Department of Mathematics
University of Toronto

Topology Comprehensive Examination | 10:00-13:00 | 6/9/2013 | BA6183

Complete all six problems. Clearly state any theorems used.

1. Let K and L be embedded submanifolds of a smooth manifold M.

   a) What does it mean for K and L to intersect transversally? State a precise definition.

   b) Prove that if K and L intersect transversally, then $K \cap L$ is an embedded submanifold, and compute its dimension in terms of $\dim K, \dim L, \dim M$.

   c) Let $S^1 \subset \mathbb{R}^2$ be given by $\{(x_1, x_2) \,:\, x_1^2 + x_2^2 = 1\}$. Write the equations defining the tangent bundle of S^1 as a subset of $T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$, using standard coordinates (x_1, x_2, u_1, u_2) for the tangent vector

   $$u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2}$$

   above the point $(x_1, x_2) \in \mathbb{R}^2$.
   Finally, prove that this inclusion $TS^1 \subset T\mathbb{R}^2$ is an embedded submanifold.

2. Let M be an n-dimensional smooth manifold, and fix a point $p \in M$.

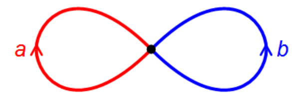
   a) Let $\mu \in \Omega^1(M, \mathbb{R})$ be a 1-form such that $d\mu = 0$ and $\mu(p) \neq 0$. Show there exists a neighbourhood U of p and coordinates $(x^1, \ldots, x^n)$ in this neighbourhood such that $\mu = dx^1$ in U.

   b) Let $\mu_i \in \Omega^1(M, \mathbb{R})$, for $i = 1, \ldots, n$. Give necessary and sufficient conditions on $\{\mu_1, \ldots, \mu_n\}$ for the existence of a coordinate system $\{x^1, \ldots, x^n\}$ near a point $p \in M$ such that $\mu_i = dx^i$ for all i.

3. **a)** Give a complete statement of Stokes' theorem for manifolds with boundary.

   b) Let M be a compact oriented 3-manifold with boundary, where the boundary is $\partial M = S^1 \times S^1$. Let $\theta_i \in \Omega^1(\partial M)$, $i = 1, 2$ be the 1-forms obtained by pulling back the standard volume form $\theta \in \Omega^1(S^1)$ by the first and second projections $\pi_i : S^1 \times S^1 \to S^1$, $i = 1, 2$. Prove that it is not possible to extend both θ_1 and θ_2 to closed 1-forms on M (i.e. one of θ_1, θ_2 must fail to extend in this way).

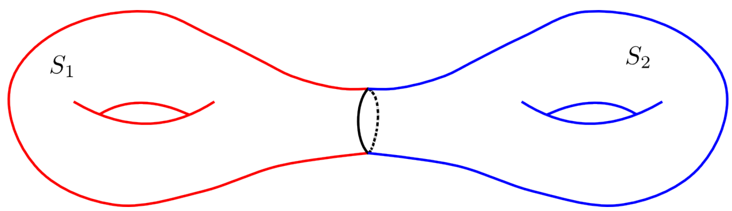
4. Let $X = S^1 \vee S^1$ be the figure eight space:



a) Describe all connected two-sheeted covering spaces of X. You may use diagrams such as the one above.

b) Find a connected three-sheeted covering space of X which is regular, and find another which is not regular. Recall that a covering $p : \widetilde{X} \to X$ is regular when $p_*\pi_1(\widetilde{X})$ is normal in $\pi_1(X)$.

5. Compute the fundamental group of the punctured torus $S = T^2 \setminus \{p\}$, where $p \in T^2$. Use this to compute the fundamental group of the genus two surface $S_1 \cup S_2$ shown below. Provide a detailed derivation.



6. Compute the cohomology of $\mathbb{R}P^n$ with coefficients in $\mathbb{Z}$ and with coefficients in $\mathbb{Z}/2\mathbb{Z}$, for all n. Describe the derivation clearly.