Department of Mathematics
Comprehensive Exam 2022
First half: Monday, September 26, 6pm-9pm
BA6183

Last name ....

First name ....

Email ....

Solve 6 questions out 14 questions. Indicate which questions you want to be graded. Passing score is 80 percent from those 6 questions.

1. (10 pts)

Take analytic function $f(z)$, $z \in \mathcal{D}$. Prove that *Maximum Module Principle* is valid: if $f \not\equiv const$, then $|f(z)|$ has no local maxima in $\mathcal{D}$. In particular if $\mathcal{D}$ is *bounded* and $f(z)$ *extends continuously to the boundary* $\partial\mathcal{D}$, then $|f(z)|$ *assumes its maximum value on the boundary.*

2. (10 pts) Let $X_i, i \in \mathbb{N}$ be a sequence of independent random variables such that $\mathbb{E}X_i = 1/2$ and $X_i \in [-1, 1]$ almost surely for all $i \in \mathbb{N}$. Show that almost surely,

$$\lim_{n\to\infty} \sum_{i=1}^{n} X_i = \infty.$$

3. (10 pts) An $n \times n$ matrix with complex valued entries, A, is Hermitian if $A^* = A$ where A^* is the matrix constructed from A via $(A^*)_{ij} = \overline{A_{ji}}$ for all $1 \le i, j \le n$.

Let A be tridiagonal and Hermitian, with all of its subdiagonal and superdiagonal entries nonzero. (That is, $A_{i,i+1}$ and $A_{i+1,i}$ nonzero for $1 \le i \le n-1$.) Prove that the eigenvalues of A are distinct.

4. (10 pts) Let $K := \mathbb{Q}(\zeta, \alpha)$ be the subfield of $\mathbb{C}$ generated by $\zeta := e^{2\pi i/5}$ and $\alpha := \sqrt[5]{3}$.

a) Show that $K/\mathbb{Q}$ is Galois and determine its degree $[K : \mathbb{Q}]$.

b) Show that the Galois group $Gal(K/\mathbb{Q})$ is a semidirect product $N \rtimes H$ for some non-trivial subgroups N and H (you do not have to determine the action of H on N).

c) Determine all fields F such that $\mathbb{Q} \subset F \subset K$ and $[F : \mathbb{Q}] = 4$.

5. (10 pts)

a) State the homology groups of the circle S^1. You do not need to provide a proof

b) Use a Mayer-Vietoris sequence to compute the homology of the 2-sphere S^2 with integer coefficients.

c) Find the cohomology ring of $S^1 \times S^1$ with rational coefficients. You need to state a basis for each cohomology group and the cup product of any two of the basis elements. You do not need to supply a proof.

6. (10 pts) The Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is a unitary operator.

a) Use the inversion theorem to prove $\mathcal{F}$ can have at most four eigenvalues.

b) Use your knowledge of its properties to Find eigenvectors corresponding to each of these four eigenvalues.

c) Use the Fourier coefficients of the function $f(x) = x$ in $L^2[0,1]$ to compute $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$.

7. (10 pts) Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain. Consider a weak solution $u \in H_0^1(\Omega)$ of the nonlinear problem

$$-\Delta u = u^3, \qquad x \in \Omega. \tag{$*$}$$

1. State a general regularity theorem for a linear problem $Lu = f$, where L is a linear, non-constant coefficients, uniformly elliptic operator, and f is given.

2. In which dimensions can you prove that a weak solution of $(*)$ is in $H^2_{\text{loc}}(\Omega)$?

3. Prove that for $n \leq 3$ a weak solution of $(*)$ is smooth.

   What can you say about $n = 4$?