Refinements of Alexandrov's estimate

The Monge-Ampere equation is a nonlinear PDE arising in differential geometry and optimal transportation that in its simplest form can be written

$$\det D^2 u = f, \qquad u : \Omega \to \mathbb{R} \text{ convex}, \qquad \Omega \subset \mathbb{R}^n. \tag{1}$$

Here D^2u is the matrix of second derivatives $(\frac{\partial^2 u}{\partial x_i \partial x_j})_{i,j=1}^n$ of the function u. Typically the equation is studied with boundary conditions such as $u = 0$ on $\partial\Omega$. The requirement that u is convex forces f to be nonnegative. The equation may be understood in a weak sense if u is not C^2.

Regularity theory for the Monge-Ampere equation asks about the exact degree of smoothness of a solution u, in terms of the smoothness of ingredients in the equation: the right-hand side f (or more nonlinear right-hand sides, depending on u and/or Du, arising for example from certain geometric questions), the boundary data, and possibly the domain geometry. It is an important topic within the broader area of regularity theory for elliptic PDE, and it has been a major focus of the work of Caffarelli (Abel Prize 2023) and Figalli (Fields Medal 2018). A basic ingredient in this regularity theory is Alexandrov's estimate (1960s), which states that if u solves (1) and $u = 0$ on $\partial\Omega$, then

$$[u]_{1/n} \leq C_n (\mathrm{diam}\Omega)^{\frac{n-1}{n}} \left(\int_\Omega f \, dx \right)^{1/n}$$

where C_n is a constant depending only on the dimension n, and

$$[u]_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

The goal of this project will be to establish a refined Alexandrov estimate, in some ways improving upon that in the preprint `https://arxiv.org/abs/2310.20612`, which grew out of an NSERC USRA from summer 2023.

Work on this project will require a solid grasp of real analysis at the level of at least MAT357; familiarity with PDEs is not needed.

This research is connected to another project for this summer, **Total Variation of the Hessian Determinant**, and students working on the two projects should expect to talk frequently and exchange ideas.

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