Problem 1. Consider the operator $Lu = 2\frac{\partial^2 u}{\partial x \partial y}$ on the domain $U := \{(x, y) \in \mathbb{R}^2 \mid y > x\}$.

1. Is this operator elliptic, parabolic or hyperbolic? (HINT: Diagonalize the matrix $a_{ij} = a_{ji}$.)

2. If we try to prescribe the boundary values of both $u$ and its outward normal derivative $\frac{\partial u}{\partial \nu}$ on the boundary of $U$ to be given, respectively, by functions $g \in C^2(\partial U)$ and $h \in C^1(\partial U)$:

   How many solutions does the equation $Lu = 0$ admit?

   How smooth will these solutions be in the interior of $U$?

Problem 2. Assume that $U$ is an open, bounded set of $\mathbb{R}^n$, with smooth boundary.

1. Let $\lambda_1 > 0$ the smallest eigenvalue of the operator $(-\Delta)$ with zero Dirichlet boundary condition on $U$. Let $u$ be a smooth solution of the diffusion equation

   $u_t - \Delta u = 0$ in $U \times (0, \infty)$

   $u = 0$ on $\partial U \times [0, \infty)$.

   $u = g$ on $U \times \{t = 0\}$.

   Prove the exponential decay estimate:

   \[\|u(\cdot, t)\|_{L^2(U)} \leq e^{-\lambda_1 t}\|g\|_{L^2(U)}.\]

2. Suppose that $u$ is a smooth solution of

   $u_t - \Delta u + q(x)u = 0$ in $U \times (0, \infty)$

   $u = 0$ on $\partial U \times [0, \infty)$.

   $u = g$ on $U \times \{t = 0\}$.

   and the function $q$ satisfies for all $x \in U$, $q(x) \geq \alpha > 0$, (\(\alpha\) constant). Prove

   \[|u(x, t)| \leq C e^{-\alpha t} \quad (x, t) \in U \times (0, T).\]

Problem 3. Consider the solution $u = u(x, t)$ of the quasilinear partial differential equation

\[u_t + a(u)u_x = 0, \quad u(x, 0) = f(x)\]  

1. Derive an implicit formula for the solution $u$.

2. Show that $u$ becomes singular for some $t > 0$ unless $a(f(s))$ is a non-decreasing function of $s$.

3. Define the concept of a classical and a weak (or integral) solution of (1).
Problem 4. Fix a bounded domain $U \subseteq \mathbb{R}^n$ with $C^1$ smooth boundary, and $f \in L^\infty(U)$. Define the functional
\[
E(u) := \int_U \left( \frac{1}{2} |Du|^2 - fu \right) dx.
\]
(1) If the functional $E(u)$ happens to be minimized on $W^{1,2}_0(U)$ by some $u \in C^2(\overline{U})$, derive the partial differential equation that will be satisfied by $u$.
(2) Show the partial differential equation derived in part (1) has at most one solution in $C^2(\overline{U})$ satisfying the boundary condition $u = 0$ on $\partial U$.
(3) Show the functional $E(u)$ has a unique minimizer in $W^{1,2}_0(U)$.

Problem 5. Let $u = u(x, t)$ be a solution of the 3+1 dimensional nonlinear Klein-Gordon equation
\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + u &= u^3 & \text{in} & \quad \mathbb{R}^3 \times (0, \infty), \\
u &= g, \quad u_t = h & \text{on} & \quad \mathbb{R}^3 \times \{0\}. \\
\end{align*}
\]
where the data $(g, h)$ are smooth and compactly supported.
(1) Define the energy functional
\[
E(u)(t) = \frac{1}{2} \|\partial_t u(t)\|_{H^2(\mathbb{R}^3)}^2 + \frac{1}{2} \|\nabla u(t)\|_{H^2(\mathbb{R}^3)}^2 + \frac{1}{2} \|u(t)\|_{H^2(\mathbb{R}^3)}^2.
\]
Prove that solutions of (KG) satisfy, for some constant $C$, the energy inequality
\[
\frac{d}{dt} E(u)(t) \leq C \left( E(u)(t) \right)^2.
\]
(2) Deduce that on some time interval $[0, T]$ for $T$ small enough depending on the energy of the initial data $E(u)(0)$, one has the estimate
\[
E(u)(t) \leq 2 E(u)(0).
\]

Problem 6. Consider the initial value problem associated to the linearized Korteweg-de Vries equation
\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} &= 0, & x \in \mathbb{R}, \ t \in \mathbb{R}^+, \\
u(x, 0) &= u_0(x).
\end{align*}
\]
Here $u(x, t)$ is real-valued. Assume $u \in S(\mathbb{R})$ (i.e. of Schwartz class).
(1) Write the differential equation satisfied by $\hat{u}(k, t)$, the Fourier transform of $u$ (in $x$), and solve it.
(2) Write $u$ in the form of a convolution of $u_0$ with a kernel $T(x, t)$, written in terms of the Airy function
\[
Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(kx + k^3/3)} dk.
\]