Problem 2. In standard coordinates \( x, y, z \), the Euler vector field on \( \mathbb{R}^3 \) is given by \( E = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \).

a) Let \( v = dx \wedge dy \wedge dz \). Show that \( i_E(v) \) pulls back to the unit sphere \( S^2 \subset \mathbb{R}^3 \) to define a volume form \( \omega \).

b) Write down vector fields \( V_1, V_2, V_3 \) on \( \mathbb{R}^3 \) whose flows are the rotations about the \( x, y, z \) axes, respectively.

c) Show that \( V_1, V_2, V_3 \) are tangent to \( S^2 \), defining vector fields on \( S^2 \).

d) Finally, find functions \( f_1, f_2, f_3 \) on \( S^2 \) such that \( i_{V_k}(\omega) = df_k, \quad k = 1, 2, 3 \).

Problem 3. Let \( X \) be a manifold and \( \phi : X \to X \) a diffeomorphism. The mapping torus of \((X, \phi)\) is defined to be the quotient manifold \( M = (X \times \mathbb{R})/\sim \), where the equivalence relation is \((x, t) \sim (\phi(x), t+1)\).

a) If \( M = U \cup V \) for open sets \( U, V \), write down the short exact sequence of cochain complexes which relates the de Rham complexes of \( M, U, V \), and \( U \cap V \), being careful to define the maps involved.

b) Using the Mayer-Vietoris long exact sequence, compute the de Rham cohomology groups of the mapping torus of \((S^n, A)\), where \( A : S^n \to S^n \) is the antipodal map \( x \mapsto -x \).

Topology II

Problem 4. Give an example of a non-trivial knot in \( \mathbb{R}^3 \), that is an embedding \( f : S^1 \to \mathbb{R}^3 \) such that \( \pi_1(\mathbb{R}^3 \setminus f(S^1)) \) is not isomorphic to \( \mathbb{Z} \). Prove your answer.

Problem 5.

a) Prove that each continuous map \( f : CP^2 \to CP^2 \) has a fixed point.

b) Prove that \( \mathbb{R}P^3 \) is not homotopy equivalent to \( \mathbb{R}P^2 \vee S^3 \).

Problem 6. Let \( M \) be a closed \( n \)-dimensional manifold such that its fundamental group is isomorphic to the free group \( F_2 \) with two generators. (Recall that \( F_2 = \mathbb{Z} * \mathbb{Z} \).

a) Determine \( H^{n-1}(M; \mathbb{Z}_2) \).

b) Prove that each two-dimensional homology class of \( M \) is spherical. (This means that for each \( h \in H_2(M; \mathbb{Z}) \) there exists a continuous map \( f : S^2 \to M \) such that \( h = f_*([S^2]) \), where \([S^2]\) denotes the fundamental homology class of the 2-dimensional sphere \( S^2 \), and \( f_* \) denotes the homomorphism \( H_2(S^2; \mathbb{Z}) \to H_2(M; \mathbb{Z}) \) induced by \( f \). Or, in other words, this means that the Hurewicz homomorphism \( \pi_2(M) \to H_2(M) \) is surjective.)