IS THE DETERMINANT UNIQUE?

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This is a proposal for summer undergraduate research at the University of Toronto.

1. Proposal

For a field $\mathbb{F}$, the determinant is the unique function

$$\{(n \times n)\text{-matrices with entries in } \mathbb{F}\} \longrightarrow \mathbb{F}$$

that sends the identity matrix to 1, and is multilinear and alternating. It is a classical result that it is given by the polynomial $\det_n = \sum_{\sigma \in \Sigma_n} (-1)^{\mathrm{sign}(\sigma)} \prod_{i=1}^n x_{i\sigma(i)}$ in variables x_{ij} standing for the (i,j)th entry of the matrix. Over a finite field different polynomials can give the same function—that is, take all the same values—so it is natural to wonder whether there exists another polynomial than $\det_n$ that gives the determinant. Over $\mathbb{F}_2$, the field with two elements, there is a particularly elegant reformulation of this question because being multilinear and alternating is equivalent to saying $\det_n$ is invariant under left- and right-multiplication by $\mathrm{GL}_n(\mathbb{F}_2)$.

Question. Is $\det_n$ the only non-zero polynomial over $\mathbb{F}_2$ in the variables x_{ij} of degree n that is invariant under left- and right-multiplication by $\mathrm{GL}_n(\mathbb{F}_2)$?

This question is a part of a subject known as *modular invariant theory.* It came up for us in the course of a computation in algebraic topology [GKRW18], where we resolved it positively for $n \leqslant 4$ using results from the early 20th century [Dic11, Kra14] (and independently by brute-force computer calculation). It is just one of a family—you can vary the field, the degree, and the group—whose answers conjecturally have the property that they "stabilise" appropriately as one increase n [SS15]. After solving the above question we can explore these conjectures.

2. Prerequisites

The prerequisites are:

- Linear algebra.
- Finite fields.

Moreover, the following may be helpful:

- Experience with computer algebra software.
- Algebra (groups, rings, modules).
- Representation theory.
- Algebraic topology.

References

[Dic11] L. E. Dickson, *A fundamental system of invariants of the general modular linear group with a solution of the form problem*, Trans. Amer. Math. Soc. **12** (1911), no. 1, 75–98. 1

[GKRW18] S. Galatius, A. Kupers, and O. Randal-Williams, *E_∞-cells and general linear groups of finite fields*, https://arxiv.org/abs/1810.11931, 2018. 1

[Kra14] W. C. Krathwohl, *Modular Invariants of Two Pairs of Cogredient Variables*, Amer. J. Math. **36** (1914), no. 4, 449–460. 1

[SS15] S. V. Sam and A. Snowden, *Stability patterns in representation theory*, Forum Math. Sigma **3** (2015), Paper No. e11, 108. MR 3376738 1