Near-pulse solutions of the FitzHugh-Nagumo equations on cylindrical surfaces

by

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Abstract

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In 1961, FitzHugh [19] suggested a model to explain the basic properties of excitability, namely the ability to respond to stimuli, as exhibited by the more complex Hodgkin-Huxley equations [24]. The following year Nagumo et al. [42] introduced another version based on FitzHugh's model. This is the model we consider in the thesis. It is called the FitzHugh-Nagumo model and describes the propagation of electrical signals in nerve axons. Many features of the system have been studied in great detail in the case where an axon is modelled as a one-dimensional object. Here we consider a more realistic geometric structure: the axons are modelled as warped cylinders and pulses propagate on their surface, as it happens in nature.

The main results in this thesis are the stability of pulses for standard cylinders of small constant radius, and existence and stability of near-pulse solutions for warped cylinders whose radii are small and vary slowly along their lengths. On the standard cylinder, we write a solution near a pulse as the superposition of a modulated pulse with a fluctuation and prove that the fluctuation decreases exponentially over time as the solution converges to a nearby translation of the pulse. On warped cylinders, we write a solution near a pulse in the same way as in standard cylinders and prove bounds on the fluctuation of near-pulse solutions.

Publications

Most of the parts of Chapters 3 and 4 appear in [47]. This joint work with Almut Burchard and Israel Michael Sigal has been submitted for publication, and the preprint is available on the arxiv.

Contents

Chapter 1

Introduction

1.1 The problem

The FitzHugh-Nagumo system [19, 42], modelling the propagation of electrical impulses in nerve axons, is a simplified version of the Hodgkin-Huxley system [24] and is given by

$$\begin{aligned}\partial_t u_1 &= \partial_x^2 u_1 + f(u_1) - u_2 \\ \partial_t u_2 &= \varepsilon(u_1 - \gamma u_2)\,.\end{aligned} \tag{1.1.1}$$

Here, u_1 and u_2 are real-valued functions that depend on $x \in \mathbb{R}$ and $t \geq 0$. The parameters ε and γ are chosen to be positive and small and the reaction term f is given by the cubic polynomial

$$f(u_1) := -u_1(u_1 - \alpha)(u_1 - 1), \tag{1.1.2}$$

for $0 < \alpha < \frac{1}{2}$. In this case, a nerve axon is represented by a straight line without an internal geometric structure. The two components, u_1 and u_2, are evolve in two different time scales. The first component is called the **excitation variable** and represents the electrical potential across the axonal membrane. The second is called the **recovery variable** and represents the current that passes through slowly adapting ion channels. We discuss in more detail the biophysical background of the FitzHugh-Nagumo equations in Section 1.3.

In this thesis, we take a further step in taking into account the geometry of the axon, namely, a cylindrical cable-like fiber, with electrical signals propagating on its surface. Thus we consider an extension of the FitzHugh-Nagumo (FHN) system on a cylindrical surface, $\mathcal{S}$. The system has the form

$$\begin{aligned} \partial_t u_1 &= \Delta_{\mathcal{S}} u_1 + f(u_1) - u_2\,, \\ \partial_t u_2 &= \varepsilon(u_1 - \gamma u_2)\,, \end{aligned} \tag{1.1.3}$$

where $\Delta_{\mathcal{S}}$ denotes the Laplace-Beltrami operator on the surface $\mathcal{S}$ and ε, γ and f are the same as above. Taking formally $\mathcal{S} = \mathbb{R}$ in Eq. (1.1.3) gives Eq. (1.1.1).

A solution to Eq. (1.1.1) which is a function of a single variable, $z = x - ct$, $c > 0$, and vanishes at infinity is called a **pulse**. One of the first results on the existence of pulses is due to Hastings [21], who showed that when $\mathcal{S}$ is the real line, $0 < \alpha < \frac{1}{2}$ and ε, γ are positive and sufficiently small, Eq. (1.1.1) has a pulse solution, whose speed depends on the parameters α and ε. The pulse is obtained as a homoclinic orbit in a related system of ordinary differential equations. We explain further the proof of the existence of the pulse in Subsection 2.1.1.

It turns out that when $\varepsilon > 0$ is sufficiently small, Eq. (1.1.1) has at least two different pulse solutions, the **fast** pulse studied by Hastings, which travels with speed $c_f(\varepsilon) = \frac{\sqrt{2}}{2}(1 - 2\alpha) + o(\varepsilon)$, and a **slow** pulse that travels with speed $c_s(\varepsilon) = O(\sqrt{\varepsilon})$. The existence of pulses has been studied by numerous authors. See, for example, the papers by Carpenter [3], Hastings [21, 22], Langer [36], Evans [17], Krupa, Sandstede and Szmolyan [35], Jones, Kopell and Langer [31], Arioli and Koch [1]. Langer [36] also proved uniqueness of the fast pulse. Jones [30], and independently, Yanagida [49], proved that the fast pulse is stable. In addition, fast pulses with oscillatory tails exist and are stable (Carter and Sandstede [5], Carter, de Rijk and Sandstede [4]). On the other hand, the slow pulse is always unstable (Flores [20], Evans [17], Ikeda, Mimura and Tsujikawa [29]).

Existence and stability for fast pulses have been studied for variants of Eq. (1.1.1), where the second equation also has a diffusion term (Cornwell and Jones [9], Chen and Choi [6], Chen and Hu [7]). Another system that admits stable fast pulses is the discrete analogue of Eq. (1.1.1) (Hupkes and Sandstede [27], Schouten-Straatman and Hupkes [45], Hupkes, Morelli, Schouten-Straatman and Van Vleck [25]).

There are a few results in higher dimensions. In $\mathbb{R}^2$, Mikhailov and Krinskii [38] and Keener [33] studied spiral solutions of Eq. (1.1.1). In n-dimensions, for $n \geq 2$, Tsujikawa, Nagai, Mimura, Kobayashi and Ikeda [48] proved that there exist fast pulse solutions propagating in a one-dimensional direction. Such solutions are stable.

In this thesis we study solutions of the FHNcyl system, Eq. (1.1.3), on infinitely long, thin cylindrical surfaces. For $\mathcal{S}$ a standard cylinder, the pulse solutions to Eq. (1.1.1) are also (angle-independent) solutions to Eq. (1.1.3) and we continue to call them the pulses. We show that

1. on a cylinder of small constant radius, the (fast) pulses are stable under general perturbations of the initial condition that depend on both spatial variables;

2. on a warped cylinder whose radius is small and varies slowly along its length, solutions that are initially close to a pulse stay near the family of pulses for all time.

In nature, there exist neurons of different types and shapes. There are also diseases that harm healthy neurons, by damaging their axons, making them unfunctional. Our extension of Eq. (1.1.1), namely Eq. (1.1.3), is more geometrical rather than biophysical. By adding the appropriate diffusion coefficients in Eq. (1.1.3), the system becomes more realistic since the second order elliptic operator describes the surface of evolution instead of the Laplacian $\Delta_{\mathcal{S}}$. The techniques we use could be modified to address such diffusion. We also expect that our approach to the problem can be used for different equations which provide more qualitative features than the FHN system.

1.2 The main results

In this thesis we consider Eq. (1.1.3) on a surface $\mathcal{S}$ of a cylinder. Let S^1 denote the unit ball. Define the standard cylinder of constant radius R centered about the x-axis in $\mathbb{R}^3$ as

$$\mathcal{S}_R := \left\{ x \in \mathbb{R},\, \theta \in S^1 \;\middle|\; (x, R\cos\theta, R\sin\theta) \right\} . \tag{1.2.1}$$

The Laplacian on this surface is defined by

$$\Delta_{\mathcal{S}_R} = \partial_x^2 + R^{-2}\partial_\theta^2 \tag{1.2.2}$$

and the Riemannian area element is $R\,d\theta dx$. For $\mathcal{S} = \mathcal{S}_R$, Eq. (1.1.3) is invariant under translations. This means that if $u(x,\theta,t) := (u_1,u_2)(x,\theta,t)$ is a solution, then so is

$$u_h(x,\theta,t) := u(x-h,\theta,t), \quad h \in \mathbb{R}. \tag{1.2.3}$$

Each pulse Φ on $\mathcal{S} = \mathbb{R}$ defines a smooth axisymmetric traveling wave solution $u(x,\theta,t) = \Phi(x-ct)$ of Eq. (1.1.3) on $\mathcal{S}_R$. Its speed c is determined by the parameters α, γ, and ε. It is a consequence of translation invariance that all translates Φ_h of Φ are pulses of the same speed c.

Our first result concerns the stability of $\mathcal{M} \subset H^{2,1}$, where $H^{2,1}$ is the mixed Sobolev space defined by

$$H^{2,1}(\mathcal{S}_R) := \{u \in L^2(\mathcal{S}_R) \mid \Delta_{\mathcal{S}_R} u_1 \in L^2(\mathcal{S}_R), u_2 \in L^2(\mathcal{S}_R)\} \tag{1.2.4}$$

We prove that solutions that are initially close to Φ converge to nearby translates of Φ as $t \to \infty$:

Theorem 1.2.1 (Stability of pulses, standard cylinder)**.** *Consider Eq.* (1.1.3) *on the cylinder $\mathcal{S}_R$ of constant radius $R \leq 1$. Fix $\alpha \in (0, \frac{1}{2})$, $\varepsilon > 0$, and $\gamma > 0$ such that the equation has a fast pulse solution $\Phi(x - ct)$. If ε is sufficiently small, then there is a neighborhood $\mathcal{U}$ of Φ in $H^{2,1}$ such that for every $u_0 \in \mathcal{U}$, the mild solution $u(t)$ with initial value $u\big|_{t=0} = u_0$ exists globally in time and satisfies*

$$\|u(t) - \Phi_{ct+h_*}\|_{2,1} \leq C_1 e^{-\xi t} \|u_0 - \Phi\|_{2,1} \qquad (t \geq 0) \tag{1.2.5}$$

for some $\xi > 0$ and $h_* \in \mathbb{R}$ (determined by u_0) with

$$|h_*| \leq C_2 \|u_0 - \Phi\|_{2,1}\,.$$

Here, C_1 and C_2 are positive constants.

Theorem 1.2.1 will be proved in Section 3.3. We show that the conclusion will be established for any decay rate $\xi < \min\{\alpha, \beta, \varepsilon\gamma\}$, where β is the exponent from Lemma 3.2.9.

The translates of Φ form a one-dimensional manifold of pulses

$$\mathcal{M} := \{\Phi_h \mid h \in \mathbb{R}\}\,. \tag{1.2.6}$$

Denote by $\operatorname{dist}(v, \mathcal{M}) := \inf_h \|v - \Phi_h\|_{2,1}$ the distance of $v \in H^{2,1}$ from the manifold. By translation invariance, the conclusion of Theorem 1.2.1 yields a tubular neighborhood $\mathcal{W} = \{w \in H^{2,1} \mid \operatorname{dist}(w, \mathcal{M}) < \eta\}$ such that

$$\operatorname{dist}(u(t), \mathcal{M}) \leq C_1 e^{-\xi t} \operatorname{dist}(u_0, \mathcal{M})$$

for all mild solutions with initial values in $\mathcal{M}$. As $t \to \infty$, each solution converges to a particular traveling pulse $\Phi(x - ct - h_*)$.

Turning to warped cylinders we define them as graphs over the standard one determined by a positive function $\rho(x)$:

$$\mathcal{S}_\rho := \left\{ x \in \mathbb{R},\, \theta \in S^1 \;\middle|\; (x, \rho(x)\cos\theta, \rho(x)\sin\theta) \right\} . \tag{1.2.7}$$

On $\mathcal{S}_\rho$, the Laplace-Beltrami operator is given by

$$\Delta_{\mathcal{S}_\rho} := \frac{1}{\sqrt{g(x)}} \partial_x \left(\frac{\rho(x)}{\sqrt{1+\rho'(x)^2}} \partial_x \right) + \rho^{-2}(x)\partial_\theta^2 \,, \tag{1.2.8}$$

with $g(x) := \rho(x)^2(1+\rho'(x)^2)$.

When ρ is non-constant, Eq. (1.1.3) has no pulse solutions. Our second result is that pulse-like solutions persist, if ρ is sufficiently close to a constant. These pulse-like solutions remain close to the manifold $\mathcal{M}$, while moving along $\mathcal{M}$ over time. For a function $v \in H^{2,1}$, denote the distance from the manifold of pulses by

$$\mathrm{dist}\,(v, \mathcal{M}) := \inf_{h \in \mathbb{R}} \|v - \Phi_h\|_{2,1}.$$

Theorem 1.2.2 (Near-pulse solutions, warped cylinder)**.** *Consider the FHN system on a cylinder $\mathcal{S}_\rho$ of variable radius, as in Eq. (1.2.7), and let α, ε and γ be as in Theorem 1.2.1. There are a constant $\delta_* > 0$ and a tubular neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $H^{2,1}$, with the following properties: If $R \le 1$ and $\delta := R^{-1}\|\rho - R\|_{C^2} \le \delta_*$, then for every $u_0 \in \mathcal{W}$, the unique mild solution $u(t)$ with initial value $u\big|_{t=0} = u_0$ exists globally in time, and satisfies*

$$\mathrm{dist}\,(u(t), \mathcal{M}) \le C_1 e^{-\xi t} \mathrm{dist}\,(u_0, \mathcal{M}) + C_2 \delta \,, \qquad (t \ge 0) \tag{1.2.9}$$

for some positive constants C_1, C_2, and ξ.

1.3 Biological interpretation of the model

In this section we give a basic introduction to neural communication and discuss the origin of the FitzHugh-Nagumo system. The discussion is based on the books of Keener and Sneyd [34] and Murray [40, 41]. For further details we refer the reader to the books of Scott [46], Müller and Kuttler [39], and Ermetrout and Terman [13].

A typical neuron consists of three principal parts: the dendrites, the soma, and the axon. Dendrites are the input stage of a neuron and receive synaptic input form other neurons. The soma contains the necessary cellular machinery such as nucleous and mitochondria. The axon is a long thin cylindrical tube which extends from the soma and electrical signals propagate along its outer membrane. At the end of the axon are synapses, which are cellular junctions specialized for transmission of signals. Therefore, a single neuron may receive input along its dendrites from a large number of other neurons, and may similarly transmit a signal along its axon to other neurons.

The classical work by Hodgkin and Huxley [24] on the propagation of electrical signals on nerve membranes was on the nerve axon of the giant squid. They received the 1963 Nobel Prize in Physiology or Medicine for this work. The electrical pulses arise because the membrane is preferentially permeable to various chemical ions with permeabilities affected by the currents and potentials present. The key elements in the system are potassium (K^+) ions and sodium (Na^+) ions. In the rest state there is a transmembrane potential difference of about $-70mV$ due to the higher concentration of K^+ ions within the axon as compared with the surrounding medium. The membrane permeability properties change when subjected to a stimulating electrical current, I. Such a current can be generated, for example, by a local depolarisation relative to the rest state.

Next we derive the Hodgkin-Huxley model [24] and the reduced analytically tractable FitzHugh-Nagumo model [19, 42] which captures the key phenomena. Assume the positive direction for the membrane current, I, is outwards from the axon. The current $I(t)$ is made up of the current due to the individual ions which pass through the membrane and the contribution from the time variation in the transmembrane potential, that is, the membrane capacitance contribution. Thus we have

$$I(t) = C\frac{dV}{dt} + I_i\,, \tag{1.3.1}$$

where C is the capacitance and I_i is the current contribution from the ion movement across the membrane. Based on experimental observations Hodgkin and Huxley [24] took

$$\begin{aligned} I_i &= I_{Na} + I_K + I_L \\ &= g_{Na}m^3h(V - V_{Na}) + g_K n^4(V - V_K) + g_L(V - V_L)\,, \end{aligned}$$

where V denotes the potential and I_{Na}, I_K and I_L denote respectively the sodium, potassium and "leakage" currents; I_L is the contribution from all the other ions which contribute constant equilibrium potentials. The m, n and h are variables, bounded by 0 and

1, which are determined by the following differential equations

$$\begin{aligned}\frac{dm}{dt} &= \alpha_m(V)(1-m) - \beta_m(V)m \\ \frac{dn}{dt} &= \alpha_n(V)(1-n) - \beta_n(V)n \\ \frac{dh}{dt} &= \alpha_h(V)(1-h) - \beta_h(V)h\,,\end{aligned} \tag{1.3.2}$$

where the α_* and β_* are given functions of the potential V. More specifically, α_n and α_m are behave like $(1+tanh(V))/2$ while α_h is behave like $(1-tanh(V))/2$. The values empirically determined by fitting the results to the data. For the explicit representations of the functions α_* and β_* we refer the interesting reader to Chapter 5 of [34].

Imposing an applying current $I_{app}(t)$, Eq. (1.3.1) becomes

$$C\frac{dV}{dt} = -g_{Na}m^3h(V-V_{Na}) - g_K n^4(V-V_K) - g_L(V-V_L) + I_{app}\,. \tag{1.3.3}$$

To summarize, the Hodgin-Huxley system consists of the four ordinary differential equations given by Eqs. (1.3.3) and (1.3.2).

Setting $I_{app} = 0$, the rest state of the Hodgkin-Huxley system is linearly stable but is excitable. This means that, if the perturbation from the steady state is sufficiently large there is a large excursion of the variables in their phase space before returning to the steady state. On the other hand, if $I_{app} \neq 0$ there is a range of values where regular repetitive firing occurs; that is, the mechanism displays limit cycle characteristics. Both types have been observed experimentally. Because of the complexity of the Hodgkin-Huxley system (1.3.2)-(1.3.3) various simpler mathematical models, which capture the key features of the full system, have been proposed, the best known is the FitzHugh-Nagumo system [19, 42], which we now derive.

In Eq. (1.3.2) the time scale for m is much faster than n and h, so it is reasonable to assume that m is sufficiently fast that it relaxes immediately to its value determined by setting $\frac{dm}{dt} = 0$ in Eq. (1.3.2). If we also set $h = h_0$, where h_0 is a constant, the system still retains many of the features experimentally observed. The resulting model is a system of two equations that can be qualitatively approximated by the dimensionless system

$$\begin{aligned}\frac{du_1}{dt} &= f(u_1) - u_2 + I_{app} \\ \frac{du_2}{dt} &= \varepsilon(u_1 - \gamma u_2)\,,\end{aligned} \tag{1.3.4}$$

where $f(u_1) = -u_1(u_1 - \alpha)(u_1 - 1)$, $0 < \alpha < \frac{1}{2}$ (same as in Eq. (1.1.2)), and ε, γ are positive constants. In this system, u_1 is like the membrane potential V, and u_2 plays the role of all three variables m, n and h in Eq. (1.3.2).

For $I_{app} = 0$, Eq. (1.3.4) is the system FitzHugh [19] originally considered. This system is a two-variable phase plane system. The excitability characteristic, discussed above for the Hodgkin-Huxley system. For example, a perturbation from 0 to a point on the u_1-axis with $u_1 > \alpha$, undergoes a large phase trajectory excursion before returning to 0.

A couple of years later, Nagumo et al [42] introduced another version based on FitzHugh's model, a reaction-diffusion partial differential equation given by Eq. (1.1.1). With the diffusion term added, the system admits traveling waves along the axon, that depend on space x and time t.

1.4 Organization of the thesis

We will use the classical semigroup theory and known spectral results to prove the above two theorems. We now describe the structure of the thesis and state the basic assumptions regarding the parameters of the FHN system.

Chapter 2 is a review of existing results about the existence and stability of pulses of the FHN system. We include results on the real line (Section 2.1) as well as in higher dimensions (Section 2.2). In Section 2.3 we give a motivation of our work and a more concrete outline of the proofs of the main results.

Chapter 3 starts with the local well-posedness of Eq. (1.1.3) on the surface of $\mathcal{S}_R$. The remaining sections of that chapter are devoted to the stability result of Theorem 1.2.1. Specifically, in Section 3.2 we develop linear semigroup estimates and in Section 3.3 we prove the nonlinear stability. In Section 3.4 we include numerical simulations to complement our theoretical results.

In Chapter 4 we show that pulse-like solutions persist on the surface of warped cylinders $\mathcal{S}_\rho$ of slowly varying radius, and remain close to the manifold $\mathcal{M}$. The key ingredient is a perturbation result of the linearization, which we prove in Section 4.2. In Section 4.3 we complete the proof of Theorem 1.2.2, and in Section 4.4 we develop numerical simulations to illustrate the result of this theorem.

We assume that the parameters ε and γ are positive and small enough that the FHN system (1.1.3) admits a fast pulse solution, Φ. Furthermore, we assume that ε is small enough so that the spectral results of Jones [30] and Yanagida [49] apply. For the warped cylinders $\mathcal{S}_\rho$ we assume that the function ρ is of class C^2, and is also positive, bounded, and bounded away from zero.

In our estimates, we will denote by either M or C any positive constant whose value, which may change from line to line, is dependent of any fixed parameter α, γ, or ε of Eq. (1.1.3). We frequently use the notation $\lesssim$ and $\gtrsim$ for the respective inequalities up to such constants. Dependence on any other parameters, including R, ρ and u_0 will be made explicit.

Chapter 2

Background of the FitzHugh-Nagumo equations

2.1 The real line

The system of reaction-diffusion equations in Eq.(1.1.1) is the FitzHugh-Nagumo system defined in one spatial dimension. We have mentioned at the beginning of Section 1.2 that the pulse Φ is a stationary solution to Eq. (1.1.1). It is a function of the single variable, $z = x - ct$, where $c > 0$ denotes its speed, and it vanishes for large values of z. In other words, the pair of solutions $(\phi_1(z), \phi_2(z))$ satisfies the system

$$\begin{aligned} -c\partial_z\phi_1 &= \partial_z^2\phi_1 + f(\phi_1) - \phi_2 \\ -c\partial_z\phi_2 &= \varepsilon(\phi_1 - \gamma\phi_2)\,, \end{aligned} \tag{2.1.1}$$

and $(\phi_1(z), \phi_2(z)) \to (0, 0)$ as $z \to \pm\infty$. In Subsection 2.1.1 we briefly describe Langer's approach [36] about the existence of the pulse. The proof is based on the geometric singular perturbation theory [18]. Langer was not the first who proved existence of solutions to Eq. (1.1.1). Earlier existence results are coming from Carpenter [3], Hastings [21], and Conley [8] where the authors used topological and fixed point arguments. The advantage is that Langer's proof answered the question of uniqueness of the pulse for fixed ε.

Jones [30] and, independently, Yanagida [49] studied the stability of pulse solutions of Eq. (1.1.1). Both proofs are based on the Evans function. In Subsection 2.1.2 we sketch the proof of Yanagida's paper [49] as it is more elementary and easy to follow.

2.1.1 Existence and uniqueness of pulses

We begin with the sketch of the proof of existence proposed by Langer [36]. Rewrite Eq. (2.1.1) as a system of three ordinary differential equations

$$\begin{aligned}\partial_z u_1 &= u_3\\ \partial_z u_2 &= -\frac{\varepsilon}{c}(u_1-\gamma u_3)\\ \partial_z u_3 &= -cu_3 - f(u_1) + u_2\,.\end{aligned} \tag{2.1.2}$$

In the (u_1,u_2,u_3) phase plane the above system has a critical point at the origin $(0,0,0)$. Langer constructed a pulse as a homoclinic orbit to the origin. A pulse is said to be **homoclinic** if

$$\lim_{z\to-\infty}(u_1(z),u_2(z),u_3(z)) = \lim_{z\to+\infty}(u_1(z),u_2(z),u_3(z))\,.$$

In Langer's proof this homoclinic orbit is constructed as $\varepsilon \to 0$. Reducing the number of differential equations to two in Eq. (2.1.2), by taking $\varepsilon = 0$ (the second equation degenerates to $u_2 = const.$), we observe that there exist $w_{\min} < 0$ and $w_{\max} > 0$, such that if $w_{\min} < w < w_{\max}$ then the system

$$\begin{aligned}\partial_z u_1 &= u_3\\ \partial_z u_3 &= -cu_3 - f(u_1) + u_2\end{aligned} \tag{2.1.3}$$

has three critical points. One critical point is the origin $(0,0,0)$, another is at $(1,0,0)$ and the third is between those two. Take $u_2 = 0$ in Eq. (2.1.3). Then there exists a nonzero constant c_f for which there is a heteroclinic orbit joining the origin $(0,0,0)$ to the critical point $(1,0,0)$. We denote this heteroclinic orbit by J_+. Similarly, for the same constant c_f there is a u_2^* for which a heteroclinic orbit exists to Eq. (2.1.3) joining the critical point $(1,0,0)$ to the origin $(0,0,0)$. This orbit is denoted by J_-.

Setting $u_3 = 0$, the second equation in (2.1.3) becomes

$$u_2 = f(u_1)\,. \tag{2.1.4}$$

For $u_2 \in [0,u_2^*]$ and u_1 being the largest root of Eq. (2.1.4) there exists an orbit denoted by E_r^*. For $u_2 \in [0,u_2^*]$ and u_1 being the smallest root of $u_2 = f(u_1)$ there exists another orbit denoted by E_l^*. Therefore, the orbit in $\mathbb{R}^3$ is given by the union of the four pieces constructed above, i.e., by $J_+ \cup E_r^* \cup J_- \cup E_l^*$. The orbit is singular as for $\varepsilon \to 0$ the u_2 equation degenerates.

Langer's existence result states that, given any neighborhood N of the singular orbit, there is an ε_0 so that there exists a solution to Eq. (2.1.2) for some $c = c(\varepsilon)$, with $0 < \varepsilon < \varepsilon_0$, which is homoclinic to the origin $(0, 0, 0)$ and lies entirely in the neighborhood N. The wavespeed $c(\varepsilon)$ converges to c_f as $\varepsilon \to 0$. Moreover, if the neighborhood N is small enough, the solution is unique for each ε.

This type of geometric singular perturbation method has been further developed by many authors [4, 5, 9, 10, 25, 26, 27].

2.1.2 Stability of pulses

Consider the space of bounded continuous functions. For the change of variables $z = x - ct$, the linearization of Eq. (1.1.1) around the pulse $\Phi(z) = (\phi_1(z), \phi_2(z))$ is given by

$$L = \begin{pmatrix} \partial_z^2 + c\partial_z + f'(\phi_1(z)) & -1 \\ \varepsilon & c\partial_z - \varepsilon\gamma \end{pmatrix}. \tag{2.1.5}$$

The domain of L is the space of bounded continuous functions. Yanagida [49] proved that there exists a positive constant β such that

$$\mathrm{spec}_{disc}(L) \cap \{\lambda \in \mathbb{C} \mid \text{ Re } \lambda \geq -\beta\} = \{0\}. \tag{2.1.6}$$

Moreover, the eigenvalue $\lambda = 0$ is simple. This implies linear stability. The nonlinear stability follows directly by a result in Evans [14].

Let $\lambda \in \mathbb{C}$, $u \in (u_1, u_2)$ and L be the linear operator of Eq. (2.1.5). Consider the eigenvalue problem

$$\lambda u = Lu\,. \tag{2.1.7}$$

By definition a number $\lambda \in \mathbb{C}$ is an eigenvalue of Eq. (2.1.7) if there is a nontrivial solution to this equation satisfying

$$\sup_{z\in\mathbb{R}} |u_1(z)| < \infty$$
$$\sup_{z\in\mathbb{R}} |u_2(z)| < \infty\,.$$

To solve the eigenvalue problem (2.1.7) Yanagida split the complex plane $\mathbb{C}$ into three subdomains. Assume $0 < r_1 < r_2$ are constants that do not depend on ε. Then, define

$$\begin{aligned} S_1 &:= \{\lambda \in \mathbb{C} \mid |\lambda| < r_1,\ \mathrm{Re}\,\lambda \geq -\beta\} \\ S_2 &:= \{\lambda \in \mathbb{C} \mid |\lambda| > r_2,\ \mathrm{Re}\,\lambda \geq -\beta\} \\ S_3 &:= \{\lambda \in \mathbb{C} \mid r_1 \leq |\lambda| \leq r_2,\ \mathrm{Re}\,\lambda \geq -\beta\}\,. \end{aligned} \tag{2.1.8}$$

Eq. (2.1.6) is verified by showing that there are no eigenvalues of Eq. (2.1.7) in the set $S_2 \cup S_3$, and that $\lambda = 0$ is the unique eigenvalue in the subdomain S_1. The main tool used here is the Evans function, $E(\lambda;\, \varepsilon)$ (Lemma 5.1, [49]). Yanagida constructed the Evans function, $E(\lambda;\, \varepsilon)$, with the following properties (Lemma 4.3, [49]):

1. it is a complex analytic function of λ, and is real-valued for real λ

2. for $\lambda = 0$, $E(0;\, \varepsilon) = 0$

3. for $\lambda = 0$, $\frac{d}{d\lambda}E(0; \varepsilon) > 0$.

Specifically the author proved that there exist no eigenvalues in S_2, if r_2 is sufficiently large. Similarly, no eigenvalues exist in $S_1 \cup S_3 \setminus \{0\}$. The fact that $\lambda = 0$ is a simple eigenvalue in S_1 follows by showing that the equation

$$\partial_z \Phi = Lu$$

has no bounded solutions as $z \to \pm\infty$ (Section 5, [49]).

2.2 Higher-dimensional domains

In n-dimensions, an extension of the FHN system is given by the differential equations in (1.1.3) with

$$\Delta_S := \sum_{j=1}^{n} \partial_{x_j}^2 \tag{2.2.1}$$

for $n \geq 2$. Tsujikawa, Nagai, Mimura, Kobayashi and Ikeda [48] proved that, under small perturbations, there exist fast pulse solutions propagating in a one-dimensional direction. In addition, the authors showed numerically that under larger perturbations the pulse is unstable and demonstrated how spiral waves can evolve. In the following subsection we explain the method they used to prove stability of the pulse.

2.2.1 Stability of pulses

Consider the FHN system (1.1.3), with Δ_S given by Eq. (2.2.1), and assume that u_1, u_2 satisfy periodic boundary conditions. Let S_n be the strip defined as

$$S_n := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \in \mathbb{R},\, x_j \in (-l_j, l_j),\, j = 2, \dots, n\}.$$

The pulse traveling with wavespeed $c = c_f$ in the x_1-direction is defined as

$$(\phi_1(x,t),\, \phi_2(x,t)) = (\phi_1(x_1 + c_f t),\, \phi_2(x_1 + c_f t)),$$

and is uniformly in the directions $x_2, \dots, x_n$. As in the one-dimensional case the pulse satisfies the property

$$(\phi_1(\pm\infty, x_2, \dots, x_n, t),\; \phi_2(\pm\infty, x_2, \dots, x_n, t)) = (0, 0).$$

Tsujikawa et. al. [48] studied the stability of such a pulse in S_n. They derived a general theorem for semilinear evolution equations and they applied this theorem to the given FHN system. In what follows we briefly discuss their theorem.

Let X be a Banach space, $T > 0$ be a fixed time and $u : [0, T] \to X$. Consider a semilinear evolution equation of the form

$$\partial_t u = Lu + N(u) \tag{2.2.2}$$

for $t \in [0, T]$, with initial condition

$$u|_{t=0} = u_0 .$$

The operator L is the infinitesimal generator of a strongly continuous semigroup e^{tL} on the space X and $N \in C^1(X; X)$ is the nonlinearity. It is assumed that the flow is locally well-posed in X. This means that for each initial value $u_0 \in X$ there is a time $T > 0$ for which there exists a unique solution $u(t) \in X$ of the initial value problem for $t \in [0, T]$. This solution satisfies the integral equation

$$u(t) = e^{tL} u_0 + \int_0^t e^{(t-s)L} N(u(s))\, ds \tag{2.2.3}$$

for $t \in [0, T]$.

To obtain stability of the pulse to perturbations of initial data, the following assumptions regarding the linear operator L are required. Rewrite L as

$$L = L_0 + K$$

such that L_0 is the infinitesimal generator of a strongly continuous semigroup e^{tL_0} on the Banach space X satisfying $Dom(L_0) = Dom(L)$, and $K : X \to X$ is a bounded linear operator. The assumptions about L are stated next:

1. The operator $K(\lambda - L_0)^{-1} : X \to X$ is compact for some λ in the resolvent set of L_0.

2. (a) Denote by $\|\cdot\|_X$ the norm in the Banach space X. There are positive constants ω and C such that
$$\|e^{tL_0}\|_X \leq Ce^{-\omega t} \tag{2.2.4}$$
for all $t > 0$.

   (b) Moreover, $Ke^{tL_0} \in C((0,\infty); B(X))$, where $B(X)$ denotes the space of bounded linear operators form X to X.

3. (a) The eigenvalue at 0 is an isolated eigenvalue of L and the nullspace of L has a basis
$$\partial_{h_1}\Phi, \ldots, \partial_{h_n}\Phi\,.$$
Here, $\Phi = \Phi(h)$ with $h = h_1, \ldots, h_n$.

   (b) Let Null(L) denote the nullspace of L and Ran$\,(L)$ denote the range of L. Then Ran$\,(L) \cap$ Null$(L) = \{0\}\,$.

4. There exists a constant $\omega_1 \in (0, \omega)$ such that the set
$$\{\lambda \in \mathbb{C} \;\;|\;\; \text{Re}\;\lambda \geq -\omega_1\} \setminus \{0\}$$
is contained in the resolvent set of L.

For the nonlinearity, N, assume that

$$N'(\Phi(0)) = 0\,, \tag{2.2.5}$$

where $'$ denotes the derivative of N. The theorem is stated next:

Theorem 2.2.1 (Tsujikawa et al. [48]). *Consider the linear problem* (2.2.2) *and suppose that the linearization* L *satisfies the assumptions* (1) – (4)*. Then there exist constants*

$a > 0$, $b > 0$ and $M > 0$ such that for any $u_0 \in X$ with $\|u_0 - \Phi(0)\|_X \leq b$ there is $h \in \mathbb{R}^n$ with $|h| \leq M\|u_0 - \Phi(0)\|_X$ satisfying

$$\|u(t) - \Phi(h)\|_X \leq Me^{-at}\|u_0 - \Phi(0)\|_X \tag{2.2.6}$$

for $t > 0$, where u is a unique global solution of Eq. (2.2.2).

Theorem 2.2.1 is an abstract version of the result in Evans [14, 16] for a more general class of systems including Eq. (1.1.1). The proof of Theorem 2.2.1 is given in Section 4 of [48]. Here we briefly discuss the assumptions of the theorem.

Assumption 2(a) implies that

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > -\omega\}$$

is contained in the resolvent set of L_0. Moreover, it gives estimates on the resolvent operator of L_0. Specifically, there is a constant $C > 0$ such that for $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > -\omega$ and $n = 1, 2, \ldots$,

$$\|(\lambda - L_0)^{-n}\|_X \leq \frac{C}{(\operatorname{Re} \lambda + \omega)^n}.$$

Assumptions 3 and 4 imply that L is a Fredholm operator with index zero and the nullspace of L has algebraic and geometric multiplicity n. The assumption on the non-linearity, Eq. (2.2.5), implies that

$$N(u) - N(\Phi) = o(\|u - \Phi\|_X). \tag{2.2.7}$$

Consider the special case where the Banach space is a Hilbert space, denoted by H, with norm $\|\cdot\|_H$. The above assumptions (1)-(4) about the linearization L appear to be the same with those in Theorem 4.3.5 of [32]. However, the assumption about the nonlinearity is different. In Theorem 4.3.5 of [32] the nonlinearity N is assumed to be quadratic in $\|\cdot\|_H$ near zero, in other words there exist a positive constant M such that

$$\|N(u)\|_H \leq M\|u\|_H^2. \tag{2.2.8}$$

Eq. (2.2.8) is stronger than Eq. (2.2.7).

Our proof of Theorem 1.2.1 could be slightly shortened by appealing to the general Theorem 4.3.5 of [32]. However, we provide a self-contained proof which uses well-known results from the linear semigroup theory as well as a spectral result of [30] and [49]. The sketch of the proof of Theorem 1.2.1 and that of Theorem 1.2.2 is discussed in the next subsection.

2.3 Thin cylindrical surfaces

Although Eq. (1.1.3) with $\Delta_{\mathcal{S}}$ given by Eq. (2.2.1) is of a great interest from the mathematical point of view, it has no biological meaning. As mentioned in Section 1.3, the motivation of the FHN system was to isolate the mathematical properties of excitation from the electrochemical properties of K^+ and Na^+ ion flow in nerve axons. But a nerve axon has roughly the shape of a long thin cylinder. Therefore, it comes naturally to study the FHN equations on cylindrical surfaces. We will consider two different cases: the case of a standard cylinder $\mathcal{S}_R$, and that of warped cylinders $\mathcal{S}_\rho$.

In the next subsection we briefly describe our results, namely Theorems 1.2.1 and 1.2.2, and we give some key features of the proofs of these theorems.

2.3.1 Our approach

We analyze the FHN system defined in Eq. (1.1.3) with Δ_S given either by Eq. (1.2.2) or (1.2.8) in a neighborhood of the one-dimensional manifold $\mathcal{M}$. On the standard cylinder $\mathcal{S}_R$, the manifold $\mathcal{M}$ consists of stationary solutions. This means that a solution of Eq. (1.1.3), with $\mathcal{S} = \mathcal{S}_R$, that starts on the manifold remains on it. While a solution of the same equation that starts close to the manifold is decomposed into a modulated pulse and a fluctuation. Under suitable values of the parameters α, γ and ε of Eq. (1.1.3), the fluctuation decreases exponentially fast over time, while the solution stays close to a nearby translate of the pulse Φ. This is the conclusion of Theorem 1.2.1. On warped cylinders $\mathcal{S}_\rho$, that is not the case. Even if a solution of Eq. (1.1.3), with $\mathcal{S} = \mathcal{S}_\rho$, starts on the manifold $\mathcal{M}$, the dynamics push this solution off the manifold. However, the solution always stays in a neighborhood of $\mathcal{M}$. Theorem 1.2.2 provides bounds on the fluctuation of near-pulse solutions in terms of the distance of the variable radius ρ from the constant radius R.

Next we discuss the two theorems in a bit more detail. In Theorem 1.2.1 we consider the Laplacian on the surface of the standard cylinder $\mathcal{S}_R$. The proof consists of two parts: we first establish the linearized stability, and then prove the nonlinear stability by using a fixed point argument (Sections 3.2 and 3.3, respectively). Fix a (fast) pulse Φ that is moving with speed c. Changing the variables to $z = x - ct$ we construct a moving frame.

In this frame the FHN system (1.1.3) has the form

$$\begin{aligned} \partial_t u_1 &= \Delta_{\mathcal{S}_R} u_1 + c\partial_z u_1 + f(u_1) - u_2 \\ \partial_t u_2 &= c\partial_z u_2 + \varepsilon(u_1 - \gamma u_2). \end{aligned} \tag{2.3.1}$$

Notice that the pulse Φ is a stationary solution of Eq. (2.3.1). This is due to the fact that Φ does not depend on the angle θ, i.e., $\frac{1}{R^2}\partial_\theta^2\Phi$, and so Eq. (2.3.1) is reduced to Eq. (2.1.1).

We start with linear estimates. Let L denote the linearized operator of Eq. (2.3.1) about the stationary solution Φ, and e^{tL}, $t \geq 0$, be the linear semigroup generated by L. The operator L is neither self-adjoint nor sectorial, making the study of the linear estimates more difficult. We show in Section 3.2 that the semigroup e^{tL} decays exponentially to the manifold $\mathcal{M}$. That is,

$$\|e^{tL}(\mathbf{1} - P)\|_{2,1} \lesssim e^{-\sigma t}$$

for $t \geq 0$, and for some positive constant σ. In the above estimate, P is the projection onto the tangent space of $\mathcal{M}$ at the pint Φ that commutes with L, and $\|\cdot\|_{2,1}$ denotes the operator norm on the Sobolev space $H^{2,1}$. The projection P is not orthogonal, because L is not a self-adjoint operator. We construct P as the spectral projection associated with the zero eigenvalue of L. In Subsection 3.2.2 we prove that the remainder of the spectrum of L lies in the left half-plane, implying the decay of the above estimate. The proof of this estimate is the most challenging part of Theorem 1.2.1.

Section 3.3 is devoted to the nonlinear stability of the pulse Φ. In particular, we show that every mild solution of Eq. (2.3.1) that starts sufficiently close to the pulse Φ converges exponentially to a translated pulse Φ_{h_*}. For a suitable choice of h, write a solution of Eq. (2.3.1) as

$$u = \Phi_h + v \tag{2.3.2}$$

where Φ_h is a modulated pulse moving on the manifold $\mathcal{M}$ and v is a fluctuation that is transversal to $\mathcal{M}$ (Subsection 3.3.1). This decomposition transforms Eq. (2.3.1) into an equation for $v(t)$:

$$\partial_t v = Lv + N(v, h)\,,$$

coupled to an ordinary differential equation for the evolution of h. In Eq. (2.3.1), $N(v, h)$ is the nonlinearity and is of order $(|h| + \|v\|_{2,1})\|v\|_{2,1}$. Using the decay estimate of the linear semigroup from Section 3.2, we prove that

$$\|v(t)\|_{2,1} \lesssim e^{-\xi t} \|v_0\|_{2,1} ,$$

i.e., the fluctuation decays exponentially, and

$$|h(t) - h(0)| \lesssim e^{-\xi t} \|v_0\|_{2,1}^2$$

for $t \geq 0$, and for any ξ with $\xi < \sigma$. Theorem 1.2.1 follows since $\|v_0\|_{2,1} \lesssim \|u_0 - \Phi\|_{2,1}$ and $|h_0| \lesssim \|u_0 - \Phi\|_{2,1}$.

In Chapter 4 we prove Theorem 1.2.2. Here the Laplace-Beltrami operator is given by Eq. (1.2.8). We consider the variable radius $\rho(x)$ of a warped cylinder as a perturbation of the constant radius R. The basis for the argument is an estimate for the linearized evolution on $\mathcal{S}_\rho$. Transforming the FHN system in the moving frame as we do on the standard cylinder $\mathcal{S}_R$, the operator $\Delta_{\mathcal{S}_{\rho(x)}}$ becomes $\Delta_{\mathcal{S}_{\rho(z+ct)}}$, in other words the principal part is time-dependent. This leads to an evolution system associated with the family of time-dependent linear operators of Eq. (1.1.3) (Section 5, [43]). But too less theory is known about evolution systems. So, to avoid this issue, we study the perturbation in the static frame, and linearize the FHN system about the zero solution instead of the pulse. The advantage is that the time-independent linearized operator, A_ρ, is sectorial. Therefore, we can represent the semigroup e^{tA_ρ} by an absolutely convergent contour integral, and control the perturbation via resolvent estimates. We use Grönwall's inequality to extend these perturbation estimates to the nonlinear evolution generated by the FHNcyl system on $\mathcal{S}_\rho$. In combination with the exponential decay of fluctuations for near-pulse solutions on $\mathcal{S}_R$ that was proved in Theorem 1.2.1, this yields Theorem 1.2.2.

Chapter 3

Pulses on standard cylinders

Consider the case of the standard cylinder $\mathcal{S} = \mathcal{S}_R$, where the radius R is fixed. On the surface of $\mathcal{S}_R$ fix a pulse Φ, and let $\mathcal{M}$ be the manifold of its translates defined in Eq. (1.2.6). By definition, Φ is an axisymmetric traveling wave solution of

$$\begin{aligned}\partial_t u_1 &= \Delta_{\mathcal{S}_R} u_1 + f(u_1) - u_2 \\ \partial_t u_2 &= \varepsilon(u_1 - \gamma u_2)\,.\end{aligned} \tag{3.0.1}$$

In the moving frame, let $G(u)$ be the right hand side of Eq. (2.3.1) given explicitly by

$$G(u) := \begin{pmatrix} \Delta_{\mathcal{S}_R} u_1 + c\partial_z u_1 + f(u_1) - u_2 \\ c\partial_z u_2 + \varepsilon(u_1 - \gamma u_2) \end{pmatrix}. \tag{3.0.2}$$

Set $u = \Phi + v$. Since Φ is a stationary solution, $G(\Phi) = 0$. Taylor expanding about Φ yields, for v, the equation

$$\begin{aligned}\partial_t v &= G(\Phi + v) \\ &= Lv + N(v)\,,\end{aligned} \tag{3.0.3}$$

where the linearization L is given by the Gâteaux derivative of G about Φ:

$$L := dG(\Phi) = \begin{pmatrix} \Delta_{\mathcal{S}_R} + c\partial_z + f'(\phi_1) & -1 \\ \varepsilon & c\partial_z - \varepsilon\gamma \end{pmatrix}, \tag{3.0.4}$$

and $N(v)$ is defined as

$$N(v) := \begin{pmatrix} v_1^2(\alpha + 1 - 3\phi_1 - v_1) \\ 0 \end{pmatrix}. \tag{3.0.5}$$

We call $N(v)$ the nonlinearity.

The linearization in Eq. (3.0.4) defines a closed, linear operator on the Hilbert space $L^2 := L^2(\mathcal{S}_R, \mathbb{C}^2)$ of two-component square integrable functions, with inner product

$$\langle u, w\rangle := \int_{\mathcal{S}_R} \left(u_1\bar{w}_1 + \varepsilon^{-1}u_2\bar{w}_2\right) R\, d\theta dz\,, \tag{3.0.6}$$

and the corresponding norm $\|\cdot\|$. The domain of L is the dense subspace

$$H^{2,1}(\mathcal{S}_R) := \{u \in L^2(\mathcal{S}_R) \mid \Delta_{\mathcal{S}_R}u_1 \in L^2(\mathcal{S}_R), u_2 \in L^2(\mathcal{S}_R)\} \tag{3.0.7}$$

with norm

$$\|u\|_{2,1} := \sum_{0\le i\le 1} \|(\Delta_{\mathcal{S}_R})^i u_1\| + \sum_{0\le j\le 1} \varepsilon^{-1}\|\partial_z^j u_2\|\,. \tag{3.0.8}$$

We will prove in Lemma 3.2.16 that the norm $\|u\|_{2,1}$ is equivalent to the graph norm of the operator L.

First we verify that the solution v of Eq. (3.0.3) is locally well-posed in time under suitable initial conditions (Section 3.1), and then we prove stability of the pulse Φ.

3.1 Local well-posedness

Consider the initial value problem

$$\begin{aligned} \partial_t v &= Lv + N(v) \\ v|_{t=0} &= v_0 \end{aligned} \tag{3.1.1}$$

on the mixed Sobolev space $H^{2,1}$, where L and $N(\cdot)$ are given by Eqs. (3.0.4) and (3.0.5), respectively. By Duhamel's formula, a classical solution $v(t)$ of Eq. (3.1.1) also solves the integral equation

$$v(t) = e^{tL}v_0 + \int_0^t e^{(t-s)L}N(v(s))\, ds\,. \tag{3.1.2}$$

Denote by $\mathcal{F}_R(v)(t)$ the right hand side of Eq. (3.1.2), i.e.,

$$\mathcal{F}_R(v)(t) := e^{tL}v_0 + \int_0^t e^{(t-s)L}N(v(s))\, ds\,. \tag{3.1.3}$$

By definition, a **mild solution** of the initial value problem (3.1.1) is a strongly continuous function $v(t)$, taking values in the space $H^{2,1}$, that solves the fixed point problem

$$v = \mathcal{F}_R(v) \tag{3.1.4}$$

in the space $C([0,T]; H^{2,1})$ for some $T > 0$.

Proposition 3.1.1 (Local well-posedness)**.** *Assume R is fixed. Then for each $v_0 \in H^{2,1}(\mathcal{S}_R)$, there exists $T = T(\|v_0\|_{2,1}) > 0$ such that Eq. (3.1.1) on $\mathcal{S}_R$ has a unique mild solution v in $C([0,T], H^{2,1}(\mathcal{S}_R))$ with initial condition $v|_{t=0} = v_0$. The solution depends continuously on $v_0 \in H^{2,1}(\mathcal{S}_R)$.*

Given an initial value $v_0 \in H^{2,1}$, fix $\eta > 0$ and $T > 0$, and define the ball

$$\mathcal{B} := \left\{ v \in C([0,T];\, H^{2,1}) \;\middle|\; \|v(t)\|_{2,1} \le \eta \text{ for all } 0 \le t \le T \right\} . \tag{3.1.5}$$

The ball $\mathcal{B}$ is a complete metric space equipped with the norm $\|v\|_T := \sup_{0\le t\le T} \|v(t)\|_{2,1}$. We prove Proposition 3.1.1 using the **contraction mapping principle**, which states that if the map $\mathcal{F}_R$ of Eq. (3.1.3) is a strict contraction in $\mathcal{B}$, then $\mathcal{F}_R$ has a unique fixed point in $\mathcal{B}$. The proof requires bounds on the nonlinearity N.

Lemma 3.1.2 (Nonlinearity)**.** *Let η be a positive constant. Then there exists a constant $C_\eta > 0$ (depending on α, γ, ε and η) such that N, defined in Eq. (3.0.5), is locally Lipschitz in $H^{2,1}$. I.e., the nonlinearity $N(v)$ satisfies*

$$\|N(v) - N(w)\|_{2,1} \le C_\eta \|v - w\|_{2,1} \tag{3.1.6}$$

for all v, w with $\|v\|_{2,1}, \|w\|_{2,1} \le \eta$.

Proof. Let $N_1(v)$ denote the first component of N. The difference is given by

$$\begin{aligned} N_1(v) - N_1(w) &= v_1^2(\alpha + 1 - 3\phi_1 - v_1) - w_1^2(\alpha + 1 - 3\phi_1 - w_1) \\ &= \big((\alpha + 1 - 3\phi_1)(v_1 + w_1) - (v_1^2 + v_1 w_1 + w_1^2)\big)(v_1 - w_1)\,. \end{aligned} \tag{3.1.7}$$

Since the cylindrical surface has dimension 2, the Sobolev space $H^2(\mathcal{S}_R)$ is a Banach algebra. By the continuity of the multiplication and the fact that $\phi_1 \in H^2$, Eq. (3.1.7) implies

$$\begin{aligned} \|N(v) - N(w)\|_{2,1} &\le C \max\{\|v_1\|_{H^2}, \|w_1\|_{H^2}\} \|v_1 - w_1\|_{H^2} \\ &\le C_\eta \|v - w\|_{2,1}\,, \end{aligned}$$

as desired.

□

An immediate consequence of Lemma 3.1.2 is that the nonlinearity N is quadratic in $\|\cdot\|_{2,1}$ near zero, i.e., there exists $\eta > 0$ and $C_\eta > 0$ such that

$$\|N(v)\|_{2,1} \leq C_\eta \|v_1\|_{H^2}^2$$

for all v with $\|v_1\|_{H^2} \leq \eta$.

Proof of Proposition 3.1.1. Given $v_0 \in H^{2,1}$, fix $\eta > 0$ and $T > 0$, and consider the ball, $\mathcal{B}$, given by Eq. (3.1.5). The time T will be specified below. We first show that the map $\mathcal{F}$ defined in Eq. (3.1.3) is Lipschitz continuous on $\mathcal{B}$. Indeed, by Lemma 3.1.2 and the fact that $H^{2,1}$ is a Banach algebra, we have

$$\begin{aligned}\|\mathcal{F}(v) - \mathcal{F}(w)\|_T &\leq \sup_{0\leq t\leq T} \int_0^t \|e^{(t-s)L}\big(N(v(s)) - N(w(s))\big)\|_{2,1}\, ds \\ &\leq \sup_{0\leq t\leq T} \int_0^t \|e^{(t-s)L}\|_{2,1} \|N(v(s)) - N(w(s))\|_{2,1}\, ds \\ &\leq T\, C_0\, C_\eta\, \|v - w\|_T\end{aligned}$$

where $C_0 := \sup_{0\leq t\leq T} \|e^{tL}\|_{2,1}$, and C_η as in Eq. (3.1.6). For this argument we do not need explicit estimates on the linear semigroup e^{tL}, $t \geq 0$. Defining the constant $C'_\eta := C_0\, C_\eta$ we obtain

$$\|\mathcal{F}(v) - \mathcal{F}(w)\|_T \leq T\, C'_\eta\, \|v - w\|_T\,.$$

Moreover, for $v = 0$ and since $N(0) = 0$:

$$\|\mathcal{F}(0)\|_T = \sup_{0\leq t\leq T} \|e^{tL} v_0\|_{2,1} \leq C_0 \|v_0\|_{2,1}\,. \tag{3.1.8}$$

Choose $\eta = 2C_0\|v_0\|_{2,1}$, and $T = (2C'_\eta)^{-1}$. Then $\mathcal{F}$ is Lipschitz continuous with Lipschitz constant $\frac{1}{2}$. Furthermore, from Eq. (3.1.8), $\mathcal{F}$ maps $\mathcal{B}$ into $\mathcal{B}$. By the contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $\mathcal{B}$. This is the mild solution of the initial value problem (3.1.1).

Next we show that the mild solution v depends continuously on v_0 in the Sobolev space $H^{2,1}$. Let $w(t)$ be another mild solution, with initial data $w|_{t=0} = w_0$ satisfying

$\|w_0\|_{2,1} < \eta$. By definition of the mild solution, their difference satisfies

$$\begin{aligned}\|v(t) - w(t)\|_{2,1} &\le \|e^{tL}(v_0 - w_0)\|_{2,1} + \int_0^t \|e^{(t-s)L}\big(N(v(s)) - N(w(s))\big)\|_{2,1}\, ds \\ &\le C_0\|v_0 - w_0\|_{2,1} + C'_\eta \int_0^t \|v(s) - w(s)\|_{2,1}\, ds\,,\end{aligned}$$

where in the second inequality we have applied Eq. (3.1.6). Using Grönwall's inequality we have

$$\|v(t) - w(t)\|_{2,1} \le C_0\, e^{C'_\eta t}\|v_0 - w_0\|_{2,1}\,.$$

Therefore, the solution depends continuously on the initial value.

□

3.2 Linear stability

This section is the core of the proof of Theorem 1.2.1. Here we study the linear stability of the pulse Φ under the flow generated by the FHN system in Eq (1.1.3) on the surface of a standard cylinder $\mathcal{S}_R$.

Specifically, we will show that L generates a strongly continuous semigroup e^{tL} on L^2. The point at zero is an eigenvalue of L, and the tangent vector $\tau = -\partial_z \Phi$ is the corresponding eigenfunction. This follows from the fact that

$$L\partial_z\Phi = 0\,. \tag{3.2.1}$$

To verify Eq. (3.2.1) recall that $G(\Phi) = 0$, since Φ is a stationary solution of Eq. (2.3.1). By translation invariance, $G(\Phi_h) = 0$ for any $h \in \mathbb{R}$. Eq. (3.2.1) follows by differentiating $G(\Phi_h) = 0$ with respect to h at $h = 0$. It turns out that the spectral projection P associated with the zero eigenvalue of L has rank one. The complementary projection $Q = \mathbf{1} - P$ is the projection onto the range of L. Both P and Q commute with L and with the semigroup e^{tL}.

The main result of this chapter is stated next:

Proposition 3.2.1 (Linearized decay)**.** *Let L be the operator defined by Eq. (3.0.4). If $\varepsilon > 0$ is sufficiently small, and $0 < R \le 1$, then there exists $\sigma > 0$ such that the semigroup e^{tL} satisfies*

$$\left\|e^{tL}Q\right\|_{2,1} \le Ce^{-\sigma t} \tag{3.2.2}$$

for some constant $C > 0$ and for all $t \ge 0$.

The proof of Proposition 3.2.1 will be given in Subsection 3.2.4. In the proof we specify the conditions on σ.

3.2.1 The linear semigroup

Consider the linearization L about the pulse Φ defined in Eq. (3.0.4) with domain $\mathrm{Dom}(L) = H^{2,1}$. Here, we show that L is the infinitesimal generator of a strongly continuous semigroup. We denote this semigroup by e^{tL}, $t \geq 0$. The idea is to write L as a generator of a strongly continuous semigroup, that is easy to study, perturbed by a bounded linear operator.

Indeed, we decompose L as $L = \bar{L} + V$ where

$$\bar{L} := \begin{pmatrix} \Delta_{\mathcal{S}_R} + c\partial_z + f'(0) & -1 \\ \varepsilon & c\partial_z - \varepsilon\gamma \end{pmatrix} \tag{3.2.3}$$

is a constant coefficient linearized operator about zero and V is the matrix multiplication operator:

$$V := \begin{pmatrix} f'(\phi_1) - f'(0) & 0 \\ 0 & 0 \end{pmatrix} . \tag{3.2.4}$$

Since f is a polynomial and ϕ_1 is a smooth, bounded function, V is a bounded operator on the space L^2 and also on $H^{2,1}$.

We say that the operator L is a **relatively compact perturbation** of $\bar{L}$ if $(L - \bar{L})(\lambda - \bar{L})^{-1} : H^{2,1} \mapsto H^{2,1}$ is compact for some λ in the resolvent set of $\bar{L}$. The next lemma is used to construct the semigroup generated by L. Moreover, it plays a role in locating the essential spectrum of L.

Lemma 3.2.2. *The operator L is a bounded, relatively compact perturbation of $\bar{L}$.*

Proof. Observe that the nonzero entry of V is continuous and decays at infinity. Also, $\bar{L}$ is a constant coefficient operator with domain $H^{2,1}$. By a standard result (Theorem 3.1.11, [32]), $V(\lambda - \bar{L})^{-1}$ is compact for fixed λ in the resolvent set of $\bar{L}$.

□

The operator $\bar{L}$ captures the behavior of L at infinity, since $\lim_{z\to\pm\infty} \Phi(z) = 0$ implies $\lim_{z\to\pm\infty} f'(\phi_1(z)) = f'(0)$ and so $V \equiv 0$. Next we show that $\bar{L}$ generates a strongly continuous semigroup. For this we introduce the notion of dissipativity. A linear operator B on a Hilbert space H is called **dissipative** if

$$\mathrm{Re}\langle Bu, u\rangle \leq 0 \, ,$$

for every $u \in \mathrm{Dom}(B)$. In Appendix A we state the Lumer-Phillips theorem which provides a characterization to the generator of a densely defined, dissipative operator.

Lemma 3.2.3 (Dissipative operators)**.** *Let B be a closed, densely defined, dissipative operator on a Hilbert space H. Then B generates a strongly continuous semigroup of contractions, e^{tB}. Its spectrum lies in the left half-plane $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \leq 0\}$, and its resolvent is bounded by*

$$\|(\lambda - B)^{-1}\| \leq \frac{1}{\mathrm{Re}\,\lambda} \qquad (\mathrm{Re}\,\lambda > 0)\,. \tag{3.2.5}$$

Proof. Since B is dissipative, the operator $\mathbf{1} + B$ is injective,

$$\begin{aligned} \|(\mathbf{1}+B)v\| &\geq |\,\mathrm{Re}\langle(\mathbf{1}+B)v, v\rangle| \\ &\geq \|v\|^2\,. \end{aligned}$$

Since the adjoint of B is injective by the same argument, it follows that the range of $\mathbf{1} + B$ is dense.

Let $v_0 \in H$ be arbitrary. Choose a sequence $(v_n) = ((\mathbf{1}+B)u_n)$ in the range of $\mathbf{1}+B$ with $\lim v_n = v_0$. Since

$$\begin{aligned} \|u_n - u_m\| &\leq \|(\mathbf{1}+B)(u_n - u_m)\| \\ &= \|v_n - v_m\|\,, \end{aligned}$$

the sequence (u_n) satisfies the Cauchy criterion, and converges to some limit, u_0. Since B is a closed operator, $v_0 = (\mathbf{1}+B)u_0$. We conclude that $\mathbf{1}+B$ is surjective.

By the Lumer-Phillips theorem A.0.3, B generates a strongly continuous semigroup of contractions on a Hilbert space H. The resolvent bound in Eq. (3.2.5) follows from Lemma A.0.2.

□

The above lemma is a consequence of the Lumer-Phillips theorem that is used frequently in the text. Taking into account Lemma 3.2.3 we prove the following:

Lemma 3.2.4. *Let $\bar{L}$ be the operator defined by Eq. (3.2.3), and $\sigma = min\{\alpha, \varepsilon\gamma\}$. Then,*

1. *$\bar{L} + \sigma$ is dissipative, i.e.,*
$$\mathrm{Re}\langle \bar{L}v, v\rangle \leq -\sigma\|v\|^2\,.$$

2. *$\bar{L}$ generates a strongly continuous semigroup of contractions, $e^{t\bar{L}}$. The semigroup*

$e^{t\bar{L}}$ satisfies the estimate

$$\|e^{t\bar{L}}\| \le e^{-\sigma t} \tag{3.2.6}$$

for all $t \ge 0$.

Proof. First we show that $\bar{L} + \sigma$ is dissipative. Since ∂_z is skew-adjoint in $L^2(\mathcal{S}_R)$, and the off-diagonal terms in $\bar{L}$ are skew-adjoint with respect to the inner product from Eq. (3.0.6), we have

$$\begin{aligned}
\operatorname{Re}\langle \bar{L}v, v\rangle &= \operatorname{Re}\int_{\mathcal{S}_R} \big((\Delta_{\mathcal{S}_R} v_1)\bar{v}_1 + f'(0)|v_1|^2 - v_1\bar{v}_2 + \bar{v}_1 v_2 - \gamma|v_2|^2\big) R\, d\theta dz \\
&= \int_{\mathcal{S}_R} \big((\Delta_{\mathcal{S}_R} v_1)\bar{v}_1 - \alpha|v_1|^2 - \gamma|v_2|^2\big) R\, d\theta dz \\
&= -\int_{\mathcal{S}_R} \big(\alpha|v_1|^2 + \gamma|v_2|^2\big) R\, d\theta dz \\
&\le -\min\{\alpha, \varepsilon\gamma\}\|v\|^2.
\end{aligned}$$

In the second inequality we used that the operator $\Delta_{\mathcal{S}_R}$ is negative semi-definite. Since the parameters α, ε and γ are positive, it follows that $\operatorname{Re}\langle \bar{L}v, v\rangle \le -\sigma\|v\|^2$ for all $v \in H^{2,1}$.

It follows directly by Lemma 3.2.3 that $\bar{L}$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^2(\mathcal{S}_R)$. We denote this semigroup by $e^{t\bar{L}}$.

Define $\sigma := \min\{\alpha, \varepsilon\gamma\}$. For $\operatorname{Re}\lambda > -\sigma$ we have

$$\begin{aligned}
\|(\lambda - \bar{L})v\| &\ge \frac{\operatorname{Re}\langle(\lambda - \bar{L})v, v\rangle}{\|v\|} \\
&\ge (\operatorname{Re}\lambda + \sigma)\|v\|.
\end{aligned}$$

Setting $v := (\lambda - \bar{L})^{-1}w$ we obtain that the resolvent operator of $\bar{L}$ is bounded by

$$\|(\lambda - \bar{L})^{-1}\| \le \frac{1}{\operatorname{Re}\lambda + \sigma}$$

for $\operatorname{Re}\lambda > -\sigma$. By (Corollary II.3.6, [12]), the above inequality implies that the semigroup $e^{t\bar{L}}$ satisfies the estimate

$$\|e^{t\bar{L}}\| \le e^{-\sigma t}$$

for all $t \ge 0$, as desired.

□

Studying further the operator $\bar{L}$, in the Fourier representation it is given by the matrix multiplication operator

$$m(k,n) := \begin{pmatrix} -k^2 - n^2R^{-2} + ick - \alpha & -1 \\ \varepsilon & ick - \varepsilon\gamma \end{pmatrix} \tag{3.2.7}$$

for $k \in \mathbb{R}$, and $n \in \mathbb{N}$. Therefore $\bar{L}$ has only essential spectrum. In particular, the spectrum contains the branch of all eigenvalues of $m(k,0)$, given by

$$\begin{aligned} \lambda_+(k,0) &= ick - \frac{1}{2}(k^2 + \alpha + \varepsilon\gamma) + \frac{1}{2}\sqrt{(k^2 + \alpha - \varepsilon\gamma)^2 - 4\varepsilon} \\ &\sim ick - \varepsilon\gamma \qquad (|k| \to \infty)\,. \end{aligned}$$

An important consequence is that $\bar{L}$ is not sectorial, and $e^{t\bar{L}}$ is not an analytic semigroup.

Combining the above results we are now able to define and estimate the linear semigroup e^{tL}.

Proposition 3.2.5. *The linear operator L generates a strongly continuous semigroup, e^{tL}, satisfying the estimate*

$$\|e^{tL}\| \le e^t \tag{3.2.8}$$

for all $t \ge 0$.

Proof. Consider the linear operators L and $\bar{L}$ with $\mathrm{Dom}(L) = \mathrm{Dom}(\bar{L}) = H^{2,1}$ and so that $L = \bar{L} + V$. Let $\sigma := \min\{\alpha, \varepsilon\gamma\}$. For all λ such that $\mathrm{Re}\,\lambda > -\sigma$ we express $\lambda - L$ as

$$\begin{aligned} \lambda - L &= \lambda - \bar{L} - V \\ &= (\mathbf{1} - V(\lambda - \bar{L})^{-1})(\lambda - \bar{L}). \end{aligned} \tag{3.2.9}$$

We claim that $\mathbf{1} - V(\lambda - \bar{L})^{-1}$ is invertible. Indeed, for $\mathrm{Re}\,\lambda > -\sigma + \|V\|$ and since $\|(\lambda - \bar{L})^{-1}\| \le \frac{1}{\mathrm{Re}\,\lambda + \sigma}$ by Lemma 3.2.4, the bounded linear operator $V(\lambda - \bar{L})^{-1}$ satisfies

$$\|V(\lambda - \bar{L})^{-1}\| \le \frac{\|V\|}{\mathrm{Re}\,\lambda + \sigma} < 1.$$

Thus, $\mathbf{1} - V(\lambda - \bar{L})^{-1}$ is invertible. For all λ in the resolvent set of L, we have by Eq. (3.2.9) that

$$\begin{aligned} (\lambda - L)^{-1} &= (\lambda - \bar{L})^{-1}(\mathbf{1} - V(\lambda - \bar{L})^{-1})^{-1} \\ &= (\lambda - \bar{L})^{-1}\sum_{j=0}^{\infty}(V(\lambda - \bar{L})^{-1})^j. \end{aligned}$$

Next we estimate the resolvent operator $(\lambda - L)^{-1}$:

$$\begin{aligned}\|(\lambda - L)^{-1}\| &\leq \frac{1}{\operatorname{Re}\lambda + \sigma} \cdot \frac{1}{1 - \frac{\|V\|}{\operatorname{Re}\lambda+\sigma}} \\ &= \frac{1}{\operatorname{Re}\lambda + \sigma - \|V\|}\end{aligned}$$

for $\operatorname{Re}\lambda > -\sigma + \|V\|$. Therefore, the operator L generates a strongly continuous semigroup of contractions, e^{tL}, for $t \geq 0$, satisfying

$$\|e^{tL}\| \leq e^{(-\sigma+\|V\|)t} \tag{3.2.10}$$

(see Corollary II.3.6 [12]). Observe that

$$\|V\| = \|f'(y) - f'(0)\| = \| - 3y^2 + 2(\alpha+1)y\| < 1 \tag{3.2.11}$$

for $y \in \mathbb{R}$ and $0 < \alpha < \frac{1}{2}$. The desired estimate follows by applying Eq. (3.2.11) to Eq. (3.2.10).

□

3.2.2 Construction of the projection

Consider the linear operator L, given by Eq. (3.0.4), and its adjoint L^*. In the discussion at the beginning of Subsection 3.2.1 we verified that zero is an eigenvalue of L. In this subsection we explore in more depth the spectrum of L. In particular, we prove the following

1. $\operatorname{spec}(L) \subset \{0\} \cup \left\{\lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda < -\sigma\right\}$ for some $\sigma > 0$;

2. 0 is a simple eigenvalue of L and L^*.

These two results will be used to construct the spectral projection Q that appears in Proposition 3.2.1.

We start our analysis with the spectrum of L. By definition, the *spectrum* consists of two sets: the *essential spectrum* and the *discrete spectrum*. More specifically, we have:

Lemma 3.2.6 (Essential spectrum of L)**.** *Define $\sigma = \min\{\alpha, \varepsilon\gamma\}$. The operator L given by Eq.* (3.0.4) *satisfies*

$$\operatorname{spec}_{ess}(L) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\,\lambda \leq -\sigma\}\,. \tag{3.2.12}$$

Proof. By Lemma 3.2.2, L is a relatively compact perturbation of the operator $\bar{L}$, where recall $\bar{L}$ is defined in Eq. (3.2.3). From Weyl's essential spectrum theorem (Theorem 2.2.6, [32]) it holds that

$$\mathrm{spec}_{ess}(L) = \mathrm{spec}_{ess}(\bar{L})\,.$$

In Lemma 3.2.4 we proved that $\sigma + \bar{L}$ is dissipative. By Lemma 3.2.3 the spectrum of $\sigma + \bar{L}$ lies in the left half plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq 0\}$ and so the relation in Eq. (3.2.12) follows.

□

Next, we determine the discrete spectrum of L. To this end, we expand functions on the surface of $\mathcal{S}_R$ as Fourier series in the angular variable, θ:

$$v(z,\theta) = \sum_{n\in\mathbb{Z}} v_n(z)e^{in\theta}\,.$$

As in Eq. (3.2.7), L splits into a direct sum $L = \bigoplus_{n\geq 0} L_n$, where

$$L_n := \begin{pmatrix} \partial_z^2 - n^2R^{-2} + c\partial_z + f'(\phi_1(z)) & -1 \\ \varepsilon & c\partial_z - \varepsilon\gamma \end{pmatrix} \tag{3.2.13}$$

is the restriction of L to the invariant subspace corresponding to the Fourier modes $n \geq 0$. We study separately the zero mode from the positive modes. The case $n = 0$ is more delicate but it coincides with the linear operator L in the one dimension defined in Eq. (2.1.5). So, we are using the existing results from Subsection 2.1.2.

Starting with $n > 0$ we have:

Lemma 3.2.7 (Discrete spectrum, $n > 0$)**.** *Consider the operator L_n in Eq. (3.2.13). On $\mathcal{S}_R$, with $0 < R \leq 1$:*

$$\mathrm{spec}_{disc}(L_n) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\,\lambda \leq -\sigma\}\,, \tag{3.2.14}$$

where $n > 0$ and $\sigma := \min\{\alpha, \varepsilon\gamma\}$.

Proof. If we show that $L_n + \sigma$ is dissipative, i.e., $\operatorname{Re}\langle (L_n + \sigma)v, v\rangle \leq 0$ for $v \in \mathrm{Dom}(L_n)$, then Eq. (3.2.14) follows by Lemma 3.2.3. Indeed,

$$\begin{aligned}\operatorname{Re}\langle L_n v, v\rangle &= \operatorname{Re}\int_{\mathbb{R}} \big\{(\partial_z^2 v_1)\bar{v}_1 + \big(-n^2R^{-2} + f'(\phi_1(z))\big)|v_1|^2 - v_1\bar{v}_2 + \bar{v}_1 v_2 - \gamma|v_2|^2\big\} R\,dz \\ &= \int_{\mathbb{R}} \big\{(\partial_z^2 v_1)\bar{v}_1 + \big(-n^2R^{-2} + f'(\phi_1(z))\big)|v_1|^2 - \gamma|v_2|^2\big\} R\,dz \\ &= \int_{\mathbb{R}} \big\{\big(-n^2R^{-2} + f'(\phi_1(z))\big)|v_1|^2 - \gamma|v_2|^2\big\} R\,dz \\ &\leq -\sigma\|v\|^2.\end{aligned}$$

To obtain the last inequality we used that $n > 0$, $R \leq 1$ and $f'(\phi_1(z)) - f'(0) \leq 1$, where recall $f'(0) = -\alpha$.

□

Lemmas 3.2.7 and 3.2.3 imply the following resolvent estimate:

Corollary 3.2.8 (Resolvent estimate, $n > 0$)**.** *For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\sigma$, and $\sigma := \min\{\alpha, \varepsilon\gamma\}$, the resolvent operator of L_n, for $n > 0$, satisfies*

$$\big\|(\lambda - L_n)^{-1}\big\| \leq \frac{1}{\operatorname{Re}\lambda + \sigma}\,. \tag{3.2.15}$$

Turning to the zero mode we have:

Lemma 3.2.9 (Discrete spectrum, $n = 0$)**.** *Let L_0 be given by Eq. (3.2.13) with $n = 0$. If $\varepsilon > 0$ is sufficiently small, then there exists $\beta > 0$ such that*

$$\operatorname{spec}_{disc}(L_0) \subset \{0\} \cup \big\{\lambda \in \mathbb{C} \;\big|\; \operatorname{Re}\lambda \leq -\beta\big\}\,. \tag{3.2.16}$$

Moreover, the eigenvalue at 0 is simple.

Proof. Let $\mathcal{M}$ be the manifold of stationary solutions defined in Eq. (1.2.6). Denote by $\tau = -\partial_z\Phi$ the tangent vector on $\mathcal{M}$ at a fixed point Φ. By the translation invariance $L\tau = 0$. This implies that 0 is an eigenvalue of L_0 with corresponding eigenfunction τ. In what follows we show that the eigenvalue 0 is simple, and that there are no other eigenvalues to the right of $\{\operatorname{Re}\lambda = -\beta,\ \beta > 0\}$.

The operator L_0 agrees with the linearization of the FHN system in the one spatial dimension defined in Eq. (2.1.5). Jones [30] and Yanagida [49] studied extensively the spectrum of the latter linearization in the space of bounded continuous functions. As discussed in Subsection 2.1.2, they proved that for $\varepsilon > 0$ sufficiently small, 0 is a simple eigenvalue of the linearization, and all other eigenvalues lie in a half-space $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < -\beta\}$.

We claim that the discrete spectrum of L_0 on the Hilbert space $L^2(\mathbb{R})$ is contained in its discrete spectrum on the space of bounded continuous functions. Indeed, any generalized eigenfunction must lie in the domain of L_0, given by the mixed Sobolev space $H^{2,1}$ (we prove this result below in Lemma 3.2.16). In particular, the generalized eigenfunctions are bounded and continuous. Therefore, the results of Jones [30] and Yanagida [49] also apply to L_0 on $L^2(\mathbb{R})$. $\square$

The following proposition gives the spectrum of the linear operator L.

Proposition 3.2.10 (Spectrum of L)**.** *Assume $\varepsilon > 0$ sufficiently small, and $\beta > 0$ (depending on ε) is the constant determined by Lemma 3.2.9. Define $\sigma := \min\{\alpha, \beta, \varepsilon\gamma\}$ and consider the linear operator L, given in Eq. (3.0.4). Then the spectrum of L is contained in*

$$\{0\} \cup \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq -\sigma\}\,. \tag{3.2.17}$$

Moreover, 0 is a simple eigenvalue of both L and its adjoint, L^*.

Proof. Eq. (3.2.17) follows directly by combining Lemmas 3.2.6, 3.2.7 and 3.2.9. It remains to verify the second argument,i.e., that the eigenvalue 0 is simple. Indeed, the linear operator $\bar{L}$, defined in Eq. (3.2.3), is Fredholm. By Lemma 3.2.4, 0 lies in its resolvent set, so the Fredholm index of $\bar{L}$ is zero. Since L is a relatively compact perturbation of $\bar{L}$, by Lemma 3.2.2, L is a Fredholm operator of the same index as $\bar{L}$, namely of index zero. Therefore, 0 is a simple eigenvalue of both L and L^*.

$\square$

Since the eigenvalue at 0 is isolated, the spectral projection to the zero eigenspace is defined by the line integral

$$P = \frac{1}{2\pi i}\oint_{\Gamma_0} (\lambda - L)^{-1} d\lambda, \tag{3.2.18}$$

where $\Gamma_0 \subset \mathbb{C}$ is any simple closed positively oriented curve that separates zero from the remainder of the spectrum of L (Subsection 2.2.4, [32]). By definition, P is a bounded linear operator that commutes with the linearization L, i.e., $PL = LP$, as well as with the generator e^{tL}. The complementary projection of P is defined as $Q := \mathbf{1} - P$, and is the projection onto the range of L. The projection Q also commutes with L and its semigroup e^{tL}.

Lemma 3.2.11 (Spectral projection)**.** *Let $\tau = -\partial_z\Phi$ be the tangent vector on the manifold $\mathcal{M}$ at a fixed point Φ, and τ^* the eigenfunction of the adjoin L^* corresponding to the* *zero eigenvalue, normalized to* $\langle \tau, \tau^* \rangle = 1$. *Then the projection in Eq.* (3.2.18) *is given by*

$$Pv = \langle v, \tau^* \rangle \tau \tag{3.2.19}$$

for $v \in L^2$.

Proof. The eigenfunction τ^* is well-defined because 0 is a simple eigenvalue for both L and L^*, by Proposition 3.2.10. In particular, since τ does not lie in the range of L, τ^* is not orthogonal to τ. The Riesz projection P is uniquely determined by its action on the nullspace and range of L, that is, the following two properties hold:

$$P\tau = \tau \quad \text{and} \quad PLv = 0$$

for all $v \in H^{2,1}$. We verify that $\langle \tau, \tau^* \rangle \tau = \tau$, and $\langle Lv, \tau^* \rangle \tau = \langle v, L^* \tau^* \rangle \tau = 0$, as required. □

We are now able to estimate the resolvent operator of L_0.

Lemma 3.2.12 (Resolvent estimate, $n = 0$). *Let* P *be the projection to the nullspace of* L *and* $Q = \mathbf{1} - P$ *be the complementary projection. Denote by* Q_0 *the restriction of* Q *to the range of* L_0 *in* L^2. *For every* $\sigma > \min\{\alpha, \beta, \varepsilon\gamma\}$, *there exists a positive constant* C *such that*

$$\|(\lambda - L_0)^{-1} Q_0\| \leq C \tag{3.2.20}$$

for $\mathrm{Re}\, \lambda \geq -\sigma$. *Here,* β *is as in Lemma 3.2.9.*

The proof of the above lemma is deferred to the next subsection.

3.2.3 Proof of Lemma 3.2.12

The idea of the proof is to estimate the resolvent operators of the linearization L_0, given by Eq. (3.2.13) with $n = 0$, in the three regions:

$$\begin{aligned} S_1 &:= \{\lambda \in \mathbb{C} \mid -\sigma \leq \mathrm{Re}\, \lambda \leq 2, \, |\mathrm{Im}\, \lambda| \leq N\}\,, \\ S_2 &:= \{\lambda \in \mathbb{C} \mid \mathrm{Re}\, \lambda \geq 2\}, \\ S_3 &:= \{\lambda \in \mathbb{C} \mid -\sigma \leq \mathrm{Re}\, \lambda \leq 2, \, |\mathrm{Im}\, \lambda| \geq N\}\,. \end{aligned} \tag{3.2.21}$$

Here, the constant σ satisfies: $\sigma < \min\{\alpha, \beta, \varepsilon\gamma\}$, and β is as in Lemma 3.2.9. The positive constant N will be specified below. Yanagida had also slit the complex plane into three regions (see Eq. (2.1.8)), however our approach is slightly different.

On the first region, S_1, we appeal to compactness.

Lemma 3.2.13 (Resolvent estimate on S_1)**.** *Let Q_0 be the restriction of $Q = \mathbf{1} - P$ to the range of L_0 in L^2. For any $N > 0$,*

$$\sup_{\lambda \in S_1} \|(\lambda - L_0)^{-1} Q_0\| < \infty \,.$$

Proof. By Lemma 3.2.9, the region S_1 intersects the spectrum of L_0 only at the simple eigenvalue 0. Since the restriction of L_0 to the range of the projection Q_0 has no spectrum in S_1, its resolvent is an analytic function of λ, and hence bounded on the compact set S_1.

□

The second region, i.e., the half-plane S_2, is treated by a dissipativity estimate similar to that in the proof of Lemma 3.2.7.

Lemma 3.2.14 (Resolvent estimate on S_2)**.** *For any $N > 0$,*

$$\sup_{\lambda \in S_2} \left\|(\lambda - L_0)^{-1}\right\| \leq 1 \,. \tag{3.2.22}$$

Proof. Setting $n = 0$ at the estimate in the proof of Lemma 3.2.7 we obtain

$$\mathrm{Re}\,\langle L_0 v, v\rangle \leq -\sigma \|v\|^2 \,,$$

for $v \in \mathrm{Dom}(L_0)$ and $\sigma := \min\{\alpha, \varepsilon\gamma\}$. By Lemma 3.2.3, $\lambda - L_0$ is invertible for $\mathrm{Re}\,\lambda > -\sigma$ and the inverse satisfies

$$\|(\lambda - L_0)^{-1}\| \leq \frac{1}{\mathrm{Re}\ \lambda + \sigma} \,, \tag{3.2.23}$$

for $\mathrm{Re}\ \lambda > -\sigma$. Since $\mathrm{Re}\,\lambda \geq 2$ on S_2, Eq. (3.2.23) implies Eq. (3.2.22).

□

The third region, S_3, requires more work to do. We prove the following explicit estimate.

Lemma 3.2.15 (Resolvent estimate on S_3)**.** *Assume $N > 0$ is sufficiently large. Then*

$$\sup_{\lambda \in S_3} \left\|(\lambda - L_0)^{-1}\right\| \leq \frac{2}{\min\{\alpha, \varepsilon\gamma\} - \sigma} \,. \tag{3.2.24}$$

Here, $\sigma < \min\{\alpha, \beta, \varepsilon\gamma\}$, and β is as in Lemma 3.2.9.

Proof. We write the linear operator L_0 as: $L_0 = \bar{L}_0 + V$, where

$$\bar{L}_0 := \begin{pmatrix} \partial_z^2 + c\partial_z + f'(0) & -1 \\ \varepsilon & c\partial_z - \varepsilon\gamma \end{pmatrix}$$

and V is the bounded linear operator given by the difference $L_0 - \bar{L}_0$. In the same way as in the proof of Lemma 3.2.4, we show that

$$\mathrm{Re}\langle \bar{L}_0 v, v\rangle \leq -\min\{\alpha, \varepsilon\gamma\}\|v\|^2$$

for $v \in L^2$. By Lemma 3.2.4, the resolvent set of $\bar{L}_0$ contains the half-plane $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq -\min\{\alpha, \varepsilon\gamma\}\}$ and the resolvent operator of $\bar{L}_0$ satisfies

$$\|(\lambda - L_0)^{-1}\| \leq \frac{1}{\mathrm{Re}\ \lambda + \min\{\alpha, \varepsilon\gamma\}}\,. \tag{3.2.25}$$

By the resolvent identity

$$(\lambda - \bar{L}_0)^{-1} - (\lambda - L_0)^{-1} = -(\lambda - \bar{L}_0)^{-1}V(\lambda - L_0)^{-1}\,. \tag{3.2.26}$$

To solve Eq. (3.2.26) with respect to $(\lambda - L_0)^{-1}$ we should verify first that $\mathbf{1} - (\lambda - \bar{L}_0)^{-1}V$ is invertible. In particular, we need to show that there exists an $N > 0$ such that

$$\sup_{\lambda \in S_3} \left\|(\lambda - \bar{L}_0)^{-1}V\right\| \leq \frac{1}{2}. \tag{3.2.27}$$

Assuming the above argument is true we solve Eq. (3.2.26) for the resolvent of L_0 to obtain

$$(\lambda - L_0)^{-1} = \left(\mathbf{1} - (\lambda - \bar{L}_0)^{-1}V\right)^{-1}(\lambda - \bar{L}_0)^{-1}. \tag{3.2.28}$$

Applying Eq. (3.2.27) into (3.2.28) we estimate

$$\begin{aligned} \|(\lambda - L_0)^{-1}\| &\leq \left(\|\mathbf{1} - (\lambda - \bar{L}_0)^{-1}V\|\right)^{-1} \|(\lambda - \bar{L}_0)^{-1}\| \\ &\leq 2\,\|(\lambda - \bar{L}_0)^{-1}\| \\ &\leq \frac{2}{\mathrm{Re}\ \lambda + \min\{\alpha, \varepsilon\gamma\}}\,, \end{aligned} \tag{3.2.29}$$

where the last line follows from Eq. (3.2.25). Recall that on the region S_3, $\mathrm{Re}\,\lambda \geq -\sigma$.

This estimate together with Eq. (3.2.29) imply

$$\|(\lambda - L_0)^{-1}\| \leq \frac{2}{-\sigma + \min\{\alpha, \varepsilon\gamma\}}\,,$$

where $\sigma < \min\{\alpha, \beta, \varepsilon\gamma\}$, as desired.

To complete the proof it suffices to verify Eq. (3.2.27). Indeed, pick $\lambda \in S_3$. Using that the $(1,2)$-entry and the $(2,2)$-entry of the operator matrix $(\lambda - \bar{L}_0)^{-1}V$ vanish and that

$$\left\| \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} \right\| \leq \|a_{11}\| + \|a_{21}\|,$$

we find

$$\|(\lambda - \bar{L}_0)^{-1}V\| \leq \left(\|((\lambda - \bar{L}_0)^{-1})_{11}\| + \|((\lambda - \bar{L}_0)^{-1})_{21}\|\right) \sup_{y\in\mathbb{R}} |f'(y) - f'(0)|\,.$$

Since the last term is bounded by Eq. (3.2.37), to verify Eq. (3.2.27) we only need to prove that

$$\lim_{N\to\infty} \sup_{\lambda\in S_3} \left\|((\lambda - \bar{L}_0)^{-1})_{i1}\right\| = 0\,, \quad i = 1, 2\,. \tag{3.2.30}$$

Moreover, since $\bar{L}_0$ has real coefficients, we may restrict the supremum over the intersection of S_3 with the upper half-plane.

The operator $\bar{L}_0$, in the Fourier representation is given by the matrix multiplication operator (as in Eq. (3.2.7))

$$m(k, 0) = \begin{pmatrix} -k^2 + ick + f'(0) & -1 \\ \varepsilon & ick - \varepsilon\gamma \end{pmatrix}$$

for $k \in \mathbb{R}$. In particular,

$$\left\|((\lambda - \bar{L}_0)^{-1})_{ij}\right\| = \sup_{k\in\mathbb{R}} \left|((\lambda - m(k,0))^{-1})_{ij}\right|. \tag{3.2.31}$$

Suppressing the dependence on k in the notation and using Cramer's rule, for the $(1,1)$-entry we have

$$((\lambda - m)^{-1})_{11} = \frac{\lambda - m_{22}}{\det(\lambda - m)}$$

and for the $(2,1)$-entry:

$$((\lambda - m)^{-1})_{21} = \frac{m_{21}}{\det(\lambda - m)}\,.$$

Passing to reciprocals, we compute for the first entry

$$\frac{1}{\big((\lambda - m)^{-1}\big)_{11}} = \lambda - m_{11} - \frac{m_{12}m_{21}}{\lambda - m_{22}} .$$

We next separate the real and imaginary parts. Since $\operatorname{Re}\lambda \geq -\sigma$ on S_3, and $\sigma \leq \min\{\alpha, \varepsilon\gamma\}$, we have that

$$\operatorname{Re}(\lambda - m_{11}) \geq k^2 , \tag{3.2.32}$$

and

$$\operatorname{Re}(\lambda - m_{22}) \geq \varepsilon\gamma - \sigma > 0 . \tag{3.2.33}$$

Since $m_{12}m_{21} = -\varepsilon < 0$, it follows that

$$\operatorname{Re}\frac{1}{((\lambda - m)^{-1})_{11}} \geq k^2 .$$

For the imaginary part, we have

$$\operatorname{Im}\frac{1}{((\lambda - m)^{-1})_{11}} \geq \operatorname{Im}\lambda - ck + \varepsilon|\operatorname{Im}\lambda - ck|^{-1} .$$

Combining the estimates of the real and imaginary parts, and assuming that $\operatorname{Im}\lambda \geq N$, we obtain

$$\begin{aligned}\frac{1}{|((\lambda - m)^{-1})_{11}|} &\geq \max\big\{k^2, |\operatorname{Im}\lambda - ck| + \varepsilon|\operatorname{Im}\lambda - ck|^{-1}\big\}\\ &\geq \max\left\{\tfrac{N^2}{4c^2}, \tfrac{N}{2} + \tfrac{2\varepsilon}{N}\right\}\\ &\to \infty \quad \text{as} \quad N \to \infty .\end{aligned}$$

The second inequality above holds since $k \geq \frac{N}{2c}$ whenever $|\operatorname{Im}\lambda - ck| \leq \frac{N}{2}$. This implies Eq. (3.2.30) for $i = 1$. Similarly,

$$\frac{1}{((\lambda - m)^{-1})_{21}} = \frac{\big(\lambda - m_{11}\big)\big(\lambda - m_{22}\big)}{m_{21}} - m_{12} .$$

As before, we separately estimate the real and imaginary parts of each of the factors in the numerator:

$$|\lambda - m_{11}| \geq \max\big\{k^2, |\operatorname{Im}\lambda - ck|\big\}$$

and

$$|\lambda - m_{22}| \geq \varepsilon\gamma - \sigma .$$

The two estimates yield

$$\begin{aligned}\frac{1}{|((\lambda - m)^{-1})_{21}|} &\geq \frac{\varepsilon\gamma - \sigma}{\varepsilon} \max\left\{k^2, |\operatorname{Im}\lambda - ck|\right\} - 1 \\ &\geq \frac{\varepsilon\gamma - \sigma}{\varepsilon} \max\left\{\tfrac{N^2}{4c^2}, \tfrac{N}{2}\right\} - 1 \\ &\to \infty \text{ as } N \to \infty\,.\end{aligned}$$

This implies Eq. (3.2.30) for $i = 2$.

□

Combining Lemmas 3.2.13 - 3.2.15, the proof of Lemma 3.2.12 follows.

3.2.4 Proof of Proposition 3.2.1

Consider the projection, P, constructed in Lemma 3.2.11, and let $Q = \mathbf{1} - P$ be the complementary projection to the range of L. Choose

$$\sigma < \min\{\alpha, \beta, \varepsilon\gamma\}\,,$$

where β is the constant from Lemma 3.2.9. We need to find a constant $M > 0$ such that the estimate

$$\left\|e^{tL}Q\right\|_{2,1} \leq Me^{-\sigma t} \tag{3.2.34}$$

holds for all $t > 0$.

First we verify Eq. (3.2.34) with the norm $\|\cdot\|_{2,1}$ replaced by $\|\cdot\|$. This follows from Prüss' theorem C.0.1 if we show that the resolvent operator of L projected onto the range of L, is uniformly bounded on the half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq -\sigma\}$. Then we use an equivalence relation of those two norms to verify Eq. (3.2.34).

Indeed, splitting L into the discrete sum $L = \bigoplus_{n\geq 0} L_n$, where recall that L_n are the Fourier modes of L given by Eq. (3.2.13), we obtain

$$\begin{aligned}\left\|(\lambda - L)^{-1}Q\right\| &= \left\| \oplus_{n\geq 0} (\lambda - L_n)^{-1}Q_n\right\| \\ &\leq \sup_{n\geq 0}\left\{\left\|(\lambda - L_n)^{-1}Q_n\right\|\right\}\,.\end{aligned}$$

Here, Q_n is the corresponding projection to the range of L_n, for each $n \geq 0$. Corollary 3.2.8 and Lemma 3.2.12 imply that there exists a constant $C > 0$ such that

$$\|(\lambda - L)^{-1}Q\| \leq C$$

for Re $\lambda \geq -\sigma$. Since Q commutes with L, this establishes the hypotheses of Prüss' theorem on the range of L.

Applying Prüss' theorem C.0.1 with B given by the restriction of L to the range of Q we obtain that

$$\|e^{tL}Q\| \leq Me^{-\sigma t} \tag{3.2.35}$$

for some constant $M > 0$.

To reduce Eq. (3.2.35) from the space L^2 to $H^{2,1}$ we prove in the following lemma that the $H^{2,1}$-norm is equivalent to the graph norm of L. The notation $\lesssim$ and $\gtrsim$ stand for the respective inequalities up to constants. The constants depend on the parameters α, ε and γ but not on the solution u or the initial value u_0.

Lemma 3.2.16. *The graph norm of the operator L, with* $\mathrm{Dom}(L) = H^{2,1}$, *satisfies*

$$\|u\|_{2,1} \lesssim \|Lu\| + \|u\| \lesssim \|u\|_{2,1} \tag{3.2.36}$$

for $u \in H^{2,1}$.

Proof. We use that

$$\|\partial_z u_1\| \leq \tfrac{1}{2c}\|\Delta_{\mathcal{S}_R} u_1\| + \tfrac{c}{2}\|u_1\|\,,$$

and set

$$b := \sup_{z\in\mathbb{R}} |f'(\phi_1(z)) - f'(0)|\,. \tag{3.2.37}$$

By the reverse triangle inequality, this yields for the lower bound

$$\|Lu\| \geq \min\left\{\tfrac{1}{2}, c\right\} \|(\Delta_{cS_R} u_1, \partial_z u_2)\| - (\tfrac{c^2}{2} + b)\|u_1\|\,,$$

which implies that

$$\begin{aligned}\left(1 + \tfrac{c^2}{2} + b\right)(\|Lu\| + \|u\|) &\geq \|Lu\| + \left(1 + \tfrac{c^2}{2} + b\right)\|u\| \\ &\geq \min\left\{\tfrac{1}{2}, c\right\}\|u\|_{2,1}\,.\end{aligned}$$

For the upper bound, the triangle inequality yields

$$\|Lu\| + \|u\| \leq \max\left\{\tfrac{3}{2}, c, 1 + \tfrac{c^2}{2} + b\right\}\|u\|_{2,1}\,.$$

□

Since L commutes with e^{tL}, it follows that

$$\begin{aligned}\|e^{tL}Qu\|_{2,1} &\lesssim \|Le^{tL}Qu\| + \|e^{tL}Qu\| \\ &\leq Me^{-\sigma t}\big(\|Lu\| + \|u\|\big) \\ &\leq Me^{-\sigma t}\|u\|_{2,1}\end{aligned}$$

for $u \in H^{2,1}$. Therefore, Eq. (3.2.34) follows by Lemma 3.2.16, and this completes the proof of Proposition 3.2.1.

□

An immediate consequence of Proposition 3.2.1 is that the semigroup generated by L is uniformly bounded.

Corollary 3.2.17. *Under the assumptions of Proposition 3.2.1, there is a constant $C > 0$ such that*

$$\|e^{tL}\|_{2,1} \leq C$$

for all $t \geq 0$.

Proof. By the triangle inequality,

$$\begin{aligned}\|e^{tL}\|_{2,1} &\leq \|e^{tL}P\|_{2,1} + \|e^{tL}Q\|_{2,1} \\ &\leq \|P\|_{2,1} + Me^{-\sigma t}\end{aligned}$$

for $t \geq 0$, where we have used that e^{tL} is constant on the nullspace of L to bound the first summand, and applied Proposition 3.2.1 to the second. This proves the claim with $C = \|P\|_{2,1} + M$. □

3.3 Nonlinear Stability

In this section we return to the nonlinear system on $\mathcal{S}_R$ in the moving frame, and prove Theorem 1.2.1. Let $G(u)$ denote the right hand side of Eq. (2.3.1). It is defined explicitly in Eq. (3.0.2). By Proposition 3.1.1, the initial value problem (3.1.1) is locally well-posed in the class of mild solutions on $H^{2,1}$.

3.3.1 Modulated pulses

Let $\mathcal{M}$ be the manifold of pulses. We decompose solutions near $\mathcal{M}$ into a nonlinear projection onto $\mathcal{M}$ that evolves slowly in time, and a fluctuation that satisfies a suitable orthogonality condition.

Let P be the projection to the tangent space of $\mathcal{M}$ at Φ from Lemma 3.2.11. By translation invariance,

$$P_h v := \langle v, \tau_h^* \rangle \, \tau_h \,, \quad v \in L^2$$

defines the corresponding projection to the tangent space of $\mathcal{M}$ at the translated pulse Φ_h. This is the spectral projection associated with the zero eigenspace of $L_h := dG(\Phi_h)$.

Proposition 3.3.1. *Under the assumptions of Proposition 3.2.1:*

1. (Projection onto $\mathcal{M}$.) *There exists a tubular neighborhood $\mathcal{W}$ of $\mathcal{M}$ in $H^{2,1}$ such that every $u \in \mathcal{W}$ has a unique decomposition as*

$$u = \Phi_h + v \quad \text{with } P_h v = 0\,. \tag{3.3.1}$$

2. (Local projection near Φ.) *There exists a neighborhood $\mathcal{U}$ of Φ in $H^{2,1}$ such that every $u \in \mathcal{U}$ has a unique decomposition*

$$u = \Phi_h + v \quad \text{with } Pv = 0\,. \tag{3.3.2}$$

In both cases, h and v are smooth functions of u.

In the proof, we show that each of Eq. (3.3.1) and Eq. (3.3.2) defines a pair of complementary non-linear projections $\mathcal{P} : u \mapsto \Phi_{h(u)}$ (onto $\mathcal{M}$) and and $\mathcal{Q} : u \mapsto v$ (onto a transversal subspace), with

$$d\mathcal{P}\Big|_{u=\Phi} = P\,, \qquad d\mathcal{Q}\Big|_{u=\Phi} = Q\,.$$

In fact, Eq. (3.3.2) defines a diffeomorphism $u \mapsto (h, v)$ from $\mathcal{U}$ onto a neighborhood of the origin in $\mathbb{R} \times \mathrm{Ran}\,(Q)$. The proof relies on the Implicit Function Theorem. The following lemma provides the requisite smoothness.

Lemma 3.3.2 (Smooth dependence on h)**.** *The manifold of pulses $\mathcal{M}$ is a smooth simple curve in $H^{2,1}$. Moreover, the tangent vector τ_h, the dual vector τ_h^*, the projections P_h, Q_h, the linearization $L_h = dG(\Phi_h)$, and the nonlinearity $N_h(v) := G(\Phi_h + v) - L_h v$ depend smoothly in $H^{2,1}$ on h, with bounded derivatives of all orders.*

Proof. The smoothness of $\mathcal{M}$ follows from the smoothness and decay of Φ and its derivatives. This also proves the smoothness of τ_h, τ_h^*, and the projections. The linearization L_h is a matrix-valued differential operator whose coefficients are smooth functions of Φ_h; the linearity $N_h(v) = G(\Phi_h + v) - L_h(\Phi_h)v$ is a cubic polynomial in v_1 whose coefficients are smooth functions of Φ_h. Since $H^2(\mathcal{S}_R)$ is a Banach algebra, $G(\Phi_h + v)$, $L_h v$, and $N_h(v)$ all depend smoothly on h.

□

Proof of Proposition 3.3.1. (i). Given u near $\mathcal{M}$, we need to find $h \in \mathbb{R}$ such that $P_h(u - \Phi_h) = 0$. Choose $h_0 \in \mathbb{R}$ with $\|u - \Phi_{h_0}\| = \operatorname{dist}(u, \mathcal{M})$. By applying the translation τ_{-h_0} to a neighborhod of u, we may assume that $h_0 = 0$. Thus we need to solve

$$\mathcal{H}(u, h) := \langle u - \Phi_h, \tau_h^* \rangle = 0$$

near $(\Phi, 0)$. Clearly, $\mathcal{H}(\Phi, 0) = 0$. Moreover, since Φ_h and τ_h^* are smooth in h, the map $\mathcal{H}$ is continuously differentiable in $h \in \mathbb{R}$ and $u \in L^2$. Since $\partial_z \Phi = -\tau$,

$$\partial_h \mathcal{H}(u, h)\Big|_{u=\Phi} = \langle \tau, \tau^* \rangle = 1.$$

By the Implicit Function Theorem, there is a unique solution $h = h(u)$ in a neighborhood $\mathcal{U}$ of Φ, which is continuously differentiable in u and satisfies $h(0) = 0$. Since Φ is smooth, also $v(u) = u - \Phi_{h(u)}$ is smooth. The tubular neighborhood $\mathcal{W}$ is the union of all translates of $\mathcal{U}$.

(ii) Apply the Implicit Function Theorem to $\mathcal{H}(u, h) := \langle u - \Phi_h, \tau^* \rangle$.

□

Fix a pulse $\Phi \in \mathcal{M}$, and let $\mathcal{U}$ be the neighborhood constructed in the second part of Proposition 3.3.1. Consider a mild solution u of Eq. (2.3.1) on $\mathcal{U}$. By Eq. (3.3.2), we can represent it uniquely as the superposition of a modulated pulse $\Phi_{h(t)}$ and a transversal fluctuation $v(t)$

$$u(t) = \Phi_{h(t)} + v(t) \quad \text{with} \quad Pv(t) = 0\,. \tag{3.3.3}$$

Since Φ_h is a stationary solution of Eq. (1.1.3), we have $G(\Phi_h) = 0$. Its Taylor expansion about Φ_h is given by $G(\Phi_h + v) = L_h v + N_h(v)$, where $L_h = dG(\Phi_h)$ is as in Eq. (3.0.4) but with Φ_h in place of Φ, and

$$N_h(v) = \begin{pmatrix} v_1^2(\alpha + 1 - 3(\phi_h)_1 - v_1) \\ 0 \end{pmatrix}. \tag{3.3.4}$$

Note that N_h differs from the nonlinearity N in Eq. (3.0.5) by a bounded multiplication operator that decays as $z \to \pm\infty$.

Assume for the moment that u is a classical solution of Eq. (2.3.1). Substituting

Eq. (3.3.3) into Eq. (3.1.1) and using that $\partial_t \Phi_h(z) = \dot{h}\tau_h$, we obtain

$$\dot{h}(t)\tau_h + \partial_t v = G(\Phi_{h(t)} + v) = L_h v + N_h(v)\,.$$

We next apply the spectral projections P and Q. Since $P\partial_t v = 0$, we have by the chain rule

$$\langle \tau_h, \tau^* \rangle \dot{h} = \langle L_h v + N_h(v), \tau^* \rangle\,. \tag{3.3.5}$$

The complementary projection yields

$$\partial_t v = Q\big(L_h v + N_h(v) - \dot{h}\tau_h\big)\,.$$

In general, if u is a mild solution of Eq. (1.1.3), we interpret v as a mild solution of the equation

$$v(t) = e^{tL}v_0 + \int_0^t e^{(t-s)L} Q\big(L_{h(s)}v(s) + N_{h(s)}(v(s)) - \dot{h}(s)\tau_{h(s)}\big)\,ds\,. \tag{3.3.6}$$

By the same argument as in Eq. (3.1.6), the nonlinearity N_h is locally Lipschitz on $H^{2,1}$. We will need the following refined estimate that takes advantage of the fact that N_h vanishes quadratically at $v = 0$.

Lemma 3.3.3 (Small Lipschitz estimate)**.** *For any $\eta > 0$ there exists a constant $C_\eta > 0$ such that the nonlinearity $N_h(v)$ defined in Eq. (3.3.4) satisfies*

$$\|N_h(v) - N_h(w)\|_{2,1} \le C_\eta \max\big\{\|v_1\|_{H^2}, \|w_1\|_{H^2}\big\}\|v_1 - w_1\|_{H^2} \tag{3.3.7}$$

for all v, w with $\|v_1\|_{H^2}, \|w_1\|_{H^2} \le \eta$ and all $h \in \mathbb{R}$.

Proof. We expand the first component of N_h as

$$(N_h(v))_1 - (N_h(w))_1 = \big((\alpha + 1 - 3(\phi_h)_1)(v_1 + w_1) - (v_1^2 + v_1 w_1 + w_1^2)\big)(v_1 - w_1)\,.$$

Eq. (3.3.7) follows directly from the continuity of the multiplication in H^2 and the fact that $\phi_1 \in H^2$.

□

Lemma 3.3.4 (Evolution inequalities)**.** *With the notation and assumptions of Proposition 3.2.1, suppose $h(t)$ and $v(t)$ satisfy Eqs.* (3.3.5)-(3.3.6) *on some interval* $[0, T]$, *and that*

$$|h(t)| \le \kappa\,, \qquad \|v(t)\|_{2,1} \le \eta$$

for all $0 \le t \le T$*, where* $\kappa > 0$ *is sufficiently small, and* $\eta > 0$*. Then there exists a constant* $C > 0$ *(depending on* κ *and* η*) such that*

$$\big|\dot h\big| \le C\big(|h| + \|v\|_{2,1}\big)\|v\|_{2,1}\,, \qquad (0 \le t \le T)\,, \tag{3.3.8}$$

and

$$\|v(t)\|_{2,1} \le C_0 e^{-\sigma t}\|v_0\|_{2,1} + C\int_0^t e^{-\sigma(t-s)}\big(|h(t)| + \|v(t)\|_{2,1}\big)\|v(t)\|_{2,1}\,ds\,. \tag{3.3.9}$$

Proof. Choose $\kappa > 0$ such that $\langle \tau_h, \tau^*\rangle \ge \frac{1}{2}$ for all h with $|h| \le \kappa$. This is possible because $\langle \tau, \tau^*\rangle = 1$, and $\langle \tau_h, \tau^*\rangle$ depends smoothly on h by Lemma 3.3.2. For $|h| \le \kappa$, Eq. (3.3.5) yields

$$|\dot h| \le 2\big(|\langle L_h v, \tau^*\rangle| + |\langle N_h(v), \tau^*\rangle|\big)\,.$$

Since $L^* v = 0$, the first summand is bounded by

$$|\langle L_h v, \tau^*\rangle| = |\langle v, (L_h^* - L^*)\tau^*\rangle| \le C|h|\,\|v\|_{2,1}$$

for some constant C. By Lemma 3.3.3, the second summand satisfies $\|N_h(v)\|_{2,1} \le C_\eta \|v\|_{2,1}^2$. Combining these two inequalities yields the bound on $|\dot h|$.

For $v(t)$, we separately estimate each term on the right hand side of Eq. (3.3.6). In the integrand, we use that $Lv = 0$ and $\|L_h - L\|_{2,1}$ is of order $|h|$ by Lemma 3.3.2. For the nonlinearity, we use Lemma 3.3.3, and for the last term, we use Eq. (3.3.8). The result is

$$\|v(t)\|_{2,1} \le \|e^{tL}Q\|_{2,1}\|v_0\|_{2,1} + C\int_0^t \|e^{-(t-s)L}Q\|_{2,1}\big(|h(s)| + \|v(s)\|_{2,1}\big)\|v(s)\|_{2,1}\,ds\,.$$

The proof is completed with Proposition 3.2.1. $\square$

We end this subsection with a differential inequality that will be used below.

Lemma 3.3.5 (Estimates on h)**.** *Let* y *be a nonnegative, nondecreasing function on* $[0,t]$*, and let* C, ξ, η *be nonnegative constants. If* h *satisfies the differential inequality*

$$|\dot h| \le C e^{-\xi t} y(|h| + y) \qquad (0 \le t \le T)$$

with $h(0) = 0$, *and* $y(T) \le \eta$, *then*

$$|h(t)| \le C_1 y(t)\,, \quad |\dot{h}(t)| \le C_2 e^{-\xi t} y(t)^2 \qquad (0 \le t \le T)\,,$$

where $C_1 = (e^{\frac{C\eta}{\xi}} - 1)$, $C_2 = Ce^{\frac{C\eta}{\xi}}$.

Proof. Fix $t_0 \in (0, T]$. Since y is non-decreasing, $|\dot{h}(s)| \le Cy(t)(|h(t)| + y(t))$ for all $0 \le s \le t$. We separate variables and integrate from $h(0) = 0$ to obtain

$$\log\Big(\frac{|h(s)| + y(t)}{y(t)}\Big) \le \frac{Cy(t)}{\xi}(1 - e^{-\xi t}) \le \frac{C\eta}{\xi} \qquad (0 \le s \le t)\,,$$

which yields the first claim after solving for $h(s)$ and setting $s = t$. The second claim follows by substituting this bound back into the differential inequality.

□

3.3.2 Proof of Theorem 1.2.1

In this section we establish the stability result of Theorem 1.2.1, that solutions with initial values close to the pulse Φ converge exponentially to a small translate Φ_{ct+h^*} of the pulse solution Φ_{ct}.

Let L be the linearization about Φ in the moving frame, defined in Eq. (3.0.4), and $\xi \in (0, \sigma)$. We will construct a neighborhood $\mathcal{U}$ of Φ in $H^{2,1}$ such that for every solution $u(t)$ with initial value $u_0 \in \mathcal{U}$ there is a real-valued function $h(t)$ such that

$$\|u(t) - \Phi_{h(t)}\| \lesssim e^{-\xi t}\|u_0 - \Phi\|_{2,1} \qquad (t \ge 0)$$

where $|h(0)| \lesssim \|u_0 - \Phi\|_{2,1}$, and there exists $h_* \in \mathbb{R}$ such that

$$|h(t) - h_*| \lesssim e^{-\xi t}\|u_0 - \Phi\|_{2,1}^2 \qquad (t \ge 0)\,.$$

Transforming back to the static frame, this will prove the theorem.

By Eq. (3.3.1) of Proposition 3.3.1, there is a neighborhood $\mathcal{U}$ of Φ such that each $u_0 \in \mathcal{U}$ can be written uniquely as $u_0 = \Phi_{h_0} + v_0$, where $P_{h_0}v = 0$. Since h_0 depends smoothly on u,

$$|h_0| \lesssim \|u_0 - \Phi\|_{2,1}\,, \quad \|v_0\|_{2,1} = \|u_0 - \Phi_{h_0}\|_{2,1} \lesssim \|u_0 - \Phi\|_{2,1}\,.$$

Replacing u_0 with its translate $(u_0)_{-h_0}$, we may assume that $h_0 = 0$, that is,

$$u_0 = \Phi + v_0 \,, \quad Pv_0 = 0 \,.$$

By the second part of Proposition 3.3.1, the solution of Eq. (2.3.1) with initial value u_0 can be written uniquely as

$$u(t) = \Phi_{h(t)} + v(t) \,, \qquad Pv(t) = 0 \,,$$

so long as $u(t) \in \mathcal{U}$. The functions $h(t)$ and $v(t)$ satisfy inequalities (3.3.5) and (3.3.6) with initial values $h(0) = 0$ and $v(0) = v_0$.

Since the map $u \mapsto (h, v)$ is a diffeomorphism from $\mathcal{U}$ to a neighborhood of the origin in $\mathbb{R} \times \mathrm{Ran}\,(Q)$, by replacing $\mathcal{U}$ with a smaller neighborhood we may assume that it has the form $\mathcal{U} = \left\{\Phi_h + v \;\middle|\; (h, v) \in \mathbb{R} \times \mathrm{Ran}\,(Q), |h| < \kappa, \|v\|_{2,1} < \eta\right\}$, where κ is so small that $\langle \tau_h, \tau^* \rangle \geq \frac{1}{2}$ whenever $|h| \leq \kappa$. The value of $\eta > 0$ will be further specified below.

Let σ be the exponent from Proposition 3.2.1, and let C_0 be the multiplicative constant. Choose $\xi \in (0, \sigma)$, define the monotonically increasing function

$$y(t) := \sup_{0 \leq s \leq t} e^{-\xi s} \|v(s)\|_{2,1} \,,$$

and let

$$T := \inf \left\{ t \geq 0 \;\middle|\; |h(s)| \geq \kappa \text{ or } \|y(s)\|_{2,1} \geq \eta \right\} \,.$$

Assume that $\|v_0\|_{2,1} < \frac{\eta}{2C}$, and apply Lemma 3.3.4. By Eq. (3.3.8),

$$|\dot{h}(t)| \leq Ce^{-\xi t}\big(|h(t)| + y(t)\big)y(t) \,, \qquad (0 \leq t \leq T) \,,$$

where we have used that $\|v(t)\|_{2,1} \leq e^{-\xi t} y(t)$ by definition of y. It follows by Lemma 3.3.5 that $h(t) \leq C_\eta y(t)$ for $0 \leq t \leq T$ for some constant C_η. Since $y(t) < \eta$ for $t < T$, by reducing the value of η we can achieve that $|h(t)| < \kappa$ for all $t \in [0, T)$. Inserting this estimate into Eq. (3.3.9) yields

$$y(t) \leq C_0 e^{-(\sigma - \xi)t} \|v_0\|_{2,1} + C \int_0^t e^{-(\sigma - \xi)(t - s)} y^2(s) \, ds \,,$$

where C is the product of C_0, C_η, and the constant from Lemma 3.3.3. Since y is nondecreasing, taking it out of the integral yields the upper bound

$$y(t) \leq C_0 \|v_0\|_{2,1} + C y^2(t) \qquad (0 \leq t \leq T) \tag{3.3.10}$$

with a suitably adjusted constant C.

Consider the quadratic polynomial $p(y) := C_0\|v_0\|_{2,1} - y + Cy^2$. If $d := 4C_0C\|v_0\|_{2,1} < 1$, then P has two positive real roots, and is positive on the interval between them. The smaller root satisfies

$$y_* = \frac{1}{2C}\left(1 - \sqrt{1 - 4C_0C\|v_0\|_{2,1}}\right) \leq 2C\|v_0\|_{2,1} < \eta\,.$$

Since $C_0 \geq 1$, we have that $\|v_0\|_{2,1} < y_*$. Eq. (3.3.10) implies, by continuity, that $y(t) \leq y_*$ for all $0 \leq t \leq T$. If $T < \infty$, then by continuity also $y(T) \leq y_* < \eta$, contradicting the definition of T. Hence $T = +\infty$, and

$$\|v(t)\|_{2,1} \leq e^{-\xi t} y(t) \leq 2C_0 e^{-\xi t}\|v_0\|_{2,1} \qquad (t \geq 0)\,.$$

Since $|h(t)| \leq \kappa$ and $\|v(t)\| < \eta$, we conclude that $\Phi_{h(t)} + v(t) \in \mathcal{U}$ for all $t \geq 0$, and $\|v(t)\|_{2,1}$ converges exponentially to zero.

To show that $h(t)$ converges as well, we use again Lemma 3.3.5 to see that

$$|\dot{h}(t)| \leq Ce^{-\xi t}\|v_0\|_{2,1}^2\,.$$

It follows that $h(t)$ converges exponentially to a limit, h_*, with $|h_*| \lesssim \|v_0\|_{2,1}^2$. Since $\|v_0\|_{2,1} \lesssim \|u_0 - \Phi\|_{2,1}$, this proves the estimate for h. The proof of the theorem is completed by shrinking the neighborhood once more, to

$$\mathcal{U} = \left\{\Phi_h + v \;\middle|\; (h, v) \in \mathbb{R} \times \mathrm{Ran}\,(Q), |h| < \kappa, \|v\|_{2,1} < \tfrac{\eta}{2C}\right\}$$

□

3.4 Numerical simulations

To strengthen our analytical results we include numerical simulations. Discretize Eq. (1.1.3) with Δ_S given by Eq. (1.2.2) uniformly in space (x, θ) and time (t). Consider Dirichlet boundary conditions in the x-direction and periodic boundary conditions in the θ-direction. In all simulations the parameters α, γ, ε and R are fixed.

For the standard cylinder $\mathcal{S}_R$ fix the radius $R = 0.8$. The first three plots of Fig. 3.1 depict images of the formation of the pulse and the last two of the propagation of the pulse. In Fig. 3.2 we pick the cross section at $\theta = \pi$ and draw the profiles of the pulses.

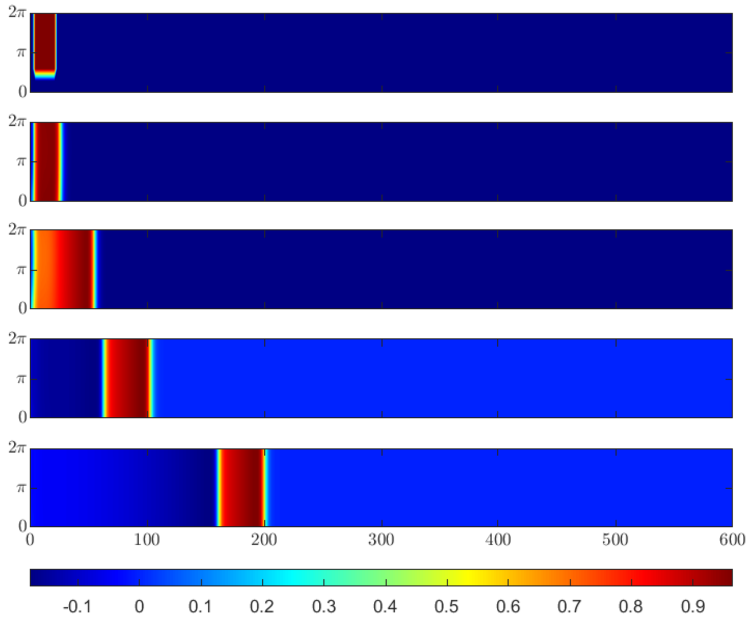


Figure 3.1: Evolution of the potential, u_1, of the FitzHugh-Nagumo system (1.1.3) for $\mathcal{S} = \mathcal{S}_R$. The x-axis and y-axis represent the variables x and θ, respectively. The snapshots are taken in time $t = 0, 50, 150, 300$ and 600 (from top to bottom). The initial condition depends on both x and θ, and the boundary conditions are Dirichlet in the x-direction and periodic in the θ-direction. The values of the parameters are: $\alpha = 0.22$, $\varepsilon = 0.001$, $\gamma = 2.5$ and $R = 0.8$.

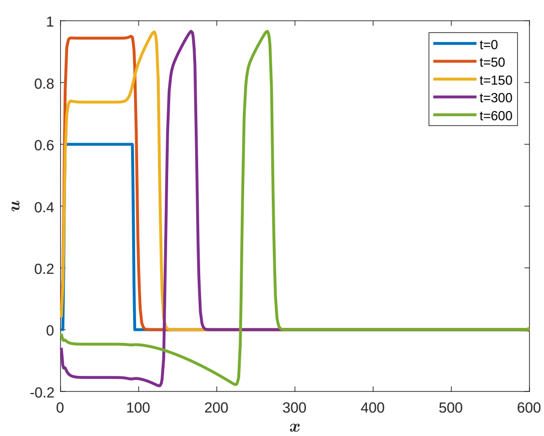


Figure 3.2: Configuration of the pulse corresponding to the images of Figure 3.1 at the cross section $\theta = \pi$.

Chapter 4

Near-pulse solutions on warped cylinders

In this chapter, we consider the case of warped cylinders $\mathcal{S} = \mathcal{S}_\rho$. We assume that the radius $\rho(x)$ is a function of class C^2, and it is also positive, bounded, and bounded away from zero. The Laplace-Beltrami operator $\Delta_{\mathcal{S}_\rho}$ is given by Eq. (1.2.8) and the FHN system on $\mathcal{S}_\rho$ by

$$\begin{aligned} \partial_t u_1 &= \Delta_{\mathcal{S}_\rho} u_1 + f(u_1) - u_2 \,, \\ \partial_t u_2 &= \varepsilon(u_1 - \gamma u_2) \,. \end{aligned} \tag{4.0.1}$$

Let $F_\rho(u)$ denote the right hand side of Eq. (4.0.1). It is defined explicitly as

$$F_\rho(u) := \begin{pmatrix} \Delta_{\mathcal{S}_\rho} u_1 + f(u_1) - u_2 \\ \varepsilon(u_1 - \gamma u_2) \end{pmatrix} \,. \tag{4.0.2}$$

The pulse Φ is not a solution of Eq. (4.0.1), so $F_\rho(\Phi) \neq 0$. This implies that expanding Eq. (4.0.1) (or the corresponding equation in the moving frame (2.3.1)) about Φ, the principal part will be time-dependent. To avoid this issue we Taylor expand Eq. (4.0.1) in the static frame about the zero solution:

$$\begin{aligned} \partial_t u &= F_\rho(u) \\ &= A_\rho u + N(u) \,. \end{aligned}$$

Here the linearization of Eq. (4.0.1) about zero is given by the Gâteaux derivative $A_\rho :=$

$dF_\rho(0)$:

$$A_\rho := \begin{pmatrix} \Delta_{\mathcal{S}_\rho} + f'(0) & -1 \\ \varepsilon & -\varepsilon\gamma \end{pmatrix} \tag{4.0.3}$$

and $N(u) := F_\rho(u) - A_\rho u$ is the nonlinearity stated explicitly as

$$N(u) := \begin{pmatrix} -u_1^3 + (\alpha+1)u_1^2 \\ 0 \end{pmatrix}. \tag{4.0.4}$$

Due to the geometry of the surface of $\mathcal{S}_\rho$, the inner product on $L^2(\mathcal{S}_\rho)$ is defined by the surface integral

$$\langle u, w\rangle_\rho := \int_{\mathcal{S}_\rho} (u_1\bar{w}_1 + \varepsilon^{-1}u_2\bar{w}_2)\sqrt{g}\, d\theta dx, \tag{4.0.5}$$

with the corresponding norm $\|\cdot\|_\rho$. In Eq. (4.0.5), $\sqrt{g}\, d\theta dx$ is the Riemannian area element with density $g = \rho^2(1+(\rho')^2)$. The linearization in Eq. (4.0.3) is an operator on $L^2(\mathcal{S}_\rho)$.

Define the mixed Sobolev spaces

$$H^{2k,\ell}(\mathcal{S}_\rho) := \left\{ u \in L^2 \;\middle|\; (\Delta_{\mathcal{S}_\rho})^k u_1 \in L^2(\mathcal{S}_\rho), (\partial_x)^\ell u_2 \in L^2(\mathcal{S}_\rho) \right\} \tag{4.0.6}$$

for $k, \ell = 0, 1$, with norms

$$\|u\|_{2k,\ell;\rho} := \sum_{0\le i\le k} \|(\Delta_{\mathcal{S}_\rho})^i u_1\|_\rho + \sum_{0\le j\le \ell} \varepsilon^{-1}\|\partial_x^j u_2\|_\rho\,. \tag{4.0.7}$$

Observe that the spaces in Eq. (4.0.6) do not impose a condition on the derivative $\partial_\theta u_2$. The space $H^{0,0}$ agrees with the Hilbert space L^2, and the $H^{2,0}$ agrees with the corresponding Sobolev space $H^2 \times L^2$. On the other hand, $H^{0,1}$ strictly contains the space $L^2 \times H^1$, and $H^{2,1}$ strictly contains $H^2 \times H^1$. In the special case of the standard cylinder $\mathcal{S}_R$, Eq. (4.0.6) with $k = l = 1$ coincides with the definition of $H^{2,1}(\mathcal{S}_R)$ in Eq. (4.0.7). We will prove in Lemma 4.2.7 that the norms $\|\cdot\|_{2k,\ell;\rho}$ are equivalent to $\|\cdot\|_{2k,\ell}$.

First we verify that the solution u of Eq. (4.1.1) with suitable initial conditions is well-posed locally in time, and then we prove Theorem 1.2.2. The proof of the first argument is similar to that of the local well-posedness on $\mathcal{S}_R$ (Section 3.1), but for the sake of completeness we include the proof in Section 4.1. As mentioned in the introduction, the proof of Theorem 1.2.2 relies on a perturbation estimate that controls the dependence of solutions on ρ. We prove this key estimate in Subsection 4.3.1, and in Subsection 4.3.2 we complete the proof of Theorem 1.2.2.

4.1 Local well-posedness

Let A_ρ and $N(u)$ be given by Eqs. (4.0.3) and (4.0.4), respectively, and consider the initial value problem

$$\begin{aligned} \partial_t u &= A_\rho u + N(u) \\ u|_{t=0} &= u_0 \end{aligned} \tag{4.1.1}$$

on the space $H^{2,1}(\mathcal{S}_\rho)$. If $u(t)$ is a classical solution of Eq. (4.1.1), then by Duhamel's formula it also solves the integral equation

$$u(t) = e^{tA_\rho} u_0 + \int_0^t e^{(t-s)A_\rho} N(u(s))\, ds =: \mathcal{F}_\rho(u)(t)\,. \tag{4.1.2}$$

As in Lemma 3.1.2, for every $\eta > 0$ there exists a constant $C_\eta > 0$ depending on α, γ, ε, and η such that

$$\|N(u) - N(w)\|_{2,1;\rho} \le C_\eta \|u - w\|_{2,1;\rho} \tag{4.1.3}$$

for all u, w with $\|u\|_{2,1;\rho}$, $\|u\|_{2,1;\rho} \le \eta$.

Proposition 4.1.1 (Local well-posedness)**.** *Assume that ρ is of class C^2, bounded, and bounded away from zero. Then for each $u_0 \in H^{2,1}(\mathcal{S}_\rho)$, there exists $T > 0$ (depending on $\|u_0\|_{2,1}$) such that Eq. (4.0.1) on $\mathcal{S}_\rho$ has a unique mild solution u in $C([0,T], H^{2,1})$ with initial condition $u|_{t=0} = u_0$. The solution depends continuously on u_0.*

Proof. Given $u_0 \in H^{2,1}(\mathcal{S}_\rho)$, fix $\eta > 0$ and $T > 0$ (to be specified below) and consider the ball

$$\mathcal{B} := \left\{ u \in C([0,T], H^{2,1}(\mathcal{S}_\rho)) \;\middle|\; \|u(t)\|_{2,1;\rho} \le \eta \text{ for all } 0 \le t \le T \right\}\,,$$

equipped with the norm $\|u\|_T := \sup_{0 \le t \le T} \|u(t)\|_{2,1;\rho}$.

The map $\mathcal{F}_\rho$ defined by Eq. (4.1.2) is Lipschitz continuous on $\mathcal{B}$,

$$\begin{aligned} \|\mathcal{F}_\rho(u) - \mathcal{F}_\rho(w)\|_T &\le \sup_{0 \le t \le T} \int_0^t \|e^{(t-s)A_\rho}\|_{2,1;\rho} \|N(u(s)) - N(w(s))\|_{2,1;\rho}\, ds \\ &\le T C_0 C_\eta \, \|u - w\|_T\,, \end{aligned}$$

where $C_0 := \sup_{t \ge 0} \|e^{tA_\rho}\|_{2,1;\rho}$, and C_η is as in Eq. (4.1.3). Moreover,

$$\|\mathcal{F}_\rho(0)\|_T = \sup_{0 \le t \le T} \|e^{tA_\rho} u_0\|_{2,1;\rho} \le C_0 \, \|u_0\|_{2,1;\rho}\,.$$

Choose $\eta = 2C_0\|u_0\|_{2,1;\rho}$, and $T = (2C_0C_\eta)^{-1}$. Then $\mathcal{F}_\rho$ has Lipschitz constant $\frac{1}{2}$ and maps $\mathcal{B}$ into itself. By the contraction mapping principle, $\mathcal{F}_\rho$ has a unique fixed point in $\mathcal{B}$, which provides the desired mild solution of Eq. (4.1.1).

Let $w(t)$ be another mild solution, whose initial value $w_0 := w\big|_{t=0}$ satisfies $\|w_0\|_{2,1;\rho} < \eta$. The difference between the solutions is bounded by

$$\begin{aligned}\|u(t) - w(t)\|_{2,1;\rho} &\leq \|e^{tA_\rho}(u_0 - w_0)\|_{2,1;\rho} + \int_0^t \|e^{(t-s)A_\rho}\big(N(u(s)) - N(w(s))\big)\|_{2,1;\rho}\, ds \\ &\leq C_0\|u_0 - w_0\|_{2,1;\rho} + C_0C_\eta \int_0^t \|u(s) - w(s)\|_{2,1;\rho}\, ds\,,\end{aligned}$$

so long as $\max\{\|u(s)\|_{2,1;\rho}, \|w(s)\|_{2,1;\rho}\} \leq \eta$ for all $0 \leq s \leq t$. By Grönwall's inequality,

$$\|u(t) - w(t)\|_{2,1;\rho} \leq C_0 e^{C_0C_\eta t}\|u_0 - w_0\|_{2,1;\rho}\,.$$

This proves continuous dependence on initial data.

□

4.2 Linear dynamics

This section is dedicated to the study of the linear part of Eq. (4.1.1). We start with some basic properties of A_ρ (Subsection 4.2.1) and then we compare the linear solutions of the FHN system on the surface of $\mathcal{S}_\rho$ with those on the surface of the standard cylinder $\mathcal{S}_R$ (Subsection 4.2.2).

4.2.1 Properties of the linear operator

The linearization A_ρ, of Eq. (4.0.1) about the zero solution, is given by Eq. (4.0.3). The domain of A_ρ is $H^{2,0}(\mathcal{S}_\rho)$.

Lemma 4.2.1 (A_ρ is dissipative)**.** *Define* $\sigma := \min\{\alpha, \varepsilon\gamma\}$. *Then*

$$\mathrm{Re}\langle A_\rho u, u\rangle_\rho \leq -\sigma\|u\|_\rho^2 \tag{4.2.1}$$

for $u \in H^{2,0}(\mathcal{S}_\rho)$.

Proof. Using that $\Delta_{\mathcal{S}_\rho}$ is negative semi-definite with respect to the Riemannian metric on the surface of $\mathcal{S}_\rho$, that $f'(0) = -\alpha < 0$, and that the off-diagonal terms of A_ρ are skew-adjoint relative to the inner product $\langle \cdot, \cdot \rangle_\rho$, we have

$$\begin{aligned}
\mathrm{Re}\langle A_\rho u, u\rangle_\rho &= \mathrm{Re}\int_{\mathcal{S}_\rho} \big((\Delta_{\mathcal{S}_\rho} u_1)\bar{u}_1 + f'(0)|u_1|^2 - u_1\bar{u}_2 + \bar{u}_1 u_2 - \gamma|u_2|^2\big)\sqrt{g}\, d\theta dx \\
&= \int_{\mathcal{S}_\rho} \big((\Delta_{\mathcal{S}_\rho} u_1)\bar{u}_1 - \alpha|u_1|^2 - \gamma|u_2|^2\big)\sqrt{g}\, d\theta dx \\
&\leq -\int_{\mathcal{S}_\rho} \big(\alpha|u_1|^2 + \gamma|u_2|^2\big)\sqrt{g}\, d\theta dx \\
&\leq -\sigma\|u\|_\rho^2.
\end{aligned}$$

Since $\sigma > 0$, the right hand side in the above inequality is negative implying that A_ρ is dissipative.

□

In the next lemma we show that the graph norm of A_ρ is equivalent to the norm $\|\cdot\|_{2,0;\rho}$.

Lemma 4.2.2. *The graph norm of the operator A_ρ satisfies*

$$\|u\|_{2,0;\rho} \lesssim \|A_\rho u\|_\rho + \|u\|_\rho \lesssim \|u\|_{2,0;\rho} \tag{4.2.2}$$

for $u \in H^{2,0}(\mathcal{S}_\rho)$.

Proof. The upper bound in Eq. (4.2.2) follows immediately from the fact that A_ρ is a second-order differential operator with twice continuously differentiable coefficients. For the lower bound, we use $\|\partial_x u_1\|_{L^2_\rho} \leq b\|\partial_x^2 u_1\|_{L^2_\rho} + \frac{1}{4b}\|u_1\|_{L^2_\rho}$ for b small enough, and we estimate the principal part of A_ρ:

$$\begin{aligned}
\|\Delta_{\mathcal{S}_\rho} u_1\|_{L^2_\rho} &\geq \sup(1+\rho'^2)^{-1}\|\partial_x^2 u_1\|_{L^2_\rho} + (\sup\rho)^{-2}\|\partial_\theta^2 u_1\|_{L^2_\rho} - C_1\|\partial_x u_1\|_{L^2_\rho} \\
&\geq C_2\|u_1\|_{H^2_\rho} - C_3\|u_1\|_{L^2_\rho}
\end{aligned}$$

for some positive constants C_1, C_2 and C_3. The rest of the terms in A_ρ are bounded operators on L^2. It follows that

$$\|A_\rho u\|_{L^2_\rho} \geq C_2\|u\|_{H^2_\rho \times L^2_\rho} - C_4\|u\|_{L^2_\rho}$$

where $C_4 > 0$. Therefore, for any $C_6 \geq 1$,

$$C_6(\|A_\rho u\|_{L^2_\rho} + \|u\|_{L^2_\rho}) \geq C_5\|u\|_{H^2_\rho \times L^2_\rho} + (C_6 - C_4)\|u\|_{L^2_\rho}.$$

Take $C_4 = C_6$. Dividing by C_6 we obtain the lower bound in Eq. (4.2.2). □

By Eq. (4.2.1) the graph norm of A_ρ is also equivalent to the norm $\|A_\rho u\|_\rho$. Combining this with the relation in Eq. (4.2.2) we have

$$\|u\|_{2,0;\rho} \lesssim \|A_\rho u\|_\rho \lesssim \|u\|_{2,0;\rho} \,. \tag{4.2.3}$$

Since A_ρ is dissipative it follows form Lemma 3.2.3 that A_ρ generates a strongly continuous semigroup, denoted by e^{tA_ρ} for $t \geq 0$, on $L^2(\mathcal{S}_\rho)$ that has $H^{2,0}(\mathcal{S}_\rho)$ as an invariant subspace.

Next we restrict A_ρ to the subspace $H^{0,1}(\mathcal{S}_\rho)$. The reason is that we want to work in the subspace $H^{2,1}$ that was used in Theorem 1.2.1, but it is not immediately obvious that this space is an invariant subspace of e^{tA_ρ}.

Lemma 4.2.3 (Domain of A_ρ in $H^{0,1}$)**.** *Let α, γ, ε be fixed positive constants, and ρ be a positive function of class C^2 on the real line. Then the operator A_ρ maps $H^{2,1}(\mathcal{S}_\rho)$ bijectively onto $H^{0,1}(\mathcal{S}_\rho)$, and*

$$\|u\|_{2,1;\rho} \lesssim \|A_\rho u\|_{0,1;\rho} \lesssim \|u\|_{2,1;\rho} \tag{4.2.4}$$

with $u \in H^{2,1}(\mathcal{S}_\rho)$.

Proof. Fix ρ as in the assumptions, and let $u \in H^{2,1}(\mathcal{S}_\rho)$. To simplify notation, we momentarily suppress the dependence of the spaces and norms on ρ in the notation.

Write $\|A_\rho u\|_{0,1} = \|A_\rho u\| + \|\partial_x(\varepsilon u_1 - \varepsilon\gamma u_2)\|$, and combine the upper bound in Eq. (4.2.3) with the estimates $\|\partial_x u_1\| \leq \|u\|_{2,1}$ and $\|\partial_x u_2\| \leq \|u\|_{0,1}$. It follows that $A_\rho u \in H^{0,1}$, and the upper bound in Eq. (4.2.4) holds. In particular, A_ρ maps $H^{2,1}$ to $H^{0,1}$. By Eq. (4.2.3), this map is injective.

To show that this map is also surjective, let $w \in H^{0,1}$. Since $w \in L^2$, the equation $A_\rho u = w$ has a unique solution $u \in H^{2,0}$. The lower bound in Eq. (4.2.3) yields $u_1 \in H^2$, and $\|u_1\|_{H^2} \lesssim \|A_\rho u\| \leq \|A_\rho u\|_{0,1}$. For the second component, we use that $\varepsilon u_1 - \varepsilon\gamma u_2 = w_2$, and estimate

$$\|\partial_x u_2\| \leq \gamma^{-1}\|\partial_x u_1\| + (\varepsilon\gamma)^{-1}\|\partial_x w_2\| \lesssim \|u\|_{2,0} + \|w\|_{0,1} \lesssim \|A_\rho u\|_{0,1} \,.$$

This proves surjectivity, and the lower bound.

□

A useful consequence of Lemma 4.2.3 is the following

Corollary 4.2.4. *Let B be a bounded linear operator on $H^{0,1}(\mathcal{S}_\rho)$ that commutes with*

A_ρ. Then

$$\|B\|_{0,1;\rho} \lesssim \|B\|_{2,1;\rho} \lesssim \|B\|_{0,1;\rho}\,.$$

We continue with one more property of the linear operator A_ρ that will be used to characterize its semigroup e^{tA_ρ} on $H^{0,1}(\mathcal{S}_\rho)$.

Lemma 4.2.5 (A_ρ is sectorial)**.** *Let $\rho \in C^2$ be a real-valued function that is bounded and bounded away from zero. Then A_ρ generates an analytic semigroup e^{tA_ρ} on $H^{2,1}(\mathcal{S}_\rho)$. The spectrum of A_ρ on $H^{0,1}(\mathcal{S}_\rho)$ is contained in the sector*

$$\Sigma := \left\{\lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\,\lambda \le -\sigma \min\{1, \varepsilon^{-\frac{1}{2}}\,|\operatorname{Im}\,\lambda|\}\right\}, \tag{4.2.5}$$

where $\sigma := \min\{\alpha, \varepsilon\gamma\}$ and α, ε, γ are fixed positive constants.

Proof. We will bound the numerical range of A_ρ with respect to a certain weighted inner product, and then apply [43, Theorem 1.3.9]. As in the proof of Lemma 4.2.3, ρ is fixed and will be suppressed in the notation. For $s > 0$, define

$$\langle u, v\rangle_s := \langle u_1, v_1\rangle + \varepsilon^{-1}\langle u_2, v_2\rangle + s\varepsilon^{-1}\langle \partial_x u_2, \partial_x v_2\rangle\,.$$

The corresponding norm $\|\cdot\|_s$ is equivalent to the norm $\|\cdot\|_{0,1}$ from Eq. (4.0.6),

$$\min\{1, s^{\frac{1}{2}}\}\|u\|_{0,1} \le \|u\|_s \le \max\{1, s^{\frac{1}{2}}\}\|u\|_{0,1}\,, \qquad (s>0)\,.$$

We compute

$$\begin{aligned}
\operatorname{Re}\langle A_\rho u, u\rangle_s &= -\alpha\|u_1\|^2 - \gamma\|u_2\|^2 + \langle \Delta_{\mathcal{S}_\rho} u_1, u_1\rangle + s\operatorname{Re}\langle \partial_x u_1 - \gamma\partial_x u_2, \partial_x u_2\rangle\\
&\le -\sigma\|u\|^2 - \frac{1}{1+\sup|\rho'|^2}\|\partial_x u_1\|^2 + s\|\partial_x u_1\|\|\partial_x u_2\| - s\gamma\|\partial_x u_2\|^2\\
&\le -\sigma\|u\|_s^2 + \frac{s^2}{4}(1+\sup|\rho'|^2)\|\partial_x u_2\|^2\,.
\end{aligned} \tag{4.2.6}$$

Note that the inner products and norms on the right hand side of the first line are the standard Riemannian ones for scalar functions in $L^2(\mathcal{S}_\rho)$. In the second line, we have integrated the Laplacian term by parts. The last step follows by completing the square. For any $q \in (0,1)$, we can achieve $\operatorname{Re}\langle A_\rho u, u\rangle_s \le -q\sigma\|u\|_s^2$ by choosing s sufficiently small. By Lemma 3.2.3, A_ρ generates a semigroup of contractions with respect to the norm $\|\cdot\|_s$. Moreover, the spectrum of A_ρ is contained in each of the half-planes $\{\operatorname{Re}\,\lambda \le -q\sigma\}$, and hence in their intersection.

Likewise,

$$\begin{aligned}\operatorname{Im}\langle A_\rho u, u\rangle_s &= 2\operatorname{Im}\langle u_1, u_2\rangle + s\operatorname{Im}\langle \partial_x u_1, \partial_x u_2\rangle \\ &\leq \sqrt{\varepsilon}\|u\|^2 + s\|\partial_x u_1\|\|\partial_x u_2\|\,.\end{aligned}$$

Comparing with the second line of Eq. (4.2.6), we see that for $s > 0$ sufficiently small

$$\operatorname{Re}\langle A_\rho u, u\rangle_s \leq -\sigma\varepsilon^{-\frac{1}{2}}|\operatorname{Im}\langle A_\rho u, u\rangle_s|\,.$$

Since the resolvent set of A_ρ contains 0, by [43, Theorem 1.3.9], it contains the entire complement of the sector $\{\operatorname{Re}\lambda \leq -\sigma\varepsilon^{-\frac{1}{2}}|\operatorname{Im}\lambda|\}$. In summary, for s sufficiently small, the numerical range of A_ρ with respect to $\langle\cdot,\cdot\rangle_s$ lies in

$$\Sigma_q := \left\{\lambda\in\mathbb{C}\;\middle|\; \operatorname{Re}\lambda \leq -\sigma\min\{q, \varepsilon^{-\frac{1}{2}}|\operatorname{Im}\lambda|\}\right\}\,.$$

□

For the same norm $\|\cdot\|_s$ as in the proof of Lemma 4.2.5, the resolvent operator of A_ρ satisfies the estimate

$$\|(\lambda - A_\rho)^{-1}\|_s \leq \frac{1}{\inf_{z\in\Sigma}\|\lambda - z\|_s}$$

for $\lambda\notin\Sigma$. In the space $H^{2,1}(\mathcal{S}_\rho)$ we have the following resolvent and semigroup estimates:

Lemma 4.2.6. *Assume $\rho\in C^2$ is as in Lemma 4.2.5. Then for all λ with $\operatorname{Re}\lambda \geq -\frac{1}{2}\sigma\min\{1, \varepsilon^{-\frac{1}{2}}\,|\operatorname{Im}\lambda|\}$ the resolvent operator $(\lambda - A_\rho)^{-1}$ satisfies*

$$\|(\lambda - A_\rho)^{-1}\|_{2,1;\rho} \leq C(1 + \sup|\rho'|)\min\left\{1, \frac{1}{|\lambda|}\right\} \tag{4.2.7}$$

for some constant $C > 0$. The semigroup e^{tA_ρ} satisfies

$$\sup_{t>0}\|e^{tA_\rho}\|_{2,1;\rho} \leq C(1 + \sup|\rho'|)e^{-\sigma t} \tag{4.2.8}$$

where $\sigma := \min\{\alpha, \varepsilon\gamma\}$.

Proof. To obtain Eq. (4.2.7) we choose $s = \gamma(1 + \sup|\rho'|^2)^{-1}$, and compare the norm $\|\cdot\|_s$ with the norm $\|\cdot\|_{0,1;\rho}$. Since the resolvent commutes with A_ρ, by Lemma 4.2.3, the estimate holds also for the norm $\|\cdot\|_{2,1;\rho}$.

The bound on the semigroup follows directly by Eq. (4.2.7) and the Corollary II.3.6 [12].

□

4.2.2 Comparison with the standard cylinder

The purpose of this subsection is to compare the semigroups e^{tA_ρ} and e^{tA_R}. The operator A_R is the corresponding linear operator of A_ρ on the standard cylinder $\mathcal{S}_R$. It is given explicitly by

$$A_R := \begin{pmatrix} \Delta_{S_R} + f'(0) & -1 \\ \varepsilon & -\varepsilon\gamma \end{pmatrix} . \tag{4.2.9}$$

These two semigroups are defined in different spaces, so we need the following equivalence relation.

Lemma 4.2.7 (Equivalence of Sobolev spaces)**.** *Let α, γ, ε be fixed positive constants, $0 < R \le 1$, and ρ a positive function of class C^2. If $\delta := R^{-1}\|\rho - R\|_{C^2} \le \frac{1}{16}$, then*

$$\frac{1}{2}\|u\|_{2k,\ell} \le \|u\|_{2k,\ell;\rho} \le 2\|u\|_{2k,\ell} \tag{4.2.10}$$

for $k, \ell \in \{0,1\}$.

Proof. Consider a scalar-valued function w on $\mathcal{S}_\rho$. By Eq. (4.0.5):

$$\|w\|_\rho^2 = \int_{\mathbb{R}}\int_{S^1} |w|^2 \sqrt{g(x)}\, d\theta dx \,,$$

where $g = \rho^2(1 + (\rho')^2)$. The bound $|\sqrt{g(x)} - R| \le 2\delta$ yields

$$\big|\|w\|_\rho - \|w\|\big| \le 2\delta\|w\| \,. \tag{4.2.11}$$

Applying Eq. (4.2.11) to $\partial_x w$ we have

$$\big|\|\partial_x w\|_\rho - \|\partial_x w\|\big| \le 2\delta\|\partial_x w\| \,. \tag{4.2.12}$$

Next we claim that

$$\big|\|\Delta_{\mathcal{S}_\rho} w\|_\rho - \|\Delta_{\mathcal{S}_R} w\|\big| \le 8\delta\big(\|\Delta_{\mathcal{S}_R} w\| + \|w\|\big) \,. \tag{4.2.13}$$

Indeed,

$$\Delta_{\mathcal{S}_\rho} - \Delta_{\mathcal{S}_R} = a(x)\partial_x^2 + b(x)\partial_x + c(x)|\rho(x) - R|R^{-2}\partial_\theta^2 \,, \tag{4.2.14}$$

where the coefficients are bounded by $0 < a(x) \le \frac{1}{2}R\delta$, $|b(x)| \le \delta + R^2\delta^2$, and $c(x) \le 6R\delta$. Since $\delta \le 1$, $R \le 1$, and

$$\|\partial_x w\| \le \frac{1}{2}\big(\|\Delta_{\mathcal{S}_R} w\| + \|w\|\big) \,, \tag{4.2.15}$$

we obtain by the triangle inequality that

$$\|(\Delta_{\mathcal{S}_\rho} - \Delta_{\mathcal{S}_R})w\| \leq 6\delta(\|\Delta_{\mathcal{S}_R} w\| + \|w\|)\,. \tag{4.2.16}$$

Using once more the triangle inequality, as well as Eq. (4.2.11), we arrive at

$$\begin{aligned}\big|\|\Delta_{\mathcal{S}_\rho} w\|_\rho - \|\Delta_{\mathcal{S}_R} w\|\big| &\leq \|(\Delta_{\mathcal{S}_\rho} - \Delta_{\mathcal{S}_R})w\| + \big|\|\Delta_{\mathcal{S}_\rho} w\|_\rho - \|\Delta_{\mathcal{S}_\rho} w\|\big| \\ &\leq 8\delta\left(\|\Delta_{\mathcal{S}_R} w\| + \|w\|\right)\,.\end{aligned}$$

When $\delta \leq \frac{1}{16}$, we can solve Eq. (4.2.11) for the norm on $L^2(\mathcal{S}_\rho)$ to obtain

$$\frac{1}{2}\|w\| \leq \|w\|_\rho \leq 2\|w\|\,.$$

Similarly, we solve Eq. (4.2.12) for $\|\partial_x w\|_\rho$, and Eq. (4.2.13) for $\|\Delta_{\mathcal{S}_\rho} w\|_\rho$. The desired result follows from the definitions of the norms in Eqs. (1.2.4) and (4.0.6).

□

A key tool is a similarity argument of the corresponding resolvent operators, $(\lambda - A_R)^{-1}$ and $(\lambda - A_\rho)^{-1}$.

Lemma 4.2.8 (Perturbation estimate for the resolvent)**.** *Let α, γ and ε be positive constants, and $0 < R \leq 1$. There exists a constant C such that if $\delta := R^{-1}\|\rho - R\|_{C^2} \leq \frac{1}{16}$, then*

$$\|(\lambda - A_\rho)^{-1} - (\lambda - A_R)^{-1}\|_{2,1} \leq C\delta \min\{1, |\lambda|^{-1}\}$$

for all λ with $\mathrm{Re}\,\lambda \geq -\frac{1}{2}\sigma \min\{1, \varepsilon^{-\frac{1}{2}}|\,\mathrm{Im}\,\lambda|\}$.

Proof. If λ is as in the statement of the lemma, then by Lemma 4.2.5 it lies in the resolvent set of both A_ρ and A_R. To estimate the difference, we write

$$A_\rho - A_R = \begin{pmatrix} \Delta_{\mathcal{S}_\rho} - \Delta_{\mathcal{S}_R} & 0 \\ 0 & 0 \end{pmatrix} =: W\,,$$

and apply the resolvent identity

$$(\lambda - A_\rho)^{-1} - (\lambda - A_R)^{-1} = (\lambda - A_R)^{-1} W (\lambda - A_\rho)^{-1}\,.$$

For the factor on the right, we use Lemmas 4.2.7 and 4.2.3 to see that

$$\begin{aligned}\|(\lambda - A_\rho)^{-1}\|_{2,1} &\lesssim \|(\lambda - A_\rho)^{-1}\|_{2,1;\rho} \\ &\lesssim \|(\lambda - A_\rho)^{-1}\|_{0,1;\rho} \\ &\lesssim \min\left\{1, \frac{1}{|\lambda|}\right\}.\end{aligned} \tag{4.2.17}$$

The second inequality holds because the resolvent commutes with A_ρ. By Eq. (4.2.16), the middle factor maps $H^{2,1}$ into $H^{0,1}$ and satisfies

$$\|Wu\|_{0,1} = \|(\Delta_{\mathcal{S}_\rho} - \Delta_{\mathcal{S}_R})u_1\| \lesssim \delta\|u\|_{2,1}$$

for all $u \in H^{2,1}$. By Lemma 4.2.3, the factor on the left maps $H^{0,1}$ back into $H^{2,1}$, and

$$\begin{aligned}\|(\lambda - A_R)^{-1}u\|_{2,1} &\lesssim \|A_R(\lambda - A_R)^{-1}u\|_{0,1} \\ &\leq \|u\|_{0,1} + |\lambda|\,\|(\lambda - A_R)^{-1}u\|_{0,1} \\ &\lesssim \|u\|_{0,1}\end{aligned}$$

for all $u \in H^{0,1}$. In the second line we have written $A_R = -(\lambda - A_R) + \lambda$ and applied the triangle inequality, and in the last line we have used that $\|(\lambda - A_R)^{-1}\|_{0,1} \lesssim \min\{1, |\lambda|^{-1}\}$ by Lemma 4.2.5.

Combining the inequalities for the three factors, we conclude that

$$\left\|\big((\lambda - A_\rho)^{-1} - (\lambda - A_R)^{-1}\big)u\right\|_{2,1} \lesssim \delta \min\{1, |\lambda|^{-1}|\}\,\|u\|_{2,1}$$

for all $u \in H^{2,1}$, proving the claim.

□

Using the result of the above lemma we prove that the semigroup generated by A_R is close to that of A_ρ.

Proposition 4.2.9 (Perturbation estimate for the semigroup)**.** *Let α, γ and ε be positive constants. There exists a constant C such that, if $0 < R \leq 1$ and $\delta := R^{-1}\|\rho - R\|_{C^2} \leq \frac{1}{16}$, then the semigroup generated by A_ρ on $H^{2,1}$ satisfies*

$$\|e^{tA_R} - e^{tA_\rho}\|_{2,1} \leq C\delta(1 + \log t^{-1})$$

for all $t \geq 0$.

Proof. Let Γ be the contour consisting of the two half-lines $\operatorname{Re}\lambda = -\frac{1}{2}\sigma\varepsilon^{-\frac{1}{2}}|\operatorname{Im}\lambda|$, traversed counterclockwise. By Lemma 4.2.5, Γ encloses the spectrum of A_ρ and A_R. Since A_ρ is sectorial, the semigroup e^{tA_ρ} is represented by the contour integral

$$e^{tA_\rho} = \frac{1}{2\pi i}\oint_\Gamma e^{\lambda t}(\lambda - A_\rho)^{-1}\, d\lambda\,,$$

and correspondingly for A_R. Parametrizing Γ by $\lambda(s) = -|s| + 2i\sigma^{-1}\sqrt{\varepsilon}$, we see that for each $t > 0$, the integral converges absolutely with respect to the operator norm on $H^{2,1}$.

We estimate the difference from e^{tA_R} by

$$\begin{aligned}
\|e^{tA_\rho} - e^{tA_R}\|_{2,1} &= \left\| \frac{1}{2\pi i}\oint_\Gamma e^{\lambda t}\big((\lambda - A_\rho)^{-1} - (\lambda - A_R)^{-1}\big)\, d\lambda \right\|_{2,1}\\
&\le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{t\operatorname{Re}\lambda(s)}\|(\lambda - A_\rho)^{-1} - (\lambda - A_R)^{-1}\|_{2,1}\,|\lambda'(s)|\, ds\\
&\lesssim \delta \int_0^\infty e^{-ts}\min\{1, s^{-1}\}\, ds\,,
\end{aligned}$$

where we have applied Lemma 4.2.8 to the integrand in the last step. For $t \ge 1$, the integral is uniformly bounded. For $t < 1$, we have

$$\begin{aligned}
\int_0^\infty e^{-ts}\min\{1, s^{-1}\}\, ds &\le 1 + \int_1^{t^{-1}} s^{-1}\, ds + \int_{t^{-1}}^\infty te^{-ts}\, ds\\
&\le 2 + \log t^{-1}\,,
\end{aligned} \tag{4.2.18}$$

proving the claim. □

4.3 Nonlinear dynamics

We extend the perturbation estimates from Section 4.2 to the nonlinear evolution generated by the FHN system on $\mathcal{S}_\rho$. To prove Theorem 1.2.2 we need a perturbation estimate that controls the dependence of solutions on ρ. The size of the perturbation is measured in terms of the parameter $\delta := R^{-1}\|\rho - R\|_{C^2}$. In Subsection 4.3.1 we state and prove that perturbation result, and in Subsection 4.3.2 we prove Theorem 1.2.2.

4.3.1 Perturbation of the radius

Suppose that $u \in C([0,T], H^{2,1})$ is a mild solution of Eq. (3.0.1) on a standard cylinder $\mathcal{S}_R$, with initial values $u|_{t=0} = u_0$. Eq. (3.0.1) is equivalent to Eq. (4.0.1), expressed in the static frame. The pulse defines a traveling wave solution $\Phi(x-ct)$ on $\mathcal{S}_R$. In the next proposition, we address the dependence of the mild solution on the variable radius, $\rho(x)$.

Proposition 4.3.1 (Perturbation of the radius)**.** *Let $u_\rho \in C([0,T], H^{2,1}(\mathcal{S}_\rho))$ be a mild solution of Eq. (4.0.1) on $\mathcal{S}_\rho$ with initial values $u_\rho|_{t=0} = u_0$. There are positive constants δ_* and C such that if $0 < R \le 1$ and $\delta := R^{-1}\|\rho - R\|_{C^2} \le \delta_*$, then u_ρ satisfies*

$$\sup_{0\le t\le T} \|u_\rho(t) - u(t)\|_{2,1} \le C\delta. \tag{4.3.1}$$

Proof. Given a solution $u(t)$ of the FHN system on $\mathcal{S}_R$, set $\eta = 2\sup_{0\le t\le T}\|u(t)\|_{2,1}$. Let $u_\rho(t)$ be the solution on $\mathcal{S}_\rho$ with the same initial value, u_0. By definition of the mild solutions,

$$u_\rho(t) - u(t) = (e^{tA_\rho} - e^{tA_R})u_0 + \int_0^t \left(e^{(t-s)A_\rho}N(u_\rho(s)) - e^{(t-s)A_R}N(u(s))\right)\, ds\,,$$

so long as both solutions exist. By Proposition 4.2.9, for $\delta \le \frac{1}{16}$ the difference of the semigroups is bounded by

$$\|e^{tA_\rho} - e^{tA_R}\| \le C_0\delta(1+\log t^{-1})\,, \qquad (t>0)$$

with some constant C_0. We use the triangle inequality on the integrand, and then apply the semigroup estimate and Eq. (4.1.3),

$$\begin{aligned}
&\|e^{(t-s)A_\rho}N(u_\rho) - e^{(t-s)A_R}N(u)\|_{2,1} \\
&\qquad\le \|(e^{(t-s)A_\rho} - e^{(t-s)A_R})N(u)\|_{2,1} + \|e^{(t-s)A_\rho}(N(u_\rho) - N(u))\|_{2,1} \\
&\qquad\le C_0\delta(1+\log t^{-1})\|N(u)\|_{2,1} + C_1\|N(u_\rho) - N(u)\|_{2,1} \\
&\qquad\le C_0C_\eta\delta(1+\log t^{-1})\|u\|_{2,1} + C_1C_\eta\|u_\rho - u\|_{2,1}\,,
\end{aligned}$$

so long as $\|u_\rho(s)\| \le \eta$. Here, $C_1 = \sup_t e^{tA_\rho}$, see Lemma 4.2.5, and C_η is the Lipschitz constant in Eq. (4.1.3). For the integral, it follows that

$$\|u_\rho(t) - u(t)\|_{2,1} \le C_2(T)\delta\|u_0\|_{2,1} + C_3\int_0^t \|u_\rho(s) - u(s)\|_{2,1}\, ds\,,$$

where $C_2(t) = C_1(1+2C_\eta t)(1+\log t^{-1})$, and $C_3 = C_2 C_\eta$. In the bound on the nonlinearity, we have used that $\|u_\rho(t)\|_{2,1} \le \frac{1}{2}\eta$ for $0 \le t \le T$. Set $C := C_2(T)e^{C_3 T}$. By Grönwall's inequality,

$$\|u_\rho(t) - u(t)\|_{2,1} \le C\delta\|u_0\|_{2,1}\,, \qquad (0 \le t \le T)$$

provided that $\sup_{0\le t\le T}\|u_\rho(t)\|_{2,1} \le \eta$. Since $\|u(t)\|_{2,1} \le \frac{1}{2}\eta$ for $0 \le t \le T$, by the triangle inequality this is guaranteed by setting $\delta_* = \min\left\{\frac{1}{16}, \frac{\eta}{2C}\right\}$.

□

4.3.2 Proof of Theorem 1.2.2

We consider the FHN system (3.0.1) on a standard cylinder $\mathcal{S}_R$ in a neighborhood of $\mathcal{M}$ as a reference. Fix a pulse $\Phi \in \mathcal{M}$. By Theorem 1.2.1, there are constants $\xi_0 > 0$ and $C_0 \ge 1$ such that

$$\mathrm{dist}\,(u(t), \mathcal{M}) \le C_0 e^{-\xi_0 t}\|u_0 - \Phi\|_{2,1}$$

for $t \ge 0$ and for every solution u with initial values in a neighborhood $\mathcal{U}$ of Φ in $H^{2,1}(\mathcal{S}_R)$. But Φ is translation invariant, so

$$\mathrm{dist}\,(u(t), \mathcal{M}) \le C_0 e^{-\xi_0 t}\mathrm{dist}\,(u_0, \mathcal{M}). \tag{4.3.2}$$

Eq. (4.3.2) holds for all u with $u_0 \in \mathcal{W}$, where $\mathcal{W}$ is a neighborhood of the manifold $\mathcal{M}$ of the form

$$\mathcal{W} = \left\{w \in H^{2,1} \;\middle|\; \mathrm{dist}\,(w, \mathcal{M}) < \eta\right\}$$

for some $\eta > 0$ with $C_0\eta \le \|\Phi\|_{2,1}$. Set $T := \frac{1}{\xi_0}\log(2C_0)$, so that $C_0 e^{-\xi_0 T} = \frac{1}{2}$. Using the triangle inequality we have:

$$\begin{aligned}\sup_{0\le t\le T}\|u(t)\|_{2,1} &\le \sup_{0\le t\le T}\mathrm{dist}\,(u(t), \mathcal{M}) + \sup_{\Phi_h\in\mathcal{M}}\|\Phi_h\|_{2,1}\\ &\le 2\|\Phi\|_{2,1}\end{aligned}$$

for all solutions on $\mathcal{S}_R$ with initial value $u\big|_{t=0} \in \mathcal{W}$.

Consider the FHN system (3.0.1) on a warped cylinder $\mathcal{S}_\rho$. Define $\delta := R^{-1}\|\rho - R\|_{C^2} \le \delta_*$, where the constant δ_* will be determined below. Given an initial condition $u_0 \in \mathcal{W}$, let $u(t)$ be the mild solution of FHN on $\mathcal{S}_R$ with initial condition $u\big|_{t=0} = u_0$. Since $\|u(t)\|_{2,1} \le 2\|\Phi\|_{2,1}$ for all $t \ge 0$, by Proposition 4.3.1 there exists $\delta_0 > 0$ and $C > 0$ such that

$$\sup_{0\le t\le T}\|u_\rho(t) - u(t)\|_{2,1} \le C\delta\,, \tag{4.3.3}$$

provided that $\delta \le \delta_0$. Choose

$$\delta_* := \min\left\{\delta_0, \frac{\eta}{2C}\right\}. \tag{4.3.4}$$

Assuming that $\delta \le \delta_*$, we observe that

$$\begin{aligned} \operatorname{dist}(u_\rho(t), \mathcal{M}) &\le \operatorname{dist}(u(t), \mathcal{M}) + \|u_\rho(t) - u(t)\|_{2,1} \\ &\le C_0 e^{-\xi_0 t} \operatorname{dist}(u_0, \mathcal{M}) + C\delta \\ &< 2C_0\eta \end{aligned} \tag{4.3.5}$$

for all $0 \le t \le T$. In Eq. (4.3.5) we first used the triangle inequality and then applied Eqs. (4.3.2) and (4.3.3). For $t = T$ and for δ_* as in Eq. (4.3.4) we have

$$\operatorname{dist}(u_\rho(T), \mathcal{M}) \le \frac{1}{2}\operatorname{dist}(u_0, \mathcal{M}) + C\delta < \eta\,. \tag{4.3.6}$$

Therefore, $u_\rho \in \mathcal{W}$ whenever $u_\rho(0) \in \mathcal{W}$.

Iterating Eq. (4.3.6) we obtain

$$\operatorname{dist}(u_\rho((k{+}1)T), \mathcal{M}) \le \frac{1}{2}\operatorname{dist}(u_\rho(kT), \mathcal{M}) + C\delta \tag{4.3.7}$$

for $k \in \mathbb{N}$. Solving the recursion we conclude that

$$\operatorname{dist}(u_\rho(kT), \mathcal{M}) \le 2^{-k}\operatorname{dist}(u_0, \mathcal{M}) + 2C\delta.$$

By Eq. (4.3.5):

$$\operatorname{dist}(u_\rho(t), \mathcal{M}) \le 2^{-k} C_0 \operatorname{dist}(u_0, \mathcal{M}) + (2 + C_0)C\delta$$

for all t with $kT \le t \le (k+1)T$ and all $k \in N$. For $T = \frac{1}{\xi_0} ln(2C_0)$ the above inequality implies

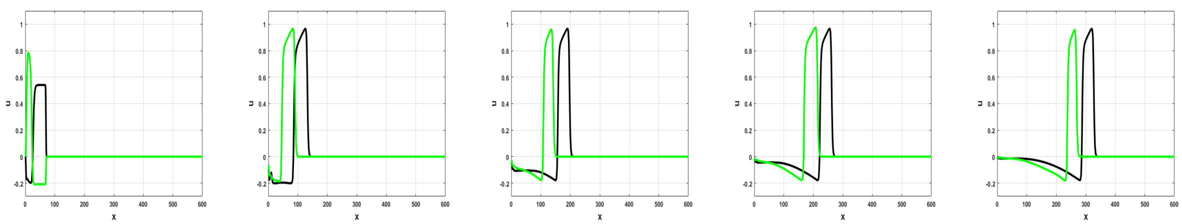
$$\operatorname{dist}(u(t), \mathcal{M}) \le C_1 e^{-\xi t}\operatorname{dist}(u_0, \mathcal{M}) + C_2\delta\,, \qquad (t \ge 0)$$

with $C_1 := 2C_0$, $C_2 := (2 + C_0)C$, $\xi = \frac{ln2}{ln(2+C_0)}\xi_0$, and all $t \le 0$.

$\square$

4.4 Numerical simulations

As in the standard cylinder we include numerical results of the simulations for $\mathcal{S} = \mathcal{S}_\rho$. Here the radius is defined by the function $\rho(x) = 0.78 + 0.023e^{\sin(4x)}$. The potential u_1 starts close to a pulse (Fig. 4.1a), and as time progresses it remains close to a translation of it, see (Fig. 4.1b−4.1e).



(a) $t = 0$ **(b)** $t = 200$ **(c)** $t = 400$ **(d)** $t = 600$ **(e)** $t = 800$

Figure 4.1: Near-pulse solution of the FitzHugh-Nagumo equation on $\mathcal{S} = \mathcal{S}_\rho$ (green pulse) compared to the constant pulse (black pulse). The radius on $\mathcal{S}_\rho$ is given by the function $\rho(x) = 0.78+0.023e^{\sin(4x)}$. The initial conditions are θ-independent and the boundary conditions are Dirichlet in the x-direction and periodic in the θ-direction for both pulses.

Observe that on the surface of a warped cylinder, the width of the pulse changes over time. For example, the pulse at the timestep $t = 800$ (Fig. 4.1e) is narrower compared to the pulse at $t = 600$ (Fig. 4.1d). On the other hand, the pulse on the surface of a constant cylinder, after the formation, propagates without changing its shape.

Appendix A

The Hille - Yosida Theorem

We state two theorems: the Hille-Yosida and the Lumer-Phillips. Both theorems characterize the linear operators that are the generator of a strongly continuous semigroup of contractions.

Recall that a semigroup e^{tB}, $t \geq 0$, of a bounded linear operator B, on a Hilbert space H, is **strongly continuous** if

$$\lim_{t \downarrow 0} e^{tB} u = u \text{ for every } u \in H. \tag{A.0.1}$$

Assume there exists a constant $\omega \in \mathbb{R}$ such that $\|e^{tB}\| \leq e^{\omega t}$, $t \geq 0$. Then e^{tB} is called a **contraction semigroup**.

Theorem A.0.1 (Hille-Yosida)**.** *For a linear operator B on a Hilbert space H, the following properties are equivalent.*

1. *B generates a strongly continuous semigroup of contractions*

2. *B is closed, densely defined, and for every $\lambda > 0$ one has λ in the resolvent set of B and*

$$\|\lambda (\lambda - B)^{-1}\| \leq 1\,. \tag{A.0.2}$$

For the proof we refer the reader to either [12] or [43]. The next lemma is a consequence of the Hille-Yoshida theorem.

Lemma A.0.2. *Let B be the infinitesimal generator of a strongly continuous semigroup of contractions. Its spectrum lies in the left half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq 0\}$ and its resolvent is bounded by*

$$\|(\lambda - B)^{-1}\| \leq \frac{1}{\operatorname{Re}\,\lambda} \qquad (\operatorname{Re}\lambda > 0). \tag{A.0.3}$$

Proof. For $\lambda > 0$ and $u \in H$ let

$$R(\lambda)u = \int_0^\infty e^{-\lambda t}\, e^{tB} u\, dt. \tag{A.0.4}$$

The above integral exists by continuity and it is well-defined for λ with $\operatorname{Re}\lambda > 0$. Taking the norms we have

$$\begin{aligned} \|R(\lambda)u\| &\leq \int_0^\infty e^{\lambda t}\, \|e^{tB}u\|\, dt \\ &\leq \frac{1}{\text{Re }\lambda}\, \|u\|. \end{aligned}$$

For $h > 0$,

$$\begin{aligned} \frac{e^{hB} - \mathbf{1}}{h}\, R(\lambda)u &= \frac{1}{h}\int_0^\infty e^{-\lambda t}\left(e^{(t+h)B}u - e^{tB}u\right) dt \\ &= \frac{e^{\lambda h} - 1}{h}\int_0^\infty e^{-\lambda t}\, e^{tB} u\, dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}\, e^{tB} u\, dt \\ &= \frac{e^{\lambda h} - 1}{h}\, R(\lambda)u - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}\, e^{tB} u\, dt. \end{aligned} \tag{A.0.5}$$

Taking the limit as $h \to 0^+$, the left-hand side of Eq. (A.0.5) converges to $BR(\lambda)u$ and the right-hand side to $\lambda R(\lambda)u - u$. That is, $R(\lambda)u \in \operatorname{Dom}(B)$ and

$$(\lambda - B)R(\lambda) = \mathbf{1}. \tag{A.0.6}$$

For $u \in \operatorname{Dom}(B)$ and since B is closed we have

$$\begin{aligned} R(\lambda)Bu &= \int_0^\infty e^{-\lambda t}\, e^{tB} Bu\, dt \\ &= \int_0^\infty e^{-\lambda t} B e^{tB} u\, dt \\ &= B\left(\int_0^\infty e^{-\lambda t}\, e^{tB} u\, dt\right) \\ &= BR(\lambda)u. \end{aligned} \tag{A.0.7}$$

Combining Eq. (A.0.6) and Eq. (A.0.7) we obtain that

$$R(\lambda)(\lambda - B)u = u \qquad (u \in \operatorname{Dom}(B)).$$

Therefore, $R(\lambda) = (\lambda - B)^{-1}$ which implies that the resolvent set of B contains the open right half-plane. The estimate (A.0.3) follows directly.

□

The second theorem gives a characterization for generators of contraction semigroups that does not require explicit knowledge of the resolvent. Let us first recall the definition of dissipativity and then we state the theorem. In a Hilbert space H, we say that a linear operator B is **dissipative** if for every $u \in \mathrm{Dom}(B) \subset H$,

$$\mathrm{Re}\langle Bu, u\rangle \leq 0\,.$$

Theorem A.0.3 (Lumer-Phillips)**.** *For a densely defined, dissipative operator B on a Hilbert space H the following estimates are equivalent.*

1. *The closure $\bar{B}$ of B generates a strongly continuous semigroup of contractions.*

2. *The range of $\lambda - B$, $\mathrm{Ran}\,(\lambda - B)$, is dense in H for all $\lambda > 0$.*

For the complete proof of Theorem A.0.3 see Theorem 1.4.3, [43].

Appendix B

Inversion formulas

In this appendix we state two inversion formulas. Let $\omega \in \mathbb{R}$. Assume there exists a constant $M \geq 1$ such that

$$\|e^{tB}\| \leq Me^{\omega t} \tag{B.0.1}$$

for all $t \geq 0$. Without requiring that the semigroup e^{tB} is bounded, we have:

Lemma B.0.1. *Let B be the infinitesimal generator of a strongly continuous semigroup on a Banach space X. Then*

$$e^{tB}u = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\omega - in}^{\omega + in} e^{\lambda t}(\lambda - B)^{-1}u \, d\lambda \tag{B.0.2}$$

for all $u \in \mathrm{Dom}(B)$ and ω such that Eq. (B.0.1) holds, with uniform convergence for t in compact intervals of $(0, \infty)$.

Notice that in Eq. (B.0.2) the integral does not converge absolutely. Hence, it is difficult to derive from it estimates on the semigroup e^{tB}. Adding more regularity on x, the representation of $e^{tB}u$ in the next lemma will converge absolutely.

Lemma B.0.2. *Let e^{tB}, $t \geq 0$, be a strongly continuous semigroup on a Banach space X. Then*

$$e^{tB}u = \frac{(k-1)!}{t^{k-1}} \frac{1}{2\pi i} \lim_{n\to\infty} \int_{\omega - in}^{\omega + in} e^{\lambda t} \left[(\lambda - B)^{-1}\right]^k u \, d\lambda \tag{B.0.3}$$

for all ω such that Eq. (B.0.1) holds, $k \in \mathbb{N}$, $t > 0$, and $x \in \mathrm{Dom}(B^2)$. Moreover, if $k \geq 2$, then the integral converges absolutely and uniformly for $t > 0$.

Complete proofs of Lemmas B.0.1 and B.0.2, as well as more inversion formulas can be found in the book [11].

Appendix C

Proof of Gearhart-Prüss Theorem

In Chapter 3 we proved exponential decay of the strongly continuous semigroup $e^{tL}Q$ on the Hilbert space $H^{2,1}$, using the Gearhart-Prüss theorem. Recall that L is the linear operator given by Eq. (3.0.4) and Q is the projection to the range of L. Notice that the Gearhart-Prüss theorem does not hold on arbitrary Banach spaces but holds on Hilbert spaces. Here we prove the Gearhart-Prüss theorem. We adopt the proof from [11].

Theorem C.0.1 (Gearhart-Prüss)**.** *Suppose that the operator $B : D(B) \subset H \to H$, where H is a Hilbert space, generates a strongly continuous semigroup. If the resolvent $(\lambda - B)^{-1}$ is uniformly bounded on the half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > 0\}$, then there exists a constant $M > 0$ such that the semigroup e^{tB} is exponentially stable for all $t \geq 0$.*

Proof. For each λ_0 in the resolvent set of B, the **spectral radius** of the resolvent operator $(\lambda_0 - B)^{-1}$ is given by

$$\operatorname{dist}(\lambda_0, \sigma(B)) \geq \frac{1}{\|(\lambda_0 - B)^{-1}\|}\,. \tag{C.0.1}$$

From Eq. (C.0.1) we have that $i\mathbb{R}$ is contained in the resolvent set of B. Hence the uniform boundedness of the resolvent $(\lambda - B)^{-1}$ extends by continuity to $\operatorname{Re}\lambda \geq 0$.

Define

$$\omega_0 := \inf\left\{\beta \in \mathbb{R} \ : \ \lim_{t\to\infty} e^{-\beta t}\|e^{tB}\| = 0\right\}.$$

Take $\beta > |\omega_0| + \iota$, for $\iota > 0$ small, and consider the semigroup $S(t) = e^{-\beta t}e^{tB}$, for $t \geq 0$. In Lemma A.0.2 we proved, for $u \in H$, that

$$(\lambda - B)^{-1}u = \int_0^{+\infty} e^{-\lambda t} e^{tB} u\, dt.$$

So, for $s \in \mathbb{R}$ we have that

$$\big((\beta + is) - B\big)^{-1} u = \big(is - (B - \beta)\big)^{-1} u = \int_0^{+\infty} e^{-ist} S(t) u \, dt.$$

For $t < 0$, set $S(t) := 0$, so that $S(t)$ is defined for all $t \in \mathbb{R}$. Using the Fourier transform we have

$$\big((\beta + is) - B\big)^{-1} u = \hat{S}(s) u.$$

Plancherel's theorem implies that

$$\begin{aligned} \int_{-\infty}^{+\infty} \left\| \big((\beta + is) - B\big)^{-1} u \right\|^2 ds &= \int_{-\infty}^{+\infty} \|\hat{S}(s) u\|^2 \, ds \\ &= 2\pi \int_0^{+\infty} \|S(t) u\|^2 \, dt \\ &\le C \|u\|^2 \end{aligned} \tag{C.0.2}$$

for some positive constant C and $u \in H$.

By the resolvent identity we have

$$(is - B)^{-1} = \big((\beta + is) - B\big)^{-1} + \beta (is - B)^{-1} \big((\beta + is) - B\big)^{-1}$$

which implies

$$\begin{aligned} \left\| (is - B)^{-1} u \right\| &\le \left\| \mathbf{1} + \beta (is - B)^{-1} \right\| \left\| \big((\beta + is) - B\big)^{-1} u \right\| \\ &\le (1 + \beta M) \left\| \big((\beta + is) - B\big)^{-1} u \right\|. \end{aligned} \tag{C.0.3}$$

for some $M > 0$, and for all $s \in \mathbb{R}$, $u \in H$. Combining Eqs. (C.0.2) and (C.0.3) we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \left\| (is - B)^{-1} u \right\|^2 ds &\le (1 + \beta M)^2 \int_{-\infty}^{+\infty} \left\| \big((\beta + is) - B\big)^{-1} u \right\|^2 ds \\ &\le (1 + \beta M)^2 \, C^2 \, \|u\|^2 \end{aligned} \tag{C.0.4}$$

for all $u \in H$. Since $\|S\| = \|S^*\|$, where S^* is the adjoint semigroup generated by the adjoint operator B^*, the resolvent $(is - B^*)^{-1}$ satisfies the same estimate as above, i.e.,

$$\int_{-\infty}^{+\infty} \left\| (is - B^*)^{-1} w \right\|^2 ds \le (1 + \beta M)^2 \, C^2 \, \|w\|^2 \tag{C.0.5}$$

for all $w \in H$.

Using the formula in Lemma B.0.2 for $k = 2$ we conclude that

$$\begin{aligned}(te^{tB}u, w) &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{(\beta+is)t}\big(\big((\beta+is-B)^{-1}\big)^2 u, w\big)\,ds \\ &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{ist}\big((is-B)^{-1}u, (-is-B^*)^{-1}w\big)\,ds\end{aligned} \tag{C.0.6}$$

for all $u \in \mathrm{Dom}(B^2)$ and $w \in H$. For the second equality we used Cauchy's integral theorem, which is applicable since $(\lambda - B)^{-1}$ is uniformly bounded for $\mathrm{Re}\,\lambda \geq 0$, and hence

$$\begin{aligned}\|(\lambda - B)^{-1}u\| &= \frac{1}{|\lambda|}\|(\lambda - B)^{-1}Bu + u\| \\ &\leq \frac{1}{|\lambda|}(M\|Bu\| + \|u\|).\end{aligned}$$

Applying the Cauchy-Schwarz inequality and using Eqs. (C.0.4) and (C.0.5), Eq. (C.0.6) gives

$$\begin{aligned}|(te^{tB}u, w)| &\leq \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty}\left\|(is-B)^{-1}u\right\|^2 ds\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}\left\|(is-B^*)^{-1}w\right\|^2 ds\right)^{\frac{1}{2}} \\ &\leq \frac{1}{2\pi}\left\|(is-B)^{-1}u\right\|\left\|(is-B^*)^{-1}w\right\| \\ &\leq \frac{1}{2\pi}(1+\beta M)^2\, C^2\,\|u\|\,\|w\|\end{aligned}$$

for all $u, w \in \mathrm{Dom}(B^2)$. Since $\mathrm{Dom}(B^2)$ is dense in H, the above inequality implies that

$$\begin{aligned}\left\|te^{tB}\right\| &= \sup\{|(te^{tB}u, w)| : u, w \in \mathrm{Dom}(B^2),\ \|u\| = \|w\| = 1\} \\ &\leq \frac{1}{2\pi}(1+\beta M)^2\, C^2, .\end{aligned}$$

The latter inequality implies that $\lim_{t\to\infty}\|S(t)\| = 0$, as desired.

□

The converse implication is also true. In other words, if the semigroup e^{tB} is exponentially stable, then $\|(\lambda - B)^{-1}\|$ is uniformly bounded in the half-plane $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq -\beta\}$. This follows directly from the fact that

$$\|(\lambda - B)^{-1}\| \leq \frac{M}{\mathrm{Re}\,\lambda + \beta}$$

when $\|e^{tB}\| \leq Me^{-\beta t}$, $t \geq 0$ in $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq -\beta\}$.

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