

MARKOFF TRIPLES AND PERIODS OF LINEAR RECURRENCE SEQUENCES

The Markoff Equation is defined by $X^2 + Y^2 + Z^2 = 3XYZ$, and the integer solutions to this equation are called Markoff triples. These triples were first introduced by Andrei Markoff in the late 1800s in the context of Diophantine approximation and quadratic forms, and over time have found applications in many different areas of mathematics, such as the study of free groups on two generators and the index of certain 4-dimensional manifolds. More recently, work has been done to study the modulo p solutions to the Markoff equation, particularly due to their cryptographic applications.

It is well-known that there are infinitely many Markoff triples, which can be arranged into a tree via the action of the Vieta group. Analogously, the modulo p solutions to the Markoff equation can be arranged into a graph via the action of the Vieta group. It can be shown that this graph is connected if and only if every modulo p solution to the Markoff equation has an integer lift.

In 2016, Bourgain, Gamburd, and Sarnak used this observation to show that every Markoff mod p solution has an integer lift, for all primes p except for those where $p^2 - 1$ is “very smooth”. To do this, the authors showed that the cycles in these graphs can be understood by studying the order of certain matrices in $\text{GL}_2(\mathbb{Z}/p\mathbb{Z})$. In this project, we’ll leverage a new observation made in [a recent project](#) that associates these matrices to families of order-two linear recurrence sequences, such as the Fibonacci sequence. Using these methods, we will investigate explicit results about the structure of the Markoff mod p graphs, as well as search for explicit families of primes which guarantee integer lifts.

Prerequisites: A first course in elementary number theory, linear algebra, and familiarity with some of the basic topics from a first course in abstract algebra (including groups, rings and fields). Coding experience is preferred, but not required.

If you are interested in this project, please submit this form:

<https://forms.office.com/r/YtXAWTraWc>

If you have any further questions, please feel free to contact me:

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