Quadratic Poisson brackets and co-Higgs fields

by

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Contents

Chapter 1

Introduction

The goal of this thesis is to develop new geometric constructions in the theory of holomorphic Poisson brackets. One of the two main objects of this thesis is a Poisson bivector σ on a total space of a holomorphic vector bundle V over a complex manifold X that is invariant with respect to the scaling action of $\mathbb{C}^*$ on the fibers of V. We are going to call such a bivector *quadratic* as it has terms of order at most two in its fiberwise Taylor expansion. In a sense, quadratic Poisson structures are one step higher in complexity than linear Poisson structures, which have been extensively studied in the framework of Lie algebroids and Lie groupoids (see [7] and references therein).

The standing assumption on a quadratic Poisson structure σ, which we are going to adopt almost everywhere below, is that the fibers of V are coisotropic. This is equivalent to requiring that the pushforward of σ onto the base manifold X is zero. Such σ naturally induces a tensor of the form $\phi \in \mathrm{Hom}(V, V \otimes \mathcal{T}_X)$ (see Lemma 3.2.1), where $\mathcal{T}_X$ is the tangent bundle of X. The latter tensor, which we call a *co-Higgs field* on V, is the second main object of this thesis. Poisson integrability of σ leads to the integrability condition $\phi \wedge \phi = 0 \in \mathrm{Hom}(V, V \otimes \wedge^2 \mathcal{T}_X)$. Co-Higgs fields are analogues of Higgs fields developed by Hitchin [16] and Simpson [34, 35], which are defined analogously using the cotangent bundle $\mathcal{T}_X^*$ instead of $\mathcal{T}_X$. Co-Higgs bundles, i.e. vector bundles with co-Higgs fields on them, play an important role in generalized complex geometry developed by Hitchin [18] and Gualtieri [13, 15], as they serve as generalized vector bundles. Rayan's PhD thesis [27] discusses the moduli space of co-Higgs bundles in great detail (also, see [28, 29]). The questions we will strive to address in the current thesis are:

- How do the properties of the co-Higgs field ϕ reflect those of the quadratic Poisson structure σ?
- What does it take to recover the quadratic Poisson structure σ from its co-Higgs field ϕ?

The motivation for posing these questions is the idea that the co-Higgs tensor ϕ as a certain simplification of the Poisson tensor σ. This can be justified as follows. Firstly, ϕ lives on a lower dimensional manifold compared to σ. Secondly, and more importantly, for constructing ϕ and describing its properties there exists a powerful geometric tool of *spectral correspondence* described by Beauville, Narasimhan, Ramanan in the case of $\dim X = 1$ [3] and Simpson in the general case [35]. In particular, instead of dealing directly with the Poisson integrability equation $[\sigma, \sigma] = 0 \in H^0(V, \wedge^3 V)$, which is typically an overdetermined system of non-linear PDEs, one can deal with spectral varieties in $\mathcal{T}_X$ and sheaves on them.

Before discussing the posed questions, let us point out that there is a parallel version of the story, where instead of a quadratic Poisson structure on a vector bundle V we consider a Poisson structure on the projectivization $\mathbb{P}(V)$. The former always induces the latter, but the converse is true only under some additional assumptions. Previously, it was shown by Bondal [4] and Polishchuk [26, Theorem 12.1] that this is true in the case when X is a point; we show the case when the total space of V is Calabi-Yau (Theorem 3.1.5), and a family of counterexamples in the case $X = \mathbb{P}^1$, rk $V = 2$ (Corollary 4.1.4). A Poisson structure on $\mathbb{P}(V)$ induces a zero trace co-Higgs field on V (see Lemma 3.2.3), and most of the questions about the correspondence between quadratic Poisson structures and co-Higgs fields may, and will, be recast in the $\mathbb{P}(V)$ setup as well.

If V has rank 1, i.e. V is a line bundle, then a co-Higgs field on V is just a vector field v on X. Each vector field v lifts uniquely to a quadratic Poisson structure σ on V via the formula $\sigma = \widetilde{v} \wedge Eul$, where $\widetilde{v}$ is any choice of a local $\mathbb{C}^*$-invariant vector field on V projecting onto v and Eul is the Euler vector field on V. For the case when $\mathbb{P}(V)$ is a projective line bundle, Polishchuk showed [26, Theorem 6.1] that any zero trace co-Higgs field ϕ on V uniquely lifts to a Poisson structure on $\mathbb{P}(V)$. It turns out that in rank $r > 1$, if one is to have any hope of finding a Poisson lift of a co-Higgs field ϕ, then on top of the usual integrability condition $\phi \wedge \phi = 0$, one should impose what we are calling the *strong integrability* condition on ϕ. Strong integrability of ϕ means that the coefficients $s_i(\phi) \in H^0(X, S^i\mathcal{T}_X)$, $i = 1, 2, ..., r$, of the characteristic polynomial of ϕ pairwise Poisson commute when viewed as fiberwise polynomial functions on $\mathcal{T}_X^*$. This imposes quite a restriction on the spectral cover of ϕ. For instance, if $X = \mathbb{P}^1$ and the spectral curve is reduced and irreducible, then one must have $s_i(\phi) = 0$, $i = 1, 2, ..., r-1$ (Corollary 4.0.2). Moreover, there may be further local obstruction to the existence of a Poisson lift near the branch points of the spectral cover. Somewhat surprisingly, co-Higgs fields of rank $r > 2$ over a small one dimensional disc whose spectral curves are smooth and connected cannot be lifted to either a Poisson structure on $\mathbb{P}(V)$ (Proposition 4.3.1), or a quadratic Poisson structure on V (Corollary 4.3.2).

Let us outline the proof of the projective bundle version of the fact above, as it strongly uses the interaction between the co-Higgs field and its Poisson lift to $\mathbb{P}(V)$, which we find particularly illustrative. For a co-Higgs field ϕ on V, one can define its variety of eigenvectors as the set of $x \in X$, $0 \neq v \in V|_x$ such that $v \wedge \phi_x(v) = 0 \in (\wedge^2 V \otimes \mathcal{T}_X)|_x$. Its projectivization defines a subvariety E of $\mathbb{P}(V)$, which we are calling the *eigenvariety* of ϕ. The latter is closely related, but generally not isomorphic, to the spectral variety $\Sigma \subset \mathcal{T}_X$ of ϕ. It turns out that the eigenvariety E of ϕ contains the zero set of any Poisson lift σ to $\mathbb{P}(V)$, and moreover, under certain mild genericity conditions the two sets are equal. Recall that the zero set of any Poisson structure carries a special vector field, called the *modular vector field*, introduced by Weinstein [37]. In the case when the Poisson structure σ lives on $\mathbb{P}(V)$ and has coisotropic fibers, we prove that under certain genericity conditions the modular vector field of σ is completely determined by its co-Higgs field (Lemma 3.3.4). Now, for the case of a co-Higgs bundle over a small one dimensional disc with smooth, connected spectral curve, one can check that the genericity conditions mentioned above do hold away from the branch points of the spectral cover. Furthermore, a quick calculation of the modular vector field shows that it has to have a pole over the branch points if the rank of the bundle is greater than 2 (see the proof of Proposition 4.3.1 for more detail).

This obstructedness result for co-Higgs fields with smooth spectral curves necessitates considering co-Higgs fields whose spectral curves have singularities. We include Appendix A, where we discuss local normal forms for (co-)Higgs bundles with singular spectral curves over a formal one dimensional disc. The results in Appendix A are derived from classification results available in the literature on Cohen-Macaulay modules over Cohen-Macaulay curves. In Subsections 4.3.1 and 4.3.2, we further apply these results to obtain classification of Poisson structures on the trivial $\mathbb{P}^2$-bundle over a small one dimensional disc $\mathcal{U}$, under a certain bound on the vanishing of the characteristic polynomial of the corresponding co-Higgs field (the bound is imposed in such a way that the co-Higgs field extends smoothly to $\mathbb{P}^1$). Subsection 4.3.1 deals with the zero trace co-Higgs fields satisfying what we call the *non-resonance* condition, namely, that generically no eigenvalue equals the average of two others. The key tool for constructing the Poisson lifts of such co-Higgs bundles (V, ϕ) is logarithmic connections on V adapted to the spectral data of ϕ. The resonant case is considered in Subsection 4.3.2, and here the key tool is the pencil technique of constructing Poisson 3-folds, described in [26, Section 13] and [25, Section 3.2].

A substantial part of this thesis is devoted to discussing Poisson structures on rank 2 vector/projective bundles over $\mathbb{P}^1$ (Chapter 4). As a byproduct of our results on Poisson rank 2 bundles over $\mathbb{P}^1$ (Theorem 4.1.1), we obtain a classification of line bundles over Hirzebruch surfaces that admit the structure of a Poisson module (Theorem 4.2.1). In Section 4.3 we discuss which rank 3 co-Higgs bundles (V, ϕ) over $\mathbb{P}^1$ admit a Poisson lift to $\mathbb{P}(V)$, and how unique such a lift is. The results of this section, in particular, contain the classification of Poisson $\mathbb{P}^2$-bundles over $\mathbb{P}^1$, under the additional assumption that the spectral curve of (the co-Higgs field of) the Poisson structure is reduced. We believe that the additional assumption is not essential, and can be dropped after developing our theory a bit further. We remark that there are only a few classification results for low dimensional holomorphic Poisson manifolds known at the moment. See [1] for classification of Poisson surfaces, [6, 19] for classification of Poisson structures on $\mathbb{P}^3$, Pym's thesis [23] (also [24]) for classification of unimodular quadratic Poisson structures on $\mathbb{C}^4$, and [19] for classification of Poisson structuress on Fano 3-folds whose Picard group has rank 1.

The final Chapter 5 is devoted to constructing a family of strongly integrable co-Higgs fields on Schwarzenberger bundles over $\mathbb{P}^d$, $d > 1$, and their elliptic analogues. These co-Higgs fields are conjecturally generalizations of the co-Higgs fields over $\mathbb{P}^2$ constructed in Rayan's thesis [27] (also, see [29]). Furthermore, we construct Poisson lifts of these co-Higgs fields, and relate the obtained Poisson structures to the family of Feigin-Odesskii Poisson structures q_n described in [21]. The Poisson map we construct has the geometric interpretation as the desingularization of secant varieties of an elliptic curve sitting inside a projective space $\mathbb{P}^{n-1}$.

Chapter 2

Preliminaries

2.1 Poisson structures

Let us collect basic facts about Poisson brackets in the holomorphic setup. The general references are [26, 25, 20].

For a smooth complex manifold X, we denote by $\mathcal{O}_X$ its *structure sheaf*, i.e. the sheaf of locally defined holomorphic functions on X. Notations $\mathcal{T}_X$ and $\mathcal{T}_X^*$ stand for the *tangent sheaf* and the *cotangent sheaf*, respectively. If $x_1, ..., x_n$ are local coordinates on X, then locally, the sheaf $\mathcal{T}_X$ is spanned over $\mathcal{O}_X$ by the vector fields $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $i = 1, ..., n$, and the sheaf $\mathcal{T}_X^*$ is spanned over $\mathcal{O}_X$ by the 1-forms dx_i, $i = 1, ..., n$.

By the *Schouten bracket* (which sometimes is also called the Nijenhuis-Schouten bracket, or Schouten-Nijenhuis bracket) we mean the $\mathbb{C}$-linear operation on the skew symmetric multivector fields

$$[\ ,\ ] : \wedge^k \mathcal{T}_X \times \wedge^m \mathcal{T}_X \longrightarrow \wedge^{k+m-1}\mathcal{T}_X,$$

which for $k = m = 1$ coincides with the Lie bracket of vector fields, and otherwise is defined by

$$[u_1 \wedge ... \wedge u_k, v_1 \wedge ... \wedge v_m] = \sum_{i=1}^{k}\sum_{j=1}^{m}(-1)^{i+j}[u_i, v_j]u_1 \wedge ...\widehat{u_i}... \wedge u_k \wedge v_1 \wedge ...\widehat{v_j}... \wedge v_m,$$

for u_i, $v_j \in \mathcal{T}_X$, and $[f, u] = -[u, f] = \iota_{df}(u)$, for $f \in \mathcal{O}_X$, $u \in \wedge\mathcal{T}_X$.

The Schouten bracket turns $\wedge\mathcal{T}_X$ into a Gerstenhaber algebra, which means that it satisfies

- (graded skew-symmetry) $[v, u] = -(-1)^{(|u|-1)(|v|-1)}[u, v]$, $u, v \in \wedge\mathcal{T}_X$,
- (graded Jacobi identity) $[[u, v], w] = [u, [v, w]] - (-1)^{(|u|-1)(|v|-1)}[v, [u, w]]$, $u, v, w \in \wedge\mathcal{T}_X$,
- (graded Leibniz rule) $[u, v \wedge w] = [u, v] \wedge w + (-1)^{|v||w|}[u, w] \wedge v$, $u, v, w \in \wedge\mathcal{T}_X$.

A *Poisson bracket* on X is a $\mathbb{C}$-linear operation $\{\ ,\ \} : \mathcal{O}_X \times \mathcal{O}_X \longrightarrow \mathcal{O}_X$ that satisfies

- (skew-symmetry) $\{g, f\} = -\{f, g\}$, $f, g \in \mathcal{O}_X$,
- (Jacobi identity) $\{\{f, g\}, h\} = \{f, \{g, h\}\} - \{g, \{f, h\}\}$, $f, g, h \in \mathcal{O}_X$,
- (Leibniz rule) $\{f, gh\} = \{f, g\}h + \{f, h\}g$, $f, g, h \in \mathcal{O}_X$.

A skew-symmetric bivector $\sigma \in H^0(X, \wedge^2 \mathcal{T}_X)$ is called *Poisson*, if it satisfied the integrability condition $[\sigma, \sigma] = 0 \in H^0(X, \wedge^3 \mathcal{T}_X)$, where $[\ ,\ ]$ denotes the Schouten bracket.

A Poisson bracket $\{\ ,\ \}$ on X can be encoded by the bivector $\sigma \in H^0(X, \wedge^2 \mathcal{T}_X)$ defined by identifying $\wedge^2 \mathcal{T}_X \cong Hom(\wedge^2 \mathcal{T}_X^*, \mathcal{O}_X)$, and letting $\sigma(df, dg) = \{f, g\}$, $f, g \in \mathcal{O}_X$. Jacobi identity implies that the bivector σ is Poisson, and vice versa. We are going to use the terms Poisson structure or Poisson manifold to indicate the presence of either a Poisson bracket, or a Poisson bivector.

Example 2.1.1. Let $X = \mathbb{C}^{2n}$ with coordinates $x_1, y_1, ..., x_n, y_n$. The bivector $\sigma = \sum_{i=1}^n \partial_{x_i} \wedge \partial_{y_i}$ is Poisson, and defines the Poisson bracket $\{f, g\} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \sum_{i=1}^n \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial y_i}$, $f, g \in \mathcal{O}_X$.

Yet another way to define a Poisson bracket on a manifold X is to specify a skew-symmetric bundle map $\sigma^\# : \mathcal{T}_X^* \to \mathcal{T}_X$. The bivector σ can be uniquely recovered from the map $\sigma^\#$ via the formula $\sigma^\#(\alpha) = \iota_\alpha(\sigma)$, $\alpha \in \mathcal{T}_X^*$, and vice versa. A 2-form $\omega \in H^0(X, \wedge^2 \mathcal{T}_X^*)$ is called *non-degenerate* if the bundle map $\omega^\flat : \mathcal{T}_X \to \mathcal{T}_X^*$ defined by $\omega^\flat(u) = \iota_u \omega$, $u \in \mathcal{T}_X$, is an isomorphism. A 2-form $\omega \in H^0(X, \wedge^2 \mathcal{T}_X^*)$ is called *closed* if $d\omega = 0$, where d is the de Rham differential. If X is even-dimensional and one has a non-degenerate 2-form $\omega \in H^0(X, \wedge^2 \mathcal{T}_X^*)$, then one can attempt to define a Poisson structure on X by declaring $\sigma^\# = (\omega^\flat)^{-1}$. The obtained bivector σ satisfies the Poisson integrability $[\sigma, \sigma] = 0$ if and only if the form ω is closed. A closed non-degenerate 2-form is called *symplectic*. Likewise, a Poisson structure is called symplectic if $\sigma^\#$ is an isomorphism (and so, $\sigma^\# = (\omega^\flat)^{-1}$ for some symplectic ω).

Example 2.1.2. For the bivector $\sigma = \sum_{i=1}^n \partial_{x_i} \wedge \partial_{y_i}$ from Example 2.1.1, one has $\sigma^\#(dx_i) = \partial_{y_i}$, $\sigma^\#(dy_i) = -\partial_{x_i}$, $i = 1, ..., n$. Such σ is symplectic, and $\sigma^\# = (\omega^\flat)^{-1}$ for $\omega = \sum_{i=1}^n dy_i \wedge dx_i$.

More generally, for any manifold X, the cotangent bundle $\mathcal{T}_X^*$ carries the *standard symplectic form* that has the coordinate expression $\omega = \sum_{i=1}^n dy_i \wedge dx_i$, where x_i is any choice of local coordinates on the base X and y_i are the fiberwise linear coordinates on $\mathcal{T}_X^*$ corresponding to the local basis dx_1, ..., dx_n.

If a Poisson bivector σ lives on a $2n$-dimensional manifold X, we define its *Pfaffian* Pf σ as the section $\wedge^n \sigma$ of the anticanonical bundle ω_X^{-1}. If σ is symplectic at a generic point of X, the divisor $\{\mathrm{Pf}\ \sigma = 0\}$ is called the *degeneracy divisor* of σ.

For a function $f \in \mathcal{O}_X$ the expression $H_f = \sigma^\#(df)$ defines a vector field on X, called the *Hamiltonian vector field* of f. Poisson integrability $[\sigma, \sigma] = 0$ implies $H_{\{f,g\}} = [H_f, H_g]$, $f, g \in \mathcal{O}_X$. Sometimes, we will also speak of a Hamiltonian vector field of a 1-form α, which means $\sigma^\#(\alpha)$. A vector field v is called Poisson with respect to σ if $[v, \sigma] = 0$. Hamiltonian vector fields are always Poisson. Poisson vector fields are tangent to the zero set $\{\sigma = 0\}$ and the degeneracy divisor $\{\mathrm{Pf}\ \sigma = 0\}$. A function $f \in \mathcal{O}_X$ is called *Casimir* if $H_f = 0$, i.e. $\{f, g\} = 0$, for each $g \in \mathcal{O}_X$.

Poisson maps and submanifolds

Let (X, σ) and (Y, π) be two *Poisson* manifolds. A holomorphic map $F : X \to Y$ is called Poisson if for each $x \in X$, one has $TF(\sigma_x) = \pi_{F(x)}$, where TF is the tangent map to F. Equivalently, F is Poisson if $\{F^*g, F^*h\}_\sigma = F^*\{g, h\}_\pi$, $g, h \in \mathcal{O}_Y$, where F^* is the pullback of functions, and the subscript near the bracket indicates the bivector that induces it.

Lemma 2.1.3. *[26] Let $F : X \to Y$ be a surjective map between two complex manifolds having compact, connected fibers. Let σ be a Poisson structure on X. Then there is a unique Poisson structure on Y rendering F a Poisson map.*

Proof. We need to prove that for any two functions g_1, g_2 defined on an analytic neighborhood $\mathcal{U} \subset Y$, one has $\{F^*g_1, F^*g_2\} = F^*h$, for some function h defined on $\mathcal{U}$. The function $\{F^*g_1, F^*g_2\}$ is defined on $F^{-1}(\mathcal{U})$, and since the F fibers are compact and connected, it has to be constant on each F fiber. So, it has to be of the form F^*h, for some function h defined on $\mathcal{U}$. □

For two Poisson manifolds (X,σ) and (Y,π), one can define the product Poisson structure on $X \times Y$ by taking the bivector $\sigma \otimes 1 + 1 \otimes \pi$. The *Weinstein splitting theorem* says that every Poisson manifold is locally isomorphic to a product of the form $X \times Y$, where X is a symplectic manifold and Y is a Poisson manifold whose Poisson tensor vanishes at a point.

Every point of x of a Poisson manifold (X,σ) defines a set $S_x \subset X$ of all points $y \in X$ such that there is a C^∞ path $\gamma : [0,1] \to X$, $\gamma(0) = x$, $\gamma(1) = y$ with $\gamma'(t) \in \mathrm{Im}(\sigma^\#) \subset \mathcal{T}_X$, for each $t \in [0,1]$. The set S_x has the structure of an (injectively immersed) submanifold of X. The Poisson structure on X induces a non-degenerate Poisson structure on S_x, which is why S_x is called the *symplectic leaf* of X containing x.

Let (X,σ) be a Poisson manifold, and Y be a manifold injectively immersed in X. We call Y a *Poisson submanifold* of X if $\mathcal{T}_Y \supset \mathrm{Im}\ \sigma^\#$. Equivalently, Y is a Poisson submanifold if and only if the inclusion map $Y \to X$ is Poisson, if and only if Y fully contains each symplectic leaf it intersects. We call Y a *coisotropic submanifold* if $\sigma^\#(\alpha) = 0$ for every 1-form $\alpha \in \mathcal{T}^*_X$ whose pullback to Y vanishes.

Remark 2.1.4. Let $p : F \to X$ be a fiber bundle, and σ be a Poisson structure on F. Then the fibers of F are coisotropic if and only if the projection map p is Poisson, where we endow X with the zero Poisson structure.

Connections

Let $p : V \to X$ be a holomorphic vector bundle over a complex manifold X. A (linear) connection is a splitting of the short exact sequence

$$0 \longrightarrow (p^*V)^{\mathbb{C}^*} \longrightarrow (\mathcal{T}_V)^{\mathbb{C}^*} \longrightarrow (p^*\mathcal{T}_X)^{\mathbb{C}^*} \longrightarrow 0$$

of sheaves on V, where the superscript $\mathbb{C}^*$ indicates invariants under the scaling action of $\mathbb{C}^*$. More concretely, a connection on V is a map $\nabla : V \to V \otimes \mathcal{T}^*_X$ that satisfies the Leibniz rule $\nabla(fs) = s \otimes df + f\nabla(s)$, $f \in \mathcal{O}_X$, $s \in V$. Let $\{\mathcal{U}_i\}_{i\in I}$ be an open cover of X, and $\mathbf{y}^{(i)} = (y_1^{(i)}, ..., y_r^{(i)}) : V|_{\mathcal{U}_i} \to \mathbb{C}^r \times \mathcal{U}_i$ be fiberwise linear coordinates on each $V|_{\mathcal{U}_i}$. Let $g_{ji} : \mathbb{C}^r \times \mathcal{U}_i \to \mathbb{C}^r \times \mathcal{U}_j$ be the transitions functions, in the sense that $\mathbf{y}^{(j)} = g_{ji}\mathbf{y}^{(i)}$, $i, j \in I$. Then over each $\mathcal{U}_i$, a connection ∇ has the form $\nabla = d + A_i$ for some connection matrix $A_i \in H^0(\mathcal{U}_i, End(\mathbb{C}^r) \otimes \mathcal{T}^*_X)$. The connection matrices A_i satisfy the *gauge equivalence* relations $A_j = (dg_{ji})g_{ji}^{-1} + g_{ji}A_i g_{ji}^{-1}$, over each $\mathcal{U}_i \cap \mathcal{U}_j$.

A connection ∇ is *flat* if $\nabla_u\nabla_v - \nabla_v\nabla_v = \nabla_{[u,v]}$, $u, v \in \mathcal{T}_X$, where the notation ∇_u means $\iota_u\nabla$. A section s of V is called *flat* with respect to ∇ if $\nabla(s) = 0$. Given a connection ∇ on V, any C^∞ path $\gamma : [0,1] \to X$ defines an isomorphism $V|_{\gamma(0)} \to V|_{\gamma(1)}$ called the *holonomy* of ∇ along γ. If ∇ is flat, then holonomy depends only on the homotopy class of γ, not on γ itself.

A holomorphic vector bundle V over a complex manifold X admits a holomorphic connection if and only if its *Atiyah class* $\mathrm{At}_V \in H^1(X, End(V) \otimes \mathcal{T}^*_X)$ vanishes. The Atiyah class At_V is defined by the Čech 1-cocycle $\left((dg_{ji})g_{ji}^{-1}\right)_{\mathcal{U}_i \cap \mathcal{U}_j}$, where g_{ji} are the transition functions of V as above. If V does admit a connection ∇, then all other connections on V are obtained by adding a section of $End(V) \otimes \mathcal{T}^*_X$ to ∇.

Poisson connections

If (X,σ) is a Poisson manifold, and $p : V \to X$ is a vector bundle, by a *Poisson connection* on V we mean a map $\nabla : V \to V \otimes \mathcal{T}_X$ that satisfies the Leibniz rule $\nabla(fs) = -s \otimes \sigma^{\#}(df) + f\nabla(s)$, $f \in \mathcal{O}_X$, $s \in V$. Given a Poisson connection ∇, we denote $\{f,s\} = \iota_{df}\nabla(s) \in V$, for $f \in \mathcal{O}_X$, $s \in V$. Then the Leibniz rule reads $\{g,fs\} = \{g,f\}s + \{g,s\}f$, $f,g \in \mathcal{O}_X$, $s \in V$.

Remark 2.1.5. A Poisson connection on V uniquely defines a Poisson connection on V^*. For a line bundle L, a Poisson connection on L gives one on $L^{\otimes k}$, and vice versa.

A Poisson connection is called *flat* if one has $\{\{f,g\},s\} = \{f,\{g,s\}\} - \{g,\{f,s\}\}$, $f,g \in \mathcal{O}_X$, $s \in V$. A vector bundle with a flat Poisson connection is called a *Poisson module*.

Proposition 2.1.6. *[26, Proposition 5.2] Let $p : L \to X$ be a line bundle over a Poisson manifold (X,σ). Then there is a one-to-one correspondence between the Poisson module structures on L^* with respect to σ and $\mathbb{C}^*$-invariant Poisson structures on the total space of L rendering the projection map p Poisson.*

The trivial line bundle $\mathcal{O}_X$ over a Poisson manifold X carries the trivial Poisson module structure. The canonical line bundle ω_X of a Poisson manifold X carries the *canonical Poisson module* structure, defined by $\{f,\mu\} = Lie_{\sigma^{\#}(df)}\mu$, $f \in \mathcal{O}_X$, $\mu \in \omega_X$, where *Lie* is the Lie derivative.

Remark 2.1.7. If the Poisson bracket on X has coordinate expression

$$\{x_i, x_j\} = f_{ij},$$

then the Poisson structure on ω_X coming from the canonical Poisson module has the bracket

$$\{x_i, x_j\} = f_{ij}, \quad \{x_k, y\} = \sum_i \frac{\partial f_{ik}}{\partial x_i} y,$$

where y is the linear fiberwise coordinate on ω_X in the trivialization given by $dx_1 \wedge \ldots \wedge dx_n$.

Remark 2.1.8. If a Poisson structure σ on X is generically symplectic, and the degeneracy divisor $\{\mathrm{Pf}\ \sigma = 0\}$ has an irreducible, reduced component D, then $\mathcal{O}(D)$ admits a canonical Poisson module structure (D is allowed to occur in $\{\mathrm{Pf}\ \sigma = 0\}$ with multiplicity). Indeed, representing the local section of $\mathcal{O}(D)$ as local functions on X that are allowed to have a simple pole over D, we define the Poisson module action of $f \in \mathcal{O}$ on $s \in \mathcal{O}(D)$ by the usual Poisson bracket $\{f,s\}$. In principle, the bracket $\{f,s\}$ could have a pole of order 2 over D, but because σ vanishes on D, the bracket $\{f,s\}$, in fact, has at most a simple pole over D, so $\{f,s\} \in \mathcal{O}(D)$.

A holomorphic vector bundle V over a Poisson manifold X admits a Poisson connection if and only if its *Poisson-Atiyah class* $\sigma^{\#}(\mathrm{At}_V) \in H^1(X, End(V) \otimes \mathcal{T}_X)$ vanishes. If V does admit a Poisson connection ∇, then all other Poisson connections on V are obtained by adding a section of $End(V) \otimes \mathcal{T}_X$ to ∇.

Proposition 2.1.9. *[14, Proposition 18.9] Let L be a line bundle over a Poisson manifold (X,σ), whose Poisson-Atiyah class $\sigma^{\#}(\mathrm{At}_V) \in H^1(X,\mathcal{T}_X)$ vanishes. Choose a C^∞ vector field v on X such that $\overline{\partial} v = \sigma^{\#}(\mathrm{At}_V)$, and denote $\rho = [\sigma, v] \in H^0(X, \wedge^2 \mathcal{T}_X)$. Then L admits a Poisson module structure with respect to σ if and only if $\rho = [\sigma, u]$ for some $u \in H^0(X, \mathcal{T}_X)$*

BV operator. Modular vector field

Let $\Omega \in H^0(X, \omega_X)$ be a volume form on a complex manifold X of dimension n. This induces the *BV operator* $\Delta : \wedge^\bullet \mathcal{T}_X \to \wedge^{\bullet-1}\mathcal{T}_X$, defined by $\Delta = I^{-1} dI$, where I is the identification $I : \wedge^\bullet \mathcal{T}_X \to \wedge^{n-\bullet}\mathcal{T}_X^*$, $u \mapsto \iota_u \Omega$, and $d : \wedge^\bullet \mathcal{T}_X^* \to \wedge^{\bullet+1}\mathcal{T}_X^*$ is the de Rham differential. We will use the notation Δ_Ω, in case we want to indicate the choice of Ω.

Example 2.1.10. Let $X = \mathbb{C}^n$ with coordinates $x_1, ..., x_n$, and $\Omega = dx_1 \wedge ... \wedge dx_n$. Then

$$\Delta_\Omega = \sum_{k=1}^{n} \nabla_{\partial_{x_k}} \otimes \iota_{dx_k},$$

where $\nabla_{\partial_{x_k}}$ is the operation of taking derivative of the coefficients f_J of a multivector $\sum_{J \subset \{1,...,n\}} f_J \partial_{x_J}$, and ι_{dx_k} is the contraction.

If $\Omega' = f\Omega$, $f \in \mathcal{O}_X^*$, is another volume form, then $\Delta_{\Omega'} = \Delta_\Omega + \iota_{d\log(f)}$. The BV operator satisfies $\Delta^2 = 0$ and is related to the Schouten bracket via the relations

$$[u, v] = (-1)^{|u|}\big(\Delta(u \wedge v) - \Delta(u) \wedge v\big) - u \wedge \Delta(v), \tag{2.1}$$

$$\Delta[u, v] = [\Delta(u), v] - (-1)^{|u|}[u, \Delta(v)], \tag{2.2}$$

where $u, v \in \wedge \mathcal{T}_X$.

If σ is a Poisson bivector on X, and Ω is a locally defined volume form on X, the vector field $\zeta = \Delta_\Omega(\sigma)$ is called the *modular vector field* of σ with respect to Ω. The modular vector field is Poisson. In particular, it is tangent to the zero set $\{\sigma = 0\}$. If $\Omega' = f\Omega$, $f \in \mathcal{O}_X^*$, is another choice of local volume form, then $\Delta_{\Omega'}(\sigma) = \zeta + \sigma^\#(d\log(f))$. In particular, the restriction of modular vector field to the zero set of σ does not depend on the choice of local volume form Ω.

Example 2.1.11. Let $X = \mathbb{C}^n$ with coordinates $x_1, ..., x_n$, and $\Omega = dx_1 \wedge ... \wedge dx_n$. The modular vector field of $\sigma = \sum_{i<j} \sigma_{ij} \partial_{x_i} \wedge \partial_{x_j}$ with respect to Ω is

$$\zeta = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial \sigma_{ij}}{\partial_{x_i}} \right) \partial_{x_j}$$

(here, if $i > j$, the notation σ_{ij} means $-\sigma_{ji}$, and if $i = j$, then $\sigma_{ij} = 0$).

For instance, for $n = 2$, the Poisson bracket $\{x_1, x_2\} = x_2$ has the modular vector field $\zeta = -\partial_{x_1}$, and the Poisson bracket $\{x_1, x_2\} = x_1 x_2$ has the modular vector field $\zeta = x_2 \partial_{x_2} - x_1 \partial_{x_1}$.

A Poisson manifold (X, σ) is called *unimodular* if it possesses a volume form $\Omega \in H^0(X, \omega_X)$ such that the modular vector field $\zeta = \Delta_\Omega(\sigma)$ is zero. If $\dim X = 2n$ and $\sigma^\# = (\omega^\flat)^{-1}$ for a symplectic form ω, then Liouville's Theorem implies that (X, σ) is unimodular with respect to $\Omega = \wedge^n \omega$.

Poisson structures on 3-folds from pencils of surfaces

A pencil on a manifold X is a surjective map $X \dashrightarrow \mathbb{P}^1$ that is allowed to be ill-defined on a codimension 2 subset B of X. The set B is called base locus of the pencil, and preimages of a points $x \in \mathbb{P}^1$ are called members of the pencil.

Let X be a 3-fold, i.e. a manifold of dimension 3. To a trivector $\tau \in \wedge^3 \mathcal{T}_X$ and a 1-form $\alpha \in \mathcal{T}_X^*$, one can bring into correspondence a bivector $\sigma = \iota_\alpha(\tau)$. The integrability condition $[\sigma,\sigma] = 0$ is equivalent to the equation $\alpha \wedge d\alpha = 0 \in \wedge^3 \mathcal{T}_X^* \cong \omega_X$.

Let $f : X \dashrightarrow \mathbb{P}^1$ be a pencil on a 3-fold X with a base locus $B \subset X$. Let $D = D_0 \cup D_\infty$, where $D_0 = f^{-1}(0) \cup B$, $D_\infty = f^{-1}(\infty) \cup B$. Suppose one can find a trivector $\tau \in H^0(X, \omega_X^{-1})$ that vanishes along D. Then the bivector $\iota_{d\log(f)}(\tau)$ has no poles, and is integrable. The meromorphic function f is a Casimir for σ (see e.g. [25, Section 3] for further discussion).

Example 2.1.12. Let $X = \mathbb{P}^3$, and f, g be two generic sections of $\mathcal{O}_{\mathbb{P}^3}(2)$. Then f/g defines a pencil $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ with the base locus $B = \{f = g = 0\}$, an elliptic curve. Since $\wedge^3 \mathcal{T}_X \cong \mathcal{O}_{\mathbb{P}^3}(4)$, one can find a trivector $\tau \in H^0(\mathbb{P}^3, \wedge^3 \mathcal{T}_X)$ that vanishes along $\{f = 0\}$ and $\{g = 0\}$. One obtains a Poisson bivector $\sigma = \iota_{d\log(f/g)}(\tau)$, which is the projectivization of the Feigin-Odesskii Poisson structure q_4 on $\mathbb{C}^4$ described in [21].

2.2 Co-Higgs fields. Spectral correspondence

Recall that a *co-Higgs sheaf* on a smooth complex manifold X over $\mathbb{C}$ is a coherent sheaf V together with a morphism $\phi \in \mathrm{Hom}(V, V \otimes \mathcal{T}_X)$ satisfying the integrability condition

$$\phi \wedge \phi = 0 \in \mathrm{Hom}(V, V \otimes \wedge^2 \mathcal{T}_B). \tag{2.3}$$

The morphism ϕ is called *co-Higgs field* on V. A *co-Higgs bundle* is a co-Higgs sheaf (V, ϕ) such that V is locally free.

Co-Higgs bundles are dual objects to Higgs bundles, which were introduced in dimension one by Hitchin [16, 17], and generalized to all dimensions by Simpson [32, 33]. The moduli space of Higgs bundles was constructed by Hitchin [16] and Simpson [34, 35].

Remark 2.2.1. A co-Higgs field $\phi \in \mathrm{Hom}(V, V \otimes \mathcal{T}_X)$ can be viewed as a Poisson connection on V over the manifold X endowed with the zero Poisson structure. The integrability condition (2.3) is equivalent to the flatness of such a connection.

Let us recall the spectral correspondence for co-Higgs sheaves. A co-Higgs field ϕ on a coherent sheaf V of $\mathcal{O}_X$-modules turns V into a sheaf of $\mathrm{Sym}(\mathcal{T}_X^*)$-modules, where a tensor $\alpha_1 \otimes \ldots \otimes \alpha_r \in (\mathcal{T}_X^*)^{\otimes r}$ acts on V by $\langle \phi, \alpha_1 \rangle \ldots \langle \phi, \alpha_r \rangle \in \mathrm{End}(V)$. The integrability condition (2.3) ensures that the action of $\otimes(\mathcal{T}_X^*)$ descends to an action of $\mathrm{Sym}(\mathcal{T}_X^*)$. Conversely, an $\mathcal{O}_X$-coherent sheaf V of $\mathrm{Sym}(\mathcal{T}_X^*)$-modules defines a co-Higgs sheaf (V, ϕ), where for each section α of $\mathcal{T}_X^*$, the tensor $\langle \phi, \alpha \rangle \in \mathrm{End}(V)$ is given by the action of α.

Furthermore, one can bring into correspondence with an $\mathcal{O}_X$-coherent sheaf V of $\mathrm{Sym}(\mathcal{T}_X^*)$-modules a coherent sheaf $\widetilde{V}$ on the completed vector bundle $\mathbb{P}(\mathcal{T}_X \oplus \mathcal{O}_X)$, whose support is disjoint from the infinity divisor $D = \mathbb{P}(\mathcal{T}_X \oplus \mathcal{O}_X) \smallsetminus \mathcal{T}_X$. In particular, if V is locally free around $x \in X$, then the support of $\widetilde{V}$ intersects the fibers $\mathcal{T}_X$ at the eigenvalues of $\phi|_x$. Conversely, if $\widetilde{V}$ is a coherent sheaf on $\mathbb{P}(\mathcal{T}_X \oplus \mathcal{O}_X)$, whose support is disjoint from the divisor D, then the pushforward $p_* \widetilde{V}$ under the natural projection map $p : \mathcal{T}_X \to X$ defines an $\mathcal{O}_X$-coherent sheaf V on X. Moreover, the pullback $p^* \mathcal{T}_X$ has a tautological section θ that over a point $y \in \mathcal{T}_X$ has value y. Multiplication by θ defines $\phi \in \mathrm{Hom}(V, V \otimes p^* \mathcal{T}_X) \cong \mathrm{Hom}(V, V \otimes \mathcal{T}_X)$, which is a co-Higgs field.

The discussion is summarized in the following:

Theorem 2.2.2. *Co-Higgs sheaves on X are in one-to-one correspondence with coherent sheaves on $\mathbb{P}(\mathcal{T}_X \oplus \mathcal{O}_X)$ whose support is disjoint from the divisor $D = \mathbb{P}(\mathcal{T}_X \oplus \mathcal{O}_X) \smallsetminus \mathcal{T}_X$.*

Theorem 2.2.2 was proved in [3] for the case when X is a curve and $\mathcal{T}_X$ is replaced with any line bundle. An analogue of Theorem 2.2.2 for Higgs sheaves was proved by Simpson in [35]. His argument may be easily modified to give the co-Higgs version stated above.

Theorem 2.2.2 gives a convenient way to construct co-Higgs bundles. The typical way to construct the coherent sheaf on $\mathbb{P}(\mathcal{T}_X^* \oplus \mathcal{O}_X)$ with support disjoint from D is to embed (or immerse) a variety $i : \Sigma \to \mathcal{T}_X$ so that it intersect each fiber $\mathcal{T}_x$ at the same number of points (counting them with multiplicity), and then give a line bundle L over Σ. Then the sheaf $i_* L$ on $\mathcal{T}_X \subset \mathbb{P}(\mathcal{T}_X \oplus \mathcal{O}_X)$ by Theorem 2.2.2 defines a co-Higgs sheaf $(V = p_*(i_* L), \phi)$ on X.

For future use, we point out yet another way to construct a co-Higgs field on X. Let $p : \Sigma \to X$ be a branched cover of smooth manifolds, L be a line bundle over Σ and v be a vector field on Σ. Then taking derivative of p we obtain a map $\mathcal{T}p : \mathcal{T}_\Sigma \to \mathcal{T}_X$, which we can precompose with $v : \Sigma \to \mathcal{T}_\Sigma$ to obtain a map $\mathcal{T}p \circ v : \Sigma \to \mathcal{T}_X$. The pushforward $(\mathcal{T}p \circ v)_* L$ then defines a coherent sheaf on $\mathcal{T}_X \subset \mathbb{P}(\mathcal{T}_X \oplus \mathcal{O}_X)$ and by Theorem 2.2.2 we obtain a co-Higgs sheaf (V, ϕ) on X. We will call the triple $(p : \Sigma \to X, L, v)$ a *spectral data*. The co-Higgs field ϕ on $V = p_* L$ can be recovered directly by the following recipe. The tensor $\phi : \mathcal{T}_X^* \otimes V \to V$ sends $\alpha \otimes v \in \mathcal{T}_X^* \otimes V$ to $\langle p^* \alpha, v \rangle s \in V$, where the multiplication of $s \in V$ by the function $\langle p^* \alpha, v \rangle \in \mathcal{O}_\Sigma$ is defined by regarding s as a section of L.

Chapter 3

Quadratic Poisson structures on vector bundles vs. Poisson structures on projective bundles vs. co-Higgs bundles

3.1 Lifting a Poisson structure on a projective bundle to a quadratic Poisson structure on a vector bundle

Let $p: V \to X$ be a vector bundle over a complex manifold X. Let σ be a Poisson structure on the total space of V, invariant under the fiberwise action of $\mathbb{C}^*$ via dilation. Let $x_1, ..., x_n$ be coordinates on a small open set $\mathcal{U} \subset X$ and $y_1, ..., y_r$ be fiberwise linear coordinates on $V|_{\mathcal{U}}$. Then a $\mathbb{C}^*$-invariant Poisson structure on $V|_{\mathcal{U}}$ has an expression of the form

$$\sigma = \sum_{i,j=1}^{n} f_{ij}\ \partial_{x_i} \wedge \partial_{x_j} + \sum_{i=1}^{n} \sum_{j,k=1}^{r} g_{ijk}\ \partial_{x_i} \wedge y_j \partial_{y_k} + \sum_{i,j,k,l=1}^{r} h_{ijkl}\ y_i y_j \partial_{y_k} \wedge \partial_{y_l},$$

where the functions f_{ij}, g_{ijk}, $h_{ijkl} \in \mathcal{O}_{\mathcal{U}}$, that is, they depend only on x variables. The presence of the quadratic terms $y_i y_j \partial_{y_k} \wedge \partial_{y_l}$ motivates our calling a $\mathbb{C}^*$-invariant Poisson structure *quadratic*.

A quadratic (= $\mathbb{C}^*$-invariant) Poisson structure on V induces a Poisson structure on the projectivization $\mathbb{P}(V)$ by pushing the Poisson tensor along the projection $V \setminus 0 \to \mathbb{P}(V)$.

Example 3.1.1. Consider the case when $V = \mathbb{C}^{n+1}$ is a vector space. If a Poisson structure on V is given by the bracket

$$\{y_i, y_j\} = f_{ij},$$

where $y_0, y_1, ..., y_n$ are linear coordinates on V, and f_{ij} are quadratic polynomials, then its projectivization on $\mathbb{P}^n = \mathbb{P}(V)$ is given by the bracket

$$\left\{\frac{y_i}{y_0}, \frac{y_j}{y_0}\right\} = \frac{y_0 f_{ij} + y_i f_{j0} + y_j f_{0i}}{y_0^3}.$$

For future use, let us record the following statement.

Lemma 3.1.2. *Let $\sigma = \sum_{i,j,k,m=0}^{n} c_{ijkm}\ y_i y_j \partial_{y_k} \wedge \partial_{y_m}$, $c_{ijkm} \in \mathbb{C}$ be a quadratic Poisson structure on $\mathbb{C}^{n+1}$, such that $c_{ijkm} = 0$ unless all the indices i, j, k, m are distinct. Then the projectivization of σ vanishes on all the points of the form $e_i = [0 : ... : 0 : 1 : 0 : ... : 0] \in \mathbb{P}^n$, $i = 0, 1, ..., n$. Moreover, the modular vector field of the projectivization of σ also vanishes at each e_i, $i = 0, 1, ..., n$.*

Proof. Without loss of generality, it is enough to prove the lemma for e_0. Let $\mathcal{U}_0 \in \mathbb{C}^{n+1}$ be the set of points $[y_0 : ... : y_n] \in \mathbb{P}^n$ with $y_0 \neq 0$. Then the fractions $z_i = \frac{y_i}{y_0}$, $i \neq 0$ define coordinates on $\mathcal{U}_0$. The point $e_0 \in \mathcal{U}_0$ is given by $z_1 = z_2 = ... = z_n = 0$. Without loss of generality, we may assume that $\sigma = y_i y_j \partial_{y_k} \wedge \partial_{y_m}$, where i, j, k, m are four distinct indices.

Case 1. All i, j, k, m are non-zero. The projectivization of σ has the expression $z_i z_j \partial_{z_k} \wedge \partial_{z_m}$ on $\mathcal{U}_0$. It vanishes at e_0, and has zero modular vector field.

Case 2. Either $i = 0$, or $j = 0$. Assume $i = 0$. The projectivization of σ has the expression $z_j \partial_{z_k} \wedge \partial_{z_m}$. Again, it vanishes at e_0, and has zero modular vector field.

Case 3. Either $k = 0$, or $m = 0$. Assume $k = 0$. The projectivization of σ has the expression $\sum_{l=1}^{n} z_i z_j z_l \partial_{z_m} \wedge \partial_{z_l}$. This bivector vanishes at e_0, and has the modular vector field $-n z_i z_j \partial_{z_m}$, which vanishes at e_0. □

Going from a Poisson structure on $\mathbb{P}(V)$ to one on V is not always possible, let alone unique. However, the situation is well understood for the case when the base X is a point, i.e. V is a vector space.

Theorem 3.1.3. *[4, 26] Let V be a $\mathbb{C}$-vector space. Any Poisson strucutre π on $\mathbb{P}(V)$ admits a lift to a quadratic Poisson structure on V. Moreover, all such lifts are parametrized by Poisson vector fields on $\mathbb{P}(V)$.*

Proof. Quadratic Poisson structures on V projecting onto π are in one-to-one correspondence with quadratic Poisson structures on the line bundle $\mathcal{O}(-1)_{\mathbb{P}(V)}$ projecting onto π. By Proposition 2.1.6, the latter are in one-to-one correspondence with Poisson module structures on $\mathcal{O}(-1)_{\mathbb{P}(V)}$. By Remark 2.1.5, the latter are in one-to-one correspondence with Poisson module structures on $\mathcal{O}(-n-1)_{\mathbb{P}(V)} = \omega_{\mathbb{P}(V)}$, where $n = \dim \mathbb{P}(V)$. If $\mathbb{P}(V)$ is endowed with a Poisson structure π, then its canonical bundle $\omega_{\mathbb{P}(V)}$ has canonical Poisson module structure, which therefore gives a preferred way to construct a $\mathbb{C}^*$-invariant Poisson structure σ on V lifting π. All other Poisson module structures on $\omega_{\mathbb{P}(V)}$ differ from the canonical one by a Poisson vector field v on $\omega_{\mathbb{P}(V)}$. Therefore all $\mathbb{C}^*$-invariant lifts of π to V are of the form $\sigma + Eul \wedge v$, where v is Poisson vector field on $\mathbb{P}(V)$. □

Remark 3.1.4. [26, Theorem 12.2] In coordinates, if a Poisson structure σ on $\mathbb{P}^n$ is given by the bracket

$$\left\{ \frac{y_i}{y_0}, \frac{y_j}{y_0} \right\} = \frac{f_{ij}}{y_0^3},$$

where $y_0, y_1, ..., y_n$ are affine coordinates, f_{ij} are homogeneous polynomials of degree 3, then there is a preferred (i.e. corresponding to the zero Poisson vector field) lift of σ to a $\mathbb{C}^*$-invariant Poisson structure on $\mathbb{C}^{n+1}$ given by the bracket

$$\{y_i, y_j\} = \frac{1}{y_0} \left(f_{ij} - \frac{y_i}{n+1} \sum_{k=1}^{n} \frac{\partial f_{kj}}{\partial y_k} + \frac{y_j}{n+1} \sum_{k=1}^{n} \frac{\partial f_{ki}}{\partial y_k} \right).$$

Let us now generalize this result to the case when V is a vector bundle, rather than a vector space.

Theorem 3.1.5. *Let V be a vector bundle over X whose total space is Calabi-Yau, i.e. $\omega_V \cong \mathcal{O}_V$. Then any Poisson structure π on $\mathbb{P}(V)$ lifts to a unimodular quadratic Poisson structure on V. If X is compact, such a lift is unique.*

Proof. We exploit the ideas of Bondal [4]. Pick a fiberwise constant volume form Ω on V, and denote by Δ the corresponding BV operator $\wedge^\bullet \mathcal{T}_V \to \wedge^{\bullet-1}\mathcal{T}_V$. Cover the base manifold X by open sets $\{\mathcal{U}_i\}_{i\in I}$ such that V is trivial over each $\mathcal{U}_i$ and $H^k(\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i}) = 0$, $k > 0$. We are going to construct the desired lift of π over each $\mathcal{U}_i$ separately, and prove that these lifts agree on double overlaps $\mathcal{U}_i \cap \mathcal{U}_j$.

Consider the short exact sequence

$$0 \longrightarrow (\mathcal{O}_{V\smallsetminus 0})^{\mathbb{C}^*} \xrightarrow{\wedge Eul} (\mathcal{T}_{V\smallsetminus 0})^{\mathbb{C}^*} \longrightarrow (p^*\mathcal{T}_{\mathbb{P}(V)})^{\mathbb{C}^*} \longrightarrow 0, \tag{3.1}$$

of sheaves on $V \smallsetminus 0$, where $p : V \smallsetminus 0 \to \mathbb{P}(V)$ is the natural projection, and the left map is multiplication by the Euler vector field Eul. Taking $\wedge^2$ of (3.1) we obtain

$$0 \longrightarrow (p^*\mathcal{T}_{\mathbb{P}(V)})^{\mathbb{C}^*} \longrightarrow (\wedge^2\mathcal{T}_{V\smallsetminus 0})^{\mathbb{C}^*} \longrightarrow (p^* \wedge^2 \mathcal{T}_{\mathbb{P}(V)})^{\mathbb{C}^*} \longrightarrow 0. \tag{3.2}$$

Over each $\mathcal{U}_i$, we have $H^1\big((V\smallsetminus 0)|_{\mathcal{U}_i}, (p^*\mathcal{T}_{\mathbb{P}(V)})^{\mathbb{C}^*}\big) = H^1(\mathbb{P}(V)|_{\mathcal{U}_i}, \mathcal{T}_{\mathbb{P}(V)}) = H^1(\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i}) = 0$, so the long exact cohomological sequence for (3.2) implies that any bivector on $\mathbb{P}(V)|_{\mathcal{U}_i}$ lifts to a $\mathbb{C}^*$-invariant bivector $\widetilde{\sigma}_i$ on $(V\smallsetminus 0)|_{\mathcal{U}_i}$. By Hartog's theorem, $\widetilde{\sigma}_i$ extends smoothly to $V|_{\mathcal{U}_i}$. Let us modify $\widetilde{\sigma}_i$ to make it Poisson. The identity (2.1) implies the identity

$$\Delta(Eul \wedge v) + Eul \wedge \Delta(v) = \Delta(Eul) \wedge v = (r+1)v, \quad v \in \wedge^\bullet \mathcal{T}_V, \tag{3.3}$$

where $r+1$ is the rank of V, and Eul is the fiberwise Euler vector field on V. It follows from (3.3) that $\sigma_i = \widetilde{\sigma}_i - \frac{1}{r+1} Eul \wedge \Delta(\widetilde{\sigma}_i)$ satisfies $\Delta(\sigma_i) = 0$. Also, note that the term $Eul \wedge -$ projects to zero on $\mathbb{P}(V)$, so σ_i is also a lift of π.

We claim that σ_i is Poisson. Since $p_*\sigma_i = \pi$, we get $p_*[\sigma_i, \sigma_i] = [\pi, \pi] = 0$, so $Eul \wedge [\sigma_i, \sigma_i] = 0$. Since

$$[\sigma_i, \sigma_i] = \Delta(\sigma_i \wedge \sigma_i) - 2\Delta(\sigma_i) \wedge \sigma_i = \Delta(\sigma_i \wedge \sigma_i),$$

we obtain

$$[\sigma_i, \sigma_i] = \frac{1}{r+1}\Big(\Delta(Eul \wedge \Delta(\sigma_i \wedge \sigma_i)) + Eul \wedge \Delta^2(\sigma_i \wedge \sigma_i)\Big) = 0.$$

Next, we claim that over $\mathcal{U}_i \cap \mathcal{U}_j$ the Poisson bivectors σ_i and σ_j coincide. Indeed, the difference $\sigma_i - \sigma_j$ is annihilated by both Δ and $Eul \wedge -$, so the identity (3.3) implies that $\sigma_i - \sigma_j = 0$. This implies that the Poisson bivectors σ_i defined on $V|_{\mathcal{U}_i}$ glue together to a global Poisson bivector σ on V that lifts π.

Finally, assuming X is compact, let us prove that any two unimodular $\mathbb{C}^*$-invariant Poisson bivectors σ, σ' lifting π must coincide. Due to compactness of the base, any two fiberwise constant volume forms on V must be scalar multiples of each other. So, without loss of generality, we may assume that σ and σ' are annihilated by a BV operator Δ coming from the same volume form Ω. Then the difference $\sigma - \sigma'$ is annihilated by Δ and $Eul \wedge -$, hence the identity (3.3) implies that $\sigma - \sigma' = 0$. ☐

Remark 3.1.6. Let V be a trivial vector bundle over an open set $\mathcal{U} \subset \mathbb{C}^m$, and let π be a Poisson structure on $\mathbb{P}(V)$ given by the bracket

$$\{x_i, x_j\} = f_{ij}, \quad \left\{x_i, \frac{y_j}{y_0}\right\} = \frac{g_{ij}}{y_0^2}, \quad \left\{\frac{y_i}{y_0}, \frac{y_j}{y_0}\right\} = \frac{h_{ij}}{y_0^3},$$

where $x_1, ..., x_m$ are coordinates on $\mathcal{U}$, $y_0, ..., y_r$ are fiberwise linear coordinates on V, and f_{ij}, g_{ij} and h_{ij} are holomorphic functions in x's and y's that are homogeneous in y's of degree 0, 2 and 3, respectively. Then π has a lift to a $\mathbb{C}^*$-invariant Poisson structure on V given by the bracket

$$\{x_i, x_j\} = f_{ij},$$

$$\{x_i, y_j\} = \frac{1}{y_0}\left(g_{ij} - \frac{y_j}{r+1}\sum_{k=1}^{r}\frac{\partial g_{ik}}{\partial y_k}\right) - \frac{y_j}{r+1}\sum_{k=1}^{m}\frac{\partial f_{ki}}{\partial x_k},$$

$$\{y_i, y_j\} = \frac{1}{y_0}\left(h_{ij} - \frac{y_i}{r+1}\sum_{k=1}^{r}\frac{\partial h_{kj}}{\partial y_k} + \frac{y_j}{r+1}\sum_{k=1}^{r}\frac{\partial h_{ki}}{\partial y_k} - \frac{y_i}{r+1}\sum_{k=1}^{r}\frac{\partial g_{kj}}{\partial x_k} + \frac{y_j}{r+1}\sum_{k=1}^{r}\frac{\partial g_{ki}}{\partial x_k}\right).$$

3.2 Constructing a co-Higgs field from a Poisson structure

Vector bundle case

Let $p : V \to X$ be a vector bundle over a manifold X. The one dimensional torus $\mathbb{C}^*$ acts on V via dilation, and generates the Euler vector field Eul. Recall that $\mathbb{C}^*$-invariant vector fields on V tangent to the fibers are in one-to-one correspondence with sections of $End(V)$. The Euler vector field in this way corresponds to the identity endomorphism of V.

Let σ be a quadratic Poisson structure on the total space of V. In local coordinates, one has

$$\sigma = \sum_{i,j=1}^{n} f_{ij}\ \partial_{x_i} \wedge \partial_{x_j} + \sum_{i=1}^{n} \sum_{j,k=1}^{r} g_{ijk}\ \partial_{x_i} \wedge y_j \partial_{y_k} + \sum_{i,j,k,l=1}^{r} h_{ijkl}\ y_i y_j \partial_{y_k} \wedge \partial_{y_l},$$

where $x_1, ..., x_n$ are coordinates on a small open set $\mathcal{U} \subset X$, $y_1, ..., y_r$ are fiberwise linear coordinates on $V|_{\mathcal{U}}$, and the functions f_{ij}, g_{ijk}, $h_{ijkl} \in \mathcal{O}_{\mathcal{U}}$ depend only on x variables.

The terms $f_{ij}\partial_{x_i} \wedge \partial_{x_j}$ alone define a Poisson structure on X, which we are going to denote by $p_*\sigma$. A coordinate free way to define $p_*\sigma$ is to projectivize σ to obtain a Poisson structure π on $\mathbb{P}(V)$, and then apply Lemma 2.1.3 to push forward π further to X.

We are often going to assume that $p_*\sigma$ is zero. According to Remark 2.1.4, this is equivalent to requiring the fibers of V to be coisotropic with respect to σ. Under this assumption, the terms $g_{ijk}\partial_{x_i} \wedge y_j \partial_{y_k}$ define a co-Higgs field on X. A coordinate free way to say this is the following:

Lemma 3.2.1. *Let $p : V \to X$ be a vector bundle and σ be a quadratic Poisson structure on V with coisotropic fibers. Then σ induces a co-Higgs field $\phi \in \mathrm{Hom}(V, V \otimes \mathcal{T}_X)$, given by the formula*

$$\langle \phi, \alpha \rangle = \sigma^{\#}(p^*\alpha), \quad \textit{for any } \alpha \in \mathcal{T}_X^*. \tag{3.4}$$

Proof. For any 1-form $\alpha \in \mathcal{T}_X^*$, the pullback $p^*\alpha \in \mathcal{T}_V^*$ is $\mathbb{C}^*$-invariant. Therefore, the Hamiltonian vector field $\sigma^{\#}(p^*\alpha)$ is $\mathbb{C}^*$-invariant as well. Moreover, since the fibers of V are coisotropic, the vector field $\sigma^{\#}(p^*\alpha)$ is tangent to the fibers. Hence, we can use the correspondence between $\mathbb{C}^*$-invariant vector fields on V tangent to the fibers, and endomorphisms of V to define the expression $\langle \phi, \alpha \rangle \in End(V)$, for each $\alpha \in \mathcal{T}_X^*$, via (3.4).

To prove that the tensor ϕ satisfies the co-Higgs integrability condition, we need to check that for $\alpha, \beta \in \mathcal{T}_X^*$, the endomorphisms $\langle \phi, \alpha \rangle$ and $\langle \phi, \beta \rangle$ commute. Equivalently, we need to check that the vector fields $\sigma^{\#}(p^*\alpha)$ and $\sigma^{\#}(p^*\beta)$ Lie commute. The latter fact follows from the assumption that the fibers of V are coisotropic with respect to σ. □

In the setup of Lemma 3.2.1, σ is called a *Poisson lift* of ϕ to V.

Remark 3.2.2. In Lemma 3.2.1, if one drops the requirement of coisotropic fibers, then instead of a co-Higgs field, one can construct a flat Poisson connection on V with respect to the Poisson structure $p_*\sigma$ on X. The latter is the dual of the following Poisson connection on V^*:

$$\begin{array}{rccc} \nabla : & \mathcal{T}_X^* \times V^* & \longrightarrow & V^*, \\ & (df, s) & \longmapsto & \{p^*f, s\}_\sigma. \end{array}$$

Projective bundle case

Recall that vector fields on a projective bundle $\mathbb{P}(V)$ tangent to the fibers are in a one-to-one correspondence with the sections of $End_0(V)$, the zero trace endomorphisms of V. Recall that to a co-Higgs field ϕ one can bring into correspondence a pair (ϕ_0, v) of a zero trace co-Higgs field and a vector field such that $v = \mathrm{Tr}(\phi)$ and $\phi_0 = \phi - \frac{1}{\mathrm{rk}\ V}\mathrm{Tr}(\phi)$. Conversely, one can recover ϕ from the pair (ϕ_0, v) in an obvious way.

Lemma 3.2.3. *Let V be a vector bundle over X, $p : \mathbb{P}(V) \to X$ be its projectivization, and π be a Poisson structure on $\mathbb{P}(V)$ with coisotropic fibers. Then π induces a unique zero trace co-Higgs field $\phi_0 \in H^0(X, End_0(V) \otimes \mathcal{T}_X)$, given by the formula*

$$\langle \phi_0, \alpha \rangle = \pi^{\#}(p^*\alpha), \quad \text{for any } \alpha \in \mathcal{T}_X^*. \tag{3.5}$$

Proof. For any 1-form $\alpha \in \mathcal{T}_X^*$, the Hamiltonian vector field $\pi^{\#}(p^*\alpha) \in \mathcal{T}_{\mathbb{P}(V)}$ is tangent to the fibers. Hence, we can use the correspondence between vector fields on $\mathbb{P}(V)$ tangent to the fibers, and zero trace endomorphisms of V to define the expression $\langle \phi_0, \alpha \rangle \in End(V)$, for each $\alpha \in \mathcal{T}_X^*$, via (3.5).

To prove that the tensor ϕ_0 satisfies the co-Higgs integrability condition, we need to check that for $\alpha, \beta \in \mathcal{T}_X^*$, the endomorphisms $\langle \phi_0, \alpha \rangle$ and $\langle \phi_0, \beta \rangle$ commute. Equivalently, we need to check that the vector fields $\pi^{\#}(p^*\alpha)$ and $\pi^{\#}(p^*\beta)$ Lie commute. The latter fact follows from the assumption that the fibers of V are coisotropic with respect to π. $\square$

In the setup of Lemma 3.2.1, π is called a *Poisson lift* of ϕ_0 to $\mathbb{P}(V)$. Sometimes, we will say that a Poisson structure π on $\mathbb{P}(V)$ lifts a co-Higgs field ϕ that does not necessarily have zero trace. This will mean that π lifts the zero trace part of ϕ.

Overall, we obtain the following commutative diagram of operations

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \text{Quadratic Poisson structures on } V \\ \text{with coistropic fibers} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{Poisson structures on } \mathbb{P}(V) \\ \text{with coistropic fibers} \end{array} \right\} \\
\downarrow & & \downarrow \\
\left\{ \text{Co-Higgs fields on } V \right\} & \longrightarrow & \left\{ \text{Zero trace co-Higgs fields on } V \right\}
\end{array}$$

Going in the directions opposite to the arrows is usually non-trivial, except for the lower horizontal arrow, and constitutes one of the main themes of this thesis. Theorem 3.1.5 above dealt with the reversing the upper horizontal arrow, without the assumption of coisotropic fibers.

3.3 Modular vector field via co-Higgs field

Recall the notion of the modular vector field of a Poisson structure from Section 2.1. In the current section we prove that for Poisson structures on projective bundles and quadratic Poisson structures on vector bundles, provided that the fibers are coisotropic, some part of the modular vector field can be read off from the underlying co-Higgs field.

Vector bundle case

Lemma 3.3.1. *Let σ be a quadratic Poisson structure on a vector bundle V over X whose fibers are coisotropic, and let ϕ be the corresponding co-Higgs field on V. Then the modular vector field of σ restricted to the zero section of V equals the negative of the trace of ϕ.*

Proof. The statement is local, so without loss of generality we may assume that $X = \mathcal{U}$ is a small open set in $\mathbb{C}^n$. Let $x_1, ..., x_n$ be coordinates on $\mathcal{U}$ and $y_1, ..., y_r$ be fiberwise linear coordinates on $V|_{\mathcal{U}}$. Let

$$\sigma = \sum_{i=1}^{n} \sum_{j,k=1}^{r} g_{ijk}\ \partial_{x_i} \wedge y_j \partial_{y_k} + \sum_{i,j,k,l=1}^{r} h_{ijkl}\ y_i y_j \partial_{y_k} \wedge \partial_{y_l}, \quad \text{for some } g_{ijk}, h_{ijkl} \in \mathcal{O}_{\mathcal{U}}.$$

Then

$$\phi = \sum_{i=1}^{n} \sum_{j,k=1}^{r} g_{ijk} \partial_{x_i} \otimes y_j \partial_{y_k},$$

$$\mathrm{Tr}\ \phi = \sum_{i=1}^{n} \sum_{j=1}^{r} g_{ijj} \partial_{x_i}.$$

Choose the volume form $\Omega = \bigwedge_{i=1}^{n} dx_i \wedge \bigwedge_{j=1}^{r} dy_j$, and let $\Delta = \sum_{i=1}^{n} \nabla_{\partial_{x_i}} \otimes \iota_{dx_i} + \sum_{j=1}^{r} \nabla_{\partial_{x_j}} \otimes \iota_{dx_j}$ be the corresponding BV operator on multivectors on V. Then the modular vector field of σ is

$$\zeta = \Delta(\sigma) = \sum_{i=1}^{n} \sum_{j,k=1}^{r} \frac{\partial g_{ijk}}{\partial x_i} y_j \partial_{y_k} - \sum_{i=1}^{n} \sum_{j=1}^{r} g_{ijj} \partial_{x_i} + \Delta\left(\sum_{i,j,k,l=1}^{r} h_{ijkl}\ y_i y_j \partial_{y_k} \wedge \partial_{y_l} \right).$$

At the zero section $\cap_{j=1}^{r} \{y_j = 0\}$ the first and last terms vanish, and the second one equals $-\mathrm{Tr}\ \phi$. ☐

Projective bundle case

Let $\phi \in H^0(X, End(V) \otimes \mathcal{T}_X)$ be a co-Higgs field. By its *eigenvariety* we mean the subvariety $E \subset \mathbb{P}(V)$ consisting of the points (x, v), $x \in X$, $0 \neq v \in V|_x$ such that $\phi_x(v) \wedge v = 0 \in \wedge^2 V|_x \otimes \mathcal{T}|_x$. The eigenvariety $E \subset \mathbb{P}(V)$ of ϕ is closely related to the spectral variety $\Sigma \subset \mathcal{T}_X$ of ϕ. However, they are not always isomorphic.

Lemma 3.3.2. *Let $\phi \in H^0(X, End(V) \otimes \mathcal{T}_X)$ be a co-Higgs field. Let $E \subset \mathbb{P}(V)$ be the eigenvariety of ϕ and $\Sigma \subset \mathcal{T}_X$ be the spectral variety of ϕ. Then there is a canonical morphism $Eig : E \to \Sigma$ sending a ϕ-eigenvector to its eigenvalue. If ϕ is regular semi-simple over $\mathcal{U} \subset X$, then the morphism Eig is an isomorphism over $\mathcal{U}$.*

Proof. Let $\mathcal{U} \subset X$ be an analytic open set. Choose a basis of $V|_{\mathcal{U}}$. Let $\mathcal{U}_{ij} \subset \mathbb{P}(V)|_{\mathcal{U}}$ be the open subset given by the points (x, v), $x \in \mathcal{U}$, $v = [v_1 : ... : v_r] \in \mathbb{P}(V)|_x$, such that $v_i \neq 0$ and $v_j \neq 0$. Then over $E \cap \mathcal{U}_{ij}$ the morphism Eig is defined by formula $Eig(x, v) = \frac{v_j}{v_i} \frac{\phi_x(v)_i}{\phi_x(v)_j}$.

If ϕ is regular semi-simple over $\mathcal{U}$, then each eigenvalue corresponds to exactly one eigenvector, up to scaling, so there is a one-to-one correspondence between the eigenvalues and the eigenvectors of ϕ. □

Lemma 3.3.3. *Let V be a vector bundle over X, $p : \mathbb{P}(V) \to X$ be its projectivization, and π be a Poisson structure on $\mathbb{P}(V)$ with coisotropic fibers. Let ϕ be the corresponding zero trace co-Higgs field on V. Then the zero set of π is contained in the eigenvariety $E \subset \mathbb{P}(V)$ of ϕ.*

Proof. If π vanishes at a point $(x, v) \in \mathbb{P}(V)$, then in particular the Hamiltonian vector field $\pi^{\#}(p^*\alpha)$ vanishes at (x, v) for each $\alpha \in \mathcal{T}^*|_x$. This implies $\langle \phi_x(v), \alpha \rangle$ is collinear with v for any $\alpha \in \mathcal{T}^*|_x$. In other words $\phi_x(v) \wedge v = 0 \in \wedge^2 V|_x \otimes \mathcal{T}|_x$. □

Lemma 3.3.4. *Let $p : V \to \mathcal{U}$ be a rank r vector bundle over an analytic open set $\mathcal{U} \subset \mathbb{C}^n$, and π be a Poisson structure on $\mathbb{P}(V)$ with coisotropic fibers. Let the corresponding zero trace co-Higgs field ϕ on V be diagonal, with diagonal vector fields $v_1, ..., v_r$. Furthermore, let $2v_k - v_i - v_j$ be a non-zero vector field, unless $i = j = k$. (In particular, we are assuming that ϕ is regular semi-simple generically on $\mathcal{U}$.)*

Then the zero set of π is equal to the eigenvariety $E \subset \mathbb{P}(V)$ of ϕ. Moreover, the modular vector field of π, restricted to the branch $s_i : \mathcal{U} \to E$ corresponding to v_i, equals $r s_{i,*} v_i$.

Proof. Using Theorem 3.1.5, let us lift the Poisson structure π on $\mathbb{P}(V)$ to a unimodular Poisson structure σ on V. The structure σ lifts a co-Higgs field of the form $\phi + vId$ on V, for some vector field v. However due to unimodularity of σ and Lemma 3.3.1, we have $v = 0$, and so σ lifts ϕ.

Let us choose coordinates $x_1, ..., x_n$ on $\mathcal{U}$, so that $v_i = \sum_{j=1}^n f_{ij} \partial_{x_j}$, for some $f_{ij} \in \mathcal{O}_{\mathcal{U}}$. Also, let us choose for each $i = 1, ..., r$ a linear fiberwise coordinate y_i on the eigenline of V with the eigenvalue v_i. Then the coordinate expression for σ must be of the form

$$\sigma = \sum_{b=1}^{n} \sum_{a=1}^{r} f_{ab}\, \partial_{x_b} \wedge y_a \partial_{y_a} + \sum_{\substack{i,j,k,m=1 \\ i \le j, k < l}}^{n} g_{ijkl}\; y_i y_j \partial_{y_k} \wedge \partial_{y_l}, \tag{3.6}$$

for some functions $g_{ijkl} \in \mathcal{O}_{\mathcal{U}}$.

First, let us prove the lemma assuming the second summand in (3.6) vanishes. The points of $E \subset \mathbb{P}(V)$ are given by those $(x_1, ..., x_n, y_1, ..., y_r)$ for which all y_i's, except one, are zero. For any $i \in \{1, 2, ..., r\}$, let $\mathcal{U}_i \subset \mathbb{P}(V)$ be the set of points $(x_1, ..., x_n, y_1, ..., y_r)$ with $y_i \ne 0$. Then $x_1, ..., x_n$ together with $z_a = \frac{y_a}{y_i}$, $a \ne i$, define coordinates on $\mathcal{U}_i$. The projectivization of first summand of (3.6) in these coordinates has the form

$$\sum_{b=1}^{n} \sum_{a \ne i} (f_{ab} - f_{ib}) \partial_{x_b} \wedge z_a \partial_{z_a}. \tag{3.7}$$

This expression vanishes on $E \cap \mathcal{U}_i = E_i$, since the latter set consists of points for which the coordinates z_a are zero for all $a \ne i$. Moreover, by choosing the volume form $\Omega = dx_1 \wedge ... \wedge dx_n \wedge (\bigwedge_{a \ne i} dz_a)$, we can calculate the contribution of (3.7) to the modular vector field of σ by applying the corresponding BV operator $\Delta_\Omega = \sum_{b=1}^n \partial_{x_b} \otimes \iota_{dx_b} + \sum_{a \ne i} \partial_{z_a} \otimes \iota_{dz_a}$:

$$\Delta_\Omega \left(\sum_{b=1}^{n} \sum_{a \ne i} (f_{ab} - f_{ib}) \partial_{x_b} \wedge z_a \partial_{z_a} \right) = \sum_{b=1}^{n} \sum_{a \ne i} \frac{\partial (f_{ab} - f_{ib})}{\partial_{x_b}} z_a \partial_{z_a} + \sum_{b=1}^{n} \sum_{a \ne i} (f_{ib} - f_{ab}) \partial_{x_b}.$$

In the latter expression, the first summand vanishes on E_i, whereas the second one gives $s_{i,*}((r - 1)v_i - \sum_{a \ne i} v_a) = r s_{i,*} v_i$.

Let us now explain why the projectivization of the second summand in (3.6) also vanishes on E, and does not give a contribution to the modular vector field of σ over E. We are going to use the fact that the vector fields v_i, $i = 1, ..., r$, pairwise Lie commute. Thid will be proved later in Lemma 3.4.2.

The $\partial_x \wedge \partial_y \wedge \partial_y$ component of the trivector $[\sigma, \sigma]$ is given by

$$2 \sum_{\substack{b,i,j,k,m=1 \\ i\leq j, k<l}}^{n} g_{ijkl} \partial_{x_b} \wedge \left[\sum_{a=1}^{n} f_{ab} t_a \partial_{y_a}, y_i y_j \partial_{y_k} \wedge \partial_{y_l} \right] = 2 \sum_{\substack{b,i,j,k,m=1 \\ i\leq j, k<l}}^{n} g_{ijkl} \partial_{x_b} \wedge (f_{ib} + f_{jb} - f_{kb} - f_{lb}) y_i y_j \partial_{y_k} \wedge \partial_{y_l}.$$

Since σ is Poisson, this component vanishes, and we see that $g_{ijkl} = 0$ unless $f_{ib} + f_{jb} - f_{kb} - f_{lb} = 0$ for all b. That is to say, $g_{ijkl} = 0$ unless $v_i + v_j - v_k - v_l = 0$. Due to the assumption of the lemma, we get that the only terms appearing in the second sum of (3.6) are the ones where all the indices i, j, k, l are distinct. Now, we apply Lemma 3.1.2 fiberwise to argue that all such summands in (3.6) vanish on E and do not contribute to the modular vector field of σ on E. □

3.4 Lifting a co-Higgs field to a Poisson structure.

3.4.1 Local analysis

Let $\phi \in H^0(X, End(V) \otimes \mathcal{T}_X)$ be a co-Higgs field on a rank r vector bundle V. Let

$$\tau_i(\phi) = \mathrm{Tr}(\phi^k) \in H^0(X, S^k\mathcal{T}_X),\ k = 1, 2, ..., r.$$

Specifically viewed as an element of $\mathrm{Hom}(S^k\mathcal{T}_X^*, \mathcal{O}_X)$, $\tau_k(\phi)$ is the composition

$$S^k\mathcal{T}_X^* \longrightarrow End(V) \xrightarrow{Trace} \mathcal{O}_X$$

$$\omega_1 \otimes \omega_1 \otimes ... \otimes \omega_k \longmapsto \langle\phi, \omega_1\rangle\langle\phi, \omega_2\rangle \dots \langle\phi, \omega_k\rangle \longmapsto \mathrm{Tr}(\langle\phi, \omega_1\rangle\langle\phi, \omega_2\rangle \dots \langle\phi, \omega_k\rangle).$$

Definition 3.4.1. We say that the co-Higgs field ϕ is *strongly integrable* if the $\tau_k(\phi)$, $k = 1, 2, ..., r$, pairwise Poisson commute when viewed as functions on the symplectic manifold $\mathcal{T}^*X$.

In addition to the functions $\tau_k(\phi)$, we consider the coefficients $s_k(\phi) \in H^0(X, S^k\mathcal{T}_X)$ of the characteristic polynomial of ϕ. Over each fiber, if $v_1, ..., v_r$ are the eigen vector fields (=$\mathcal{T}_X$-valued eigenvalues) of ϕ counted with multiplicity, then

$$s_1(\phi) = v_1 + v_2 + ... + v_2,$$

$$s_2(\phi) = \sum_{1 \le i < j \le r} v_i v_j,$$

$$s_3(\phi) = \sum_{1 \le i < j < k \le r} v_i v_j v_k,$$

$$\vdots$$

$$s_r(\phi) = v_1 v_2 ... v_r.$$

Recall that the spectral variety of ϕ is the subvariety of $\mathcal{T}_X$ cut out by the section

$$\chi_\phi(\theta) = \det(\theta - \phi) = \theta^r - s_1(\phi)\theta^{r-1} + ... + (-1)^{r-1}s_{r-1}(\phi)\theta + (-1)^r s_r(\phi)$$

of $p^*S^r\mathcal{T}_X$, where $p : \mathcal{T}_X \to X$ is the natural projection, and θ is the tautological section of $p^*\mathcal{T}_X$.

The functions $\tau_k(\phi)$ and $s_k(\phi)$ are related by the classical Newton's identities

$$\tau_k(\phi) = (-1)^{k-1}ks_k(\phi) + \sum_{i=1}^{k-1}(-1)^{k-i+1}s_{k-i}(\phi)\tau_i(\phi),\ 1 \le k \le r. \tag{3.8}$$

Formulas (3.8) imply, by induction, that the $\tau_k(\phi)$, $k = 1, 2, .., r$, pairwise Poisson commute if and only if the $s_k(\phi)$, $k = 1, 2, ..., r$, do.

Lemma 3.4.2. *Let $\phi \in H^0(X, End(V) \otimes \mathcal{T}_X)$ be a co-Higgs field. Let $\mathcal{U} \subset X$ an analytic open set on which ϕ can be diagonalized:*

$$\phi = \begin{pmatrix} v_1 & 0 & \dots & 0 \\ 0 & v_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & v_r \end{pmatrix}, \ v_i \in H^0(\mathcal{U}, \mathcal{T}_X), \ i = 1, 2, \dots, r. \tag{3.9}$$

Then

a) ϕ is strongly integrable over $\mathcal{U}$ if and only if the vector fields v_i, $i = 1, 2, \dots, r$, pairwise Lie commute.

b) ϕ admits a lift to a quadratic Poisson structure on $V|_{\mathcal{U}}$ if and only if ϕ is strongly integrable.

Proof. a) Over $\mathcal{U}$, one has $\tau_k(\phi) = \sum_{i=1}^r v_i^k$. The Lie bracket of vector fields is the same as the Poisson bracket of the corresponding fiberwise linear functions on $\mathcal{T}^*X$, so if the v_i pairwise Lie commute, then the τ_k pairwise Poisson commute over $\mathcal{U}$. Conversely, assume that the τ_k pairwise Poisson commute, and let us prove that the v_i pairwise Lie commute.

By passing to a smaller $\mathcal{U}$, if necessary, we ensure that for each pair of indices $i \neq j$, we have either $v_i(z) = v_j(z)$, for all $z \in \mathcal{U}$, or $v_i(z) \neq v_j(z)$, for all $z \in \mathcal{U}$. After doing this, let us renumerate the v_i so that $v_1, v_2, ..., v_{r'}$, for some $r' \le r$, are all distinct. For each $1 \le j \le r'$, let $m_i \ge 1$ be the multiplicity of this particular v_j in the list. One has $m_1 + m_2 + ... + m_{r'} = r$.

For any $k, l \in \{1, 2, ..., r\}$ one has

$$0 = \{\tau_k(\phi), \tau_l(\phi)\} = \sum_{i,j=1}^{r} \{v_i^k, v_j^l\} = \sum_{i,j=1}^{r'} m_i m_j \{v_i^k, v_j^l\} = kl \sum_{i,j=1}^{r'} m_i m_j \{v_i, v_j\} v_i^{k-1} v_j^{m-1}.$$

Denoting by P the $r' \times r'$ matrix whose (i,j)-th entry is $\{v_i, v_j\}$ one obtains that

$$M(VPV^T)M = 0,$$

where V is the Vandermonde $r' \times r'$ matrix whose (i,j)-th entry is v_j^{i-1}, and M is the diagonal $r' \times r'$ matrix whose (i,i)-th entry is m_i. The matrix M is invertible. Also, since $v_i \neq v_j$, for $i, j \le r'$, $i \neq j$, the Vandermonde matrix V is invertible, too, and therefore P is the zero matrix.

b) Let $y_1, y_2, ..., y_r$ be the fiberwise linear coordinates on $V|_{\mathcal{U}}$ given by the trivialization of $V|_{\mathcal{U}}$ in which ϕ has the form (5.1). Then by part a), the v_i pairwise Lie commute, and so the bivector $\sigma_0 = \sum_{i=1}^r v_i \wedge y_i \frac{\partial}{\partial y_i}$ lifting ϕ is Poisson. Conversely, let us assume that ϕ is not strongly integrable. Then the lift σ_0 is not Poisson, because

$$[\sigma_0, \sigma_0] = 2 \sum_{1 \le i < j \le r} [v_i, v_j] \wedge y_i \frac{\partial}{\partial y_i} \wedge y_j \frac{\partial}{\partial y_j} \neq 0.$$

Moreover, we claim that no other lift σ of ϕ is Poisson. Indeed, let

$$\sigma = \sum_{i=1}^r v_i \wedge y_i \frac{\partial}{\partial y_i} + \sum_{k,l,m,n=1}^r h_{klmn} \, y_k y_l \frac{\partial}{\partial y_m} \wedge \frac{\partial}{\partial y_n},$$

where $h_{klmn} \in \mathcal{O}_{\mathcal{U}}$. Then

$$[\sigma, \sigma] = 2 \sum_{1 \le i < j \le r} [v_i, v_j] \wedge y_i \frac{\partial}{\partial y_i} \wedge y_j \frac{\partial}{\partial y_j} + \sum_{i,k,l,m,n=1}^r h_{klmn} \, v_i \wedge \left[y_i \frac{\partial}{\partial y_i}, y_k y_l \frac{\partial}{\partial y_m} \wedge \frac{\partial}{\partial y_n} \right] + \dots, \tag{3.10}$$

where the omitted terms are purely vertical. Note that

$$\left[y_i \frac{\partial}{\partial y_i}, y_k y_l \frac{\partial}{\partial y_m} \wedge \frac{\partial}{\partial y_n}\right] = (\delta_{ik} + \delta_{il} - \delta_{im} - \delta_{in})\, y_k y_l \frac{\partial}{\partial y_m} \wedge \frac{\partial}{\partial y_n}.$$

Therefore, the second sum in (3.10) will never cancel out the first one. The only terms that have a chance of doing so are the terms with $k = m$, $l = n$, and the terms with $k = n$, $l = m$, but they get multiplied by zero. □

Example 3.4.3. Let C be a smooth curve, v be a vector field, and $d \geq 1$. Let ϕ be the co-Higgs field coming from the spectral data $(p : C \times S^{d-1}C \to S^d C, (v, 0, ..., 0), \mathcal{L})$, where p is the symmetrization map, and $\mathcal{L}$ is any line bundle over $C \times S^{d-1}C$. The the co-Higgs field ϕ is strongly integrable. Indeed, if we choose d disjoint analytic open subsets $\mathcal{U}_1, ..., \mathcal{U}_d$ of C, then over the open subset $\mathcal{U}_1 \times ... \times \mathcal{U}_d \subset S^d C$, the eigen vector fields v_i from Lemma 3.4.2 equal $(0, \dots, 0, \underset{i\text{-th}}{v}, 0 \dots, 0)$, and clearly pairwise Lie commute.

Remark 3.4.4. For part (a) of Lemma 3.4.2, it is not essential that the co-Higgs field ϕ is diagonalizable. Passing to a smaller open set $\mathcal{U}$, if necessary, we can always define the eigen vector fields v_i as local sections of the spectral variety $p : \Sigma \to X$. Then the proof of part (a) works verbatim for these v_i.

Remark 3.4.5. In part (b) of Lemma 3.4.2, it **is** essential that the co-Higgs field ϕ is diagonalizable, as the following example shows.

Example 3.4.6. Let $\mathcal{U}$ be an analytic open subset of $\mathbb{C}$ containing 0, and consider the co-Higgs field

$$\phi = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \partial_x$$

on the rank 2 vector bundle $V = \mathcal{O}_\mathcal{U} \oplus \mathcal{O}_\mathcal{U}$. The co-Higgs field ϕ is strongly integrable, because $\tau_1(\phi) = 2\partial_x$, $\tau_2(\phi) = 2\partial_x^{\otimes 2}$. Denoting by y_1 and y_2 the fiberwise linear coordinates on the first and the second summand of V, respectively, we notice that any lift of ϕ to a $\mathbb{C}^*$-invariant bivector on V has to be of the form

$$\sigma = \partial_x \wedge y_1\partial_{y_1} + \partial_x \wedge y_2\partial_{y_2} + x\partial_x \wedge y_2\partial_{y_1} + f(x)y_1\partial_{y_1} \wedge y_1\partial_{y_2} + g(x)y_1\partial_{y_1} \wedge y_2\partial_{y_2} + h(x)y_2\partial_{y_1} \wedge y_2\partial_{y_2},$$

where $f, g, h \in \mathcal{O}_\mathcal{U}$. Then one has

$$[\sigma, \sigma] = 4xf(x)\partial_x \wedge y_1\partial_{y_1} \wedge y_2\partial_{y_2} - 2(xg(x) + 1)\partial_x \wedge y_2\partial_{y_1} \wedge y_2\partial_{y_2}.$$

We see that no choice of f, g, h produces an integrable lift σ.

Corollary 3.4.7. *Let ϕ be a diagonalizable zero trace co-Higgs field on a vector bundle V over an open set $\mathcal{U} \subset \mathbb{C}$. Then ϕ admits a lift to a Poisson structure on $\mathbb{P}(V)$ if and only it is strongly integrable.*

Proof. If ϕ is strongly integrable, then by Lemma 3.4.2.b), it admits a lift to a quadratic Poisson structure σ on V. Then projectivization of σ gives a Poisson lift of ϕ to $\mathbb{P}(V)$.

Conversely, let ϕ admit a Poisson lift π to $\mathbb{P}(V)$. By Theorem 3.1.5, π admits a further lift to a quadratic unimodular Poisson structure σ on V. Then σ lifts a co-Higgs field $\psi = \phi + vId$, for some vector field $v \in H^0(\mathcal{U}, \mathcal{T}_\mathcal{U})$. By Lemma 3.3.1, the co-Higgs field ψ has zero trace, just as ϕ does. Therefore, $v = 0$, and so σ is a Poisson lift of ϕ. Lemma 3.4.2.b) then implies that ϕ is strongly integrable. □

3.4.2 Global analysis.

On top of the local obstructions discussed in Subsection 3.4.1, there may be further obstructions for the existence of a Poisson lift of a co-Higgs field. In the current subsection, we discuss the obstruction coming from the Atiyah class of the underlying vector bundle. Prior to this, let us discuss two low rank unobstructed cases.

Remark 3.4.8. If $V = L$ is a line bundle over a manifold X, then a co-Higgs field on L is the same as a vector field on X. Each vector field $v \in H^0(X, \mathcal{T}_X)$ viewed as a co-Higgs field on L admits a unique Poisson lift to L in the following way. The quadratic Poisson structure on L lifting v is $\sigma = \widetilde{v} \wedge Eul$, where $\widetilde{v}$ is a local choice of a $\mathbb{C}^*$-invariant vector field on L projecting onto v, and Eul is the Euler vector field. Note that since the fibers of L are one dimensional, different choices of $\widetilde{v}$ will amount to the same σ.

Theorem 3.4.9. *[26] Let V be a rank 2 vector bundle over a base manifold X. Then there is a one-to-one correspondence between the Poisson structures on $\mathbb{P}(V)$ with coisotropic fibers and the zero trace co-Higgs fields on V.*

For a detailed proof of Theorem 3.4.9, see [26, Theorem 6.1], which is stated and proved without the assumption of coisotropic fibers. Let us present an explicit formula for the unique Poisson lift π of a zero trace co-Higgs field ϕ on a rank 2 vector bundle $V \to X$. Let

$$\phi = \sum_{i=1}^{n} \Big(f_{i11}\ \partial_{x_i} \otimes y_1\partial_{y_1} + f_{i12}\ \partial_{x_i} \otimes y_1\partial_{y_2} + f_{i21}\ \partial_{x_i} \otimes y_2\partial_{y_1} - f_{i11}\ \partial_{x_i} \otimes y_2\partial_{y_2} \Big),$$

where $f_{ijk} \in \mathcal{O}_{\mathcal{U}}$, $x_1, ..., x_n$ are coordinates on $\mathcal{U} \subset X$, and y_1, y_2 are fiberwise linear coordinates on $V|_{\mathcal{U}}$. Then the Poisson lift of ϕ to $\mathbb{P}(V|_{\mathcal{U}})$ is

$$\pi = \sum_{i=1}^{n} \Big(2f_{i11}\ \partial_{x_i} \wedge \widetilde{y}_1\partial_{\widetilde{y}_1} + f_{i12}\ \partial_{x_i} \wedge {\widetilde{y}_1}^2\partial_{\widetilde{y}_1} + f_{i21}\ \partial_{x_i} \wedge \partial_{\widetilde{y}_1} \Big),$$

where $\widetilde{y}_1$ is the affine coordinate $\frac{y_1}{y_2}$.

Let $\phi \in H^0(X, \mathcal{T}_X \otimes End(V))$ be a co-Higgs tensor (not necessarily integrable). Consider the short exact sequence of $\mathcal{O}_X$-sheaves

$$0 \longrightarrow S^2V^* \otimes \wedge^2 V \longrightarrow (\wedge^2\mathcal{T}_V)^{\mathbb{C}^*}_{cois} \longrightarrow \mathcal{T}_X \otimes End(V) \longrightarrow 0,$$

where $(\wedge^2\mathcal{T}_V)^{\mathbb{C}^*}_{cois}$ denotes the $\mathbb{C}^*$-invariant bivectors on V, with respect to which the fibers of V are coisotropic, and the right hand side map is given by Lemma 3.2.1.

From the corresponding long exact cohomological sequence

$$0 \longrightarrow H^0(X, S^2V^* \otimes \wedge^2V) \longrightarrow H^0(X, (\wedge^2\mathcal{T}_V)_{cois})^{\mathbb{C}^*} \longrightarrow H^0(X, \mathcal{T}_X \otimes End(V)) \xrightarrow{\ \delta\ }$$

$$\xrightarrow{\ \delta\ } H^1(X, S^2V^* \otimes \wedge^2V) \longrightarrow \dots$$

we see that ϕ admits a lift to a $\mathbb{C}^*$-invariant bivector on V if and only if its image under the connecting homomorphism $\delta(\phi)$ represents the zero element of $H^1(X, S^2V^* \otimes \wedge^2V)$. One can express the element

$\delta(\phi) = \langle \mathrm{At}_V \wedge \phi \rangle$ as the pairing of the Atiyah class $\mathrm{At}_V \in H^1(X, \mathcal{T}_X^* \otimes End(V))$ with the co-Higgs tensor $\phi \in H^0(X, \mathcal{T}_X \otimes End(V))$ coming from the following bundle map

$$\begin{aligned} (\mathcal{T}_X^* \otimes End(V)) \otimes (\mathcal{T}_X \otimes End(V)) &\xrightarrow{\langle\, ,\, \rangle} End(V) \otimes End(V) \xrightarrow{\quad\wedge\quad} S^2V^* \otimes \wedge^2 V, \\ \Big(dx_i \otimes y_j \partial_{y_k}\Big) \otimes \Big(\partial_{x_a} \otimes y_b \partial_{y_c}\Big) &\longmapsto \delta_{ia}(y_j \partial_{y_k} \otimes y_b \partial_{y_c}) \longmapsto \delta_{ia} y_j y_b \otimes \partial_{y_k} \wedge \partial_{y_c}, \end{aligned} \tag{3.11}$$

where x's are local coordinates on X, y's are fiberwise linear coordinates on V, and δ_{ia} equals 1 if $i = a$, and 0 if $i \neq a$. We have arrived at the following:

Proposition 3.4.10. *Let $\phi \in H^0(X, \mathcal{T}_X \otimes End(V))$ be a (not necessarily integrable) co-Higgs tensor. Then ϕ lifts to a (not necessarily integrable) $\mathbb{C}^*$-invariant bivector on V if and only if the pairing $\langle \mathrm{At}_V \wedge \phi \rangle$ defined by* (3.11) *gives the zero cohomology class in $H^1(X, S^2V^* \otimes \wedge^2 V)$.*

If the obstruction $\langle \mathrm{At}_V \wedge \phi \rangle$ does vanish, the set of lifts of ϕ to $H^0(X, (\wedge^2 \mathcal{T}_V)_{cois})^{\mathbb{C}^*}$ is a torsor over $H^0(X, S^2V^* \otimes \wedge^2 V)$.

If ϕ is a strongly integrable co-Higgs field, one might hope to find a lift of ϕ that is Poisson. To do this, one needs to solve the Maurer-Cartan equation

$$[\widetilde{\sigma} + \beta, \widetilde{\sigma} + \beta] = 0,$$

where $\widetilde{\sigma}$ is a fixed, not necessarily integrable lift of ϕ, and β runs over $H^0(X, S^2V^* \otimes \wedge^2 V)$ and is viewed as a purely vertical bivector on V.

For future use, let us state and prove the following:

Lemma 3.4.11. *Let ϕ be a co-Higgs field on $V = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(k-2)$ with $k \neq 0$. Then ϕ lifts to a $\mathbb{C}^*$-invariant (not necessarily Poisson) bivector on V if and only if the* $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(k), \mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{T}_{\mathbb{P}^1})$ *component of ϕ is zero.*

Proof. Let

$$\phi = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$$

where $v_1, v_4 \in H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1})$, $v_2 \in H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}(2))$, $v_3 \in H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}(-2)) \cong \mathbb{C}$. We claim that to be able to lift ϕ to a $\mathbb{C}^*$-invariant bivector on V one must assume that the constant v_3 is zero. To this end, let us calculate the obstruction $\langle \mathrm{At}_V \wedge \phi \rangle \in H^1(\mathbb{P}^1, S^2V^* \otimes \wedge^2 V)$ provided by Proposition 3.4.10. Note that

$$S^2V^* \otimes \wedge^2 V = \Big(\mathcal{O}_{\mathbb{P}^1}(-2k) \oplus \mathcal{O}_{\mathbb{P}^1}(2-2k) \oplus \mathcal{O}_{\mathbb{P}^1}(4-2k)\Big) \otimes \mathcal{O}_{\mathbb{P}^1}(2k-2) = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2),$$

so $H^1(\mathbb{P}^1, S^2V^* \otimes \wedge^2 V) \cong \mathbb{C}$, and only the first summand contributes to the H^1.

Let

$$\begin{aligned} v_1 &= (a_{10} + a_{11}x + a_{12}x^2)\partial_x, \\ v_2 &= (a_{20} + a_{21}x + a_{22}x^2 + a_{23}x^3 + a_{24}x^4)\partial_x, \\ v_3 &= a_{30}\partial_x, \\ v_4 &= (a_{40} + a_{41}x + a_{42}x^2)\partial_x, \end{aligned}$$

where $a_{ij} \in \mathbb{C}$ and x is an affine coordinate on $\mathcal{U}_0 \subset \mathbb{P}^1$. Also, choose fiberwise linear coordinates y_1 and y_2 on the first and the second summand of V, respectively. Then with respect to the Čech cover $\mathcal{U}_0$, $\mathcal{U}_1 = \mathbb{P}^1 \smallsetminus \{x = 0\}$, the Atiyah class At_V has the 1-cocycle representative

$$\mathrm{At}_V = k y_1 \partial_{y_1} \otimes \frac{dx}{x} + (k-2) y_2 \partial_{y_2} \otimes \frac{dx}{x} \in (End(V) \otimes \mathcal{T}^*_{\mathbb{P}^1})|_{\mathcal{U}_0 \cap \mathcal{U}_1}.$$

Also, let us express

$$\phi = y_1 \partial_{y_1} \otimes v_1 + y_2 \partial_{y_1} \otimes v_2 + y_1 \partial_{y_2} \otimes v_3 + y_2 \partial_{y_2} \otimes v_4 \in H^0(\mathbb{P}^1, End(V) \otimes \mathcal{T}_{\mathbb{P}^1}).$$

Then

$$\begin{aligned} \langle \mathrm{At}_V \ \text{⸼}\ \phi \rangle \quad = \quad & k y_1 \partial_{y_1} \wedge (y_1 \partial_{y_2} \tfrac{\langle v_3, dx \rangle}{x} + y_2 \partial_{y_2} \tfrac{\langle v_4, dx \rangle}{x}) + \\ & (k-2) y_2 \partial_{y_2} \wedge (y_1 \partial_{y_1} \otimes \tfrac{\langle v_1, dx \rangle}{x} + y_2 \partial_{y_1} \otimes \tfrac{\langle v_2, dx \rangle}{x}) \quad \in (S^2 V^* \otimes \wedge^2 V)|_{\mathcal{U}_0 \cap \mathcal{U}_1}. \end{aligned}$$

The summand contributing to the $\mathcal{O}_{\mathbb{P}^1}(-2)$ component of $S^2 V^* \otimes \wedge^2 V$ is

$$k \ y_1^2 \partial_{y_1} \wedge \partial_{y_2} \frac{\langle v_3, dx \rangle}{x} = k \frac{a_{30}}{x} y_1^2 \partial_{y_1} \wedge \partial_{y_2}.$$

Since $k \neq 0$, this will give a zero class in $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$ if and only if $a_{30} = 0$. Therefore, by Proposition 3.4.10, the co-Higgs field lifts to a $\mathbb{C}^*$-invariant bivector on V if and only if $v_3 = 0$. □

3.4.3 Lifting a co-Higgs field via logarithmic connection.

For a co-Higgs type tensor $\phi \in H^0(X, \mathcal{T}_X \otimes End(V))$ (not necessarily integrable) and a connection ∇ on V (not necessarily flat), let us denote by $Lift_\nabla(\phi)$ the $\mathbb{C}^*$-invariant bivector on V lifting ϕ given by applying

$$\mathcal{T}_X \otimes End(V) \xrightarrow{\nabla \otimes 1} (\mathcal{T}_V)^{\mathbb{C}^*} \otimes End(V) \xrightarrow{\wedge} (\wedge^2 \mathcal{T}_V)^{\mathbb{C}^*}.$$

Let $x_1, ..., x_n$ be coordinates on an open set $\mathcal{U} \subset X$ and $y_1, ..., y_r$ be fiberwise linear coordinates on $V|_{\mathcal{U}}$, and let

$$\phi = \sum_{i=1}^n \sum_{j,k=1}^r \phi_{ijk} \ \partial_{x_i} \otimes y_j \partial_{y_k},$$

$$\nabla = d + \sum_{a=1}^n \sum_{b,c=1}^r A_{abc} \ dx_a \otimes y_b \partial_{y_c},$$

for some $\phi_{ijk}, A_{abc} \in \mathcal{O}_{\mathcal{U}}$. Then

$$Lift_\nabla(\phi) = \sum_{i=1}^n \sum_{j,k,b,c=1}^r (\partial_{x_i} + A_{ibc} \ y_b \partial_{y_c}) \wedge (\phi_{ijk} \ y_j \partial_{y_k}). \tag{3.12}$$

Remark 3.4.12. Two different connections ∇_1, ∇_2 may produce the same lifts $Lift_{\nabla_1}(\phi)$, $Lift_{\nabla_2}(\phi)$. Specifically, if $\nabla_1 - \nabla_2 = \omega \in H^0(X, \mathcal{T}^*_X \otimes End(V))$, then

$$Lift_{\nabla_1}(\phi) - Lift_{\nabla_2}(\phi) = \langle \omega \ \text{⸼}\ \phi \rangle \in S^2 V^* \otimes \wedge^2 V,$$

where we are using the pairing $\langle \ \text{⸼}\ \rangle$ defined by (3.11).

Suppose now that the connection ∇ has logarithmic poles along a divisor $D \subset X$. Then we can lift ϕ to a $\mathbb{C}^*$-invariant bivector $\sigma = Lift_\nabla(\phi)$ on V away from the divisor D. Let us discuss under what conditions the bivector σ extends smoothly over $V|_D$. By Hartog's theorem, it is enough to check when σ extends smoothly over the fibers $V|_{D^o}$ where D^o is the smooth part of D.

Let us denote by $\mathcal{N}_{X/D^o} = (\mathcal{T}_X|_{D^o})/\mathcal{T}_{D^o}$ the normal bundle of D^o, and by $pr_{\mathcal{N}_{X/D^o}} : \mathcal{T}_X|_{D^o} \to \mathcal{N}_{X/D^o}$ the canonical projection. Let $pr_{\mathcal{N}_{X/D^o}}(\phi) \in H^0(D, End(V) \otimes \mathcal{N}_{X/D^o})$ be the normal projection of ϕ defined over D^o. Let $res_\nabla \in H^0(D^o, End(V))$ be the residue of ∇.

Lemma 3.4.13. *Let $\phi \in H^0(X, \mathcal{T}_X \otimes End(V))$. Let ∇ be a $\mathcal{T}_X(-\log D)$-connection for a divisor D. Then the bivector $Lift_\nabla(\phi)$ on $V|_{X\setminus D}$ extends smoothly over D if and only if over the smooth part D^o of D*

$$res_\nabla \wedge pr_{\mathcal{N}_{X/D^o}}(\phi) = 0 \in H^0(D^o, S^2V^* \otimes \wedge^2 V \otimes \mathcal{N}_{X/D^o}).$$

Proof. Let z be a smooth point of D. Let us choose local coordinates $x_1, x_2, ..., x_n$ on X around z in such a way that $D = \{x_1 = 0\}$. Let us also choose a trivialization of V around z, so that we have fiberwise linear coordinates $y_1, ..., y_r$ on V. Let

$$\phi = \sum_{i=1}^{n} \sum_{j,k=1}^{r} \phi_{ijk}\ \partial_{x_i} \otimes y_j \partial_{y_k},$$

$$\nabla = d + \sum_{b,c=1}^{r} A_{1bc}\ \frac{dx_1}{x_1} \otimes y_b \partial_{y_c} + \sum_{a=2}^{n} \sum_{b,c=1}^{r} A_{abc}\ dx_a \otimes y_b \partial_{y_c},$$

for some $\phi_{ijk}, A_{abc} \in \mathcal{O}_\mathcal{U}$. Then

$$Lift_\nabla(\phi) = \sum_{j,k,b,c=1}^{r} \left(\partial_{x_1} + \frac{1}{x_1} A_{1bc}\ y_b \partial_{y_c}\right) \wedge (\phi_{1jk}\ y_j \partial_{y_k}) + \sum_{i=2}^{n} \sum_{j,k,b,c=1}^{r} (\partial_{x_i} + A_{ibc}\ y_b \partial_{y_c}) \wedge (\phi_{ijk}\ y_j \partial_{y_k}).$$

We see that the only term that gives a pole is the one involving $\frac{1}{x_1}$. The coefficient at $\frac{1}{x_1}$ equals

$$\sum_{j,k,b,c=1}^{r} A_{1bc}\ y_b \partial_{y_c} \wedge \phi_{1jk}\ y_j \partial_{y_k}.$$

When restricted to $\{x_1 = 0\}$, the coefficient gives $res_\nabla \wedge pr_{\mathcal{N}_{X/D^o}}(\phi)$. □

Example 3.4.14. If $\mathrm{rk}(V) = 1$, then the condition in Lemma 3.4.13 is vacuously true. This agrees with the fact that in the $\mathrm{rk}(V) = 1$ case a co-Higgs field always admits a Poisson lift to V (Remark 3.4.8).

Example 3.4.15. If the co-Higgs tensor ϕ is tangent to the divisor D, i.e. $\phi \in H^0(X, End(V) \otimes \mathcal{T}_X(-\log D))$, then $pr_{\mathcal{N}_{X/D^o}}(\phi)$ vanishes. That is to say that $Lift_\nabla(\phi)$ is always smooth if ∇ is logarithmic with respect to D and ϕ is tangent to D.

Example 3.4.16. Consider the symmetrization map $p : \mathbb{C}^2 \to S^2(\mathbb{C}) = \mathbb{C}^2$, $p(x_1, x_2) = (x_1 + x_2, x_1 x_2)$. Let V be the rank 2 vector bundle $p_* \mathcal{O}_{\mathbb{C}^2}$ over $X = S^2(\mathbb{C})$. The vector field $\frac{\partial}{\partial x_1}$ on $\mathbb{C}^2$, by the spectral correspondence, gives a co-Higgs field ϕ on $V = p_* \mathcal{O}_{\mathbb{C}^2}$. Moreover, the canonical trivialization of $\mathcal{O}_{\mathbb{C}^2}$ gives a trivialization of V away from the branch locus of p, that is, away from $\{\alpha_1^2 - 4\alpha_2 = 0\}$, where $\alpha_1 = x_1 + x_2$, $\alpha_2 = x_1 x_2$ are coordinates on $X = S^2(\mathbb{C})$. This trivialization gives a flat connection ∇ on V away from $\{\alpha_1^2 - 4\alpha_2 = 0\}$. Let us check if the bivector $Lift_\nabla(\phi)$ extends over the divisor $\{\alpha_1^2 - 4\alpha_2 = 0\}$.

Let us choose the basis $\langle 1,\ x_1 - x_2\rangle$ of $\mathbb{C}[x_1, x_2]$ viewed as a module of $\mathbb{C}[\alpha_1, \alpha_2]$, so that we have the decomposition $\mathbb{C}[x_1, x_2] = \mathbb{C}[\alpha_1, \alpha_2] \oplus (x_1 - x_2)\mathbb{C}[\alpha_1, \alpha_2]$. This gives a trivialization of $V = p_*\mathcal{O}_{\mathbb{C}^2}$. In the coordinates α_1, α_2 and above trivialization, the connection ∇ has the form

$$\nabla = d - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{dx_1 - dx_2}{x_1 - x_2} = d - \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{d(\alpha_1^2 - 4\alpha_2)}{\alpha_1^2 - 4\alpha_2}.$$

So, ∇ has a logarithmic pole along D with residue

$$res_\nabla = -\frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The normal projection of the co-Higgs field equals

$$pr_{\mathcal{N}_{X/D}}(\phi) = \langle \phi, d(\alpha_1^2 - 4\alpha_2)\rangle = \text{multiplication by } 2(x_1 - x_2) = 2 \begin{pmatrix} 0 & \alpha_1^2 - 4\alpha_2 \\ 1 & 0 \end{pmatrix}\Bigg|_D = 2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

If we denote by β_1, β_2 the fiberwise coordinates on V corresponding to the trivialization $\langle 1,\ x_1 - x_2\rangle$ of $\mathbb{C}[x_1, x_2]$, then res_∇ corresponds to the vector field $\frac{1}{2}\beta_2 \frac{\partial}{\partial \beta_2}$ and $pr_{\mathcal{N}_{X/D}}(\phi)$ to the vector field $2\beta_1 \frac{\partial}{\partial \beta_2}$. These vector fields are collinear, so the condition of Lemma 3.4.13 holds, and therefore the bivector $Lift_\nabla(\phi)$ extends smoothly over the divisor D.

We remark that Example 3 illustrates that the condition in Lemma 3.4.13 is not preserved under dualization of the vector bundle V: while the vector fields $\frac{1}{2}\beta_2 \frac{\partial}{\partial \beta_2}$ and $2\beta_1 \frac{\partial}{\partial \beta_2}$ are collinear, the vector fields $\frac{1}{2}\beta_2 \frac{\partial}{\partial \beta_2}$ and $2\beta_2 \frac{\partial}{\partial \beta_1}$ are not.

3.4.4 Lifting a co-Higgs field via diagonal logarithmic connection.

Recall that if V is a vector bundle over a base X and D is a divisor on X, then by a $\mathcal{T}_X(-\log D)$-connection on V we mean a connection that is allowed to have logarithmic poles along D. If one has a splitting V into a direct sum of line bundles $\mathcal{L}_1, ..., \mathcal{L}_r$, then a connection ∇ is called *diagonal* with respect to such a splitting, if everywhere locally it has a diagonal matrix in some (equivalently, all) trivialization(s) of V given by non-vanishing sections s_i of $\mathcal{L}_i$, $i = 1, ..., r$.

Definition 3.4.17. If ϕ is generically regular semi-simple co-Higgs on V, then by a *logarithmic ϕ-diagonal connection* on V we mean a connection ∇ on V such that ∇ is allowed to have only logarithmic poles and only at the points of branching of the spectral cover $\Sigma \to X$, and ∇ is diagonal with respect to the splitting of V into a direct sum of eigen line bundles of ϕ away from branch points of the spectral cover.

Lemma 3.4.18. *Let $\phi \in H^0(X, End(V) \otimes \mathcal{T}_X)$ be a strongly integrable co-Higgs field that is generically regular semi-simple. Let ∇ be a logarithmic ϕ-diagonal connection on V. Then the bivector $Lift_\nabla(\phi)$ is a meromorphic Poisson structure.*

Proof. It is enough to check integrability of the bivector $Lift_\nabla(\phi)$ on a small open set. Let $\mathcal{U} \subset X$ be an analytic open set over which ∇ has no poles, and ϕ can be brought to the diagonal form $\phi = \operatorname{diag}(v_1, ..., v_r)$. Let $V|_\mathcal{U} = \oplus_{i=1}^r \mathcal{L}_i$ be the decomposition of V into the eigen line bundles of ϕ, and let for each $i = 1, ..., r$, y_i be a fiberwise linear coordinate on $\mathcal{L}_i$ such that the section $\{y_i = 1\}$ of $\mathcal{L}_i$ is ∇-flat.

Then one has $Lift_\nabla(\phi) = \sum_{i=1}^r v_i \wedge y_i\partial_{y_i}$. From the strong integrability of the co-Higgs field ϕ we get by Lemma 3.4.2.a) that the v_i pairwise Lie commute, so $[Lift_\nabla(\phi), Lift_\nabla(\phi)] = 0$. □

Let $(p : \Sigma \to X, \mathcal{L}, v)$ be a co-Higgs spectral data, that is, $p : \Sigma \to X$ is an r-to-1 branched cover of smooth varieties, $\mathcal{L}$ is a line bundle over Σ, and v is a vector field on Σ. Let $\phi \in H^0(X, End(V) \otimes \mathcal{T}_X)$ be the corresponding co-Higgs field on $V = p_*\mathcal{L}$, which we assume to be strongly integrable (Definition 3.4.1). Let $R \subset \Sigma$ be the ramification divisor of p, and $D_{br} \subset X$ be the branch divisor of p. We are going to assume that $p : \Sigma \to X$ has *simple branching*, i.e., a generic point $x \in D_{br}$ has a small open neighborhood $\mathcal{U} \ni x$ such that $p^{-1}(\mathcal{U})$ is a disjoint union of $\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_{r-1}$, such that $p : \mathcal{U}_1 \to \mathcal{U}$ is a 2-to-1 branched cover and $p : \mathcal{U}_i \to \mathcal{U}$ is an isomorphism for $i = 2, 3, ..., r-1$.

In this subsection, we explore the following method of lifting the co-Higgs field ϕ on V to a quadratic Poisson structure σ on V:

- Choose a meromorphic section s of $\mathcal{L}$.
- Construct the logarithmic flat connection ∇^s on $\mathcal{L}$ by declaring s to be ∇^s-flat. Specifically,

$$\nabla^s = d - \frac{ds}{s}.$$

- Away from the branch divisor $D_{br} \subset X$ of p, create the diagonal logarithmic flat connection $\nabla = p_*(\nabla^s)$ on V using the direct sum decomposition

$$V|_{\mathcal{U}} \cong \oplus_{i=1}^r \mathcal{L}|_{\mathcal{U}_i}.$$

- Define the lift

$$\sigma = Lift_{p_*(\nabla^s)}(\phi) \tag{3.13}$$

  where we are using the formula (3.12). The bivector σ is well defined and smooth away from the divisors $p(\{s = 0\})$, $p(\{s = \infty\})$ and D_{br}. We proved in Lemma 3.4.18 that the bivector is Poisson wherever defined.

- Check if σ extends smoothly over the whole X.

We say that a co-Higgs spectral data $(p : \Sigma \to X, \mathcal{L}, v)$ is *admissible for a diagonal lift*, if the vector field v is tangent to $p^{-1}(p(R)) \smallsetminus R$, where $R \subset \Sigma$ is the ramification divisor of p.

Example 3.4.19. Let C be a smooth curve, v be a vector field, and $d \geq 1$. Consider the co-Higgs spectral data $(p : C \times S^{d-1}C \to S^dC, (v, 0, ..., 0), \mathcal{L})$, where p is the symmetrization map, and $\mathcal{L}$ is any line bundle over $C \times S^{d-1}C$. Let us check that this spectral data is admissible for a diagonal lift. Indeed, let $\{z_1, z_2, ..., z_d\} \in S^dC$ be a generic point on the branch divisor of p, that is, z_1 coincides with z_2, but z_2, z_3, ... , z_d are pairwise distinct. The point $\{z_1, z_2, ..., z_d\} \in S^dC$ has $d-1$ preimages: $(z_1, \{z_2, z_3, ..., z_d\}) \in C \times S^{d-1}C$ and $(z_i, \{z_1, z_2, ..., \widehat{z_i}, ..., z_d\}) \in C \times S^{d-1}C$, $i = 3, 4, ..., d$. The vector field $(v, 0, ..., 0)$ is tangent to the ramification locus $\{z_1 = z_2\}$ at each preimage of the form $(z_i, \{z_1, z_2, ..., \widehat{z_i}, ..., z_d\}) \in C \times S^{d-1}C$, $i = 3, 4, ..., d$.

We say that a divisor $D \in \Sigma$ is *adapted to the spectral data* $(p : \Sigma \to X, \mathcal{L}, v)$, if the following conditions hold:

- (away from the ramification divisor R) for any smooth points z_1, z_2 of $D_{red} \setminus R$ with $p(z_1) = p(z_2)$, the vector $n_1 p_*(v|_{z_2}) - n_2 p_*(v|_{z_1})$ is tangent to $p(D)$ at $p(z)$, where n_i is the multiplicity of D at z_i, $i = 1, 2$.

- (on the ramification divisor R) for each irreducible component R_1 of R such that v is not tangent to R_1, the divisor D_{red} is transverse to $p^{-1}(p(R_1))$.

Example 3.4.20. Let C be a smooth curve, v be a vector field, and $d \geq 1$. Consider the co-Higgs spectral data $(p : C \times S^{d-1}C \to S^d C, (v, 0, ..., 0), \mathcal{L})$, where p is the symmetrization map, and $\mathcal{L}$ is any line bundle over $C \times S^{d-1}C$. Any divisor of the form $\{(z,\xi) \in C \times S^{d-1}C : z = z_1\}$, where z_1 is a fixed point of C, is adapted to the above spectral data. A divisor of the form $\{(z,\xi) \in C \times S^{d-1}C : \xi \ni z_1\}$ is adapted to the spectral data if and only if z_1 is a zero of the vector field v. The ramification divisor $R = \{(z,\xi) \in C \times S^{d-1}C : \xi \ni z\}$ is not adapted to the spectral data, because the vector field $(v, 0, ..., 0)$ is not tangent to R.

Proposition 3.4.21. *Let $\phi \in H^0(X, End(V) \otimes \mathcal{T}_X)$ be a strongly integrable co-Higgs field given by a spectral data $(p : \Sigma \to X, \mathcal{L}, v)$, where p has simple branching. Let s be a meromorphic section of $\mathcal{L}$ with the zero-pole divisor D. Let $\sigma = Lift_{p_*(\nabla^s)}(\phi)$. Then*

a) The bivector σ is Poisson wherever defined.

b) The Poisson structure σ extends smoothly over the divisor $p(D) \setminus D_{br}$, where $D_{br} \subset X$ is the branch divisor of p, if and only if D is adapted to the spectral data $(p : \Sigma \to X, \mathcal{L}, v)$ away from the ramification divisor R.

c) The Poisson structure σ extends smoothly over the branch divisor D_{br} of p if and only if the spectral data $(p : \Sigma \to X, \mathcal{L}, v)$ is admissible for a diagonal lift, and D is adapted to the spectral data $(p : \Sigma \to X, \mathcal{L}, v)$ on the ramification divisor R.

Proof. a) Follows from Lemma 3.4.18.

b) By Hartog's theorem it is enough to check when σ extends smoothly over an open dense subset of $D \setminus D_{br}$. Let z be a smooth point of $D \setminus D_{br}$. Let $\mathcal{U}$ be an analytic neighborhood of z in X whose p-preimage is a union of disjoint open sets $\mathcal{U}_1 \ni z, \mathcal{U}_2, ..., \mathcal{U}_r \subset \Sigma$. Let $D_1 = D \cap \mathcal{U}_1$, $D_i = p^{-1}(p(D)) \cap \mathcal{U}_i$, $i > 1$. Let $2 \leq m \leq r$ be such that that $D_i \subset D$, $1 \leq i \leq m$, and $D_i \cap D = \varnothing$, $m < i \leq r$. Let us choose coordinates $x_1, ..., x_n$ on $\mathcal{U} \subset X$ and fiberwise coordinates y_i on $\mathcal{L}|_{\mathcal{U}_i}$, $i = 1, .., r$ in such a way that $D \cap \mathcal{U} = \{x_1 = 0\}$, $s|_{\mathcal{U}_i} = x_1^{n_i}$, $1 \leq i \leq r$, where n_i are integer numbers ($n_i \geq 1$ if s has a zero along D_i, and $n_i \leq -1$ if s has a pole along D_i, and $n_i = 0$ for $i > m$). Then on $\mathcal{L}|_{\mathcal{U}_i}$ one has

$$\nabla^s = d - \frac{ds}{s} = d - n_i \frac{dx_1}{x_1},$$

and so the connection $\nabla = p_*(\nabla^s)$ on $V|_{\mathcal{U}}$ has the expression

$$\nabla = d - \frac{dx_1}{x_1} \begin{pmatrix} n_1 & 0 & \dots & 0 \\ 0 & n_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n_r \end{pmatrix} \implies res_{\nabla} = - \begin{pmatrix} n_1 & 0 & \dots & 0 \\ 0 & n_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n_r \end{pmatrix}.$$

Moreover, if $v_i = \sum_{k=1}^n v_i^k \frac{\partial}{\partial x_i}$, then $p_*(v_i) = v_i^1$ modulo $\mathcal{T}_{p(D)_{red}}$, and so

$$pr_{\mathcal{N}_{\{x_1=0\}}}(\phi) = \begin{pmatrix} v_1^1 & 0 & \dots & 0 \\ 0 & v_2^1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_r^1 \end{pmatrix}.$$

We see that the condition of Lemma 3.4.13 holds over $p(D)_{red} \cap \mathcal{U}$ if and only if D is adapted to the spectral data $(p : \Sigma \to X, \mathcal{L}, v)$ over $\mathcal{U}$.

$c)$ Let z be a smooth point of R, and $\mathcal{U}$ be a small connected analytic neighborhood of $p(z)$ in X. The p-preimage of $\mathcal{U}$ consists of several connected open sets $\mathcal{U}_1 \ni z, \mathcal{U}_2, ..., \mathcal{U}_{r-1}$. Without loss of generality, we may assume that $X = \mathcal{U}$ and $\Sigma = \cup_{i=1}^{r-1} \mathcal{U}_i$. Let us choose coordinates $x_1, .., x_n$ on $\mathcal{U}$ centered at $p(z)$, and also coordinates $x_1^i, ..., x_n^i$ on each $\mathcal{U}_i$ so that the map $p : \mathcal{U}_i \to \mathcal{U}$ has coordinate expression

$$(x_1, x_2, ..., x_n) = ((x_1^i)^2, x_2^i, ..., x_n^i), \quad \text{if } i = 1,$$

$$(x_1, x_2, ..., x_n) = (x_1^i, x_2^i, ..., x_n^i), \quad \text{if } 2 \le i \le r-1.$$

Over $\mathcal{U}_1$, we choose the basis $1, x_1^1$ for $\mathcal{O}_{\mathcal{U}_1}$ as a module over $\mathcal{O}_{\mathcal{U}}$. For $i = 2, ..., r-1$, we identify $\mathcal{O}_{\mathcal{U}_i} \cong \mathcal{O}_{\mathcal{U}}$. In these bases, the normal projection of the co-Higgs field ϕ will have the matrix expression

$$pr_{\mathcal{N}_{\{x_1=0\}}}(\phi) = \langle v, p^*(dx_1)\rangle|_{x_1=0} = \left.\left(\begin{array}{cc|cccc} 0 & 0 & & & \\ 2v_1^1 & 0 & & & \\ \hline & & v_2^1 & & \\ & & & \ddots & \\ & & & & v_{r-1}^1 \end{array}\right)\right|_{x_1=0} \bigotimes \frac{\partial}{\partial x_1},$$

where we assume that $v|_{\mathcal{U}_i}$ has the coordinate expession $\sum_{j=1}^n v_i^j \partial_{x_j^i}$, for each $i = 1, 2, ..., r-1$.

Let $s|_{\mathcal{U}_i} = (x_1^i)^{l_i} \tilde{s}_i$, $1 \le i \le r-1$, where $\tilde{s}_i$ is non-vanishing on $\mathcal{U}_i$. Then

$$\nabla = d - \left(\begin{array}{cc|ccc} \frac{d\tilde{s}_1}{\tilde{s}_1} & 0 & & & \\ 0 & \frac{d\tilde{s}_1}{\tilde{s}_1} & & & \\ \hline & & \frac{d\tilde{s}_2}{\tilde{s}_2} & & \\ & & & \ddots & \\ & & & & \frac{d\tilde{s}_{r-1}}{\tilde{s}_{r-1}} \end{array}\right) - \left(\begin{array}{cc|ccc} l_1 \frac{dx_1^1}{x_1^1} & 0 & & & \\ 0 & (l_1 - 1)\frac{dx_1^1}{x_1^1} & & & \\ \hline & & l_2 \frac{dx_1^2}{x_1^2} & & \\ & & & \ddots & \\ & & & & l_{r-1} \frac{dx_1^{r-1}}{x_1^{r-1}} \end{array}\right),$$

$$res_\nabla = -\left(\begin{array}{cc|ccc} \frac{l_1}{2} & 0 & & & \\ 0 & \frac{l_1-1}{2} & & & \\ \hline & & l_2 & & \\ & & & \ddots & \\ & & & & l_{r-1} \end{array}\right).$$

From the coordinate expressions, we see that $pr_{\mathcal{N}_{\{x_1=0\}}}(\phi) \wedge res_\nabla$ vanishes along the branch divisor $\{x_1 = 0\}$ if and only if the following two conditions hold:

1) $v_i^1\big|_{\{x_1=0\}} = 0$ for each $i = 2, 3, ..., r-1$, and

2) either $v_1^1\big|_{\{x_1=0\}} = 0$, or $l_i = 0$ for $i = 1, 2, .., r-1$.

Condition 1) means precisely that the spectral data $(p : \Sigma \to X, \mathcal{L}, v)$ is admissible for a diagonal lift, and condition 2) means precisely that D is adapted to the spectral data $(p : \Sigma \to X, \mathcal{L}, v)$ on the ramification divisor R. Applying Lemma 3.4.13 finishes the proof. □

Example 3.4.22. Let C be a curve of genus ≤ 1, u be vector field on C, L be a line bundle over C, and $d \geq 2$. Consider the spectral data $(p : \Sigma \to X, \mathcal{L}, v)$, where $\Sigma = C \times S^{d-1}C$, $X = S^d C$, p is the symmetrization map, $\mathcal{L} = L \boxtimes \mathcal{O}_{S^{d-1}C}$, and $v = (u, 0)$. As we checked in Example 3.4.19, this spectral data is admissible for a diagonal lift. Let s be a meromorphic section of L whose zeros and poles are simple. Then the zero-pole divisor of the section $(s, 0)$ of $\mathcal{L}$ is adapted to the spectral data $(p : \Sigma \to X, \mathcal{L}, v)$ (Example 3.4.20). Proposition 3.4.21 implies that the co-Higgs field ϕ generated by the spectral data $(p : \Sigma \to X, \mathcal{L}, v)$ admits the Poisson lift $\sigma = Lift_{p_*(\nabla^{(s,0)})}(\phi)$ to V.

Chapter 4

Co-Higgs bundles over $\mathbb{P}^1$

In this chapter, we study Poisson lifts of co-Higgs bundles over $\mathbb{P}^1$. First, we develop tools for constructing such lifts. Then we proceed to a detailed discussion of rank 2 co-Higgs bundles over $\mathbb{P}^1$ (Section 4.1). In Section 4.2, we apply our results to obtain a classification of line bundles over Poisson Hirzebruch surfaces that admit a Poisson module structure. In Section 4.3, we study projective Poisson lifts of co-Higgs bundles of rank 3 over $\mathbb{P}^1$. In particular, we obtain a classification of Poisson $\mathbb{P}^2$-bundles over $\mathbb{P}^1$ whose spectral curve is reduced.

By $\mathbb{P}^1$ we denote the one dimensional complex projective space, that is, the space of one dimensional linear subspaces of $W = \mathbb{C}^2$. By $\mathcal{O}_{\mathbb{P}^1}$ we denote the structure sheaf of $\mathbb{P}^1$, and by $\mathcal{O}_{\mathbb{P}^1}(-1)$ we denote the tautological line bundle over $\mathbb{P}^1$, that is, the line bundle whose fiber over $\ell \in \mathbb{P}^1$ is the line $\ell \subset \mathbb{C}^2$ itself. For $k > 0$, the notation $\mathcal{O}_{\mathbb{P}^1}(-k)$ stands for the tensor power $\mathcal{O}_{\mathbb{P}^1}(-1)^{\otimes k}$, and the notation $\mathcal{O}_{\mathbb{P}^1}(k)$ stands for the dual bundle $\mathcal{O}_{\mathbb{P}^1}(-k)^*$. For $k = 0$, the notation $\mathcal{O}_{\mathbb{P}^1}(k)$ means $\mathcal{O}_{\mathbb{P}^1}$. Recall that the sheaf cohomology $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))$ is isomorphic to the symmetric tensor power $S^k W^*$ if $k \geq 0$, and vanishes if $k < 0$. The sheaf cohomology $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))$ is related to H^0 via Serre duality: $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2-k))^*$.

Recall that every holomorphic line bundle over $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(k)$, for a unique $k \in \mathbb{Z}$. Also, every holomorphic vector bundle over $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles (Grothendieck-Birkhoff Theorem).

Recall that a co-Higgs bundle over $\mathbb{P}^1$ is a pair (V, ϕ), where V is a holomorphic vector bundle over $\mathbb{P}^1$ and $\phi \in H^0(\mathbb{P}^1, End(V) \otimes \mathcal{T}_{\mathbb{P}^1})$ (note that no integrability condition is imposed on ϕ, because $\dim \mathbb{P}^1 = 1$). Rayan's work [27, 28] describes the moduli space of co-Higgs bundles over $\mathbb{P}^1$. In this chapter, we are dealing with the question of when ϕ admits a Poisson lift to V or $\mathbb{P}(V)$ (in the sense of Lemmas 3.2.1 and 3.2.3), and how unique such a lift is.

Recall that *strong integrability* of ϕ (Definition 3.4.1) is a necessary condition for the existence of a Poisson lift (see Lemma 3.4.2 and Corollary 3.4.7). We start the discussion with the following result that shows that strong integrability over $\mathbb{P}^1$ imposes quite a constraint on the spectral curve of the co-Higgs field.

Proposition 4.0.1. *Let ϕ be a co-Higgs bundle of rank r on $\mathbb{P}^1$. Then ϕ is strongly integrable if and only if its characteristic polynomial has the form*

$$\chi_\phi(\theta) = \theta^{r-km}\prod_{i=1}^{m}(\theta^k - \lambda_i\rho), \tag{4.1}$$

for some $1 \le k \le r$, $0 \le m \le r$ with $km \le r$, $\lambda_i \in \mathbb{C}$, $i = 1, 2, ..., m$, and $\rho \in H^0(\mathbb{P}^1, S^k\mathcal{T}_{\mathbb{P}^1})$.

Proof. To prove sufficiency, note that the coefficients of the characteristic polynomial of the form (4.1) will all be of the form $s_j(\phi) = C_j\rho^{j'}$ for some $C_j \in \mathbb{C}$, $j' \ge 0$. Any two such expressions pairwise Poisson commute when viewed as functions on $\mathcal{T}^*_{\mathbb{P}^1}$.

To prove necessity, fix a strongly integrable co-Higgs bundle (V, ϕ) of rank r over $\mathbb{P}^1$. Let $s_i(\phi)$ and $s_j(\phi)$ be any two coefficients of the characteristic polynomial χ_ϕ. We claim that there is $\widetilde{\rho} \in H^0(\mathbb{P}^1, S^k\mathcal{T}_{\mathbb{P}^1})$ for $\widetilde{k} = g.c.d.(i, j)$ such that $s_i(\phi) = \widetilde{C}_i\widetilde{\rho}^{i/\widetilde{k}}$ and $s_j(\phi) = \widetilde{C}_j\widetilde{\rho}^{j/\widetilde{k}}$ for some constants $\widetilde{C}_i, \widetilde{C}_j \in \mathbb{C}$. If $s_i(\phi) = 0$ or $s_j(\phi) = 0$, then there is nothing to prove, so we assume that both $s_i(\phi), s_j(\phi)$ are non-zero. Choosing an affine coordinate x on $\mathbb{P}^1$, we can express

$$s_i(\phi) = f(x)\partial_x^i,\ s_j(\phi) = g(x)\partial_x^j,$$

for some polynomials f and g of degrees $2i$ and $2j$, respectively. The strong integrability of ϕ implies

$$0 = \{f\partial_x^i, g\partial_x^j\} = (ifg' - jf'g)\partial_x^{i+j-1},$$

$$i\log(g)' = j\log(f)' \implies g^i = Cf^j, \text{ for some constant } C \in \mathbb{C}.$$

Therefore, there is a polynomial h of degree $2\widetilde{k}$, $\widetilde{k} = g.c.d.(i, j)$, such that $f = \widetilde{C}_ih^{i/\widetilde{k}}$, $g = \widetilde{C}_jh^{j/\widetilde{k}}$, for some constants $\widetilde{C}_i, \widetilde{C}_j \in \mathbb{C}$. Setting $\rho = h\partial_x^{\widetilde{k}}$ proves the claim we made above.

Applying the claim for each pair of coefficients of χ_ϕ, we obtain $\rho \in H^0(\mathbb{P}^1, S^k\mathcal{T}_{\mathbb{P}^1})$ and $k = g.c.d.(i : s_i(\phi) \ne 0)$ such that each $s_i(\phi) = C_i\rho^{i/k}$ for some constant $C_i \in \mathbb{C}$. It follows that

$$\chi_\phi(\theta) = \sum_{i:k|i}\theta^{r-i}(-1)^{r-i}C_i\rho^{i/k} = \theta^r p(\rho/\theta^k),$$

where $p(z) \in \mathbb{C}[z]$ is a polynomial in one variable with constant coefficients such that $p(0) = 1$. Decomposing $p(z) = \prod_{i=1}^m(1 - \lambda_i z)$, we arrive at the expression (4.1) for $\chi_\phi(\theta)$. ☐

Corollary 4.0.2. *Let ϕ be a co-Higgs bundle of rank r on $\mathbb{P}^1$ with reduced, irreducible spectral curve. Then ϕ is strongly integrable if and only if the characteristic polynomial $\chi_\phi(\theta) = \theta^r + (-1)^r s_r(\phi)$ for some $s_r(\phi) \in H^0(\mathbb{P}^1, S^r\mathcal{T}_{\mathbb{P}^1})$ if and only if $\mathrm{Tr}(\phi^k) = 0$ for $k = 1, 2, ..., r-1$.*

Proof. The former equivalence follows from Proposition 4.0.1, as the characteristic polynomial in (4.1) is irreducible if and only if $m = 1$ and $k = r$. The latter equivalence follows directly from Newton's identities (3.8). ☐

Remark 4.0.3. The spectral curve cut out by the characteristic polynomial $\chi_\phi(\theta) = \theta^r + (-1)^r s_r(\phi)$ is smooth if and only if the section $s_r(\phi) \in H^0(\mathbb{P}^1, S^r\mathcal{T}_{\mathbb{P}^1})$ does not have repeated zeroes.

Lemma 4.0.4. *Let $\mathcal{U} \subset \mathbb{C}$ be a simply connected open set, $V = \mathcal{O}_{\mathcal{U}}^{\oplus r}$, and let*

$$\phi = \begin{pmatrix} v_1 & 0 & \dots & 0 \\ 0 & v_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & v_r \end{pmatrix}, \ v_i \in H^0(\mathcal{U}, \mathcal{T}_{\mathcal{U}}), \ i = 1, 2, \dots, r, \tag{4.2}$$

be a co-Higgs field on V that is strongly integrable.

Then:

1. *There is a vector field $v \in H^0(\mathcal{U}, \mathcal{T}_{\mathcal{U}})$ and constants $\lambda_i \in \mathbb{C}$, $i = 1, 2, ..., r$, such that $v_i = \lambda_i v$,*
2. *Each lift of ϕ to a quadratic Poisson structure on V is of the form*

$$\sigma = v \wedge \left(\sum_{i=1}^{r} \lambda_i y_i \partial_{y_i} \right) + \sum_{\substack{i,j,k,m=1 \\ i \le j, k < l}}^{r} f_{ijkl}(x) y_i y_j \partial_{y_k} \wedge \partial_{y_l}, \tag{4.3}$$

where x is a coordinate on $\mathcal{U}$ and y_i is a fiberwise linear coordinate on the i-th summand of V. Moreover, the function $f_{ijkl}(x)$ vanishes unless $\lambda_i + \lambda_j = \lambda_k + \lambda_l$.

Proof. Lemma 3.4.2.a) implies that $[v_i, v_j] = 0$, for each $1 \le i, j \le r$. Assuming v_i is not a zero vector field, we can express $v_j = g_j(x) v_i$, for some $g_j \in \mathcal{O}_{\mathcal{U}}$. Then the condition $[v_i, v_j] = 0$ implies that the Lie derivative $\mathrm{Lie}_{v_i}(g_j)$ vanishes. Since $\mathcal{U}$ is one dimensional, this implies that g_j is a constant. Repeating this argument for each pair of non-zero vector fields v_i, v_j, we get the first claim of the lemma.

For the second claim, the only non-trivial part is to show that $f_{ijkl} = 0$ unless $\lambda_i + \lambda_j = \lambda_k + \lambda_l$. Let us take π as in (4.3), and calculate the Schouten bracket:

$$[\sigma, \sigma] = 2v \wedge \left(\sum_{\substack{i,j,k,m=1 \\ i \le j, k < l}}^{r} (\lambda_i + \lambda_j - \lambda_k - \lambda_k) f_{ijkl}(x) y_i y_j \partial_{y_k} \wedge \partial_{y_l} \right) + \dots.$$

Here we have omitted the purely vertical terms, i.e. the ones of the form $\partial_y \wedge \partial_y \wedge \partial_y$. The terms of $[\sigma, \sigma]$ written above cannot cancel each other, nor can they be cancelled by the omitted terms. Therefore, in order to have $[\sigma, \sigma] = 0$, it is necessary that for each quadruple i, j, k, l, either $\lambda_i + \lambda_j = \lambda_k + \lambda_l$ or $f_{ijkl} = 0$. □

Lemma 4.0.5. *Let $p : V \to \mathbb{P}^1$ be a vector bundle. Then the canonical bundle ω_V is isomorphic to $p^* \det V^* \otimes p^* \mathcal{O}_{\mathbb{P}^1}(-2)$.*

Proof. The claim follows by taking the determinant of the short exact sequence

$$0 \longrightarrow p^* V \longrightarrow \mathcal{T}_V \longrightarrow p^* \mathcal{T}_{\mathbb{P}^1} \longrightarrow 0.$$

□

Proposition 4.0.6. *Let $p : V \to \mathbb{P}^1$ be a vector bundle of rank r. Then one has*

$$\omega_{\mathbb{P}(V)}^{-1} = p^* \det(V) \otimes p^* \mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}(V)}(r),$$

where $\omega_{\mathbb{P}(V)}^{-1}$ *is the anticanonical bundle of* $\mathbb{P}(V)$ *and* $\mathcal{O}_{\mathbb{P}(V)}(r) = \mathcal{O}_{\mathbb{P}(V)}(-1)^{-r}$*, where* $\mathcal{O}_{\mathbb{P}(V)}(-1)$ *is the tautological line bundle whose fiber over* $(x,l) \in \mathbb{P}(V)$ *is the line* $l \subset V$*.*

Proof. Taking the determinant of the fiberwise Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \longrightarrow p^* \det(V) \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \longrightarrow \mathcal{T}_v\mathbb{P}(V) \longrightarrow 0,$$

we obtain $\det \mathcal{T}_v\mathbb{P}(V) = p^* \det(V) \otimes \mathcal{O}_{\mathbb{P}(V)}(r)$. Tensoring this with $\mathcal{T}_h\mathbb{P}(V) = p^*\mathcal{O}_{\mathbb{P}^1}(2)$, we obtain the desired formula. □

For a vector bundle V over $\mathbb{P}^1$, we denote by $\deg(V) \in \mathbb{Z}$ its degree, or equivalently its first Chern class. If $V \cong \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^1}(k_i)$, then $\deg(V) = \sum_{i=1}^{l} k_i$.

Corollary 4.0.7. *Let* $p : V \to \mathbb{P}^1$ *be a vector bundle over* $\mathbb{P}^1$*. If* $\deg(V) = -2$*, then any Poisson structure on* $\mathbb{P}(V)$ *lifts to a unimodular quadratic Poisson structure on* V*. If* $\deg(V) \neq -2$*, then a Poisson structure* π *on* $\mathbb{P}(V)$ *lifts to a quadratic Poisson structure on* V *if and only if the line bundle* $p^*\mathcal{O}_{\mathbb{P}^1}(1)$ *admits a Poisson module structure with respect to* π*.*

Proof. If $\deg(V) = -2$, then Lemma 4.0.5 implies that $\omega_V \cong \mathcal{O}_V$. Theorem 3.1.5 then implies that any Poisson structure on $\mathbb{P}(V)$ lifts to a unimodular quadratic Poisson structure on V.

If $\deg(V) = k \neq -2$, then Proposition 4.0.6 implies that $\omega_{\mathbb{P}(V)} = p^*\mathcal{O}_{\mathbb{P}^1}(2-k) \otimes \mathcal{O}_{\mathbb{P}(V)}(-r)$. Let π be a Poisson structure on $\mathbb{P}(V)$. The line bundle $\omega_{\mathbb{P}(V)}$ admits the canonical Poisson module structure, so according to Remark 2.1.5, the line bundle $p^*\mathcal{O}_{\mathbb{P}^1}(1)$ admits a Poisson module structure if and only if $\mathcal{O}_{\mathbb{P}(V)}(-1)$ does. It remains to note that the line bundle $\mathcal{O}_{\mathbb{P}(V)}(-1)$ admits a Poisson module structure with respect to π on $\mathbb{P}(V)$ if and only if the total space of $\mathcal{O}_{\mathbb{P}(V)}(-1)$ admits a quadratic Poisson structure projecting onto π. □

Recall that for a co-Higgs bundle (V,ϕ), the eigenvariety is defined as the subvariety $E \subset \mathbb{P}(V)$ consisting of the points (x,v), $x \in X$, $0 \neq v \in V|_x$ such that $\phi_x(v) \wedge v = 0 \in \wedge^2 V|_x \otimes \mathcal{T}|_x$.

Lemma 4.0.8. *Let* $(p : V \to \mathbb{P}^1, \phi)$ *be a co-Higgs bundle whose spectral curve* $\Sigma \subset \mathcal{T}_{\mathbb{P}^1}$ *is smooth. Then the eigenvariety* $E \subset \mathbb{P}(V)$ *is isomorphic to* Σ*.*

Proof. Note that, since Σ is smooth (in particular, reduced), the co-Higgs ϕ is regular semi-simple over a Zariski open set $\mathcal{U} \subset \mathbb{P}^1$. We will prove that the morphism $Eig : E|_{\mathcal{U}} \to \Sigma|_{\mathcal{U}}$ given by Lemma 3.3.2 extends smoothly to a global isomorphism between E and Σ. It is enough to check the last statement in any analytic neighborhood of any point $x_0 \in \mathbb{P}^1 \smallsetminus \mathcal{U}$. Moreover, it is enough to assume that around x_0, the characteristic polynomial of ϕ equals $y^r - x$, for some choice of local coordinate x centered at x_0. The sheaf $\mathcal{O}_\Sigma = \mathbb{C}\{x,y\}/(y^r - x)$ has rank r over $\mathcal{O}_{\mathcal{U}} = \mathbb{C}\{x\}$, with a basis given by $1, y, y^2, ..., y^{r-1}$. In the trivialization of $V = p_*\mathcal{O}_\Sigma$ given by this basis, the co-Higgs field has the matrix

$$\phi = \left(\begin{array}{ccccc|c} & & & & & x \\ \hline 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 1 & \end{array}\right)$$

The equation $\phi_x(v) \wedge v = 0$ can be rewritten as

$$(xt_r, t_1, t_2, ..., t_{r-1}) \wedge (t_1, t_2, ..., t_r) = 0,$$

where $v = (t_1, ..., t_r)$. The latter equation cuts out the smooth curve $(x, t_1, t_2, ..., t_r) = (y^r, y^{r-1}, y^{r-2}, ..., y, 1)$, where y runs over a small analytic neighborhood of $0 \in \mathbb{C}$. □

4.1 Rank 2 co-Higgs bundles over $\mathbb{P}^1$

Recall that any rank 2 vector bundle over $\mathbb{P}^1$ is of the form $V = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(m)$, where $m, k \in \mathbb{Z}$. Recall that by Lemma 4.0.5, the total space of V is Calabi-Yau if and only if $k + m = -2$.

Theorem 4.1.1. *Let ϕ be a co-Higgs field on $V = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(m)$, where $m, k \in \mathbb{Z}$.*

1. *If $k + m = -2$, then any zero trace co-Higgs field ϕ on V lifts to a unimodular quadratic Poisson structure on V.*
2. *Let $k + m \neq -2$, and let ϕ have reduced, irreducible spectral curve. Then ϕ lifts to a quadratic Poisson structure on V if and only if ϕ has zero trace and vanishes at a point $z \in \mathbb{P}^1$.*

Remark 4.1.2. As it will be seen from the proof, the "if" direction for the second part does not require the assumption on the spectral curve.

Proof. The first part of the theorem follows from Theorem 3.4.9 and Theorem 3.1.5. By Theorem 3.4.9, the co-Higgs field ϕ lifts to a Poisson structure π on $\mathbb{P}(V)$. Then by Theorem 3.1.5, π further lifts to a unimodular quadratic Poisson structure σ on V. The structure σ lifts a co-Higgs $\phi_1 = \phi + v\mathrm{Id}$, for some vector field v on $\mathbb{P}^1$. However, due to Lemma 3.3.1, unimodularity of σ implies $v = 0$. So, σ lifts ϕ. This proves the first claim.

Let us present an explicit formula for lifting a co-Higgs field

$$\phi = \begin{pmatrix} v_1 & v_2 \\ v_3 & -v_1 \end{pmatrix}$$

on $V = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k-2)$, where $v_1 \in H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1})$, $v_2 \in H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}(2k+2))$, $v_3 \in H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}(-2k-2))$. Let us choose an affine coordinate x on $\mathcal{U} \subset \mathbb{P}^1$ and express

$$v_i = f_i(x)\partial_x, \ i = 1, 2, 3,$$

where f_1, f_2 and f_3 are polynomials in x of degrees ≤ 2, $\le 2k+4$ and $\le -2k$ respectively. Note that we use the convention that the zero polynomial has degree $-\infty$.

Then the unimodular quadratic Poisson structure on V lifting ϕ is given by

$$\sigma = \sigma_0 - \frac{1}{2} Eul \wedge \Delta(\sigma_0), \tag{4.4}$$

where

$$\sigma_0 = v_1 \wedge y_1\partial_{y_1} + v_2 \wedge y_2\partial_{y_1} + v_3 \wedge y_1\partial_{y_2} - v_1 \wedge y_2\partial_{y_2},$$

$$Eul = y_1\partial_{y_1} + y_2\partial_{y_2},$$

and Δ is the BV operator on multivectors on V coming from the volume form $\Omega = dx \wedge dy_1 \wedge dy_2$. One has

$$\begin{aligned} Eul \wedge \Delta(\sigma_0) &= (y_1\partial_{y_1} + y_2\partial_{y_2}) \wedge (\operatorname{div}(v_1)y_1\partial_{y_1} + \operatorname{div}(v_2)y_2\partial_{y_1} + \operatorname{div}(v_3)y_1\partial_{y_2} - \operatorname{div}(v_1)y_2\partial_{y_2}) \\ &= -2\operatorname{div}(v_1)y_1\partial_{y_1} \wedge y_2\partial_{y_2} - \operatorname{div}(v_2)y_2\partial_{y_1} \wedge y_2\partial_{y_2} + \operatorname{div}(v_3)y_1\partial_{y_1} \wedge y_1\partial_{y_2}, \end{aligned}$$

where $\operatorname{div}(v_i) = f_i'(x)$.

For the second part of the theorem, sufficiency is proved analogously to the first part. Even though, in general, one cannot find a holomorphic volume form on V, Lemma 4.0.5 guarantees that one can always find a section Ω of ω_V that is non-zero away from the fiber $D = V|_z$ and has a zero or a pole (possibly of high order) along D. Then one gets the BV operator $\Delta : \wedge^i \mathcal{T}_V(-\log D) \to \wedge^{i-1}\mathcal{T}_V$ corresponding to Ω. Then the formula (4.4) produces a holomorphic lift of ϕ, because the non-integrable lift σ_0 is tangent to D. In other words, if one chooses the coordinate x on $\mathbb{P}^1$ so that the point z is $x = \infty$, then the formula (4.4) gives the desired lift of ϕ.

To show necessity of the second part of the theorem, note that if ϕ does admit a lift to $\mathbb{C}^*$-invariant Poisson structure on V, then ϕ is strongly integrable (Lemma 3.4.2.b)). Then Corollary 4.0.2 implies that ϕ must have zero trace. Then Lemma 4.1.3 below shows that ϕ must vanish at a point $z \in \mathbb{P}^1$. □

Lemma 4.1.3. *Let V be a rank 2 vector bundle over $\mathbb{P}^1$, and let σ be a $\mathbb{C}^*$-invariant Poisson structure on V. Let the co-Higgs field $\phi \in \operatorname{Hom}(V, V \otimes \mathcal{T}_{\mathbb{P}^1})$ induced by σ be traceless, and $\phi(z) \neq 0$, for all $z \in \mathbb{P}^1$.*

Then there exists a fiberwise constant holomorphic volume form Ω on V that is invariant under all σ-Hamiltonian vector fields. Moreover, such a volume form is unique up to a multiplicative constant. In particular, the total space of V is Calabi-Yau.

Proof. Let us prove uniqueness first. Let Ω and Ω' be two fiberwise constant holomorphic volume forms on $V|_{\mathcal{U}}$, for some open connected $\mathcal{U} \subset \mathbb{P}^1$, that are invariant under all σ-Hamiltonian flows. The latter condition means that $\Delta_\Omega(\sigma) = 0 = \Delta_{\Omega'}(\sigma)$, where Δ_Ω and $\Delta_{\Omega'}$ are BV operators generated by Ω and Ω', respectively. One can find $f \in \mathcal{O}^*_{\mathcal{U}}$ such that $\Omega' = f\Omega$. Then

$$\Delta_{\Omega'}(\sigma) = \Delta_\Omega(\sigma) + H_{\log f} \implies H_{\log f} = 0.$$

Moreover, the Hamiltonian vector field of $\log f$ can be expressed purely in terms of ϕ as $H_{\log f} = \langle \phi, d\log f\rangle \in End(V)$. Since ϕ is a non-zero co-Higgs tensor, we get that $d\log f = 0$, which implies that f is a constant.

Now that we have proved uniqueness, we prove existence by constructing the required volume forms Ω_i, $i = 0,1$, over each of $\mathcal{U}_0 = \mathbb{P}^1 \setminus \{0\}$ and $\mathcal{U}_1 = \mathbb{P}^1 \setminus \{\infty\}$. Uniqueness implies that $\Omega_0 = \lambda\Omega_1$ on $\mathcal{U}_0 \cap \mathcal{U}_1$ for a constant $\lambda \in \mathbb{C}^*$. Hence, one can replace Ω_1 with $\lambda\Omega_1$, so that Ω_i, $i = 0,1$, glue together to give a volume form Ω on the whole V.

To this end, for an open simply connected $\mathcal{U} \subset \mathbb{C}$, let us construct a fiberwise constant volume form Ω on $V|_{\mathcal{U}}$ such that $\Delta_\Omega(\sigma) = 0$. Since the bundle V is trivial over $\mathcal{U}$, we can pick a fiberwise constant volume form Ω' on $V|_{\mathcal{U}}$. The modular vector field $v = \Delta_{\Omega'}(\sigma)$ is $\mathbb{C}^*$-invariant, because σ is, and by Lemma 3.3.1 it is tangent to the fibers of V. Therefore, $v \in End(V)|_{\mathcal{U}}$. Moreover, since $\Delta_{\Omega'}(v) = 0$, the endomorphism of $V|_{\mathcal{U}}$ representing the vector field v has zero trace. Furthermore, since $[v, \sigma] = 0$ we deduce that $[v, \langle \phi, df\rangle] = 0$ for any $f \in \mathcal{O}_{\mathcal{U}}$. Now, we use the elementary fact that if two traceless 2×2 matrices commute, one of them has to be a constant multiple of the other. Using this fact and the assumption that ϕ does not vanish at any point of $\mathcal{U}$, we find $f \in \mathcal{O}_{\mathcal{U}}$ such that $v = \langle \phi, df \rangle$. In other words, we have shown that the modular vector field $v = \Delta_{\Omega'}(\sigma)$ is actually Hamiltonian, i.e. $v = H_f$. Now, we can let $\Omega = e^{-f}\Omega'$, and obtain $\Delta_\Omega(\sigma) = v - H_f = 0$. □

The following corollary shows that the Calabi-Yau assumption in Theorem 3.1.5 was essential.

Corollary 4.1.4. *Let π be a Poisson structure on $\mathbb{P}(V)$, $V = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(m)$, $m + k \neq -2$, whose zero locus is reduced and irreducible. Then π is not the projectivization of a quadratic Poisson structure on V.*

Proof. We are going to apply Theorem 4.1.1.2 to the traceless co-Higgs field ϕ on V generated by π. We need to check that ϕ does not vanish on $\mathbb{P}^1$ and that the spectral curve $\Sigma \subset \mathcal{T}_{\mathbb{P}^1}$ of ϕ is reduced and irreducible. Both these claims follow from the assumption that the zero locus $\{\pi = 0\}$ is reduced and irreducible. Indeed, if $\phi(x) = 0$ for some $x \in \mathbb{P}^1$, then the fiber $\mathbb{P}(V|_x)$ would form an irreducible component of $\{\pi = 0\}$. Also, Lemmas 3.3.2 and 3.3.4 imply that the there is a canonical map from $\{\pi = 0\}$ to Σ that is an isomorphism, possibly away from finitely many points, so Σ is reduced and irreducible. □

4.2 Poisson modules over Hirzebruch surfaces.

This subsection contains an application of the obtained results about lifting of a co-Higgs field on a rank 2 vector bundle V over $\mathbb{P}^1$ to a quadratic Poisson structure on V (Theorem 4.1.1).

Recall that the m-th Hirzebruch surface $\mathbb{F}_m$, $m \geq 0$, is defined as the total space of the projective bundle $\mathbb{P}(V)$, $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)$. The line bundles over $\mathbb{F}_m$ are given by $\mathrm{Pic}(\mathbb{F}_m) = \{\mathcal{O}(aH + bF) : a, b \in \mathbb{Z}\} \cong \mathbb{Z}^2$, where H is the divisor given by the section $\mathcal{O}_{\mathbb{P}^1} \oplus 0$ of $\mathbb{P}(V)$ and F is the divisor given by a fiber of $\mathbb{P}(V)$ [2]. The intersection numbers of the generating divisors H, F are $H.H = m$, $H.F = 1$, $F.F = 0$. Given a Poisson structure π on $\mathbb{F}_m$, one may ask which of the line bundles on $\mathbb{F}_m$ admit a Poisson module structure with respect to π. We denote by $\mathrm{Pic}_\pi(\mathbb{F}_m)$ the collection of line bundles that do admit a Poisson module structure. Obvious examples of these include the anticanonical bundle $\omega^{-1}_{\mathbb{F}_m} = \mathcal{O}(2H + (m+2)F)$ and $\mathcal{O}(D)$ for any irreducible component of the reduced part of the zero divisor $\{\pi = 0\}$ (see Remark 2.1.8). Additionally, we recall that $\mathrm{Pic}_\pi(\mathbb{F}_m)$ is closed under tensor product and under taking a root of any order (whenever the latter is defined). The following statement says that $\mathrm{Pic}_\pi(\mathbb{F}_m)$ does not contain any other line bundles, except the above.

Theorem 4.2.1. *Let π be a Poisson structure on $\mathbb{F}_m$, $m \geq 0$. Let $\mathrm{Pic}_\pi(\mathbb{F}_m)$ be the subgroup of $\mathrm{Pic}(\mathbb{F}_m) \cong \mathbb{Z}^2$ of the line bundles admitting a Poisson module structure with respect to π. Then $\mathrm{Pic}_\pi(\mathbb{F}_m) \otimes \mathbb{Q} = G \otimes \mathbb{Q}$, where G is the subgroup of $\mathrm{Pic}(\mathbb{F}_m)$ generated by the irreducible components of the divisor $\{\pi = 0\}$.*

Proof. As we pointed out above, the inclusion $\supseteq$ follows from Remark 2.1.8. Let us prove the inclusion $\subseteq$.

First, let us assume that the zero divisor $\{\pi = 0\}$ is reduced irreducible. Then the group $G \otimes \mathbb{Q} \subset \mathbb{Q}^2$ is generated by a single element $\mathcal{O}(2H + (m+2)F)$. To show the inclusion $\mathrm{Pic}_\pi(\mathbb{F}_m) \otimes \mathbb{Q} \subseteq G \otimes \mathbb{Q}$, it is enough to check that $\mathcal{O}(F)$ does not admit a Poisson module structure with respect to π. By Corollary 4.0.7, this is equivalent to saying that π does not lift to a $\mathbb{C}^*$-invariant Poisson structure on $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)$, which follows from Corollary 4.1.4. This finishes the proof in the case when $\{\pi = 0\}$ is reduced irreducible.

Now, we assume that the divisor $\{\pi = 0\}$ has multiple irreducible components (that may or may not coincide). If one of these components represents a divisor $aH + bF$, with (a, b) not collinear with $(2, m+2)$, then $G \otimes \mathbb{Q} = \mathbb{Q}^2$, and the proof is finished. So, we may assume that every irreducible component of $\{\pi = 0\}$ corresponds to a divisor $aH + bF$, with (a, b) being collinear with $(2, m+2)$. This can only happen when m is even and $\{\pi = 0\}$ has two components, each giving the divisor $H + \frac{m+2}{2}F$. We claim that this can only happen when $m = 0$ or $m = 2$. Indeed, when $m > 2$, for the co-Higgs field ϕ on $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)$ corresponding to π, the component $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(m), \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{T}_{\mathbb{P}^1})$ of ϕ must vanish, which implies that the subbundle $\mathcal{O}_{\mathbb{P}^1} \oplus 0 \subset V$ is an eigenbundle of ϕ. Therefore, when $m > 2$, the Poisson structure π vanishes on the section $H = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus 0) \subset \mathbb{P}(V)$, and this section forms an irreducible component of $\{\pi = 0\}$ (possibly with multiplicity). So, in the case $m > 2$, the zero locus $\{\pi = 0\}$ cannot have exactly two components, each giving the divisor $H + \frac{m+2}{2}F$. The remaining cases $m = 0$ and $m = 2$ are considered separately below.

Assume that $m = 2$ and that the zero locus $\{\pi = 0\}$ has exactly two components, each giving the divisor $H + \frac{m+2}{2}F$. To finish the proof in this case, it is enough to show that $\mathcal{O}(F)$ does not admit a Poisson module structure with respect to π. To seek a contradiction, let us suppose that $\mathcal{O}(F)$ does admit a Poisson module structure with respect to π. Then Corollary 4.0.7 implies that π admits a lift to a Poisson structure $\widetilde{\pi}$ on $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)$. Then, for the co-Higgs field $\widetilde{\phi}$ on $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)$ corresponding to $\widetilde{\pi}$, the component $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(m), \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{T}_{\mathbb{P}^1})$ of $\widetilde{\phi}$ must vanish by Lemma 3.4.11. Then, for the co-Higgs field ϕ on $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)$ corresponding to π, the component $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(m), \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{T}_{\mathbb{P}^1})$ of ϕ must vanish as well (this is because ϕ and $\widetilde{\phi}$ may differ by a co-Higgs of the form vId for some vector field v on $\mathbb{P}^1$, but that does not affect the discussed component). Therefore, the subbundle $\mathcal{O}_{\mathbb{P}^1} \oplus 0$ is an eigenbundle of ϕ, which forces π to vanish on $H = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus 0) \subset \mathbb{P}(V)$. This contradicts the assumption that $\{\pi = 0\}$ has exactly two components, each giving the divisor $H + \frac{m+2}{2}F$.

Finally, let us assume that $m = 0$, i.e. $\mathbb{F}_m = \mathbb{P}^1 \times \mathbb{P}^1$, and that the zero locus $\{\pi = 0\}$ can be expressed as a union of D_1 and D_2, each of which is a divisor of type $\mathcal{O}(1, 1)$ (we allow $D_1 = D_2$). We need to show that the line bundle $\mathcal{O}(0, 1)$ does not admit a Poisson module structure. Each $D_i \subset \mathbb{P}^1 \times \mathbb{P}^1$ can be viewed as a graph of an automorphism $f_i : \mathbb{P}^1 \to \mathbb{P}^1$. Without loss of generality, assume that f_1 is the graph of the identity map, i.e. D_1 is the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. Then f_2 either has two distinct fixed points, or one fixed point of multiplicity two, or f_2 is identity. Up to a choice of coordinate, we obtain three cases: $f_2(x) = 2x$, $f_2(x) = x + 1$, and $f_2(x) = x$. So, we need to consider the following three Poisson structures:

$$\pi_1 = (x_2 - x_1)(x_2 - 2x_1)\partial_{x_1} \wedge \partial_{x_2} = 2x_1^2 \partial_{x_1} \wedge \partial_{x_2} - 3x_1 \partial_{x_1} \wedge x_2 \partial_{x_2} + \partial_{x_1} \wedge x_2^2 \partial_{x_2},$$

$$\pi_2 = (x_2 - x_1)(x_2 - x_1 - 1)\partial_{x_1} \wedge \partial_{x_2} = x_1(x_1 + 1)\partial_{x_1} \wedge \partial_{x_2} - (2x_1 + 1)\partial_{x_1} \wedge x_2 \partial_{x_2} + \partial_{x_1} \wedge x_2^2 \partial_{x_2},$$

$$\pi_3 = (x_2 - x_1)^2 \partial_{x_1} \wedge \partial_{x_2} = x_1^2 \partial_{x_1} \wedge \partial_{x_2} - 2x_1 \partial_{x_1} \wedge x_2 \partial_{x_2} + \partial_{x_1} \wedge x_2^2 \partial_{x_2}.$$

To conclude the proof, we need to show that for each of π_i, $i = 1, 2, 3$, the line bundle $\mathcal{O}(0, 1)$ does not admit a Poisson module structure. This follows by a direct computation from the technical Lemma 4.2.2 below. □

Lemma 4.2.2. *Let π be a Poisson structure on $\mathbb{P}^1 \times \mathbb{P}^1$ that has coordinate expression*

$$\pi = u(x_1) \wedge \partial_{x_2} + v(x_1) \wedge x_2 \partial_{x_2} + w(x_1) \wedge x_2^2 \partial_{x_2},$$

where x_1 and x_2 are affine coordinates on the first and second copy of $\mathbb{P}^1$, respectively, and u, v, w are linearly independent vector fields on the first copy of $\mathbb{P}^1$. Then in order for $\mathcal{O}(0,1)$ to admit a Poisson module structure it is necessary that the Lie bracket $[u,w]$ be a linear combination of u, w, $[u,v]$ and $[v,w]$.

Proof. We are going to use Proposition 2.1.9 that provides obstructions for existence of a Poisson module structure on a line bundle. The Atiyah class of $\mathcal{O}(0,1)$ can be represented as a Dolbeault 1-cocycle

$$\mathrm{At}_{\mathcal{O}(0,1)} = \frac{dx_2 \wedge d\bar{x}_2}{(1+x_2\bar{x}_2)^2}.$$

Then the first obstruction to $\mathcal{O}(0,1)$ being a Poisson module is

$$\pi^{\#}(\mathrm{At}_{\mathcal{O}(0,1)}) = \frac{1}{(1+x_2\bar{x}_2)^2}\big(u(x_1)\otimes d\bar{x}_2 + v(x_1)\otimes x_2 d\bar{x}_2 + w(x_1)\otimes x_2^2 d\bar{x}_2\big).$$

This 1-cocycle is in fact a coboundary (which is expected, since $H^1(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1\times\mathbb{P}^1}) = 0$) with the $\bar{\partial}$-primitive

$$\chi^{1,0} = -\frac{1}{(1+x_2\bar{x}_2)}\big(u(x_1)\bar{x}_2 - v(x_1) - w(x_1)x_2\big).$$

By evaluating

$$[\pi, \chi^{1,0}] = -[u,v]\wedge\partial_{x_2} - [u,w]\wedge x_2\partial_{x_2},$$

we obtain a holomorphic bivector. By Proposition 2.1.9, the line bundle $\mathcal{O}(0,1)$ admits a Poisson module structure if and only if $[\pi,\chi^{1,0}] = [\pi,\tau]$ for some holomorphic vector field τ on $\mathbb{P}^1\times\mathbb{P}^1$. Choose the basis of $H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{T})$ consisting of $u(x_1), v(x_1), w(x_1), \partial_{x_2}, x_2\partial_{x_2}, x_2^2\partial_{x_2}$, and apply $[\pi,-]$ to each of the element in the basis:

$$\begin{array}{lllll}
[\pi,\partial_{x_2}] & = & -v\wedge\partial_{x_2} & -2w\wedge x_2\partial_{x_2} & \\
[\pi,x_2\partial_{x_2}] & = & u\wedge\partial_{x_2} & & -w\wedge x_2^2\partial_{x_2} \\
[\pi,x_2^2\partial_{x_2}] & = & & 2u\wedge x_2\partial_{x_2} & +v\wedge x_2\partial_{x_2} \\
[\pi,u] & = & & [u,v]\wedge x_2\partial_{x_2} & +[u,w]\wedge x_2^2\partial_{x_2} \\
[\pi,v] & = & -[u,v]\wedge\partial_{x_2} & & +[v,w]\wedge x_2^2\partial_{x_2} \\
[\pi,w] & = & -[u,w]\wedge\partial_{x_2} & -[v,w]\wedge x_2\partial_{x_2} &
\end{array}$$

By examining the terms at $x_2\partial_{x_2}$, we arrive at the statement of the lemma. □

4.3 Rank 3 co-Higgs bundles over $\mathbb{P}^1$

Let us start with the following statement that sets the tone for the rest of the section.

Proposition 4.3.1. *Let (V, ϕ) be a rank $r > 2$ co-Higgs bundle over an analytic simply connected open set $\mathcal{U} \subset \mathbb{C}$ whose spectral curve is smooth connected. Then ϕ does not lift to a Poisson structure on $\mathbb{P}(V)$.*

Proof. The obstruction to the existence of the lift will have a local nature. By Corollary 4.0.2, we can assume that the characteristic equation of ϕ is $\theta^r = f$, where $f \in S^r\mathcal{T}_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}$, and θ is the tautological section of the pullback of $\mathcal{T}_{\mathcal{U}}$ to the total space of $\mathcal{T}_{\mathcal{U}}$. Since the spectral curve is smooth, f has only simple zeroes (Remark 3.4.12), and since the spectral curve is connected, f has at least one zero. Let us choose a coordinate x on $\mathcal{U}$ and fiberwise linear coordinate y on $\mathcal{T}_{\mathcal{U}}$, passing to smaller $\mathcal{U}$ if needed, so that the spectral curve Σ of ϕ has equation $y^r = x$. By Lemma 4.0.8, the eigenvariety $E \subset \mathbb{P}(V)$ of ϕ is smooth and locally parametrized by y. Over any non-zero x, the co-Higgs field ϕ can be diagonalized with diagonal entries $x^{1/r}\partial_x, \zeta x^{1/r}\partial_x, \zeta^2 x^{1/r}\partial_x, ..., \zeta^{r-1}x^{1/r}\partial_x$, where $x^{1/r}$ is a choice of r-th root of x, and $\zeta = e^{2\pi i/r}$. These diagonal entries satisfy the conditions of Lemma 3.3.4, so if ϕ were to lift to a Poisson structure π on $\mathbb{P}(V)$, this Poisson structure would have to vanish on $E \subset \mathbb{P}(V)$ and moreover the modular vector field at a point $y = \zeta^k x^{1/r}$ of E would have to be equal to

$$ry\partial_x = ry\frac{d(\zeta^k x^{1/r})}{dx}\partial_y = ry\zeta^k(x^{1/r})^{1-r}\partial_y = y^{2-r}\partial_y.$$

This modular vector field has a pole at $y = 0$, since $r > 2$. So, the lift π cannot be smooth at the fiber over $x = 0$. □

Corollary 4.3.2. *Let (V, ϕ) be a rank $r > 2$ co-Higgs bundle over an analytic simply connected open set $\mathcal{U} \subset \mathbb{C}$ whose spectral curve is smooth connected. Then ϕ does not lift to a quadratic Poisson structure on V.*

Proof. If σ were a quadratic Poisson structure on V lifting ϕ, then its projectivization π would be a Poisson structure on $\mathbb{P}(V)$ lifting the zero trace part of ϕ. Since the spectral curve of ϕ is smooth and connected, Corollary 4.0.2 implies that ϕ has zero trace to start with. Hence, the Poisson structure π, in fact, would be lifting ϕ itself, which would be in contradiction with Proposition 4.3.1. □

Proposition 4.3.1, in particular, says that if we want to look for Poisson structures on $\mathbb{P}^{r-1}$-bundles over $\mathbb{P}^1$, for $r > 2$, then we should consider co-Higgs bundles over $\mathbb{P}^1$ whose spectral curves are not smooth. In the rest of the section, we will take a closer look at the case $r = 3$.

In Subsections 4.3.1 and 4.3.2, we consider a zero-trace co-Higgs field ϕ on $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$ over a simply connected analytic open subset $\mathcal{U} \subset \mathbb{C}$ and study the conditions ϕ should have to admit a lift to a Poisson structure on $\mathbb{P}(V)$. Note that due to Theorem 3.1.5, this is equivalent to ϕ admitting a lift to a quadratic Poisson structure on V. For simplification, we will assume throughout the section that the *spectral curve of ϕ is reduced*. Equivalently, we assume that over an open subset $\widetilde{\mathcal{U}} \subset \mathcal{U}$, the co-Higgs field has the form $\phi = \mathrm{diag}(v_1, v_2, v_3)$ for some pairwise distinct vector fields v_i. To expect liftability of ϕ to a Poisson structure on $\mathbb{P}(V)$, we must assume strong integrability of ϕ, i.e. that the Lie bracket $[v_i, v_j]$ vanishes for all i, j (Lemma 3.4.2). Under such an assumption, Lemma 4.0.4 guarantees that, over $\widetilde{\mathcal{U}}$, the v_i are constant multiples of the same vector field v, and that liftability of ϕ is going to depend drastically on whether a certain combinatorial condition on the above constants takes place. Motivated by this, we are giving the following:

Definition 4.3.3. Let ϕ be a strongly integrable co-Higgs field on a rank 3 vector bundle V over an analytically open subset $\mathcal{U} \subset \mathbb{P}^1$, whose spectral curve is reduced. Then ϕ is called *resonant* (with respect to its Poisson liftability properties) if generically one has $2v_i = v_j + v_k$ where v_i, v_j, v_k are three distinct (locally defined) eigen vector fields of ϕ. Otherwise, ϕ is called *non-resonant*.

One can see that if the resonance condition $2v_i = v_j + v_k$ holds, it does so for exactly one v_i. Lemma 4.0.4 suggests how one should define resonance for rank > 3 co-Higgs fields, should one pursue the question of Poisson liftability of these.

Let us restate Lemma 4.0.4 for the rank 3 case.

Lemma 4.3.4. *Let $\phi = \mathrm{diag}(\lambda_1 v, \lambda_2 v, \lambda_3 v)$, where λ_i's are pairwise distinct complex numbers adding up to zero and $v \in \mathcal{T}_{\mathcal{U}}$, be a co-Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$ over an analytic simply connected open $\mathcal{U} \subset \mathbb{C}$.*

1) If ϕ is non-resonant, then all the lifts of ϕ to a Poisson structure on $\mathbb{P}(V)$ are given by

$$\pi = v \wedge u_1 + f(x) u_1 \wedge u_2, \tag{4.5}$$

where $f(x)$ is any function in $\mathcal{O}_{\mathcal{U}}$, and $u_i = (\lambda_2^i - \lambda_1^i)\widetilde{y_2}\partial_{\widetilde{y_2}} + (\lambda_3^i - \lambda_1^i)\widetilde{y_3}\partial_{\widetilde{y_3}}$, $i = 1,2$, where $\widetilde{y_2} = \frac{y_2}{y_1}$, $\widetilde{y_3} = \frac{y_3}{y_1}$, and y_1, y_2, y_3 are fiberwise linear coordinates on the corresponding summands of V.

2) If ϕ is resonant, i.e. w.l.o.g. $2\lambda_1 = \lambda_2 + \lambda_3$, then all the lifts of ϕ to a Poisson structure on $\mathbb{P}(V)$ are given by

$$\pi = v \wedge u_1 + f(x) u_1 \wedge u_2 + g(x)\partial_{\widetilde{y_2}} \wedge \partial_{\widetilde{y_3}}, \tag{4.6}$$

where $f(x)$, $g(x)$ are any functions in $\mathcal{O}_{\mathcal{U}}$, and $u_i = (\lambda_2^i - \lambda_1^i)\widetilde{y_2}\partial_{\widetilde{y_2}} + (\lambda_3^i - \lambda_1^i)\widetilde{y_3}\partial_{\widetilde{y_3}}$, $i = 1,2$, where $\widetilde{y_2} = \frac{y_2}{y_1}$, $\widetilde{y_3} = \frac{y_3}{y_1}$, and y_1, y_2, y_3 are fiberwise linear coordinates on the corresponding summands of V.

Proof. Let π be a lift of ϕ to a Poisson structure on $\mathbb{P}(V)$. By Theorem 3.1.5, there is a further lift of π to a quadratic Poisson structure σ on V. By Lemma 4.0.4, the bivector σ must be of the form

$$\sigma = v \wedge \Lambda + f(x)\Lambda \wedge \Lambda^2 + f_1(x) Eul \wedge \Lambda + f_2(x) Eul \wedge \Lambda^2 + g(x) y_1^2 \partial_{y_2} \wedge \partial_{y_3},$$

where $\Lambda = \sum_{i=1}^3 \lambda_i y_i \partial_{y_i}$, $\Lambda^2 = \sum_{i=1}^3 \lambda_i^2 y_i \partial_{y_i}$, $f, f_1, f_2, g \in \mathcal{O}_{\mathcal{U}}$, and the last summand can only occur in the resonant case $2\lambda_1 = \lambda_2 + \lambda_3$. By projectivizing σ we obtain the expression (4.5) (in the non-resonant case) or (4.6) (in the resonant case).

Conversely, one checks that if π is given by (4.5) (in the non-resonant case) or (4.6) (in the resonant case), then the Schouten bracket $[\pi, \pi]$ is purely vertical, and therefore has to vanish by dimensionality. □

Lemma 4.3.5. *Let π be a Poisson structure on $\mathbb{P}(V)$, for a rank 3 vector bundle V over an analytic open $\mathcal{U} \subset \mathbb{P}^1$. Then for any $f \in \mathcal{O}_{\mathcal{U}}$, the bivector $f\pi$ is also Poisson (where we denote the pullback of f to $\mathbb{P}(V)$ by f, too). The zero-trace co-Higgs field on V corresponding to $f\pi$ equals $f\phi$, where ϕ is the zero-trace co-Higgs field on V corresponding to π.*

Proof. To check that the trivector $[f\pi, f\pi]$ vanishes, it is enough to check $\iota_{df}[f\pi, f\pi] = 0$. Note that $[f\pi, f\pi] = 2f\pi \wedge [f, \pi] = -2f\pi \wedge \iota_{df}(\pi)$. Therefore $\iota_{df}[f\pi, f\pi] = -2f\big(\iota_{df}(\pi) \wedge \iota_{df}(\pi) + \pi \wedge \iota_{df}\iota_{df}(\pi)\big) = 0$.

The last claim follows directly from the definition of the co-Higgs induced by $f\pi$ (Lemma 3.2.3). □

4.3.1 Local Poisson lifts. Non-resonant case.

Recall that if V is a vector bundle over a base B and D is a divisor on B, then by a log D-connection on V that is allowed to have logarithmic poles along D. If one has a splitting V into direct sum of line bundles $\mathcal{L}_1, ..., \mathcal{L}_r$, then a connection ∇ is called diagonal with respect to such a splitting, if everywhere locally it has diagonal matrix in some (equivalently, all) trivialization of V given by non-vanishing sections s_i of $\mathcal{L}_i$, $i = 1, ..., r$. If ϕ is generically semi-simple co-Higgs on V, then by a logarithmic diagonal with respect to ϕ connection on V we mean a connection ∇ on V such that ∇ is allowed to have logarithmic poles only at the points of branching of the spectral cover $\Sigma \to B$ and ∇ is diagonal with respect to splitting of V into direct sum of eigen line bundles away from branch points of the spectral cover.

Lemma 4.3.6. *Let $\phi \in \mathrm{Hom}(V, V \otimes \mathcal{T}_{\mathcal{U}})$ be strongly integrable co-Higgs field on a rank 3 vector bundle V over an connected, simply connected, open $0 \in \mathcal{U} \subset \mathbb{C}$. Let the spectral curve of ϕ be reduced and smooth away from 0. Let ϕ be non-resonant (Definition 4.3.3).*

Let ∇ be a logarithmic connection diagonal with respect to ϕ, and let $\mathrm{res}_{x=0}\nabla \in \mathrm{End}(V)|_0$ be its residue at 0.

Then ϕ admits a lift to a Poisson structure on $\mathbb{P}(V)$ if and only if there is a constant $\mu \in \mathbb{C}$ such that

$$Eul \wedge \varphi|_0 \wedge \mathrm{res}_{x=0}\nabla = \mu \lim_{x\to 0} x^{-s}(Eul \wedge \varphi|_x \wedge \varphi^2|_x), \tag{4.7}$$

where both sides of the equality are interpreted as $\mathbb{C}^*$-invariant trivectors on the fiber $V|_0$; Eul is the Euler vector field on $V|_0$; $\varphi = \langle \phi, dx\rangle \in \mathrm{End}(V)$; $\mathrm{res}_{x=0}\nabla$, $\varphi|_x$ and $\varphi^2|_x$ are viewed as the vector fields on $V|_0$ or $V|_x$ given by the corresponding matrices; and s is the unique non-negative integer for which the limit in the right hand side is a finite and non-zero.

Proof. The bivector $\sigma = Lift_\nabla(\phi)$ on V defined by (3.12) is Poisson by Lemma 3.4.18, but it may or may not have a simple pole over $x = 0$. If the right hand side of (4.7) vanishes, the projectivization π of σ defines a smooth Poisson bivector on $\mathbb{P}(V)$ (with no pole over $x = 0$). If the right hand side of (4.7) does not vanish, then π has a pole over $x = 0$, but one may try to correct the bivector π by adding a purely vertical bivector π_1 to it so that the sum $\pi + \pi_1$ does not have a pole over $x = 0$. By Lemma 4.3.4.1), in order to ensure integrability of $\pi + \pi_1$, the summand π_1 must be projectivization of $f(x)\varphi \wedge \varphi^2$ for some function f defined on $\mathcal{U} \setminus \{0\}$. Such correction π_1 making the sum $\pi + \pi_1$ smooth exists if and only if the condition (4.7) holds true (in which case one can set $f(x) = -\mu x^{-s-1}$). ☐

A_2 singularity

Lemma 4.3.7. *Let ϕ be a co-Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$ over an analytic neighborhood $\mathcal{U}$ of $0 \in \mathbb{C}$ whose spectral curve is $\Sigma = \{y^3 = x^2\}$. Let ϕ be brought to one of the two possible normal forms given by Theorem A.0.2, that is,*

$$\phi = \Phi_0 = \left(\begin{array}{cc|c} & & x^2 \\ \hline 1 & & \\ & 1 & \end{array}\right), \quad or \quad \phi = \Phi_1 = \left(\begin{array}{cc|c} & & x \\ \hline x & & \\ & 1 & \end{array}\right).$$

Then there are logarithmic connection ∇_i on V diagonal with respect to $\phi = \Phi_i$, $i = 0, 1$, having the *following residues at* 0

$$\operatorname{res}_{x=0}\nabla_0 = \begin{pmatrix} 0 & & \\ & -\frac{2}{3} & \\ & & -\frac{4}{3} \end{pmatrix}, \qquad \operatorname{res}_{x=0}\nabla_1 = \begin{pmatrix} 0 & & \\ & \frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix}$$

(we write the matrices for residues in the same basis in which the co-Higgs matrices are written).

Proof. First, consider the case of Φ_0. Let $\mathbf{y} = (y_1, y_2, y_3)$ be the fiberwise linear coordinates on V given by the basis e_1, e_2, e_3 in which the co-Higgs Φ_0 is written. Everywhere away from $x = 0$, the section $e_1 = \{y_1 = 1, y_2 = 0, y_3 = 0\}$ of V locally gives trivialization of each of the eigen line bundle of Φ_0. This defines a diagonal connection ∇_0 on V away from $x = 0$. Let us check that this connection has logarithmic pole at $x = 0$ and calculate its residue.

Let $\widetilde{\mathcal{U}} \subset \mathcal{U}$ be a simply connected open set not containing $x = 0$. Then one can define the coordinate $\widetilde{x} = x^{1/3}$ on $\widetilde{\mathcal{U}}$ by choosing a branch of the cube root. One checks that the transition matrix

$$T = \begin{pmatrix} 1 & \widetilde{x}^2 & \widetilde{x}^4 \\ 1 & \zeta\widetilde{x}^2 & \zeta^2\widetilde{x}^4 \\ 1 & \zeta^2\widetilde{x}^2 & \zeta\widetilde{x}^4 \end{pmatrix}, \qquad \zeta = \exp(2\pi i/3),$$

diagonalizes Φ_0 over $\widetilde{\mathcal{U}}$, in the sense that $T\Phi_0 T^{-1}$ is diagonal.

Let us define fiberwise coordinates $\widetilde{\mathbf{y}} = (\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3)$ on V over $\widetilde{\mathcal{U}}$ by $\widetilde{\mathbf{y}}^t = T\mathbf{y}^t$. Then the section $\mathbf{y} = (1, 0, 0)$ is expressed as $\widetilde{\mathbf{y}} = (1, 1, 1)$, and therefore in coordinates $\widetilde{x}, \widetilde{\mathbf{y}}$ the connection ∇_0 has zero matrix. Hence in coordinates $x, \mathbf{y}$ the connection ∇_0 has matrix

$$-T^{-1}dT = \begin{pmatrix} 0 & & \\ & -2 & \\ & & -4 \end{pmatrix} \frac{d\widetilde{x}}{\widetilde{x}} = \begin{pmatrix} 0 & & \\ & -\frac{2}{3} & \\ & & -\frac{4}{3} \end{pmatrix} \frac{dx}{x}.$$

We see that the latter expression for ∇_0 does not depend on the chosen open set $\widetilde{\mathcal{U}}$, and that ∇_0 has logarithmic pole at $x = 0$ with the announced residue.

The case of Φ_1 is considered analogously to Φ_1, with the only modification that the transition matrix diagonalizing Φ_1 is

$$T = \begin{pmatrix} 1 & \widetilde{x} & \widetilde{x}^{-1} \\ 1 & \zeta\widetilde{x} & \zeta^2\widetilde{x}^{-1} \\ 1 & \zeta^2\widetilde{x} & \zeta\widetilde{x}^{-1} \end{pmatrix}, \qquad \zeta = \exp(2\pi i/3).$$

□

Proposition 4.3.8. *Let ϕ be a zero-trace strongly integrable co-Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$ over an open $\mathcal{U} \subset \mathbb{C}$, whose spectral curve has an A_2-singularity. Then ϕ does not admit a lift to Poisson structure on $\mathbb{P}(V)$.*

Proof. It follows from Proposition 4.0.1 that near the A_2 singularity there is a choice of coordinate x on the curve and a fiberwise linear coordinate y on $\mathcal{T}_{\mathcal{U}}$ so the the spectral curve Σ of ϕ is cut out by the equation $y^3 = x^2$. So, we only need to consider the two co-Higgs fields from Theorem A.0.2

$$\Phi_0 = \left(\begin{array}{cc|c} & & x^2 \\ \hline 1 & & \\ & 1 & \end{array}\right), \quad \text{and} \quad \Phi_1 = \left(\begin{array}{cc|c} & & x \\ \hline x & & \\ & 1 & \end{array}\right).$$

We are going to apply the criterion from Lemma 4.3.6 and the calculated residues from Lemma 4.3.7. We are going to denote by y_1, y_2, y_3 the fiberwise linear coordinates on V given by the basis in which the co-Higgs fields are written.

Case Φ_0. The left hand side of (4.7) equals

$$(y_1\partial_{y_1} + y_2\partial_{y_2} + y_3\partial_{y_3}) \wedge (y_1\partial_{y_2} + y_2\partial_{y_3}) \wedge (-\frac{2}{3}y_2\partial_{y_2} - \frac{4}{3}y_3\partial_{y_3}) = \frac{2}{3}y_1(-2y_1y_3 + y_2^2)\partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_3}.$$

The right hand side of (4.7) equals

$$\mu \lim_{x\to 0} x^{-s}\left((y_1\partial_{y_1} + y_2\partial_{y_2} + y_3\partial_{y_3}) \wedge (y_1\partial_{y_2} + y_2\partial_{y_3} + x^2y_3\partial_{y_1}) \wedge (y_1\partial_{y_3} + x^2y_2\partial_{y_1} + x^2y_3\partial_{y_2})\right) =$$

$$= \mu \lim_{x\to 0} x^{-s}\left(y_1^3\partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_3} + o(1)\right),$$

here $o(1)$ denotes the terms that go to 0 when $x \to 0$. We have $s = 0$, and so the right hand side of (4.7) equals

$$\mu y_1^3\partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_3},$$

and so the equality in (4.7) cannot be achieved for any $\mu \in \mathbb{C}$. Lemma 4.3.6 implies that Φ_0 does not lift to a Poisson structure on $\mathbb{P}(V)$.

Case Φ_1. The left hand side of (4.7) equals

$$(y_1\partial_{y_1} + y_2\partial_{y_2} + y_3\partial_{y_3}) \wedge (y_2\partial_{y_3}) \wedge (\frac{1}{3}y_2\partial_{y_2} - \frac{1}{3}y_3\partial_{y_3}) = -\frac{1}{3}y_1y_2^2\partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_3}.$$

The right hand side of (4.7) equals

$$\mu \lim_{x\to 0} x^{-s}\left((y_1\partial_{y_1} + y_2\partial_{y_2} + y_3\partial_{y_3}) \wedge (xy_1\partial_{y_2} + y_2\partial_{y_3} + xy_3\partial_{y_1}) \wedge (xy_1\partial_{y_3} + xy_2\partial_{y_1} + x^2y_3\partial_{y_2})\right) =$$

$$= \mu \lim_{x\to 0} x^{-s}\left(xy_2^3\partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_3} + o(x)\right),$$

here $o(1)$ denotes the terms that have order of vanishing > 1 when $x \to 0$. We have $s = 1$, and so the right hand side of (4.7) equals

$$\mu y_2^3\partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_3},$$

and so the equality in (4.7) cannot be achieved for any $\mu \in \mathbb{C}$. Lemma 4.3.6 implies that Φ_1 does not lift to a Poisson structure on $\mathbb{P}(V)$. □

Corollary 4.3.9. *Let V is a rank 3 vector bundle over $\mathbb{P}^1$, and let π be a Poisson structure on $\mathbb{P}(V)$. Then the spectral curve of the co-Higgs field ϕ corresponding to π cannot be reduced, irreducible.*

Proof. Assuming the spectral curve is reduced, irreducible, Corollary 3.4.7 and Proposition 4.0.1 imply that the characteristic polynomial of ϕ has to be of the form $\theta^3 - q = 0$, where $q \in H^0(S^3\mathcal{T}_{\mathbb{P}^1})$. Since $S^3\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(6)$, the tensor q has 6 zeros counting multiplicity. By Proposition 4.3.1, q cannot have isolated zeros, and by Proposition 4.3.8, it cannot have zeros of multiplicity two. So, either q has two zeros each of multiplicity three, or it has one zero of multiplicity six. Either way, one can express $q = v^3$ for a vector field $v \in H^0(\mathcal{T}_{\mathbb{P}^1})$. This contradicts irreducibility of the characteristic polynomial, since $\theta^3 - q = (\theta - v)(\theta - \zeta v)(\theta - \zeta^2 v)$, for $\zeta = \exp(2\pi i/3)$. □

D_4 singularity

Lemma 4.3.10. *Let $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$, where $\mathcal{U}$ is an analytic neighborhood of $0 \in \mathbb{C}$. Let Σ be a curve inside $\mathcal{T}_{\mathcal{U}}$ cut out by the equation $\{\prod_{i=1}^3 (y - \lambda_i x) = 0\}$ for some pairwise distinct $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ (it has D_4 singularity). For each of the following co-Higgs fields $\Phi_i^{D_4}$, $0 \le i \le 6$, on V with the spectral curve Σ, there is a logarighmic connection $\nabla_i^{D_4}$ on V diagonal with respect to $\Phi_i^{D_4}$ having the specified residue at $x = 0$ (each residue matrix is written in the same basis as the corresponding co-Higgs field):*

$$\Phi_0^{D_4} = \begin{pmatrix} \lambda_1 x & & \\ 1 & \lambda_2 x & \\ & 1 & \lambda_3 x \end{pmatrix}, \qquad \mathrm{res}_{x=0} \nabla_0^{D_4} = \begin{pmatrix} 0 & & \\ & -1 & \\ & & -2 \end{pmatrix},$$

$$\Phi_1^{D_4} = \begin{pmatrix} \lambda_1 x & & \\ x & \lambda_2 x & \\ & 1 & \lambda_3 x \end{pmatrix}, \qquad \mathrm{res}_{x=0} \nabla_1^{D_4} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -1 \end{pmatrix},$$

$$\Phi_2^{D_4} = \begin{pmatrix} \lambda_1 x & & \\ 1 & \lambda_2 x & \\ & x & \lambda_3 x \end{pmatrix}, \qquad \mathrm{res}_{x=0} \nabla_2^{D_4} = \begin{pmatrix} 0 & & \\ & -1 & \\ & & -1 \end{pmatrix},$$

$$\Phi_{2+i}^{D_4} = \left(\begin{array}{c|cc} \lambda_i x & & \\ \hline & \lambda_{i+1} x & \\ & 1 & \lambda_{i+2} x \end{array}\right), \quad \mathrm{res}_{x=0} \nabla_{2+i}^{D_4} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -1 \end{pmatrix}, \quad i = 1, 2, 3,$$

$$\Phi_6^{D_4} = \begin{pmatrix} \lambda_1 x & & \\ & \lambda_2 x & \\ & & \lambda_3 x \end{pmatrix}, \qquad \mathrm{res}_{x=0} \nabla_6^{D_4} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}.$$

Proof. Let us consider the cases $\Phi_i^{D_4}$, for $i = 0, 1, 2$ (the remaining cases are easier, and can be considered analogously). One checks that over the punctured neighborhood $\{x \neq 0\}$ the matrix $T_i \Phi_i^{D_4} T_i^{-1}$ is diagonal, where

$$T_i = \begin{pmatrix} 1 & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) x^{k_i} & 0 \\ 1 & (\lambda_3 - \lambda_1) x^{k_i} & (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) x^{l_i} \end{pmatrix},$$

and $k_0 = 1, l_0 = 2$, $k_1 = 0, l_1 = 1$, $k_2 = 1, l_2 = 1$.

Let $\mathbf{y} = (y_1, y_2, y_3)$ be the fiberwise linear coordinates on V in which the matrix Φ^{D_4} is written, and let $\widetilde{\mathbf{y}} = (\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3)$ on V be the coordinates on V over $\{x \neq 0\}$ defined by $\widetilde{\mathbf{y}}^t = T_i \mathbf{y}^t$. For each $i = 0, 1, 2$, we define the connection $\nabla_i^{D_4}$ as the one having the zero matrix in coordinates $x, \widetilde{\mathbf{y}}$. Then in coordinates

$x, \mathbf{y}$ the connection $\nabla_i^{D_4}$ has the matrix

$$-T_i^{-1} dT_i = \begin{pmatrix} 0 & & \\ & -k_i & \\ & & -l_i \end{pmatrix} \frac{dx}{x}.$$

□

Proposition 4.3.11. *Let $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$, where $\mathcal{U}$ is an analytic neighborhood of $0 \in \mathbb{C}$. Let Σ be a curve inside $\mathcal{T}_{\mathcal{U}}$ cut out by the equation $\{\prod_{i=1}^{3}(y - \lambda_i x) = 0\}$ for some pairwise distinct $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ (Σ has D_4 singularity). Furthermore, let $2\lambda_i \neq \lambda_j + \lambda_k$, unless $i = j = k$ (non-resonance condition).*

Then among the co-Higgs fields Φ^{D_4}, $0 \leq i \leq 6$, having spectral curve Σ, the co-Higgs fields $\Phi_0^{D_4}$ and $\Phi_2^{D_4}$ do not admit a Poisson lift to $\mathbb{P}(V)$, while the remaining ones do admit such Poisson lifts.

Proof. Follows by direct calculation using the criterion from Lemma 4.3.6 and the calculated residues from Lemma 4.3.10. □

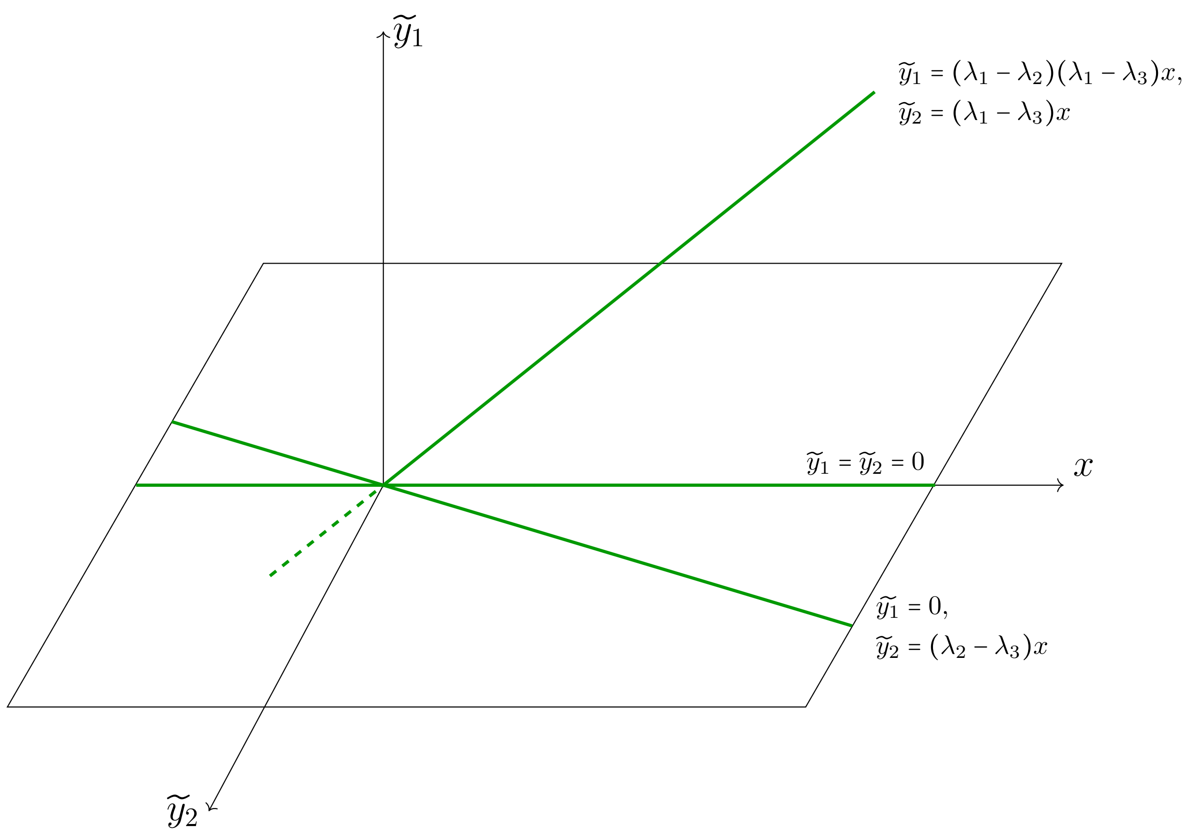
Below we list the general forms of the local Poisson lifts to $\mathbb{P}(V)$ for the non-resonant co-Higgs fields with D_4 singularity, for the cases when they exist. We denote by y_1, y_2, y_3 the fiberwise linear coordinates on V given by the basis in which the co-Higgs fields are written.

$$\Phi_1^{D_4} = \begin{pmatrix} \lambda_1 x & & \\ x & \lambda_2 x & \\ & 1 & \lambda_3 x \end{pmatrix} \partial_x,$$

$$\pi \quad = \quad \partial_x \wedge \varphi + \frac{1}{x}(\widetilde{y}_1 \partial_{\widetilde{y}_1} + \widetilde{y}_2 \partial_{\widetilde{y}_2}) \wedge \varphi + \frac{1}{x^2} f(x) \varphi \wedge \varphi^2,$$

$$\begin{aligned} \varphi &= \langle \Phi_1^{D_4}, dx \rangle &&= (\lambda_1 - \lambda_3) x \widetilde{y}_1 \partial_{\widetilde{y}_1} + (\lambda_2 - \lambda_3) x \widetilde{y}_2 \partial_{\widetilde{y}_2} + x \widetilde{y}_1 \partial_{\widetilde{y}_2} - \widetilde{y}_2 (\widetilde{y}_1 \partial_{\widetilde{y}_1} + \widetilde{y}_2 \partial_{\widetilde{y}_2}), \\ \varphi^2 &= \langle (\Phi_1^{D_4})^2, dx^2 \rangle &&= (\lambda_1^2 - \lambda_3^2) x^2 \widetilde{y}_1 \partial_{\widetilde{y}_1} + (\lambda_2^2 - \lambda_3^2) x^2 \widetilde{y}_2 \partial_{\widetilde{y}_2} + (\lambda_1 + \lambda_2) x^2 \widetilde{y}_1 \partial_{\widetilde{y}_2} \\ & && \quad -(\lambda_2 + \lambda_3) x \widetilde{y}_2 (\widetilde{y}_1 \partial_{\widetilde{y}_1} + \widetilde{y}_2 \partial_{\widetilde{y}_2}) - x \widetilde{y}_1 (\widetilde{y}_1 \partial_{\widetilde{y}_1} + \widetilde{y}_2 \partial_{\widetilde{y}_2}), \end{aligned}$$

where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, and f is any locally defined holomorphic function in x.



Zero set of a local Poisson lift of $\Phi_1^{D_4}$

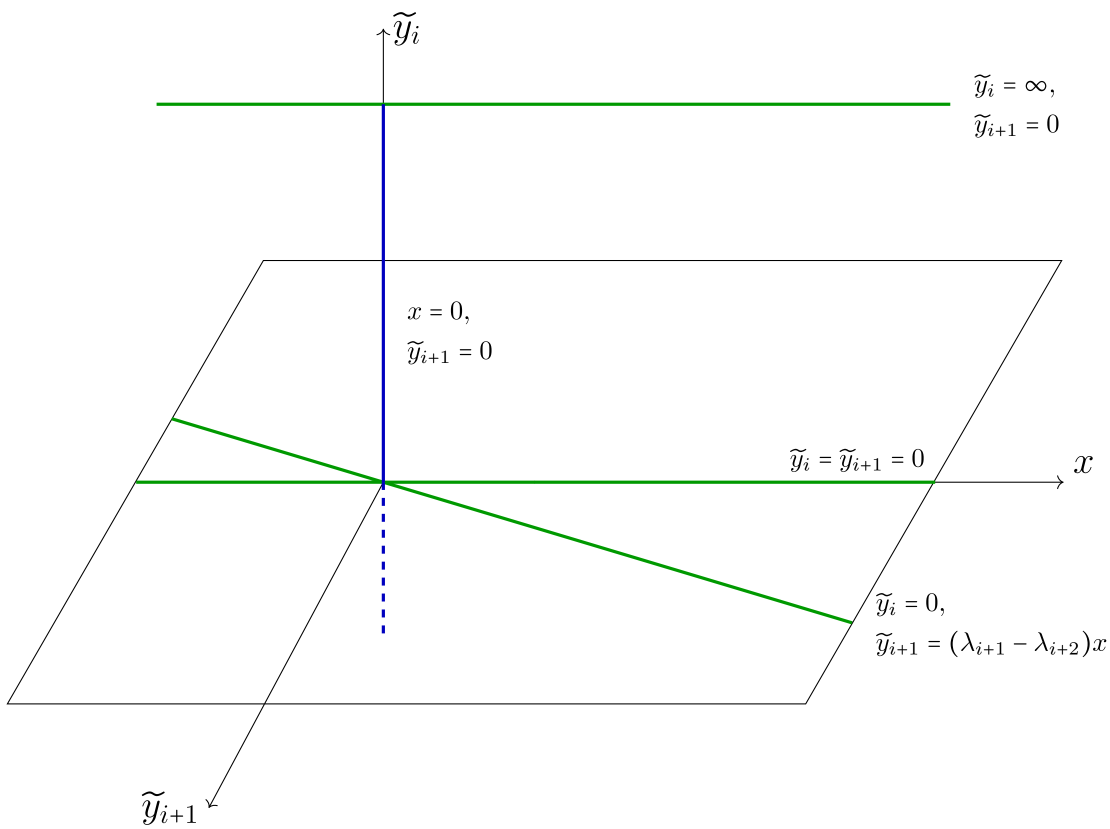
$$\Phi^{D_4}_{2+i} = \begin{pmatrix} \lambda_i x & & \\ & \lambda_{i+1} x & \\ & 1 & \lambda_{i+2} x \end{pmatrix} \partial_x, \quad i = 1,2,3,$$

$$\pi = \partial_x \wedge \varphi + \frac{1}{x}(\widetilde{y}_i \partial_{\widetilde{y}_i} + \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}) \wedge \varphi + \frac{1}{x^2} f(x) \varphi \wedge \varphi^2,$$

$$\varphi = \langle \Phi^{D_4}_{2+i}, dx \rangle = (\lambda_i - \lambda_{i+2}) x \widetilde{y}_i \partial_{\widetilde{y}_i} + (\lambda_{i+1} - \lambda_{i+2}) x \widetilde{y}_{i+1} \partial_{\widetilde{y}_{i+1}} - \widetilde{y}_{i+1}(\widetilde{y}_i \partial_{\widetilde{y}_i} + \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}),$$

$$\varphi^2 = \langle (\Phi^{D_4}_{2+i})^2, dx^2 \rangle = (\lambda_i^2 - \lambda_{i+2}^2) x^2 \widetilde{y}_i \partial_{\widetilde{y}_i} + (\lambda_{i+1}^2 - \lambda_{i+2}^2) x^2 \widetilde{y}_{i+1} \partial_{\widetilde{y}_{i+1}} - (\lambda_{i+1} + \lambda_{i+2}) x \widetilde{y}_{i+1}(\widetilde{y}_i \partial_{\widetilde{y}_i} + \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}),$$

where $\widetilde{y}_i = \frac{y_i}{y_{i+2}}$, $\widetilde{y}_{i+1} = \frac{y_{i+1}}{y_{i+2}}$, and f is any locally defined holomorphic function in x.



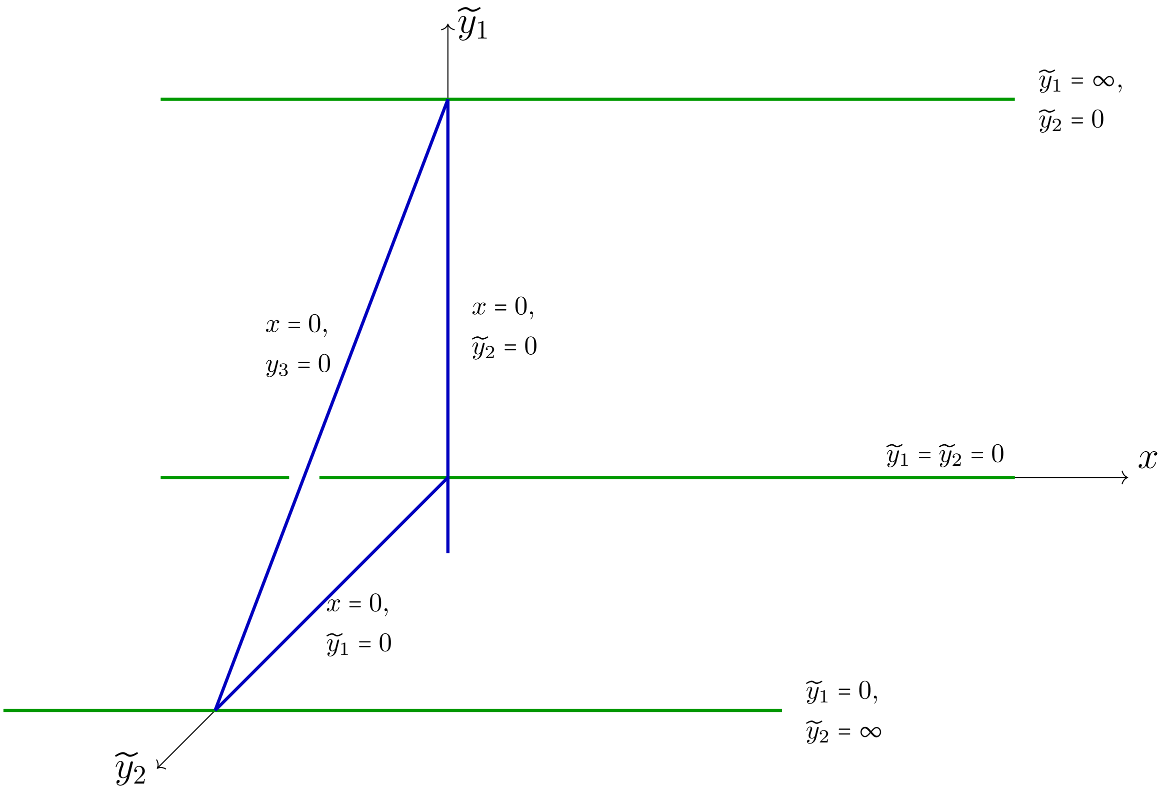
Zero set of a local Poisson lift of $\Phi^{D_4}_{2+i}$, $i = 1,2,3$

$$\Phi_6^{D_4} = \begin{pmatrix} \lambda_1 x & & \\ & \lambda_2 x & \\ & & \lambda_3 x \end{pmatrix} \partial_x,$$

$$\pi = \partial_x \wedge \varphi + \frac{1}{x^3} f(x) \varphi \wedge \varphi^2,$$

$$\begin{aligned} \varphi &= \langle \Phi_6^{D_4}, dx \rangle &&= (\lambda_1 - \lambda_3) x \widetilde{y}_1 \partial_{\widetilde{y}_1} + (\lambda_2 - \lambda_3) x \widetilde{y}_2 \partial_{\widetilde{y}_2}, \\ \varphi^2 &= \langle (\Phi_6^{D_4})^2, dx^2 \rangle &&= (\lambda_1^2 - \lambda_3^2) x^2 \widetilde{y}_1 \partial_{\widetilde{y}_1} + (\lambda_2^2 - \lambda_3^2) x^2 \widetilde{y}_2 \partial_{\widetilde{y}_2}, \end{aligned}$$

where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, and f is any locally defined holomorphic function in x.



Zero set of a local Poisson lift of $\Phi_6^{D_4}$

The figure is drawn under assumption $f(0) \neq 0$.
If $f(0) = 0$, then additionally the plane $\{x = 0\}$ is contained in the zero set.

T_{36} **singularity**

Lemma 4.3.12. *Let $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$, where $\mathcal{U}$ is an analytic neighborhood of $0 \in \mathbb{C}$. Let Σ be a curve inside $\mathcal{T}_{\mathcal{U}}$ cut out by the equation $\{\prod_{i=1}^{3}(y - \lambda_i x^2) = 0\}$ for some pairwise distinct $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ (it has T_{36} singularity). For each of the following co-Higgs fields $\Phi_i^{T_{36}}$, $0 \le i \le 13$, on V with the spectral curve Σ, there is a logarighmic connection $\nabla_i^{T_{36}}$ on V diagonal with respect to $\Phi_i^{T_{36}}$ having the specified residue at $x = 0$ (each residue matrix is written in the same basis as the corresponding co-Higgs field):*

$$\Phi_0^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ 1 & \lambda_2 x^2 & \\ & 1 & \lambda_3 x^2 \end{pmatrix}, \qquad \mathrm{res}_{x=0} \nabla_0^{T_{36}} = \begin{pmatrix} 0 & & \\ & -2 & \\ & & -4 \end{pmatrix},$$

$$\Phi_i^{T_{36}} = \begin{pmatrix} \lambda_i x^2 & & \\ x^3 & \lambda_{i+1} x^2 & \\ & 1 & \lambda_{i+2} x^2 \end{pmatrix}, \qquad \mathrm{res}_{x=0} \nabla_i^{T_{36}} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad i = 1, 2, 3,$$

$$\Phi_{3+i}^{T_{36}} = \begin{pmatrix} \lambda_i x^2 & & \\ 1 & \lambda_{i+1} x^2 & \\ & x^3 & \lambda_{i+2} x^2 \end{pmatrix}, \qquad \mathrm{res}_{x=0} \nabla_{3+i}^{T_{36}} = \begin{pmatrix} 0 & & \\ & -2 & \\ & & -1 \end{pmatrix}, \quad i = 1, 2, 3,$$

$$\Phi_7^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ x^2 & \lambda_2 x^2 & \\ & 1 & \lambda_3 x^2 \end{pmatrix}, \qquad \mathrm{res}_{x=0} \nabla_7^{T_{36}} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -2 \end{pmatrix},$$

$$\Phi_8^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ 1 & \lambda_2 x^2 & \\ & x^2 & \lambda_3 x^2 \end{pmatrix}, \qquad \mathrm{res}_{x=0} \nabla_8^{T_{36}} = \begin{pmatrix} 0 & & \\ & -2 & \\ & & -2 \end{pmatrix},$$

$$\Phi_9^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ x & \lambda_2 x^2 & \\ & 1 & \lambda_3 x^2 \end{pmatrix}, \qquad \mathrm{res}_{x=0} \nabla_9^{T_{36}} = \begin{pmatrix} 0 & & \\ & -1 & \\ & & -3 \end{pmatrix},$$

$$\Phi_{10}^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ 1 & \lambda_2 x^2 & \\ & x & \lambda_3 x^2 \end{pmatrix}, \qquad \mathrm{res}_{x=0} \nabla_{10}^{T_{36}} = \begin{pmatrix} 0 & & \\ & -2 & \\ & & -3 \end{pmatrix},$$

$$\Phi_{10+i}^{T_{36}} = \begin{pmatrix} \lambda_i x^2 & & \\ & \lambda_{i+1} x^2 & \\ & 1 & \lambda_{i+2} x^2 \end{pmatrix}, \qquad \mathrm{res}_{x=0} \nabla_{10+i}^{T_{36}} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -2 \end{pmatrix}, \quad i = 1, 2, 3.$$

Likewise, for each $\beta \in \mathbb{C} \setminus \{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}$, the co-Higgs field

$$\Psi_\beta^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ x & \lambda_2 x^2 & \\ \frac{1}{\beta - \lambda_1 - \lambda_3} & x & \lambda_3 x^2 \end{pmatrix},$$

admits a logarithmic connection $\nabla_\beta^{T_{36}}$ *on* V *diagonal with respect to* $\Psi_\beta^{T_{36}}$ *that has the residue*

$$\mathrm{res}_{x=0}\nabla_\beta^{T_{36}} = \begin{pmatrix} 0 & & \\ & -1 & \\ & & -2 \end{pmatrix}.$$

Proof. The proof of this lemma mimics the proof of Lemma 4.3.10. For fixed $j,k \in \mathbb{Z}_{\geq 0}$, the co-Higgs field

$$\phi = \begin{pmatrix} \lambda_1 x^2 & & \\ x^j & \lambda_2 x^2 & \\ & x^k & \lambda_3 x^2 \end{pmatrix}$$

can be diagonalized over $\{x \neq 0\}$ via the matrix

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & (\lambda_2-\lambda_1)x^{2-j} & 0 \\ 1 & (\lambda_3-\lambda_1)x^{2-j} & (\lambda_3-\lambda_1)(\lambda_3-\lambda_2)x^{4-j-k} \end{pmatrix},$$

in the sense that $T\phi T^{-1}$ is diagonal. Let $\mathbf{y} = (y_1, y_2, y_3)$ be the fiberwise linear coordinates on V in which the matrix ϕ is written, and let $\widetilde{\mathbf{y}} = (\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3)$ on V be the coordinates on V over $\{x \neq 0\}$ defined by $\widetilde{\mathbf{y}}^t = T\mathbf{y}^t$. We define the connection ∇ as the one having the zero matrix in coordinates $x, \widetilde{\mathbf{y}}$. Then in coordinates $x, \mathbf{y}$ the connection ∇ has the matrix

$$-T_i^{-1}dT_i = \begin{pmatrix} 0 & & \\ & j-2 & \\ & & j+k-4 \end{pmatrix}\frac{dx}{x}.$$

This argument covers the cases $\Phi_0^{T_{36}}$ through $\Phi_{T_{36}10}$. The cases $\Phi_{11}^{T_{36}}$, $\Phi_{12}^{T_{36}}$, $\Phi_{13}^{T_{36}}$ are covered by splitting V into direct sum of line bundle $\{y_2 = y_3 = 0\}$ and rank 2 vector bundle $\{y_1 = 0\}$, and handle each summand separately. Finally, for the case $\Psi_\beta^{T_{36}}$, one has to use the same argument as above with the diagonlizing transition matrix T replaced by

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & (\lambda_2-\lambda_1)x & 0 \\ 1 & (\lambda_3-\lambda_1)\frac{\beta-\lambda_1-\lambda_3}{\beta-\lambda_1-\lambda_2}x & (\lambda_3-\lambda_2)(\lambda_3-\lambda_1)\frac{\beta-\lambda_1-\lambda_3}{\beta-\lambda_1-\lambda_2}x^2 \end{pmatrix}$$

□

Proposition 4.3.13. *Let* $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$, *where* $\mathcal{U}$ *is an analytic neighborhood of* $0 \in \mathbb{C}$. *Let* Σ *be a curve inside* $\mathcal{T}_{\mathcal{U}}$ *cut out by the equation* $\{\prod_{i=1}^{3}(y-\lambda_i x^2) = 0\}$ *for some pairwise distinct* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ *(*Σ *has* T_{36} *singularity). Furthermore, let* $2\lambda_i \neq \lambda_j + \lambda_k$, *unless* $i = j = k$ *(non-resonance condition).*

Then among the co-Higgs fields $x\Phi^{D_4}$, $0 \leq i \leq 6$, $\Phi_i^{T_{36}}$, $0 \leq i \leq 13$, $\Psi_\beta^{T_{36}}$, $\beta \in \mathbb{C}\setminus\{\lambda_1+\lambda_2, \lambda_1+\lambda_3, \lambda_2+\lambda_3\}$, *having spectral curve* Σ, *the co-Higgs fields* $\Phi_i^{T_{36}}$, $i = 0,4,5,6,8,9,10$ *and* $\Psi_\beta^{T_{36}}$, $\beta \in \mathbb{C}\setminus\{\lambda_1+\lambda_2, \lambda_1+\lambda_3, \lambda_2+\lambda_3\}$, *do not admit a Poisson lift to* $\mathbb{P}(V)$, *while the co-Higgs fields* $x\Phi_i^{D_4}$, $0 \leq i \leq 6$, *and* $\Phi_i^{T_{36}}$, $i = 1,2,3,7,11,12,13$ *do admit such Poisson lifts.*

Proof. Follows by direct calculation using the criterion from Lemma 4.3.6 and the calculated residues from Lemma 4.3.12. □

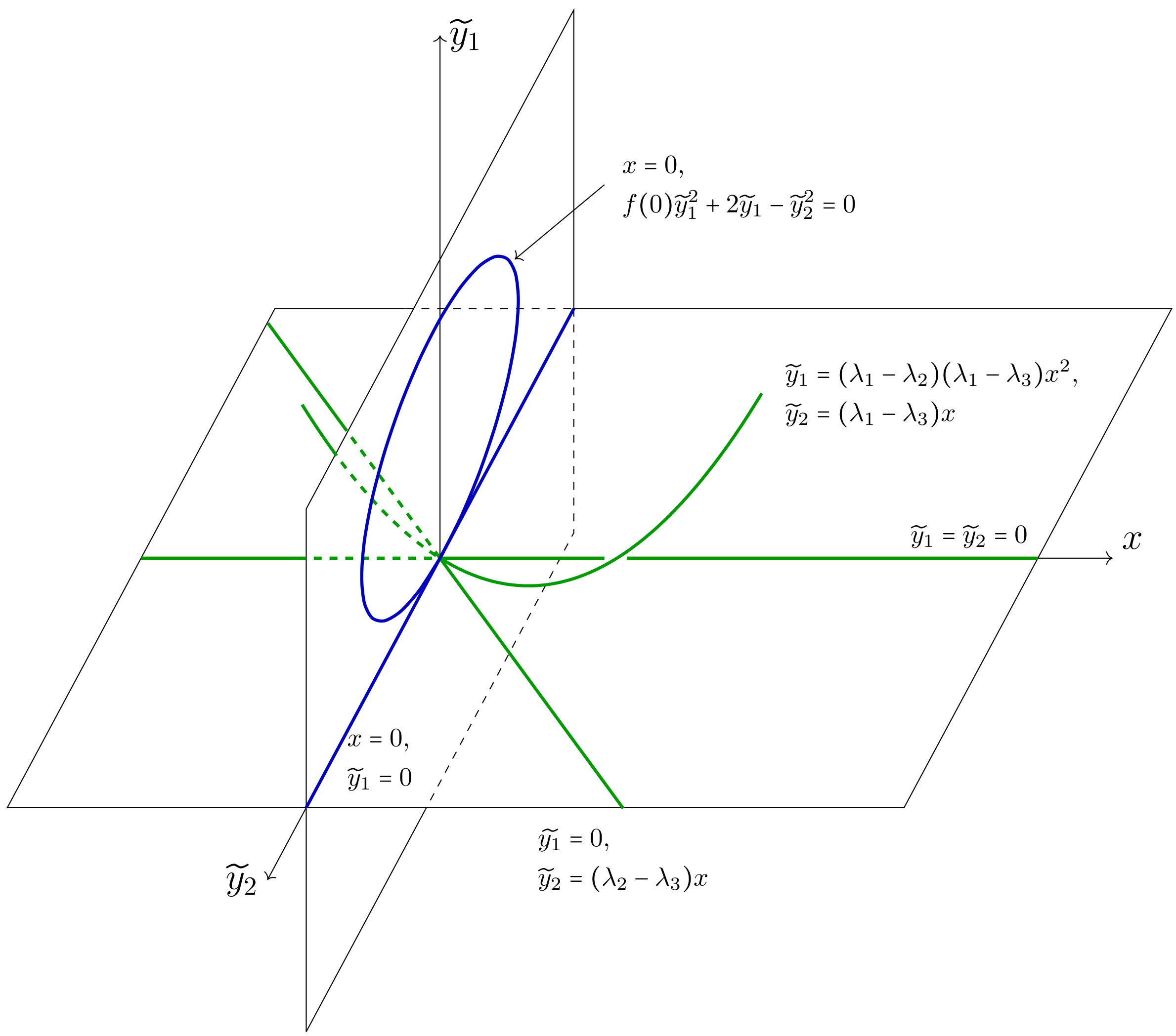
Below we list the general forms of the local Poisson lifts to $\mathbb{P}(V)$ for the non-resonant co-Higgs fields with T_{36} singularity, for the cases when they exist. We denote by y_1, y_2, y_3 the fiberwise linear coordinates on V given by the basis in which the co-Higgs fields are written.

$$x\Phi_0^{D_4} = x\begin{pmatrix}\lambda_1 x & & \\ 1 & \lambda_2 x & \\ & 1 & \lambda_3 x\end{pmatrix}\partial_x,$$

$$\pi = x\partial_x \wedge \varphi + (2\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}) \wedge \varphi + f(x)\varphi \wedge \varphi^2,$$

$$\begin{aligned}\varphi &= \langle \Phi_0^{D_4}, dx\rangle &&= (\lambda_1-\lambda_3)x\widetilde{y}_1\partial_{\widetilde{y}_1} + (\lambda_2-\lambda_3)x\widetilde{y}_2\partial_{\widetilde{y}_2} + \widetilde{y}_1\partial_{\widetilde{y}_2} - \widetilde{y}_2(\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}),\\ \varphi^2 &= \langle (\Phi_0^{D_4})^2, dx^2\rangle &&= (\lambda_1^2-\lambda_3^2)x^2\widetilde{y}_1\partial_{\widetilde{y}_1} + (\lambda_2^2-\lambda_3^2)x^2\widetilde{y}_2\partial_{\widetilde{y}_2} + (\lambda_1+\lambda_2)x\widetilde{y}_1\partial_{\widetilde{y}_2}\\ & && \quad -(\lambda_2+\lambda_3)x\widetilde{y}_2(\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}) - \widetilde{y}_1(\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}),\end{aligned}$$

where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, and f is any locally defined holomorphic function in x.



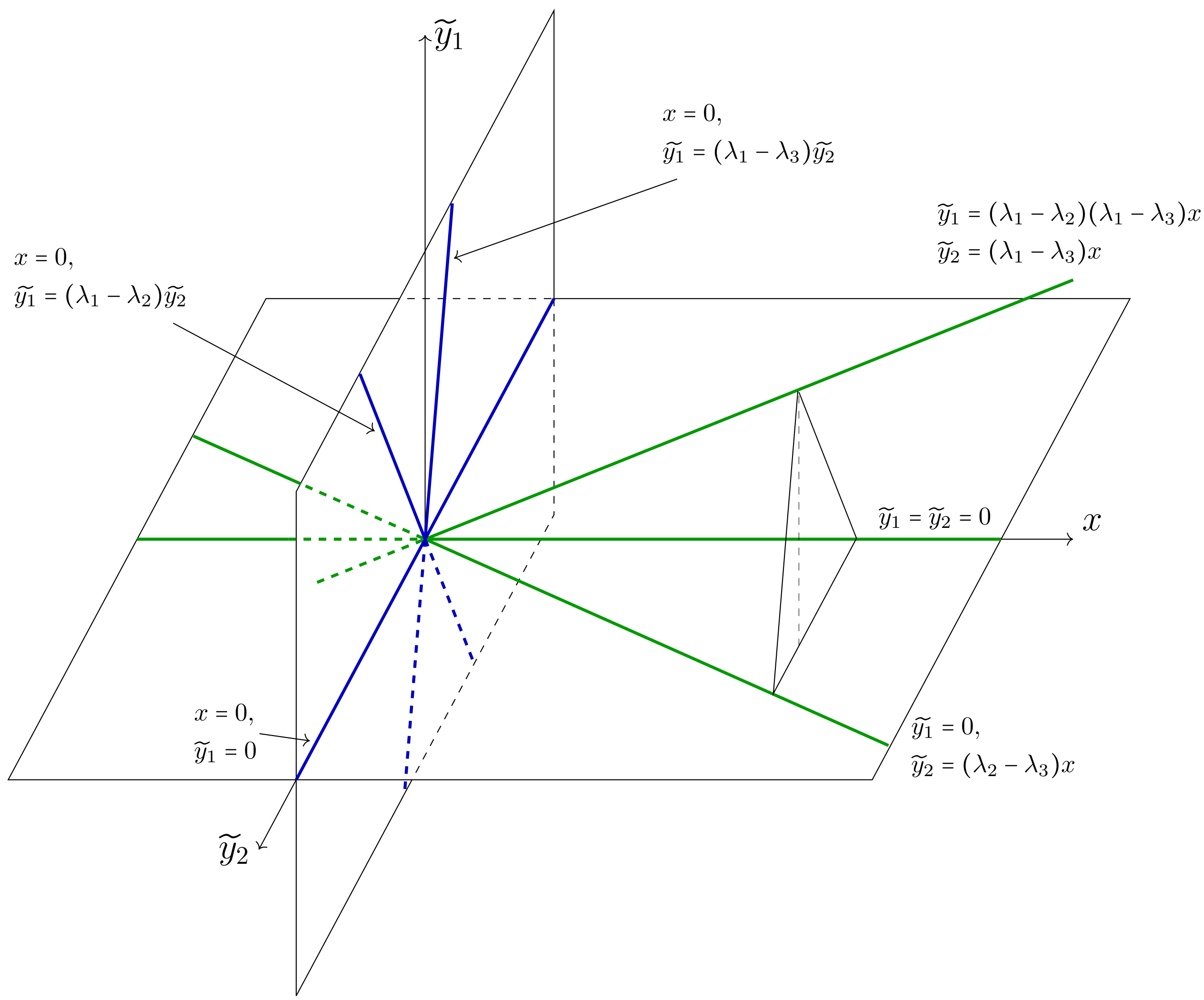
Zero set of a local Poisson lift of $x\Phi_0^{D_4}$

$$x\Phi_1^{D_4} = x\begin{pmatrix}\lambda_1 x & & \\ x & \lambda_2 x & \\ & 1 & \lambda_3 x\end{pmatrix}\partial_x,$$

$$\pi = x\partial_x \wedge \varphi + (\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}) \wedge \varphi + \frac{1}{x^2} f(x)\varphi \wedge \varphi^2,$$

$$\begin{aligned}
\varphi &= \langle \Phi_1^{D_4}, dx\rangle &&= (\lambda_1-\lambda_3)x\widetilde{y}_1\partial_{\widetilde{y}_1} + (\lambda_2-\lambda_3)x\widetilde{y}_2\partial_{\widetilde{y}_2} + x\widetilde{y}_1\partial_{\widetilde{y}_2} - \widetilde{y}_2(\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}),\\
\varphi^2 &= \langle (\Phi_1^{D_4})^2, dx^2\rangle &&= (\lambda_1^2-\lambda_3^2)x^2\widetilde{y}_1\partial_{\widetilde{y}_1} + (\lambda_2^2-\lambda_3^2)x^2\widetilde{y}_2\partial_{\widetilde{y}_2} + (\lambda_1+\lambda_2)x^2\widetilde{y}_1\partial_{\widetilde{y}_2}\\
& &&\quad -(\lambda_2+\lambda_3)x\widetilde{y}_2(\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}) - x\widetilde{y}_1(\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}),
\end{aligned}$$

where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, and f is any locally defined holomorphic function in x.



Zero set of a local Poisson lift of $x\Phi_1^{D_4}$

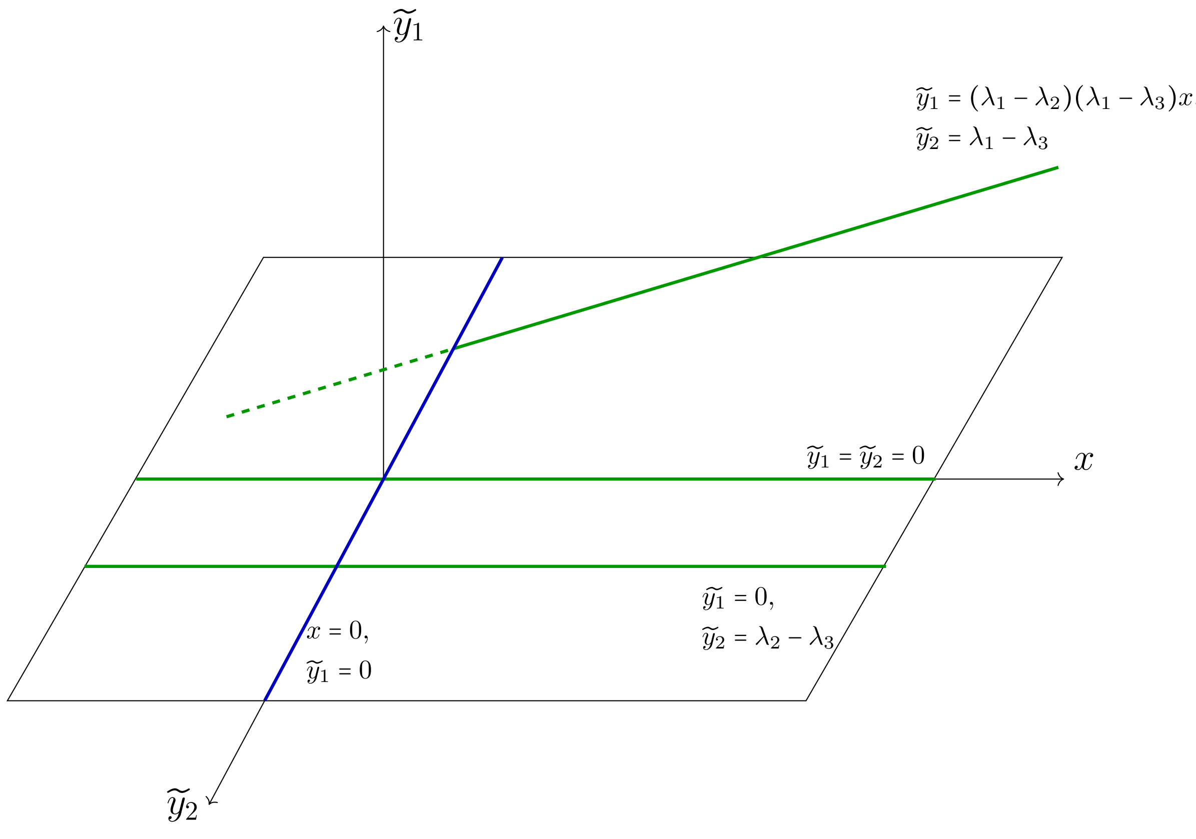
The figure is drawn under assumption $f(0) \neq 0$. If $f(0) = 0$, then additionally the plane $\{x = 0\}$ is contained in the zero set.

$$x\Phi_2^{D_4} = x\begin{pmatrix}\lambda_1 x & & \\ 1 & \lambda_2 x & \\ & x & \lambda_3 x\end{pmatrix}\partial_x,$$

$$\pi = x\partial_x \wedge \varphi + \widetilde{y}_1\partial_{\widetilde{y}_1} \wedge \varphi + \frac{1}{x}f(x)\varphi\wedge\varphi^2,$$

$$\begin{aligned}\varphi &= \langle \Phi_2^{D_4}, dx\rangle &&= (\lambda_1-\lambda_3)x\widetilde{y}_1\partial_{\widetilde{y}_1} + (\lambda_2-\lambda_3)x\widetilde{y}_2\partial_{\widetilde{y}_2} + \widetilde{y}_1\partial_{\widetilde{y}_2} - x\widetilde{y}_2(\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}),\\ \varphi^2 &= \langle (\Phi_2^{D_4})^2, dx^2\rangle &&= (\lambda_1^2-\lambda_3^2)x^2\widetilde{y}_1\partial_{\widetilde{y}_1} + (\lambda_2^2-\lambda_3^2)x^2\widetilde{y}_2\partial_{\widetilde{y}_2} + (\lambda_1+\lambda_2)x\widetilde{y}_1\partial_{\widetilde{y}_2}\\ & && -(\lambda_2+\lambda_3)x^2\widetilde{y}_2(\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}) - x\widetilde{y}_1(\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}),\end{aligned}$$

where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, and f is any locally defined holomorphic function in x.



Zero set of a local Poisson lift of $x\Phi_2^{D_4}$

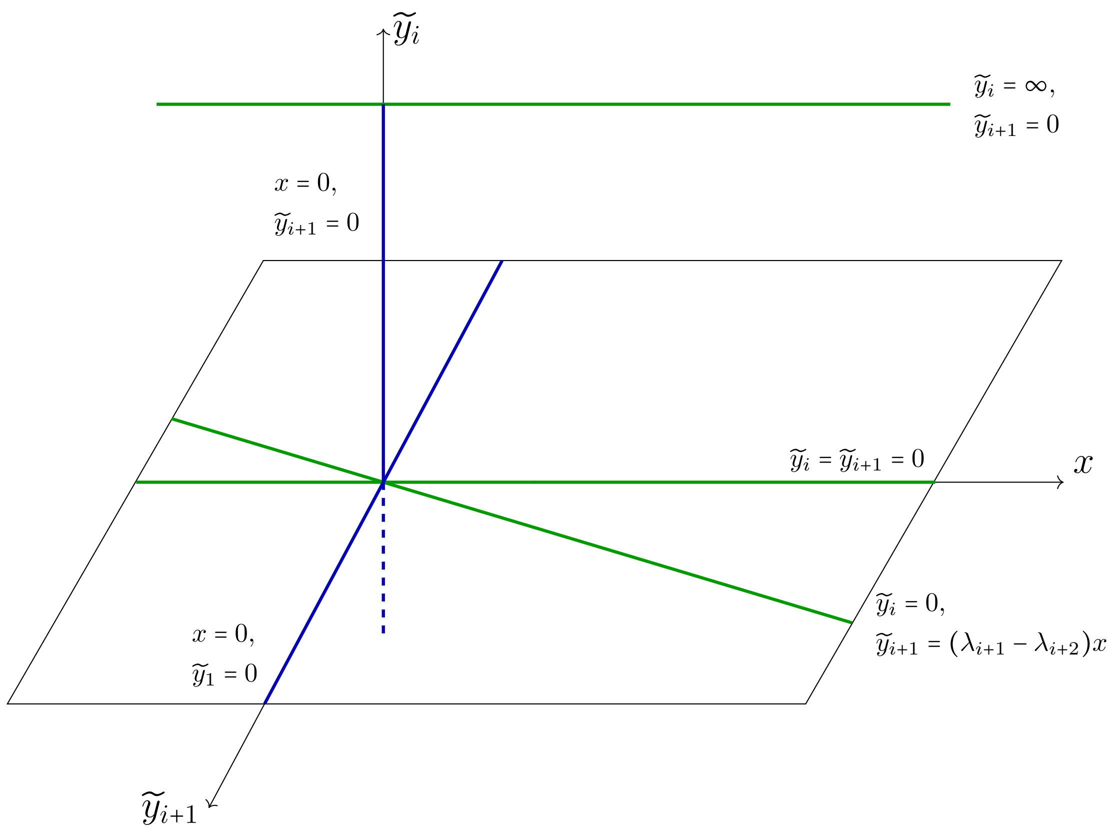
$$x\Phi^{D_4}_{2+i} = x\begin{pmatrix} \lambda_i x & & \\ & \lambda_{i+1}x & \\ & 1 & \lambda_{i+2}x \end{pmatrix}\partial_x, \quad i = 1,2,3,$$

$$\pi = x\partial_x \wedge \varphi + (\widetilde{y}_i\partial_{\widetilde{y}_i} + \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}) \wedge \varphi + \frac{1}{x^2}f(x)\varphi\wedge\varphi^2,$$

$$\varphi = \langle \Phi^{D_4}_{2+i}, dx\rangle = (\lambda_i - \lambda_{i+2})x\widetilde{y}_i\partial_{\widetilde{y}_i} + (\lambda_{i+1}-\lambda_{i+2})x\widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}} - \widetilde{y}_{i+1}(\widetilde{y}_i\partial_{\widetilde{y}_i} + \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}),$$

$$\varphi^2 = \langle (\Phi^{D_4}_{2+i})^2, dx^2\rangle = (\lambda_i^2 - \lambda_{i+2}^2)x^2\widetilde{y}_i\partial_{\widetilde{y}_i} + (\lambda_{i+1}^2-\lambda_{i+2}^2)x^2\widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}} - (\lambda_{i+1}+\lambda_{i+2})x\widetilde{y}_{i+1}(\widetilde{y}_i\partial_{\widetilde{y}_i} + \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}),$$

where $\widetilde{y}_i = \frac{y_i}{y_{i+2}}$, $\widetilde{y}_{i+1} = \frac{y_{i+1}}{y_{i+2}}$, and f is any locally defined holomorphic function in x.



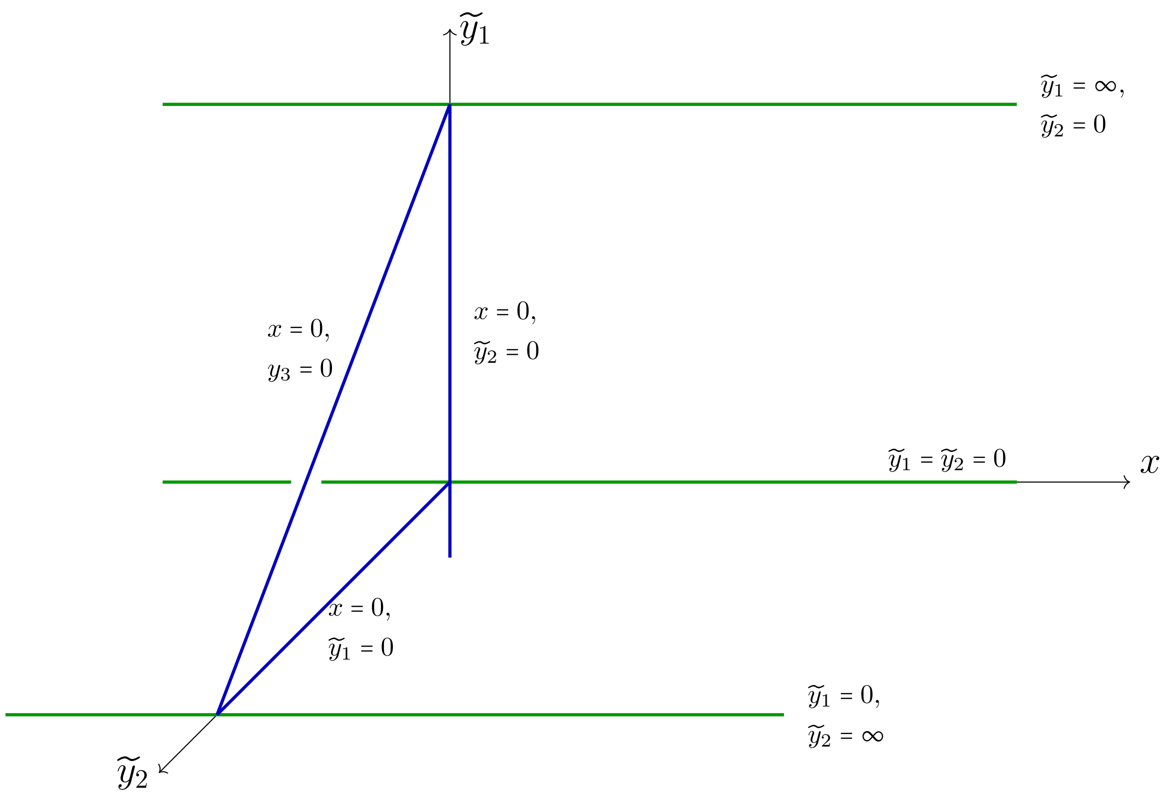
Zero set of a local Poisson lift of $x\Phi^{D_4}_{2+i}$, $i = 1,2,3$
The figure is drawn under assumption $f(0) \neq 0$.
If $f(0) = 0$, then additionally the plane $\{x = 0\}$ is contained in the zero set.

$$x\Phi_6^{D_4} = x\begin{pmatrix}\lambda_1 x & & \\ & \lambda_2 x & \\ & & \lambda_3 x\end{pmatrix}\partial_x,$$

$$\pi = x\partial_x \wedge \varphi + \frac{1}{x^3} f(x)\varphi\wedge\varphi^2,$$

$$\begin{aligned}\varphi &= \langle \Phi_6^{D_4}, dx\rangle &&= (\lambda_1-\lambda_3)x\widetilde{y}_1\partial_{\widetilde{y}_1} + (\lambda_2-\lambda_3)x\widetilde{y}_2\partial_{\widetilde{y}_2},\\ \varphi^2 &= \langle (\Phi_6^{D_4})^2, dx^2\rangle &&= (\lambda_1^2-\lambda_3^2)x^2\widetilde{y}_1\partial_{\widetilde{y}_1} + (\lambda_2^2-\lambda_3^2)x^2\widetilde{y}_2\partial_{\widetilde{y}_2},\end{aligned}$$

where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, and f is any locally defined holomorphic function in x.



Zero set of a local Poisson lift of $x\Phi_6^{D_4}$

The figure is drawn under assumption $f(0) \neq 0$.
If $f(0) = 0$, then additionally the plane $\{x = 0\}$ is contained in the zero set.

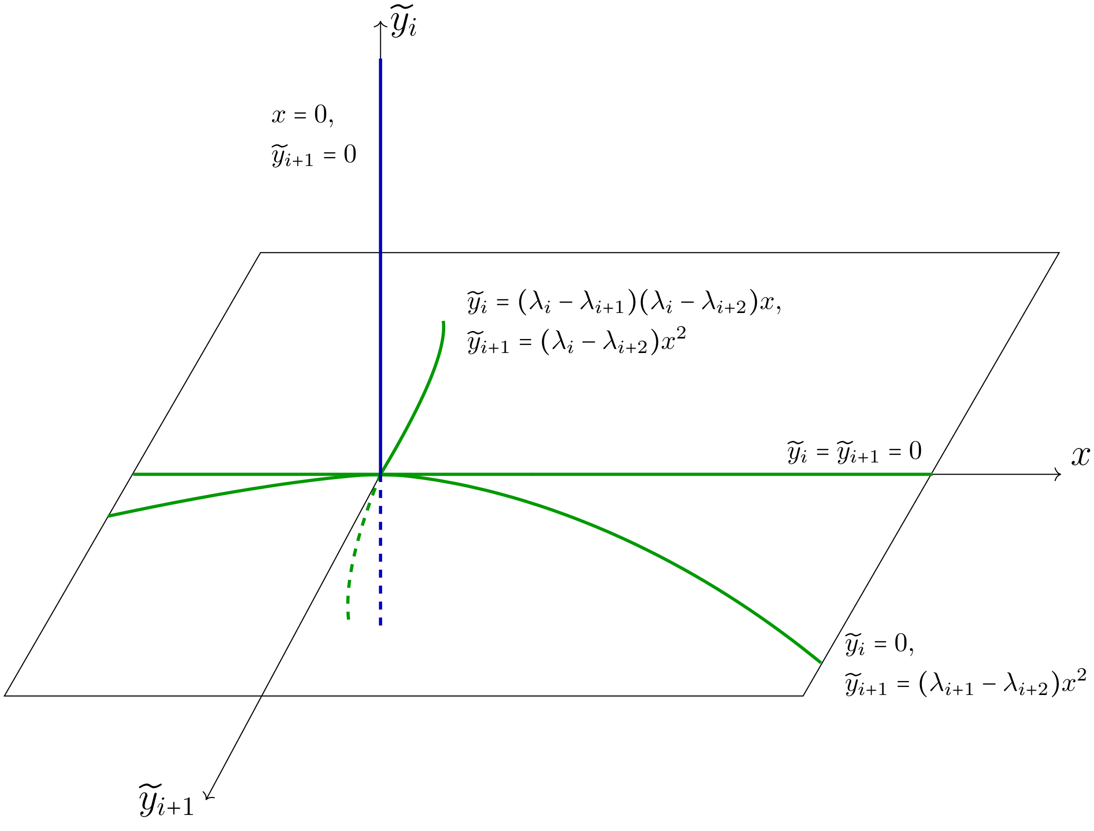
$$\Phi_i^{T_{36}} = \begin{pmatrix} \lambda_i x^2 & & \\ x^3 & \lambda_{i+1}x^2 & \\ & 1 & \lambda_{i+2}x^2 \end{pmatrix} \partial_x, \quad i = 1,2,3,$$

$$\pi = \partial_x \wedge \varphi + \frac{1}{x}(\widetilde{y}_i\partial_{\widetilde{y}_i} + 2\widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}) \wedge \varphi - \frac{1}{x^5}\frac{1}{(\lambda_i - \lambda_{i+2})(\lambda_i - \lambda_{i+1})}\varphi \wedge \varphi^2 + \frac{1}{x^4} f(x)\varphi \wedge \varphi^2,$$

$$\varphi = \langle \Phi_i^{T_{36}}, dx\rangle = (\lambda_i - \lambda_{i+2})x^2\widetilde{y}_i\partial_{\widetilde{y}_i} + (\lambda_{i+1} - \lambda_{i+2})x^2\widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}} + x^3\widetilde{y}_i\partial_{\widetilde{y}_{i+1}} - \widetilde{y}_{i+1}(\widetilde{y}_i\partial_{\widetilde{y}_i} + \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}),$$

$$\begin{aligned}\varphi^2 = \langle (\Phi_i^{T_{36}})^2, dx^2\rangle = {} & (\lambda_i^2 - \lambda_{i+2}^2)x^4\widetilde{y}_i\partial_{\widetilde{y}_i} + (\lambda_{i+1}^2 - \lambda_{i+2}^2)x^4\widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}} + (\lambda_i + \lambda_{i+1})x^5\widetilde{y}_i\partial_{\widetilde{y}_{i+1}} \\ & -(\lambda_{i+1} + \lambda_{i+2})x^2\widetilde{y}_{i+1}(\widetilde{y}_i\partial_{\widetilde{y}_i} + \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}) - x^3\widetilde{y}_i(\widetilde{y}_i\partial_{\widetilde{y}_i} + \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}),\end{aligned}$$

where $\widetilde{y}_i = \frac{y_i}{y_{i+2}}$, $\widetilde{y}_{i+1} = \frac{y_{i+1}}{y_{i+2}}$, and f is any locally defined holomorphic function in x.



Zero set of a local Poisson lift of $\Phi_i^{T_{36}}$, $i = 1,2,3$

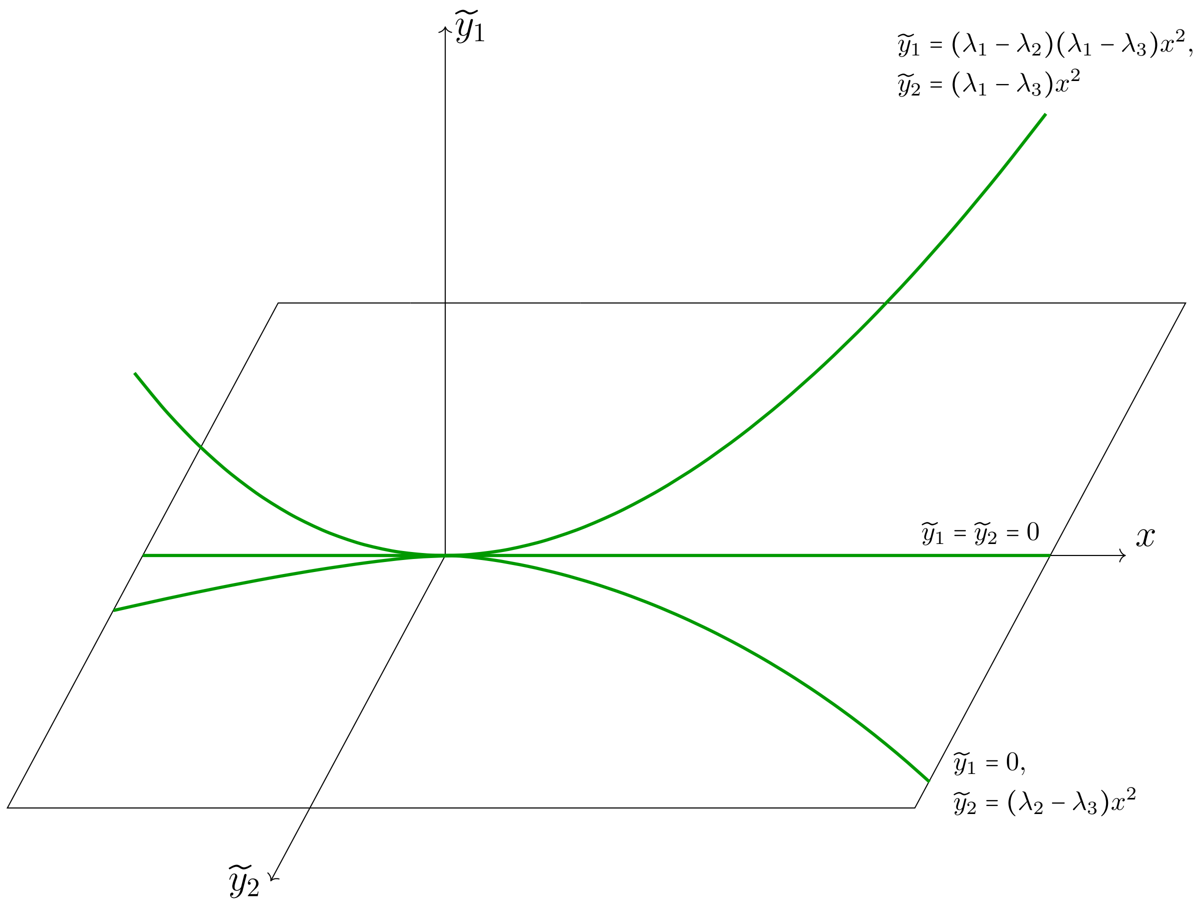
$$\Phi_7^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ x^2 & \lambda_2 x^2 & \\ & 1 & \lambda_3 x^2 \end{pmatrix} \partial_x,$$

$$\pi = \partial_x \wedge \varphi + \frac{2}{x}(\widetilde{y}_1 \partial_{\widetilde{y}_1} + \widetilde{y}_2 \partial_{\widetilde{y}_2}) \wedge \varphi + \frac{1}{x^4} f(x) \varphi \wedge \varphi^2,$$

$$\varphi = \langle \Phi_7^{T_{36}}, dx \rangle = (\lambda_1 - \lambda_3) x^2 \widetilde{y}_1 \partial_{\widetilde{y}_1} + (\lambda_2 - \lambda_3) x^2 \widetilde{y}_2 \partial_{\widetilde{y}_2} + x^2 \widetilde{y}_1 \partial_{\widetilde{y}_2} - \widetilde{y}_2 (\widetilde{y}_1 \partial_{\widetilde{y}_1} + \widetilde{y}_2 \partial_{\widetilde{y}_2}),$$

$$\begin{aligned} \varphi^2 = \langle (\Phi_7^{T_{36}})^2, dx^2 \rangle = & (\lambda_1^2 - \lambda_3^2) x^4 \widetilde{y}_1 \partial_{\widetilde{y}_1} + (\lambda_2^2 - \lambda_3^2) x^4 \widetilde{y}_2 \partial_{\widetilde{y}_2} + (\lambda_1 + \lambda_2) x^4 \widetilde{y}_1 \partial_{\widetilde{y}_2} \\ & -(\lambda_2 + \lambda_3) x^2 \widetilde{y}_2 (\widetilde{y}_1 \partial_{\widetilde{y}_1} + \widetilde{y}_2 \partial_{\widetilde{y}_2}) - x^2 \widetilde{y}_1 (\widetilde{y}_1 \partial_{\widetilde{y}_1} + \widetilde{y}_2 \partial_{\widetilde{y}_2}), \end{aligned}$$

where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, and f is any locally defined holomorphic function in x.



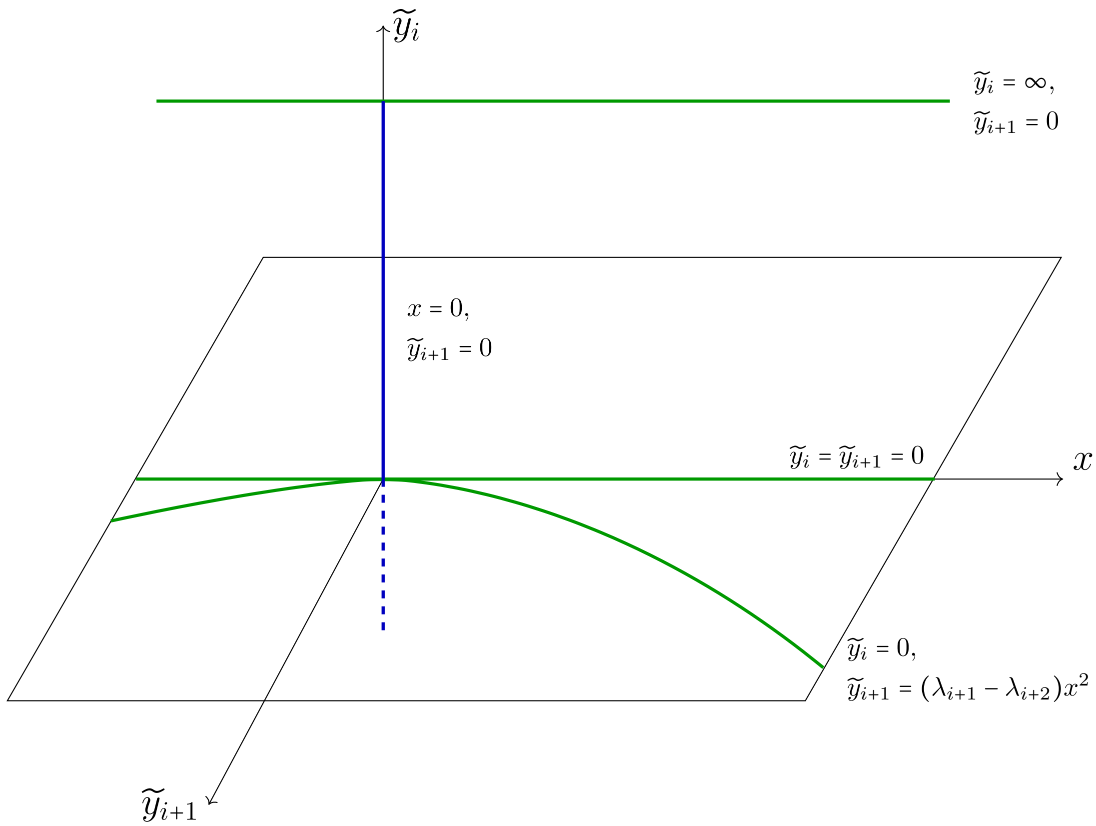
Zero set of a local Poisson lift of $\Phi_7^{T_{36}}$
The figure is drawn under assumption $f(0) \neq 0$.
If $f(0) = 0$, then additionally the line $\{x = \widetilde{y}_2 = 0\}$ is contained in the zero set.

$$\Phi^{T_{36}}_{10+i} = \begin{pmatrix} \lambda_i x^2 & & \\ & \lambda_{i+1} x^2 & \\ & 1 & \lambda_{i+2} x^2 \end{pmatrix} \partial_x, \quad i = 1, 2, 3,$$

$$\pi = \partial_x \wedge \varphi + \frac{2}{x}(\widetilde{y}_i \partial_{\widetilde{y}_i} + \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}) \wedge \varphi + \frac{1}{x^4} f(x) \varphi \wedge \varphi^2,$$

$$\begin{aligned} \varphi &= \langle \Phi^{T_{36}}_{10+i}, dx\rangle &&= (\lambda_i - \lambda_{i+2})x^2 \widetilde{y}_i \partial_{\widetilde{y}_i} + (\lambda_{i+1} - \lambda_{i+2})x^2 \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}} - \widetilde{y}_{i+1}(\widetilde{y}_i \partial_{\widetilde{y}_i} + \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}), \\ \varphi^2 &= \langle (\Phi^{T_{36}}_{10+i})^2, dx^2\rangle &&= (\lambda_i^2 - \lambda_{i+2}^2)x^4 \widetilde{y}_i \partial_{\widetilde{y}_i} + (\lambda_{i+1}^2 - \lambda_{i+2}^2)x^4 \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}} \\ & && \quad -(\lambda_{i+1} + \lambda_{i+2})x^2 \widetilde{y}_{i+1}(\widetilde{y}_i \partial_{\widetilde{y}_i} + \widetilde{y}_{i+1}\partial_{\widetilde{y}_{i+1}}), \end{aligned}$$

where $\widetilde{y}_i = \frac{y_i}{y_{i+2}}$, $\widetilde{y}_{i+1} = \frac{y_{i+1}}{y_{i+2}}$, and f is any locally defined holomorphic function in x.



Zero set of a local Poisson lift of $\Phi^{T_36}_{10+i}$, $i = 1, 2, 3$

4.3.2 Local Poisson lifts. Resonant case

The goal of this subsection is to construct Poisson $\mathbb{P}^2$-bundles over a small open set $0 \in \mathcal{U} \subset \mathbb{C}$ out of rank 3 co-Higgs bundles over $\mathcal{U}$ that are resonant (Definition 4.3.3). The main tool for this is the pencils, that is one parameter families of divisors. Recall that a pencil on a manifold X is a surjective map $X \dashrightarrow \mathbb{P}^1$ that is allowed to be ill-defined on a codimension 2 subset B of X. The set B is called base locus of the pencil, and preimages of a points $x \in \mathbb{P}^1$ are called members of the pencil.

Lemma 4.3.14. *Let $\mathcal{U}$ be an analytic open subset of a smooth projective curve, and (V, ϕ) be a traceless rank 3 co-Higgs bundle over $\mathcal{U}$ that is strongly integrable, resonant. Then there is a $\mathbb{P}^1$-bundle Q over $\mathcal{U}$ and a fiber bundle map $F : \mathbb{P}(V) \dashrightarrow Q$, that is allowed to be ill-defined on a codimension 2 subset, such that for any $x \in \mathcal{U}$, $\alpha \in \mathcal{T}^*_x$, the vector field $\langle \phi_x, \alpha \rangle$ is tangent to the preimages $F^{-1}(q)$, $q \in Q|_x$, and the latter preimages are quadrics inside $\mathbb{P}(V)|_x$. We are going to call Q a family of pencils of quadrics on fibers of $\mathbb{P}(V)$.*

Proof. The rank 6 vector bundle S^2V^* carries the co-Higgs field $Lie_\phi \in \mathrm{Hom}(S^2V^*, S^2V^* \otimes \mathcal{T}_\mathcal{U})$ given by $Lie_\phi(s_1 \odot s_2) = \phi^*(s_1) \odot s_2 + s_1 \odot \phi^*(s_2)$, where $\phi^* \in \mathrm{Hom}(V^*, V^* \otimes \mathcal{T}_\mathcal{U})$ is the dual co-Higgs (one can think of Lie_ϕ as differentiation of the fiberwise quadric along the fiberwise linear vector field given by ϕ). Let K be the kernel of the sheaf morphism $Lie_\phi : S^2V^* \to S^2V^* \otimes \mathcal{T}_\mathcal{U}$. Being a subsheaf of a vector bundle over a curve, K itself has to be a vector bundle. To determine the rank of K it is enough to look at a small open subset $\widetilde{\mathcal{U}} \subset \mathcal{U}$ where ϕ is diagonalizable. Let us choose a coordinate x on $\widetilde{\mathcal{U}}$ and a trivializaiton of V over $\widetilde{\mathcal{U}}$, so that $\phi = \mathrm{diag}(0, \lambda\partial_x, -\lambda\partial_x)$, $\lambda \in \mathbb{C} \setminus \{0\}$. Then in the fiberwise linear coordinates y_1, y_2, y_3 given by the trivialiation of V, the action of Lie_ϕ become the Lie derivative with respect to the vector field $\lambda y_2\partial_{y_2} - \lambda y_3 \partial_{y_3}$. In these coordinates, the sheaf K has two generators y_1^2 and $y_2 y_3$, and therefore has rank 2.

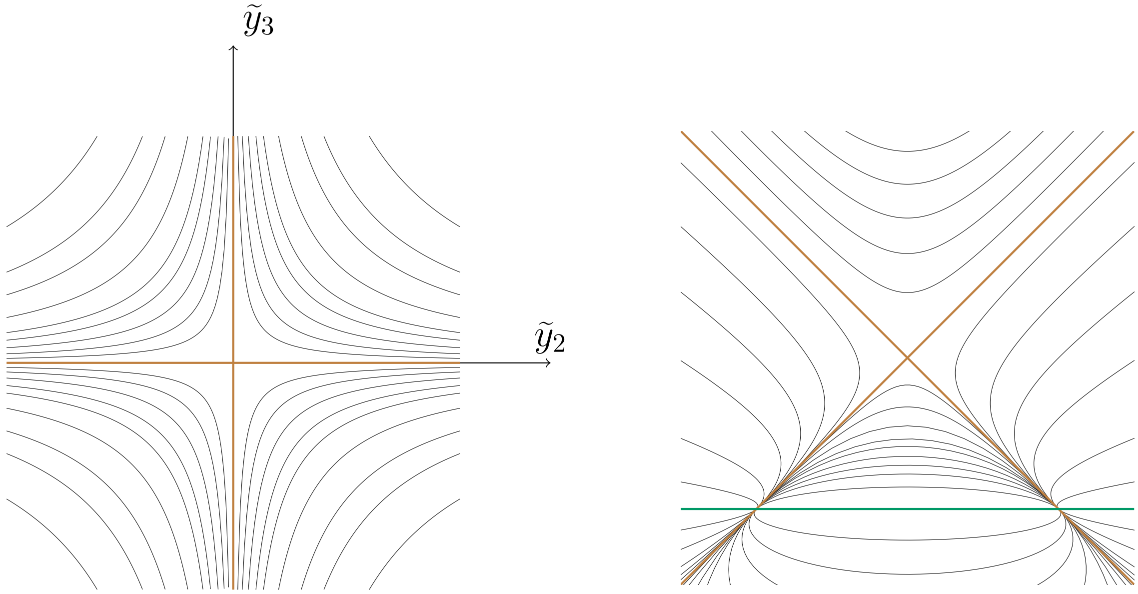
The projectivization $Q = \mathbb{P}(K) \subset \mathbb{P}(S^2V^*)$ is the desired family of pencils of quadrics. □

Note the the kernel K in the proof of Lemma 4.3.14 contains a distinguished line subbundle $\ker(\phi^* : V^* \to V^* \otimes \mathcal{T}_\mathcal{U})^{\otimes 2}$. Therefore, the $\mathbb{P}^1$-bundle Q contains a distinguished section $\mathbb{P}(\ker(\phi^*)^{\otimes 2})$.

Recall that a generic pencil of quadrics on $\mathbb{P}^2$ has three singular members, each of which is a union of two straight lines intersecting at a point. We remark that even on a generic fiber of $\mathbb{P}(V)$ the obtained pencil of quadric $\mu y_1^2 + \nu y_2 y_3 = 0$, $[\mu : \nu] \in \mathbb{P}^1$, is not a generic pencil of quadrics. Instead, it has one singular member $y_2 y_3 = 0$ and one non-reduced member $y_1^2 = 0$. A convenient way to depict this pencil would be to choose the affine chart with coordinates $\widetilde{y}_2 = \frac{y_2}{y_1}$, $\widetilde{y}_3 = \frac{y_3}{y_1}$ (Figure 4.1a). However, that would leave out the base locus $\{[0:1:0], [0:0:1]\}$. Sometimes we will want to visualize the pencil of quadrics in such a way that the two points of base locus and the singular point of the unique reduced singular member of the pencil are all visible (Figure 4.1b).

At a point $x_0 \in \mathbb{P}^1$ where the co-Higgs field ϕ has repeated eigenvalues, they all have to be zero due to strong integrability of ϕ, and the pencil of fiberwise quadrics built in Lemma 4.3.14 will degenerate further. If rank of ϕ_{x_0} is zero, then one can express $\phi = (x - x_0)^s \psi$, for some $s \geq 1$ and another strongly integrable, resonant co-Higgs field ψ. One can replace ϕ with ψ without changing the family Q of fiberwise pencils of quadrics, and thus reduce discussion to the case of rank ϕ_{x_0} being two/one.

If rank of ϕ_{x_0} is two, then in appropriate basis of V one has



(a) In the chart $\widetilde{y}_2 = \frac{y_2}{y_1}$, $\widetilde{y}_3 = \frac{y_3}{y_1}$.

(b) In a chart that contains the base locus and the singular point of $y_2 y_3 = 0$.

Figure 4.1: Pencil of fiberwise quadrics $\mu y_1^2 + \nu y_2 y_3 = 0$, $[\mu : \nu] \in \mathbb{P}^1$.

$$\phi_{x_0} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \partial_x.$$

Then the action of Lie_ϕ on $S^2 V^*$ corresponds to the Lie derivative along $y_1 \partial_{y_2} + y_2 \partial_{y_3}$, where y_1, y_2, y_3 are the fiberwise linear coordinates on V in which the matrix ϕ_{x_0} above is written. Then Q is the pencil $\mu y_1^2 + \nu(y_2^2 - 2y_1 y_3) = 0$, $[\mu : \nu] \in \mathbb{P}^1$ (Figure 4.2a).

If rank of ϕ_{x_0} is one, then in appropriate basis one has

$$\phi_{x_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \partial_x.$$

Then the action of Lie_ϕ on $S^2 V^*$ corresponds to the Lie derivative along $y_2 \partial_{y_3}$, where y_1, y_2, y_3 are the fiberwise linear coordinates on V in which the matrix ϕ_{x_0} above is written. In this case, the pencil Q over $x = x_0$ is not completely determined by the value of ϕ_{x_0}, however one can tell that each member of the pencil has to be of the form $(\alpha_1 y_1 - \beta_1 y_2)(\alpha_2 y_1 - \beta_2 y_2) = 0$, for some $[\alpha_1 : \beta_1], [\alpha_2 : \beta_2] \in \mathbb{P}^1$. In other words, over $x = x_0$ each member of the pencil Q is a union of two lines, which may or may not coincide (Figure 4.2b). Moreover, the number of double lines in such a pencil equals either two or one. After a change of basis, in the former case one obtains the pencil $\mu y_1^2 + \nu y_2^2 = 0$, $[\mu : \nu] \in \mathbb{P}^1$, while in the latter case one obtains $\mu y_2^2 + \nu y_1 y_2 = 0$, $[\mu : \nu] \in \mathbb{P}^1$.

Recall that a connection on a fiber bundle $p : X \to B$, is a bundle map $j : p^* \mathcal{T}_B \to \mathcal{T}_X$ such that $Tp(j(p^* v)) = v$, for each $v \in \mathcal{T}_B$. A connection j is called flat, if the distribution $\mathrm{Im} j \subset \mathcal{T}_X$ is Frobenius integrable. If $\dim B = 1$, then any connection on a fiber bundle over B is automatically flat.

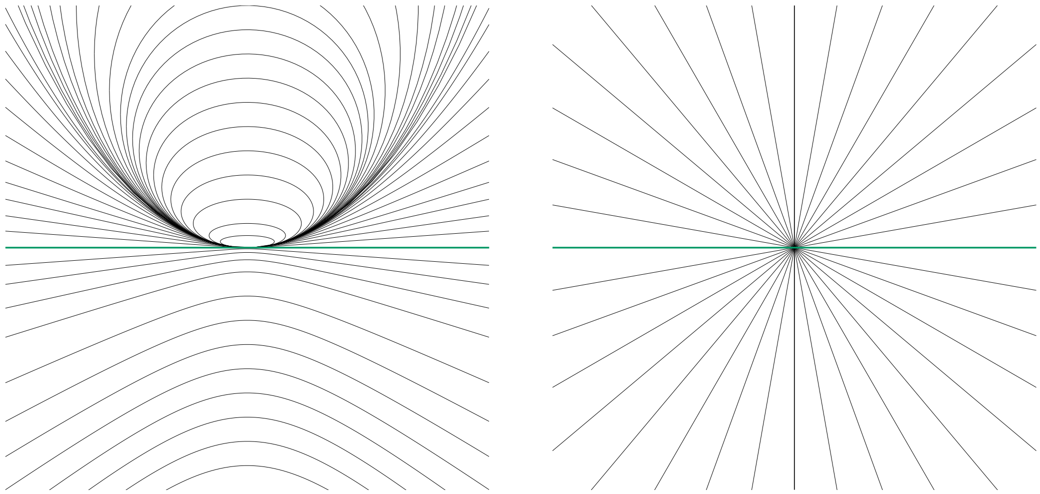

(a) At a point where ϕ is nilpotent of rank 2. (b) At a point where ϕ is nilpotent of rank 1.

Figure 4.2: Degenerations of the fiberwise pencil of quadrics

Proposition 4.3.15. *Let $\mathcal{U}$ be a simply connected subset of $\mathbb{C}$, V be a vector bundle over $\mathcal{U}$, and $\phi \in \operatorname{Hom}(V, V \otimes \mathcal{T}_{\mathcal{U}})$ be a traceless strongly integrable, resonant co-Higgs field on V that does not vanish at any point of $\mathcal{U}$. Let Q be the $\mathbb{P}^1$-bundle over $\mathcal{U}$ constructed in Lemma 4.3.14.*

Then a lift of ϕ to a Poisson structure π on $\mathbb{P}(V)$ uniquely determines a connection on the $\mathbb{P}^1$-bundle Q that is tangent to the distinguished section $\mathbb{P}(\ker(\phi^*)^{\otimes 2})$, and vice versa, a connection on Q tangent to $\mathbb{P}(\ker(\phi^*)^{\otimes 2})$ uniquely determines a Poisson structure π on $\mathbb{P}(V)$ lifting ϕ.

This connection is uniquely determined by the property that any flat local section q of Q, when viewed as a meromorphic function on $\mathbb{P}(V)$, is Casimir with respect to π.

Proof. Let π be a Poisson structure on $\mathbb{P}(V)$. Let σ be a unimodular quadratic Poisson structure on V lifting π, whose extistence is guaranteed by Theorem 3.1.5. As in the proof of Lemma 4.3.14, let $K = \ker(Lie_\phi : S^2V^* \to S^2V^* \otimes \mathcal{T}_{\mathcal{U}})$, and also let $K_1 = \ker(\phi^* : V^* \to V^* \otimes \mathcal{T}_{\mathcal{U}})^{\otimes 2} \subset K$. We claim that σ-Hamiltonian vector fields of any local section of K (resp. K_1) is a vertical cubic vector field, and when viewed as a section of $S^3V^* \otimes V$ it lies inside the rank 2 subbundle $K \otimes \langle\varphi\rangle$ (resp. rank 1 subbundle $K_1 \otimes \langle\varphi\rangle$), where $\langle\varphi\rangle$ is the line subbundle of $V^* \otimes V$ spanned by the vertical vector field $\varphi = \langle\phi, \alpha\rangle$, for a non-vanihsing $\alpha \in \mathcal{T}^*_{\mathcal{U}}$. It is enough to check the claim on an open subset $\widetilde{\mathcal{U}} \subset \mathcal{U}$ where ϕ is diagonalizable. Let x, y_1, y_2, y_3 be coordinates on $V|_{\widetilde{\mathcal{U}}}$ such that $\phi = (y_2\partial_{y_2} - y_3\partial_{y_3}) \otimes \partial_x$. Then $K = \langle y_1^2, y_2y_3\rangle \mathcal{O}_{\widetilde{\mathcal{U}}} \subset S^2V^*$, and $K_1 = \langle y_1^2\rangle \mathcal{O}_{\widetilde{\mathcal{U}}} \subset S^2V^*$. The Poisson structure π, by Lemma 4.3.4.2), must be of the form

$$\pi = \partial_x \wedge (\widetilde{y_2}\partial_{\widetilde{y_2}} - \widetilde{y_3}\partial_{\widetilde{y_3}}) + f(x)\widetilde{y_2}\partial_{\widetilde{y_2}} \wedge \widetilde{y_3}\partial_{\widetilde{y_3}} + g(x)\partial_{\widetilde{y_2}} \wedge \partial_{\widetilde{y_3}},$$

where $f(x)$, $g(x)$ are any functions in $\mathcal{O}_{\widetilde{\mathcal{U}}}$. Then

$$\sigma = \partial_x \wedge (y_2\partial_{y_2} - y_3\partial_{y_3}) + f(x)y_2\partial_{y_2} \wedge y_3\partial_{y_3} + g(x)y_1^2\partial_{y_2} \wedge \partial_{y_3} + \frac{1}{3}\left(f(x) + \frac{h'(x)}{h(x)}\right) Eul \wedge (y_2\partial_{y_2} - y_3\partial_{y_3}),$$

where $h \in \mathcal{O}_{\widetilde{\mathcal{U}}}$ is such that the volume form $h(x)dx \wedge dy_1 \wedge dy_2 \wedge dy_3$ is rendering σ unimodular.

Using the expression for σ we easily check the claim:

$$\sigma^{\#}(d(k(x)y_1^2)) = \left(\left(\frac{2}{3}f(x) + \frac{2}{3}\frac{h'(x)}{h(x)}\right)k(x) + k'(x)\right)y_1^2(y_2\partial_{y_2} - y_3\partial_{y_3}) \in K_1 \otimes \langle\varphi\rangle,$$

$$\sigma^{\#}(d(k(x)y_2y_3)) = \left(\left(-\frac{1}{3}f(x)y_2y_3 - g(x)y_1^2 + \frac{2}{3}\frac{h'(x)}{h(x)}y_2y_3\right)k(x) + k'(x)y_2y_3\right)(y_2\partial_{y_2} - y_3\partial_{y_3}) \in K \otimes \langle\varphi\rangle.$$

We have shown that if s is section of K_1, then $\sigma^{\#}(ds) = k_1(x)s\varphi$. Replacing s with $\exp(-\int k_1(x)dx)s$, we can assume $\sigma^{\#}(ds) = 0$. Let us choose a section t of K that is linearly independent from s everywhere in $\mathcal{U}$. We have shown that $\sigma^{\#}(dt) = (k_2(x)s + k_3(x)t)\varphi$. Replacing t with $\exp(-\int k_3(x)dx)t$ we can assume that $\sigma^{\#}(dt) = k_2(x)s\varphi$. Further replacing t with $t-(\int k_2(x)dx)s$, we can assume that $\sigma^{\#}(dt) = 0$. Therefore, we can find two linearly independent sections $s \in H^0(\mathcal{U}, K_1) \subset H^0(\mathcal{U}, K)$ and $t \in H^0(\mathcal{U}, K)$ such that $\sigma^{\#}(ds) = \sigma^{\#}(dt) = 0$. In other words, s and t are two quadratic Casimir funtions for σ. The level sets of the rational function t/s define trivialization of the bundle $Q = \mathbb{P}(K)$, which uniquely defines a connection on Q. Note that the level sets of t/s are submanifolds of $\mathbb{P}(V)$ that are Poisson with respect to π, the projectivization of σ.

Going in the opposite direction, assume that the $\mathbb{P}^1$-bundle Q has a connection tangent to the section $\mathbb{P}(\ker(\phi^*)^{\otimes 2})$. Let us define a Poisson structure π on $\mathbb{P}(V)$ lifting ϕ such that for any flat (with respect to the connection) section q of Q the preimage of q under the rational map $\mathbb{P}(V) \dashrightarrow Q$ is Poisson with respect to π. Since $\mathcal{U}$ is simply connected, the connection on Q defines a trivialization of Q. Let us choose two linearly independent sections $s \in H^0(\mathcal{U}, K_1) \subset H^0(\mathcal{U}, K)$, $t \in H^0(\mathcal{U}, K)$ whose projectivizations are flat section of $Q = \mathbb{P}(K)$. The section t/s viewed as meromorphic function on $\mathbb{P}(V)$ has zero of order two along the divisor given by the rank 2 subbundle $\ker(\phi)$ of V, and a simple pole along the divisor $\ker(t : S^2V \to \mathcal{O}_{\mathcal{U}})$. The former divisor intersects each fiber of $\mathbb{P}(V)$ at a divisor of degree 1 (i.e. straight line), while the latter divisor intersects each fiber of $\mathbb{P}(V)$ at a divisor of degree 2 (i.e. fiberwise quadric). Therefore, one can choose a trivector τ on $\mathbb{P}(V)$ having simple zeros precisely along the zero-pole divisor of t/s. Then the bivector $\pi_1 = \iota_{d\log(t/s)}\tau$ is smooth everywhere, and defines a Poisson structure on $\mathbb{P}(V)$. We claim that the Poisson structure π_1 lifts a co-Higgs field $\phi_1 = h\phi$, for some nowhere vanishing $h \in \mathcal{O}_{\mathcal{U}}$ (function h depends on the chosen trivector τ). In order to check that the co-Higgs field of π_1 is a multiple of ϕ, it is enough to consider an open subset $\widetilde{\mathcal{U}}$ where ϕ is diagonalizable. Let x, y_1, y_2, y_3 be coordinates on $V|_{\widetilde{\mathcal{U}}}$ such that $\phi = (y_2\partial_{y_2} - y_3\partial_{y_3}) \otimes \partial_x$. Then $s = k_1(x)y_1^2$, $t = k_2(x)y_1^2 + k_3(x)y_2y_3$, for some $k_1, k_3 \in \mathcal{O}^*_{\mathcal{U}}$, $k_2 \in \mathcal{O}_{\mathcal{U}}$. Then $\tau = h_1(x)k_1(x)(k_2(x) + k_3(x)\widetilde{y}_2\widetilde{y}_3)\partial_x \wedge \partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3}$ for some $h_1 \in \mathcal{O}^*_{\mathcal{U}}$, where $\widetilde{y}_2 = \frac{y_2}{y_1}$, $\widetilde{y}_3 = \frac{y_3}{y_1}$.

$$\pi_1 = \iota_{d\log(t/s)}\tau = \frac{1}{s^2}\iota_{sdt-tds}\tau = h_1(x)k_1'(x)(k_2(x) + k_3(x)\widetilde{y}_2\widetilde{y}_3)\partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3} +$$

$$+ h_1(x)k_1(x)(k_2'(x) + k_3'(x)\widetilde{y}_2\widetilde{y}_3)\partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3} + h_1(x)k_1(x)k_3(x)\partial_x \wedge (\widetilde{y}_2\partial_{\widetilde{y}_2} - \widetilde{y}_3\partial_{\widetilde{y}_3}).$$

This shows that the co-Higgs field of π_1 is $h\phi$, where $h = h_1k_1k_3$. The fact that h vanishes nowhere (even at the branch points of the spectral cover of ϕ) follows from the fact that s and t are linearly independent at each fiber. Since π_1 is a Poisson structure on $\mathbb{P}(V)$ lifting $h\phi$, Lemma 4.3.5 implies that $\pi = \frac{1}{h}\pi_1$ is a Poisson structure on $\mathbb{P}(V)$ lifting ϕ.

□

Corollary 4.3.16. *Let $\phi \in \operatorname{Hom}(V, V \otimes \mathcal{T}_{\mathcal{U}})$ be a traceless strongly integrable, resonant co-Higgs field on a rank 3 vector bundle V over a simply connected open $\mathcal{U} \subset \mathbb{C}$. Then ϕ admits a lift to a Poisson structure on $\mathbb{P}(V)$.*

Proof. Let $\phi = h\phi_1$, where $h \in \mathcal{O}_{\mathcal{U}}$ and $\phi_1 \in \operatorname{Hom}(V, V \otimes \mathcal{T}_{\mathcal{U}})$ does not vanish at any point of $\mathcal{U}$. The co-Higgs field ϕ_1 is again strongly integrable and resonant. By Proposition 4.3.15, to find a Poisson lift π_1 of ϕ_1, it is enough to find a connection on the $\mathbb{P}^1$-bundle Q tangent to the section $\mathbb{P}(\ker(\phi_1^*)^{\otimes 2})$. Since $\mathcal{U} \subset \mathbb{C}$ is simply connected, such a connection exists for any $\mathbb{P}^1$-bundle over $\mathcal{U}$. Hence Proposition 4.3.15 guarantees existence of a Poisson lift π_1 of ϕ_1, and then Lemma 4.3.5 implies that $\pi = h\pi_1$ is a Poisson lift of ϕ. □

Let us discuss how the described pencil technique works over a set $\mathcal{U}$ where the strongly integrable, resonant co-Higgs field ϕ is diagonalizable. Let x, y_1, y_2, y_3 be coordinates on $V|_{\widetilde{\mathcal{U}}}$ such that $\phi = (y_2\partial_{y_2} - y_3\partial_{y_3}) \otimes \partial_x$. The projective bundle $Q \subset \mathbb{P}(S^2V^*)$ is spanned by two sections y_1^2, y_2y_3 of S^2V^*. Let Q have a connection whose flat sections are projectivizations of $s = k(x)y_1^2$ and $t = l(x)y_1^2 + y_2y_3$, $k \in \mathcal{O}^*_{\mathcal{U}}$, $l \in \mathcal{O}_{\mathcal{U}}$ (and any linear combinations of these two sections). Let $\alpha = d\log(t/s)$, and τ be a trivector on $\mathbb{P}(V)$ having simple zeros along the divisors $\{y_1 = 0\}$ and $\{t = 0\}$. In coordinates $\widetilde{y}_2 = \frac{y_2}{y_1}$, $\widetilde{y}_3 = \frac{y_3}{y_1}$, one has

$$\alpha = d\log\left(l(x) + \widetilde{y}_2\widetilde{y}_3\right) - d\log(k(x)),$$

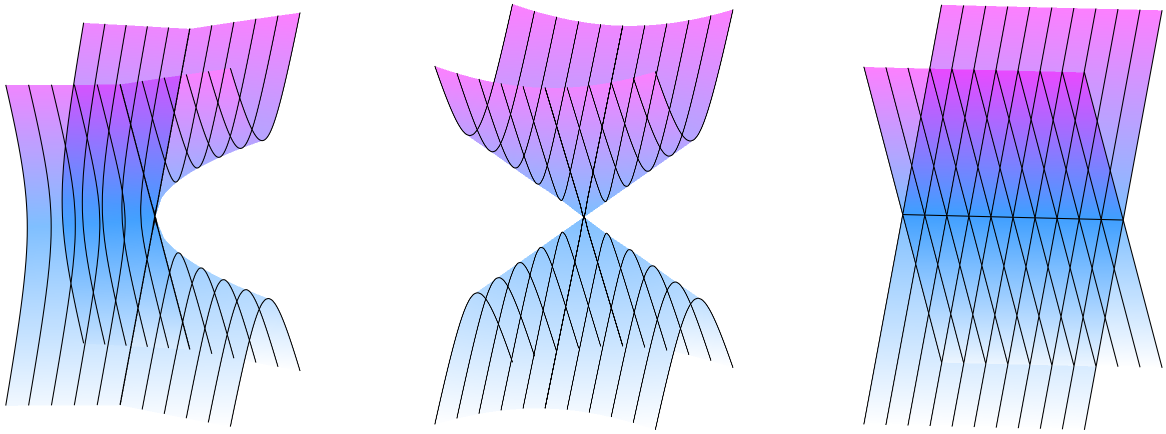
$$\tau = (l(x) + \widetilde{y}_2\widetilde{y}_3)\partial_x \wedge \partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3}.$$

Then the Poisson bivector $\pi = \iota_\alpha(\tau)$ has the expression

$$\pi = \partial_x \wedge (\widetilde{y}_2\partial_{\widetilde{y}_2} - \widetilde{y}_3\partial_{\widetilde{y}_3}) + l'(x)\partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3} - \frac{k'(x)}{k(x)}(l(x) + \widetilde{y}_2\widetilde{y}_3)\partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3}.$$

One can see that π vanishes on the two branches $[0:1:0]$ and $[0:0:1]$ of the eigen-variety of ϕ. As for the third branch $[1:0:0]$ (the one corresponding to the zero eigenvalue), π typically will not vanish there. It will vanish however over the points x_0 such that $\left(\frac{l}{k}\right)'(x_0) = 0$. Invariant way to say this is that π vanishes on the unique singular point of the unique singular quadric of the fiberwise pencil Q_{x_0} if and only if the connection on Q constructed in Proposition 4.3.15 is tangent at $x = x_0$ to section q_{sing} of Q, whose value over x is the unique singular quadric of Q_x. (If π does not vanish at the branch $[1:0:0]$ of the eigen-variety of ϕ, then this branch still has Poisson-geometric meaning. Specifically, the three branches of the eigen-variety of ϕ contain all the points where a fiber of $\mathbb{P}(V)$ is tangent to the symplectic foliation of π.)

Two dimensional leaves of the constructed Poisson lift π intersect each fiber $\mathbb{P}(V|_{x_0})$ along a quadric from the fiberwise family Q_{x_0}. If this quadric is smooth, then the symplectic leaf near the fiber $\mathbb{P}(V|_{x_0})$ will $\mathbb{C}^*$-fiber bundle over the base $\mathcal{U}$. On the other hand, there are special two dimensional symplectic leaves that for some x_0 intersect $\mathbb{P}(V_{x_0})$ at the unique singular quadric $\{y_2y_3 = 0\}$. The geometry of such symplectic leaf depends on the interplay between the projective connection on Q and the section q_{sing} of Q consisting of the singular quadrics (see Figure 4.3 describing different scenarios).



(a) Near a point where the projective connection on Q is transverse to q_{sing}

(b) Near a point where the projective connection on Q is tangent to q_{sing}

(c) If the projective connection on Q is tangent to q_{sing} everywhere

Figure 4.3: Poisson lift of a diagonal resonant co-Higgs field.
Symplectic leaves intersecting a singular fiberwise quadric

Let us now discuss the local Poisson lifts of traceless resonant, strongly integrable co-Higgs bundles ϕ of rank 3 over $\mathbb{P}^1$, near the singularities of the spectral curve. The spectral curve of such ϕ must be of the form $\{\theta(\theta^2 - w) = 0\}$, where w is a symmmetric bivector on $\mathbb{P}^1$. Since, $S^2\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(4)$, the tensor w has 4 zeros counting multiplicities. Therefore, we have the following possible singularities for such a spectral curve

(A_1) $y(y^2 - x) = 0,$

(D_4) $y(y - x)(y + x) = 0,$

(E_7) $y(y^2 - x^3) = 0,$

(T_{36}) $y(y - x^2)(y + x^2) = 0.$

A_1 **singularity.**

$$\Phi_0^{A_1} = \left(\begin{array}{c|cc} 0 & & \\ \hline 1 & & x \\ & 1 & \end{array}\right)\partial_x,$$

Vertical linear vector field: $\varphi = y_1\partial_{y_2} + y_2\partial_{y_3} + xy_3\partial_{y_2}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2y_1y_3 - xy_3^2$.
Poisson lift to $\mathbb{P}(V)$: $\pi = \partial_x \wedge \varphi + \frac{1}{2}\widetilde{y}_1\partial_{\widetilde{y}_1} \wedge \partial_{\widetilde{y}_2} + f(x)\varphi \wedge \varphi^2 + g(x)\widetilde{y}_1^3\partial_{\widetilde{y}_1} \wedge \partial_{\widetilde{y}_2}$,
where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, and f, g are holomorphic function in x defined near $x = 0$.

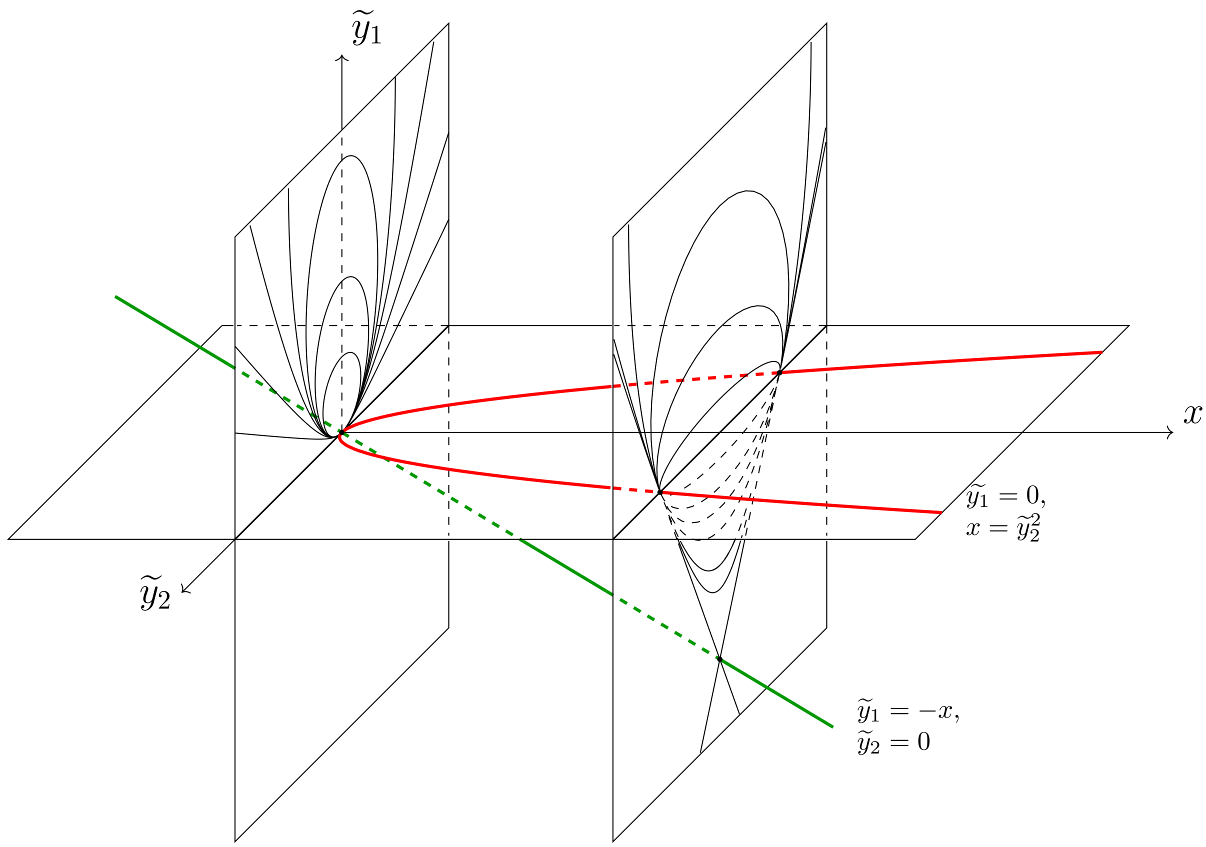


Figure 4.4: Family of fiberwise quadrics for $\Phi_0^{A_1}$

$$\Phi_1^{A_1} = \left(\begin{array}{c|cc} 0 & & \\ \hline & & x \\ & 1 & \end{array}\right) \partial_x,$$

Vertical linear vector field: $\varphi = y_2\partial_{y_3} + xy_3\partial_{y_2}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - xy_3^2$.
Poisson lift to $\mathbb{P}(V)$: $\pi = \partial_x \wedge \varphi + \frac{1}{2}\widetilde{y}_1\partial_{\widetilde{y}_1} \wedge \partial_{\widetilde{y}_2} + f(x)\frac{1}{x}\varphi \wedge \varphi^2 + g(x)\widetilde{y}_1^3\partial_{\widetilde{y}_1} \wedge \partial_{\widetilde{y}_2}$,
where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, and f, g are holomorphic function in x defined near $x = 0$.

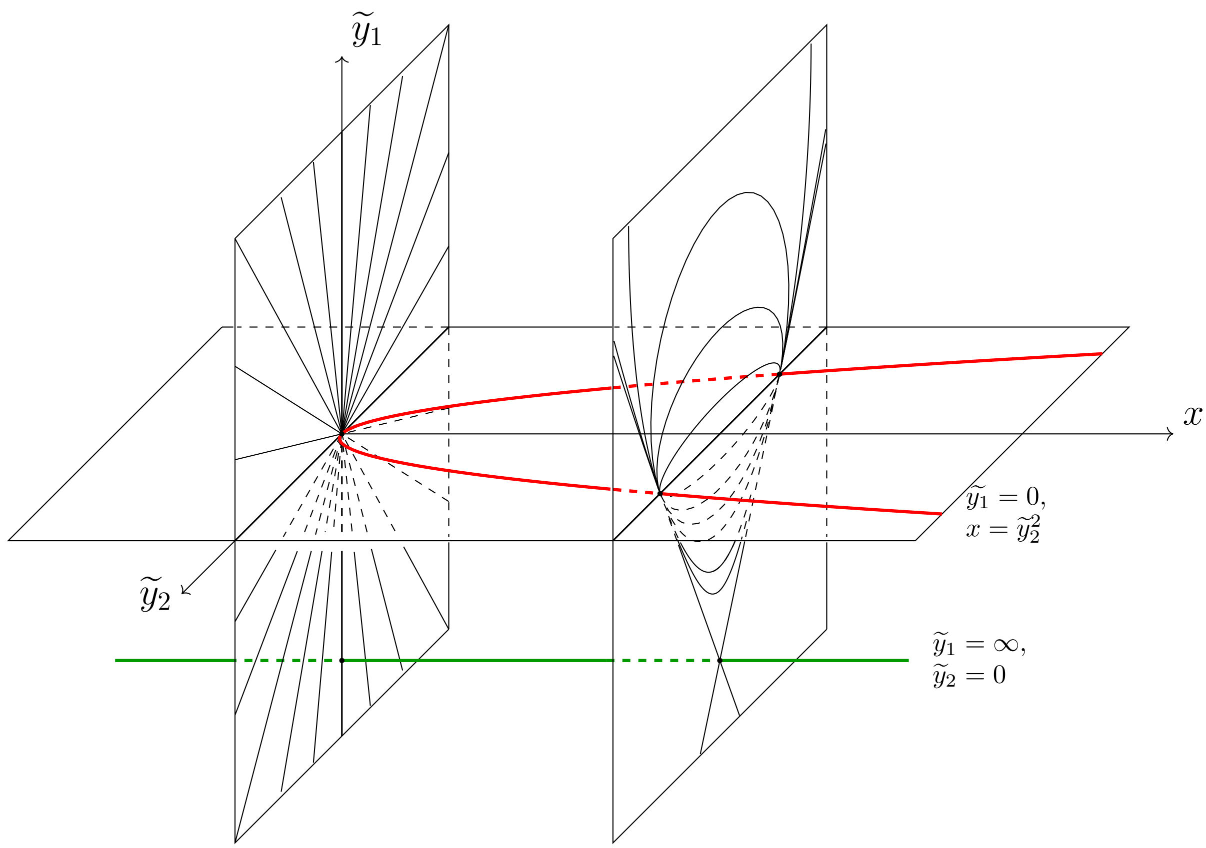


Figure 4.5: Family of fiberwise quadrics for $\Phi_1^{A_1}$

D_4 singularity.

For each of the isomorphism classes of co-Higgs field over a small neighborhood of $0 \in \mathbb{C}$ with spectral curve $y(y^2 - x^2) = 0$ (Corollary A.0.4), we present the generators for fiberwise pencil of quadrics constructed in Lemma 4.3.14.

$$\Phi_0^{D_4} = \begin{pmatrix} 0 & & \\ 1 & x & \\ & 1 & -x \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = y_1\partial_{y_2} + xy_2\partial_{y_2} + y_2\partial_{y_3} - xy_3\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2y_1y_3 - 2xy_2y_3$.

$$\Phi_1^{D_4} = \begin{pmatrix} 0 & & \\ x & x & \\ & 1 & -x \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = xy_1\partial_{y_2} + xy_2\partial_{y_2} + y_2\partial_{y_3} - xy_3\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2xy_1y_3 - 2xy_2y_3$.

$$\Phi_2^{D_4} = \begin{pmatrix} 0 & & \\ 1 & x & \\ & x & -x \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = y_1\partial_{y_2} + xy_2\partial_{y_2} + xy_2\partial_{y_3} - xy_3\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $xy_2^2 - 2y_1y_3 - 2xy_2y_3$.

$$\Phi_3^{D_4} = \begin{pmatrix} 0 & & \\ & x & \\ & 1 & -x \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = xy_2\partial_{y_2} + y_2\partial_{y_3} - xy_3\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2xy_2y_3$.

$$\Phi_4^{D_4} = \begin{pmatrix} 0 & & \\ & x & \\ 1 & & -x \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = xy_2\partial_{y_2} + y_1\partial_{y_3} - xy_3\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_1y_2 - xy_2y_3$.

$$\Phi_5^{D_4} = \begin{pmatrix} 0 & & \\ 1 & x & \\ & & -x \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = xy_2\partial_{y_2} + y_1\partial_{y_2} - xy_3\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_1y_3 + xy_2y_3$.

$$\Phi_6^{D_4} = \begin{pmatrix} 0 & & \\ & x & \\ & & -x \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = xy_2\partial_{y_2} - xy_3\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, y_2y_3.

E_7 singularity.

For each of the isomorphism classes of co-Higgs field over a small neighborhood of $0 \in \mathbb{C}$ with spectral curve $y(y^2 - x^3) = 0$ (Theorem A.0.8), we present the generators for fiberwise pencil of quadrics constructed in Lemma 4.3.14.

$$\Phi_0^{E_7} = \left(\begin{array}{c|cc} 0 & & \\ \hline 1 & & x^3 \\ & 1 & \end{array}\right) \partial_x,$$

Vertical linear vector field: $\varphi = y_1\partial_{y_2} + y_2\partial_{y_3} + x^3y_3\partial_{y_2}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2y_1y_3 - x^3y_3^2$.

$$\Phi_1^{E_7} = \left(\begin{array}{c|cc} 0 & & \\ \hline & & x^3 \\ & 1 & \end{array}\right) \partial_x,$$

Vertical linear vector field: $\varphi = y_2\partial_{y_3} + x^3y_3\partial_{y_2}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - x^3y_3^2$.

$$\Phi_2^{E_7} = \left(\begin{array}{c|cc} 0 & & \\ \hline x & & x^2 \\ & x & \end{array}\right) \partial_x,$$

Vertical linear vector field: $\varphi = xy_1\partial_{y_2} + xy_2\partial_{y_3} + x^2y_3\partial_{y_2}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2y_1y_3 - xy_3^2$.

$$\Phi_3^{E_7} = \left(\begin{array}{c|cc} 0 & & \\ \hline & & x^2 \\ & x & \end{array}\right) \partial_x,$$

Vertical linear vector field: $\varphi = xy_2\partial_{y_3} + x^2y_3\partial_{y_2}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - xy_3^2$.

$$\Phi_4^{E_7} = \left(\begin{array}{c|cc} 0 & & \\ \hline 1 & & x \\ & x^2 & \end{array}\right) \partial_x,$$

Vertical linear vector field: $\varphi = y_1\partial_{y_2} + x^2y_2\partial_{y_3} + xy_3\partial_{y_2}$.
Fiberwise φ-invariant quadrics: y_1^2, $x^2y_2^2 - 2y_1y_3 - xy_3^2$.

$$\Phi_5^{E_7} = \left(\begin{array}{c|cc} 0 & & \\ \hline x^2 & & x^3 \\ & 1 & \end{array}\right)\partial_x,$$

Vertical linear vector field: $\varphi = x^2 y_1 \partial_{y_2} + y_2 \partial_{y_3} + x^3 y_3 \partial_{y_2}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2x^2 y_1 y_3 - x^3 y_3^2$.

$$\Phi_6^{E_7} = \left(\begin{array}{c|cc} 0 & & \\ \hline 1 & & x^2 \\ & x & \end{array}\right)\partial_x,$$

Vertical linear vector field: $\varphi = y_1 \partial_{y_2} + x y_2 \partial_{y_3} + x^2 y_3 \partial_{y_2}$.
Fiberwise φ-invariant quadrics: y_1^2, $x y_2^2 - 2y_1 y_3 - x^2 y_3^2$.

$$\Phi_7^{E_7} = \left(\begin{array}{c|cc} 0 & & \\ \hline x & & x^3 \\ & 1 & \end{array}\right)\partial_x,$$

Vertical linear vector field: $\varphi = x y_1 \partial_{y_2} + y_2 \partial_{y_3} + x^3 y_3 \partial_{y_2}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2x y_1 y_3 - x^3 y_3^2$.

T_{36} singularity.

For each of the isomorphism classes of co-Higgs field over a small neighborhood of $0 \in \mathbb{C}$ with spectral curve $y(y^2 - x^4) = 0$ (Theorem A.0.10), we present the generators for fiberwise pencil of quadrics constructed in Lemma 4.3.14. The seven isomorphism classes with representatives $x\Phi_i^{D_4}$, $0 \le i \le 6$, have the same fiberwise pencils of quadrics as their D_4 counterparts. The remaining T_{36} isomorphism classes behave as follows.

$$\Phi_0^{T_{36}} = \begin{pmatrix} 0 & & \\ 1 & x^2 & \\ & 1 & -x^2 \end{pmatrix}\partial_x,$$

Vertical linear vector field: $\varphi = y_1 \partial_{y_2} + x^2 y_2 \partial_{y_2} + y_2 \partial_{y_3} - x^2 y_3 \partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2y_1 y_3 - 2x^2 y_2 y_3$.

$$\Phi_1^{T_{36}} = \begin{pmatrix} 0 & & \\ x^3 & x^2 & \\ & 1 & -x^2 \end{pmatrix}\partial_x,$$

Vertical linear vector field: $\varphi = x^3 y_1 \partial_{y_2} + x^2 y_2 \partial_{y_2} + y_2 \partial_{y_3} - x^2 y_3 \partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2x^3 y_1 y_3 - 2x^2 y_2 y_3$.

$$\Phi_2^{T_{36}} = \begin{pmatrix} x^2 & & \\ x^3 & -x^2 & \\ & 1 & 0 \end{pmatrix}\partial_x,$$

Vertical linear vector field: $\varphi = x^3 y_1 \partial_{y_2} + x^2 y_1 \partial_{y_1} + y_2 \partial_{y_3} - x^2 y_2 \partial_{y_2}$.
Fiberwise φ-invariant quadrics: $(y_2 - xy_1 + x^2 y_3)^2$, $xy_1^2 - 2y_1 y_2$.

$$\Phi_3^{T_{36}} = \begin{pmatrix} -x^2 & & \\ x^3 & 0 & \\ & 1 & x^2 \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = x^3 y_1 \partial_{y_2} - x^2 y_1 \partial_{y_1} + y_2 \partial_{y_3} + x^2 y_3 \partial_{y_3}$.
Fiberwise φ-invariant quadrics: $(y_2 + xy_1)^2$, $2y_1 y_2 + 2x^2 y_1 y_3 + xy_1^2$.

$$\Phi_4^{T_{36}} = \begin{pmatrix} -x^2 & & \\ 1 & x^2 & \\ & x^3 & 0 \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = y_1 \partial_{y_2} - x^2 y_1 \partial_{y_1} + x^3 y_2 \partial_{y_3} + x^2 y_2 \partial_{y_2}$.
Fiberwise φ-invariant quadrics: $(y_1 + x^2 y_2 - xy_3)^2$, $2y_1 y_3 + 2x^2 y_2 y_3 - xy_3^2 - x^3 y_2^2$.

$$\Phi_5^{T_{36}} = \begin{pmatrix} 0 & & \\ 1 & -x^2 & \\ & x^3 & x^2 \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = y_1 \partial_{y_2} - x^2 y_2 \partial_{y_2} + x^3 y_2 \partial_{y_3} + x^2 y_3 \partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $2y_1 y_3 - 2x^2 y_2 y_3 - x^3 y_2^2$.

$$\Phi_6^{T_{36}} = \begin{pmatrix} x^2 & & \\ 1 & 0 & \\ & x^3 & -x^2 \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = y_1 \partial_{y_2} + x^2 y_1 \partial_{y_1} + x^3 y_2 \partial_{y_3} - x^2 y_3 \partial_{y_3}$.
Fiberwise φ-invariant quadrics: $(y_1 - x^2 y_2)^2$, $2y_1 y_3 - x^3 y_2^2$.

$$\Phi_7^{T_{36}} = \begin{pmatrix} 0 & & \\ x^2 & x^2 & \\ & 1 & -x^2 \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = x^2 y_1 \partial_{y_2} + x^2 y_2 \partial_{y_2} + y_2 \partial_{y_3} - x^2 y_3 \partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2x^2 y_1 y_3 - 2x^2 y_2 y_3$.

$$\Phi_8^{T_{36}} = \begin{pmatrix} 0 & & \\ 1 & x^2 & \\ & x^2 & -x^2 \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = y_1 \partial_{y_2} + x^2 y_2 \partial_{y_2} + x^2 y_2 \partial_{y_3} - x^2 y_3 \partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $x^2 y_2^2 - 2y_1 y_3 - 2x^2 y_2 y_3$.

$$\Phi_9^{T_{36}} = \begin{pmatrix} 0 & & \\ x & x^2 & \\ & 1 & -x^2 \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = xy_1\partial_{y_2} + x^2y_2\partial_{y_2} + y_2\partial_{y_3} - x^2y_3\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2xy_1y_3 - 2x^2y_2y_3$.

$$\Phi_{10}^{T_{36}} = \begin{pmatrix} 0 & & \\ 1 & x^2 & \\ & x & -x^2 \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = y_1\partial_{y_2} + x^2y_2\partial_{y_2} + xy_2\partial_{y_3} - x^2y_3\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $xy_2^2 - 2y_1y_3 - 2x^2y_2y_3$.

$$\Phi_{11}^{T_{36}} = \begin{pmatrix} 0 & & \\ & x^2 & \\ & 1 & -x^2 \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = x^2y_2\partial_{y_2} + y_2\partial_{y_3} - x^2y_3\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2^2 - 2x^2y_2y_3$.

$$\Phi_{12}^{T_{36}} = \begin{pmatrix} 0 & & \\ & x^2 & \\ 1 & & -x^2 \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = x^2y_2\partial_{y_2} + y_1\partial_{y_3} - x^2y_3\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_2y_3 - x^2y_1y_2$.

$$\Phi_{13}^{T_{36}} = \begin{pmatrix} 0 & & \\ 1 & x^2 & \\ & & -x^2 \end{pmatrix} \partial_x,$$

Vertical linear vector field: $\varphi = x^2y_2\partial_{y_2} + y_1\partial_{y_2} - x^2y_3\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $y_1y_3 + x^2y_2y_3$.

$$\Psi_{\beta}^{T_{36}} = \begin{pmatrix} 0 & & \\ x & x^2 & \\ \frac{1}{\beta+1} & x & -x^2 \end{pmatrix} \partial_x,, \qquad \beta \in \mathbb{C} \smallsetminus \{1, -1, 0\},$$

Vertical linear vector field: $\varphi = xy_1\partial_{y_2} + x^2y_2\partial_{y_2} + xy_2\partial_{y_3} - x^2y_3\partial_{y_3} + \frac{1}{\beta+1}y_1\partial_{y_3}$.
Fiberwise φ-invariant quadrics: y_1^2, $xy_2^2 - 2xy_1y_3 - 2x^2y_2y_3 + \frac{2}{\beta+1}y_1y_2$.

4.3.3 Global Poisson lifts. Non-resonant case.

Throughout the subsection, ϕ is a co-Higgs field on a rank 3 vector bundle V over $\mathbb{P}^1$ that is strongly integrable, non-resonant (Definition 4.3.3), and $\Sigma \subset \mathcal{T}_{\mathbb{P}^1}$ is its spectral curve. We are going to discuss when such ϕ admits a lift to a Poisson structure on $\mathbb{P}(V)$. Recall from Subsection 4.3.1 that in order to admit such a lift, the spectral curve Σ has to be cut out by the equation $\prod_{i=1}^{3}(\theta - \lambda_i v) = 0$, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ are distinct, and $v \in H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1})$. In particular, Σ has three irreducible components, each isomorphic to $\mathbb{P}^1$, and either has two D_4 singularities (if v has two distinct zeros) or one T_{36} singularity (if v has one zero of multiplicity two). Moreover, out of all local types of co-Higgs fields with D_4 or T_{36} singularities described in Appendix, the only ones allowing possibility of Poisson lift are

$$\begin{array}{ll} (D_4) & \Phi_i^{D_4}, i = 1,3,4,5,6, \\ (T_{36}) & \Phi_i^{T_{36}}, i = 1,2,3,7,11,12,13 \text{ and } x\Phi_i^{D_4}, 0 \le i \le 6. \end{array} \tag{4.8}$$

We are going to say that ϕ has admissible singularities, if around each point where the spectral curve of ϕ is singular, the co-Higgs ϕ is isomorphic to one of the local normal forms (4.8).

We will denote by $(\Phi_i^{D_4}) - (\Phi_j^{D_4})$ a strongly integrable, non-resonant rank 3 co-Higgs field ϕ over $\mathbb{P}^1$ whose spectral curve has two D_4 singularities and such that ϕ is isomrophic to $\Phi_i^{D_4}$ near one of them and to $\Phi_j^{D_4}$ near the other. Similarly, we will denote by $(\Phi_i^{T_{36}})$ (resp. $(x\Phi_i^{D_4})$) a strongly integrable, non-resonant rank 3 co-Higgs field ϕ over $\mathbb{P}^1$ whose spectral curve has one singularity of type T_{36} and such that ϕ is isomorhic to $(\Phi_i^{T_{36}})$ (resp. $(x\Phi_i^{D_4})$) near this singularity.

Given a strongly integrable, non-resonant rank 3 co-Higgs field ϕ over $\mathbb{P}^1$, we introduce the line bundle $L_\phi \subset \wedge^2\mathcal{T}_{ver}\mathbb{P}(V)$ given by projectivization of the image of $\phi \wedge \phi^2 : S^3\mathcal{T}^*_{\mathbb{P}^1} \longrightarrow \wedge^2 End(V) \cong \wedge^2\mathcal{T}_{ver}V$. One can characterize L_ϕ as the set of local sections of $\wedge^2\mathcal{T}_{ver}\mathbb{P}(V)$ that Schouten commute with the vertical vector fields $\langle\phi,\alpha\rangle \in End(V)$, for all $\alpha \in \mathcal{T}^*_{\mathbb{P}^1}$.

Lemma 4.3.17. *Let ϕ be strongly integrable, non-resonant rank 3 co-Higgs bundle (V,ϕ) over $\mathbb{P}^1$ having admissible singularities* (4.8)*. Then there is a cohomology class $\mathrm{Obs}_\phi \in H^1(\mathbb{P}^1, L_\phi)$ that is an obstruction to lifting ϕ to a Poisson structure on $\mathbb{P}(V)$. In other words, ϕ admits a lift to a Poisson structure on $\mathbb{P}(V)$ if and only if the cohomology class Obs_ϕ vanishes.*

If the obstruction Obs_ϕ does vanish, then all the lifts of ϕ form an affine space over $H^0(\mathbb{P}^1, L_\phi)$.

Proof. First, let us explain the construciton of the obstruction class Obs_ϕ. Choose a Čech cover $\{\mathcal{U}_i\}_{i\in I}$ of $\mathbb{P}^1$ such that over each $\mathcal{U}_i$, the co-Higgs field ϕ admits a lift to a Poisson structure π_i on $\mathbb{P}(V|_{\mathcal{U}_i})$ (here we use that ϕ has admissible singularities). Over double overlaps $\mathcal{U}_i \cap \mathcal{U}_j$, the Poisson structure π_i and π_j may differ, however, by Lemma 4.3.4.1), the difference $\pi_i - \pi_j \in H^0(\mathcal{U}_i \cap \mathcal{U}_j, L_\phi)$. So, we obtain the Čech 1-cocycle $\{\pi_i|_{\mathcal{U}_i\cap\mathcal{U}_j} - \pi_j|_{\mathcal{U}_i\cap\mathcal{U}_j}\}_{i,j\in I}$. By Lemma 4.3.4.1), different choices of the local lifts π_i produce the same cohomology class $\mathrm{Obs}_\phi \in H^1(\mathbb{P}^1, L_\phi)$. The remaining claims also follow directly from Lemma 4.3.4.1). □

The line bundle L_ϕ is abstractly isomorphic to $\mathcal{O}_{\mathbb{P}^1}(k-6)$, wehre k is the number zeros of the tensor $\phi\wedge\phi^2 \in \mathrm{Hom}(S^3\mathcal{T}^*, End(V))$ (counting with multiplicities). Contribution of each of the admissible local normal forms (4.8) to the number of zeros of $\phi \wedge \phi^2$ are presented in Table 4.1.

Table 4.1 shows that for a stronly integrable, non-resonant rank 3 co-Higgs ϕ over $\mathbb{P}^1$ one has $L_\phi \cong \mathcal{O}_{\mathbb{P}^1}(-l)$, where $l = 0,1,2$ or 3. By applying Lemma 4.3.17, we get that a Poisson lift of ϕ, if

ϕ	$\Phi_1^{D_4}$	$\Phi_3^{D_4}$	$\Phi_4^{D_4}$	$\Phi_5^{D_4}$	$\Phi_6^{D_4}$		
Order of vanishing	2	2	2	2	3		
ϕ	$x\Phi_0^{D_4}$	$x\Phi_1^{D_4}$	$x\Phi_2^{D_4}$	$x\Phi_3^{D_4}$	$x\Phi_4^{D_4}$	$x\Phi_5^{D_4}$	$x\Phi_6^{D_4}$
Order of vanishing	3	5	4	5	5	5	6
ϕ	$\Phi_1^{T_{36}}$	$\Phi_2^{T_{36}}$	$\Phi_3^{T_{36}}$	$\Phi_7^{T_{36}}$	$\Phi_{11}^{T_{36}}$	$\Phi_{12}^{T_{36}}$	$\Phi_{13}^{T_{36}}$
Order of vanishing	4	4	4	4	4	4	4

Table 4.1: Order of vanishing of $\phi \wedge \phi^2$ at $x = 0$.

it exists, is unique, with the exception of co-Higgs fields the types $(\Phi_6^{D_4}) - (\Phi_6^{D_4})$ and $(x\Phi_6^{D_4})$ (in which case there is $\mathbb{C}$-worth of such lifts). Let us now consider each of the cases $L_\phi \cong \mathcal{O}_{\mathbb{P}^1}(-l)$ in a more detail.

Case $L_\phi \cong \mathcal{O}_{\mathbb{P}^1}$

The types falling into this category are $(\Phi_6^{D_4}) - (\Phi_6^{D_4})$ and $(x\Phi_6^{D_4})$. The co-Higgs field ϕ here is globally daigonalizable, and has the form $\phi = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)v$, where $v = x\partial_x$ or $v = x^2\partial_x$. All the Poisson lifts of ϕ are of the form $\pi = \partial_x \wedge \varphi + \frac{C}{x^3}\varphi \wedge \varphi^2$ (case $(\Phi_6^{D_4}) - (\Phi_6^{D_4})$) or $\pi = \partial_x \wedge \varphi + \frac{C}{x^6}\varphi \wedge \varphi^2$ (case $(x\Phi_6^{D_4})$), where $\varphi = \langle \phi, dx \rangle$, $\varphi^2 = \langle \phi^2, dx^2 \rangle$, and $C \in \mathbb{C}$.

Case $L_\phi \cong \mathcal{O}_{\mathbb{P}^1}(-1)$

The types in question are $(\Phi_i^{D_4}) - (\Phi_6^{D_4})$, $i = 1,3,4,5$, and $(x\Phi_i^{D_4})$, $1 = 1,3,4,5$. For these cases one has $H^k(\mathbb{P}^1, L_\phi) = 0$, $k = 0,1$, and hence Lemma 4.3.17 implies that ϕ admits a unique Poisson lift to $\mathbb{P}(V)$.

Example 4.3.18. (Type $(\Phi_1^{D_4}) - (\Phi_6^{D_4})$)

Let us choose a coordinate x on $\mathbb{P}^1$ so that the spectral curve of ϕ has the form $\Sigma = \{\prod_{i=1}^3(\theta - \lambda_i x\partial_x) = 0\}$. Let $\overline{\Sigma} \cong \bigsqcup_{i=1}^3 \Sigma_i$ be the normalization of Σ. Each Σ_i is isomorphic to $\mathbb{P}^1$, so let us introduce a coordinate x_i on Σ_i that is pullback of x from the base. The partial normalization $\widetilde{\Sigma}$ corresponding to the eigensheaf $\mathcal{F}$ of ϕ near $x = \infty$ is isomorphic to Spec $\bigoplus_{i=1}^3 \mathbb{C}\{x_i^{-1}\}$ (full normalization), and near $x = 0$ is isomorphic to Spec $\widetilde{R}$, $\widetilde{R} = (1,1,1)\mathbb{C} + \bigoplus_{i=1}^3 x_i\mathbb{C}\{x_i\}$.

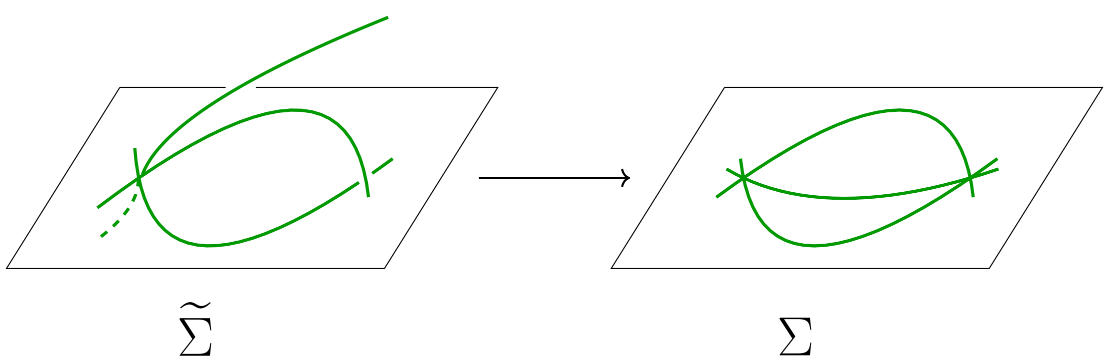


Figure 4.6: Spectral curve Σ and its partial normalization $\widetilde{\Sigma}$ for co-Higgs field of type $(\Phi_1^{D_4}) - (\Phi_6^{D_4})$

The eigensheaf $\mathcal{F}$ of ϕ near $x = 0$ is isomorphic to the ideal sheaf $(x,y)\widetilde{R}$, where $x = (x_1, x_2, x_3)$, $y = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)$. Such a sheaf is completely determined by the triple $(d_1, d_2, d_3) \in \mathbb{Z}^3$, such that $\mathcal{F}|_{\Sigma_i} \cong \mathcal{O}_{\Sigma_i}(d_i)$, $i = 1,2,3$. Moreover, the pushforward $V = p_*\mathcal{F}$ under the projection $p : \Sigma \to \mathbb{P}^1$ can be shown to be isomorphic to $\mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2) \oplus \mathcal{O}_{\mathbb{P}^1}(d_3 - 1)$, where we assume $d_1 \leq d_2 \leq d_3$. Then ϕ is isomorphic to the following

$$\phi = \begin{pmatrix} \lambda_1 x & 0 & 0 \\ x & \lambda_2 x & 0 \\ 0 & 1 & \lambda_3 x \end{pmatrix} \partial_x$$

where ϕ is written with repspect to the decomposition of V above, in the affine chart $\{x \neq \infty\}$.

The unique Poisson lift of ϕ to $\mathbb{P}(V)$ in the chart $\{x \neq \infty\}$ is given by the formula

$$\pi = \partial_x \wedge \varphi + \frac{1}{x}(\widetilde{y}_1 \partial_{\widetilde{y}_1} + \widetilde{y}_2 \partial_{\widetilde{y}_2}) \wedge \varphi,$$

$$\varphi = \langle \phi, dx \rangle = (\lambda_1 - \lambda_3) x \widetilde{y}_1 \partial_{\widetilde{y}_1} + (\lambda_2 - \lambda_3) x \widetilde{y}_2 \partial_{\widetilde{y}_2} + x \widetilde{y}_1 \partial_{\widetilde{y}_2} - \widetilde{y}_2 (\widetilde{y}_1 \partial_{\widetilde{y}_1} + \widetilde{y}_2 \partial_{\widetilde{y}_2}),$$

where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$.

We obtain a family of Poisson structure parametrized by a a dense open subset of $\mathbb{Z}^3 \times \mathbb{C}^2$. The $\mathbb{Z}^3$ part corresponds to the degree (d_1, d_2, d_3) of the eigensheaf of ϕ. The $\mathbb{C}^2$ part corresponds to the triples of constants $\lambda_1, \lambda_2, \lambda_3$ adding up to zero.

Case $L_\phi \cong \mathcal{O}_{\mathbb{P}^1}(-2)$

This case covers all the remaining types, except $(x\Phi_0^{D_4})$. Specifically, the types fitting into the case $L_\phi \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ are $(\Phi_i^{D_4})$–$(\Phi_j^{D_4})$, $i = 1, 3, 4, 5$, $(x\Phi_2^{D_4})$, and $(\Phi_i^{T_{36}})$, $i = 1, 2, 3, 7, 11, 12, 13$. In all these case, except $(\Phi_1^{T_{36}})$, $(\Phi_2^{T_{36}})$, $(\Phi_3^{T_{36}})$, the obstruction class Obs_ϕ constructed in Lemma 4.3.17 can be computed as the projectivization of the pairing $\langle \mathrm{At}_V \,\hat{,}\, \phi \rangle$, of ϕ with the Atiyah class $\mathrm{At}_V \in H^1(\mathbb{P}^1, \mathcal{T}^*_{\mathbb{P}^1} \otimes End(V))$, defined by (3.11) (see the explanation of this fact in the proof of Lemma 4.3.19).

Recall that, given a rank one torsion free sheaf $\mathcal{F}$ on $\Sigma = \{\prod_{i=1}^3 (\theta - \lambda_i v) = 0\}$, by the degree of $\mathcal{F}$ we mean the triple $(d_1, d_2, d_3) \in \mathbb{Z}^3$, where d_i is the degree of the restriction of $\mathcal{F}$ to the irreducible component $\{\theta - \lambda_i v = 0\} \cong \mathbb{P}^1$.

Lemma 4.3.19. *Let ϕ be strongly integrable, non-resonant rank 3 co-Higgs bundle (V, ϕ) over $\mathbb{P}^1$ of one of the types $(\Phi_i^{D_4}) - (\Phi_j^{D_4})$, $i = 1, 3, 4, 5$, $(x\Phi_2^{D_4})$, and $(\Phi_i^{T_{36}})$, $i = 7, 11, 12, 13$. In particular, $L_\phi \subset \mathcal{T}_{ver}\mathbb{P}(V)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2)$.*

Then ϕ admits a Poisson lift to $\mathbb{P}(V)$ if and only if the following matrix has zero determinant

$$\begin{pmatrix} d_1 & d_2 & d_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{pmatrix}, \tag{4.9}$$

where $(d_1, d_2, d_3) \in \mathbb{Z}^3$ is the degree of the eigensheaf $\mathcal{F}$ of ϕ, and $\lambda_i \in \mathbb{C}$ are the constants in the characteristic equation $\prod_{i=1}^3 (\theta - \lambda_i v) = 0$ of ϕ (which are unique up to an overall multiple).

Proof. Let us cover the base $\mathbb{P}^1$ with two open sets $\mathcal{U}_0 = \{|x| < 2\}$ and $\mathcal{U}_1 = \{|x| > 1/2\}$, where the coordinate x is chosen so that the spectral curve Σ of ϕ has singularities only over $x = 0$ and possibly over $x = \infty$. Note that x also defines a coordinate on each irreducible component $\Sigma_i = \{\theta - \lambda_i v = 0\} \cong \mathbb{P}^1$. Let $\widetilde{\mathcal{U}}_j \subset \Sigma$, $j = 0, 1$ be the preimage of $\mathcal{U}_j$ under the projection. Choose an identification of $\mathcal{F}|_{\widetilde{\mathcal{U}}_j}$ with an ideal sheaf inside the structure sheaf of the normalization $\mathcal{I}_j \subset \oplus_{i=1}^3 \mathcal{O}_{\Sigma_i \cap \widetilde{\mathcal{U}}_j}$. Let h be an identification $\mathcal{I}_0|_{\widetilde{\mathcal{U}}_0 \cap \widetilde{\mathcal{U}}_1} \xrightarrow{\sim} \mathcal{I}_1|_{\widetilde{\mathcal{U}}_0 \cap \widetilde{\mathcal{U}}_1}$, such that $\mathcal{I}_0$ glued to $\mathcal{I}_1$ via h is isomorphic to $\mathcal{F}$. Let y_i be fiberwise linear coordinate on $\mathcal{O}_{\Sigma_i\cap\widetilde{\mathcal{U}}_0}$, $i = 1,2,3$, and z_i be fiberwise linear coordinate on $\mathcal{O}_{\Sigma_i\cap\widetilde{\mathcal{U}}_1}$, $i = 1,2,3$. Let h be given by three functions $h_i \in \mathcal{O}_{\Sigma_i\cap\widetilde{\mathcal{U}}_0\cap\widetilde{\mathcal{U}}_1}$ such that $z_i = h_i(x)y_i$, $i = 1,2,3$. Without loss of generality, we may assume that each h_i is a Laurent polynomial in x, and let

$$h_i(x) = x^{p_i}\prod_{j=1}^{q_i}(x-a_j^i)\prod_{j=1}^{r_i}(x-b_j^i), \quad \text{where } |a_j^i| < 1/2, \quad |b_j^i| > 2. \tag{4.10}$$

Since we assumed that $\mathcal{F}|_{\Sigma_i} \cong \mathcal{O}_{\Sigma_i}(d_i)$, we have $d_i = -p_i - q_i$, $i = 1,2,3$.

Now, let us turn to computing the obstruction $\mathrm{Obs}_\phi \in H^1(\mathbb{P}^1, L_\phi)$ constructed in Lemma 4.3.17. Since ϕ has type $(\Phi_i^{D_4}) - (\Phi_j^{D_4})$, $i = 1,3,4,5$, $(x\Phi_2^{D_4})$, and $(\Phi_i^{T_{36}})$, $i = 7,11,12,13$, the Poisson lifts π_0 over $\mathcal{U}_0$ and π_1 over $\mathcal{U}_1$ can be chosen so that over punctured neigborhoods $\mathcal{U}_0 \setminus \{x=0\}$ and $\mathcal{U}_1 \setminus \{x=\infty\}$ they have expressions

$$\pi_0 = v \wedge ((\lambda_1-\lambda_3)\widetilde{y}_1\partial_{\widetilde{y}_1} + (\lambda_2-\lambda_3)\widetilde{y}_2\partial_{\widetilde{y}_2}),$$

$$\pi_1 = v \wedge ((\lambda_1-\lambda_3)\widetilde{z}_1\partial_{\widetilde{z}_1} + (\lambda_2-\lambda_3)\widetilde{z}_2\partial_{\widetilde{z}_2}),$$

where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, $\widetilde{z}_1 = \frac{z_1}{z_3}$, $\widetilde{z}_2 = \frac{z_2}{z_3}$, and v equals either $x\partial_x$ (in the case of type $(\Phi_i^{D_4}) - (\Phi_j^{D_4})$, $i = 1,3,4,5$), or $x^2\partial_x$ (in the case of type $(x\Phi_2^{D_4})$, or $(\Phi_i^{T_{36}})$, $i = 7,11,12,13$). Then over $\mathcal{U}_0 \cap \mathcal{U}_1$, the difference $\pi_0 - \pi_1$ will be equal to the projectivization of $\langle \mathrm{At}_V \curlywedge \phi\rangle$ defined by (3.11). Let us calculate the latter.

Over $\mathcal{U}_0 \cap \mathcal{U}_1$, in coordinates x, y_1, y_2, y_3 one has the following expressions for ϕ and At_V

$$\phi = x\partial_x \otimes \left(\sum_{i=1}^{3}\lambda_i y_i \partial_{y_i}\right), \quad \text{if } \Sigma \text{ has two } D_4 \text{ singularities,} \tag{4.11}$$

$$\phi = x^2\partial_x \otimes \left(\sum_{i=1}^{3}\lambda_i y_i \partial_{y_i}\right), \quad \text{if } \Sigma \text{ has one } T_{36} \text{ singularity.} \tag{4.12}$$

$$\mathrm{At}_V = \sum_{i=1}^{3} d\log(h_i(x)) \otimes y_i\partial_{y_i}. \tag{4.13}$$

In the projectivized corrdinates x, $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, on $\mathbb{P}(V)|_{\mathcal{U}_0\cap\mathcal{U}_1}$ the expressions above yield

$$\phi = x\partial_x \otimes ((\lambda_1-\lambda_3)\widetilde{y}_1\partial_{\widetilde{y}_1} + (\lambda_2-\lambda_3)\widetilde{y}_2\partial_{\widetilde{y}_2}), \quad \text{if } \Sigma \text{ has two } D_4 \text{ singularities,} \tag{4.14}$$

$$\phi = x^2\partial_x \otimes ((\lambda_1-\lambda_3)\widetilde{y}_1\partial_{\widetilde{y}_1} + (\lambda_2-\lambda_3)\widetilde{y}_2\partial_{\widetilde{y}_2}), \quad \text{if } \Sigma \text{ has one } T_{36} \text{ singularities.} \tag{4.15}$$

$$\mathrm{At}_V = (d\log(h_1(x)) - d\log(h_3(x))) \otimes \widetilde{y}_1\partial_{\widetilde{y}_1} + (d\log(h_2(x)) - d\log(h_3(x))) \otimes \widetilde{y}_2\partial_{\widetilde{y}_2}. \tag{4.16}$$

Let s be a section of L_ϕ having a simple pole over $x = 0$ and another simple pole over $x = \infty$ (and no other poles). If Σ has type two D_4 singularities, then we can choose $s = \langle \phi \wedge \phi^2, \frac{dx^3}{x^3}\rangle$. In coordinates x, $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, the section s has the expression $s = \widetilde{y}_1\partial_{\widetilde{y}_1} \wedge \widetilde{y}_2\partial_{\widetilde{y}_2}$ (up to a multiplicative constant, which we omit). If Σ has one T_{36} singularity, then we can choose $s = \langle \phi \wedge \phi^2, \frac{dx^3}{x^5}\rangle$. In coordinates x, $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, the section s has the expression $s = x\widetilde{y}_1\partial_{\widetilde{y}_1} \wedge \widetilde{y}_2\partial_{\widetilde{y}_2}$ (up to a multiplicative constant, which we omit).

Now, let us calculate the Čech 1-cocyle of L_ϕ with respect to the cover $\{\mathcal{U}_0, \mathcal{U}_1\}$ that represents the obstruction Obs_ϕ from Lemma 4.3.17. In the case when ϕ has two D_4 singularities, we have

$$\mathrm{Obs}_\phi = \langle \mathrm{At}_V \wedge \phi \rangle = f(x)\widetilde{y_1}\partial_{\widetilde{y}_1} \wedge \widetilde{y_2}\partial_{\widetilde{y}_2},$$

where

$$f(x) = -(\lambda_1 - \lambda_3)\left(\frac{xh_2'(x)}{h_2(x)} - \frac{xh_3'(x)}{h_3(x)}\right) + (\lambda_2 - \lambda_3)\left(\frac{xh_1'(x)}{h_1(x)} - \frac{xh_3'(x)}{h_3(x)}\right). \tag{4.17}$$

In the case when Σ has one T_{36} singularity, we similarly obtain

$$\mathrm{Obs}_\phi = \langle \mathrm{At}_V \wedge \phi \rangle = xf(x)\widetilde{y_1}\partial_{\widetilde{y}_1} \wedge \widetilde{y_2}\partial_{\widetilde{y}_2},$$

where $f(x)$ is the same as in (4.29).

We see that in both the cases, Obs_ϕ has Čech 1-cocycle $fs|_{\mathcal{U}_0\cap\mathcal{U}_1}$. Pick an isomorphism $\varrho : L_\phi \xrightarrow{\sim} \mathcal{T}^*_{\mathbb{P}^1}$ sending s to $\frac{dx}{x}$. Under ϱ, the cocycle $fs|_{\mathcal{U}_0\cap\mathcal{U}_1}$ will get sent to

$$f(x)\frac{dx}{x}\Big|_{\mathcal{U}_0\cap\mathcal{U}_1} = \Big(-(\lambda_1-\lambda_3)(d\log(h_2(x)) - d\log(h_3(x))) - (\lambda_2-\lambda_3)(d\log(h_1(x)) + d\log(h_3(x)))\Big)\Big|_{\mathcal{U}_0\cap\mathcal{U}_1}.$$

The 1-form above represents zero cohomology class in $H^1(\mathbb{P}^1, \mathcal{T}^*_{\mathbb{P}^1})$ if and only if the sum of its residues inside $\mathcal{U}_0$ equals zero. From the expressions (4.26) for h_i's we get that the sum of the residues is equal to

$$(\lambda_1 - \lambda_3)(d_2 - d_3) - (\lambda_2 - \lambda_3)(d_1 - d_3) = -\det\begin{pmatrix} d_1 & d_2 & d_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{pmatrix}.$$

Applying Lemma 4.3.17 finishes the proof. □

Example 4.3.20. (Type $(\Phi_1^{D_4}) - (\Phi_1^{D_4})$)

Let us choose a coordinate x on $\mathbb{P}^1$ so that the spectral curve of ϕ has the form $\Sigma = \{\prod_{i=1}^3(\theta - \lambda_i x\partial_x) = 0\}$. Let $\overline{\Sigma} \cong \bigsqcup_{i=1}^3 \Sigma_i$ be the normalization of Σ. Let $L_i \cong \mathcal{O}_{\Sigma_i}(d_i)$ be the restriction of the eigensheaf $\mathcal{F}$ of ϕ to Σ_i. Then $\mathcal{F}$ is isomorphic to the subsheaf of sections $(s_1, s_2, s_3) \in L_1 \oplus L_2 \oplus L_3$ such that $\sum_{i=1}^3 \alpha_i s_i(0) = 0$ and $\sum_{i=1}^3 \beta_i s_i(\infty) = 0$, for some $\alpha_i, \beta_i \in \mathbb{C}^*$. By applying automorphisms of L_i's, if necessary, we assume that $\beta_i = 1$, $i = 1, 2, 3$. Then the triple $\alpha_1, \alpha_2, \alpha_3$ up to an overall scaling, gives an element of $\mathbb{C}^* \times \mathbb{C}^*$.

A degree $(d_1, d_2, d_3) \in \mathbb{Z}^3$ satisfies vanishing of the determinant (4.9) for some non-resonant triple $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ if and only if d_i's are either all pairwise equal or pairwise distinct. If $d_1 = d_2 = d_3$, then without loss of generality we may assume they all equal 0. Then $V = p_*\mathcal{F} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, unless $\alpha_1 = \alpha_2 = \alpha_3$ (in which case $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$). By Lemma 4.3.19, we obtain a four dimensional family of Poisson structures on $\mathbb{P}(V)$, $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, parametrized by α_2/α_1, α_3/α_1, $\lambda_3 - \lambda_1$, $\lambda_2 - \lambda_1$. Moreover, since the total space of V is Calabi-Yau, by Theorem 3.1.5, all of these Poisson structures lift to quadratic Poisson strucutres on V.

Another way to obtain Poisson lifts of ϕ of type $(\Phi_1^{D_4}) - (\Phi_1^{D_4})$ is to choose degrees $d_1 < d_2 < d_3$. The constants λ_i's are determined uniquely, up to scaling, from vanishing of the determinant (4.9), if we require $\sum_{i=1}^3 \lambda_i = 0$. To obtain a non-resonant triple $(\lambda_1, \lambda_2, \lambda_3)$, we must additionally insist that $d_2 - d_1 \neq d_3 - d_2$. By Lemma 4.3.19, we obtain a three dimensional family of Poisson structures on $\mathbb{P}(V)$, for $V = p_*\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2) \oplus \mathcal{O}_{\mathbb{P}^1}(d_3 - 2)$, parametrized by α_2/α_1, α_3/α_1, $\lambda_2 - \lambda_1$. Note that since there are infinitely many triples $(d_1, d_2, d_3) \in \mathbb{Z}^3$ satisfying $d_1 < d_2 < d_3$, $d_2 - d_1 \neq d_3 - d_2$, we obtain infinitely many of such three dimensional families of Poisson structures.

The only cases with $L_\phi \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ we have not covered so far are the types $(\Phi_i^{T_{36}})$, $i = 1, 2, 3$. Up to permutation of indices, it is enough to consider the case $(\Phi_1^{T_{36}})$.

Lemma 4.3.21. *Let ϕ be strongly integrable, non-resonant rank 3 co-Higgs bundle (V, ϕ) over $\mathbb{P}^1$ of $(\Phi_1^{T_{36}})$. In particular, $L_\phi \subset \mathcal{T}_{ver}\mathbb{P}(V)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2)$.*

Then ϕ admits a Poisson lift to $\mathbb{P}(V)$ if and only if the following matrix has zero determinant

$$\begin{pmatrix} d_1 - 1 & d_2 & d_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{pmatrix}, \tag{4.18}$$

where $(d_1, d_2, d_3) \in \mathbb{Z}^3$ is the degree of the eigensheaf $\mathcal{F}$ of ϕ, and $\lambda_i \in \mathbb{C}$ are the constants in the characteristic equation $\prod_{i=1}^3 (\theta - \lambda_i v) = 0$ of ϕ (which are unique up to an overall multiple).

Proof. Let us cover the base $\mathbb{P}^1$ with two open sets $\mathcal{U}_0 = \{|x| < 2\}$ and $\mathcal{U}_1 = \{|x| > 1/2\}$, where the coordinate x is chosen so that the spectral curve Σ of ϕ has the only singularity over $x = 0$. Note that x also defines a coordinate on each irreducible component $\Sigma_i = \{\theta - \lambda_i x^2 \partial_x = 0\} \cong \mathbb{P}^1$. Let $\widetilde{\mathcal{U}}_j \subset \Sigma$, $j = 0, 1$ be the preimage of $\mathcal{U}_j$ under the projection. Choose an identification of $\mathcal{F}|_{\widetilde{\mathcal{U}}_1}$ with $\mathcal{I}_1 = \mathcal{O}_{\mathcal{U}_1} = \oplus_{i=1}^3 \mathcal{O}_{\Sigma_i \cap \mathcal{U}_1}$, and also an identification of $\mathcal{F}|_{\widetilde{\mathcal{U}}_0}$ with the eigensheaf $\mathcal{I}_0$ of $\Phi_1^{T_{36}}$. Let h be an identification $\mathcal{I}_0|_{\widetilde{\mathcal{U}}_0 \cap \widetilde{\mathcal{U}}_1} \xrightarrow{\sim} \mathcal{I}_1|_{\widetilde{\mathcal{U}}_0 \cap \widetilde{\mathcal{U}}_1}$, such that $\mathcal{I}_0$ glued to $\mathcal{I}_1$ via h is isomorphic to $\mathcal{F}$. Let y_i be fiberwise linear coordinate on $\mathcal{O}_{\Sigma_i \cap \widetilde{\mathcal{U}}_0}$, $i = 1, 2, 3$, and z_i be fiberwise linear coordinate on $\mathcal{O}_{\Sigma_i \cap \widetilde{\mathcal{U}}_1}$, $i = 1, 2, 3$. Let h be given by three functions $h_i \in \mathcal{O}_{\Sigma_i \cap \widetilde{\mathcal{U}}_0 \cap \widetilde{\mathcal{U}}_1}$ such that $z_i = h_i(x) y_i$, $i = 1, 2, 3$. Without loss of generality, we may assume that each h_i is a Laurent polynomial in x, and let

$$h_i(x) = x^{p_i} \prod_{j=1}^{q_i} (x - a_j^i) \prod_{j=1}^{r_i} (x - b_j^i), \quad \text{where } |a_j^i| < 1/2, \quad |b_j^i| > 2. \tag{4.19}$$

Since we assumed that $\mathcal{F}|_{\Sigma_i} \cong \mathcal{O}_{\Sigma_i}(d_i)$, we have $d_i = -p_i - q_i$, $i = 1, 2, 3$.

Now, let us turn to computing the obstruction $\mathrm{Obs}_\phi \in H^1(\mathbb{P}^1, L_\phi)$ constructed in Lemma 4.3.17. The Poisson lifts π_0 over $\mathcal{U}_0$ and π_1 over $\mathcal{U}_1$ can be chosen so that over $\mathcal{U}_0 \setminus \{x = 0\}$ and $\mathcal{U}_1$ they have expressions

$$\pi_0 = x^2 \partial_x \wedge ((\lambda_1 - \lambda_3)\widetilde{y}_1 \partial_{\widetilde{y}_1} + (\lambda_2 - \lambda_3)\widetilde{y}_2 \partial_{\widetilde{y}_2}) + (\lambda_2 - \lambda_3) x \widetilde{y}_1 \partial_{\widetilde{y}_1} \wedge \widetilde{y}_2 \partial_{\widetilde{y}_2},$$

$$\pi_1 = x^2 \partial_x \wedge ((\lambda_1 - \lambda_3)\widetilde{z}_1 \partial_{\widetilde{z}_1} + (\lambda_2 - \lambda_3)\widetilde{z}_2 \partial_{\widetilde{z}_2}),$$

where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, $\widetilde{z}_1 = \frac{z_1}{z_3}$, $\widetilde{z}_2 = \frac{z_2}{z_3}$. Then over $\mathcal{U}_0 \cap \mathcal{U}_1$, the difference $\pi_0 - \pi_1$ will be equal to the sum of the projectivization of $\langle \mathrm{At}_V \wedge \phi \rangle$ defined by (3.11) and the second (purely vertical) term in the expression for π_0. Let us calculate this.

Over $\mathcal{U}_0 \cap \mathcal{U}_1$, in corrdinates x, $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, on $\mathbb{P}(V)|_{\mathcal{U}_0 \cap \mathcal{U}_1}$ one has

$$\phi = x^2 \partial_x \otimes ((\lambda_1 - \lambda_3)\widetilde{y}_1 \partial_{\widetilde{y}_1} + (\lambda_2 - \lambda_3)\widetilde{y}_2 \partial_{\widetilde{y}_2}), \tag{4.20}$$

$$\mathrm{At}_V = (d\log(h_1(x)) - d\log(h_3(x))) \otimes \widetilde{y}_1 \partial_{\widetilde{y}_1} + (d\log(h_2(x)) - d\log(h_3(x))) \otimes \widetilde{y}_2 \partial_{\widetilde{y}_2}. \tag{4.21}$$

Now, let us calculate the Čech 1-cocyle of L_ϕ with respect to the cover $\{\mathcal{U}_0, \mathcal{U}_1\}$ that represents the obstruction Obs_ϕ from Lemma 4.3.17

$$\mathrm{Obs}_\phi = \langle \mathrm{At}_V \overset{\wedge}{,} \phi\rangle + (\lambda_2 - \lambda_3)x\widetilde{y_1}\partial_{\widetilde{y_1}} \wedge \widetilde{y_2}\partial_{\widetilde{y_2}} = (f(x) + \lambda_2 - \lambda_3)x\widetilde{y_1}\partial_{\widetilde{y_1}} \wedge \widetilde{y_2}\partial_{\widetilde{y_2}},$$

where

$$f(x) = -(\lambda_1 - \lambda_3)\left(\frac{xh_2'(x)}{h_2(x)} - \frac{xh_3'(x)}{h_3(x)}\right) + (\lambda_2 - \lambda_3)\left(\frac{xh_1'(x)}{h_1(x)} - \frac{xh_3'(x)}{h_3(x)}\right). \tag{4.22}$$

Note that the meromorphic section $s = x\widetilde{y_1}\partial_{\widetilde{y_1}} \wedge \widetilde{y_2}\partial_{\widetilde{y_2}}$ of L_ϕ has a simple pole over $x = 0$ and another simple pole over $x = \infty$ (and no other poles). Pick an isomorphism $\varrho : L_\phi \xrightarrow{\sim} \mathcal{T}^*_{\mathbb{P}^1}$ sending s to $\frac{dx}{x}$. Under ϱ, the cocycle $(f(x) + \lambda_2 - \lambda_3)s|_{\mathcal{U}_0\cap\mathcal{U}_1}$ will get sent to

$$(f(x) + \lambda_2 - \lambda_3)\frac{dx}{x}\bigg|_{\mathcal{U}_0\cap\mathcal{U}_1}$$

The 1-form above represents zero cohomology class in $H^1(\mathbb{P}^1, \mathcal{T}^*_{\mathbb{P}^1})$ if and only if the sum of its residues inside $\mathcal{U}_0$ equals zero. From the expressions (4.26) for h_i's we get that the sum of the residues is equal to

$$(\lambda_1 - \lambda_3)(d_2 - d_3) - (\lambda_2 - \lambda_3)(d_1 - d_3) + (\lambda_2 - \lambda_3) = -\det\begin{pmatrix} d_1 - 1 & d_2 & d_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{pmatrix}.$$

Applying Lemma 4.3.17 finishes the proof. □

Case $L_\phi \cong \mathcal{O}_{\mathbb{P}^1}(-3)$

In this case, ϕ must have the type $(x\Phi_0^{D_4})$. Since $H^1(\mathbb{P}^1, L_\phi) \cong \mathbb{C}^2$, we should come as no surprise that vanishing of the obstruction $\mathrm{Obs}_\phi \in H^1(\mathbb{P}^1, L_\phi)$ encapsulates two conditions (see Lemma 4.3.22 below). One of the conditions involves degree of the eigensheaf and λ_i's, just as in the previous cases. To discuss the second condition, recall that the eigensheaf $\mathcal{F}$ of ϕ is pushforward of a line bundle on the partial normalization $\widetilde{\Sigma}$ of the spectral curve $\Sigma = \{\prod_{i=1}^3(\theta - \lambda_i x^2\partial_x) = 0\}$ that above a small neighborhood of $x = 0$ is isomorphic to Spec $\mathbb{C}\{x_1\} \oplus \mathbb{C}\{x_2\} \oplus \mathbb{C}\{x_3\}/((x_1, x_2, x_3), (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3))$.

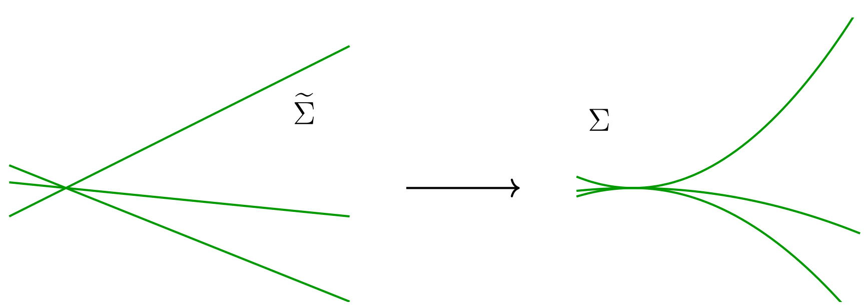


Figure 4.7: Spectral curve Σ and its partial normalization $\widetilde{\Sigma}$ for co-Higgs field of type $(x\Phi_0^{D_4})$

Note that a function $(f_1(x_1), f(x_2), f(x_3)) \in \mathbb{C}\{x_1\} \oplus \mathbb{C}\{x_2\} \oplus \mathbb{C}\{x_3\}$ belongs to $\mathcal{O}_{\widetilde{\Sigma}}$ if and only if

$$f_1(0) = f_2(0) = f_3(0) \quad \text{and} \quad \det\begin{pmatrix} f_1'(0) & f_2'(0) & f_3'(0) \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{pmatrix} = 0.$$

For a divisor $D = \sum_{j=1}^{d_1} b_j^1 + \sum_{j=1}^{d_2} b_j^2 + \sum_{j=1}^{d_3} b_j^3$ on $\widetilde{\widetilde{\Sigma}}$, where each b_j^i lies on the irreducible component $\Sigma_i = \{\theta - \lambda_i x^2 \partial_x = 0\}$ and away from $x = 0$, we define the following invariant

$$\tau_D = \det \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{where} \quad \beta_i = \sum_{j=1}^{d_i} \frac{1}{b_j^i}, \tag{4.23}$$

where in the last summation we use the choice of the coordinate x on $\mathbb{P}^1$. One can check that for fixed d_1, d_2, d_3, the divisors D and D' are linearly equivalent if and only if $\tau_D = \tau_{D'}$. This shows that the space of line bundles on $\widetilde{\widetilde{\Sigma}}$ with fixed degree is non-canonically isomorphic to $\mathbb{C}$ (the isomorphism depends on the choice of coordinate x). Moreover, for each degree (d_1, d_2, d_3) that makes the determinant of (4.24) vanish, there is a distinguished divisor $d_1 \cdot b^{(1)} + d_2 \cdot b^{(2)} + d_3 \cdot b^{(3)}$, where $b^{(i)}$ denotes the divisor $\{x_i = b\}$ on Σ_i, and $b \in \mathbb{P}^1 \setminus \{0\}$ (the linear equivalence class of this divisor is independent of b in this case). In terms of the parameter τ_D defined by (4.23), the distinguished divisor is characterized by $\tau_D = 0$. By taking the line bundle on $\widetilde{\widetilde{\Sigma}}$ corresponding to this divisor, and pushing it forward to Σ we obtain a distinguished torsion free sheaf in degree (d_1, d_2, d_3).

Lemma 4.3.22. *Let ϕ be strongly integrable, non-resonant rank 3 co-Higgs bundle (V, ϕ) over $\mathbb{P}^1$ of type $(x\Phi_0^{T_{D_4}})$. In particular, $L_\phi \subset \mathcal{T}_{ver}\mathbb{P}(V)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-3)$.*

Then ϕ admits a Poisson lift to $\mathbb{P}(V)$ if and only if the following two conditions hold

1) the following matrix has zero determinant

$$\begin{pmatrix} d_1 & d_2 & d_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{pmatrix}, \tag{4.24}$$

where $(d_1, d_2, d_3) \in \mathbb{Z}^3$ is the degree of the eigensheaf $\mathcal{F}$ of ϕ, and $\lambda_i \in \mathbb{C}$ are the constants in the characteristic equation $\prod_{i=1}^3 (\theta - \lambda_i x^2 \partial_x) = 0$ of ϕ (which are unique up to an overall multiple).

2) the eigensheaf $\mathcal{F}$ of ϕ corresponds to the distinguished divisor $d_1 \cdot b^{(1)} + d_2 \cdot b^{(2)} + d_3 \cdot b^{(3)}$, for some $b \in \mathbb{P}^1 \setminus \{0\}$, on the partial normalization $\widetilde{\widetilde{\Sigma}}$ of Σ defined by $\mathcal{F}$.

Proof. Multiplication by the section $x \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ gives the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-3) \xrightarrow{x} \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2)|_{\{x=0\}} \longrightarrow 0.$$

By taking long exact cohomological sequence we obtain

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)|_{\{x=0\}}) & \longrightarrow & H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-3)) & \xrightarrow{x} & H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) & \longrightarrow 0 \\ & \| & & \| & & \| & \\ & \mathbb{C} & \longrightarrow & \mathbb{C}^2 & \longrightarrow & \mathbb{C} & \end{array} \tag{4.25}$$

The plan is to prove that the condition 1) of the lemma is equivalent to vanishing the projection of Obs_ϕ onto $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$ in (4.25). Under this condition, the class Obs_ϕ represents and element of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)|_{\{x=0\}})$, and we are going to prove that the condition 2) of the lemma is equivalent to vanishing of this element.

Let us cover the base $\mathbb{P}^1$ with two open sets $\mathcal{U}_0 = \{|x| < 2\}$ and $\mathcal{U}_1 = \{|x| > 1/2\}$, where the coordinate

x is chosen so that the spectral curve Σ of ϕ has the only singularity over $x = 0$. Note that x also defines a coordinate on each irreducible component $\Sigma_i = \{\theta - \lambda_i x^2 \partial_x = 0\} \cong \mathbb{P}^1$. Let $\widetilde{\mathcal{U}}_j \subset \Sigma$, $j = 0, 1$ be the preimage of $\mathcal{U}_j$ under the projection. Choose an identification of $\mathcal{F}|_{\widetilde{\mathcal{U}}_1}$ with $\mathcal{I}_1 = \mathcal{O}_{\mathcal{U}_1} = \oplus_{i=1}^3 \mathcal{O}_{\Sigma_i \cap \mathcal{U}_1}$, and also an identification of $\mathcal{F}|_{\widetilde{\mathcal{U}}_0}$ with the eigensheaf $\mathcal{I}_0$ of $x\Phi_0^{D_4}$. Let h be an identification $\mathcal{I}_0|_{\widetilde{\mathcal{U}}_0 \cap \widetilde{\mathcal{U}}_1} \xrightarrow{\sim} \mathcal{I}_1|_{\widetilde{\mathcal{U}}_0 \cap \widetilde{\mathcal{U}}_1}$, such that $\mathcal{I}_0$ glued to $\mathcal{I}_1$ via h is isomorphic to $\mathcal{F}$. Let y_i be fiberwise linear coordinate on $\mathcal{O}_{\Sigma_i \cap \widetilde{\mathcal{U}}_0}$, $i = 1, 2, 3$, and z_i be fiberwise linear coordinate on $\mathcal{O}_{\Sigma_i \cap \widetilde{\mathcal{U}}_1}$, $i = 1, 2, 3$. Let h be given by three functions $h_i \in \mathcal{O}_{\Sigma_i \cap \widetilde{\mathcal{U}}_0 \cap \widetilde{\mathcal{U}}_1}$ such that $z_i = h_i(x) y_i$, $i = 1, 2, 3$. Without loss of generality, we may assume that each h_i is a Laurent polynomial in x, and let

$$h_i(x) = x^{p_i} \prod_{j=1}^{q_i} (x - a_j^i) \prod_{j=1}^{r_i} (x - b_j^i), \quad \text{where } |a_j^i| < 1/2, \quad |b_j^i| > 2. \tag{4.26}$$

Since we assumed that $\mathcal{F}|_{\Sigma_i} \cong \mathcal{O}_{\Sigma_i}(d_i)$, we have $d_i = -p_i - q_i$, $i = 1, 2, 3$. The divisor on $\widetilde{\Sigma}$ corresponding to $\mathcal{F}$ is

$$D = (d_1 - r_1) \cdot \infty^{(1)} + (d_2 - r_2) \cdot \infty^{(2)} + (d_3 - r_3) \cdot \infty^{(3)} + \sum_{j=1}^{r_1} b_j^1 + \sum_{j=1}^{r_2} b_j^2 + \sum_{j=1}^{r_3} b_j^3.$$

Now, let us turn to computing the obstruction $\mathrm{Obs}_\phi \in H^1(\mathbb{P}^1, L_\phi)$ constructed in Lemma 4.3.17. The Poisson lifts π_0 over $\mathcal{U}_0$ and π_1 over $\mathcal{U}_1$ can be chosen so that over $\mathcal{U}_0 \setminus \{x = 0\}$ and $\mathcal{U}_1$ they have expressions

$$\pi_0 = x^2 \partial_x \wedge ((\lambda_1 - \lambda_3)\widetilde{y}_1 \partial_{\widetilde{y}_1} + (\lambda_2 - \lambda_3)\widetilde{y}_2 \partial_{\widetilde{y}_2}),$$

$$\pi_1 = x^2 \partial_x \wedge ((\lambda_1 - \lambda_3)\widetilde{z}_1 \partial_{\widetilde{z}_1} + (\lambda_2 - \lambda_3)\widetilde{z}_2 \partial_{\widetilde{z}_2}),$$

where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, $\widetilde{z}_1 = \frac{z_1}{z_3}$, $\widetilde{z}_2 = \frac{z_2}{z_3}$. Then over $\mathcal{U}_0 \cap \mathcal{U}_1$, the difference $\pi_0 - \pi_1$ is equal to the projectivization of $\langle \mathrm{At}_V \wedge \phi \rangle$ defined by (3.11). Let us calculate this.

Over $\mathcal{U}_0 \cap \mathcal{U}_1$, in corrdinates x, $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$, on $\mathbb{P}(V)|_{\mathcal{U}_0 \cap \mathcal{U}_1}$ one has

$$\phi = x^2 \partial_x \otimes ((\lambda_1 - \lambda_3)\widetilde{y}_1 \partial_{\widetilde{y}_1} + (\lambda_2 - \lambda_3)\widetilde{y}_2 \partial_{\widetilde{y}_2}), \tag{4.27}$$

$$\mathrm{At}_V = (d \log(h_1(x)) - d \log(h_3(x))) \otimes \widetilde{y}_1 \partial_{\widetilde{y}_1} + (d \log(h_2(x)) - d \log(h_3(x))) \otimes \widetilde{y}_2 \partial_{\widetilde{y}_2}. \tag{4.28}$$

Now, let us calculate the Čech 1-cocyle of L_ϕ with respect to the cover $\{\mathcal{U}_0, \mathcal{U}_1\}$ that represents the obstruction Obs_ϕ from Lemma 4.3.17

$$\mathrm{Obs}_\phi = \langle \mathrm{At}_V \wedge \phi \rangle = f(x) x \widetilde{y}_1 \partial_{\widetilde{y}_1} \wedge \widetilde{y}_2 \partial_{\widetilde{y}_2},$$

where

$$f(x) = -(\lambda_1 - \lambda_3)\left(\frac{x h_2'(x)}{h_2(x)} - \frac{x h_3'(x)}{h_3(x)}\right) + (\lambda_2 - \lambda_3)\left(\frac{x h_1'(x)}{h_1(x)} - \frac{x h_3'(x)}{h_3(x)}\right). \tag{4.29}$$

Note that the meromorphic section $s = x\widetilde{y}_1 \partial_{\widetilde{y}_1} \wedge \widetilde{y}_2 \partial_{\widetilde{y}_2}$ of L_ϕ has a double pole over $x = 0$ and a simple pole over $x = \infty$ (and no other poles). Also, the meromorphic section xs of $L_\phi(1)$ has a simple pole over $x = 0$ and another simple pole over $x = \infty$. Pick an isomorphism $\varrho : L_\phi(1) \xrightarrow{\sim} \mathcal{T}^*_{\mathbb{P}^1}$ sending xs to $\frac{dx}{x}$. Under ϱ, the cocycle $x f(x) s|_{\mathcal{U}_0 \cap \mathcal{U}_1}$ will get sent to

$$f(x) \frac{dx}{x}\bigg|_{\mathcal{U}_0 \cap \mathcal{U}_1} = \Big(-(\lambda_1 - \lambda_3)(d \log(h_2(x)) - d \log(h_3(x))) + (\lambda_2 - \lambda_3)(d \log(h_1(x)) + d \log(h_3(x))) \Big)\Big|_{\mathcal{U}_0 \cap \mathcal{U}_1}.$$

The 1-form above represents zero cohomology class in $H^1(\mathbb{P}^1, \mathcal{T}^*_{\mathbb{P}^1})$ if and only if the sum of its residues inside $\mathcal{U}_0$ equals zero. From the expressions (4.26) for h_i's we get that the sum of the residues is equal to

$$(\lambda_1 - \lambda_3)(d_2 - d_3) - (\lambda_2 - \lambda_3)(d_1 - d_3) = -\det \begin{pmatrix} d_1 & d_2 & d_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{pmatrix}.$$

This shows that the image $x\mathrm{Obs}_\phi \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$, as in (4.25), vanishes if and only if the condition 1) of the lemma holds. Assuming $x\mathrm{Obs}_\phi$ does vanish in $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$, we can interpret Obs_ϕ as the the element $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)|_{x=0}) \cong \mathbb{C}$ by taking the value $f_0(0)$, where f_0 is the meromorphic function appearing in a decomposition

$$f(x)\frac{dx}{x} = f_0(x)dx - f_1(x)dx,$$

where $f_i(x)dx$ has no poles inside $\mathcal{U}_i$. One way to obtain such a decomposition is to take

$$f_0(x) = -(\lambda_1 - \lambda_3)\left(\sum_{j=1}^{r_2} \frac{1}{x - b_j^2} + \sum_{j=1}^{r_3} \frac{1}{x - b_j^3}\right) + (\lambda_2 - \lambda_3)\left(\sum_{j=1}^{r_1} \frac{1}{x - b_j^1} + \sum_{j=1}^{r_3} \frac{1}{x - b_j^3}\right),$$

$$f_1(x) = (\lambda_1 - \lambda_3)\frac{p_2 - p_3}{x} - (\lambda_2 - \lambda_3)\frac{p_1 - p_3}{x} -$$

$$(\lambda_1 - \lambda_3)\left(\sum_{j=1}^{q_2} \frac{1}{x - a_j^2} + \sum_{j=1}^{q_3} \frac{1}{x - a_j^3}\right) - (\lambda_2 - \lambda_3)\left(\sum_{j=1}^{q_1} \frac{1}{x - a_j^1} + \sum_{j=1}^{q_3} \frac{1}{x - a_j^3}\right).$$

We obtain $f_0(0) = -\tau_D$ where τ_D is the parameter defined by (4.23). It remains to use that $\tau_D = 0$ if and only if the condition 2) of the lemma holds. ☐

Example 4.3.23. The sheaf $\mathcal{O}_{\widetilde{\Sigma}}$ satisfies both the conditions 1) and 2) of Lemma 4.3.22. The corresponding co-Higgs field on $V = p_*\mathcal{O}_{\widetilde{\Sigma}} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ has the expression

$$\phi = \begin{pmatrix} \lambda_1 x^2 & 0 & 0 \\ x & \lambda_2 x^2 & 0 \\ 0 & x & \lambda_3 x^2 \end{pmatrix} \partial_x,$$

over $\{x \neq \infty\}$. The Poisson lift of ϕ to $\mathbb{P}(V)$ guaranteed by Lemma 4.3.22 has the expression

$$\pi \quad = \quad x\partial_x \wedge \varphi + (2\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}) \wedge \varphi,$$

where

$$\varphi = (\lambda_1 - \lambda_3)x\widetilde{y}_1\partial_{\widetilde{y}_1} + (\lambda_2 - \lambda_3)x\widetilde{y}_2\partial_{\widetilde{y}_2} + \widetilde{y}_1\partial_{\widetilde{y}_2} - \widetilde{y}_2(\widetilde{y}_1\partial_{\widetilde{y}_1} + \widetilde{y}_2\partial_{\widetilde{y}_2}),$$

where $\widetilde{y}_1 = \frac{y_1}{y_3}$, $\widetilde{y}_2 = \frac{y_2}{y_3}$.

4.3.4 Global Poisson lifts. Resonant case.

Throughout this subsection, $\phi \in \mathrm{Hom}(V, V \otimes \mathcal{T}_{\mathbb{P}^1})$ denotes a traceless, strongly integrable co-Higgs field on a rank 3 bector bundle V over $\mathbb{P}^1$, whose spectral curve Σ is reduced and satisfies the resonance condition (Definition 4.3.3). Recall that ϕ being resonant means that generically $2\lambda_1 = \lambda_2 + \lambda_3$, where λ_i's are eigenvalues of ϕ. Since we assume $\sum_{i=1}^{3} \lambda_i = 0$, we get $\lambda_1 = 0$ (we will always denote by λ_1 the zero eigenvalue). As a result the spectral curve Σ has an irreducible component $\Sigma_1 \cong \mathbb{P}^1$ given by the zero section of $\mathcal{T}_{\mathbb{P}^1}$. The two other eigenvalues λ_2, λ_3 form either another subcurve $\Sigma_{23} \subset \Sigma$, which may either be irreducible, or split into two components Σ_2, Σ_3. We are going to study when ϕ admits a lift to Poisson structure on $\mathbb{P}(V)$.

Case of nowhere vanishing ϕ.

Proposition 4.3.24. *Let C be a smooth curve over $\mathbb{C}$, V be a vector bundle over C, and $\phi \in \mathrm{Hom}(V, V \otimes \mathcal{T}_C)$ be a traceless strongly integrable, resonant co-Higgs field on V that does not vanish at any point of C. Let Q be the $\mathbb{P}^1$-bundle over C constructed in Lemma 4.3.14.*

Then a lift of ϕ to a Poisson structure π on $\mathbb{P}(V)$ uniquely determines a connection on the $\mathbb{P}^1$-bundle Q that is tangent to the distinguished section $\mathbb{P}(\ker(\phi^*)^{\otimes 2})$, and vice versa, a connection on Q tangent to $\mathbb{P}(\ker(\phi^*)^{\otimes 2})$ uniquely determines a Poisson structure π on $\mathbb{P}(V)$ lifting ϕ.

This connection is uniquely determined by the property that any flat local section q of Q, when viewed as a meromorphic function on $\mathbb{P}(V)$, is Casimir with respect to π.

Proof. This is a direct consequence of the local version of the statement, that is, Proposition 4.3.15. □

The proposition above, in particular, shows that if C is a non-compact curve, then every traceless, strongly integrable, resonant co-Higgs field on a rank 3 vector bundle V over C admits a Poisson lift to $\mathbb{P}(V)$. From here on, we will specialize to the case $C = \mathbb{P}^1$.

Recall that $Q = \mathbb{P}(K)$, where K is the rank 2 subbundle over S^2V^* defined as the kernel of the sheaf morphism $Lie_\phi : S^2V^* \to S^2V^* \otimes \mathcal{T}_{\mathbb{P}^1}$. The vector bundle K has a distinguished line subbundle $K_1 = \ker(\phi^*)^{\otimes 2}$. Denote by K_2 the line bundle K/K_1.

Lemma 4.3.25. *In the above notation, the $\mathbb{P}^1$ bundle $Q = \mathbb{P}(K)$ over $\mathbb{P}^1$ admits a connection tangent to the section $\mathbb{P}(K_1)$ if and only if $\deg K_2 = \deg K_1$. If such connection exists then it is unique.*

Proof. The condition $\deg K_2 = \deg K_1$ on $\mathbb{P}^1$ is equivalent to $K_2 \cong K_1$. If it does hold, then the short exact sequence $0 \to K_1 \to K \to K_2 \to 0$ splits. Therefore, $Q = \mathbb{P}(K_1 \oplus K_1) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$ is trivial and has unique connection.

Conversely, assume that Q admits a connection tangent to the section $\mathbb{P}(K_1)$. Since $\mathbb{P}^1$ is simply connected, this implies that Q is a trivial $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. By taking a flat section of Q disjoint from the section $\mathbb{P}(K_1)$, we obtain a line subbunde K' of K such that $K \cong K_1 \oplus K'$. One must then have $K' \cong K_2$. It remains to note that the bundle $\mathbb{P}(K_1 \oplus K_2)$ is trivial only if $\deg K_2 = \deg K_1$. □

Let us now discuss how to calculate $\deg K_2$. We are going to construct a line subbundle $K_{sing} \subset K$, whose degree can be read off the spectral data, and calculate the number of intersections of $\mathbb{P}(K_{sing})$ with $\mathbb{P}(K_1)$ (counting with multiplicities). To construct such K_{sing}, first consider the restriction morphism $\alpha : \mathcal{F} \to \mathcal{F}|_{\Sigma_{23}}$, where $\mathcal{F}$ is the eigensheaf of ϕ and Σ_{23} is the subcurve of the spectral curve Σ obtained by removing its zero section component Σ_1. By pushing α forward to the base $\mathbb{P}^1$, we obtain a surjective morphism $p_*\alpha : V \to W$, where $W = p_*(\mathcal{F}|_{\Sigma_{23}})$ is a rank 2 bundle. Let us denote $\beta = S^2(p_*\alpha) : S^2V \to S^2W$. Note that ϕ induces a co-Higgs field ψ on W, whose spectral curve is Σ_{23} and the eigensheaf is $\mathcal{F}|_{\Sigma_{23}}$. The curve Σ_{23} can have only A_n singularities. Therefore, the rank 1 torsion free sheaf $\mathcal{F}|_{\Sigma_{23}}$ is a pushforward of a line bundle L on a partial normalizaiton $\widetilde{\Sigma}_{23}$ of Σ_{23}. Let $\widetilde{\iota} : \widetilde{\Sigma}_{23} \to \widetilde{\Sigma}_{23}$ be the involution of $\widetilde{\Sigma}_{23}$ obtained by flipping the two branches of the cover $\widetilde{p} : \widetilde{\Sigma}_{23} \to \mathbb{P}^1$. Then the morphism $W \otimes_{\mathcal{O}_{\mathbb{P}^1}} W \longrightarrow \widetilde{p}_*(L \otimes_{\mathcal{O}_{\widetilde{\Sigma}_{23}}} \widetilde{\iota}^* L)$, $\widetilde{p}_* s_1 \otimes \widetilde{p}_* s_2 \longmapsto \widetilde{p}_*(s_1 \widetilde{\iota}^*(s_2))$, induces a surjective morphism $\gamma : S^2W \to \mathcal{L}$, where $\mathcal{L}$ is the line bundle over $\mathbb{P}^1$ whose local sections are $\mathbb{Z}/2\mathbb{Z}$-invariant section of $L \otimes \widetilde{\iota}^* L$. Precomposing γ with $\beta : S^2V \to S^2W$ constructed earlier we obtain a surjective bundle morphism $S^2V \longrightarrow \mathcal{L}$. Finally, we define $K_{sing} = \mathcal{L}^*$ and the embedding $\beta^*\gamma^* : K_{sing} \longrightarrow S^2V^*$. Over a small analytic $\mathcal{U} \subset \mathbb{P}^1$, if we choose coordinates x, y_1, y_2, y_3 so that $\phi = \partial_x \wedge (\lambda_2 y_2 \partial_{y_2} - \lambda_2 y_3 \partial_{y_3})$, the line bundle $K_{sing} \subset S^2V^*$ is spanned by the monomial $y_2 y_3$. This explains that $K_{sing} \subset K$. Also, it explains the notation K_{sing}. Recall that $\{y_2 y_3 = 0\}$ is the unique reduced singular member of the fiberwise pencil of quadrics Q constructed in Lemma 4.3.14. The section $q_{sing} = \mathbb{P}(K_{sing})$ of Q has been playing an important role in the previous section.

Assuming the number of intersection of $\mathbb{P}(K_{sing})$ with $\mathbb{P}(K_1)$, counting with multiplicities, equals m, we can calculate $\deg K_2 = \deg K_{sing} + m$. The condition $\deg K_1 = \deg K_2$ from Proposition 4.3.24 is equivalent to $\deg K_1 = \deg K_{sing} + m$. Both degrees in the latter equality are easily calculated from the spectral data $(\Sigma, \mathcal{F})$ of ϕ. Let d_1 be the degree of $\mathcal{F}|_{\Sigma_1}$, and d_{23} be the degree of $\mathcal{F}|_{\Sigma_{23}}$ (if Σ_{23} has two irreducible components Σ_2, Σ_3, then d_{23} denotes the sum of degrees of $\mathcal{F}|_{\Sigma_2}$ and $\mathcal{F}|_{\Sigma_3}$). Then $\deg K_1 = -2d_1$, $\deg K_{sing} = -d_{23}$. So, the condition $\deg K_1 = \deg K_2$ from Proposition 4.3.24 is equivalent to

$$2d_1 = d_{23} - m. \tag{4.30}$$

To calculate the number of intersections of $\mathbb{P}(K_{sing})$ with $\mathbb{P}(K_1)$, we need to calculate the order of contact of $\mathbb{P}(K_{sing})$ with $\mathbb{P}(K_1)$ at each singularity of Σ and add them all up.

As an example, let us calculate the order of contact of $\mathbb{P}(K_{sing})$ with $\mathbb{P}(K_1)$ for the co-Higgs $\Phi_0^{A_1}$. As we have calculated in Subsection 4.3.2, the generators for K locally around the singularity $x = 0$ are given by y_1^2 and $y_2^2 - 2y_1y_3 - xy_3^2$. The subbundle K_1 is spanned by y_1^2. The subbundle K_{sing} is spanned by $y_1^2 - xy_2^2 + 2xy_1y_3 + x^2y_3^2$ (the singular point of such a quadric over $x \neq 0$ is $\{y_2 = 0, y_1 = -xy_3\}$, as shown on Figure 4.4). Therefore, the order of contact of $\mathbb{P}(K_{sing})$ with $\mathbb{P}(K_1)$ at $x = 0$ equals one. On the other hand, for $\Phi_1^{A_1}$ the generators are y_1^2 (for K_1) and $y_2^2 - xy_3^2$ (for K_{sing}). At $x = 0$ these two do not intersect. So, the order of contact here is zero (cf. Figure 4.5). Using the explicit expressions for the generators of K calculated in Subsection 4.3.2, we calculate the order of conatact of $\mathbb{P}(K_{sing})$ with $\mathbb{P}(K_1)$ for each of the relevant local normal forms in Table 4.2.

This leads us to the classification of the zero trace resonant rank 3 co-Higgs bundles (V, ϕ) over $\mathbb{P}^1$ that admit a Poisson lift to $\mathbb{P}(V)$, assuming ϕ vanishes nowhere in $\mathbb{P}^1$. We remark that by Proposition 4.3.24 and Lemma 4.3.25, such Poisson lift is unique, whenever it exists.

The co-Higgs field ϕ is specified by the types and locations of singularities of the spectral curve Σ. One can have either four A_1 singularities, or two A_1 plus one D_4, or two D_4, or one E_7 plus one A_1, or one T_{36} singularity. This fixes the spectral curve Σ uniquely, up to rescaling in the fiber direction in $\mathcal{T}^1_{\mathbb{P}}$. Then for each of the singularities of Σ one has to specify the local normal form of the co-Higgs field from Table 4.2. This fixes the isomorphism type of the spectral sheaf $\mathcal{F}$ of ϕ near each singularity. Finally, one has to specify the spectral sheaf $\mathcal{F}$ on Σ so that its degree (d_1, d_{23}) satisfies the equation (4.30), where m is the sum of orders of contact of $\mathbb{P}(K_{sing})$ with $\mathbb{P}(K_1)$ over all singularities.

ϕ	$\Phi_0^{A_1}$	$\Phi_1^{A_1}$						
Order of contact	1	0						
ϕ	$\Phi_0^{D_4}$	$\Phi_1^{D_4}$	$\Phi_2^{D_4}$	$\Phi_3^{D_4}$	$\Phi_4^{D_4}$	$\Phi_5^{D_4}$		
Order of contact	2	0	1	0	0	0		
ϕ	$\Phi_0^{E_7}$	$\Phi_1^{E_7}$	$\Phi_4^{E_7}$	$\Phi_5^{E_7}$	$\Phi_6^{E_7}$	$\Phi_7^{E_7}$		
Order of contact	3	0	1	0	2	1		
ϕ	$\Phi_0^{T_{36}}$	$\Phi_1^{T_{36}}$	$\Phi_2^{T_{36}}$	$\Phi_3^{T_{36}}$	$\Phi_4^{T_{36}}$	$\Phi_5^{T_{36}}$	$\Phi_6^{T_{36}}$	
Order of contact	4	2	0	0	1	1	1	
ϕ	$\Phi_7^{T_{36}}$	$\Phi_8^{T_{36}}$	$\Phi_9^{T_{36}}$	$\Phi_{10}^{T_{36}}$	$\Phi_{11}^{T_{36}}$	$\Phi_{12}^{T_{36}}$	$\Phi_{13}^{T_{36}}$	$\Psi_\beta^{T_{36}}$
Order of contact	0	2	2	3	0	0	0	1

Table 4.2: Order of contact of $\mathbb{P}(K_{sing})$ with $\mathbb{P}(K_1)$

Example 4.3.26. Consider the case $\Sigma = \{\theta(\theta^2 - w) = 0\}$, where w is a symmetric bivector on $\mathbb{P}^1$ with four distinct zeros x_1, x_2, x_3, x_4. Such Σ has four A_1 singularities, one per each x_i. Let ϕ be isomorphic to $\Phi_0^{A_1}$ around each x_i, i.e. the spectral sheaf $\mathcal{F}$ is a line bundle over Σ. Without loss of generality, we may assume that $d_1 = 0$. Then the equality (4.30) says that to ensure existence of a Poisson lift to $\mathbb{P}(V)$, we must insist that $d_{23} = 4$. So, the spectral sheaf $\mathcal{F}$ is obtained from $L_1 \cong \mathcal{O}_{\Sigma_1}$ and a degree 4 line bundle L_{23} over the elliptic curve Σ_{23} via gluing each fiber $L_{23}|_{x_i}$ to the corresponding fiber $L_1|_{x_i}$. We obtain a six dimensional family of Poisson structures on $\mathbb{P}(V)$. One parameter controls the cross-ratio $(x_3 - x_1)(x_4 - x_2)/(x_3 - x_2)(x_4 - x_1)$, one parameter controls the isomorphism class of L_{23}, three parameters control the gluings $L_1|_{x_i} \xrightarrow{\sim} L_{23}|_{x_i}$, and one parameter controls fiberwise scaling of $\Sigma \subset \mathcal{T}_{\mathbb{P}^1}$.

For a generic $\mathcal{F}$ in the six dimensional family, the line bundle L_{23} is not isomorphic to $p^*\mathcal{O}_{\mathbb{P}^1}$. For such $\mathcal{F}$, the pushforward $p_*\mathcal{F}$ is $V \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Moreover, one has the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow V \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0,$$

where the surjection $V \longrightarrow \mathcal{O}_{\mathbb{P}^1}$ is the pushforward of restriction morphism $\mathcal{F} \longrightarrow \mathcal{F}|_{\Sigma_1}$, and the kernel $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ coincides with the image of $\phi : \mathcal{T}^*_{\mathbb{P}^1} \otimes V \longrightarrow V$. One can choose a splitting of the short exact sequence, and this will make the matrix of ϕ to have the block triangular form

$$\phi = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline u_1 & v_1 & v_2 \\ u_2 & v_3 & -v_1 \end{array}\right),$$

where $u_1, u_2 \in H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}(-1))$, $v_1, v_2, v_3 \in H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1})$. Over an affine chart of $\mathbb{P}^1$ with coordinate x, expressing $v_i = f_i(x)\partial_x$, $u_i = g_i(x)\partial_x$, for some polynomials f_i of degree ≤ 2 and g_i of degree ≤ 1, we can write down the two generators for $K \subset S^2V^*$: $s = y_1^2$ and $t = f_3y_2^2 - f_2y_3^2 - 2f_1y_2y_3 + 2g_2y_1y_2 - 2g_1y_1y_3$. Let $\alpha = d\log(t/s)$, and τ be a trivector on $\mathbb{P}(V)$ vanishing on $\{y_1 = 0\}$ and $\{t = 0\}$. In coordinates $\widetilde{y}_2 = \frac{y_2}{y_1}$,

$\widetilde{y}_3 = \frac{y_3}{y_1}$, one has

$$\begin{aligned}
&\widetilde{t} = \frac{t}{s} = f_3\widetilde{y}_2^2 - f_2\widetilde{y}_3^2 - 2f_1\widetilde{y}_2\widetilde{y}_3 + 2g_2\widetilde{y}_2 - 2g_1\widetilde{y}_3,\\
&\alpha = \frac{d\widetilde{t}}{\widetilde{t}},\\
&\tau = -\tfrac{1}{2}\widetilde{t}\,\partial_x \wedge \partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3},\\
&\pi = \iota_\alpha(\tau) = \partial_x \wedge (f_3\widetilde{y}_2\partial_{\widetilde{y}_3} + f_2\widetilde{y}_3\partial_{\widetilde{y}_2} + f_1\widetilde{y}_2\partial_{\widetilde{y}_2} - f_1\widetilde{y}_3\partial_{\widetilde{y}_3} + g_2\partial_{\widetilde{y}_3} + g_1\partial_{\widetilde{y}_2}) -\\
&-\tfrac{1}{2}(f_3'\widetilde{y}_2^2 - f_2'\widetilde{y}_3^2 - 2f_1'\widetilde{y}_2\widetilde{y}_3 + 2g_2'\widetilde{y}_2 - 2g_1'\widetilde{y}_3)\partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3}.
\end{aligned}$$

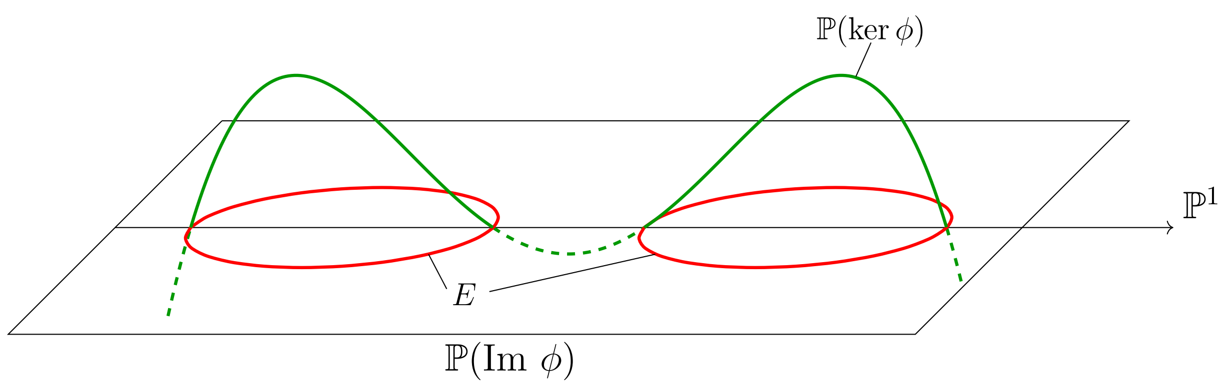


Figure 4.8: Geometry of Poisson structure on $\mathbb{P}(V)$ coming from a co-Higgs with four A_1 singularities.

The constructed Poisson structure vanishes on an elliptic curve $E \subset \mathbb{P}(\mathrm{Im}\ \phi) \subset \mathbb{P}(V)$ (see Figure 4.8). This elliptic curve is isomorphic to the component Σ_{23} of the spectral curve via the isomorphism Eig in Lemma 3.3.2. The section $\mathbb{P}(\ker\phi) \subset \mathbb{P}(V)$ consists of "fake zeros" of the Poisson structure, in the sense that the contribution from the co-Higgs tensor vanishes at the points of $\mathbb{P}(\ker\phi)$, but the whole Poisson tensor π generically does not. Poisson tensor does vanish, however, at the points of $\mathbb{P}(\ker\phi)$, where the connection on the $\mathbb{P}^1$ bundle Q (discussed in Lemma 4.3.14 and Proposition 4.3.15) is tangent to the section $\mathbb{P}(K_{sing})$. To caluclate the number of such points, note that the normal bundle of $\mathbb{P}(K_{sing}) \cong \mathbb{P}^1$ in Q is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(8)$, and the connection on Q gives a sheaf morphism $\mathcal{T}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(8)$. Such a morphism has to vanish at six points (counting multiplicities). So, there are six points on $\mathbb{P}(\ker\phi)$, where the Poisson tensor π vanishes.

One final fact about the Poisson lifts of resonant nowhere vanishing rank 3 co-Higgs fields is the following topological restriction on the $\mathbb{P}^2$-bundles over $\mathbb{P}^1$ that arise this way.

Proposition 4.3.27. *Let ϕ be a traceless resonant nowhere vanishing co-Higgs field on a rank 3 vector bundle $p : V \to \mathbb{P}^1$. Let ϕ admit a Poisson lift π to $\mathbb{P}(V)$. Then $\deg V \equiv -2 \mod 3$.*

In other words, after twisting V with $p^*\mathcal{O}_{\mathbb{P}^1}(k)$ for some $k \in \mathbb{Z}$, the total space of V is Calabi-Yau.

Proof. Consider the fiberwise Euler sequence

$$0 \longrightarrow \mathcal{O}_V(-1) \longrightarrow p^*V \longrightarrow \mathcal{T}_{\mathbb{P}(V)}(-1) \longrightarrow 0.$$

By twisting it with $p^*\mathcal{T}_{\mathbb{P}^1}$ and taking the determinant, we obtain that the anticanonical bundle ω_V^{-1} is isomorphic to $\omega_{\mathbb{P}(V)}^{-1} \otimes \mathcal{O}_{\mathbb{P}(V)}(-3)$. Let us prove that the latter bundle is trivial, possibly after replacing V with $V \otimes p^*\mathcal{O}_{\mathbb{P}^1}(k)$ for some $k \in \mathbb{Z}$. The rest of the proposition will then follow from Proposition 4.0.6.

By Proposition 4.3.24, the $\mathbb{P}^1$-bundle Q constructed in Lemma 4.3.14 has a flat connection tangent to $\mathbb{P}(K_1)$. Then Lemma 4.3.25 implies that, possibly after replacin V with $V \otimes p^*\mathcal{O}_{\mathbb{P}^1}(k)$ for some $k \in \mathbb{Z}$, one has $K = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$, $K_1 = \mathcal{O}_{\mathbb{P}^1}$. Choose two disjoint nowhere vanishing sections $s \in H^0(\mathbb{P}^1, K_1)$ and $t \in H^0(\mathbb{P}^1, K)$. Let us view $s, t \in H^0(\mathbb{P}^1, S^2V^*)$ as fiberwise quadratic functions on V. Then the 1-form $\alpha = d\log(t/s)$ is Casimir for π, has no zeros, at least away from a codimension two locus, and has logarithmic poles along the subbundle $\mathbb{P}(\mathrm{Im}\ \phi) \subset \mathbb{P}(V)$ and along the fiberwise family of quadrics $\{t = 0\}$. The fact that $\iota_\alpha(\pi) = 0$ implies that $\pi = \iota_\alpha(\tau)$, for some $\tau \in H^0(\mathbb{P}(V), \wedge^3\mathcal{T}_{\mathbb{P}(V)})$. The trivector τ has order 1 zeros along $\mathbb{P}(\mathrm{Im}\ \phi)$ and $\{t = 0\}$, so it gives a trivialization of the bundle $\omega_{\mathbb{P}(V)}^{-1} \otimes \mathcal{O}_{\mathbb{P}(V)}(-3)$. □

Case of ϕ having a zero.

Note that ϕ can have at most two zeros. If ϕ has two zeros counting multiplicities, then ϕ has the type either $(\Phi_6^{D_4}) - (\Phi_6^{D_4})$, or $(x\Phi_6^{D_4})$. Then V splits globally as $V = L_1 \oplus L_2 \oplus L_3$, and $\phi = \mathrm{diag}(0, 1, -1) \otimes v$, for some $v \in H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1})$. By Lemma 4.3.4.b), all global Poisson lifts of such ϕ are of the form

$$\pi = v \wedge (\widetilde{y}_2\partial_{\widetilde{y}_2} - \widetilde{y}_3\partial_{\widetilde{y}_3}) + f(x)\widetilde{y}_2\widetilde{y}_3\partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3} + g(x)\partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3},$$

where $\widetilde{y}_2 = \frac{y_2}{y_1}$, $\widetilde{y}_3 = \frac{y_3}{y_1}$, f is a constant function in x, and g is a polynomial of degree at most $d_2 + d_3 - 2d_1$, where $d_i = \deg L_i$ (if $d_2 + d_3 - 2d_1 < 0$, then g must vanish).

From now on we are assuming that ϕ has exactly one simple zero. If C is a smooth (not necessarily compact) curve over $\mathbb{C}$ and $x_0 \in C$, by a $\mathcal{T}_C(-\log x_0)$-connection on a fiber bundle $p : F \to C$ we mean a splitting of the short exact sequence of sheaves on F

$$0 \longrightarrow \mathcal{T}_{vert}F \longrightarrow \mathcal{T}_F(-\log x_0) \longrightarrow p^*\mathcal{T}_C(-\log x_0) \longrightarrow 0,$$

where $\mathcal{T}_{vert}F$ is the sheaf of vector fields on F tangent to the fibers of F, $\mathcal{T}_F(-\log x_0)$ is the sheaf of vector fields on F tangent to the fiber $F|_{x_0}$ and $\mathcal{T}_C(-\log x_0)$ is the sheaf of vector fields on C vanishing at x_0.

Proposition 4.3.28. *Let C be a smooth curve over $\mathbb{C}$, V be a rank 3 vector bundle over C, and $\phi \in \mathrm{Hom}(V, V \otimes \mathcal{T}_C)$ be a traceless strongly integrable, resonant co-Higgs field on V that has the only zero at $x_0 \in C$ (which is simple). Let Q be the $\mathbb{P}^1$-bundle over C constructed in Lemma 4.3.14.*

Then a lift of ϕ to a Poisson structure π on $\mathbb{P}(V)$ uniquely determines a $\mathcal{T}_C(-\log x_0)$-connection on the $\mathbb{P}^1$-bundle Q that is tangent to the distinguished section $\mathbb{P}(K_1)$, and vice versa, a $\mathcal{T}_C(-\log x_0)$-connection on Q tangent to $\mathbb{P}(K_1)$ uniquely determines a Poisson structure π on $\mathbb{P}(V)$ lifting ϕ.

This connection is uniquely determined by the property that any flat local section q of Q, when viewed as a meromorphic function on $\mathbb{P}(V)$, is Casimir with respect to π.

Proof. Assuming that the Poisson lift π on $\mathbb{P}(V)$ exists, let us construct the logarithmic connection.

Proposition 4.3.15 ensures existence and uniqueness of such connection over $C \setminus \{x_0\}$. We only need to check that such a connection has a logarithmic pole over x_0. Choose a local coordinate x on C so that x_0 is $x = 0$, and express $\phi = x\phi_1$ for some nowhere vanishing ϕ_1. In the course of proof of Proposition 4.3.15, we checked that if σ is a unimodular lift of π to V, then for any section s of $K \subset S^2V^*$ one has $\sigma^\#(ds) \subset K \otimes \langle\varphi_1\rangle$, where $\langle\varphi_1\rangle$ is the line subbundle of $V^* \otimes V$ spanned by the vertical vector field $\varphi_1 = \langle\phi_1, \alpha\rangle$, for a non-vanihsing $\alpha \in \mathcal{T}_C^*$. The co-Higgs field ϕ gives a sheaf isomorphism map $\mathcal{T}_C^*(\log x_0) \longrightarrow \langle\varphi_1\rangle$, inverting which we obtain an isomorphism $\langle\varphi_1\rangle \longrightarrow \mathcal{T}_C^*(\log x_0)$. Composing the operation of taking σ-Hamiltonian vector field with the latter isomorphism, we obtain a linear logarithmic connection $\nabla : K \longrightarrow K \otimes \mathcal{T}_C^*(\log x_0)$. Projectivizing ∇, we get a projective logarithmic connection on $Q = \mathbb{P}(K)$.

Now, let us assume that we are given a $\mathcal{T}_C(-\log x_0)$-connection j on Q tangent to $\mathbb{P}(K_1)$. Let us construct the Poisson lift π of ϕ. Proposition 4.3.15 ensures existence and uniqueness of such a lift π over $C \setminus \{x_0\}$. We only need to check that π extends smoothly over the fiber $\mathbb{P}(V|_{x_0})$. Choose two linearly independent sections s, t of K defined in a neighborhood of x_0, so that s is a section of $K_1 \subset K$. Choose a local coordinate x on C so that x_0 is $x = 0$, and express $\phi = x\phi_1$ for some nowhere vanishing ϕ_1. Let π_1 be the Poisson lift of ϕ_1 to $\mathbb{P}(V)$ such that t/s is Casimir for π (such π_1 exists and is unique by Proposition 4.3.15). Let j be projectivization of a linear connection on K whose matrix in the trivialization defined by s, t is

$$\begin{pmatrix} 0 & g(x) \\ 0 & f(x) \end{pmatrix} \frac{dx}{x},$$

where f, g are holomorphic functions on C defined in a neighborhood of x_0. Then the Poisson lift π of ϕ corresponding to the connection j has the expression

$$\pi = x\pi_1 - f(x)\frac{t}{s}\beta - g(x)\beta,$$

where β is the bivector on $\mathbb{P}(V)$ tangent to the fibers and uniquely determined by the condition $\iota_{d(t/s)}\beta = \varphi_1$, where $\varphi_1 = \langle\phi_1, dx\rangle$. Note that β vanishes to order 3 along $\mathbb{P}(K_1)$, while s vanishes to order 2 along $\mathbb{P}(K_1)$, so, the expression $\frac{t}{s}\beta$ defines a smooth bivector. Therefore, π is smooth. □

Lemma 4.3.29. *Let K be a rank 2 vector bundle over $\mathbb{P}^1$ and K_1 be its rank 1 subbundle, and let $x_0 \in \mathbb{P}^1$. Then the projective bundle $Q = \mathbb{P}(K)$ admits a $\mathcal{T}_{\mathbb{P}^1}(-\log x_0)$-connection tangent to $\mathbb{P}(K_1)$ if and only if $K \cong K_1 \oplus K_2$, where K_2 is the line bundle K/K_1.*

If $K \cong K_1 \oplus K_2$, then the set of all $\mathcal{T}_{\mathbb{P}^1}(-\log x_0)$-connections on $\mathbb{P}(K)$ tangent to $\mathbb{P}(K_1)$ forms an affine space over $\mathrm{Hom}(K_2, K_1 \otimes \mathcal{T}_{\mathbb{P}^1}^*(\log x_0)) \cong \mathbb{C}^l$, *where* $l = \min(0, \deg K_1 - \deg K_2)$.

Proof. Without loss of generality we may assume $K_1 \cong \mathcal{O}_{\mathbb{P}^1}$. Let $K_2 \cong \mathcal{O}_{\mathbb{P}^1}(n)$, where $n \in \mathbb{Z}$.

First, let us assume that $K \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ and describe all $\mathcal{T}_{\mathbb{P}^1}(-\log x_0)$-connections on $\mathbb{P}(K)$ tangent to $\mathbb{P}(K_1)$. Let us choose a coordinate x on $\mathbb{P}^1$ so that x_0 is $x = 0$. Let us also choose fiberwise linear coordinates y_1, y_2 over $\mathcal{U}_0 = \{x \neq \infty\}$, and $\widetilde{y}_1$, $\widetilde{y}_2$ over $\mathcal{U}_1 = \{x \neq 0\}$ so that

$$\begin{pmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \end{pmatrix} = T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & x^{-n} \end{pmatrix}.$$

Define a $\mathcal{T}_{\mathbb{P}^1}(-\log x_0)$-connection on $\mathbb{P}(K)$ over $\mathcal{U}_0$ by the connection matrix

$$A = \begin{pmatrix} 0 & g(x) \\ 0 & f(x) \end{pmatrix} \frac{dx}{x}.$$

Then in the chart $\mathcal{U}_1$ with coordinates $\widetilde{x} = x^{-1}$, $\widetilde{y}_1$, $\widetilde{y}_2$ the connection has the matrix

$$(dT)T^{-1} + TAT^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & -n \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} 0 & x^n g(x) \\ 0 & f(x) \end{pmatrix} \frac{dx}{x} = \begin{pmatrix} 0 & -\widetilde{x}^{-n} g\left(\frac{1}{\widetilde{x}}\right) \\ 0 & n - f\left(\frac{1}{\widetilde{x}}\right) \end{pmatrix} \frac{d\widetilde{x}}{\widetilde{x}}$$

Such a matrix has no poles over $\widetilde{x} = 0$ if and only if $f(x) \equiv n$ and $g(x)$ is a polynomial of degree at most $-n-1$. The expression $g(x)\frac{dx}{x}$ can be interpreted an element of $\mathrm{Hom}(K_2, K_1 \otimes \mathcal{T}^*_{\mathbb{P}^1}(\log x_0))$.

Next, let us assume that $K \not\cong K_1 \oplus K_2$, and prove that there is no $\mathcal{T}_{\mathbb{P}^1}(-\log x_0)$-connection on $\mathbb{P}(K)$ tangent to $\mathbb{P}(K_1)$. Let us choose a coordinate x on $\mathbb{P}^1$ so that x_0 is $x = 0$. Let us also choose fiberwise linear coordinates y_1, y_2 over $\mathcal{U}_0 = \{x \neq \infty\}$, and $\widetilde{y}_1$, $\widetilde{y}_2$ over $\mathcal{U}_1 = \{x \neq 0\}$ so that

$$\begin{pmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \end{pmatrix} = T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & p(x) \\ 0 & x^{-n} \end{pmatrix},$$

where $p(x) = \sum_{k=1}^{n-1} a_k x^{-k} \not\equiv 0$. Note that this case occurs only if $n \geq 2$.

A $\mathcal{T}_{\mathbb{P}^1}(-\log x_0)$-connection on $\mathbb{P}(K)$ tangent to $\mathbb{P}(K_1)$, over $\mathcal{U}_0$ has the connection matrix of the form

$$A = \begin{pmatrix} 0 & g(x) \\ 0 & f(x) \end{pmatrix} \frac{dx}{x}.$$

Then in the chart $\mathcal{U}_1$ with coordinates $\widetilde{x} = x^{-1}$, $\widetilde{y}_1$, $\widetilde{y}_2$ the connection has the matrix

$$(dT)T^{-1} + TAT^{-1} = \begin{pmatrix} 0 & x^{n+1}p'(x) \\ 0 & -n \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} 0 & x^n g(x) + x^n p(x) f(x) \\ 0 & f(x) \end{pmatrix} \frac{dx}{x} =$$

$$= \begin{pmatrix} 0 & -\widetilde{x}^{-n}\left(\frac{1}{\widetilde{x}} p'\left(\frac{1}{\widetilde{x}}\right) + g\left(\frac{1}{\widetilde{x}}\right) + p\left(\frac{1}{\widetilde{x}}\right) f\left(\frac{1}{\widetilde{x}}\right)\right) \\ 0 & n - f\left(\frac{1}{\widetilde{x}}\right) \end{pmatrix} \frac{d\widetilde{x}}{\widetilde{x}}.$$

The latter matrix has a pole over $\widetilde{x} = 0$ for any choice of $g, h \in \mathcal{O}_{\mathcal{U}_0}$. ☐

Recall that d_1 denotes the degree of the restriction $\mathcal{F}|_{\Sigma_1}$, and d_{23} is the degree of the restriction $\mathcal{F}|_{\Sigma_{23}}$. In the next proposition, we are assuming that $d_1 = 0$, that is, $\mathcal{F}|_{\Sigma_1} \cong \mathcal{O}_{\Sigma_1}$. This can always be achieved by twisting the sheaf $\mathcal{F}$ with $p^*\mathcal{O}_{\mathbb{P}^1}(-d_1)$, where p is the projection $\Sigma \to \mathbb{P}^1$.

Proposition 4.3.30. *Let (V, ϕ) be a traceless strongly integrable, resonant rank 3 co-Higgs bundle over $\mathbb{P}^1$. Let ϕ have a simple zero at $x_0 \in \mathbb{P}^1$ and no other zeros. Let m be the number of intersections of $\mathbb{P}(K_1)$ with $\mathbb{P}(K_{sing})$ counting multiplicities. Then $m \in \{0, 1, 2\}$. Furthermore,*

1. *If $m = 0$, then ϕ admits a Poisson lift to $\mathbb{P}(V)$.*
2. *If $m = 1$, then ϕ admits a Poisson lift to $\mathbb{P}(V)$ if and only if $d_{23} \geq 0$.*
3. *If $m = 2$ and $d_{23} < 0$, then ϕ does not admit a Poisson lift to $\mathbb{P}(V)$.*
4. *If $m = 2$ and $d_{23} > 0$, then ϕ admits a Poisson lift to $\mathbb{P}(V)$.*

5. *If $m = 2$ and $d_{23} = 0$, then ϕ admits a Poisson lift to $\mathbb{P}(V)$ if and only if $\mathcal{F}$ is the pushforward of a line bundle L on the partial normalization $\widetilde{\Sigma} \to \Sigma$ defined by $\mathcal{F}$ that satisfies $L \otimes \widetilde{\iota}^* L \cong \mathcal{O}_{\widetilde{\Sigma}}$, where $\widetilde{\iota}$ is the involution of $\widetilde{\Sigma}$ that flips the two branches of $\widetilde{\Sigma}_{23}$ and leaves $\widetilde{\Sigma}_1$ fixed.*

In the cases when the Poisson lifts to $\mathbb{P}(V)$ do exist, they form an l-dimensional family, where $l = \min(0, d_{23} - m)$.

Proof. The partial normalization $\widetilde{\Sigma}$ maps into the total space of $\mathcal{T}_{\mathbb{P}^1}(-\log x_0) \cong \mathcal{O}_{\mathbb{P}^1}(1)$, therefore $\widetilde{\Sigma}$ has either one D_4 singularity, or at most two A_1 singularities. By examining Table 4.2, we see that $m \leq 2$. In each of the cases below, Lemma 4.3.29 and Proposition 4.3.30 reduce the question of existence of a Poisson lift of ϕ to whether the short exact sequence $0 \to K_1 \to K \to K_2 \to 0$ splits. Whenever the short exact sequence does split, the different choices of the Poisson lift correspond to different choices of the logarithmic connection, which are described in Lemma 4.3.29.

If $m = 0$, then the section $\mathbb{P}(K_{sing})$ is disjoint from $\mathbb{P}(K_1)$, and $K_{sing} \cong K_2$. Therefore, one has $K \cong K_1 \oplus K_2$, and so, there is Poisson structure on $\mathbb{P}(V)$ lifting ϕ.

If $m = 1$, then $\mathbb{P}(K_{sing})$ intersects transversely $\mathbb{P}(K_1)$, and $K_2 \cong K_{sing} \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. If $-d_{23} = \deg K_{sing} \leq 0$, the $\deg K_2 \leq 1$, and the short exact sequence $0 \to K_1 \to K \to K_2 \to 0$ necessarily splits for degree reasons. On the other hand if $-d_{23} = \deg K_{sing} \geq 1$, then K cannot be isomorphic to $K_1 \oplus K_2 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-d_{23} + 1)$, because K has a subbundle of degree $-d_{23}$.

If $m = 2$, then $K_2 \cong K_{sing} \otimes \mathcal{O}_{\mathbb{P}^1}(2)$. If $-d_{23} = \deg K_{sing} < 0$, the $\deg K_2 < 2$, and the short exact sequence $0 \to K_1 \to K \to K_2 \to 0$ necessarily splits for degree reasons. On the other hand if $-d_{23} = \deg K_{sing} > 0$, then K cannot be isomorphic to $K_1 \oplus K_2 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-d_{23} + 2)$, because K has a subbundle of degree $-d_{23}$.

Let $m = 2$ and $d_{23} = 0$. One can verify that $K^* \cong (\widetilde{p}_*(L \otimes \widetilde{\iota}^* L))^{S_2}$, where L and $\widetilde{\iota}$ are as stated in the lemma, $\widetilde{p}$ is the projection $\widetilde{\Sigma} \to \mathbb{P}^1$, and the superscript S_2 means the sheaf of the local sections invariant under the involution $\widetilde{\iota}$. Since $K_1 \cong \mathcal{O}_{\mathbb{P}^1}$ and $K_2 \cong \mathcal{O}_{\mathbb{P}^1}(2)$, the short exact sequence $0 \to K_1 \to K \to K_2 \to 0$ splits if and only if the vector bundle K^* has a non-trivial global section. The latter is equivalent to $L \otimes \widetilde{\iota}^* L$ having an $\widetilde{\iota}$-invariant global section. Since $L \otimes \widetilde{\iota}^* L$ has zero degree over each component of $\widetilde{\Sigma}$, it has a non-trivial global section if and only if $L \otimes \widetilde{\iota}^* L \cong \mathcal{O}_{\widetilde{\Sigma}}$. In the case $L \otimes \widetilde{\iota}^* L \cong \mathcal{O}_{\widetilde{\Sigma}}$, every global section of $L \otimes \widetilde{\iota}^* L$ is constant and, in particular, $\widetilde{\iota}$-invariant. To sum it up, we have obtained that the short exact sequence $0 \to K_1 \to K \to K_2 \to 0$ splits if and only if $L \otimes \widetilde{\iota}^* L \cong \mathcal{O}_{\widetilde{\Sigma}}$. □

Example 4.3.31. (Type $(\Phi_0^{A_1}) - (\Phi_6^{D_4}) - (\Phi_0^{A_1})$)

Consider the case when $\Sigma = \{\theta(\theta - q) = 0\}$, where $q = x(x - x_0)^2 \partial_x^{\otimes 2}$, $x_0 \neq 0, \infty$. Assume ϕ has type $\Phi_0^{A_1}$ near $x = 0$ and $x = \infty$, and vanishes at $x = x_0$. Let $\widetilde{\Sigma}$ be the partial normalizaition of Σ corresponding to the spectral sheaf $\mathcal{F}$ of ϕ.

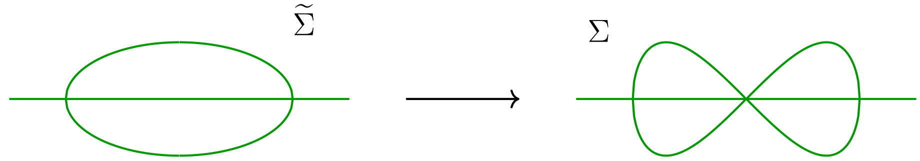


Figure 4.9: Spectral curve Σ and its partial normalization $\widetilde{\Sigma}$ for co-Higgs of type $(\Phi_0^{A_1}) - (\Phi_6^{D_4}) - (\Phi_0^{A_1})$

Each $\Phi_0^{A_1}$ singularity contributes 1 to the number m (see Table 4.2), so we are in the case $m = 2$. Let us assume that $d_1 = d_{23} = 0$, so that we are in the case 5 of Proposition 4.3.30. Then the $\mathcal{F}$ is the pushforward of a line bundle L of degree $(0,0)$ on $\widetilde{\Sigma}$. The line bundle L is determined by the monodromy $\alpha \in \mathbb{C}^*$ around the loop that generates $\pi_1(\widetilde{\Sigma}) \cong \mathbb{Z}$. One has $L \cong \mathcal{O}_{\widetilde{\Sigma}}$ if and only if $\alpha = 1$. Note that $L \otimes \widetilde{\iota}^* L$ has the monodromy α^2. So Proposition 4.3.30 says that ϕ lifts to a Poisson structure on $\mathbb{P}(V)$ if and only if $\alpha = \pm 1$.

Let $\alpha = 1$, i.e. $L \cong \mathcal{O}_{\widetilde{\Sigma}}$. In this case $V = p_* \mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, and in this decomposition one has

$$\phi = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & x \\ 0 & 1 & 0 \end{pmatrix} (x - x_0)\partial_x.$$

The bundle $K \subset S^2 V^*$ of ϕ-invariant fiberwise quadrics has generators $s = y_1^2$, $t = y_2^2 - 2y_1 y_3 - x y_3^2$. Note that s is nowhere vanishing, while t has a double zero at $x = \infty$, so $K \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. The unique $\mathcal{T}_{\mathbb{P}^1}(-\log x_0)$-connection on K tangent to $\mathbb{P}(K_1) = \mathbb{P}(\langle s \rangle)$ has the following matrix in the basis s, t:

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \frac{dx}{x - x_0}.$$

The Poisson bivector on $\mathbb{P}(V)$, guaranteed by Proposition 4.3.28, has the following expression in the coordinates x, $\widetilde{y}_2 = \frac{y_2}{y_1}$, $\widetilde{y}_3 = \frac{y_3}{y_1}$:

$$\pi = (x - x_0)\partial_x \wedge \varphi + \frac{1}{2}(x - x_0)\widetilde{y}_3^2 \partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3} - \widetilde{t} \partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3},$$

where

$$\varphi = \langle \phi, dx \rangle = \partial_{\widetilde{y}_2} + \widetilde{y}_2 \partial_{\widetilde{y}_3} + x \widetilde{y}_3 \partial_{\widetilde{y}_2},$$

$$\widetilde{t} = \frac{t}{s} = \widetilde{y}_2^2 - 2\widetilde{y}_3 - x\widetilde{y}_3^2.$$

Now, let us consider the case of non-trivial monodromy $\alpha \neq 1$. In this case $V = p_* \mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and one can bring ϕ to the following form:

$$\phi = \begin{pmatrix} 0 & 0 & -\frac{1}{\alpha - 1} x \\ 0 & 0 & \frac{1}{\alpha - 1} \\ 1 & \alpha x & 0 \end{pmatrix} (x - x_0)\partial_x.$$

The line bundle $K_1 = \ker(\phi^*)^{\otimes 2} \subset S^2 V^*$ is spanned by the nowhere vanishing section $(y_1 + x y_2)^2$ and therefore is isomorphic to $\mathcal{O}_{\mathbb{P}^1}$. If $\alpha \in \mathbb{C}^* \setminus \{1, -1\}$, then the bundle $K \subset S^2 V^*$ of ϕ-invariant fiberwise quadrics has generators $s = (\alpha + 1) y_1^2 + 2\alpha x y_1 y_2 + \frac{1}{\alpha - 1} x y_3^2$, $t = (\alpha + 1) x y_2^2 - \frac{1}{\alpha - 1} y_3^2 + 2 y_1 y_2$. Both s and t have exactly one zero (at $x = \infty$), and so in this case $K \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Applying Lemma 4.3.29, we see that there is no $\mathcal{T}_{\mathbb{P}^1}$-connection on K tangent to $\mathbb{P}(K_1)$. This shows, once again, that if we want to look for Poisson lifts to $\mathbb{P}(V)$, we must assume $\alpha = -1$. If $\alpha = -1$, then K is spanned by $s = (y_1 + x y_2)^2$, $t = y_3^2 + 4 y_1 y_2$. The section t has a double zero at $x = \infty$, while s is nowhere vanishing, so $K \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ in this case. The unique $\mathcal{T}_{\mathbb{P}^1}(-\log x_0)$-connection on K tangent to $\mathbb{P}(K_1) = \mathbb{P}(\langle s \rangle)$

has the following matrix in the basis s, t:

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \frac{dx}{x - x_0}.$$

The Poisson bivector on $\mathbb{P}(V)$, guaranteed by Proposition 4.3.28, has the following expression in the coordinates x, $\widetilde{y}_2 = \frac{y_2}{y_1}$, $\widetilde{y}_3 = \frac{y_3}{y_1}$:

$$\pi = (x - x_0)\partial_x \wedge \varphi - \frac{1}{2}(\widetilde{y}_3^2 + 4\widetilde{y}_2)(1 + x_0\widetilde{y}_2)\partial_{\widetilde{y}_2} \wedge \partial_{\widetilde{y}_3},$$

where

$$\varphi = \langle \phi, dx \rangle = \partial_{\widetilde{y}_3} - x\widetilde{y}_2\partial_{\widetilde{y}_3} - \frac{1}{2}x\widetilde{y}_3(\widetilde{y}_2\partial_{\widetilde{y}_2} + \widetilde{y}_3\partial_{\widetilde{y}_3}) - \frac{1}{2}\widetilde{y}_3\partial_{\widetilde{y}_2}.$$

Chapter 5

Application to Feigin-Odesskii Poisson structures

Let C be a smooth projective curve over $\mathbb{C}$ of genus ≤ 1. Let v be a holomorphic line field on C and L be a very ample vector bundle over C. This data determines a Kodaira embedding $C \hookrightarrow \mathbb{P}(H^0(C,L)^*)$ and a $\mathbb{C}^*$-invariant Poisson structure on the vector space $H^0(C,L)^*$ known as a Feigin-Odesskii structure. This Poisson structure is the quasi-classical limit q_n of the algebra $Q_n(\mathcal{E},\eta)$ in [21].

The goal of this chapter is to construct natural desingularizations of secant varieties of $C \subset \mathbb{P}(H^0(C,L)^*)$ and a Poisson structure on the desingularization that would project onto the Feigin-Odesskii structure on $\mathbb{P}(H^0(C,L)^*)$. The desingularization procedure we describe is a standard method in the literature (e.g. [31]).

5.1 Co-Higgs fields on Schwarzenberger bundles

For $d \geq 1$, let $Sec_d(C)$ be the d-secant variety of C inside $\mathbb{P}(H^0(C,L)^*)$, that is, the closure of the union of all linear $\mathbb{P}^{d-1}$-subspaces of $\mathbb{P}(H^0(C,L)^*)$ passing through d distinct points of C. Let $n = \dim H^0(C,L)^*$. Note that if $2d < n$, then the dimension of $Sec_d(C)$ is $2d-1$ and the singular locus of $Sec_d(C)$ equals $Sec_{d-1}(C)$. For $d \geq 1$, let $K_d \subset H^0(C,L)^*$ be the affine cone over $Sec_d(C)$. If $2d < n$, then K_d is a $2d$-dimensional variety and the singular locus of K_d is K_{d-1}.

Resolution of singularities. Fix $d \geq 1$. Let S^dC be the d-th symmetric power of C, or, equivalently, the Hilbert scheme of d points of C. Let $U_d \subset C \times S^dC$ be the universal Hilbert scheme of d points of C, that is $U_d = \{(x,\xi) \in C \times S^dC : x \in \xi\}$. Let $q : U_d \to C$ and $p : U_d \to S^dC$ be the natural projections (see Figure 5.1). Note that the map p is generically d to 1. Define the rank d vector bundle $V_d = p_*(q^*L)$ on S^dC. By relative Serre duality, $V_d^* = p_*(q^*(L^*) \otimes \mathcal{O}(R))$, where $R \subset U_d$ is the ramification divisor of p.

Remark 5.1.1. The vector bundle V_d in the case $C = \mathbb{P}^1$ is known as a *Schwarzenberger bundle.* We are going to use this term for the case when C is a genus one curve as well.

Let us construct maps $\Pi_d : V_d^* \to H^0(C,L)^*$, $d \geq 1$. Let $s \in H^0(C,L)$ be a linear function on $H^0(C,L)^*$. Then one can bring into correspondence a natural section $p_*(q^*s)$ of $V_d = p_*(q^*L)$. For each $\xi \in S^dC$, define a linear map $H^0(C,L) \to V_d|_\xi$ by $s \mapsto p_*(q^*s)|_\xi$. Dualizing, we obtain a linear map $V_d^*|_\xi \to H^0(C,L)^*$ for each $\xi \in S^dC$. This defines a map $\Pi_d : V_d^* \to H^0(C,L)^*$.

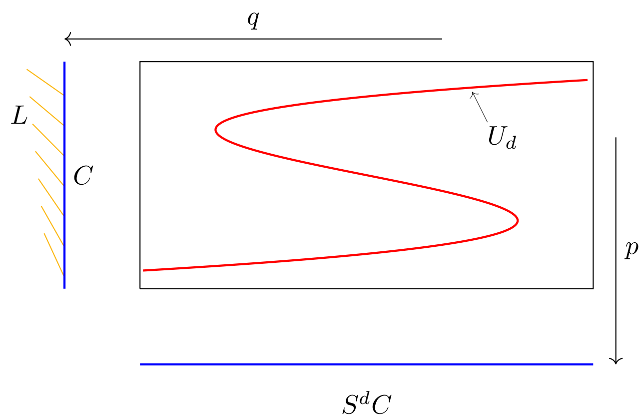


Figure 5.1: Setup for defining Schwarzenberger bundles

The image of the map $\Pi_d : V_d^* \to H^0(C,L)^*$ is K_d. Moreover, if $2d < n$, then it is a resolution of singularities of K_d. If $d \le n$, then the preimage $\Pi_d^{-1}(0)$ is precisely the zero section of V_d^*, and one can projectivize Π_d to obtain a map $\mathbb{P}(V_d^*) \to \mathbb{P}(H^0(C,L)^*)$, which we also denote by Π_d. For $d < 2n$, the image of the projectivized Π_d is the secant variety $Sec_d(C)$.

A co-Higgs field on V_d^*. The vector field v on C induces the vector field $(v,0,...,0)$ on $U_d = C\times S^{d-1}C$. Using this vector field and the finite map $p : U_d \to S^dC$, via the spectral correspondence, we obtain a co-Higgs field on $V_d^* = p_*(q^*(L^*)\otimes \mathcal{O}(R))$. Specifically, for any locally defined 1-form $\alpha \in \mathcal{T}^*_{S^dC}$, the V_d^*-endormorphism $\langle\phi,\alpha\rangle$ is given by

$$V_d^* \cong q^*(L^*)\otimes\mathcal{O}(R) \xmapsto{\text{multiplication by } \langle v, p^*\alpha\rangle} q^*(L^*)\otimes\mathcal{O}(R) \cong V_d^*.$$

Over an open subset of $C\times S^{d-1}C$ consisting of points of the form $(x_1,x_2,...,x_d)$ with all the x_i distinct, one has

$$\phi = \begin{pmatrix} v_{x_1} & 0 & \dots & 0 \\ 0 & v_{x_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_{x_d} \end{pmatrix} \tag{5.1}$$

We remark that the co-Higgs field ϕ is strongly integrable as we saw in Example 3.4.3.

5.2 Poisson lifts. $\mathbb{P}^1$ case.

In this subsection, we assume that $C = \mathbb{P}^1$, $L = \mathcal{O}_C(n-1)$. To construct a Poisson lift of the co-Higgs field ϕ on $V_d^* = p_*(q^*(L^*)\otimes\mathcal{O}(R))$, we will present a meromorphic section s of the line bundle $q^*(L^*)\otimes\mathcal{O}(R)$ on $\mathbb{P}^1\times\mathbb{P}^{d-1}$ and define $\sigma = Lift_{p_*(\nabla^s)}(\phi)$ as in (3.13). From Example 3.4.19 we know that the spectral data $(p : C\times S^{d-1}C \to S^dC, (v,0,...,0), \mathcal{L} = q^*(L^*)\otimes\mathcal{O}(R))$ is admissible for a diagonal lift. Moreover, Example 3.4.20 tells us what kind of divisor the zero-pole divisor of s should be to guarantee that the lift σ extends smoothly over the whole S^dC.

Without loss of generality, let us assume that v vanishes at $\infty \in \mathbb{CP}^1$. Let us fix any non-zero section $s_0 \in H^0(C,\mathcal{O}_C(n-1))$, and the section $\lambda \in H^0(\mathbb{P}^1\times S^{d-1}(\mathbb{P}^1),\mathcal{O}(R))$ cutting out R. Consider the rational function

$$f(x_1, x_2, ..., x_d) = \prod_{j=2}^{d} \frac{1}{x_1 - x_j}$$

on $\mathbb{P}^1 \times S^{d-1}(\mathbb{P}^1)$. The zero-pole divisor of f equals $div(f) = (d-1)D_1 + D_2 - R$, where $D_1 = \{\infty\} \times S^{d-1}(\mathbb{P}^1)$, $D_2 = \{(z, \xi) : \xi \ni \{\infty\}\}$. Then the meromorphic section

$$s = \frac{f\lambda}{q^* s_0}$$

of $p_*(q^*(L^*) \otimes \mathcal{O}(R))$ has the zero-pole divisor $div(s) = (d-1)D_1 + D_2 - div(s_0) \times S^{d-1}(\mathbb{P}^1)$. By Example 3.4.20, $div(s)$ is adapted to the spectral data $(p : C \times S^{d-1}C \to S^d C, (v, 0, ..., 0), \mathcal{L} = q^*(L^*) \otimes \mathcal{O}(R))$, and by Proposition 3.4.21, the lift $\widetilde{\sigma} = Lift_{p_*(\nabla^s)}(\phi)$ defines a quadratic Poisson structure on V_d^*. One can projectivize it and obtain a Poisson structure σ on $\mathbb{P}(V_d^*)$.

We remark that the choice of a section $s_0 \in H^0(C, \mathcal{O}_C(n-1))$ does not affect the obtained Poisson structure on V_d^*. Indeed, let s_0' be another non-zero section of $H^0(C, \mathcal{O}_C(n-1))$ producing a meromorphic section $s' = f\lambda/(q^* s_0')$ of $p_*(q^*(L^*) \otimes \mathcal{O}(R))$. Then over an open subset of $C \times S^{d-1}C$ consisting of points of the form $(x_1, x_2, ..., x_d)$ with all the x_i distinct, one has that $\omega = p_*(\nabla^s) - p_*(\nabla^{s'})$ equals the diagonal element

$$\begin{pmatrix} d\log(s_0(x_1)) - d\log(s_0'(x_1)) & 0 & \dots & 0 \\ 0 & d\log(s_0(x_2)) - d\log(s_0'(x_2)) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d\log(s_0(x_d)) - d\log(s_0'(x_d)) \end{pmatrix}$$

of $End(V_d^*) \otimes \mathcal{T}^*$. From this, together with the expression (5.1) for the co-Higgs field ϕ, we see that the pairing $\langle \phi \curlywedge \omega \rangle$ defined by (3.11) vanishes, and so, Remark 3.4.12 implies that $Lift_{p_*(\nabla^s)}(\phi) = Lift_{p_*(\nabla^{s'})}(\phi)$.

5.3 Poisson lifts. Elliptic case.

In this subsection, we assume that $C = E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, $Im(\tau) > 0$, is an elliptic curve, L is a degree $n \geq 3$ line bundle over E, v is the vector field on E coming from the constant vector field $\frac{\partial}{\partial x}$ on $\mathbb{C}$. We are going to lift the strongly integrable co-Higgs field ϕ on $V_d^* = p_*(q^*(L^*) \otimes \mathcal{O}(R))$ to a Poisson structure on the projectivization $\mathbb{P}(V_d^*)$.

Let $pr : \mathbb{C} \to E$ be the natural projection. Consider the commutative diagram of maps

$$\begin{array}{ccc} \mathbb{C} \times S^{d-1}\mathbb{C} & \xrightarrow{\alpha = (pr, S^{d-1}pr)} & E \times S^{d-1}E \\ \big\downarrow \widetilde{p} & & \big\downarrow p \\ S^d\mathbb{C} & \xrightarrow{\beta = S^d pr} & S^d E \end{array}$$

where $\widetilde{p}$ is the symmetrization map. This diagram is not a Cartesian square, so there is no canonical isomorphism between $\widetilde{p}_* \alpha^*(q^*(L^*) \otimes \mathcal{O}(R))$ and $\beta^* p_*(q^*(L^*) \otimes \mathcal{O}(R))$. However, there is such an isomorphism

- over $S^d\mathbb{C} \setminus \{\{x_1, ..., x_d\} : x_i = x_j \mod \mathbb{Z} + \tau\mathbb{Z}\}$, and
- over $S^d\mathcal{U}$, where $\mathcal{U}$ is a small ball in $\mathbb{C}$ of diameter less than $\min\{\frac{1}{2}, \frac{\tau}{2}\}$.

The line bundle $\alpha^*(q^*(L^*)\otimes\mathcal{O}(R))$ is trivial. One can recover $q^*(L^*)\otimes\mathcal{O}(R)$ as the quotient of the trivial bundle over $\mathbb{C}\times S^{d-1}\mathbb{C}$ by the relations:

$$\begin{aligned}&\mathbb{C}_{(x_1,\{x_2,\dots,x_i+1,\dots,x_d\})}\sim\mathbb{C}_{(x_1,\{x_2,\dots,x_i,\dots,x_d\})},\ 1\le i\le d\\&\mathbb{C}_{(x_1+\tau,\{x_2,\dots,x_d\})}\sim f(x_1,x_2,\dots,x_d)\mathbb{C}_{(x_1,\{x_2,\dots,x_d\})}\\&\mathbb{C}_{(x_1,\{x_2,\dots,x_i+\tau,\dots,x_d\})}\sim g(x_1,x_i)\mathbb{C}_{(x_1,\{x_2,\dots,x_i,\dots,x_d\})},\ 2\le i\le d,\end{aligned}\tag{5.2}$$

where

$$\begin{aligned}f(x_1,x_2,\dots,x_d)&=(-1)^{-n+d-1}e^{-2\pi i((-n+d-1)x_1+c-x_2-x_3-\dots-x_d)}\\g(x_1,x_i)&=-e^{-2\pi i(x_i-x_1)},\end{aligned}$$

and $c\in\mathbb{C}$ is the sum of n zeroes of a section of L, which is well defined modulo $\mathbb{Z}+\tau\mathbb{Z}$.

The vector field $\widetilde{v}=\frac{\partial}{\partial x}$ on $\mathbb{C}\times S^{d-1}\mathbb{C}$ by spectral correspondence gives a strongly integrable co-Higgs field $\widetilde{\phi}$ on the vector bundle $\widetilde{V}=\widetilde{p}_*\mathcal{O}_{\mathbb{C}\times S^{d-1}\mathbb{C}}$. The section $\widetilde{s}=1$ of the trivial bundle $\mathcal{O}_{\mathbb{C}\times S^{d-1}\mathbb{C}}$ provides the lift $\widetilde{\sigma}=Lift_{\nabla^{\widetilde{s}}}(\widetilde{\phi})$ as explained in Example 3.4.22.

Let us check that over $S^d\mathbb{C}\setminus\{\{x_1,\dots,x_d\}:x_i=x_j \mod \mathbb{Z}+\tau\mathbb{Z}\}$, the Poisson structure $\widetilde{\sigma}$ descends to a Poisson structure on $\mathbb{P}(V_d^*)$ over $S^dE\setminus\{\{x_1,\dots,x_d\}:x_i=x_j\}$. Locally, around a point $\{x_1^0,\dots,x_d^0\}\in S^d\mathbb{C}$, such that $x_i^0\neq x_j^0 \mod \mathbb{Z}+\tau\mathbb{Z}$, we can order the points of $\mathbb{Z}$ and obtain local coordinates $x_1,\dots,x_d$ on $S^d\mathbb{C}$. Let y_i be the canonical fiberwise coordinate on the fiber $\mathbb{C}_{(x_i,\{x_1,\dots,\widehat{x_i},\dots,x_d\})}$. Then in the coordinates $x_1,\dots,x_d,y_1,\dots,y_1$, the Poisson structure has the expression $\widetilde{\sigma}=\sum_{j=1}^d\frac{\partial}{\partial x_j}\wedge y_j\frac{\partial}{\partial y_j}$. Let us fix $i\in\{1,2,\dots,d\}$ and check that this expression is invariant, up to the terms of the form $(-)\wedge Eul$, under the transformations

$$\begin{aligned}x_i&\mapsto x_i+1,&\\x_j&\mapsto x_j,&j\neq i,\\y_j&\mapsto y_j,&1\le j\le d\end{aligned}$$

and

$$\begin{aligned}x_i&\mapsto x_i'=x_i+\tau,&\\x_j&\mapsto x_j'=x_j,&j\neq i,\\y_i&\mapsto y_i'=f(x_i,x_1,\dots,\widehat{x_i},\dots,x_d)y_i,&\\y_j&\mapsto y_j'=g(x_j,x_i)y_j,&j\neq i.\end{aligned}$$

The former coordinate transformation does not change $\widetilde{\sigma}$, because the vector field $\frac{\partial}{\partial x_i}$ is translation-invariant. For the latter coordinate transformation, we do the following computation:

$$\begin{aligned}\frac{\partial}{\partial x_i}&\longmapsto\frac{\partial}{\partial x_i'}-2\pi\sqrt{-1}(-n+d-1)y_i'\frac{\partial}{\partial y_i'}-2\pi\sqrt{-1}\sum_{k\neq i}y_k'\frac{\partial}{\partial y_k'},&\\\frac{\partial}{\partial x_j}&\longmapsto\frac{\partial}{\partial x_j'}+2\pi\sqrt{-1}y_i'\frac{\partial}{\partial y_i'}+2\pi\sqrt{-1}y_j'\frac{\partial}{\partial y_j'},&j\neq i\\y_j\frac{\partial}{\partial y_j}&\longmapsto y_j'\frac{\partial}{\partial y_j'}&1\le j\le d\end{aligned}$$

$$\begin{aligned}&\sum_{j=1}^d\frac{\partial}{\partial x_j}\wedge y_j\frac{\partial}{\partial y_j}\longmapsto\sum_{j=1}^d\frac{\partial}{\partial x_j'}\wedge y_j'\frac{\partial}{\partial y_j'}+\\&+2\pi\sqrt{-1}\left(-\left(y_i'\frac{\partial}{\partial y_i'}+\sum_{k\neq i}y_k'\frac{\partial}{\partial y_k'}\right)\wedge y_i'\frac{\partial}{\partial y_i'}+\sum_{j\neq i}\left(y_i'\frac{\partial}{\partial y_i'}+y_j'\frac{\partial}{\partial y_j'}\right)\wedge y_j'\frac{\partial}{\partial y_j'}\right)=\\&=\sum_{j=1}^d\frac{\partial}{\partial x_j'}\wedge y_j'\frac{\partial}{\partial y_j'}+4\pi\sqrt{-1}y_i'\frac{\partial}{\partial y_i'}\wedge Eul.\end{aligned}$$

The calculation shows that the projectivization of the Poisson structure $\widetilde{\sigma}$ on $\mathbb{P}(\widetilde{V})$ over $S^d\mathbb{C} \setminus \{\{x_1, ..., x_d\} : x_i = x_j \mod \mathbb{Z} + \tau\mathbb{Z}\}$ descends to a Poisson structure σ on $\mathbb{P}(V_d^*)$ over $S^d E \setminus \{\{x_1, ..., x_d\} : x_i = x_j\}$. We claim that σ extends smoothly over the divisor $\mathbb{P}(V_d^*)|_D$, where $D = \{\{x_1, ..., x_d\} : x_i = x_j\} \subset S^d E$. Indeed, the divisor $\mathbb{P}(V_d^*)|_D$ is irreducible, and we know that σ has no poles over $\mathbb{P}(V_d^*)|_{D \cap S^d\mathcal{U}}$, where $\mathcal{U}$ is a small neighborhood of $0 \in E$. Therefore, σ has no poles over $\mathbb{P}(V_d^*)|_D$.

5.4 Desingularization map is Poisson

Theorem 5.4.1. *Let C be a smooth curve of genus ≤ 1, v be a vector field on C, L be a line bundle over C, and $n = \dim H^0(C, L)$. Let V_d be the Schwarzenberger bundle constructed in Section 5.1, and σ be the Poisson structure on $\mathbb{P}(V_d^*)$ constructed in Sections 5.2 and 5.3.*

Then there is a unique Poisson structure ρ on $\mathbb{P}(H^0(C, L)^*)$ such that the projectivized desingularization map $\Pi_d : \mathbb{P}(V_d^*) \to \mathbb{P}(H^0(C, L)^*)$ is Poisson for each $1 \le d \le n$.

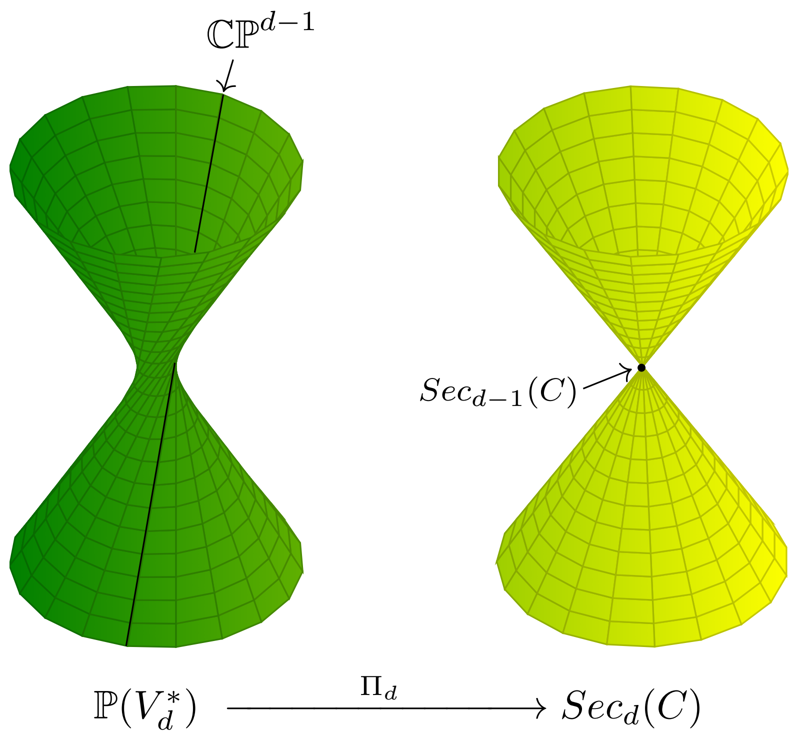


Figure 5.2: Illustration to Theorem 5.4.1 for $d < \frac{n}{2}$

When C has genus 1, the Poisson structure on $\mathbb{P}(H^0(C, L)^*) \cong \mathbb{P}^{n-1}$ constructed in Theorem 5.4.1 is the Feigin-Odesskii Poisson structure q_n [21]. This can be checked using the semiclassical version of the calculation in [21, Proposition 6]. Theorem 5.4.1 can be viewed as an alternative geometric way of defining the Poisson bracket q_n of Feigin-Odesskii. We remark that Theorem 5.4.1, in particular, shows that the secant varieties $Sec_d(C)$, which are images of Π_d, are Poisson submanifolds of $\mathbb{P}(H^0(C, L)^*)$. This matches the known description of symplectic leaves of Feigin-Odesskii Poisson structures.

Before proving Theorem 5.4.1, let us prove a few auxiliary statements, which we formulate in the notations of the theorem.

Lemma 5.4.2. *There is a family of Poisson embeddings* $I_{d_1,d_2} : \mathbb{P}(V_{d_1}^*) \longrightarrow \mathbb{P}(V_{d_2}^*)$, $d_1 < d_2$, *such that for* $d_1 < d_2 \leq n$, *one has* $\Pi_{d_1} = \Pi_{d_2} \circ I_{d_1,d_2}$.

Proof. It is enough to consider the case $d_2 = d_1 + 1$. To construct an embedding $I_{d_1,d_1+1} : \mathbb{P}(V_{d_1}^*) \longrightarrow \mathbb{P}(V_{d_1+1}^*)$, we need to fix a point $x \in E$. Then one obtains an embedding $\iota : S^{d_1}C \longrightarrow S^{d_1+1}C$, by sending $\xi \in S^{d_1}C$ to the union $\xi \cup \{x\} \in S^{d_1+1}C$. Note that $V_{d_1}|_\xi = H^0(\xi, L|_\xi)$ and $V_{d_1}|_{\xi\cup\{x\}} = H^0(\xi \cup \{x\}, L|_{\xi\cup\{x\}})$, so there is a canonical restriction map $r : V_{d_1+1}|_{\xi\cup\{x\}} \longrightarrow V_{d_1}|_\xi$. Dualizing, we obtain $r^* : V_{d_1}^*|_\xi \longrightarrow V_{d_1+1}^*|_{\xi\cup\{x\}}$. Combining ι with r^*, we obtain an embedding $V_{d_1}^* \longrightarrow V_{d_1+1}^*$, which we can projectivize to get the desired I_{d_1,d_1+1}. It is straightfoward to check that $\Pi_{d_1} = \Pi_{d_1+1} \circ I_{d_1,d_1+1}$, and that I_{d_1,d_1+1} is Poisson. □

Lemma 5.4.3. *The map* $\Pi_n : \mathbb{P}(V_n^*) \longrightarrow \mathbb{P}(H^0(C,L)^*)$ *has connected fibers.*

Proof. Recall that, for $\xi \in S^nC$, the map Π_n sends the fiber $\mathbb{P}(V_n^*|_\xi)$ to the hyperplane spanned by $\xi \subset C \subset \mathbb{P}(H^0(C,L)^*)$. Let $x \in \mathbb{P}(H^0(C,L)^*)$. If $C = \mathbb{P}^1$, then C has degree n, and so $\xi \in S^nC$ can be recovered uniquely from the the hyperplane $\Pi_n(\mathbb{P}(V_n^*|_\xi))$. Moreover, the map Π_n is one-to-one on each fiber $\mathbb{P}(V_n^*|_\xi)$. So, $\Pi_n^{-1}(x)$ is the set of hyperplanes in $\mathbb{P}(H^0(C,L)^*)$ containing x. In particular, $\Pi_n^{-1}(x)$ is connected.

If $C = E$ has genus 1, then E has degree $n+1$, and therefore $\xi \in S^nE$ cannot be recovered uniquely from the the hyperplane $\Pi_n(\mathbb{P}(V_n^*|_\xi))$. More precisely, the map $\varphi : S^nE \longrightarrow \mathbb{P}(H^0(E,L))$ that sends $\xi \in S^nE$ to the hyperplane $\Pi_n(\mathbb{P}(V_n^*|_\xi)) \subset \mathbb{P}(H^0(E,L)^*)$ is a branched cover of degree $n+1$. Note that $\Pi_n^{-1}(x) \cong \varphi^{-1}(H_x)$, where $H_x \subset \mathbb{P}(H^0(C,L))$ consists of all the hyperplanes in $\mathbb{P}(H^0(C,L)^*)$ containing x. We need to show that $\varphi^{-1}(H_x)$ is connected. The restriction of φ to $\varphi^{-1}(H_x)$ gives a branched covering $\varphi^{-1}(H_x) \longrightarrow H_x$ of degree $n+1$. Since $H_x \cong \mathbb{P}^{n-1}$ is connected, in order to check that $\varphi^{-1}(H_x)$ is connected, it is enough to check that for a generic $P \in H_x$, any two points in $\varphi^{-1}(P)$ are connected by a path that lies within $\varphi^{-1}(H_x)$.

Let $P \ni x$ be a hyperplane in $\mathbb{P}(H^0(C,L)^*)$ that intersects E in $n+1$ distinct points $\{x_1, x_2, ..., x_{n+1}\}$. Pick any two elements $\xi_0, \xi_1 \in S^nE$ in $\varphi^{-1}(P)$, and let us show that they can be connected by a continuous path within $\varphi^{-1}(H_x)$. Without loss of generality, we can assume $\xi_0 = \{x_1, x_2, ...x_{n-1}, x_n\}$, $\xi_1 = \{x_1, x_2, ..., x_{n-1}, x_{n+1}\}$. Let $\widetilde{\gamma} : [0,1] \to E$ be a continuous path such that $\widetilde{\gamma}(0) = x_{n+1}$, $\widetilde{\gamma}(1) = x_n$. Define a path $\widehat{\gamma} : [0,1] \to H_x$, by declaring $\widehat{\gamma}(t)$, $t \in [0,1]$, to be the linear span of $x_1, x_2, ..., x_{n-2}, \widetilde{\gamma}(t) \in E \in \mathbb{P}(H^0(C,L)^*)$, and x. Since the branch locus of φ has real codimension 2 in H_x, we can adjust $\widehat{\gamma}$ if necessary, and assume that for all $t \in [0,1]$ the hyperplane $\widehat{\gamma}(t)$ intersects E in $n+1$ distinct points. There is a unique continuous path $\gamma : [0,1] \to \varphi^{-1}(H_x)$ such that $\varphi \circ \gamma = \widehat{\gamma}$, $\gamma(0) = \xi_0$. We claim that $\gamma(1) = \xi_1$. Indeed, $\gamma(1)$ is obtained from $\gamma(0)$ by a permutation that sends x_{n+1} to x_n. So, the permutation sends $\xi_0 = \{x_1, x_2, ..., x_{n+1}\} \setminus \{x_{n+1}\}$ to $\xi_1 = \{x_1, x_2, ..., x_{n+1}\} \setminus \{x_n\}$. □

Proof of Theorem 5.4.1. By Lemma 5.4.2, it is enough to prove that Π_n is Poisson for a uniquely defined Poisson structure on $\mathbb{P}(H^0(C,L)^*)$. For this, by Lemma 2.1.3, it is enough to check that the fibers of Π_n are compact and connected. Compactness of the fibers is automatic, because the domain of Π_n is a projective variety. Connectedness follows from Lemma 5.4.3. □

Example 5.4.4. Consider the quadratic Poisson structure on $\mathbb{C}^n$ given by the formula

$$\{x_k, x_{k+m}\} = \sum_{i=0}^{m-1} x_{k+i}x_{k+m-i}, \quad 1 \le k < k+m \le n. \tag{5.3}$$

This is a rational degeneration of the family of Feigin-Odesskii Poisson structures [22]. It corresponds to the rational Veronese curve $C = \mathbb{P}^1 \ni [s:t] \mapsto [s^n : s^{n-1}t : .... : st^{n-1} : t^n] \in \mathbb{P}^{n-1}$ and the vector field $v = \frac{1}{2}s\partial_s - \frac{1}{2}t\partial_t$ on C.

Consider the desingularization map $\Pi_d : V^* \to K_d \subset H^0(C, L)^* \cong \mathbb{C}^n$. Here $L = \mathcal{O}_C(n-1)$, and V is the rank n vector bundle over the symmetric power S^dC given by $p_*(q^*L \otimes \mathcal{O}(B))$, where $U_d \subset C \times S^dC$ is the universal Hilbert scheme of d points of C, $q : U_d \to C$ and $p : U_d \to S^dC$ are the projections and $R \subset U_d$ is the ramification divisor of p.

Let us write down the map Π_d in coordinates. Let z be the affine coordinate s/t on $C = \mathbb{P}^1$. The vector field v in this coordinate will assume the form $z\partial_z$. Let w be a fiberwise linear coordinate on L^* over the affine piece of C. Then over an analytic neighborhood of a generic point $\{z_1, ..., z_d\} \in S^dC$, the total space of V^* will have the coordinates $z_1, ..., z_d, w_1, ..., w_d$. The map Π_d in these coordinates assumes the form

$$(z_1, ..., z_d, w_1, ..., w_d) \longmapsto \sum_{i=1}^{n} \frac{w_i}{\prod_{j \ne i}(z_i - z_j)}\left(1, z_i, z_i^2, ..., z_i^{n-1}\right)$$

The Poisson bracket σ on $\mathbb{P}(V_d^*)$ constructed in Section 5.2, is the projectivization of the bracket

$$\{w_i, z_i\} = z_i w_i, \ \{z_i, z_j\} = \{w_i, w_j\} = \{z_i, w_j\} = 0, \quad 1 \le i < j \le d. \tag{5.4}$$

Let us verify in coordinates that Π_d sends bracket (5.4) to the bracket (5.3), for the case $d = 2$, $n = 4$. The map Π_2 in coordinates has the expression

$$\begin{aligned}(z_1, z_2, w_1, w_2) \longmapsto \quad & w_1\left(\frac{1}{z_1 - z_2}, \frac{z_1}{z_1 - z_2}, \frac{z_1^2}{z_1 - z_2}, \frac{z_1^3}{z_1 - z_2}\right) \quad + \\ & w_2\left(\frac{1}{z_2 - z_1}, \frac{z_2}{z_2 - z_1}, \frac{z_2^2}{z_2 - z_1}, \frac{z_2^3}{z_2 - z_1}\right)\end{aligned} \tag{5.5}$$

For $k = 0, 1$ or 2 we have

$$\begin{aligned}\{x_k, x_{k+1}\} = &\left\{w_1\frac{z_1^k}{z_1 - z_2} + w_2\frac{z_2^k}{z_2 - z_1}, w_1\frac{z_1^{k+1}}{z_1 - z_2} + w_2\frac{z_2^{k+1}}{z_2 - z_1}\right\} = \\ &\left\{w_1\frac{z_1^k}{z_1 - z_2}, w_1\frac{z_1^{k+1}}{z_1 - z_2}\right\} + \left\{w_1\frac{z_1^k}{z_1 - z_2}, w_2\frac{z_2^{k+1}}{z_2 - z_1}\right\} + \\ &\left\{w_2\frac{z_2^k}{z_2 - z_1}, w_1\frac{z_1^{k+1}}{z_1 - z_2}\right\} + \left\{w_2\frac{z_2^k}{z_2 - z_1}, w_2\frac{z_2^{k+1}}{z_2 - z_1}\right\} = \\ & w_1\frac{z_1^k}{z_1 - z_2}w_1\frac{z_1^{k+1}}{z_1 - z_2} + w_1\frac{z_1^k}{z_1 - z_2}w_2\frac{z_2^{k+1}}{z_2 - z_1}\left(\frac{z_1}{z_2 - z_1} - \frac{z_2}{z_1 - z_2}\right) + \\ & w_2\frac{z_2^k}{z_2 - z_1}w_1\frac{z_1^{k+1}}{z_1 - z_2}\left(\frac{z_2}{z_1 - z_2} - \frac{z_1}{z_2 - z_1}\right) + w_2\frac{z_2^k}{z_2 - z_1}w_2\frac{z_2^{k+1}}{z_2 - z_1} = \\ &\left(w_1\frac{z_1^k}{z_1 - z_2} + w_2\frac{z_2^k}{z_2 - z_1}\right)\left(w_1\frac{z_1^{k+1}}{z_1 - z_2} + w_2\frac{z_2^{k+1}}{z_2 - z_1}\right) = x_k x_{k+1}\end{aligned}$$

This so far agrees with (5.3). Next, for $k = 0$ or 1 we have

$$
\begin{gathered}
\{x_k, x_{k+2}\} = \left\{w_1\frac{z_1^k}{z_1 - z_2} + w_2\frac{z_2^k}{z_2 - z_1}, w_1\frac{z_1^{k+2}}{z_1 - z_2} + w_2\frac{z_2^{k+2}}{z_2 - z_1}\right\} = \\
\left\{w_1\frac{z_1^k}{z_1 - z_2}, w_1\frac{z_1^{k+2}}{z_1 - z_2}\right\} + \left\{w_1\frac{z_1^k}{z_1 - z_2}, w_2\frac{z_2^{k+2}}{z_2 - z_1}\right\} + \\
\left\{w_2\frac{z_2^k}{z_2 - z_1}, w_1\frac{z_1^{k+2}}{z_1 - z_2}\right\} + \left\{w_2\frac{z_2^k}{z_2 - z_1}, w_2\frac{z_2^{k+2}}{z_2 - z_1}\right\} = \\
2w_1\frac{z_1^k}{z_1 - z_2}w_1\frac{z_1^{k+2}}{z_1 - z_2} + w_1\frac{z_1^k}{z_1 - z_2}w_2\frac{z_2^{k+2}}{z_2 - z_1}\left(\frac{z_1}{z_2 - z_1} - \frac{z_2}{z_1 - z_2}\right) + \\
w_2\frac{z_2^k}{z_2 - z_1}w_1\frac{z_1^{k+2}}{z_1 - z_2}\left(\frac{z_2}{z_1 - z_2} - \frac{z_1}{z_2 - z_1}\right) + 2w_2\frac{z_2^k}{z_2 - z_1}w_2\frac{z_2^{k+2}}{z_2 - z_1} = \\
2w_1\frac{z_1^k}{z_1 - z_2}w_1\frac{z_1^{k+2}}{z_1 - z_2} + w_1\frac{z_1^k}{z_1 - z_2}w_2\frac{z_2^k}{z_2 - z_1}\left(\frac{-z_1z_2^2 - z_2^3 + z_1^2z_2 + z_1^3}{z_1 - z_2}\right) + 2w_2\frac{z_2^k}{z_2 - z_1}w_2\frac{z_2^{k+2}}{z_2 - z_1} = \\
2w_1\frac{z_1^k}{z_1 - z_2}w_1\frac{z_1^{k+2}}{z_1 - z_2} + w_1\frac{z_1^k}{z_1 - z_2}w_2\frac{z_2^k}{z_2 - z_1}\left(z_1^2 + 2z_1z_2 + z_2^2\right) + 2w_2\frac{z_2^k}{z_2 - z_1}w_2\frac{z_2^{k+2}}{z_2 - z_1} = \\
\left(w_1\frac{z_1^k}{z_1 - z_2} + w_2\frac{z_2^k}{z_2 - z_1}\right)\left(w_1\frac{z_1^{k+2}}{z_1 - z_2} + w_2\frac{z_2^{k+2}}{z_2 - z_1}\right) + \left(w_1\frac{z_1^{k+1}}{z_1 - z_2} + w_2\frac{z_2^{k+1}}{z_2 - z_1}\right)^2 = x_kx_{k+2} + x_{k+1}^2
\end{gathered}
$$

This also agrees with (5.3). Finally, for $k = 0$ we have

$$
\begin{gathered}
\{x_k, x_{k+3}\} = \left\{w_1\frac{z_1^k}{z_1 - z_2} + w_2\frac{z_2^k}{z_2 - z_1}, w_1\frac{z_1^{k+3}}{z_1 - z_2} + w_2\frac{z_2^{k+3}}{z_2 - z_1}\right\} = \\
\left\{w_1\frac{z_1^k}{z_1 - z_2}, w_1\frac{z_1^{k+3}}{z_1 - z_2}\right\} + \left\{w_1\frac{z_1^k}{z_1 - z_2}, w_2\frac{z_2^{k+3}}{z_2 - z_1}\right\} + \\
\left\{w_2\frac{z_2^k}{z_2 - z_1}, w_1\frac{z_1^{k+3}}{z_1 - z_2}\right\} + \left\{w_2\frac{z_2^k}{z_2 - z_1}, w_2\frac{z_2^{k+3}}{z_2 - z_1}\right\} = \\
3w_1\frac{z_1^k}{z_1 - z_2}w_1\frac{z_1^{k+3}}{z_1 - z_2} + w_1\frac{z_1^k}{z_1 - z_2}w_2\frac{z_2^{k+3}}{z_2 - z_1}\left(\frac{z_1}{z_2 - z_1} - \frac{z_2}{z_1 - z_2}\right) + \\
w_2\frac{z_2^k}{z_2 - z_1}w_1\frac{z_1^{k+3}}{z_1 - z_2}\left(\frac{z_2}{z_1 - z_2} - \frac{z_1}{z_2 - z_1}\right) + 3w_2\frac{z_2^k}{z_2 - z_1}w_2\frac{z_2^{k+3}}{z_2 - z_1} = \\
3w_1\frac{z_1^k}{z_1 - z_2}w_1\frac{z_1^{k+3}}{z_1 - z_2} + w_1\frac{z_1^k}{z_1 - z_2}w_2\frac{z_2^k}{z_2 - z_1}\left(\frac{-z_1z_2^3 - z_2^4 + z_1^3z_2 + z_1^4}{z_1 - z_2}\right) + 3w_2\frac{z_2^k}{z_2 - z_1}w_2\frac{z_2^{k+3}}{z_2 - z_1} = \\
3w_1\frac{z_1^k}{z_1 - z_2}w_1\frac{z_1^{k+3}}{z_1 - z_2} + w_1\frac{z_1^k}{z_1 - z_2}w_2\frac{z_2^k}{z_2 - z_1}\left(z_1^3 + 2z_1^2z_2 + 2z_1z_2^2 + z_2^3\right) + 3w_2\frac{z_2^k}{z_2 - z_1}w_2\frac{z_2^{k+3}}{z_2 - z_1} = \\
\left(w_1\frac{z_1^k}{z_1 - z_2} + w_2\frac{z_2^k}{z_2 - z_1}\right)\left(w_1\frac{z_1^{k+3}}{z_1 - z_2} + w_2\frac{z_2^{k+3}}{z_2 - z_1}\right) + \\
2\left(w_1\frac{z_1^{k+1}}{z_1 - z_2} + w_2\frac{z_2^{k+1}}{z_2 - z_1}\right)\left(w_1\frac{z_1^{k+2}}{z_1 - z_2} + w_2\frac{z_2^{k+2}}{z_2 - z_1}\right) = x_kx_{k+3} + 2x_{k+1}x_{k+2}.
\end{gathered}
$$

This again agrees with (5.3). This illustrates that Π_2 is a Poisson map.

Appendix A

Local normal forms for (co-)Higgs bundles over one-dimensional formal disc

Throughout the section, $\mathcal{U} = \text{Spec } \mathbb{C}[[x]]$ denotes the formal neighborhood of $0 \in \mathbb{C}$, $V = \mathcal{O}_{\mathcal{U}}^{\oplus r}$ denotes a rank r vector bundle over $\mathcal{U}$, $\phi \in \text{End}(V)$ denotes a Higgs field on V, and $\Sigma \subset \text{Spec } \mathbb{C}[[x]][y]$ denotes the spectral curve of ϕ. A natural question is to determine how many isomorphism classes of ϕ with a given spectral curve Σ are there and what they look like. By spectral correspondence, this question is equivalent to describing the isomorphism classes of torsion-free rank one sheaves on Σ. Note that torsion-free sheaves on reduced curves are also known as maximal Cohen-Macaulay modules, or simply Cohen-Macaulay modules (e.g. see [38]). Without loss of generality, one may assume that Σ is connected and contains the origin $x = 0, y = 0$. Let Σ be given by the equation $f(x,y) = y^r + g_{r-1}(x)y^{r-1} + ... + g_1(x)y + g_0(x) = 0$, for some $g_i \in \mathbb{C}[[x]]$, $i = 0, 1, .., r-1$.

Recall that if Σ is smooth, then every torsion-free rank one sheaf on Σ is isomorphic to $\mathcal{O}_\Sigma$. One can choose the basis $1, y, y^2, ..., y^{r-1}$ of $\mathcal{O}_\Sigma$ over $\mathcal{O}_\mathcal{U}$, and the Higgs field ϕ which is the multiplication by y. Then in this basis, ϕ has the matrix (here and throughout the section we omit zero entries in a matrix):

$$\phi = \left(\begin{array}{ccccc|c} & & & & & -g_0(x) \\ \hline 1 & & & & & -g_1(x) \\ & 1 & & & & -g_2(x) \\ & & \ddots & & & \vdots \\ & & & 1 & & -g_{r-2}(x) \\ & & & & 1 & -g_{r-1}(x) \end{array}\right)$$

If Σ is not smooth, describing the torsion-free rank one sheaves on Σ is much more complicated and does not appear to have a universal answer. However, for certain types of singularities, the answer is contained in the literature on Cohen-Macaulay modules [38, 30, 8]. For the purpose of this thesis, we will need the description of torsion-free rank one sheaves over singularities of ADE and T_{pq} types defined below. We remark that all the local normal form results below that are stated over Spec $\mathbb{C}[[x]]$ also hold over Spec $\mathbb{C}\{x\}$, where the latter means convergent power series.

Recall that a singularity $C = \operatorname{Spec} \mathbb{C}[[x,y]]/(f(x,y))$ is said to have *ADE type* if, after an analytic change of variables, f appears in the following list:

(A_n) $x^2 - y^{n+1}$ $(n \geq 1)$,

(D_n) $x^2y - y^{n-1}$ $(n \geq 4)$,

(E_6) $x^3 - y^4$,

(E_7) $x^3y - y^3$,

(E_8) $x^3 - y^5$.

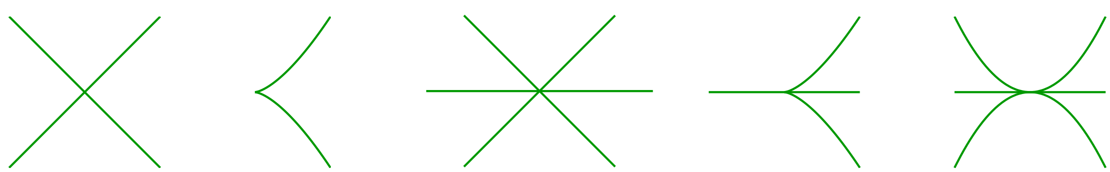
A singularity $C = \operatorname{Spec} \mathbb{C}[[x,y]]/(f(x,y))$ is said to have *type* T_{pq}, $p,q \in \mathbb{Z}_{\geq 3}$, $1/q + 1/q \leq 1/2$, if, after an analytic change of variables, f appears in the following list:

(T_{36}) $y(y-x^2)(y-ax^2)$, $a \in \mathbb{C} \smallsetminus \{0,1\}$,

(T_{44}) $yx(y-x)(y-ax)$, $a \in \mathbb{C} \smallsetminus \{0,1\}$,

(T_{pq}) $x^p + x^2y^2 + y^q$, $1/p + 1/q < 1/2$.

The following representatives of the families are of particular interest for this thesis:



$A_1 : y^2 = x^2$ $A_2 : y^2 = x^3$ $D_4 : y^3 = x^2 y$ $E_7 : y^3 = x^3 y$ $T_{36} : y^3 = x^4 y$

The *ADE* singularities are prominent because they have *finite Cohen-Macaulay type*, in the sense that for each such singularity, the category of torsion-free sheaves has finitely many indecomposables. Conversely, any curve singularity of finite Cohen-Macaulay type is a partial normalization of an *ADE* singularity [10, 12, 11]. The singularities of the type T_{pq} are prominent because they have *tame Cohen-Macaulay type*, in the sense that for each such singularity, the indecomposable torsion-free sheaves of any fixed rank form a finite number of 1-parameter families, together with possibly an additional finite number of indecomposable sheaves. Conversely, any curve singularity of tame Cohen-Macaulay type is a partial normalization of a T_{pq} singularity [9]. Finally, any curve singularity that is not a partial normalization of an *ADE* or T_{pq} singularity has *wild Cohen-Macaulay type*, in the sense that for any such singularity, the classification of its torsion-free sheaves is equivalent to the classification of modules for all algebras [9]. Informally speaking, the classification problem is hopeless in the case of wild Cohen-Macaulay type. Despite this, some of the curve singularities of wild Cohen-Macaulay type still admit a description of their torsion-free sheaves of a fixed, or bounded, rank. An interested reader should consult [30, 8, 36] for further results in this direction.

By a curve singularity, we mean Spec R, where R is the completion of a local ring of a point in a reduced, but not necessarily irreducible, algebraic curve over $\mathbb{C}$. Let F be the ring of fractions of R.

Recall that a torsion-free sheaf over a curve singularity Spec R is an R-module M such that the natural map $M \to M \otimes F$ is injective. A torsion-free sheaf M is said to have rank one if $M \otimes F$ is a free rank one F-module, that is, $M \otimes F$ is non-canonically isomorphic to F. Note that when we say that a torsion-free sheaf has rank one, we require that it has full support. For instance, in the case $R = \mathbb{C}[[x,y]]/(y^2 - x^2)$, a torsion-free sheaf that is completely supported on the irreducible component $\{x = y\}$ would not be considered to have rank one.

There are two ways of thinking of torsion-free rank one sheaves over a curve singularity. First, we can take a list of generators $a_i = \frac{g_i}{h_i} \in M \subset F$, $i = 1,..,k$, where $g_i, h_i \in R$, and apply an automorphism of F (namely, multiplication by the least common multiple of the h_i) to replace the a_i with new generators $b_i \in R$, $i = 1,...,k$. This way, we can view M as the ideal sheaf $(b_1,...,b_k)R$. Conversely, any ideal sheaf $I \subset R$ defines a torsion-free sheaf, and if the annihilator $\{a \in M : ab = 0, \text{ for every } b \in I\}$ is trivial, then I defines a torsion-free sheaf of rank one.

Another way to view a torsion-free rank one sheaf M is to embed M into $M \otimes \overline{R}$, where $R \subset \overline{R} \subset F$ is the integral closure of R. Since Spec R is smooth, the torsion-free rank one sheaf $M \otimes \overline{R}$ is isomorphic to $\overline{R}$. After applying an automorphism of $\overline{R}$, if necessary, we can make sure that the image M in $M \otimes \overline{R} \cong \overline{R}$ contains 1. So, a torsion-free rank one sheaf M can be embedded into $\overline{R}$ so that $R \subset M \subset \overline{R}$. One can check that two R-modules $R \subset M, N \subset \overline{R}$ define isomorphic torsion-free rank one sheaves if and only if there is an invertible element $x \in \overline{R}^*$ such that $xM = N$. Such a viewpoint allows us to reformulate the classification of torsion-free rank one sheaves over Spec R as the classification of R-invariant linear subspaces M with $R \subset M \subset \overline{R}$, up to the equivalence relation described above. This way, one can often calculate explicitly the isomorphism classes of torsion-free rank one sheaves for a given curve singularity (see [30] for more details and examples on this method).

For a torsion-free rank one sheaf M with $R \subset M \subset \overline{R}$, one can associate a partial normalization $\widetilde{\Sigma}$ of $\Sigma = \text{Spec } R$ by letting $\widetilde{\Sigma} = \text{Spec End}_R(M)$ $(= \text{Spec } \{r \in \overline{R} : rM \subset M\})$. Note that M canonically defines a sheaf $\widetilde{M}$ over $\widetilde{\Sigma}$ and the pushforward of $\widetilde{M}$ under the partial normalization map $\widetilde{\Sigma} \to \Sigma$ is precisely M. This allows one to draw pictures while classifying Higgs fields with a fixed spectral curve having relatively uncomplicated singularities.

A singularities.

Proposition A.0.1. *For $R = \mathbb{C}[[x,y]]/(x^2 - y^{n+1})$, each torsion-free rank one sheaf on $\Sigma = \text{Spec } R$ is isomorphic to exactly one of the ideal sheaves $(x, y^m)R$, $m = 0,1,...,\lfloor \frac{n+1}{2} \rfloor$, where $\lfloor\ \rfloor$ stands for the integer part.*

For each $m = 0,1,...,\lfloor \frac{n-1}{2} \rfloor$, the partial normalization of Σ corresponding to the ideal sheaf $(x,y^m)R$ has an A_{n-2m} singularity. The ideal sheaf $(x,y^m)R$, for $m = \lfloor \frac{n+1}{2} \rfloor$, corresponds to the full normalization of Σ.

Proof. First, let us assume that $n+1$ is odd. Then the integral closure $\overline{R} = \mathbb{C}[[t]]$, and R embeds into $\overline{R}$ by $x \mapsto t^{n+1}$, $y \mapsto t^2$. The vector space $\overline{R}/R$ has basis $t, t^3, t^5, ..., t^{n-1}$. One can check that each R-invariant vector space M with $R \subset M \subset \overline{R}$ can be brought to the form $M_k = t^{2k+1}R + R$, $0 \le k \le n/2$, via multiplication by an element $x \in \overline{R}^*$, and that different k's correspond to non-isomorphic torsion-free rank one sheaves over Spec R. Note that, for each $0 \le k \le n/2$, the sheaf M_k is isomorphic to the ideal sheaf $(x, y^{n/2-k})R$.

Now, let us consider the case when $n+1$ is even. The integral closure $\overline{R} = \mathbb{C}[[t_1]] \oplus \mathbb{C}[[t_2]]$

and R embeds into $\overline{R}$ by $x \mapsto (t_1^{(n+1)/2}, -t_2^{(n+1)/2})$, $y \mapsto (t_1, t_2)$. The vector space $\overline{R}/R$ has basis $(0,1)$,$(0,t_2)$,$(0,t_2^2)$,..., $(0,t_2^{(n-1)/2})$. One can check that each R-invariant vector space M with $R \subset M \subset \overline{R}$ can be brought to the form $M_k = (0,t_2^k)R + R$, $0 \le k \le (n+1)/2$, via multiplication by an element $x \in \overline{R}^*$, and that different k's correspond to non-isomorphic torsion-free rank one sheaves over Spec R. Note that, for each $0 \le k \le (n+1)/2$, the sheaf M_k is isomorphic to the ideal sheaf $(x, y^{(n+1)/2-k})R$. □

We remark that for ADE singularities, much more detailed results are available. For instance, see [38], where classification of torsion-free sheaves of all ranks is obtained using the matrix factorization technique, and the category of such sheaves is mapped out in the form of an Auslander-Reiten quiver, which includes information about indecomposables, Hom's and Ext's of the category.

Theorem A.0.2. *(Local normal forms for Higgs fields with spectral curve having A_n singularity)*
1) Let ϕ be a Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus(n+1)}$, $n \ge 1$, for $\mathcal{U} =$ Spec $\mathbb{C}[[x]]$, with the spectral curve $\Sigma = \{y^{n+1} = x^2\}$. Then ϕ is isomorphic to exactly one of the Higgs fields Φ_k, $k = 0,1,2,...,\lfloor \frac{n+1}{2} \rfloor$, where

$$\Phi_0 = \left(\begin{array}{ccccccc|c} & & & & & & & x^2 \\ \hline 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{array}\right), \qquad \Phi_k = \left(\begin{array}{ccccccc|c} & & & & & & & x \\ \hline 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & x & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & 1 & \end{array}\right)$$

(if $k > 0$, Φ_k has x in the $(n+1,1)$-th and $(k,k+1)$-th spots).
2) Let ϕ be a Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus 2}$, for $\mathcal{U} =$ Spec $\mathbb{C}[[x]]$, with the spectral curve $\Sigma = \{y^2 = x^{n+1}\}$. Then ϕ is isomorphic to exactly one of the Higgs fields $\widetilde{\Phi}_k$, $k = 0,1,2,...,\lfloor \frac{n+1}{2} \rfloor$, where

$$\widetilde{\Phi}_k = \begin{pmatrix} & x^{n+1-k} \\ x^k & \end{pmatrix}.$$

Proof. 1) According to the spectral correspondence, the isomorphism classes of ϕ with the fixed spectral curve $\{y^{n+1} = x^2\}$ are in one-to-one correspondence with the isomorphism classes of torsion-free rank one sheaves on Spec R, $R = \mathbb{C}[[x,y]]/(x^2 - y^{n+1})$. By Proposition A.0.1, each such sheaf is isomorphic to the ideal sheaf $(x,y^k)R$ for some $k = 0,1,2,...,\lfloor \frac{n+1}{2} \rfloor$. If $k = 0$, then the ideal $(x,y^0)R = (x,1)R$ is the whole ring R, the elements $1,y,y^2,...,y^n$ form a basis of R over $\mathbb{C}[[x]]$, and the matrix of multiplication by y in this basis is precisely Φ_0. If $0 < k \le \lfloor \frac{n+1}{2} \rfloor$, then the ideal sheaf $(x,y^k)R$ has basis $x,xy,xy^2,...,xy^{k-1},y^k,y^{k+1},...,y^n$ over $\mathbb{C}[[x]]$, and multiplication by y gives matrix Φ_k.

2) The proof goes similarly to the above, except that one swaps the roles of x and y. The ideals of $\mathbb{C}[[x,y]]/(y^2 - x^{n+1})$ are given by (y,x^k), $k = 0,1,2,...,\lfloor \frac{n+1}{2} \rfloor$. For each such k, the ideal $(y,x^k)R$ has basis x^k,y over $\mathbb{C}[[x]]$, and the multiplication by y has matrix $\widetilde{\Phi}_k$. □

D singularities.

Theorem A.0.3. *(Local normal forms for Higgs fields with spectral curve having D_n singularity)*

Let ϕ be a Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus(n-1)}$, $n \geq 4$, for $\mathcal{U} = \operatorname{Spec} \mathbb{C}[[x]]$, with the spectral curve $\Sigma = \{y^{n-1} = x^2 y\}$. Then ϕ is isomorphic to exactly one of the following Higgs fields

1. Ψ_k, $k = 0, 1, ..., n-2$, *where*

$$\Psi_0 = \left(\begin{array}{cccccccc|c} &&&&&&&& 0 \\ \hline 1 &&&&&&&& x^2 \\ & 1 &&&&&&& \\ && 1 &&&&&& \\ &&& \ddots &&&&& \\ &&&& \ddots &&&& \\ &&&&& \ddots &&& \\ &&&&&& 1 && \\ &&&&&&& 1 & \end{array}\right), \qquad \Psi_k = \left(\begin{array}{cccccccc|c} &&&&&&&& 0 \\ \hline 1 &&&&&&&& x \\ & \ddots &&&&&&& \\ && 1 &&&&&& \\ &&& x &&&&& \\ &&&& 1 &&&& \\ &&&&& \ddots &&& \\ &&&&&& \ddots && \\ &&&&&&& 1 & \end{array}\right)$$

(if $k > 0$, Ψ_k has x in the $(n-1, 2)$-th and $(k, k+1)$-th spots),

2. $0_1 \oplus \Phi_k$, $k = 0, 1, ..., \lfloor \frac{n-2}{2} \rfloor$, *where 0_1 is the zero matrix of size 1×1, and Φ_k is the matrix in the list of local normal forms, specified in Theorem A.0.2, for Higgs fields with spectral curve having the A_{n-3} singularity $\{y^{n-2} = x^2\}$,*

3. *(This case occurs only if n is even)* $\Phi'_{\pm} \oplus \Phi''_{\mp}$, *where*

$$\Phi'_{\pm} = \left(\begin{array}{cccccccc|c} &&&&&&&& 0 \\ \hline 1 &&&&&&&& \pm x \\ & 1 &&&&&&& \\ && 1 &&&&&& \\ &&& \ddots &&&&& \\ &&&& \ddots &&&& \\ &&&&& \ddots &&& \\ &&&&&& 1 && \\ &&&&&&& 1 & \end{array}\right), \qquad \Phi''_{\mp} = \left(\begin{array}{ccccccc|c} &&&&&&& \mp x \\ \hline 1 &&&&&&& \\ & 1 &&&&&& \\ && \ddots &&&&& \\ &&& \ddots &&&& \\ &&&& \ddots &&& \\ &&&&& 1 && \\ &&&&&& 1 & \end{array}\right).$$

(Here Φ' has size $\frac{n}{2} \times \frac{n}{2}$, Φ'' has size $(\frac{n}{2} - 1) \times (\frac{n}{2} - 1)$; if one has $+$ in Φ', then Φ'' must have $-$, and vice versa).

Proof. It is possible to carry out the classification of torsion-free rank one sheaves over Spec R, $R = \mathbb{C}[[x,y]]/(y^{n-1} - x^2 y)$, directly as we did in the proof of Theorem A.0.2. Instead, we refer the reader to [38, Chapter 9], where such classification is carried out without restriction to the rank one case. The isomorphism classes of rank one torsion-free sheaves on Spec R are given by the following:

- R itself. In the basis $1, y, ..., y^{n-2}$ of this module over $\mathbb{C}[[x]]$, the multiplication by y has matrix Φ_0.
- Ideals $(x, y^k)R$, $k = 1, 2, ..., n-2$. For each such k, the choice of basis $x, xy, xy^2, ..., xy^{k-1}, y^k, y^{k+1}, ..., y^{n-2}$ produces Φ_k.

- Ideals $(y^{n-2}-x^2, xy, y^{k+1})R$, $k = 0, 1, ..., \lfloor \frac{n-2}{2} \rfloor$. For each such k, the choice of basis $y^{n-2}-x^2, xy, xy^2, ..., xy^k$, $y^{k+1}, y^{k+2}, ..., y^{n-2}$ produces the matrix $0_1 \oplus \Phi_k$.
- Ideals $(y(y^{(n-2)/2}-x), y^{(n-2)/2}+x)R$ and $(y(y^{(n-2)/2}+x), y^{(n-2)/2}-x)R$, assuming n is even. The basis $a, ay, ay^2, ..., ay^{(n-2)/2}, b, by, ..., by^{n/2}$ produces a matrix of the form $\Phi' \oplus \Phi''$, where $a = y(y^{(n-2)/2} \mp x)$, $b = y^{(n-2)/2} \pm x$.

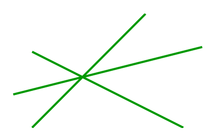
□

Among all D_n singularities, the one that we need for this thesis the most is D_4. Let us have a closer look at it. The equation $y^3 = x^2y$ can be rewritten as $y(y-x)(y+x) = 0$, so it cuts out three lines on a plane intersecting at a common point. One can check that for any distinct $\lambda_i \in \mathbb{C}$, $i = 1, 2, 3$, the singularity $(y-\lambda_1 x)(y-\lambda_2 x)(y-\lambda_3 x) = 0$ is analytically isomorphic to $y(y-x)(y+x) = 0$, so it is also a D_4 singularity. Let us use this symmetric form of the equation, to emphasize the symmetry of the three branches of the singularity. Theorem A.0.3 says that for D_4, there are $3+2+2 = 7$ isomorphism classes of Higgs fields. Let us rewrite them in the symmetric form.

Corollary A.0.4. *Let ϕ be a Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$, for $\mathcal{U} = \operatorname{Spec} \mathbb{C}[[x]]$, with the spectral curve $\Sigma = \{(y-\lambda_1 x)(y-\lambda_2 x)(y-\lambda_3 x) = 0\}$, where $\lambda_i \in \mathbb{C}$, $i = 1, 2, 3$, are fixed and pairwise distinct.*

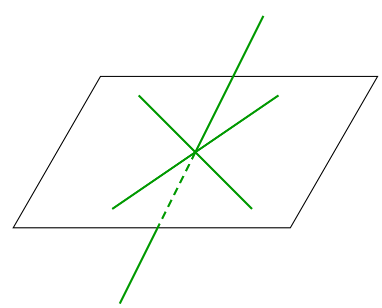
Then ϕ is isomorphic to exactly one of the following seven Higgs fields (below we include the picture of the partial normalizations of Σ that correspond to the eigensheaf of the Higgs field):

$$\Phi_0^{D_4} = \begin{pmatrix} \lambda_1 x & & \\ 1 & \lambda_2 x & \\ & 1 & \lambda_3 x \end{pmatrix}$$

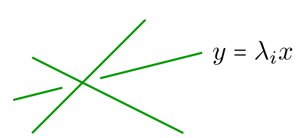


$$\Phi_1^{D_4} = \begin{pmatrix} \lambda_1 x & & \\ x & \lambda_2 x & \\ & 1 & \lambda_3 x \end{pmatrix}$$

$$\Phi_2^{D_4} = \begin{pmatrix} \lambda_1 x & & \\ 1 & \lambda_2 x & \\ & x & \lambda_3 x \end{pmatrix}$$

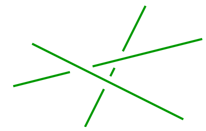


$$\Phi_{2+i}^{D_4} = \left(\begin{array}{c|cc} \lambda_i x & & \\ \hline & \lambda_{i+1} x & \\ & 1 & \lambda_{i+2} x \end{array}\right)$$



$i = 1, 2, 3$

$$\Phi_6^{D_4} = \begin{pmatrix} \lambda_1 x & & \\ & \lambda_2 x & \\ & & \lambda_3 x \end{pmatrix}$$



Proof. The seven options guaranteed by Theorem A.0.3 are Ψ_0, Ψ_1, Ψ_2, $0_1 \oplus \Phi_0$, $0_1 \oplus \Phi_1$ and $\Phi'_\pm \oplus \Phi''_\mp$.

Let $R = \mathbb{C}[[x,y]]/(\prod_{i=1}^3 (y - \lambda_i x))$. Let $\overline{R} = \mathbb{C}[[t_1]] \oplus \mathbb{C}[[t_2]] \oplus \mathbb{C}[[t_3]]$ be the integral closure of R, where R embeds into $\overline{R}$ by $x \mapsto (t_1, t_2, t_3)$, $y \mapsto (\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3)$.

The Higgs Φ_0 corresponds to the structure sheaf R. If instead of choosing the obvious basis $1, y, y^2$ of R over $\mathcal{U}$, we choose the basis $1, y - \lambda_1 x, (y - \lambda_1 x)(y - \lambda_2 x)$, we obtain the matrix $\Phi_0^{D_4}$.

The Higgs fields Ψ_1 and Ψ_2 correspond to the ideals $(x,y)R$ and $(x,y^2)R$, respectively. The basis $x, y - \lambda_1 x, (y - \lambda_1 x)(y - \lambda_2 x)$ in the former case gives matrix $\Phi_1^{D_4}$, and the basis $x, x(y - \lambda_1 x), (y - \lambda_1 x)(y - \lambda_2 x)$ in the latter case gives matrix $\Phi_2^{D_4}$. The ideal sheaf $(x,y^2)R$ is isomorphic to the sheaf given by the R-module $M_2 \subset \overline{R}$ spanned by 1 and $y^2/x = (\lambda_1^2 t_1, \lambda_2^2 t_2, \lambda_3^2 t_3)$. It follows that $M_2 = 1\mathbb{C} \oplus t_1 \mathbb{C}[[t_1]] \oplus t_2 \mathbb{C}[[t_2]] \oplus t_3 \mathbb{C}[[t_3]]$ forms a ring itself. One can see that $M_2 = \mathbb{C}[[\widetilde{x}, \widetilde{y}, \widetilde{z}]]/(\widetilde{x}\widetilde{y}, \widetilde{x}\widetilde{z}, \widetilde{y}\widetilde{z})$, where $\widetilde{x} = (t_1, 0, 0)$, $\widetilde{y} = (0, t_2, 0)$, $\widetilde{z} = (0,0,t_3)$, that is, $\Sigma_2 = \mathrm{Spec}\, M_2$ is the singularity of three coordinate lines in $\mathbb{C}^3$ meeting together at the origin. This singularity can be viewed as a partial normalization of Spec R, and M_2 can be viewed as the pushforward of the structure sheaf of Σ_2 onto Σ. The ideal sheaf $(x,y)R$ is isomorphic to the R-module $M_1 \subset \overline{R}$ spanned by 1 and $y/x = (\lambda_1, \lambda_2, \lambda_3)$. One can see that M_1 contains the ring M_2, but does not form a ring itself. So, M_1 can be viewed as a pushforward under the normalization map $\Sigma_2 \to \Sigma$ of a certain torsion-free sheaf over Σ_2 (in fact, it happens to be the dualizing sheaf of Σ_2).

The Higgs fields $0_1 \oplus \Phi_0$ and $\Phi'_\pm \oplus \Phi''_\mp$ correspond to the three ideals $(y - \lambda_i x, (y - \lambda_{i+1} x)(y - \lambda_{i+2} x))R$, $i = 1,2,3$, (the correspondence depends on the explicit choice of isomorphism of $\Pi_1^3 (y - \lambda_i x) = 0$ with $y^3 - x^2 y = 0$). For each $i = 1,2,3$, the basis $(y - \lambda_{i+1} x)(y - \lambda_{i+2} x), y - \lambda_i x, (y - \lambda_i x)(y - \lambda_{i+1} x)$ produces the matrix $\Phi_{2+i}^{D_4}$. For each $i = 1,2,3$, the ideal $(y - \lambda_i x, (y - \lambda_{i+1} x)(y - \lambda_{i+2} x))R$ is isomorphic to the R-module $M_{2+i} \subset \overline{R}$ given by $\mathbb{C}[[t_i]] \oplus (1\mathbb{C} \oplus t_{i+1}\mathbb{C}[[t_{i+1}]] \oplus t_{i+2}\mathbb{C}[[t_{i+2}]])$, which itself forms a ring. The curve singularity Spec M_{2+i} corresponds to separating the branch $\{y = \lambda_i x\}$ from the tripod.

Finally, the Higgs field $0_1 \oplus \Phi_1$ corresponds to the ideal $(x^2, xy, y^2)R$. The basis $(y - \lambda_2 x)(y - \lambda_3 x)$, $(y - \lambda_1 x)(y - \lambda_3 x)$, $(y - \lambda_1 x)(y - \lambda_2 x)$ produces $\Phi_6^{D_4}$. The ideal sheaf $(x^2, xy, y^2)R$ is isomorphic to the pushforward of the structure sheaf of the full normalization of Σ (where all three irreducible components are being "taken apart"). □

Example A.0.5. Consider the Higgs field

$$\phi = \begin{pmatrix} \lambda_1 x & & \\ 1 & \lambda_2 x & \\ x & 1 & \lambda_3 x \end{pmatrix}.$$

This Higgs fields has spectral curve $\Sigma = \{\prod_{i=1}^3 (y - \lambda_i x) = 0\}$, and at $x = 0$ has rank 2. Out of the seven options given in Corollary A.0.4, only $\Phi_0^{D_4}$ has rank 2 at $x = 0$. This guarantees that ϕ is isomorphic to $\Phi_0^{D_4}$.

Example A.0.6. Consider the Higgs field

$$\phi = \begin{pmatrix} \lambda_1 x & & \\ x & \lambda_2 x & \\ 1 & x & \lambda_3 x \end{pmatrix}.$$

Now, the job of identifying the isomorphism class of ϕ is not as easy as in the previous example, because five out of seven options given by Corollary A.0.4 have the same rank at $x = 0$ as ϕ. Let e_1, e_2, e_3 be the basis of $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$ in which the matrix ϕ is written. Then, over the formal punctured disc Spec $\mathbb{C}((x))$, we can identify the generators of three eigeinlines of ϕ as

$$\begin{array}{rcllll} s_1 &=& xe_1 &+ \frac{x}{\lambda_1-\lambda_2}e_2 &+& \left(\frac{1}{\lambda_1-\lambda_3} + \frac{x}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\right)e_3, \\ s_2 &=& & e_2 &+& \frac{1}{\lambda_2-\lambda_3}e_3, \\ s_3 &=& & & & e_3. \end{array}$$

Solving this for e_1, e_2, e_3, we obtain

$$\begin{array}{rcllll} e_1 &=& \frac{1}{x}s_1 &- \frac{1}{\lambda_1-\lambda_2}s_2 &+& \left(-\frac{1}{(\lambda_1-\lambda_3)x} + \frac{1}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}\right)s_3, \\ e_2 &=& & s_2 &-& \frac{1}{\lambda_2-\lambda_3}s_3, \\ e_3 &=& & & & s_3. \end{array}$$

Recall that the fraction ring of $R = \mathbb{C}[[x,y]]/(\Pi_1^3(y-\lambda_i x))$ is $F = \mathbb{C}((t_1)) \oplus \mathbb{C}((t_2)) \oplus \mathbb{C}((t_3))$, and R embeds into F by $x \mapsto (t_1, t_2, t_3)$, $y \mapsto (\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3)$. The above calculation shows that the eigensheaf of ϕ is isomorphic to the R-span M of the following three elements inside F:

$$\begin{array}{rcl} e_1 &=& \left(\frac{1}{t_1},\quad -\frac{1}{\lambda_1-\lambda_2},\quad -\frac{1}{(\lambda_1-\lambda_3)t_3} + \frac{1}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}\right), \\ e_2 &=& \left(0,\quad 1,\quad -\frac{1}{\lambda_2-\lambda_3}\right), \\ e_3 &=& \left(0,\quad 0,\quad 1\right). \end{array}$$

After simplifying, we obtain $M = \langle (1,0,1), (0,1,0) \rangle R$. So, the eigensheaf of ϕ is isomorphic to the pushforward of the structure sheaf of the partial normalization Spec $\big(\mathbb{C}[[x,y]]/(y-\lambda_2 x) \oplus \mathbb{C}[[x,y]]/((y-\lambda_1 x)(y-\lambda_3 x))\big)$, that is, ϕ is isomorphic to $\Phi_4^{D_4}$ from Corollary A.0.4.

E singularities.

Theorem A.0.7. *(Local normal forms for Higgs fields with spectral curve having E_6 singularity)*

1) Let ϕ be a Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$, for $\mathcal{U} =$ Spec $\mathbb{C}[[x]]$, with the spectral curve $\Sigma = \{y^3 = x^4\}$. Then ϕ is isomorphic to exactly one of the following five Higgs fields:

$$\Phi_0^{E_6} = \left(\begin{array}{cc|c} & & x^4 \\ \hline 1 & & \\ & 1 & \end{array}\right), \qquad \Phi_1^{E_6} = \left(\begin{array}{cc|c} & & x^3 \\ \hline x & & \\ & 1 & \end{array}\right), \qquad \Phi_2^{E_6} = \left(\begin{array}{cc|c} & & x^3 \\ \hline 1 & & \\ & x & \end{array}\right)$$

$$\Phi_3^{E_6} = \left(\begin{array}{cc|c} & & x^2 \\ \hline 1 & & \\ & x^2 & \end{array}\right), \qquad \Phi_4^{E_6} = \left(\begin{array}{cc|c} & & x^2 \\ \hline x & & \\ & x & \end{array}\right).$$

2) Let ϕ be a Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus 4}$, for $\mathcal{U} = \operatorname{Spec} \mathbb{C}[[x]]$, with the spectral curve $\Sigma = \{y^4 = x^3\}$. Then ϕ is isomorphic to exactly one of the following five Higgs fields:

$$\Psi_0^{E_6} = \left(\begin{array}{ccc|c} & & & x^3 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \end{array}\right), \qquad \Psi_1^{E_6} = \left(\begin{array}{ccc|c} & & & x^2 \\ \hline x & & & \\ & 1 & & \\ & & 1 & \end{array}\right), \qquad \Psi_2^{E_6} = \left(\begin{array}{ccc|c} & & & x \\ \hline x^2 & & & \\ & 1 & & \\ & & 1 & \end{array}\right)$$

$$\Psi_3^{E_6} = \left(\begin{array}{ccc|c} & & & x \\ \hline 1 & & & \\ & x^2 & & \\ & & 1 & \end{array}\right), \qquad \Psi_4^{E_6} = \left(\begin{array}{ccc|c} & & & x \\ \hline x & & & \\ & x & & \\ & & 1 & \end{array}\right).$$

Proof. 1) The torsion-free rank one sheaves over $\Sigma = \operatorname{Spec} R$, $R = \mathbb{C}[[x,y]]/(y^3 - x^4)$, are R, $(x,y)R$, $(x,y^2)R$, $(x^2,y^2)R$ and $(x^2,xy,y^2)R$ (see [38, Chapter 9] for instance).

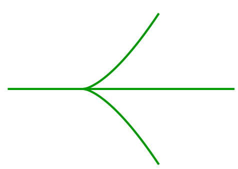
For the sheaf R, multiplication by y has matrix $\Phi_0^{E_6}$ in the basis $1, y, y^2$ over $\mathbb{C}[[x]]$. For the sheaf $(x,y)R$, one can choose the basis x, y, y^2 to obtain $\Phi_1^{E_6}$. Likewise, for the sheaves $(x,y)R$, $(x,y^2)R$, $(x^2,y^2)R$ and $(x^2,xy,y^2)R$ one can choose the bases $\{x,y,y^2\}$, $\{x,xy,y^2\}$, $\{x^2,y,y^2\}$ and $\{x^2,xy,y^2\}$, respectively, to obtain the matrices $\Phi_1^{E_6}$, $\Phi_2^{E_6}$, $\Phi_4^{E_6}$ and $\Phi_5^{E_6}$.

Part 2) is proved analogously to 1), with the roles of x and y swapped. □

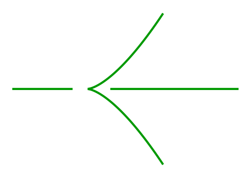
Theorem A.0.8. *(Local normal forms for Higgs fields with spectral curve having E_7 singularity)*

Let ϕ be a Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$, for $\mathcal{U} = \operatorname{Spec} \mathbb{C}[[x]]$, with the spectral curve $\Sigma = \{y^3 = x^3 y\}$. Then ϕ is isomorphic to exactly one of the following eight Higgs fields (below we include the picture of the partial normalization of Σ that corresponds to the eigensheaf of the Higgs field):

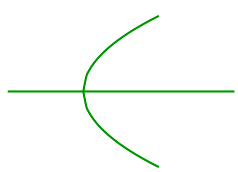
$$\Phi_0^{E_7} = \left(\begin{array}{c|cc} 0 & & \\ \hline 1 & & x^3 \\ & 1 & \end{array}\right)$$



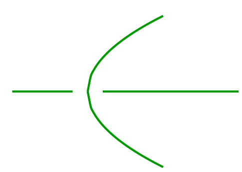
$$\Phi_1^{E_7} = \left(\begin{array}{c|cc} 0 & & \\ \hline & & x^3 \\ & 1 & \end{array}\right)$$



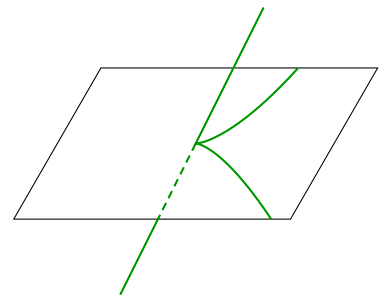
$$\Phi_2^{E_7} = \left(\begin{array}{c|cc} 0 & & \\ \hline x & & x^2 \\ & x & \end{array}\right)$$



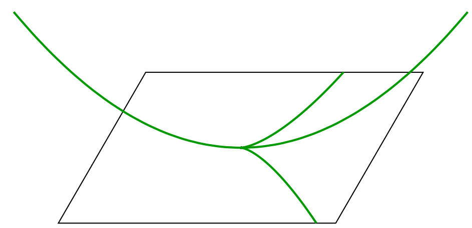
$$\Phi_3^{E_7} = \left(\begin{array}{c|cc} 0 & & \\ \hline & & x^2 \\ & x & \end{array}\right)$$



$$\left.\begin{aligned} \Phi_4^{E_7} &= \left(\begin{array}{c|cc} 0 & & \\ \hline 1 & & x \\ & x^2 & \end{array}\right) \\ \Phi_5^{E_7} &= \left(\begin{array}{c|cc} 0 & & \\ \hline x^2 & & x^3 \\ & 1 & \end{array}\right) \end{aligned}\right\}$$



$$\left.\begin{aligned} \Phi_6^{E_7} &= \left(\begin{array}{c|cc} 0 & & \\ \hline 1 & & x^2 \\ & x & \end{array}\right) \\ \Phi_7^{E_7} &= \left(\begin{array}{c|cc} 0 & & \\ \hline x & & x^3 \\ & 1 & \end{array}\right) \end{aligned}\right\}$$



Proof. Torsion-free sheaves over Σ are classified e.g. in [38, Chapter 9]. The rank one torsion-free sheaves are given by the ideals $I_0 = R$, $I_1 = (y^2 - x^3, y)R$, $I_2 = (x^2, xy, y^2)R$, $I_3 = (y^2 - x^3, xy, y^2)R$, $I_4 = (x^2, y^2, x^2y)R$, $I_5 = (yx^2 + x^3 - y^2, y^2, x^3y)R$, $I_6 = (x, xy, y^2)R$ and $I_7 = (x, y, y^2)R$. One can check that for each $j = 0, 1, ..., 7$, the ideal sheaf given by I_j has a basis over $\mathbb{C}[[x]]$ in which multiplication by y has matrix $\Phi_j^{E_7}$ (for $j = 0$ one needs to choose the basis $1, y, y^2$, for $j = 1$ one takes the basis $y^2 - x^3, y, y^2$, and for other j's one takes the presented generators of the ideal I_j as the basis).

A straightforward calculation shows that the rings $R_j = \mathrm{End}_R(I_j)$ are as follows: $R_0 = R$, $R_1 \cong \mathbb{C}[[x]] \oplus \mathbb{C}[[x, y]]/(y^2 - x^3)$, $R_2 \cong \mathbb{C}[[x, y]]/(xy)$, $R_3 \cong \mathbb{C}[[x]] \oplus \mathbb{C}[[y]]$, $R_4 \cong R_5 \cong \mathbb{C}[[x, y, z]]/(y^2 - x^3, yz, xz)$ (note that Spec R_4 has two irreducible components, $\{y^2 = x^3, z = 0\}$ and $\{x = y = 0\}$) and $R_6 \cong R_7 \cong \mathbb{C}[[x, y, z]]/(y(y^2 - x^3), yz, z(z - x^2))$ (note that Spec R_6 has two irreducible components, $\{y^2 = x^3, z = 0\}$ and $\{z = x^2, y = 0\}$). □

Theorem A.0.9. *(Local normal forms for Higgs fields with spectral curve having E_8 singularity)*

1) Let ϕ be a Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$, for $\mathcal{U} = \mathrm{Spec}\ \mathbb{C}[[x]]$, with the spectral curve $\Sigma = \{y^3 = x^5\}$. Then ϕ is isomorphic to exactly one of the following seven Higgs fields:

$$\Phi_0^{E_8} = \left(\begin{array}{cc|c} & & x^5 \\ \hline 1 & & \\ & 1 & \end{array}\right), \qquad \Phi_1^{E_8} = \left(\begin{array}{cc|c} & & x^4 \\ \hline x & & \\ & 1 & \end{array}\right), \qquad \Phi_2^{E_8} = \left(\begin{array}{cc|c} & & x^3 \\ \hline x^2 & & \\ & 1 & \end{array}\right),$$

$$\Phi_3^{E_8} = \left(\begin{array}{cc|c} & & x^2 \\ \hline x^3 & & \\ & 1 & \end{array}\right), \qquad \Phi_4^{E_8} = \left(\begin{array}{cc|c} & & x \\ \hline x^4 & & \\ & 1 & \end{array}\right),$$

$$\Phi_5^{E_8} = \left(\begin{array}{cc|c} & & x^3 \\ \hline x & & \\ & x & \end{array}\right), \qquad \Phi_6^{E_8} = \left(\begin{array}{cc|c} & & x^2 \\ \hline x^2 & & \\ & x & \end{array}\right).$$

2) Let ϕ be a Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus 5}$, for $\mathcal{U} = \operatorname{Spec} \mathbb{C}[[x]]$, with the spectral curve $\Sigma = \{y^5 = x^3\}$. Then ϕ is isomorphic to exactly one of the following seven Higgs fields:

$$\Psi_0^{E_8} = \left(\begin{array}{cccc|c} & & & & x^3 \\ \hline 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{array}\right), \quad \Psi_1^{E_8} = \left(\begin{array}{cccc|c} & & & & x^2 \\ \hline x & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{array}\right), \quad \Phi_2^{E_8} = \left(\begin{array}{cccc|c} & & & & x^2 \\ \hline 1 & & & & \\ & x & & & \\ & & 1 & & \\ & & & 1 & \end{array}\right),$$

$$\Psi_3^{E_8} = \left(\begin{array}{cccc|c} & & & & x^2 \\ \hline 1 & & & & \\ & 1 & & & \\ & & x & & \\ & & & 1 & \end{array}\right), \quad \Phi_4^{E_8} = \left(\begin{array}{cccc|c} & & & & x^2 \\ \hline 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & x & \end{array}\right),$$

$$\Psi_5^{E_8} = \left(\begin{array}{cccc|c} & & & & x \\ \hline x & & & & \\ & x & & & \\ & & 1 & & \\ & & & 1 & \end{array}\right), \quad \Phi_6^{E_8} = \left(\begin{array}{cccc|c} & & & & x \\ \hline x & & & & \\ & 1 & & & \\ & & x & & \\ & & & 1 & \end{array}\right).$$

Proof. 1) Torsion-free sheaves over Σ are classified e.g. in [38, Chapter 9]. The rank one torsion-free sheaves are given by the ideals $I_j = (x^j, y)R$, $j = 0, 1, ..., 4$, $I_5 = (x^2, xy, y^2)R$, $I_6 = (x^2, xy^2, y^3)R$.

For the ideal I_j, $j = 0, 1, ..., 4$, multiplication by y has matrix $\Phi_j^{E_8}$ in the basis x^j, y, y^2 over $\mathbb{C}[[x]]$. For the sheaves I_5 and I_6, the presented generators of the ideals form a basis over $\mathbb{C}[[x]]$, and multiplication by y in these bases gives $\Phi_5^{E_8}$ and $\Phi_6^{E_8}$, respectively.

Part 2) is proved analogously to 1), with the roles of x and y swapped. □

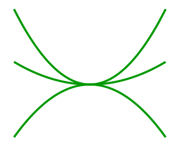
T_{36} singularities.

Recall that a T_{36} singularity is the one given by the equation $y(y-x^2)(y-ax^2) = 0$, for some $a \in \mathbb{C} \setminus \{0, 1\}$. To make the three branches more symmetric, let us express it as $\Sigma = \operatorname{Spec} \mathbb{C}[[x, y]]/(\prod_{i=1}^{3}(y-\lambda_i x^2))$, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ are pairwise distinct. To start the discussion of possible Higgs fields with such spectral curve Σ, we note that whenever ϕ is a Higgs field over $\mathcal{U} = \operatorname{Spec} \mathbb{C}[[x]]$ whose spectral curve is $\prod_{i=1}^{3}(y - \lambda_i x) = 0$ (D_4 singularity), the Higgs field $x\phi$ has the spectral curve Σ. It is easy to see that all isomorphism classes of Higgs fields with spectral curve Σ that vanish at $x = 0$ are obtained this way. Therefore, Corollary A.0.4 provides seven isomorphism classes of Higgs fields with the T_{36} spectral curve Σ. The next theorem shows what the rest of the isomorphism classes are.

Theorem A.0.10. *(Local normal forms for Higgs fields with spectral curve having T_{36} singularity)*

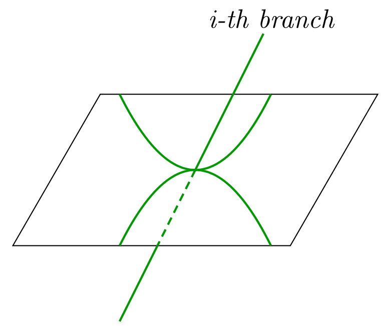
Let ϕ be a Higgs field on $V = \mathcal{O}_{\mathcal{U}}^{\oplus 3}$, for $\mathcal{U} = \operatorname{Spec}\mathbb{C}[[x]]$, with the spectral curve $\Sigma = \{\prod_{i=1}^{3}(y-\lambda_i x^2) = 0\}$, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ are pairwise distinct. Then ϕ is isomorphic either to $x\Phi_i^{D_4}$, $i = 0, 1, ..., 6$, where $\Phi_i^{D_4}$'s are the Higgs fields from Corollary A.0.4, (in which case the corresponding partial normalizations of Σ are exactly the same as the ones specified in Corollary A.0.4), or ϕ is isomrophic to exactly one of the following fourteen Higgs fields (below we include the picture of the partial normalization of Σ that corresponds to the eigensheaf of the Higgs field):

$$\Phi_0^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ 1 & \lambda_2 x^2 & \\ & 1 & \lambda_3 x^2 \end{pmatrix}$$



$$\Phi_i^{T_{36}} = \begin{pmatrix} \lambda_i x^2 & & \\ x^3 & \lambda_{i+1} x^2 & \\ & 1 & \lambda_{i+2} x^2 \end{pmatrix},$$
$i = 1, 2, 3,$

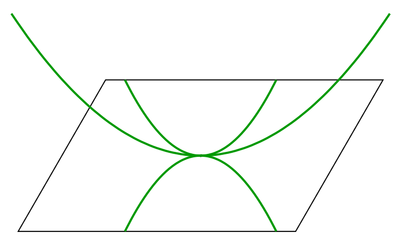
$$\Phi_{i+3}^{T_{36}} = \begin{pmatrix} \lambda_{i+2} x^2 & & \\ 1 & \lambda_{i+1} x^2 & \\ & x^3 & \lambda_i x^2 \end{pmatrix},$$
$i = 1, 2, 3,$



(one of the branches is transverse to the plane containing the other two branches)

$$\Phi_7^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ x^2 & \lambda_2 x^2 & \\ & 1 & \lambda_3 x^2 \end{pmatrix},$$

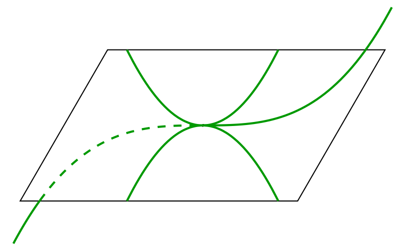
$$\Phi_8^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ 1 & \lambda_2 x^2 & \\ & x^2 & \lambda_3 x^2 \end{pmatrix},$$



(one of the branches has 1^{st} order contact with, i.e. tangent to, the plane containing the other two branches)

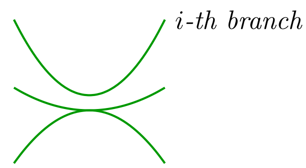
$$\Phi_9^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ x & \lambda_2 x^2 & \\ & 1 & \lambda_3 x^2 \end{pmatrix},$$

$$\Phi_{10}^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ 1 & \lambda_2 x^2 & \\ & x & \lambda_3 x^2 \end{pmatrix},$$



(one of the branches has 2^{nd} order contact with the plane containing the other two branches)

$$\Phi_{10+i}^{T_{36}} = \begin{pmatrix} \lambda_i x^2 & & \\ & \lambda_{i+1} x^2 & \\ & 1 & \lambda_{i+2} x^2 \end{pmatrix},$$
$i = 1, 2, 3,$



or ϕ is isomorphic to exactly one member of the following one-parameter family of Higgs fields:

$$\Psi_\beta^{T_{36}} = \begin{pmatrix} \lambda_1 x^2 & & \\ x & \lambda_2 x^2 & \\ \frac{1}{\beta - \lambda_1 - \lambda_3} & x & \lambda_3 x^2 \end{pmatrix}, \qquad \beta \in \mathbb{C} \smallsetminus \{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\},$$

(the partial normalization corresponding to the eigensheaf of $\Psi_\beta^{T_{36}}$ is the D_4 singularity Spec $\mathbb{C}[[x,z]]/((z-\gamma_1 x)(z-\gamma_2 x)(z-\gamma_3 x))$*, where $\gamma_i = \lambda_i^2 - \beta\lambda_i$, $i = 1,2,3$, and $z = \frac{y^2}{x^3} - \beta\frac{y}{x}$).*

Sketch of proof. Classification of torsion-free rank one sheaves on the T_{36} singularity Σ is carried out via methods of [30] and [8]. Let us outline how this lengthy, albeit straightforward, calculation goes.

The integral closure of $R = \mathbb{C}[[x,y]]/(\prod_{i=1}^3 (y - \lambda_i x^2))$ is $\overline{R} = \mathbb{C}[[t_1]] \oplus \mathbb{C}[[t_2]] \oplus \mathbb{C}[[t_3]]$ and R embeds into $\overline{R}$ by $x \mapsto (t_1, t_2, t_3)$, $y \mapsto (\lambda_1 t_1^2, \lambda_2 t_2^2, \lambda_3 t_3^2)$. Each torsion-free rank one sheaf over R is given by an R-invariant subspace $M \subset \overline{R}$ containing R, and two different such subspaces $R \subset M_1, M_2 \subset \overline{R}$ give isomorphic sheaves if and only if there is $r \in \overline{R}^*$ such that $rM_1 = M_2$. The goal is to classify all the R-modules M with $R \subset M \subset \overline{R}$, up to this isomorphism. One can either tackle this combinatorial challenge directly, or use the following simplifications.

First, one checks that each such M, except $M = R$ (that corresponds to the Higgs field $\Phi_0^{T_{36}}$), contains the ring $S = \mathrm{End}_R(m)$, where m is the maximal ideal $m = (x,y)R$. So, the problem is reduced to calculating torsion-free rank one sheaves over the ring

$$S = (1,1,1)\mathbb{C} + (t_1, t_2, t_3)\mathbb{C} + (t_1^2, t_2^2, t_3^2)\mathbb{C} + (\lambda_1 t_1^2, \lambda_2 t_2^2, \lambda_3 t_3^2)\mathbb{C} + (t_1^3, t_2^3, t_3^3)\overline{R}.$$

Next, one considers the ring $T = \mathrm{End}_S(n)$, where n is the maximal ideal $(x,y)S$. Explicitly, one has

$$T = (1,1,1)\mathbb{C} + (t_1, t_2, t_3)\mathbb{C} + (\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3)\mathbb{C} + (t_1^2, t_2^3, t_3^2)\overline{R}.$$

In particular, Spec T has D_4 singularity, because $R = \mathbb{C}[[x,w]]/(\prod_{i=1}^{3}(w-\lambda_i x))$, where $x = (t_1,t_2,t_3)$, $w = (\lambda_1 t_1, \lambda_2 t_2, \lambda_3 t_3)$. So, the torsion-free rank one sheaves over Spec T have already been classified (Corollary A.0.4), and the isomorphism classes of T-modules N with $T \subset N \subset \overline{R} = \overline{T}$ are given by

$$\begin{aligned} N_0 &= T, \\ N_1 &= T + (\lambda_1,\lambda_2,\lambda_3)T, \\ N_2 &= T + (0,0,t_3)T, \\ N_3 &= T + (1,0,0)T, \\ N_4 &= T + (0,1,0)T, \\ N_5 &= T + (0,0,1)T, \\ N_6 &= \overline{T}, \end{aligned}$$

(where the sheaf N_i corresponds to the Higgs field $\Phi_i^{D_4}$ in Corollary A.0.4). Having the classification of torsion-free rank one sheaves over Spec T, one can deduce such classification for Spec S in the following way.

Whenever one has an S-module M with $S \subset M \subset \overline{R}$, one can tensor M with T and obtain a T-module N with $T \subset N \subset \overline{R}$. Up to isomorphism, one can assume that $N = N_i$, $0 \le i \le 6$. Now, going through each N_i, $0 \le i \le 6$, one finds the isomorphism classes of M that give this particular N_i when tensoring M with T. The Higgs fields one gets in each of the seven cases are as follows:

N_0: $x\Phi_0^{D_4}$, $\Phi_8^{T_{36}}$, $\Phi_9^{T_{36}}$, $\Phi_{10}^{T_{36}}$,

N_1: $x\Phi_1^{D_4}$, $\Phi_7^{T_{36}}$,

N_2: $x\Phi_2^{D_4}$, $\Phi_4^{T_{36}}$, $\Phi_5^{T_{36}}$, $\Phi_6^{T_{36}}$, $\Psi_\beta^{T_{36}}$,

N_3: $x\Phi_3^{D_4}$, $\Phi_1^{T_{36}}$, $\Phi_{11}^{T_{36}}$,

N_4: $x\Phi_4^{D_4}$, $\Phi_2^{T_{36}}$, $\Phi_{12}^{T_{36}}$,

N_5: $x\Phi_5^{D_4}$, $\Phi_3^{T_{36}}$, $\Phi_{13}^{T_{36}}$,

N_6: $x\Phi_6^{D_4}$.

□

Bibliography

[1] C. Bartocci and E. Macri, *Classification of Poisson surfaces*, Commun. Contemp. Math. 7 (2005), no. 1, 89–95.

[2] A. Beauville, *Complex algebraic surfaces*, London Mathematical Society Student Texts, 34. Cambridge University Press, Cambridge, 1996. x+132 pp.

[3] A. Beauville, M. S. Narasimhan, and S. Ramanan, *Spectral curves and the generalised theta divisor*, J. Reine Angew. Math. 398 (1989), 169–179.

[4] A. I. Bondal, *Non-commutative deformations and Poisson brackets on projective spaces*, Max-Planck-Institute Preprint (1993), no. 93-67

[5] F. Bottacin, *Poisson structures on Hilbert schemes of points of a surface and integrable systems.* Manuscripta Math. 97 (1998), no. 4, 517-527.

[6] D. Cerveau and A. Lins Neto, *Irreducible components of the space of holomorphic foliations of degree two in CP(n), $n \geq 3$*, Ann. of Math. (2) 143 (1996), no. 3, 577–612.

[7] M. Crainic and R. L. Fernandes, *Integrability of Lie brackets*, Ann. of Math. (2) 157 (2003), no. 2, 575–620.

[8] Yu. A. Drozd and G.-M. Greuel, *On Schappert's characterization of strictly unimodal plane curve singularities*, Singularities (Oberwolfach, 1996), 3–26, Progr. Math., 162, Birkhäuser, Basel, 1998.

[9] Yu. A. Drozd and G.-M. Greuel, *Cohen-Macaulay module type* Compositio Math. 89 (1993), no. 3, 315–338.

[10] Ju. A. Drozd and A.V. Roiter, *Commutative rings with a finite number of indecomposable integral representations*, Izv. Akad. Nauk SSSR Ser. Mat. 31, 1967, 783–798.

[11] G.-M. Greuel and H. Knörrer, *Einfache Kurvensingularitäten und torsionsfreie Moduln*, Math. Ann. 270 (1985), no. 3, 417–425.

[12] H. Jacobinski, *Sur les ordres commutatifs avec un nombre fini de réseaux indécomposables*, Acta Math., 118, 1–31 (1967).

[13] M. Gualtieri, *Generalized complex structures*, Ph.D. thesis, Univeristy of Oxford, 2003.

[14] M. Gualtieri, *Branes on Poisson varieties*, The many facets of geometry, 368–394, Oxford Univ. Press, Oxford, 2010.

[15] M. Gualtieri, *Generalized complex geometry*, Ann. of Math. (2) 174 (2011), no. 1, 75–123.

[16] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3), 55 (1987), 59-126.

[17] N. J. Hitchin, *Stable bundles and integrable systems*, Duke Math. J., 54 (1987), 91-114.

[18] N. Hitchin, *Generalized Calabi-Yau manifolds*, Q. J. Math. 54 (2003), no. 3, 281–308.

[19] F. Loray, J.V.Pereira, and F. Touzet, *Foliations with trivial canonical bundle on Fano 3-folds*, Math. Nachr. 286 (2013), no. 8-9, 921–940.

[20] C. Laurent-Gengoux, A. Pichereau, and P. Vanhaecke, *Poisson structures*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 347. Springer, Heidelberg, 2013. xxiv+461 pp.

[21] A. V. Odesskii, *Elliptic algebras*, Uspekhi Mat. Nauk 57 (2002), no. 6(348), 87–122; translation in Russian Math. Surveys 57 (2002), no. 6, 1127–1162

[22] A. V. Odesski, *Rational degeneration of elliptic quadratic algebras*, Infinite analysis, Part A, B (Kyoto, 1991), 773–779,

[23] B. Pym, *Poisson Structures and Lie Algebroids in Complex Geometry*, Ph.D. thesis, University of Toronto, 2014.

[24] B. Pym, *Quantum deformations of projective three-space*, Adv. Math. 281 (2015), 1216–1241.

[25] B. Pym, *Constructions and classifications of projective Poisson varieties*, Lett. Math. Phys. 108 (2018), no. 3, 573–632.

[26] A. Polishchuk. *Algebraic geometry of Poisson brackets.* Algebraic geometry, 7. J. Math. Sci. (New York) 84 (1997), no. 5, 1413–1444.

[27] S. Rayan, *Geometry of co-Higgs bundles*, Ph.D. thesis, University of Oxford, 2011.

[28] S. Rayan, *Co-Higgs bundles on P1*, New York J. Math. 19 (2013), 925–945.

[29] S. Rayan, *Constructing co-Higgs bundles on CP2*, Q. J. Math. 65 (2014), no. 4, 1437–1460.

[30] A. Schappert, *A characterisation of strictly unimodular plane curve singularities*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), 168–177, Lecture Notes in Math., 1273, Springer, Berlin, 1987.

[31] R. L. E. Schwarzenberger, *The secant bundle of a projective variety*, Proc. London MAth. Soc. (3) 14, 1964, 369–384.

[32] C. T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to unlformization*, J. Amer. Math. Soc. 1 (1988), no. 4, 867–918.

[33] C. T. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. No. 75 (1992), 5–95.

[34] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. No. 79 (1994), 47–129.

[35] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. II*, Inst. Hautes Études Sci. Publ. Math. No. 80 (1994), 5–79 (1995).

[36] R. V. Skuratovskii, *Ideals of one-branch singularities of curves of type W*, Ukraïn. Mat. Zh. 61 (2009), no. 9, 1257–1266; translation in Ukrainian Math. J. 61 (2009), no. 9, 1487–1498.

[37] A. Weinstein, *The modular automorphism group of a Poisson manifold*, J. Geom. Phys. 23 (1997), no. 3-4, 379–394.

[38] Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, 146. Cambridge University Press, Cambridge, 1990. viii+177 pp.