Department of Mathematics
Comprehensive Exam 2022
Second half: Tuesday, September 27, 6pm-9pm
BA1200

Last name ....

First name ....

Email ....

Solve 6 questions out 14 questions. Indicate which questions you want to be graded. Passing score is 80 percent from those 6 questions.

1. (10 pts)

Denote

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \mathcal{M} \setminus \{0\}} \left[\frac{1}{(z-\omega)} - \frac{1}{\omega} - \frac{z}{\omega^2} \right],$$

where $\mathcal{M}$ is an elliptic module.

(i) Consider the series

$$\mathcal{Q} = \sum_{\omega \in \mathcal{M} \setminus \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

Take arbitrary path C from 0 to $z \notin \mathcal{M}$ away from the set $z \notin \mathcal{M}$. Show that

$$\mathsf{q}(z) := -\int_C \mathcal{Q}(\mathsf{z}) d\mathsf{z} := -\int_0^z \mathcal{Q}(\mathsf{z}) d\mathsf{z}$$

does not depend on C.

(ii) In (i) one can integrate termwise and the series of integrals with z varying in a small disk away from the set $\mathcal{M}$ converges absolutely and uniformly, i.e.

$$\mathsf{q}(z) = -\sum_{\omega \in \mathcal{M} \setminus \{0\}} \int_0^z \left[\frac{1}{(\mathsf{z}-\omega)^2} - \frac{1}{\omega^2} \right] d\mathsf{z} = \sum_{\omega \in \mathcal{M} \setminus \{0\}} \left[\frac{1}{(z-\omega)} - \frac{1}{\omega} - \frac{z}{\omega^2} \right]$$

(iii) $\mathsf{q}'(z) = -\mathcal{Q}(z)$, $\zeta'(z) = -\wp(z)$.

(iv) $\eta_i := \zeta(z+\omega_i) - \zeta(z) = const.$

(v) Pick c so $0 \in \mathsf{P}_c$ is the only point of $\mathcal{M}$ inside the contour $\partial \mathsf{P}_c$. Then

$$\frac{1}{2\pi i} \oint_{\partial \mathsf{P}_c} \zeta(z) dz = 1$$

(vi) Legendre Relation: $\omega_2\eta_1 - \omega_1\eta_2 = 1$. In particular, $\zeta(z)$ is *not double periodic*. In other words, thought the derivative of an elliptic function is always an elliptic function, *the anti-derivative, provided it exists, i.e all the residues are zero, might be not elliptic function.*

2. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. The Newton-Raphson method is an iterative method for trying to find solutions of $f(x) = 0$. Given a first guess, x_0, it constructs a sequence $\{x_n\}$ via the rule

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \qquad i \geq 0$$

Assume $p \in (a, b)$ is such that $f(p) = 0$ and that f, f', and f'' are all continuous on (a, b). Show that if $f'(p) \neq 0$ then there exists $\delta > 0$ such that if $x_0 \in (p - \delta, p + \delta)$ then the Newton-Raphson method generates a sequence that converges to p and that the convergence is quadratic:

$$|x_{i+1} - p| \leq M|x_i - p|^2 \qquad \text{for some } M < \infty \text{ and all } i \geq 1.$$

3. (10 pts) Consider the group A_4 of even permutations on four elements. All representations in this question are over $\mathbb{C}$.

a) Show that A_4 has three distinct one-dimensional representations. (It may help to consider a cyclic quotient group of A_4.)

b) Construct the character table of A_4, explaining your reasoning. You may use that the conjugacy classes have representatives e (the identity), $(1\ 2)(3\ 4)$, $(1\ 2\ 3)$, $(1\ 3\ 2)$.

c) Consider the representation V of A_4 on $\mathbb{C}^4$, given by permuting the coordinates. In other words, $\sigma(z_1, \ldots, z_4) = (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(4)})$ for $\sigma \in A_4$ and $z_i \in \mathbb{C}$. Decompose V into irreducible representations.

4. (10 pts) Two immersions $F_0 : M \to N$ and $F_1 : M \to N$ are regular homotopic if there exists homotopy $F : [0,1] \times M \to N$ between them, such that $F_t(x) = F(t,x)$ is an immersion for all $t \in [0,1]$. Consider two maps $f_0, f_1 : S^1 \to S^2$ defined by

$$f_0(\cos(\theta), \sin(\theta)) = (\cos(\theta), \sin(\theta), 0)$$

$$f_1(\cos(\theta), \sin(\theta)) = (\cos(2\theta), \sin(2\theta), 0)$$

a) Prove that both maps are immersions.

b) Are they regular homotopic?

5. **(10 pts)** Let Z be a standard normal random variable and $f : \mathbb{R} \to \mathbb{R}$ a function with bounded first derivative. Prove,

$$\mathbb{E}\left[f'(Z)\right]^2 \leq \mathrm{V}ar(f(Z)) \leq \mathbb{E}\left[f'(Z)^2\right]$$

6. (10 pts) Let $u : (x,t) \in \mathbb{R}^n \times [0,\infty)$ be a solution of the $n+1$ dimensional Klein-Gordon equation

$$\begin{cases} u_{tt} - \Delta u + mu = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ \\ u = g, \quad u_t = h & \text{on} \quad \mathbb{R}^n \times \{0\}. \end{cases} \tag{KG}$$

where $m > 0$ is a mass parameter.

a. Let

$$e[u](t) := \frac{1}{2} \int_{\mathbb{R}^d} \left(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2 + mu^2 \right) dx.$$

Prove that $e[u](t)$ is constant for sufficiently smooth solutions u of (KG) with $f = 0$

b. Do solutions of (KG) with $f = 0$ enjoy finite speed of propagation as in the case $m = 0$?

c. Write a conserved energy for the nonlinear problem (KG) with $f = u^3$.

d. Suppose you are given two bounded solutions u_1 and u_2 of the nonlinear problem (KG) with $f = u^3$, with compactly supported initial data.

Differentiate $e[u_1 - u_2](t)$ and derive a Gronwall's type inequality for it.

What consequences can you deduce from such an inequality?

7. (10 pts)

a) Let d be a positive integer. Establish the Vitali covering lemma on $\mathbb{R}^d$; that is, show that if $B_1, \cdots, B_p$ are a finite sequence of open balls in $\mathbb{R}^d$, then there exists a subsequence $B_{i_1}, \cdots, B_{i_l}$ so that

i. $j \neq k \Rightarrow B_{i_j} \cap B_{i_k} = \emptyset$.

ii. $\cup_{i=1}^{p} B_i \subset \cup_{j=1}^{l} \widetilde{B_{i_j}}$.

Here $\widetilde{B_{i_j}}$ denotes the open ball with the same center as B_{i_j} and with a radius three times as big as B_{i_j}.

b) For any $x \in \mathbb{R}^d$, let $B_r(x)$ denote the open ball of radius r around x. We recall the Hardy–Littlewood maximal function:

$$Hf\,(x) \doteq \sup_{r>0} \frac{1}{m\,(B_r(x))} \int_{B_r(x)} |f\,(x)|\;dx,$$

where m denotes the Lebesgue measure on $\mathbb{R}^d$. Use the Vitali covering lemma to prove that there exists a constant $C > 0$ such that for every measurable function f and constant $\alpha > 0$, we have

$$m\,(x : Hf(x) > \alpha) \leq C\alpha^{-1}\,||f||_{L^1\left(\mathbb{R}^d\right)}\,.$$

c) State and prove the Lebesgue differentiation theorem.