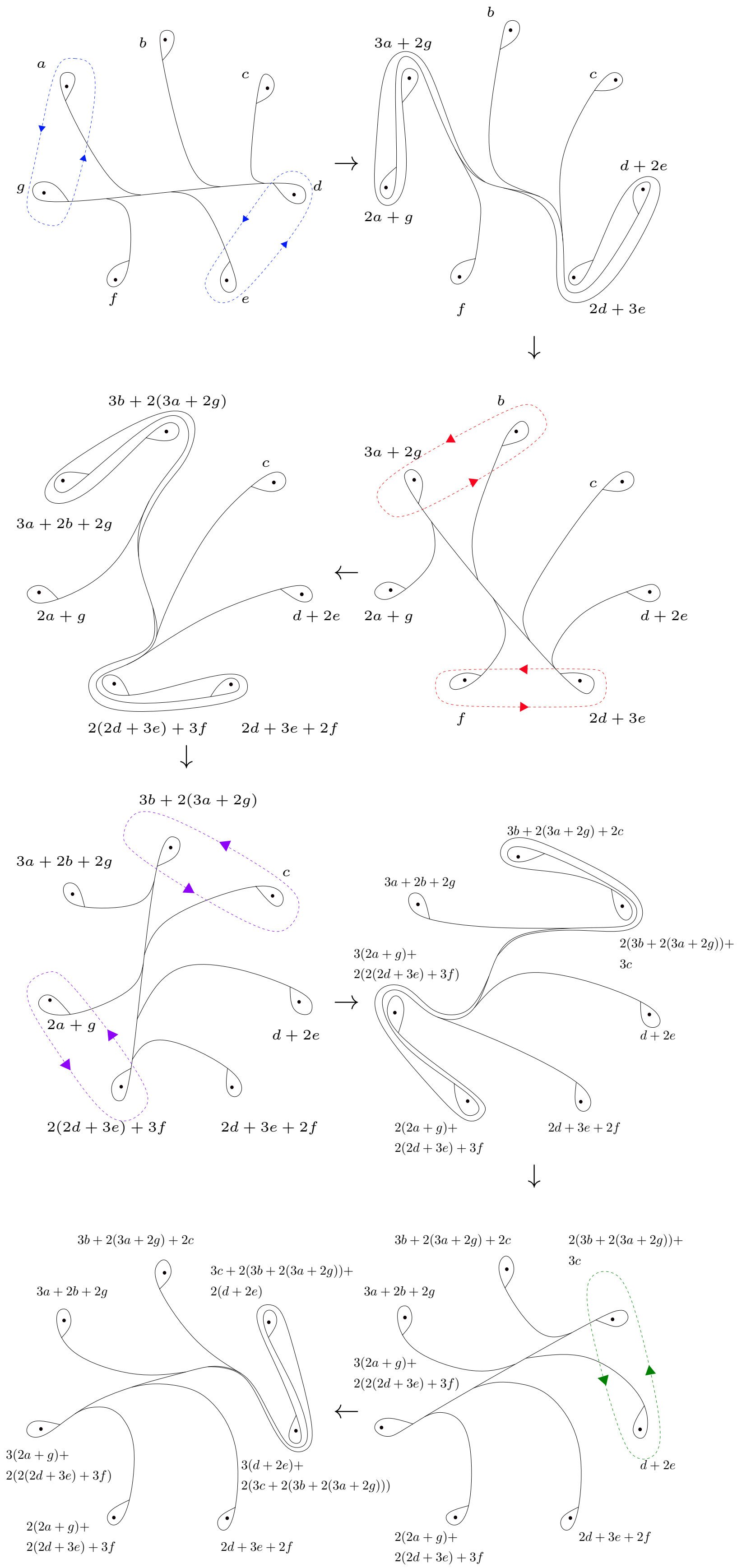
Figure 2.7: Constructing the train-track for the map ψ.

We begin by proving that $\Phi' = D^2_{\mu'_4} D^2_{\mu'_3} D^2_{\mu'_2} D^2_{\mu'_1}$ is a pseudo-Anosov on $S_{0,7}$. First, we describe how to construct the train-track associated to Φ', denoted τ', from the train-track τ associated to the map Φ from the previous example. Consider the train-track τ and place an extra puncture between the punctures labelled 1 and 2 in the previous example. Relabel the punctures so that the labelling is as in Theorem 2.4 (see Figure 2.7 (a)). Therefore, we have a train-track without a branch around the puncture labelled 2, but the rest of the train-track is as in Example 2.17 (up to relabelling). We construct a branch around the puncture labelled 2 which will turn tangentially towards the two valent spine, turning left towards the puncture labelled 3 (see Figure 2.7 (b)).

The series of images in Figure 2.8 depict the train-track τ' and its images under successive applications of the Dehn twists associated to Φ'. These images prove that $\Phi'(\tau')$ is carried by τ'. By keeping track of the weights on τ', we calculate that the induced action on the space of weights on τ' is given by the following matrix:

$$C = \begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 0 & 2 \\ 6 & 3 & 2 & 0 & 0 & 0 & 4 \\ 12 & 6 & 3 & 2 & 0 & 0 & 8 \\ 24 & 12 & 6 & 3 & 6 & 0 & 16 \\ 0 & 0 & 0 & 2 & 3 & 2 & 0 \\ 4 & 0 & 0 & 4 & 6 & 3 & 2 \\ 6 & 0 & 0 & 8 & 12 & 6 & 3 \end{pmatrix}$$

The space of admissible weights on τ' is the subset of $\mathbb{R}^7$ given by the positive real numbers a, b, c, d, e, f, and g such that $a + b + d + f = c + e + g$. The linear map described above preserves this subset. The square of the matrix C is strictly positive, which implies that the matrix is a Perron-Frobenius. Additionally, the top eigenvalue is approximately 22.08646



a
b
c
d
e
f
g
3a + 2g
2a + g
d + 2e
2d + 3e
3b + 2(3a + 2g)
3a + 2b + 2g
2(2d + 3e) + 3f
2d + 3e + 2f
3b + 2(3a + 2g) + 2c
2(3b + 2(3a + 2g))+
3c
3(2a + g)+
2(2(2d + 3e) + 3f)
2(2a + g)+
2(2d + 3e) + 3f
3c + 2(3b + 2(3a + 2g))+
2(d + 2e)
3(d + 2e)+
2(3c + 2(3b + 2(3a + 2g)))

which is associated to a unique irrational measured lamination F carried by τ' which is fixed by Φ'. As the train-track is large, generic, and birecurrent, we may apply Lemma 2.12 to finish the proof that this map is pseudo-Anosov.

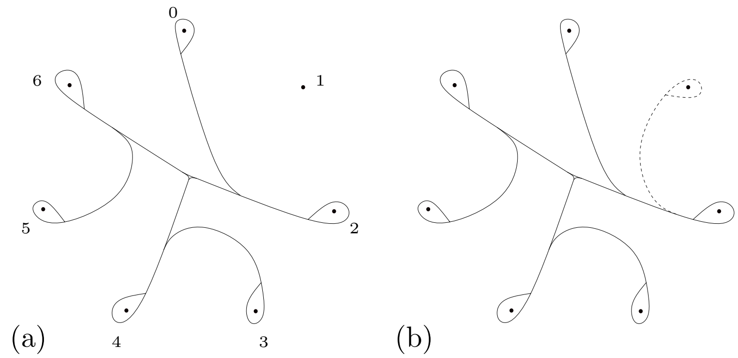


Figure 2.9: Constructing the train-track for the map $\bar{\phi}'$.

We now show that the homeomorphism $\bar{\tau}' = D^2_{\bar{\mu}_3'}D^2_{\bar{\mu}_2'}D^2_{\bar{\mu}_1'}$ is a pseudo-Anosov map. To construct the train-track, we will consider the train-track $\bar{\tau}$ associated to the map $\bar{\Phi}$ from the previous example. Consider the train-track $\bar{\tau}$ and place an extra puncture between the punctures labelled 0 and 1 in the previous example. Relabel the punctures so that the labelling is as in Theorem 2.4 (see Figure 2.9 (a)). Therefore, we have a train-track on $S_{0,7}$ which does not have a branch around the puncture labelled 1, and the rest of the train-track is as found in Example 2.17 (up to relabelling). We then construct a branch around the puncture labelled 1 which will turn tangentially into the three valent spine, turning left towards the puncture labelled 2 (see Figure 2.9 (b)).

The series of images in Figure 2.10 depict the train-track $\bar{\tau}'$ and its images under successive applications of the Dehn twists associated to $\bar{\Phi}'$.

Figure 2.10 shows that $\bar{\Phi}'(\bar{\tau}')$ is indeed carried by $\bar{\tau}'$. By keeping track of the weights on $\bar{\tau}'$, we calculate that the induced action on the space of weights on $\bar{\tau}'$ is given by the following matrix:

$$D = \begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 0 & 2 \\ 6 & 3 & 2 & 4 & 0 & 0 & 4 \\ 12 & 6 & 3 & 6 & 0 & 0 & 8 \\ 0 & 0 & 2 & 3 & 2 & 4 & 0 \\ 0 & 0 & 4 & 6 & 3 & 6 & 0 \\ 4 & 0 & 0 & 0 & 2 & 3 & 2 \\ 6 & 0 & 0 & 0 & 4 & 12 & 3 \end{pmatrix}$$

The space of admissible weights on $\bar{\tau}'$ is the subset of $\mathbb{R}^7$ given by the positive real numbers $a, b, c, d, e, f,$ and g such that $c - b - a, e - d,$ and

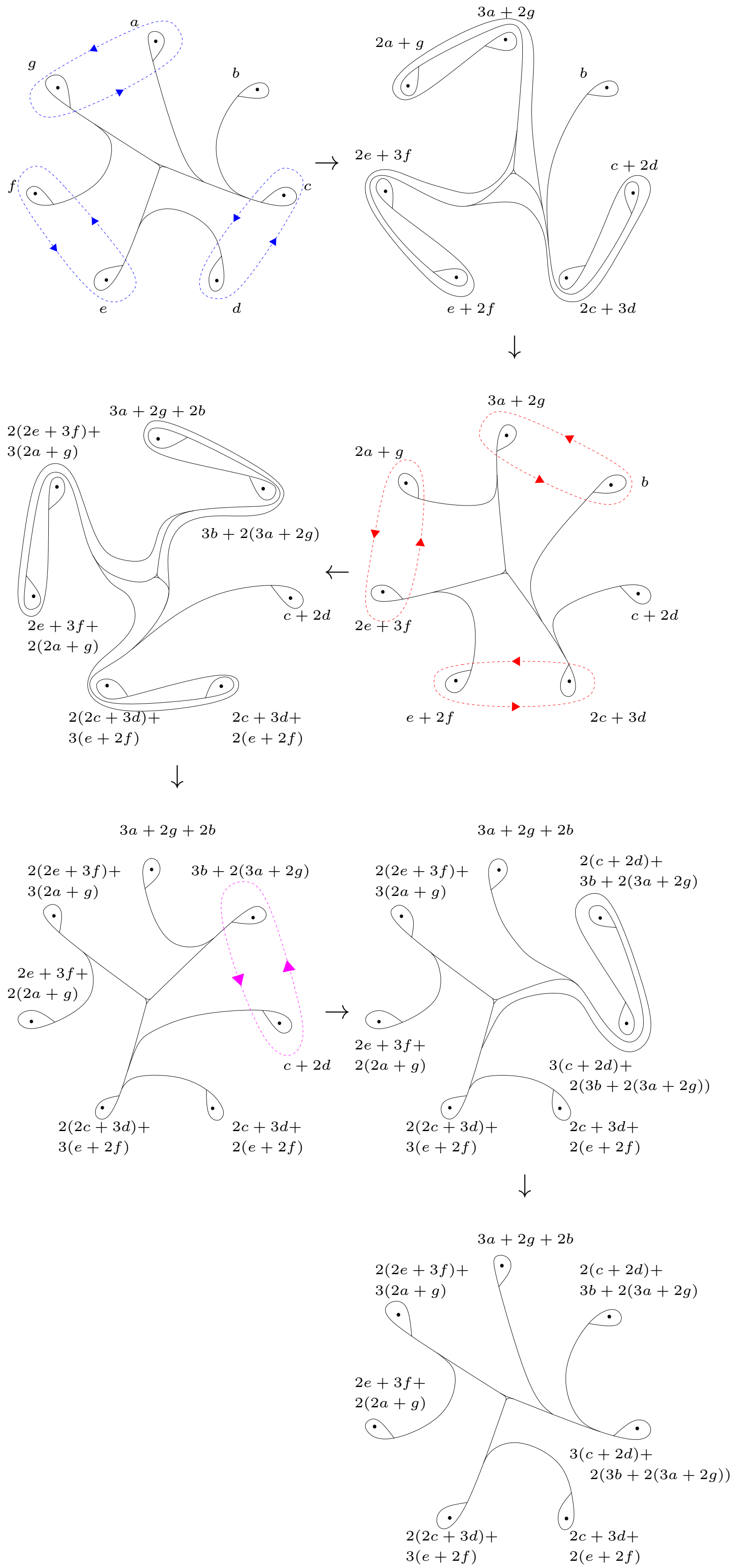


Figure 2.10: The train-track $\bar{\phi}'(\bar{\tau}')$ is carried by $\bar{\tau}'$.

$g - f$ are all positive and satisfy the triangle inequalities. The linear map described above preserves this subset. The square of the matrix D is strictly positive, which implies that the matrix is a Perron-Frobenius. The top eigenvalue of this matrix is

$$5 + \frac{1}{3}\sqrt[3]{2916 - 12\sqrt{93}} + \left(\frac{2}{3}\right)^{2/3}\sqrt[3]{243 + \sqrt{93}},$$

which is associated to a unique irrational measured lamination F carried by $\bar{\tau}'$ which is fixed by $\bar{\Phi}'$. As the train-track is large, generic, and birecurrent, we may apply Lemma 2.12 to finish the proof that this map is pseudo-Anosov.

2.4 MODIFICATIONS OF CONSTRUCTION

By considering the constructions described in Section 2.2, we notice that there are additional modifications one can make to the construction to obtain more pseudo-Anosov mapping classes.

The first modification we notice is that it is possible to apply the modification outlined in Theorem 2.6 more than once to construct additional pseudo-Anosov maps. In fact, we may continue to apply the modification step ad Infinitum to continue to find maps on punctured spheres.

Furthermore, one can apply the modification step from Theorem 2.6 in a slightly different manner in order to construct other pseudo-Anosov maps. Notice that the first application of Theorem 2.6 can allow a puncture to be placed between any of the punctures located on the k-valent spine and the puncture directly counter clockwise in the labelling. We may modify the initial partition by including a set which contains up to $k - 1$ of the k punctures which may be placed between a puncture on the spine and the element directly counter clockwise in the labelling. One may not include all k of the punctures as this map will be equivalent to a map found using Theorem 2.4. Up to relabelling the punctures, we will have defined a pseudo-Anosov map. The proof to show this is a pseudo-Anosov map follows the proof of Theorem 2.6 but with a modified train-track where an additional branch turns tangentially into the branch of the k-valent spine directly clockwise in the labelling of the punctures.

Both modifications described above can be made ad Infinium to any of the maps described in Theorem 2.4, as long as any of the new sets created never contain the same number of elements as any of the sets from the initial partition.

To obtain a third modification, we consider a map ϕ from any of the possible maps found though Theorem 2.6 or any of the modifications outlined above. It is possible to find an additional pseudo-Anosov map which has the same train-track as ϕ. Since the train-track rotates by $\frac{2\pi}{n}$ for the first k applications of $D^{q'_i}_{\mu'_i}$, we are able to define a map that will continue to rotate the train-track by $\frac{2\pi}{n}$ in place of doing the final twist(s) $D^{q'_{k+1}}_{\mu'_{k+1}}$.

For example, consider the first map from Example 2.18. After applying $D^2_{\mu'_3}D^2_{\mu'_2}D^2_{\mu'_1}$, where $\mu' = \{\{1,5\},\{2,6\},\{3,7\}\} = \{\mu'_1,\mu'_2,\mu'_3\}$, the train-track has rotated by $\frac{6\pi}{7}$. We can apply the rotations associated to punctures 4 and 1 next, then around puncture 5 and 2, around punctures 6 and 3, and finally around punctures 7 and 4, which will have rotated our train-track by a full rotation. In other words, you will obtain a new "partition" containing the sets $\mu''_i = \{i, i + \lceil\frac{7}{3}\rceil\}$ for $1 \le i \le 7$. More precisely, we obtain the following additional construction:

Theorem 2.19. *Consider the surface $S_{0,n}$. Consider one of the maps from Theorem 2.4, in particular, consider a partition of the n punctures into $1 < k < n$ sets $\{\mu_i\}_{i=1}^k$ such that the partition is evenly spaced. Apply any number of applications of Theorem 2.6 to obtain a new "partition", $\mu' = \{\mu'_1,\ldots,\mu'_k,\mu'_{k+1},\ldots,\mu'_{k+l}\}$, where $|\mu_{k+j}| < |\mu_1|$ for all $1 \le j \le l$ which defines a map on the p-sphere, where $p = \sum_{i=1}^{k+l}|\mu_i|$. Consider the train-track τ' associated to this map. Define the partition μ'' to be the partition containing the sets $\mu''_i = \{i, i + \lceil\frac{p}{k}\rceil \ldots, i + (|\mu_1|-1)\lceil\frac{p}{k}\rceil\}$, where $1 \le i \le p$. μ'' defines the pseudo-Anosov mapping class*

$$\phi' = \prod_{i=1}^{p} D^{q''_i}_{\mu''_i} = D^{q''_p}_{\mu''_p}\cdots D^{q''_2}_{\mu''_2}D^{q''_1}_{\mu''_1}$$

on $S_{0,p}$, where $q''_j = \{q''_{j_1},\ldots,q''_{j_l}\}$ is the set of powers associated to each μ''_i.

Remark 2.20. The proofs for the modifcations found in this section follow the same format as the proofs for the theorems found in Section **??**.

2.5 NUMBER-THEORETIC PROPERTIES

In this section, we will discuss the number-theoretic properties for some of the maps we are able to construct from Theorems 2.4 and 2.6.

The *trace field* of a linear group is the field generated by the traces of its elements. In particular the trace field of a group $\Gamma \subset SL_2(\mathbb{R})$ is the subfield of $\mathbb{R}$ generated by $tr(A)$, $A \in \Gamma$. Kenyon and Smillie proved that if the affine automorphism group of a surface contains an orientation preserving pseudo-Anosov element f with largest eigenvalue λ, then the trace field is $\mathbb{Q}(\lambda + \lambda^{-1})$ [KS00].

There has already been a considerable amount of research regarding the number-theoretic properties for both Penner and Thurston's constructions. For Penner's construction, Shin and Strenner were able to show that the Galois conjugates of the stretch factor are never on the unit circle [SS15]. For Thurston's construction, Hubert and Lanneau were able to show that the field $\mathbb{Q}(\lambda + 1/\lambda)$ is totally real [HL06].

In this section, we will show that some of the maps resulting from the constructions outlined in Theorem 2.4 and Theorem 2.6 are such that the Galois conjugates of the stretch factor are on the unit circle, and that the field $\mathbb{Q}(\lambda + 1/\lambda)$ is not totally real. This result shows that the construction outlined above cannot come from neither Penner nor Thurston's construction. However, we show it is also possible to find maps resulting from these constructions where $\mathbb{Q}(\lambda + 1/\lambda)$ is totally real and that the Galois conjugates of the stretch factors are not on the unit circle, where $\mathbb{Q}(\lambda + 1/\lambda)$ is totally real and that the Galois conjugates of the stretch factors are on the unit circle, or that $\mathbb{Q}(\lambda + 1/\lambda)$ is not totally real and the Galois conjugates of the stretch factors are not on the unit circle.

We begin by examining the algebraic properties of the first map introduced in Example 2.17. For this map, we will see that the Galois conjugates are never on the unit circle, and that the field $\mathbb{Q}(\lambda + 1/\lambda)$ is totally real.

Example 2.21. Consider the pseudo-Anosov map ϕ on $S_{0,6}$ introduced in Example 2.17. The characteristic polynomial of this map is

$$p_\phi = (x-1)^2(x+1)^2(x^2 - 18x + 1).$$

We notice that the leading eigenvalue λ_ϕ is a root of the factor

$$p_{\phi,\lambda}(x) = x^2 - 18x + 1,$$

which is an irreducible polynomial with real roots. Since $1 < \lambda_\phi \in \mathbb{R}$ is not on the unit circle, and therefore λ^{-1} is also not on the unit circle, that the Galois conjugates of the stretch factor are not on the unit circle.

To show that $\mathbb{Q}(\lambda + 1/\lambda)$ is totally real, we notice that we are able to write

$$\frac{p_\phi}{x} = \left(x + \frac{1}{x}\right) - 18 = q\left(x + \frac{1}{x}\right).$$

By considering the roots of $q(y) = y - 18$, we notice that the only root is $y = 18$ which implies that the field $\mathbb{Q}(\lambda + 1/\lambda)$ is totally real.

Next, we will provide an example where the Galois conjugates are never on the unit circle, and that the field $\mathbb{Q}(\lambda + 1/\lambda)$ is not totally real.

Example 2.22. We consider the map $\bar{\Phi}'$ from Example 2.18. We compute that the characteristic polynomial of this map is

$$p_{\bar{\Phi}} = (x+1)(x^3 - 15x^2 + 7x - 1)(x^3 - 7x^2 + 15x - 1).$$

We notice that λ is a root of the polynomial

$$p_{\bar{\Phi},\lambda}(x) = x^3 - 15x^2 + 7x - 1.$$

The roots of this polynomial are

$$5 + \frac{1}{3}\sqrt[3]{2916 - 12\sqrt{19}} + \left(\frac{2}{3}\right)^{2/3} \sqrt[3]{243 + \sqrt{93}},$$

$$5 + \frac{1}{3}(-1 + i\sqrt{3})\sqrt[3]{2916 - 12\sqrt{19}} - \frac{(1 + i\sqrt{3})(\sqrt[3]{1/2(243 + \sqrt{93})})}{3^{2/3}},$$

and

$$5 - \frac{1}{3}(-1 + i\sqrt{3})\sqrt[3]{2916 - 12\sqrt{19}} + \frac{(1 + i\sqrt{3})(\sqrt[3]{1/2(243 + \sqrt{93})})}{3^{2/3}}.$$

By unique factorization, we see that

$$p_{\bar{\Phi},\lambda}(x) = x^3 - 15x^2 + 7x - 1$$

is irreducible over $\mathbb{Q}$. None of the roots of $p_{\bar{\Phi},\lambda}(x)$ are on the unit circle, so we have that there are no Galois conjugates of the stretch factor on the unit circle. We now consider the polynomial

$$\left(\frac{1}{x^3}\right)(x^3 - 15x^2 + 7x - 1)(x^3 - 7x^2 + 15x - 1) = x^3 - 22x^2 + 127x - 276.$$

We are able to rewrite this polynomial as

$$q\left(x + \frac{1}{x}\right) = \left(x + \frac{1}{x}\right)^3 - 22\left(x + \frac{1}{x}\right)^2 + 124\left(x + \frac{1}{x}\right) - 232.$$

We calculate that the roots of the polynomial $q(y)$ are

$$\frac{1}{3}\left(22 + \sqrt[3]{1801 - 9\sqrt{26554}} + \sqrt[3]{1801 + 9\sqrt{26554}}\right),$$

$$\frac{1}{6}\left(44 + i(\sqrt{3} + i)\sqrt[3]{1801 - 9\sqrt{26554}} + (-1 - i\sqrt{3})\sqrt[3]{1801 + 9\sqrt{26554}}\right),$$

and

$$\frac{1}{6}\left(44+(-1-i\sqrt{3})\sqrt[3]{1801-9\sqrt{26554}}+i(\sqrt{3}+i)\sqrt[3]{1801+9\sqrt{26554}}\right).$$

By unique factorization, $q(y)$ is irreducible. Additionally, since two of the roots are imaginary, the field $\mathbb{Q}(\lambda+1/\lambda)$ is not totally real.

We now provide an example where there are Galois conjugates of the stretch factor on the unit circle, and that the field $\mathbb{Q}(\lambda+1/\lambda)$ is totally real.

Example 2.23. We now consider the pseudo-Anosov map

$$\psi = D_5^2 D_4^2 D_8^2 D_3^2 D_7^2 D_2^2 D_6^2 D_1^2,$$

which is the first map from Example 2.17 with the modification from Theorem 2.6 applied twice so that there are two partitions with one element each. The train-track η where $\psi(\eta)$ is carried by ψ is depicted in Figure 2.11 . The matrix associated to this map is

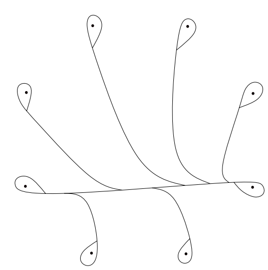


Figure 2.11: The train-track associated to ψ.

$$M = \begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\ 6 & 3 & 2 & 4 & 0 & 0 & 0 & 4 \\ 12 & 6 & 3 & 6 & 0 & 0 & 0 & 8 \\ 0 & 0 & 2 & 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 6 & 3 & 2 & 0 & 0 \\ 0 & 0 & 8 & 12 & 6 & 3 & 2 & 0 \\ 4 & 0 & 16 & 24 & 12 & 6 & 3 & 2 \\ 6 & 0 & 32 & 48 & 24 & 12 & 6 & 3 \end{pmatrix}$$

which has the characteristic polynomial

$$p(x) = (x+1)^4(x^4-28x^3+6x^2-28x+1).$$

Our leading eigenvalue λ is a root of

$$p_{\lambda,\psi}(x) = x^4 - 28x^3 + 6x^2 - 28x + 1.$$

The roots of this polynomial are

$$\lambda^{-1} = 7 + 4\sqrt{3} - 2\sqrt{24 + 14\sqrt{3}},$$

$$\lambda = 7 + 4\sqrt{3} + 2\sqrt{24 + 14\sqrt{3}},$$

$$x_1 = 7 - 4\sqrt{3} - 2i\sqrt{14\sqrt{3} - 24},$$

and

$$x_2 = 7 - 4\sqrt{3} + 2i\sqrt{14\sqrt{3} - 24}.$$

By unique factorization, we know that

$$p_{\lambda,\psi}(x) = x^4 - 28x^3 + 6x^2 - 28x + 1$$

is irreducible over $\mathbb{Q}$. Notice that $|x_1| = 1$, and $|x_2|$=1, which we can verify by direct computation or by applying by Theorem 1 of [KM04]. This implies that there are Galois conjugates of the stretch factor on the unit circle. We now show that the field $\mathbb{Q}(\lambda + 1/\lambda)$ is totally real by writing

$$\frac{p_{\lambda,\psi}}{x} = \left(x + \frac{1}{x}\right)^2 - 28\left(x + \frac{1}{x}\right) + 4 = q\left(x + \frac{1}{x}\right).$$

We notice that the roots of $q(y) = y^2 - 28y + 4$ are

$$14 - 8\sqrt{3}$$

and

$$14 + 8\sqrt{3},$$

which implies that $q(y)$ is irreducible by unique factorization. Additionally, since both roots are real we find that the field $\mathbb{Q}(\lambda + 1/\lambda)$ is totally real.

Lastly, we provide two examples where there are Galois conjugates of the stretch factor on the unit circle, and where the field $\mathbb{Q}(\lambda + 1/\lambda)$ is not totally real.

Example 2.24. We consider the second map from Example 2.17 and apply the modification from Theorem 2.6 twice so that there are two partitions with one element each. This induces the map

$$\psi' = D_4^2 D_3^2 D_8^2 D_6^2 D_2^2 D_7^2 D_5^2 D_1^2.$$

The train-track η' where $\psi'(\eta')$ is carried by ψ' is depicted in Figure 2.12 . The matrix associated to this map is

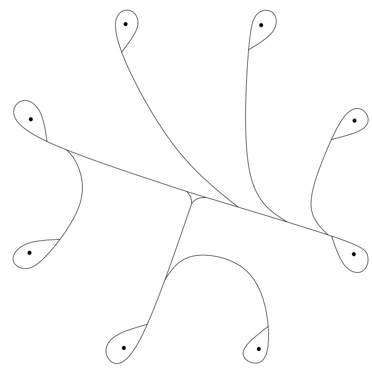


Figure 2.12: The train-track associated to ψ.

$$M = \begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\ 6 & 3 & 2 & 0 & 0 & 0 & 0 & 4 \\ 12 & 6 & 3 & 2 & 4 & 0 & 0 & 8 \\ 24 & 12 & 6 & 3 & 6 & 0 & 0 & 16 \\ 0 & 0 & 0 & 2 & 3 & 2 & 4 & 0 \\ 0 & 0 & 0 & 4 & 6 & 3 & 6 & 0 \\ 4 & 0 & 0 & 0 & 0 & 2 & 3 & 2 \\ 6 & 0 & 0 & 0 & 0 & 4 & 6 & 3 \end{pmatrix}$$

which has the characteristic polynomial

$$p_{\psi'}(x) = x^8 - 24x^7 + 156x^6 - 424x^5 - 186x^4 - 424x^3 + 156x^2 - 24x + 1.$$

This polynomial is irreducible we use the following fact, a proof for which is found in [Con].

Fact 2.25. If $f(x) \in \mathbb{Z}[x]$ is primitive of degree $d \geq 1$ and there are at least $2d+1$ different integers a such that $|f(a)|$ is 1 or a prime number then $f(x)$ is irreducible in $\mathbb{Q}[x]$

Therefore, it suffices to find 17 values of x such that $p_{\psi'}(x)$ is prime. Indeed, the following list of tuples $(x, p_{\psi'}(x))$ contains 17 x-values such that $p_{\psi'}(x)$ is prime.

$$(-160, 496582824202141441)$$
$$(-102, 14653782370731169)$$
$$(-90, 5537981240501761)$$
$$(-76, 1495649690458849)$$
$$(-52, 81377571089569)$$
$$(-46, 32071763417569)$$
$$(-40, 11167704826561)$$
$$(-22, 134573887009)$$
$$(0, 1)$$
$$(8, -7522751)$$
$$(26, 59124433057)$$
$$(72, 502376857985089)$$
$$(86, 2218259932983937)$$
$$(90, 3237148147105441)$$
$$(120, 34853759407811521)$$
$$(158, 331770565001360449)$$
$$(164, 449704327465370209)$$

Therefore, we have that $p_{\psi'}(x)$ is irreducible. By applying Theorem 1 of [KM04] we find that there are roots of this polynomial which are on the unit circle. We now write

$$\frac{p_{\psi'}}{x^4} = \left(x + \frac{1}{x}\right)^4 - 24\left(x + \frac{1}{x}\right)^3 + 152\left(x + \frac{1}{x}\right)^2 - 352\left(x + \frac{1}{x}\right) - 496.$$

We are able to prove that $q(y) = y^4 - 24y^3 + 152y^2 - 352y - 496$ is an irreducible polynomial as follows. By Gauss's lemma, a primitive polynomial is irreducible over the integers if and only if it is irreducible over the rational numbers. Since $q(y)$ is primitive, it suffices to show that $q(y)$ is irreducible over the integers. The rational root theorem gives us that $q(y)$ has no roots, so if it is reducible then $q(y) = (y^2 + ay + b)(y^2 + cy + d)$.

Therefore, suppose that $q(y) = (y^2 + ay + b)(y^2 + cy + d)$. Expanding gives rise to the system of equations

$$\begin{aligned} a + c &= -24 \\ ac + b + d &= 152 \\ ad + bc &= -352 \\ bd &= -496 \end{aligned} \tag{2.4}$$

Substituting $a = -24 - c$ and $b = \frac{-496}{d}$ into the second and third equations give

$$\begin{aligned} -24c - c^2 - \frac{496}{d} + d &= 152 \\ (-24 - c)d - \frac{496c}{d} &= -352 \end{aligned} \tag{2.5}$$

We solve for c in the second equation to find

$$c = \frac{24d^2 - 352d}{-d^2 - 496}.$$

Substituting this into the first equation gives

$$-24\left(\frac{24d^2 - 352d}{-d^2 - 496}\right)d - \left(\frac{24d^2 - 352d}{-d^2 - 496}\right)^2 d + d^2 - 152d - 496 = 0,$$

which has no integer roots. Therefore, there is no d satisfying the conditions we require, therefore $q(y)$ is irreducible. Finally, by using the formulas for the roots of a quartic equation, we see that $q(y)$ has two imaginary roots, therefore the field $\mathbb{Q}(\lambda + 1/\lambda)$ is not totally real.

Example 2.26. For this example, we will apply the modification from Theorem 2.6 to the map associated to the partition

$$\mu_{3,3,3} = \{\{1,4,7\},\{2,5,8\},\{3,6,9\}\}$$

to obtain the partition

$$\mu_{3,3,3,1} = \{\{1,5,8\},\{2,6,9\},\{3,7,10\},\{4\}\}.$$

This induces the map

$$\psi_{3,3,3,1} = D_4 D_{10} D_7 D_3 D_9 D_6 D_2 D_8 D_5 D_1.$$

The train-track $\tau_{3,3,3,1}$ where $\psi_{3,3,3,1}(\tau_{3,3,3,1})$ is carried by $\psi_{3,3,3,1}$ is depicted in Figure 2.13 . The matrix associated to this map is

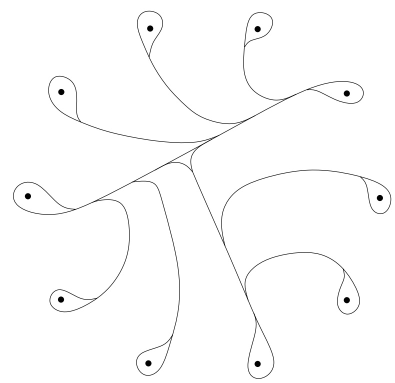


Figure 2.13: The train-track associated to ψ.

$$M = \begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 6 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 12 & 6 & 3 & 2 & 4 & 0 & 0 & 0 & 0 & 8 \\ 24 & 12 & 6 & 3 & 6 & 0 & 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 2 & 3 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 6 & 3 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 8 & 12 & 6 & 3 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 2 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 4 & 6 & 3 & 2 \\ 6 & 0 & 0 & 0 & 0 & 0 & 8 & 12 & 6 & 3 \end{pmatrix}$$

which has the characteristic polynomial

$$\begin{aligned} p_{\psi_{3,3,3,1}}(x) = x^{10} - 30x^9 + 285x^8 - 1864x^7 - 30x^6 + 204x^5 \\ - 30x^4 - 1864x^3 + 285x^2 - 30x + 1 \end{aligned}$$

To show that $p_{\psi_{3,3,3,1}}(x)$ is irreducible, it suffices to find 21 values of x such that $p_{\psi_{3,3,3,1}}(x)$ is prime by Fact 2.25. Indeed, the following list of tuples $(x, p_{\psi_{3,3,3,1}}(x))$ contains 21 x-values such that $p_{\psi_{3,3,3,1}}(x)$ is prime:

$$(-224, 362492550167302377983553)$$
$$(-208, 174468986764240268082529)$$
$$(-168, 21295969891012540121329)$$
$$(-160, 13184110259956391044801)$$
$$(-76, 93111155064177745721)$$
$$(-72, 55283878442850559 21)$$
$$(-54, 351062674729634953)$$
$$(-52, 245103960106710121)$$
$$(-12, 405908347321)$$
$$(-10, 87091192801)$$
$$(0, 1)$$
$$(2, -189671)$$
$$(6, -285219647)$$
$$(24, 6967292292721)$$
$$(74, 31613941134619 23721)$$
$$(186, 41960187610521563501353)$$
$$(204, 107295840626496890031721)$$
$$(216, 191678753902872238701553)$$
$$(234, 431604542240603942600521)$$
$$(258, 1160267613906359066071321)$$
$$(264, 1464201236305006987479121)$$

Therefore, we have that $p_{\psi'}(x)$ is irreducible. Applying Theorem 1 of [KM04] we find that there are roots of this polynomial which are on the unit circle. We now write

$$\frac{p_{\psi_{3,3,3,1}}}{x^5} = \left(x+\frac{1}{x}\right)^5 - 30\left(x+\frac{1}{x}\right)^4 + 280\left(x+\frac{1}{x}\right)^3 + 1744\left(x+\frac{1}{x}\right)^2 - 880\left(x+\frac{1}{x}\right) - 3307.$$

We rewrite the above as $q(y) = y^5 - 30y^4 + 280y^3 + 1744y^2 - 880y - 3307$. To see that $q(y)$ is irreducible over $\mathbb{Q}$, it suffices to show that there are 11 values of y such that $q(y)$ is prime. Indeed, the following list of tuples $(y, q(y))$ contains 11 y-values such that $q(y)$ is prime:

$$(-12, -1096363)$$
$$(-10, -500107)$$
$$(-6, -42379)$$
$$(-4, 1493)$$
$$(-2.2677)$$
$$(0, -3307)$$
$$(2, 3701)$$
$$(4, 32341)$$
$$(14, 479861)$$
$$(16, 658453)$$
$$(22, 1928821)$$

Finally, we notice that the discriminant of the polynomial $q(y)$ is calculated to be

$$-103017075043340205442l9.$$

As the discriminant is negative, we know that there must exists non-real roots, therefore the field $\mathbb{Q}(\lambda + 1/\lambda)$ is not totally real.

Proof of Theorem 2.8. Consider the maps from Examples 2.24 and 2.26. For Thurston's construction Hubert and Lanneau proved that the field $\mathbb{Q}(\lambda + 1/\lambda)$ is always totally real [HL06], and for Penner's construction Shin and Strenner proved the Galois conjugates of the stretch factor are never on the unit circle [SS15]. In Examples 2.24 and 2.26, it was shown that the maps ψ' and $\psi_{3,3,3,1}$ have trace fields which are not totally real and there are Galois conjugates of both stretch factors on the unit circle. Therefore, this mapping class is unable to come from either Thurston's or Penner's constructions. □

2.6 CONSTRUCTION ON SURFACES OF HIGHER GENUS

In this section, we show that we can lift the constructed pseudo-Anosov mapping classes on $2g+2$-punctured spheres to pseudo-Anosov mapping classes on surfaces of genus $g > 0$ through a branched cover. In order to lift the mapping classes, we will apply the work done by Birman and Hilden [BH73]. For an overview of Birman-Hilden theory, see [MW17].

We can see that $S_{0,2g+2}$ and $S_{g,0}$ are related by a two-fold branched covering map $S_{g,0} \to S_{0,2g+2}$ (see Figure 2.14 for an example). The $2g+2$

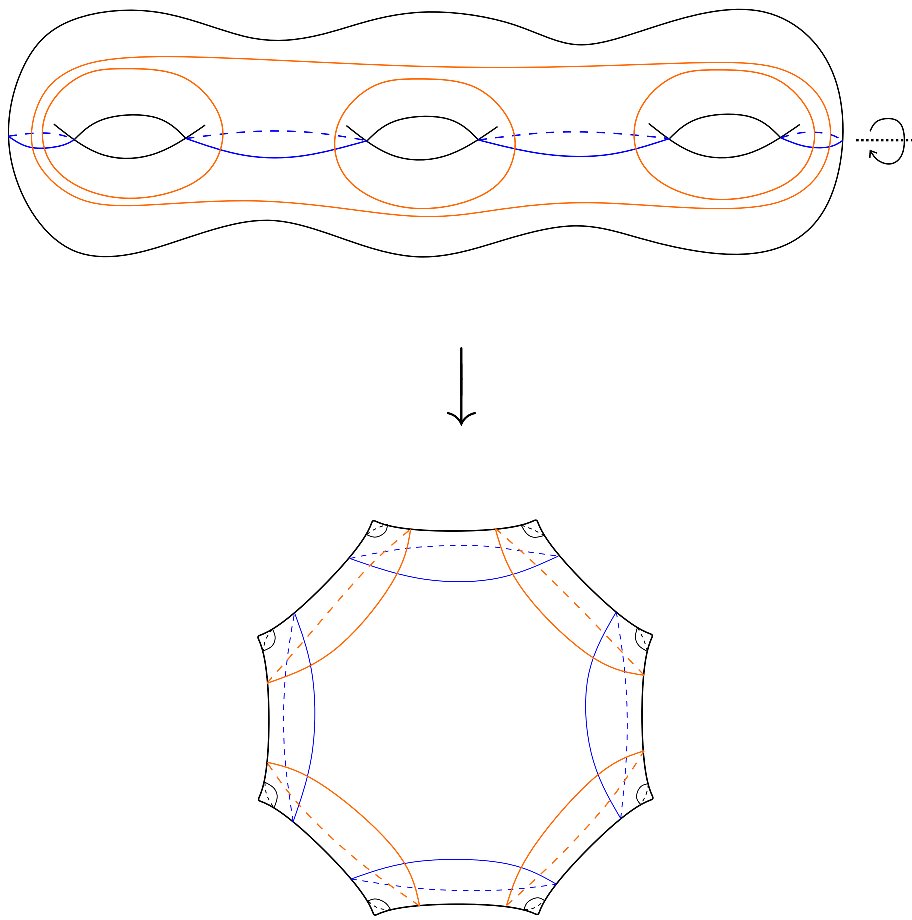


Figure 2.14: Two-fold branched covering map from $S_{3,0}$ to $S_{0,8}$.

punctures on the sphere are the branch points, and the deck transformation is the *hyperelliptic involution* of $S_{g,0}$, denoted ι. Since every element of $\mathrm{Map}(S_{g,0})$ has a representative which commutes with ι, there exists a map

$$\Theta\colon\ \mathrm{Map}(S_{g,0}) \to \mathrm{Map}(S_{0,2g+2}).$$

The cyclic group of order two generated by the involution ι is the kernel of the map Θ. Each generator for $\mathrm{Map}(S_{0,2g+2})$ lifts to $\mathrm{Map}(S_{g,0})$, therefore Θ is surjective, and we have the following short exact sequence:

$$1 \to \langle \iota \rangle \to \mathrm{Map}(S_{g,0}) \xrightarrow{\Theta} \mathrm{Map}(S_{0,2g+2}).$$

This means that a presentation for $\mathrm{Map}(S_{0,2g+2})$ can be lifted to a presentation for $\mathrm{Map}(S_{g,0})$. Since the elements of the mapping class group are defined up to isotopy, we would need to show that the isotopies respect the hyperelliptic involution in order to prove that the map Θ is well defined.

By the work of Birman and Hilden, we will see that all isotopies can be chosen to respect the hyperelliptic involution.

Definition 2.27 (Fiber preserving)**.** Let $p\colon S \to X$ be a covering map of surfaces, possibly branched, possibly with boundary. We say that $f\colon S \to S$ is *fiber preserving* if for each $x \in X$ there is a $s \in X$ so that $f(p^{-1}(x)) = p^{-1}(y)$, ie. f takes fibers to fibers.

If any two homotopic fiber-preserving mapping classes of S are homotopic through fiber-preserving homeomorphisms, then we say that the covering map p has the *Birman-Hilden property*. Equivalently, whenever a fiber-preserving homeomorphism is homotopic to the identity, it is homotopic to the identity through fiber preserving homeomorphisms [BH73].

Theorem 2.28 (Birman-Hilden)**.** *Let $p\colon S \to X$ be a finite-sheeted regular branched covering map where S is a hyperbolic surface. Assume that p is either unbranched or solvable. Then p has the Birman-Hilden property.*

Maclachlan and Harvey gave the following generalization of Theorem 2.28 [MH75]:

Theorem 2.29 (Maclachlan-Harvey)**.** *Let $p\colon S \to X$ be a finite-sheeted regular branched covering map where S is a hyperbolic surface. Then p has the Birman-Hilden property.*

Thus, we apply Theorem 2.29 to the branched covering map $S_{g,0} \to S_{0,2g+2}$ to find that the map Θ is well defined. Therefore, every pseudo-Anosov mapping class on $S_{0,2g+2}$ lifts to a pseudo-Anosov mapping class on $S_{g,0}$. Indeed, consider a pseudo-Anosov map ϕ which was found using one of the constructions in either Section 2.2 or Section 2.4. This map lifts to a map ψ on $S_{g,0}$. As all of the branch points are the punctures of the sphere this implies that ψ has the same local properties of ϕ. In particular, the stable and unstable foliations for ψ are the preimages under p for those of ϕ. Therefore, ψ is a pseudo-Anosov mapping class on $S_{g,0}$.

Remark 2.30*.* Similarly, using the same argument, it is possible to find that $S_{0,2g+3}$ and $S_{g,2}$ are related by a two-fold branched covering map $S_{g,2} \to S_{0,2g+3}$, so we may also lift the maps from odd-times punctured spheres to surfaces with higher genus. We also notice that the argument holds for any finite-sheeted regular branched covering map using the same argument.

As we are able to lift the pseudo-Anosov mapping classes from punctured spheres to surfaces with genus, we are also able to find new examples of pseudo-Anosov mapping classes on surfaces with genus by lifting the maps found in Examples 2.24 and 2.26. To determine precisely which surfaces we can lift these maps to, we will employ the Riemann-Hurwitz-Hasse formula, which is a generalization of the Riemann-Hurwitz formula [Has35]. We start by defining the ramification index.

Definition 2.31 (Ramification index)**.** Let $f\colon X \to Y$ be a regular branched cover. For a point $y \in Y$ with $f(y) = x$, the *ramification index* of f at y is a positive integer e_y such that there is an open neighborhood U of y so that x has only one preimage in U and for all other points $z \in f(U)$, $\#f^{-1}(z) = e_y$. In other words, the map from U to $f(U)$ is e_y to 1 except at y. At all but finitely many points of Y, we have that $e_y = 1$.

Let $\delta_P = e_P - 1$, and let $d = \deg(f\colon X \to Y)$. The following theorem is a generalization of the Riemann-Hurwitz Theorem, and it describes the relationship of the Euler characteristics of two surfaces when one is a regular branched cover of the other.

Theorem 2.32 (Riemann-Hurwitz-Hasse)**.** *Let $f\colon X \to Y$ be a regular branched cover, and let g and g' be the genus of X and Y, respectively. Then*

$$2g - 2 = d(2g' - 2) + \delta(f) \tag{2.6}$$

where $\delta(f) = \sum_{P \in X} \delta_P$.

Applying Theorem 2.32, we determine that we can lift the maps from Examples 2.24 and 2.26 to surfaces of genus $g = 3(d-1)$ and $g = 4(d-1)$, where $d \geq 2$.

Theorem 2.33. *For surfaces of genus $g = 3(d-1)$ and $g = 4(d-1)$, where $d \geq 2$, there exist pseudo-Anosov mapping classes which cannot be obtained from either the Thurston or the Penner constructions.*

Proof. Consider the map from Example 2.24. We notice that for a regular branched covering map of degree d, $f\colon S_{g,0} \to S_{0,8}$, that $e_p = d$. We additionally notice that there are only 8 points where the map is d to 1, namely the points which map to the 8 punctures of $S_{0,8}$. From Equation 2.6, we find

$$\begin{aligned} 2g - 1 &= d(2(0) - 2) + \sum_{p \in S_{g,0}} \delta_P \\ &= d(-2) + 8(d-1) \end{aligned} \tag{2.7}$$

which implies that $g = 3(d-1)$. Thus, using Birman-Hilden as above, we can lift the map from Example 2.24 to a map on surfaces of genus $g(d-1)$, where d is the degree of the regular branched cover.

We are able to apply the same argument for the map from Example 2.26, to find that this map lifts to surfaces of genus $4(d-1)$, where d is the degree of the regular branched covering map.

As these two maps are unique from the maps from the Penner and Thurtson constructions, the lifts of these maps to surfaces with genus will also be unique from the maps from the Penner and Thurston constructions. Indeed, as the unstable and stable foliations lift, we obtain the same stretch factor on the surfaces with genus. As the two algebraic properties we studied are dependant only on the value of the stretch factor, we see that the maps on surfaces with genus $g = 3(d-1)$ and $g = 4(d-1)$, where $d \geq 2$, will also have that the trace field is not totally real, and that there exist Galois conjugates of the stretch factor on the unit circle. □

3 STRONG CONTRACTIBILITY OF GEODESICS IN THE MAPPING CLASS GROUP

3.1 BACKGROUND AND MOTIVATION

Let S be a surface of finite type and let $\mathrm{PMap}(S)$ denote the (pure) mapping class group of S, that is, the group of orientation preserving self homeomorphisms of S fixing the punctures of S, up to isotopy. Recall that $\mathrm{PMap}(S)$ is a finitely generated group [Del96] and, after choosing a generating set, the word length turns $\mathrm{PMap}(S)$ into a metric space. A geodesic in the mapping class group is a globally distance minimizing curve.

The geometry of $\mathrm{PMap}(S)$ has been a subject of extensive study. Two important works in this area are written by Masur and Minsky. In these papers they study the geometry of the complex of curves, see Definition 1.11. In [MM99], Masur and Minsky were able to show that the complex of curves is an infinite diameter δ-hyperbolic space in all but a finite number of trivial cases. This theorem was partially motivated by the need to understand the extent in which the incomplete analogy between the geometry of the Teichmüller space and that of δ-hyperbolic spaces. In many cases this analogy holds, but Masur was able to show that the Teichmüller metric is unable to be curved in a negative sense.

Masur and Minsky's result that the complex of curves is δ-hyperbolic suggests that one should try to apply the techniques of δ-hyperbolic spaces to understand the complex of curves and its $\mathrm{Map}(S)$-action. However, there are barriers to using these techniques. In [MM00], Masur and Minsky developed tools to lift these barriers.

Links of vertices in the complex of curves are themselves complexes associated to subsurfaces. The geometry of each link is tied to the geometry of the complex of curves through a family of subsurface projection maps, which we can think of as being analogous to closest point projections to horoballs in a classic hyperbolic space. We now have that the complex of curves has a layered structure which contains hyperbolicity at each level. The main construction in Masur and Minsky's paper is a combinatorial device which ties each of these levels together, called a hierarchy of tight geodesics. The starting point of the construction of a hierarchy path is a geodesic in the curve graph of S. Hence, by construction, the shadow of a hierarchy path to the curve graph is nearly a geodesic.

It may seem intuitive that any geodesic in the mapping class group should also have this property, considering that similar statements have been shown to be true in other settings. For example, it is known that the shadow of a geodesic in Teichmüller space with respect to the Teichmüller metric is a re-parametrized quasi-geodesic in the curve graph [MM99]. The same is true for any geodesic in Teichmüller space with respect to the Thurston metric [LRT15], for any line of minima in Teichmüller space [CRS08], for a grafting ray [CDR11], or for the set of short curves in a hyperbolic 3-manifold homeomorphic to $S \times \mathbb{R}$ [Min01]. However, it is difficult to construct explicit examples of geodesics in $\mathrm{PMap}(S)$ and so far, all estimates for the word length of an element have been up to a multiplicative error.

In this Chapter, we argue that one should not expect geodesics in $\mathrm{PMap}(S)$ to be well-behaved in general. Changing the generating set changes the metric on $\mathrm{PMap}(S)$ significantly and a geodesic with respect to one generating set is only a quasi-geodesic with respect to another generating set. Since $\mathrm{PMap}(S)$ is not Gromov hyperbolic (it contains flats; to see this, consider the subgroup of $\mathrm{PMap}(S)$ generated by two disjoint Dehn twists), its quasi-geodesics are not well behaved in general. Similarly, one should not expect that the geodesics with respect to an arbitrary generating set to behave well either.

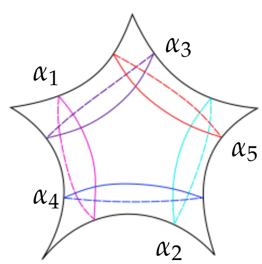


Figure 3.1: The curves $\alpha_1, \ldots, \alpha_5$ used to generate $\mathcal{S}_n$.

We make this explicit in the case where $S = S_{0,5}$ is the five-times punctured sphere. Consider the curves $\alpha_1, \ldots, \alpha_5$ depicted in Figure 3.1. Fix an integer $n \gg 1$ (to be determined in the proof of Theorem 3.3), and consider the following generating set for $\mathrm{PMap}(S)$

$$\mathcal{S}_n = \left\{ D_{\alpha_i}, s_{i,j} : i, j \in \mathbb{Z}_5, |i - j| = 1 \mod 5 \right\}$$

where $s_{i,j} = D^n_{\alpha_i} D^{-1}_{\alpha_j}$, and D_α is a Dehn twist around a curve α. Since we are considering the pure mapping class group, the set $\{D_{\alpha_i}\}_{i=1}^5$ already generates $\mathrm{PMap}(S)$. We denote the distance on $\mathrm{PMap}(S)$ induced by the generating set $\mathcal{S}_n$ by $d_{\mathcal{S}_n}$. By an $\mathcal{S}_n$-geodesic, we mean a geodesic with respect to this metric.

In the following theorem, we consider a geodesic in the mapping class group. We map this geodesic to the complex of curves using the shadow map. The image of the endpoints of the geodesic under the shadow map end up being very close together (the image of the geodesic is essentially backtracking), which is unable to be reparametrized to be a quasi-geodesic.

Theorem 3.1. *There is an $n \gg 1$ so that, for every $K, C > 0$, there exists an $\mathcal{S}_n$-geodesic*

$$\mathcal{G} \colon [0, m] \to \mathrm{Map}(S)$$

so that the shadow of $\mathcal{G}$ to the curve graph $\mathcal{C}(S)$ is not a re-parametrized (K, C)-quasi-geodesic.

Even though the mapping class group is not Gromov hyperbolic, it does have hyperbolic directions. There are different ways to make this precise. For example, Behrstock proved [Beh06] that in the direction of every pseudo-Anosov, the divergence function in $\mathrm{Map}(S)$ is super-linear. Another way to make this notion precise is to examine whether geodesics in $\mathrm{Map}(S)$ have the *contracting property*.

This notion is defined analogously with Gromov hyperbolic spaces where, for every geodesic $\mathcal{G}$ and any ball disjoint from $\mathcal{G}$, the closest point projection of the ball to $\mathcal{G}$ has a uniformly bounded diameter. However, often it is useful to work with different projection maps. We call a map

$$\mathrm{Proj} \colon X \to \mathcal{G}$$

from a metric space X to any subset $\mathcal{G} \subset X$ a *(d_1, d_2)-projection map*, $d_1, d_2 > 0$, if for every $x \in X$ and $g \in \mathcal{G}$, we have

$$d_X\big(\mathrm{Proj}(x), g\big) \le d_1 \cdot d_X(x, g) + d_2.$$

This is a weak notion of projection since Proj is not even assumed to be coarsely Lipschitz. By the triangle inequality, the closest point projection is always a $(2, 0)$-projection.

Definition 3.2. A subset $\mathcal{G}$ of a metric space X is said to have the *contracting property* if there is a constant $\rho < 1$, a constant $B > 0$ and a projection map

$\text{Proj}\colon X \to \mathcal{G}$ such that, for any ball $B(x, R)$ of radius R disjoint from $\mathcal{G}$, the projection of a ball of radius ρR, $B(x, \rho R)$, has a diameter at most B,

$$\text{diam}_{\mathcal{S}_n}\left(\text{Proj}(B(x, \rho R))\right) \leq B.$$

We say $\mathcal{G}$ has the *strong contracting property* if ρ can be taken to be 1.

In other words, a subset $\mathcal{G}$ of a metric space X is said to have the strong contracting property if for any ball disjoint from $\mathcal{G}$, the projection of this ball to $\mathcal{G}$ has diameter at most B.

The axis of a pseudo-Anosov element has the contracting property in many settings. This has been shown to be true in the setting of Teichmüller space by Minsky [Min96], in the setting of the pants complex by Brock, Masur, and Minsky [BMM09] and in the setting of the mapping class group by Duchin and Rafi [DR09].

In work by Arzhantseva, Cashen and Tao, they asked if the axis of a pseudo-Anosov element in the mapping class group has the strong contracting property and showed that a positive answer would imply that the mapping class group is growth tight [ACT15]. Additionally, using the work of Yang [Yan17], one can show that if one pseudo-Anosov element has a strongly contracting axis with respect to some generating set, then a generic element in mapping class group has a strongly contracting axis with respect to this generating set. Similar arguments would also show that the mapping class group with respect to this generating set has purely exponential growth.

However, using our specific generating set, we show that this does not always hold:

Theorem 3.3. *For every $d_1, d_2 > 0$, there exists an $n \gg 1$, a pseudo-Anosov map ϕ, a constant $c_n > 0$, a sequence of elements $w_k \in \text{Map}(S)$ and a sequence of radii $r_k > 0$ where $r_k \to \infty$ as $k \to \infty$ such that the following holds. Let $\mathfrak{a}_\phi$ be a quasi-axis for ϕ in $\text{Map}(S)$ and let $\text{Proj}_{\mathfrak{a}_\phi}\colon \text{Map}(S) \to \mathfrak{a}_\phi$ be any (d_1, d_2)-projection map. Then the ball of radius r_k centered at w_k, $B(w_k, r_k)$, is disjoint from $\mathfrak{a}_\phi$ and*

$$\text{diam}_{\mathcal{S}_n}\left(\text{Proj}_{\mathfrak{a}_\phi}\left(B(w_k, r_k)\right)\right) \geq c_n \log(r_k).$$

We remark that, since $\mathfrak{a}_\phi$ has the contracting property [DR09], the diameter of the projection can grow at most logarithmically with respect to the radius r_k (see Corollary 3.19), hence the lower-bound achieved by the above theorem is sharp.

Outline of proof

To find an exact value for the word-length of an element $f \in \mathrm{Map}(S)$, we construct a homomorphism

$$h\colon\ \mathrm{Map}(S) \to \mathbb{Z},$$

where a large value for $h(f)$ guarantees a large value for the word length of f. At times, this lower bound is realized and an explicit geodesic in $\mathrm{Map}(S)$ is constructed (see Section 3.2). The pseudo-Anosov element ϕ is defined as

$$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}.$$

In Section 3.3 we find an explicit invariant train-track for ϕ to show that ϕ is a pseudo-Anosov. In Section 3.4, we use the geodesics constructed in Section 3.2 to show that the shadows of geodesics in $\mathrm{Map}(S)$ are not necessarily quasi-geodesics in the curve complex. In Section 3.5, we begin by showing that ϕ acts loxodromically on $\mathrm{Map}(S)$, that is, it has a quasi-axis $\mathfrak{a}_\phi$ which fellow travels the path $\{\phi^i\}$. We finish Section 3.5 by showing that the bound in our main theorem is sharp. In Section 3.6, we set up and complete the proof of 3.3.

3.2 FINDING EXPLICIT GEODESICS

In this section, we develop the tools needed to show that certain paths in $\mathrm{PMap}(S)$ are geodesics. We emphasize again that, in our paper, S is the five-times punctured sphere and $\mathrm{PMap}(S)$ is the pure mapping class group. That is, all homeomorphisms are required to fix the punctures point-wise.

By a *curve* on S we mean a free homotopy class of a non-trivial, non-peripheral simple closed curve. Fix a labelling of the 5 punctures of S with elements of $\mathbb{Z}_5$, the cyclic group of order 5. Any curve γ on S cuts the surface into two surfaces; one copy of $S_{0,3}$ containing two of the punctures from S and one copy of $S_{0,4}$ which contains three of the punctures from S.

Definition 3.4 ((i,j)-curve)**.** We say that a curve γ on S is an (i,j)-*curve*, $i,j \in \mathbb{Z}_5$, if the component of $(S \backslash \gamma)$ that is a three-times punctured sphere contains the punctures labeled i and j. Furthermore, if $|i-j| \equiv 1 \mod 5$ we say that γ *separates two consecutive punctures*, and if $|i-j| \equiv 2 \mod 5$ we say that γ *separates two non-consecutive punctures.*

In [Luo97], Luo gave a simple presentation of the mapping class group where the generators are the set of all Dehn twists

$$\mathcal{S} = \{D_\gamma : \gamma \text{ is a curve}\}$$

and the relations are of a few simple types. In our setting, we only have the following two relations:

- (Conjugating relation) For any two curves β and γ,

$$D_{D_\gamma(\beta)} = D_\gamma D_\beta D_\gamma^{-1}.$$

Proof. Notice that D_γ^{-1} takes a regular neighborhood of $D_\gamma(\beta)$ to β, D_β twists the neighborhood of β, and finally D_γ sends β to $D_\gamma(\beta)$. The result is a Dehn twist about $D_\gamma(\beta)$. □

- (The lantern relation) Let i, j, k, l, m be distinct elements in $\mathbb{Z}_5$ and $\gamma_{i,j}$, $\gamma_{j,k}$ $\gamma_{k,i}$ and $\gamma_{l,m}$ be curves of the type indicated by the indices. Further assume that each pair of curves among $\gamma_{i,j}$, $\gamma_{j,k}$ and $\gamma_{k,i}$ intersect twice and that they are all disjoint from $\gamma_{l,m}$. Then

$$D_{\gamma_{i,j}} D_{\gamma_{j,k}} D_{\gamma_{k,i}} = D_{\gamma_{l,m}}.$$

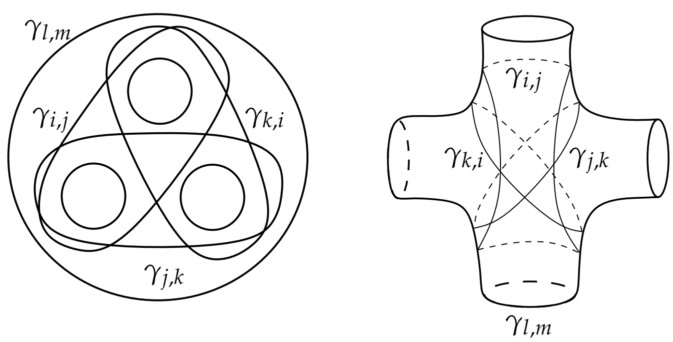


Figure 3.2: Two views of the lantern relation.

Proof. Any embedding of a compact surface S' into a surface S induces a homeomorphism $\text{Map}(S') \longrightarrow \text{Map}(S)$ by Theorem 1.9. Since relations are preserved by homomorphisms, it suffices to show that the lantern relation holds in $\text{Map}(S_{0,4})$.

To check that the lantern relation holds in $\text{Map}(S_{0,4})$, we cut $S_{0,4}$ into a disk using three arcs and apply the Alexander Method, Theorem 1.5. We show the computation in Figure 3.3, which completes the proof. Recall that positive Dehn twists are twists to the right, and we apply the elements in the relation from right to left.

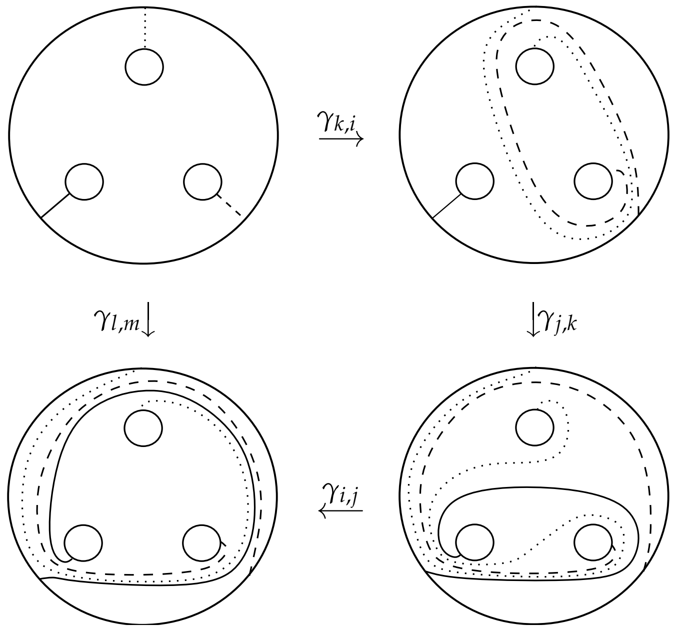


Figure 3.3: Computation of the lantern relation.

□

Using this presentation, we construct a homomorphism from $\mathrm{Map}(S)$ into $\mathbb{Z}$.

Theorem 3.5. *There exists a homomorphism* $h\colon \mathrm{PMap}(S) \to \mathbb{Z}$ *whose restriction to the generating set* $\mathcal{S}$ *is as follows:*

$$
\begin{aligned}
D_\gamma &\longmapsto 1 && \textit{if } \gamma \textit{ separates two consecutive punctures}\\
D_\gamma &\longmapsto -1 && \textit{if } \gamma \textit{ separates two non-consecutive punctures}
\end{aligned}
$$

Proof. To show that h extends to a homomorphism, it suffices to show that h preserves the relations stated above.

First, we check the conjugating relation. Let β and γ be a pair of curves. Since, D_γ is a homeomorphism fixing the punctures, if β is an (i,j)–curve, so is $D_\gamma(\beta)$. In particular, $h(D_{D_\gamma(\beta)}) = h(\beta)$. Hence,

$$
\begin{aligned}
h\big(D_{D_\gamma(\beta)}\big) = h(D_\beta) &= h(D_\gamma) + h(D_\beta) - h(D_\gamma)\\
&= h(D_\gamma) + h(D_\beta) + h\big(D_\gamma^{-1}\big) = h\big(D_\gamma D_\beta D_\gamma^{-1}\big).
\end{aligned}
$$

We now show that h preserves the lantern relation. For any three punctures of S labeled $i,j,k \in \mathbb{Z}_5$, two of these punctures are consecutive. Without loss of generality, suppose $|i-j| = 1 \mod 5$. There are two cases:

- Assume k is consecutive to one of i or j. That is, without loss of generality, suppose $|j-k| = 1 \mod 5$. Then $|i-k| = 2 \mod 5$ and the remaining two punctures, l and m, are consecutive: $|l-m| = 1 \mod 5$. Thus

$$\begin{aligned} h(D_{\gamma_{i,j}} D_{\gamma_{j,k}} D_{\gamma_{k,i}}) &= h(D_{\gamma_{i,j}}) + h(D_{\gamma_{j,k}}) + h(D_{\gamma_{k,i}}) \\ &= 1 + 1 + (-1) \\ &= 1 = h(D_{\gamma_{l,m}}). \end{aligned}$$

- Otherwise, $|j-k| = 2 \mod 5$ and $|i-k| = 2 \mod 5$, so that the remaining two punctures, l and m, are nonconsecutive: $|l-m| = 2 \mod 5$. Thus

$$\begin{aligned} h(D_{\gamma_{i,j}} D_{\gamma_{j,k}} D_{\gamma_{k,i}}) &= h(D_{\gamma_{i,j}}) + h(D_{\gamma_{j,k}}) + h(D_{\gamma_{k,i}}) \\ &= 1 + (-1) + (-1) \\ &= (-1) = h(D_{\gamma_{l,m}}). \end{aligned}$$

Thus, h preserves the lantern relation. □

Now, we switch back to the generating set $\mathcal{S}_n$ given in Section 3.1. The homomorphism of Theorem 3.5 gives a lower bound on the word length of elements in $\mathrm{PMap}(S)$. Note that

$$h(s_{i,j}) = (n-1) \qquad \text{and} \qquad h(D_{\alpha_i}) = 1.$$

Lemma 3.6. *For any $f \in \mathrm{PMap}(S)$, let*

$$h(f) = q(n-1) + r$$

for integer numbers q and r where $0 \leq |r| < \frac{n-1}{2}$. Then $\|f\|_{\mathcal{S}_n} \geq |q| + |r|$.

Proof. First we show that, if $h(f) = a(n-1) + b$ for integers a and b, then $|a| + |b| \geq |q| + |r|$. To see this, consider such a pair a and b where $|a| + |b|$ is minimized. If $a < q$, then $|b| \geq |(n-1) + r| > \frac{n-1}{2}$. Therefore, we can increase a by 1 and decrease b by $n-1$ to decrease the quantity $|a| + |b|$, which is a contradiction. Similarly, if $a > q$, then $|b| > (n-1)/2$ and we can decrease a by 1 and increase b by $n-1$ to decrease the quantity $|a| + |b|$, which again is a contradiction. Hence, $a = q$ and subsequently $b = r$.

Now, write $f = g_1 g_2 \dots g_k$, where $g_i \in \mathcal{S}_n$ or $g_i^{-1} \in \mathcal{S}_n$ and $k = \|f\|_{\mathcal{S}_n}$. For each g_i, $h(g_i)$ takes either value 1, (-1), $(n-1)$ or $(1-n)$. Hence, there are integers a' and b' so that

$$h(f) = h(g_1) + h(g_2) + \ldots + h(g_k) = a'(n-1) + b',$$

where $k \geq |a'| + |b'|$. But, as we saw before, we also have $|a'| + |b'| \geq |q| + |r|$. Hence $k \geq |q| + |r|$. □

This lemma allows us to find explicit geodesics in $\mathrm{PMap}(S)$. We demonstrate this with an example.

Example 3.7. Let $f = D_{\alpha_1}^{n^k - 1} \in \mathrm{Map}(S)$. We have,

$$h(f) = n^k - 1 = (n-1)(n^{k-1} + n^{k-2} + \cdots + n^2 + n + 1).$$

Therefore, by Lemma 3.6 $\|f\|_{\mathcal{S}_n} \geq n^{k-1} + n^{k-2} + \cdots + n^2 + n + 1$. On the other hand, (assuming k is even to simplify notation), we have

$$\begin{aligned} D_{\alpha_1}^{n^k-1} &= \left(D_{\alpha_1}^{n^k} D_{\alpha_2}^{-n^{(k-1)}}\right)\left(D_{\alpha_2}^{n^{(k-1)}} D_{\alpha_1}^{-n^{(k-2)}}\right)\ldots\left(D_{\alpha_1}^{n^2} D_{\alpha_2}^{-n}\right)\left(D_{\alpha_2}^{n} D_{\alpha_1}^{-1}\right) \\ &= s_{1,2}^{n^{(k-1)}}\, s_{2,1}^{n^{(k-2)}} \ldots s_{1,2}^{n}\, s_{2,1}. \end{aligned}$$

Since we used exactly $(n^{k-1} + n^{k-2} + \ldots + n + 1)$ elements in $\mathcal{S}_n$, we have found a geodesic path. However, notice there is a second geodesic path from the identity to f (which works for every k), namely:

$$\begin{aligned} D_{\alpha_1}^{n^k-1} &= \left(D_{\alpha_1}^{n^k} D_{\alpha_2}^{-n^{(k-1)}}\right)\left(D_{\alpha_2}^{n^{(k-1)}} D_{\alpha_3}^{-n^{(k-2)}}\right)\ldots\left(D_{\alpha_{(k-1)}}^{n^2} D_{\alpha_k}^{-n}\right)\left(D_{\alpha_k}^{n} D_{\alpha_{(k+1)}}^{-1}\right) \\ &= s_{1,2}^{n^{(k-1)}}\, s_{2,3}^{n^{(k-2)}} \ldots s_{(k-1),k}^{n}\, s_{k,k+1}. \end{aligned}$$

This shows that geodesics are not unique in $\mathrm{PMap}(S)$. Either way, we have established that

$$\|D_{\alpha_1}^{n^k-1}\|_{\mathcal{S}_n} = n^{(k-1)} + n^{(k-2)} + \cdots + n + 1. \tag{3.1}$$

We now use a similar method to compute certain word lengths that will be useful later in this chapter. Define

$$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}.$$

We will show in the next subsection that ϕ is a pseudo-Anosov element of $\mathrm{PMap}(S)$. We also use the notation

$$\phi^{k/5} = D_{\alpha_k} D_{\alpha_{k-1}} \ldots D_{\alpha_1}$$

where indices are considered to be in $\mathbb{Z}_5$. This is accurate when k is divisible by 5 but we use it for any integer k. For a positive integer k, define

$$m_k = n^k + n^{k-1} + \ldots + n + 1$$

and

$$\ell_k = n^k - n^{k-1} - n^{k-2} - \ldots - n - 1,$$

and let $w_k = D_{\alpha_1}^{m_k}$ and $u_k = D_{\alpha_1}^{\ell_k}$. Additionally, we will define

$$v_k = D_{\alpha_1}^{-\frac{k+1}{2}} D_{\alpha_2}^{-\frac{k+1}{2}}.$$

We will show that u_k and w_k are closer to a large power of ϕ than the identity even though they are both just a power of a Dehn twist.

Proposition 3.8. For u_k and w_k as above, we have

$$\left\| w_k\, \phi^{-(k+1)/5} \right\|_{\mathcal{S}_n} = \| w_k\, v_k \|_{\mathcal{S}_n} = n^{k-1} + 2n^{k-2} + \ldots + (k-1)n + k,$$

and

$$\left\| \phi^{k/5}\, u_k \right\|_{\mathcal{S}_n} = n^{k-1} - n^{k-3} - 2n^{k-4} - \ldots - (k-3)n - (k-2) + 1.$$

Proof. Note that

$$\begin{aligned} h\Big(w_k \phi^{-(k+1)/5}\Big) &= (n^k + n^{k-1} + \ldots + n + 1) - (k+1) \\ &= (n-1)(n^{k-1} + 2n^{k-2} + \ldots + (k-1)n + k). \end{aligned}$$

Lemma 3.6 implies that

$$\left\| w_k \phi^{-(k+1)/5} \right\|_{\mathcal{S}_n} \geq n^{k-1} + 2n^{k-2} + \ldots + (k-1)n + k.$$

On the other hand, since $m_k - 1 = n\, m_{k-1}$, we have

$$\begin{aligned} w_k\, \phi^{-(k+1)/5} &= D_{\alpha_1}^{m_k} \big(D_{\alpha_1}^{-1} D_{\alpha_2}^{-1} \ldots D_{\alpha_{k+1}}^{-1}\big) \\ &= D_{\alpha_1}^{(m_k-1)} \big(D_{\alpha_2}^{-1} D_{\alpha_3}^{-1} \ldots D_{\alpha_{k+1}}^{-1}\big) \\ &= s_{1,2}^{m_{k-1}} D_{\alpha_2}^{m_{k-1}} \big(D_{\alpha_2}^{-1} D_{\alpha_2}^{-1} \ldots D_{\alpha_{k+1}}^{-1}\big) \\ &= s_{1,2}^{m_{k-1}} D_{\alpha_1}^{(m_{k-1}-1)} \big(D_{\alpha_3}^{-1} D_{\alpha_4}^{-1} \ldots D_{\alpha_{k+1}}^{-1}\big) \\ &= s_{1,2}^{m_{k-1}}\, s_{2,3}^{m_{k-2}}\, D_{\alpha_3}^{m_{k-2}} \big(D_{\alpha_3}^{-1} D_{\alpha_4}^{-1} \ldots D_{\alpha_{k+1}}^{-1}\big) \\ &\quad \ldots \\ &= s_{1,2}^{m_{k-1}}\, s_{2,3}^{m_{k-2}} \ldots s_{k-1,k}^{m_1}\, s_{k,k+1}. \end{aligned}$$

Therefore,

$$\|w_k\,\phi^{-(k+1)/5}\|_{\mathcal{S}_n} = m_{k-1} + \cdots + m_1 + 1 = n^{k-1} + 2n^{k-2} + \ldots (k-1)n + k.$$

To show that

$$\|w_k\,v_k\|_{\mathcal{S}_n} = n^{k-1} + 2n^{k-2} + \ldots (k-1)n + k$$

is as above, but in place of applying $s_{i,i+1}$ for $1 \le i \le k$, we alternate between applying $s_{1,2}$ and $s_{2,1}$ to find

$$w_k\,v_k = s_{1,2}^{m_{k-1}}\,s_{2,1}^{m_{k-2}} \ldots s_{1,2}^{m_1}\,s_{2,1},$$

which proves our claim. Similarly, we have

$$\begin{aligned} h\Big(\phi^{k/5}u_k\Big) &= k + \big(n^k - n^{k-1} + \ldots - n - 1\big) \\ &= (n-1)\big(n^{k-1} - n^{k-3} - 2n^{k-4} - \ldots - (k-3)n - (k-2)\big) + 1, \end{aligned}$$

and Lemma 3.6 implies

$$\|\phi^{k/5}u_k\|_{\mathcal{S}_n} \ge n^{k-1} - n^{k-3} - 2n^{k-4} - \ldots - (k-3)n - (k-2) + 1.$$

On the other hand, since $\ell_k + 1 = n\ell_{k-1}$, we have

$$\begin{aligned} \phi^{k/5}u_k &= \big(D_{\alpha_k} \ldots D_{\alpha_2} D_{\alpha_1}\big) D_{\alpha_1}^{\ell_k} \\ &= \big(D_{\alpha_k} \ldots D_{\alpha_3} D_{\alpha_2}\big) D_{\alpha_1}^{(\ell_k+1)} \\ &= \big(D_{\alpha_k} \ldots D_{\alpha_3} D_{\alpha_2}\big) D_{\alpha_2}^{\ell_{k-1}}\,s_{1,2}^{\ell_{k-1}} \\ &= \big(D_{\alpha_k} \ldots D_{\alpha_4} D_{\alpha_3}\big) D_{\alpha_2}^{(\ell_{k-1}+1)}\,s_{1,2}^{\ell_{k-1}} \\ &= \big(D_{\alpha_k} \ldots D_{\alpha_4} D_{\alpha_3}\big) D_{\alpha_3}^{\ell_{k-2}}\,s_{2,3}^{\ell_{k-2}}\,s_{1,2}^{\ell_{k-1}} \\ &\ldots \\ &= D_{\alpha_k} D_{\alpha_k}^{\ell_1}\,s_{k-1,k}^{\ell_1} \ldots s_{2,3}^{\ell_{k-2}}\,s_{1,2}^{\ell_{k-1}} \\ &= t_{k+1}\,s_{k,k+1}\,s_{k-1,k}^{\ell_1} \ldots s_{2,3}^{\ell_{k-2}}\,s_{1,2}^{\ell_{k-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u_k\phi^{k/5}\|_{\mathcal{S}_n} &= \ell_{k-1} + \ldots \ell_1 + 2 \\ &= n^{k-1} - n^{k-3} - 2n^{k-4} - \ldots - (k-3)n - (k-2) + 1. \end{aligned}$$

This is because the coefficient of n^i is 1 is ℓ_i and is (-1) in $\ell_k, \ldots, \ell_{i+1}$. Summing up, we get $-(k-i-1)$ as the coefficient of n^i. □

3.3 THE PSEUDO-ANOSOV MAP ϕ

In this section, we introduce the pseudo-Anosov map ϕ which will be used in the proof of Theorem 3.3. Define

$$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}.$$

We check that ϕ is, in fact, a pseudo-Anosov.

Theorem 3.9. *The map ϕ is pseudo-Anosov.*

Proof. To prove that ϕ is a pseudo-Anosov map, we begin by finding a train-track τ on S so that $\phi(\tau)$ is carried by τ and show that the matrix representation of ϕ in the coordinates given by τ is a Perron-Frobenius matrix. We will then apply Lemma 2.12 from Chapter 2 to complete the proof.

The series of images in Fig. 3.4 depict the train-track τ and its images under successive applications of Dehn twists associated to ϕ. We see that $\phi(\tau)$ is indeed carried by τ and, keeping track of weights on τ, we calculate that the induced action on the space of weights on τ is given by the following matrix.

$$A = \begin{pmatrix} 3 & 2 & 0 & 0 & 2 \\ 6 & 3 & 6 & 4 & 0 \\ 4 & 2 & 3 & 2 & 0 \\ 12 & 8 & 6 & 3 & 6 \\ 6 & 4 & 4 & 2 & 3 \end{pmatrix}$$

Note that the space of admissible weights on τ is the subset of $\mathbb{R}^5$ given by positive real numbers a, b, c, d and e such that $a + b + e = c + d$. The linear map described above preserves this subset. The square of the matrix A is strictly positive, which implies that the matrix is a Perron-Frobenius matrix. We notice that the train-track τ is large, generic and birecurrent, so by Lemma 2.12 we have shown that ϕ is pseudo-Anosov. □

The work in this chapter, Chapter 3, is work which was done prior to the work found in Chapter 2. The pseudo-Anosov map ϕ from Theorem 3.9 was the inspiration for the construction of pseudo-Anosov mapping classes found in Chapter 2. As such, there is an alternate proof for Theorem 3.9 which does not use Lemma 2.12 and we include this alternate proof here.

Alternate proof of Theorem 3.9. In order to prove that ϕ is a pseudo-Anosov map, we find a train-track τ on S so that $\phi(\tau)$ is carried by τ and show that the matrix representation of ϕ in the coordinates given by τ is a Perron-Frobenius matrix.

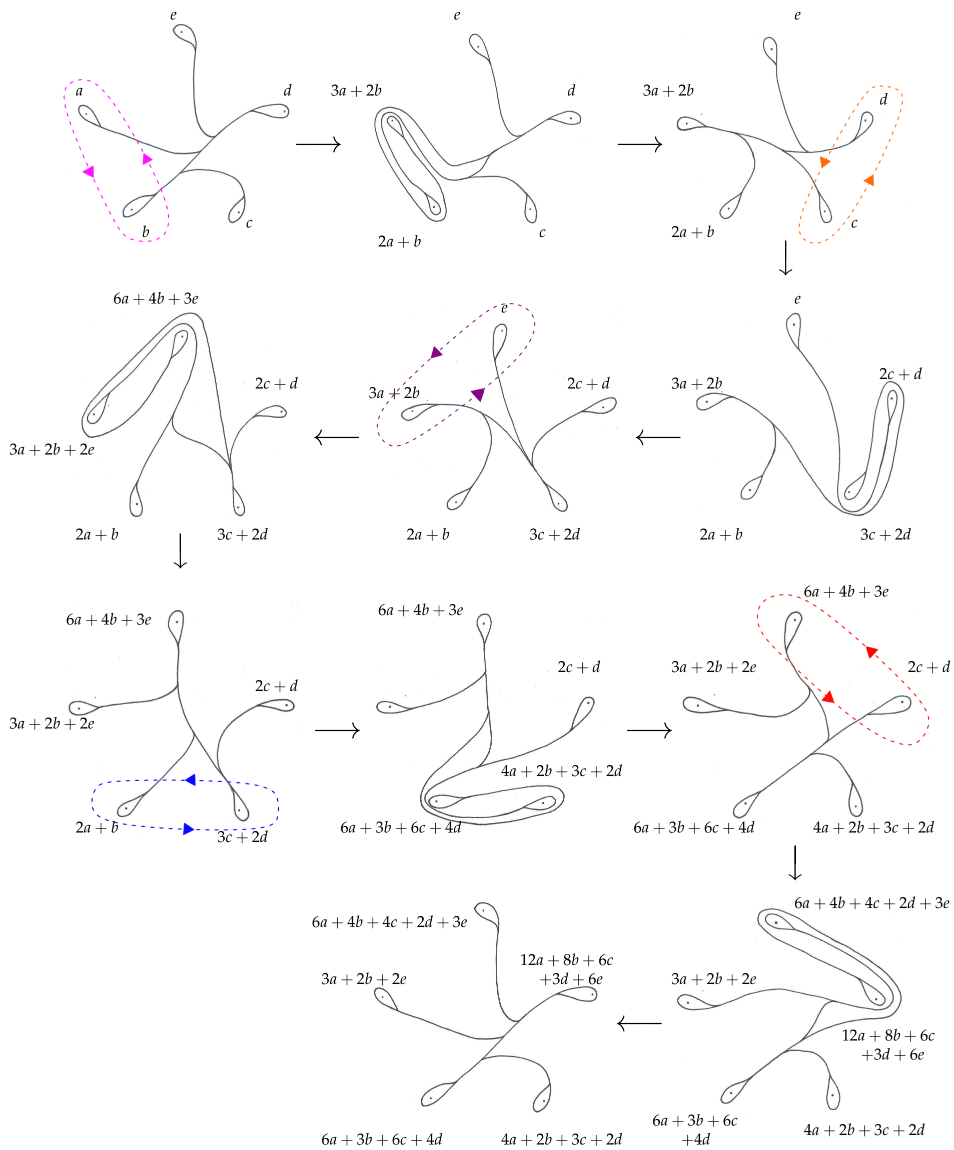


Figure 3.4: The train-track $\phi(\tau)$ is carried by τ.

□

The series of images in Figure 3.4 depict the train-track τ and its images under successive applications of Dehn twists associated to ϕ. We see that $\phi(\tau)$ is indeed carried by τ and, keeping track of weights on τ, we calculate that the induced action on the space of weights on τ is given by the following matrix.

$$A = \begin{pmatrix} 3 & 2 & 0 & 0 & 2 \\ 6 & 3 & 6 & 4 & 0 \\ 4 & 2 & 3 & 2 & 0 \\ 12 & 8 & 6 & 3 & 6 \\ 6 & 4 & 4 & 2 & 3 \end{pmatrix}$$

Note that the space of admissible weights on τ is the subset of $\mathbb{R}^5$ given by positive real numbers a, b, c, d and e such that $a + b + e = c + d$. The linear map described above preserves this subset. The square of the matrix A is strictly positive, which implies that the matrix is a Perron-Frobenius matrix. In fact, the top eigenvalue is

$$\lambda = \sqrt{13} + 2\sqrt{2\sqrt{13} + 7} + 4$$

that is associated to a unique irrational measured lamination F carried by τ that is fixed by ϕ. We now argue that F is filling. Note that, curves on S are in one-to-one association with simple arcs connecting one puncture to another. We say an arc is carried by τ if the associated curve is carried by τ. If F is not filling, it is disjoint from some arc ω connecting two of the punctures. Modifying ω outside of a small neighborhood of τ, we can produce an arc that is carried by τ. In fact, for any two cusps of the train-track τ, either an arc going clock-wise or counter-clockwise connecting these two cusps can be pushed into τ. Hence, we can replace the portion of ω that is outside of a small neighborhood of τ with such an arc to obtain an arc ω' that is still disjoint from F but is also carried by τ. Hence, if F is not filling, it is disjoint from some arc (and thus some curve) carried by τ. But F is the unique lamination carried by τ that is fixed under ϕ which is a contradiction. This implies that ϕ is pseudo-Anosov.

3.4 SHADOW TO CAYLEY GRAPH

We recall that the curve graph $\mathcal{C}(S)$ is a graph whose vertices are curves on S and whose edges are pairs of disjoint curves. We assume every edge has length one turning $\mathcal{C}(S)$ into a metric space. This means that, for a pair of curves α and β, $d_{\mathcal{C}(S)}(\alpha, \beta) = n$ if

$$\alpha = \gamma_0, \ldots, \gamma_n = \beta$$

is the shortest sequence of curves on S such that the successive γ_i are disjoint. Masur-Minsky showed that $\mathcal{C}(S)$ is an infinite diameter Gromov hyperbolic space [MM99]. In particular, Masur and Minsky showed the following.

Theorem 3.10 (Masur-Minsky)**.** *Let S be an oriented surface of finite type. The curve graph, $\mathcal{C}(S)$ is a δ-hyperbolic metric space, where δ depends on S. Except when S is a sphere with 3 or fewer punctures, $\mathcal{C}(S)$ has infinite diameter.*

We also talk about the distance between subsets of $\mathcal{C}(S)$ using the same notation. That is, for two sets of curves $\mu_0, \mu_1 \subset \mathcal{C}(S)$ we define

$$d_{\mathcal{C}(S)}(\mu_0, \mu_1) = \max_{\gamma_0 \in \mu_0, \gamma_1 \in \mu_1} d_{\mathcal{C}(S)}(\gamma_0, \gamma_1).$$

Definition 3.11 (Shadow map)**.** The *shadow map* from the mapping class group to the curve complex is the map defined as:

$$\begin{aligned} \Upsilon\colon\ \mathrm{Map}(S) &\to \mathcal{C}(S) \\ f &\to f(\alpha_1). \end{aligned}$$

The shadow map from $\mathrm{Map}(S)$ equipped with $d_{\mathcal{S}_n}$ to the curve complex is 4-Lipschitz:

Lemma 3.12. *For any $f \in \mathrm{Map}(S)$, we have*

$$d_{\mathcal{C}(S)}(\alpha_1, f\alpha_1) \leq 4\|f\|_{\mathcal{S}_n}. \tag{3.2}$$

In particular, the Lipschitz constant of the shadow map is independent of n.

Proof. It is sufficient to prove the lemma for elements of $\mathcal{S}_n$. Consider $D_{\alpha_i} \in \mathcal{S}_n$. If $i(\alpha_i, \alpha_1) = 0$ then

$$d_{\mathcal{C}(S)}(\alpha_1, D_{\alpha_i}(\alpha_1)) = d_{\mathcal{C}(S)}(\alpha_1, \alpha_1) = 0.$$

If $i(\alpha_i, \alpha_1) = 2$, then there is a curve α_j that disjoint from both α_1 and α_i and hence α_j is also disjoint $D_{\alpha_i}(\alpha_1)$. Therefore, $d_{\mathcal{C}(S)}(\alpha_1, D_{\alpha_i}(\alpha_1)) = 2$.

Now consider the element $s_{i,i+1} \in \mathcal{S}_n$. Note that $s_{i,i+1}^{-1}\alpha_i = \alpha_i$. Hence,

$$\begin{aligned} d_{\mathcal{C}(S)}(\alpha_1, s_{i,i+1}\alpha_1) &\leq d_{\mathcal{C}(S)}(\alpha_1, \alpha_i) + d_{\mathcal{C}(S)}(\alpha_i, s_{i,i+1}\alpha_1) \\ &\leq 2 + d_{\mathcal{C}(S)}(s_{i,i+1}^{-1}\alpha_i, \alpha_1) \\ &\leq 2 + d_{\mathcal{C}(S)}(\alpha_i, \alpha_1) \leq 2 + 2 = 4 \end{aligned}$$

Thus, we have proven our claim. □

We will also need to make use of the following theorem from Masur and Minsky [MM99]:

Theorem 3.13 (Masur-Minsky)**.** *For a non-sporadic surface S (ie. $2g+b+n \geq 5$) there exists a $c>0$ such that, for any pseudo-Anosov $f \in \mathrm{Map}(S)$, any $\gamma \in \mathcal{C}(S)$, and any $n \in \mathbb{Z}$,*

$$d_{\mathcal{C}(S)}(f^n(\gamma), \gamma) \geq c|n|.$$

Using this Lemma 3.12, Theorem 3.13, and the previous theorems in this chapter, we show that the shadow of geodesics from the mapping class group to the curve complex are not always quasi-geodesics.

Theorem 3.14. *For all $K \geq 1$, $C \geq 0$, there exists a geodesic in the mapping class group, whose shadow to the curve complex is not a (K,C)–quasi-geodesic.*

Proof. Recall that, for a positive integer k, we have

$$m_k = n^{k-1} + n^{k-2} + \ldots + n + 1, \qquad \ell_k = n^k - n^{k-1} - n^{k-2} - \ldots - n - 1,$$

$w_k = D_{\alpha_1}^{m_k}$ and $u_k = D_{\alpha_1}^{\ell_k}$. Note that $m_{k-1} + \ell_k = n^k$. Hence, we can write

$$D_{\alpha_1}^{n^k} = \left(w_{k-1}\phi^{-k/5}\right)\left(\phi^{k/5}u_k\right).$$

Also,

$$h\left(D_{\alpha_1}^{n^k}\right) = n^k = (n-1)\left(n^{k-1} + n^{k-2} + \ldots + n + 1\right) + 1.$$

Therefore by Lemma 3.6

$$\|D_{\alpha_1}^{n^k}\|_{\mathcal{S}_n} \geq n^{k-1} + n^{k-2} + \ldots + n + 2. \tag{3.3}$$

But, from Theorem 3.8 we have

$$\|w_{k-1}\phi^{-k/5}\|_{\mathcal{S}_n} = n^{k-2} + 2n^{k-3} + \ldots + (k-2)n + (k-1).$$

and

$$\|u_k\phi^{k/5}\|_{\mathcal{S}_n} = n^{k-1} - n^{k-3} - 2n^{k-4} - \ldots - (k-3)n - (k-2) + 1.$$

The sum of the word lengths of the two elements is

$$n^{k-1} + n^{k-2} + \ldots + n + 2$$

which is equal to the lower bound found in Equation 3.3. Thus

$$\|D_{\alpha_1}^{n^k}\|_{\mathcal{S}_n} = \|w_{k-1}\phi^{-k/5}\|_{\mathcal{S}_n} + \|\phi^{k/5}u_k\|_{\mathcal{S}_n}$$

which means there is a geodesic connecting $D_{\alpha_1}^{n^k}$ to the identity that passes through $\phi^{k/5}u_k$.

Since ϕ is a pseudo-Anosov map, there is a lower-bound on its translation distance along the curve graph by Theorem 3.13. Namely, there is a constant $\sigma > 0$ so that, for every m,

$$d_{\mathcal{C}(S)}\big(\alpha_1, \phi^m \alpha_1\big) \geq \sigma\, m. \tag{3.4}$$

Also, $u_k \alpha_1 = \alpha_1$ which implies

$$d_{\mathcal{C}(S)}\big(\alpha_1, \phi^{k/5} u_k \alpha_1\big) = d_{\mathcal{C}(S)}\big(\alpha_1, \phi^{k/5}\alpha_1\big) \geq \sigma\, \frac{k}{5}.$$

That is,

$$\Upsilon(\mathrm{id}) = \Upsilon(D_{\alpha_1}^{n^k}) = \alpha_1.$$

However, $\Upsilon\big(\phi^{k/5}u_k\big)$ is at least distance $\frac{\sigma k}{5}$ away from α_1. Therefore, choosing k large compared with σ, K and C, we see that the shadow of this geodesic (the one connecting id to $D_{\alpha_1}^{n^k}$ which passes through $\phi^{k/5}u_k$) to $\mathcal{C}(S)$ is not a (K, C)–quasi-geodesic. □

3.5 axis of a pseudo-anosov in the mapping class group

Consider the path

$$\mathcal{A}_\phi \colon \mathbb{Z} \to \mathrm{Map}(S), \qquad i \mapsto \phi^i.$$

Since $\|\phi\|_{\mathcal{S}_n} \leq 5$, then $\|\phi^i\|_{\mathcal{S}_n} \leq 5i$. Also, using Lemma 3.12 and Equation (3.4) we get

$$\|\phi^i\|_{\mathcal{S}_n} \geq \frac{1}{4} d_{C(S)}\big(\alpha_1, \phi^i \alpha_1\big) \geq \frac{i\,\sigma}{4}.$$

Therefore,

$$\frac{i\sigma}{4} \leq \|\phi^i\|_{\mathcal{S}_n} \leq 5i.$$

This proves the following lemma.

Lemma 3.15. *The path $\mathcal{A}_\phi$ is a quasi-geodesic in $(\mathrm{Map}(S), d_{\mathcal{S}_n})$ for every n with uniform constants.*

We abuse notation and allow $\mathcal{A}_\phi$ to denote both the map, and the image of the map in $\mathrm{Map}(S)$. For $i, j \in \mathbb{Z}$, let $\mathfrak{g} = \mathfrak{g}_{i,j}$ be a geodesic in

$(\mathrm{Map}(S), d_{\mathcal{S}_n})$ connecting ϕ^i to ϕ^j. Let $\mathcal{G} = \Upsilon \circ \mathfrak{g}$ be the shadow of $\mathfrak{g}$ to the curve complex and let

$$\mathrm{Proj}_{\mathcal{G}} \colon \mathrm{Map}(S) \to \mathcal{G}$$

be the composition of Υ and the closest point projection from $\mathcal{C}(S)$ to $\mathcal{G}$. The following theorem, proven in more generality by Duchin and Rafi [DR09, Theorem 4.2], is stated for geodesics $\mathfrak{g}_{i,j}$ and the path $\mathcal{G}$.

Theorem 3.16. *The path $\mathcal{G}$ is a quasi-geodesic in $\mathcal{C}(S)$. Furthermore, there exists a constant B_n which depends on n and ϕ, and a constant B depending only on ϕ such that the following holds. For $x \in \mathrm{Map}(S)$ with $d_{\mathcal{S}_n}(x, \mathfrak{g}) > B_n$, let $r = d_{\mathcal{S}_n}(x, \mathfrak{g})/B_n$ and let $B(x,r)$ be the ball of radius r centered at x in $(\mathrm{Map}(S), d_{\mathcal{S}_n})$. Then*

$$\mathrm{diam}_{\mathcal{C}(S)}\Big(\mathrm{Proj}_{\mathcal{G}}\big(B(x,r)\big)\Big) \le B.$$

In the proof of [DR09, Theorem 4.2], it can be seen that B_n (B_1 in their notation) is dependent on the generating set since B_n is taken to be large with respect to the constants from the Masur and Minsky distance formula which depend on the generating set [MM00]. Let $\mathcal{S}$ be a fixed generating set for $\mathrm{Map}(S)$. Then the word lengths of elements in $\mathcal{S}_n$ in terms of $\mathcal{S}$ grow linearly in n with respect to $\mathcal{S}$. Hence, the constants involved in the Masur-Minsky distance formula also change linearly in n. That is, $B_n \asymp n$. Also, one can see that the constant B (B_2 in their proof) depends only on ϕ and the hyperbolicity constant of the curve graph, but not the generating set.

Since, $\mathcal{A}_\phi$ is a quasi-geodesic, Theorem 3.16 and the usual Morse argument implies the following.

Proposition 3.17. The paths $\mathcal{A}_\phi[i,j]$ and $\mathfrak{g}_{i,j}$ fellow travel each other and the constant depends only on n. That is, there is a bounded constant δ_n depending on n such that

$$\delta_n \ge \max\left(\max_{p \in \mathcal{A}_\phi[i,j]} \min_{q \in \mathfrak{g}_{i,j}} d_{\mathcal{S}_n}(p,q), \max_{p \in \mathfrak{g}_{i,j}} \min_{q \in \mathcal{A}_\phi[i,j]} d_{\mathcal{S}_n}(p,q) \right).$$

Proof. Let $\mathcal{A}_\phi$ denote both the map and the image of the map

$$\mathcal{A}_\phi \colon \mathbb{Z} \to \mathrm{Map}(S), \qquad i \mapsto \phi^i.$$

We let $\mathcal{A}_\phi[i,j]$ be the section of the map $\mathcal{A}_\phi$ which connects Φ^i to Φ^j. Let $\mathfrak{g}_{i,j}$ denote the geodesic connecting Φ^i to Φ^j. We let R be a number such that $R \ge B_n$, where B_n is the constant in Theorem 3.16, and such that

$R \geq \frac{10B}{\sigma}$ where B is the constant in Theorem 3.16 and σ is the translation constant from Theorem 3.13 for the pseudo-Anosov homeomorphism c. Notice that the constants B_n, B, and σ are constants which depend on n and the surface.

On $\mathcal{A}_\phi[i,j]$, we let k and l be points such that k is the first point such that $d_{S_n}(k, P(k))$, where $P(k)$ is the closest point projection from k to $\mathfrak{g}_{i,j}$, is greater than R. Similarly, l is the final point on $\mathcal{A}_\phi[i,j]$ such that $d_{S_n}(l, P(l))$, where $P(l)$ is the closest point projections from l to $\mathfrak{g}i, j$, is greater than R. We let P denote the distance from $P(k)$ to $P(l)$ on $\mathfrak{g}_{i,j}$. See Figure 3.5 for a visual representation.

Since the distance of the section of $\mathcal{A}_\phi$ between k and l is bounded above by $5(l-k)$ by Lemma 3.15, we can cover this section of $\mathcal{A}_\phi[i,j]$ by $\frac{5(l-k)}{B_n}$ balls of radius B_n, where B_n is the constant from Theorem 3.16. Using Theorem 3.16, we are able to project these balls down to the curve graph, where the diameter of the projection of these balls is bounded above by $\frac{5(l-k)B}{B_n}$. Notice that we are also able to use the shadow map Υ to map k, l, $P(k)$, and $P(l)$ to the curve graph. See Figure 3.5 for a visual representation. In Figure 3.5, we note that D' is the distance between $\Upsilon(P(k))$ and $\Upsilon(P(l))$ on $\mathcal{G}$, and D is the distance between $P(\Upsilon(k))$ and $P(\Upsilon(l))$ on $\mathcal{G}$.

Applying the triangle inequality and making note that we are projecting $\frac{5(l-k)}{B_n}$ balls of radius B_n to $\mathcal{C}(S)$, we see that

$$D \leq 4R + \frac{5(l-k)B}{R}. \tag{3.5}$$

By the definition of the shadow map Υ and by Lemma 3.15 we have

$$d_{\mathcal{C}(S)}(\Upsilon(k), \Upsilon(P(k))) \leq 4d_{S_n}(k, P(k)) \leq 4R, \tag{3.6}$$

and similarly

$$d_{\mathcal{C}(S)}(\Upsilon(l), \Upsilon(P(l))) \leq 4d_{S_n}(l, P(l)) \leq 4R. \tag{3.7}$$

Using the triangle inequality and Equations 3.6 and 3.7, we have

$$d_{\mathcal{C}(S)}(\Upsilon(k), \Upsilon(l)) \leq 8R + D'. \tag{3.8}$$

Using the triangle inequality and Lemma 3.15, we notice that

$$(l-k) \leq \frac{d_{\mathcal{C}(S)}(\Phi^l \alpha, \Phi^k(\alpha))}{\sigma} = \frac{d_{\mathcal{C}(S)}(\Upsilon(l), \Upsilon(l))}{\sigma}. \tag{3.9}$$

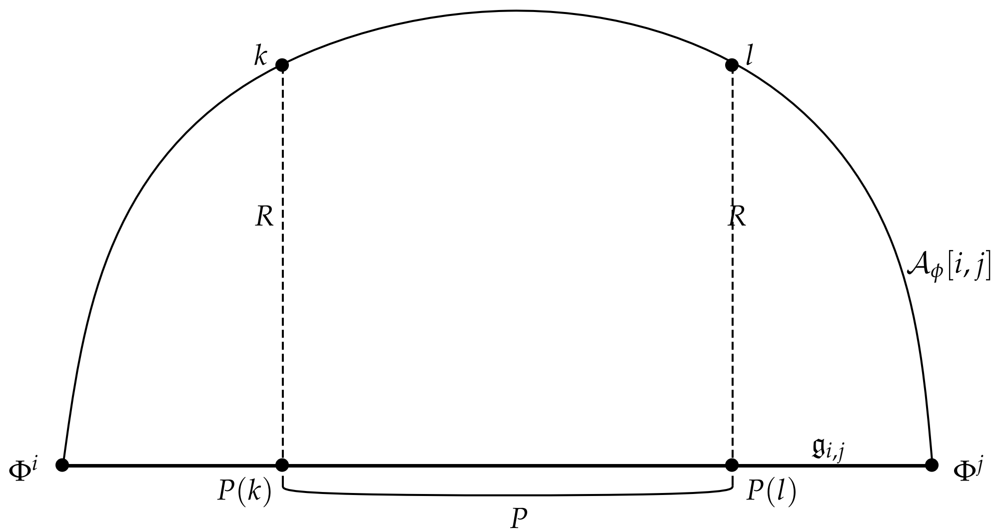


Figure 3.5: The set up for the proof of Proposition 3.17. All points are in the Cayley graph.

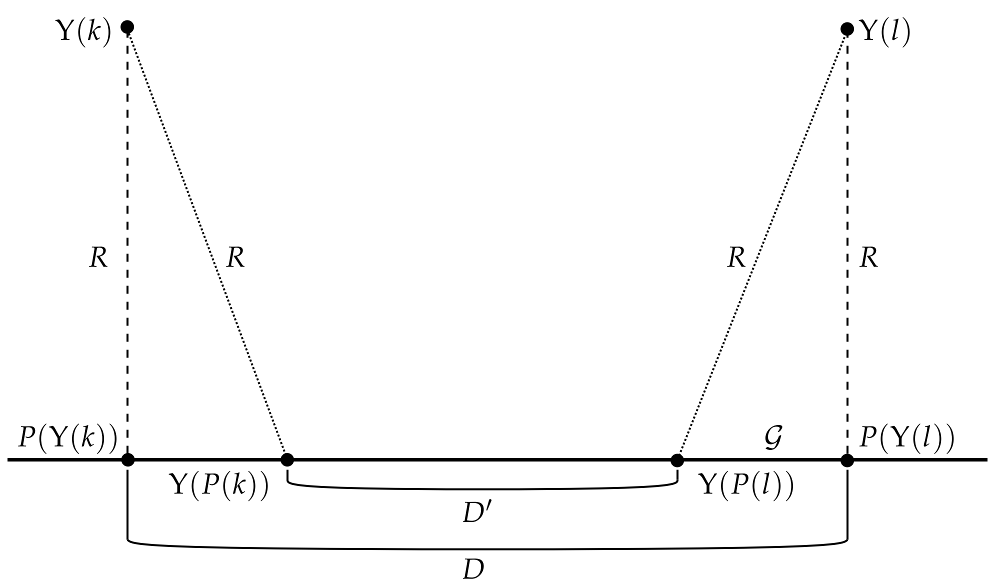


Figure 3.6: The shadow of Figure 3.5 to the curve graph.

Combining Equations 3.8 and 3.9, we find

$$(l-k) \leq \frac{8R + D'}{\sigma}. \tag{3.10}$$

By the triangle inequality, we have

$$D' \leq D + 4R \tag{3.11}$$

which applied to Equation 3.10 implies

$$(l-k) \leq \frac{12R + D}{\sigma}. \tag{3.12}$$

Combining Equations 3.5 and 3.12, we find

$$\begin{aligned} (k-l) \leq \frac{12R + D}{\sigma} &\leq \frac{16R + \frac{5(l-k)B}{R}}{\sigma} \\ &= \frac{16R}{\sigma} + \frac{5(l-k)B}{R\sigma}. \end{aligned} \tag{3.13}$$

Since we have that $R \geq \frac{10B}{\sigma}$, this along with Equation 3.13 implies that

$$\frac{(l-k)}{2} \leq \frac{16R}{\sigma}. \tag{3.14}$$

We recall that along $\mathcal{A}_\phi$, the distance from k to l is bounded by $5(l-k)$. But since $(l-k)$ is bounded above by values which depend only on the surface and on n. This means that for any point on $\mathcal{A}_\phi$, it is some value depending only on n away from a point on $\mathfrak{g}_{i,j}$, which proves our claim. □

We now show that ϕ acts loxodromically in $(\mathrm{Map}(S), d_{\mathcal{S}_n})$. That is, there exists a geodesic $\mathfrak{a}_\phi$ in $(\mathrm{Map}(S), d_{\mathcal{S}_n})$ that is preserved by a power of ϕ. This is folklore theorem, but we were unable to find a reference for it in the literature. The proof given here follows the arguments in [Bow07, Theorem 1.4] where Bowditch showed that ϕ acts loxodromically on the curve graph, which is more difficult since the curve graph is not locally finite. Bowditch's proof in turn follows the arguments of Delzant [Del96] for a hyperbolic group.

Proposition 3.18. There is a geodesic

$$\mathfrak{a}_\phi \colon \mathbb{Z} \to \mathrm{Map}(S)$$

that is preserved by some power of ϕ. We call the geodesic $\mathfrak{a}_\phi$ the quasi-axis for ϕ.

Proof. The statement is true for the action of any pseudo-Anosov homeomorphism in any mapping class group equipped with any word metric coming from a finite generating set. We only sketch the proof since it is a simpler version of the argument given in [Bow07].

Let $\mathcal{L}(i,j)$ be the set of all geodesics connecting ϕ^i to ϕ^j. Note that every point on every path in $\mathcal{L}(i,j)$ lies in the δ_n-neighborhood of $\mathcal{A}_\phi$. Letting $i \to \infty$, $j \to -\infty$ and using a diagonal limit argument ($\mathrm{Map}(S)$ is locally finite) we can find bi-infinite geodesics that are the limits of geodesic segments in sets $\mathcal{A}_\phi[i,j]$. Let $\mathcal{L}$ be the set of all such bi-infinite geodesics. Then $\phi(\mathcal{L}) = \mathcal{L}$ and every geodesic in $\mathcal{L}$ is also contained in the δ_n-neighborhood of $\mathcal{A}_\phi$. Let $\mathcal{L}/\phi$ represent the set of edges which appear in a geodesic in $\mathcal{L}$ up to the action of ϕ. Then $\mathcal{L}/\phi$ is a finite set.

Choose an order for $\mathcal{L}/\phi$. We say a geodesic $\mathfrak{g} \in \mathcal{L}$ is lexicographically least if for all vertices $x, y \in \mathfrak{g}$, the sequence of ϕ-classes of directed edges in the segment $\mathfrak{g}_0 \subset \mathfrak{g}$ between x and y is lexicographically least among all geodesic segments from x to y that are part of a geodesic in $\mathcal{L}$. Let $\mathcal{L}_L$ be the set lexicographically least elements of $\mathcal{L}$. We will show that every element of $\mathcal{L}_L$ is preserved by a power of ϕ.

Let P be the cardinality of a ball of radius δ_n in $(\mathrm{Map}(S), d_{\mathcal{S}_n})$. We claim that $|\mathcal{L}_L| \leq P^2 + 1$. Otherwise, we can find P^2+1 elements of $\mathcal{L}_L$ which all differ in some sufficiently large compact subset $N_{\delta_n}(\mathcal{A}_\phi)$, the δ_n-neighborhood of $\mathcal{A}_\phi$. In particular, we can find $x, y \in N_{\delta_n}(\mathcal{A}_\phi)$ so that each of these P^2+1 geodesics has a subsegment connecting a point in $N_{\delta_n}(x)$ to a point in $N_{\delta_n}(y)$, and these subsegments are all distinct. But then, at least two such segments must share the same endpoints, which means they cannot both be lexicographically least.

Since ϕ permutes elements of $\mathcal{L}_L$, each geodesic in $\mathcal{L}_L$ is preserved by ϕ^{P^2+1}. □

As before, we use the notation $\mathfrak{a}_\phi$ to denote both the map and the image of the map in $\mathrm{Map}(S)$. We now show that the projection of a ball that is disjoint from $\mathfrak{a}_\phi$ to $\mathfrak{a}_\phi$ grows at most logarithmically with the radius of the ball proving that Theorem 3.3 is sharp.

Corollary 3.19. *There are uniform constants $c_1, c_2 > 0$ so that, for $x \in \mathrm{Map}(S)$ and $R = d_{\mathcal{S}_n}(x, \mathfrak{a}_\phi)$, we have*

$$\mathrm{diam}_{\mathcal{C}(S)}\Big(\mathrm{Proj}_{\mathcal{G}_\phi}\big(\mathrm{Ball}(x,R)\big)\Big) \leq c_1 \log(n)\log(R) + c_2 n.$$

Proof. Consider $y \in \mathrm{Ball}(x, R - B_n)$. Let N be the smallest number so that there is a sequence of points along the geodesic connecting x to y

$$x = x_0, x_1, \ldots, x_N = y$$

so that

$$d_{\mathcal{S}_n}(x_i, x_{i+1}) \le \frac{d_{\mathcal{S}_n}(x_i, \mathfrak{a}_\phi)}{B_n}.$$

Then,

$$\begin{aligned} d_{\mathcal{S}_n}(x_{i+1}, \mathfrak{a}_\phi) &\ge d_{\mathcal{S}_n}(x_i, \mathfrak{a}_\phi) - d_{\mathcal{S}_n}(x_i, x_{i+1}) \\ &\ge d_{\mathcal{S}_n}(x_i, \mathfrak{a}_\phi) - \frac{d_{\mathcal{S}_n}(x_i, \mathfrak{a}_\phi)}{B_n} \ge d_{\mathcal{S}_n}(x_i, \mathfrak{a}_\phi)\left(1 - \frac{1}{B_n}\right). \end{aligned}$$

Hence,

$$d_{\mathcal{S}_n}(x_i, \mathfrak{a}_\phi) \ge R\left(1 - \frac{1}{B_n}\right)^i.$$

Since N is minimum

$$d_{\mathcal{S}_n}(x_i, x_{i+1}) + 1 \ge \frac{d_{\mathcal{S}_n}(x_i, \mathfrak{a}_\phi)}{B_n},$$

which implies

$$d_{\mathcal{S}_n}(x_i, x_{i+1}) \ge \frac{R}{B_n}\left(1 - \frac{1}{B_n}\right)^i - 1.$$

Since $d_{\mathcal{S}_n}(x, y) \le R - B_n$,

$$d_{\mathcal{S}_n}(x_i, \mathfrak{a}_\phi) \ge R - d_{\mathcal{S}_n}(x, x_i) \ge R - d_{\mathcal{S}_n}(x, y) \ge B_n.$$

Applying Theorem 3.16 to $r_i = d_{\mathcal{S}_n}(x_i, \mathfrak{a}_\phi)/B_n$ and $x_{i+1} \in \mathrm{Ball}(x_i, r_i)$ we get

$$d_{\mathcal{C}(S)}\big(\mathrm{Proj}_{\mathcal{G}_\phi}(x_i), \mathrm{Proj}_{\mathcal{G}_\phi}(x_{i+1})\big) \le B,$$

and hence,

$$d_{\mathcal{C}(S)}\big(\mathrm{Proj}_{\mathcal{G}_\phi}(x), \mathrm{Proj}_{\mathcal{G}_\phi}(y)\big) \le Bc'_n \log R. \tag{3.15}$$

Now, for any $y' \in \mathrm{Ball}(x, R)$ there is a $y \in \mathrm{Ball}(x, R - B_n)$ with $d_{\mathcal{S}_n}(y, y') \le B_n$. But Y is 4-Lipschitz and the closest point projection from $\mathcal{C}(S)$ to $\mathcal{G}_\phi$ is also Lipschitz with a Lipschitz constant depending on the hyperbolicity constant of $\mathcal{C}(S)$. Therefore,

$$d_{\mathcal{C}(S)}\big(\mathrm{Proj}_{\mathcal{G}_\phi}(y), \mathrm{Proj}_{\mathcal{G}_\phi}(y')\big) \le c'' B_n, \tag{3.16}$$

where c'', the Lipschitz constant for $\mathrm{Proj}_{\mathcal{G}_\phi}$, is a uniform constant. By letting

$$c_n = \max(Bc'_n, B_n c'') \asymp \log(n),$$

the Corollary follows from Equation (3.15) and Equation (3.16) and the triangle inequality. □

3.6 THE LOGARITHMIC LOWER-BOUND

In this section, we will show that the quasi-axis $\mathfrak{a}_\phi$ of the pseudo-Anosov map ϕ does not have the strongly contracting property proving Theorem 3.3 from the introduction.

Definition 3.20 ((d_1, d_2)-projection map)**.** Given a metric space (X, d_X), a subset $\mathcal{G}$ of X and constants $d_1, d_2 > 0$, we say a map $\operatorname{Proj}\colon X \to \mathcal{G}$, a (d_1, d_2)-projection map if for every $x \in X$ and $g \in \mathcal{G}$,

$$d_X\big(\operatorname{Proj}(x), g\big) \le d_1 \cdot d_X(x, g) + d_2.$$

To prove this theorem, notice first that the geodesic found in Section 2 may not determine the nearest point of $\mathcal{A}_\phi$ to $w_k = D_{\alpha_1}^{m_k}$, where $m_k = n^k + n^{k-1} + \ldots + n + 1$.

Lemma 3.21. *If ϕ^{p_k} is the nearest point of $\mathcal{A}_\phi$ to w_k, then $p_k \ge k/5$.*

Proof. Consider a point ϕ^m on $\mathcal{A}_\phi$ where $m < k/5$. Applying the homomorphism h we have

$$h(w_k \phi^{-m}) = (m_k - 5\,m) > (m_k - k) = h(w_k \phi^{-k/5}).$$

But $(m_k - k)$ is divisible by $(n-1)$. Hence, if we write $(m_k - m) = q(n-1) + r$, where $|r| \le \frac{(n-1)}{2}$, we have

$$|q| \ge \frac{m_k - k}{n-1}, \qquad \text{and} \qquad |r| \ge 0.$$

Lemma 3.6 implies that $\|w_k \phi^{-m}\|_{\mathcal{S}_n} > \|w_k \phi^{-k/5}\|_{\mathcal{S}_n}$, which means the closest point in $\mathcal{A}_\phi$ to w_k is some point ϕ^{p_k} where $p_k \ge k/5$. □

Let $R_k = d_{\mathcal{S}_n}(w_k, \phi^{p_k}) = d_{\mathcal{S}_n}(w_k, \mathcal{A}_\Phi)$ and $\Delta_k = d_{\mathcal{S}_n}(w_k, \phi^{(k+1)/5})$.

Proof of Theorem 3.3. For fixed $d_1, d_2 > 0$, let $\operatorname{Proj}_{\mathfrak{a}_\phi}\colon \operatorname{Map}(S) \to \mathfrak{a}_\phi$ be any (d_1, d_2)-projection map. Fix n large enough so that

$$\sigma > \frac{5\,d_1}{n-1}. \tag{3.17}$$

Choose the sequence $\{k_i\} = \big\{2n^i - 3\big\}$ and recall that

$$v_{k_i} = D_{\alpha_1}^{\frac{k_i+1}{2}} D_{\alpha_2}^{\frac{k_i+1}{2}}.$$

By Example 3.7 (notice that $\frac{k_i+1}{2} = n^i - 1$)

$$d_{\mathcal{S}_n}\left(v_{k_i}, \mathrm{id}\right) = \|v_{k_i}\|_{\mathcal{S}_n} = \frac{k_i+1}{n-1}. \tag{3.18}$$

and by Proposition 3.8, we have

$$d_{\mathcal{S}_n}(w_{k_i}, v_{k_i}) = \Delta_{k_i}.$$

Consider a ball $B(w_{k_i}, r_i)$ of radius $r_i = R_{k_i} - (\delta_n + 1)$ around w_{k_i}. This ball is disjoint from $\mathfrak{a}_\phi$ since $\mathcal{A}_\phi$ and $\mathfrak{a}_\phi$ are δ_n-fellow-travellers by 3.17, and $R_{k_i} = d_{\mathcal{S}_n}(w_{k_i}, \mathcal{A}_\Phi)$. For the rest of the proof, we refer to Figure 3.7.

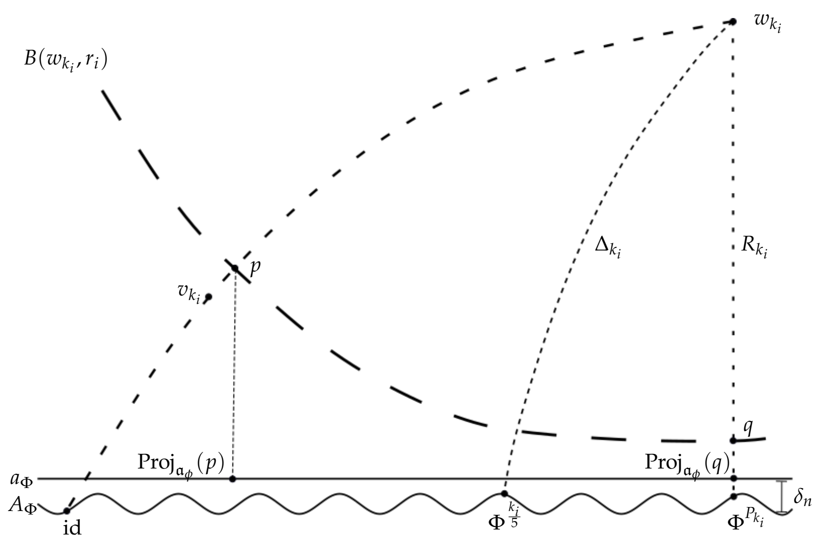


Figure 3.7: Setup for the proof of Theorem 3.3

Since h is a homomorphism, we have

$$h(w_{k_i}\phi^{k_i/5}) = h(w_{k_i}\phi^{-p_{k_i}}) + h(\phi^{p_{k_i}}\phi^{-k_i/5})$$

Theorem 3.8 showed

$$h(w_{k_i}\phi^{k_i/5}) = (n-1)\Delta_{k_i},$$

from Lemma 3.6, we have

$$h(w_k\phi^{-p_{k_i}}) \leq (n-1)R_{k_i}$$

and since $\|\phi\|_{\mathcal{S}_n} = 5$, we have

$$h(\phi^{p_{k_i}}\phi^{-k_i/5}) \leq 5p_{k_i} - k_i.$$

The above equations imply:

$$\Delta_{k_i} - R_{k_i} \leq \frac{5p_{k_i} - k_i}{n-1}.$$

Consider a point p on the geodesic from w_{k_i} to v_{k_i} such that

$$d_{\mathcal{S}_n}(w_{k_i}, p) = r_i,$$

ie. such that

$$d_{\mathcal{S}_n}\left(p, v_{k_i}\right) = \Delta_{k_i} - r_i = \Delta_{k_i} - (R_{k_i} - \delta_n - 1) \leq \frac{5p_{k_i} - k_i}{n-1} + \delta_n + 1.$$

This and Equation (3.18) imply

$$\begin{aligned} d_{\mathcal{S}_n}(\mathrm{id}, p) &\leq \frac{k_i+1}{n-1} + \frac{5p_{k_i} - k_i}{n-1} + \delta_n + 1 \\ &= \frac{5p_{k_i}+1}{n-1} + \delta_n + 1. \end{aligned}$$

Since $\mathfrak{a}_\phi$ and $\mathcal{A}_\Phi$ are δ_n-fellow-travellers by 3.17, there exists a point $x_0 \in \mathfrak{a}_\phi$ in the δ_n neighborhood of the identity. Thus $d_{\mathcal{S}_n}(p, x_0) \leq \frac{5p_{k_i}+1}{n-1} + 2\delta_n + 1$ and

$$\begin{aligned} d_{\mathcal{S}_n}(\mathrm{id}, \mathrm{Proj}_{\mathfrak{a}_\phi}(p)) &\leq d_{\mathcal{S}_n}(\mathrm{id}, x_0) + d_{\mathcal{S}_n}(x_0, \mathrm{Proj}_{\mathfrak{a}_\phi}(p)) \\ &\leq d_{\mathcal{S}_n}(p, x_0) + d_1 \cdot d_{\mathcal{S}_n}(x_0, p) + d_2 \\ &\leq \frac{5\, d_1\, p_{k_i}}{n-1} + A_p. \end{aligned} \tag{3.19}$$

where A_p is a constant depending on δ_n, d_1 and d_2 but is independent of k_i. Similarly, we consider a point q on the geodesic from w_{k_i} to $\phi^{p_{k_i}}$ such that $d_{\mathcal{S}_n}(w_{k_i}, q) = r_i$. Again, since $\mathfrak{a}_\phi$ and $\mathcal{A}_\Phi$ are δ_n-fellow-travellers by 3.17, there exists an $x_1 \in \mathfrak{a}_\phi$ such that $d_{\mathcal{S}_n}(\phi^{p_{k_i}}, x_1) \leq \delta_n$, and thus $d_{\mathcal{S}_n}(q, x_1) \leq 2\delta_n + 1$. Therefore

$$\begin{aligned} d_{\mathcal{S}_n}(\phi^{p_{k_i}}, \mathrm{Proj}_{\mathfrak{a}_\phi}(q)) &\leq d_{\mathcal{S}_n}(\phi^{p_{k_i}}, x_1) + d_{\mathcal{S}_n}(x_1, \mathrm{Proj}_{\mathfrak{a}_\phi}(q)) \\ &\leq \delta_n + d_1 \cdot (2\delta_n + 1) + d_2 \leq A_q \end{aligned} \tag{3.20}$$

where, again, A_q depends on δ_n, d_1 and d_2 but is independent of k_i. Since $p, q \in B(w_{k_i}, r_i)$, we have

$$\begin{aligned} \mathrm{diam}_{\mathcal{S}_n}\left(\mathrm{Proj}_{\mathfrak{a}_\phi}\left(B(w_{k_i}, r_i)\right)\right) &\geq d_{\mathcal{S}_n}\left(\mathrm{Proj}_{\mathfrak{a}_\phi}(p), \mathrm{Proj}_{\mathfrak{a}_\phi}(q)\right) \\ &\geq d_{\mathcal{S}_n}\left(\mathrm{id}, \phi^{p_{k_i}}\right) - d_{\mathcal{S}_n}\left(\mathrm{id}, \mathrm{Proj}_{\mathfrak{a}_\phi}(p)\right) \\ &\quad - d_{\mathcal{S}_n}\left(\mathrm{Proj}_{\mathfrak{a}_\phi}(q), \phi^{p_{k_i}}\right) \end{aligned}$$

But $d_{\mathcal{S}_n}(\mathrm{id}, \phi^{p_{k_i}}) \geq \sigma p_{k_i}$. By combining this fact and equations 3.19 and 3.20 we find

$$\begin{aligned} \mathrm{diam}_{\mathcal{S}_n}\left(\mathrm{Proj}_{\mathfrak{a}_\phi}\left(B(w_{k_i}, r_i)\right)\right) &\geq \sigma p_{k_i} - \frac{5\, d_1\, p_{k_i}}{n-1} - A_p - A_q \\ &= p_{k_i}\left(\sigma - \frac{5 d_1}{n-1}\right) - A_p - A_q. \end{aligned} \tag{3.21}$$

By our assumption on n (Equation (3.17)) this expression is positive and goes to infinity at $p_{k_i} \to \infty$. But, for n large enough, $r_i \leq R_{k_i} \leq \Delta_{k_i} \leq n^{k_i}$. Also, $p_{k_i} \geq \frac{k_i}{5}$. Hence,

$$\frac{5\, p_{k_i}}{\log n} \geq \log(r_i).$$

Hence, there is a constant c_n so that

$$\mathrm{diam}_{\mathcal{S}_n}\left(\mathrm{Proj}_{\mathfrak{a}_\phi}\left(B(w_{k_i}, r_i)\right)\right) \geq c_n \log r_i.$$

This finishes the proof. □

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