UNIVERSITY OF TORONTO
FACULTY OF ARTS & SCIENCE | Department of Mathematics

Departmental PhD Thesis Exam

Thursday, August 21, 2025 at 10:00 a.m. (sharp)
via Zoom / BA6183

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Supervisor : Stevo Todorčević

Thesis title : Proper Forcing, The P-Ideal Dichotomy, and the S-space Problem

Abstract

In this thesis, we explore the relationship between the S-space problem and the Proper Forcing Axiom (PFA). In particular, we prove that many consequences of PFA are compatible with the existence of various S-spaces.

We start by studying the notion of solid graph, first introduced by Soukup, and show this can be used to encode many objects whose existence follows from $\diamondsuit$ or CH. We prove that Neeman iterations preserve solid graphs, providing a general tool for preserving such objects while forcing consequences of PFA.

Next, we then introduce two new types of graph, namely (m,n)-solid graphs and HF graphs. These new notions can encode many different S-spaces, including strong HFD spaces, strong HFD_w spaces, and first countable strong O-spaces. We then prove that much of the theory of solid graphs extends to these new notions, including the preservation by Neeman iterations.

From here, we are able to construct novel models of ZFC with many consequences of PFA plus the aforementioned S-spaces. Using (m,n)-solid graphs, we construct models with $\mathfrak{p} = \aleph_2$, the Mapping Reflection Principle, the Open Graph Axiom, Baumgartner's Axiom, and all Aronszajn trees are club isomorphic. Using HF graphs, we construct models with $\mathfrak{s} = \aleph_1$, $\mathrm{add}(\mathscr{M}) = \aleph_2$, the Mapping Reflection Principle, and the P-Ideal Dichotomy. Most notably, our work provides a partial negative answer to a question of Todorčević about whether the P-Ideal Dichotomy and $\mathfrak{b} > \aleph_1$ implies there are no S-spaces.