1. (a) Suppose that $G$ is a (non-trivial) finite group and let $p$ be the smallest prime divisor of the order of $G$. Show that any normal subgroup of $G$ of order $p$ is contained in the centre of $G$.

(b) Suppose that $G$ is a finite simple group and $p$ a prime number such that $p^2$ divides the order of $G$. Show that any proper subgroup $H$ of $G$ has index $(G : H)$ at least $2p$. (Hint: use a suitable group action.)

(c) In the setting of part (b), give an example where equality can hold when $G = A_6$ (i.e. find $p$ and $H$ such that $p^2$ divides $|G|$ and $(G : H) = 2p$).

2. Prove that every discrete normal subgroup of a connected topological group is abelian.

3. Prove the following version of the Dominated Convergence Theorem: Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f, f_1, f_2, \ldots$ be measurable functions on $(X, \mathcal{M})$. If

- $f_n \to f$ for $\mu$ a.e. $x$,

and there exist $g, g_1, g_2, \ldots \in L^1(X, \mu)$ such that

- $|f_n| \leq |g_n|$, and
- $\|g_n - g\|_1 \to 0$ as $n \to \infty$,

then $\|f_n - f\|_1 \to 0$ as $n \to \infty$. 
4. The purpose of this problem is to prove that, if $P(z)$ is a non-constant complex polynomial, then the zeros of $P'(z)$ lie in the convex hull of the set of zeros of $P$.

   (a) Suppose that $P(z)$ has degree $n \geq 1$ and zeros $b_1, \ldots, b_n$ (each zero listed as many times as its multiplicity). Show that
   \[
   \frac{P'(z)}{P(z)} = \sum_{k=1}^{n} \frac{1}{z - b_k}.
   \]

   (b) Show that, if $P'(z) = 0$, then
   \[
   \left( \sum_{k=1}^{n} \frac{1}{|z - b_k|^2} \right) \overline{z} = \sum_{k=1}^{n} \frac{\overline{b_k}}{|z - b_k|^2}.
   \]

   (c) Deduce that, if $P'(z) = 0$, then $z$ lies in the convex hull of the points $b_k$.

5. Let $S^2$ be the 2-sphere and $A \subset S^2$ a subset of cardinality 3. We denote by $S^2/A$ the space obtained by contracting $A$ to a point.

   (a) What is the fundamental group of $S^2/A$?

   (b) Compute the singular cohomology groups $H^*(S^2/A; \mathbb{Z})$ and the cup product structure. Justify your answers with proofs.

6. Let $G = \text{Gal}(K/F)$ be the Galois group of the splitting field of a monic, integral polynomial $f(x) \in \mathbb{Z}[x]$ of degree $n$.

   (a) Prove that $G$ acts by permutation on the roots of $f(x)$ in $K$.

   (b) Under what condition on $G$ is $f(x)$ irreducible over $\mathbb{Q}$?

   (c) More generally, under what conditions on $G$ do the irreducible factors of $f$ have degrees $(n_1, \ldots, n_r)$?

   (d) What about irreducibility over $\mathbb{Z}$?