DEPARTMENT OF MATHEMATICS
University of Toronto

Real Analysis Comprehensive Exam
2 hours

September, 5 2018

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Make sure to justify all your work. If you make a reference to a result in the textbook, please make sure to state it explicitly (and correctly!).

Problem 1

Each of the following items can be solved independently

(a) Let X, Y be Banach spaces and $S, T : X \to Y$ two bounded linear operators; show that if T is a bijection, there exists $\delta > 0$ so that if $\|S - T\| < \delta$ then S is a bijection.

(b) Is it possible to find uncountably many disjoint measurable subsets of $\mathbb{R}$ with strictly positive Lebesgue measure? Either give an example or show that this is impossible.

(c) Let $p > 1/2$ and assume $(x^p + x^{-p})f \in L^2(0, \infty)$; show that $f \in L^1(0, \infty)$.

Problem 2

Let $\mathcal{M}$ denote the set of finite signed Borel measures on $\mathbb{R}$; given $z \in \mathbb{R}$ and $r > 0$, denote with $B(z, r) = \{y \in \mathbb{R} \text{ s.t. } |z - y| < r\}$ the ball of radius r centered at z. Given two $\mu, \mu' \in \mathcal{M}$, define

$$(\mu, \mu')_r = \int_{\mathbb{R}} \mu(B(z, r))\mu'(B(z, r))dz.$$

(a) show that $\|\mu\|_r = \sqrt{(\mu, \mu)_r}$ is a norm on $\mathcal{M}$ and deduce that

$$(\mu, \mu')_r \le \|\mu\|_r \|\mu'\|_r.$$

(b) Let $\mu \in \mathcal{M}$ be so that $\mu(\mathbb{R}) = 1$ and define

$$\ell_r(\mu) = \liminf_{r \to 0^+} \frac{\|\mu\|_r}{r}.$$

Show that if $\ell_r(\mu) < \infty$, then $\mu \ll \text{Leb}$ and

$$\left\|\frac{d\mu}{dx}\right\|_{L^2} \le \ell_r(\mu)$$

[Hint: For $r > 0$ define $g_r(x) = r^{-1}\mu(B(x, r))$; show that there exists a sequence $r_k \to 0^+$ so that $\lim_{k\to\infty} \|g_{r_k}\|_{L^2} \le \ell_r(\mu)$ and g_{r_k} converges weakly to some $\bar{g}$. Show that $\frac{d\mu}{dx} = \bar{g}$.]

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Problem 3

(a) State the Hahn Banach theorem, the open mapping theorem, the uniform boundedness principle and Alaoglu's theorem.
(b) Define the weak topology on a Banach space $\mathcal{X}$, and the weak-$*$ topology on its dual $\mathcal{X}^*$.
(c) Suppose that B is the closed unit ball in $\mathcal{X}$, and that $\mathcal{X}$ is reflexive. Prove that B is compact in the weak topology on $\mathcal{X}$.

Problem 4

a) What is the general formula for the Fourier transform of a function f on $\mathbb{R}$? The Fourier inversion formula? The Plancherel formula?
b) Explain what conditions you can put on f in order to make these formulas valid.
c) (Optional question on distribution theory)
Can you define the Fourier transform of the function

$$f(x) = 1 + 2x + 3x^2 + ... + (m+1)x^m?$$

What is it?

— End of the exam —