DEPARTMENT OF MATHEMATICS
Topology Comprehensive Exam 2025
September 17, 2025 - 6:00 - 9:00 p.m.
BA6183

NO AIDS ALLOWED. Passing score is 80 percent.

Last name ....

First name ....

Email ....

Question 1.
Fix an $n \times n$ matrix $A = (a_i^j)$, determining the following vector field on $\mathbb{R}^n$:

$$X_A = \sum_{i,j=1}^{n} a_i^j x^i \frac{\partial}{\partial x^j}.$$

1. What is the flow φ_t^A of the above vector field? Is it complete? Justify your answer.

2. Compute the pullback
$$(\varphi_t^A)^* v,$$
where $v = dx^1 \wedge \cdots \wedge dx^n$. Under what condition on A do we obtain $(\varphi_t^A)^* v = v$ for all t?

3. Under what condition on the $n \times n$ matrices A, B would we have that X_A and X_B are f-related by a diffeomorphism f of $\mathbb{R}^n$? Justify your answer.

Question 2. The Euler characteristic of a smooth n-manifold M is defined as the alternating sum

$$\chi(M) = \sum_{i=1}^{n} (-1)^i \dim H^i(M, \mathbb{R}).$$

1. What are the possible values of the Euler characteristic of a compact 3-manifold, and why?
2. Provide an example of a compact 4-manifold M such that $\chi(M) = 3$. Explain why it satisfies the condition.
3. Provide a sequence M_k of compact 4-manifolds such that $\chi(M_k)$ is unbounded as $k \to \infty$, with proof, stating clearly any results you need to use.

Question 3. Suppose M is a compact n-manifold with a smooth map to the circle $f : M \to S^1$ which has no critical points. Show that M is a mapping torus, i.e. M is diffeomorphic to $(X \times \mathbb{R})/\mathbb{Z}$, where X is a compact $(n-1)$-manifold, ϕ is a diffeomorphism of X, and $k \in \mathbb{Z}$ acts via $k \cdot (x, t) = (\phi^k(x), t + k)$.

Question 4.

1. Define "G is a topological group".
2. Prove that if G is a topological group then $\pi_1(G, e)$ is Abelian (for convenience, we take the basepoint of G to be its identity element e).

Question 5. Write a sketch of the proof of the following statement, including all the relevant definitions:

> If $f_0 \colon X \to Y$ and $f_1 \colon X \to Y$ are homotopic, then for every $k \geq 0$, the maps $f_{0*} \colon H_k(X) \to H_k(Y)$ and $f_{1*} \colon H_k(X) \to H_k(Y)$ are the same.

Question 6. Let n be an even positive integer.

1. Show that for any continuous map $f\colon S^n \to S^n$ there is a point $x \in S^n$ such that $f(x) = \pm x$.

2. Show that any continuous map $g\colon \mathbb{R}P^n \to \mathbb{R}P^n$ has a fixed point, where $\mathbb{R}P^n$ denotes the n-dimensional real projective space $\mathbb{R}P^n := S^n/(x \sim -x)$.