DEPARTMENT OF MATHEMATICS
Partial Differential Equations Comprehensive Exam 2025
September 19, 2025 - 6:00 - 9:00 p.m.
BA6183

NO AIDS ALLOWED. Passing score is 80 percent.

Last name ....

First name ....

Email ....

1. Given a domain $U \subset \mathbb{R}^n$, let

$$C_1^2(U \times (0,T]) = \{u : U \times (0,T] \to \mathbb{R}, \text{ such that } u, D_x u, D_x^2 u, u_t \in C(U \times (0,T])\}.$$

Suppose $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

and satisfies the growth estimate

$$u(x,t) \leq A e^{a|x|^2} \quad (x \in \mathbb{R}^n, 0 \leq t \leq T)$$

for constants $A, a > 0$. Prove that then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g.$$

2. Prove the Rellich-Kondrachov compactness theorem: assume that U is a bounded open subset of $\mathbb{R}^n$ and that ∂U is C^1. Suppose $1 \leq p < n$. Then, for each $1 \leq q < p^*$,

$$W^{1,p}(U) \subset\subset L^q(U).$$

Recall that for Banach spaces X and Y with $X \subset Y$, $X \subset\subset Y$ (X is compactly embedded in Y) means that

(i) $\| u \|_Y \leq C \| u \|_X$ for some constant C and all $u \in X$, and

(ii) each bounded sequence in X is precompact in Y.

3. Let U be an open, connected and bounded subset of $\mathbb{R}^n$, and $u : \overline{U} \to \mathbb{R}$ be a $C^2(U) \cap C^1(\overline{U})$ function. Consider the second order operator L

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu \ ,$$

where a^{ij}, b^i, c are continuous, $a^{ij} = a^{ji}$ for $i, j = 1, \ldots, n$, which is (uniformly) elliptic, i.e. there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$.

(i) Prove Hopf's lemma: if $c \equiv 0$ in U, $Lu \leq 0$ in U, there exists a point $x^0 \in \partial U$ such that $u(x^0) > u(x)$ for all $x \in U$, and U satisfies the interior ball condition at x^0 (i.e. there exists an open ball $B \subset U$ with $x^0 \in \partial B$), then

$$\frac{\partial u}{\partial \nu}(x^0) > 0 \ ,$$

where ν is the outer unit normal to B at x^0.

(ii) As a consequence, prove the following strong maximum principle: if $u \in C^2(U) \cap C(\overline{U})$, $c \equiv 0$ in U, $Lu \leq 0$ in U, and u attains its maximum over $\overline{U}$ at an interior point, then u is constant within U.

4. Suppose that $u \in W^{1,2}(B_1) \cap L^\infty(B_1)$ satisfies $\partial_i(a^{ij}u_j) = 0$ in the weak sense, where $\lambda\delta^{ij} < a^{ij} < \Lambda a^{ij}$ for $0 < \lambda \leq \Lambda < +\infty$, and we assume that a^{ij} is symmetric and measurable. Prove that

$$\|u\|_{L^\infty(B_{\frac{1}{2}})} \leq C\|u\|_{L^2(B_1)}$$

for a constant C depending only on n, λ, Λ.

5. Suppose $u \in C^2(B_1) \cap C(\overline{B_1})$ solves $a^{ij}u_{ij} = f$ in $B_1 \subset \mathbb{R}^n$, where $\lambda\delta^{ij} < a^{ij} < \Lambda\delta^{ij}$ for $0 < \lambda \leq \Lambda < +\infty$, and we assume that a^{ij} is symmetric and measurable. Suppose furthermore that $f \in L^n(B_1)$. Assume that $u\big|_{\partial B_1} \leq 0$. Prove that

$$\sup_{B_1} u \leq C(n)\left\|\frac{f^-}{D^*}\right\|_{L^n(\Gamma^+)}$$

for a constant C depending only on n. Here $f^- = \min\{f, 0\}$, $D^* = \left(\det a^{ij}\right)^{1/n}$, and Γ^+ is the upper contact set of u:

$$\Gamma^+ = \{x \in B_1 : u(x) = \widehat{u}(x)\}$$

where $\widehat{u}$ is the concave envelope of u.

6. Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is smooth and $\|D^2u\|_{L^\infty(\mathbb{R}^n)} < +\infty$. Suppose u solves $F(D^2u) = 0$ where $F(M)$ is smooth, uniformly elliptic and convex as a function of symmetric $n \times n$ matrices. Show that u is a quadratic polynomial.