The totally asymmetric exclusion process and the KPZ fixed point in the half-space

by

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Abstract

The thesis encapsulates the central project of my PhD studies. In this work, we present the multi-point probability distribution of the totally asymmetric exclusion process (TASEP) in a half-space, starting from a general deterministic initial condition. More precisely, let $h(t,x)$ denote the height function of TASEP at position x and time t; we provide an explicit formula for

$$\mathbb{P}(h(t,y_1) \leq s_1, \ldots, h(t,y_m) \leq s_m), \quad t \geq 0.$$

The formula presented is well-suited for scaling limit analysis. By applying a 1:2:3 scaling, we derive the probability distribution for the half-space KPZ fixed point, which is conjectured to represent the universal process for the limit of the KPZ universality models restricted to a half-space.

Additionally, we introduce a new formula for the full-space TASEP, starting from a general initial condition, as originally derived in [MQR21].

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Contents

Chapter 1

Introduction

1.1 Lateral growth process and the KPZ universality class

Many problems in physics involve modeling surface growth, such as modeling the growth of bacterial surfaces, the accumulation of crystals, and the spread of fire. All these surface growths share some common features. First, the dynamics are local, i.e., object at one point only feels the interaction from its neighbors; the high points propagate to further places; the surface is rough (experimentally).

A famous discrete model for modeling such lateral growth is called *ballistic aggregation model*. This is a continuous-time surface growth model on an integer lattice with the following rules. Let $h(x,t) \in \mathbb{Z}$ be the height function of the process at position $x \in \mathbb{Z}$ and at time $t \in \mathbb{R}$. Above every position x, there is an independent Poisson process $N(t)_x$. When the Poisson clock rings, there is a box dropping from the air, and one of the following cases happens: 1. It stacks on top of the box at x, so $h(t,x) = h(t_-,x)+1$, or 2. It sticks to one of the neighboring boxes, so $h(t,x) = \max\{h(t_-,x-1), h(t_-,x+1)\}$. So the total rule is just that $h(t,x) = \max\{h(t_-,x-1), h(t_-,x)+1, h(t_-,x+1)\}$, where $h(t_-,x)$ is the left limit at t. Although the model description is easy, it is a model that is hard to study, and only very few results are known about it.

Kardar, Parisi, and Zhang [KPZ86] introduced the following nonlinear stochastic PDE for modeling a growth height field $h(x,t)$:

$$\partial_t h = \lambda(\partial_x h)^2 + \nu \partial_x^2 h + \sqrt{D}\xi,$$

where λ, ν, D are parameters of the model, and $\xi(t,x), t \geq 0$ is the space-time white noise, i.e., it is the distribution-valued Gaussian process with mean zero and covariance:

$$\mathbb{E}[\xi(t_1,x_1)\xi(t_2,x_2)] = \delta(t_1-t_2)\delta(x_1-x_2).$$

More precisely,

$$\mathbb{E}[\int_{\mathbb{R}_++\times\mathbb{R}} \xi(t,x)f_1(t,x)dtdx \int_{\mathbb{R}_+\times\mathbb{R}} \xi(t,x)f_2(t,x)dtdx] = \int_{\mathbb{R}_+\times\mathbb{R}} f_1(t,x)f_2(t,x)dtdx,$$

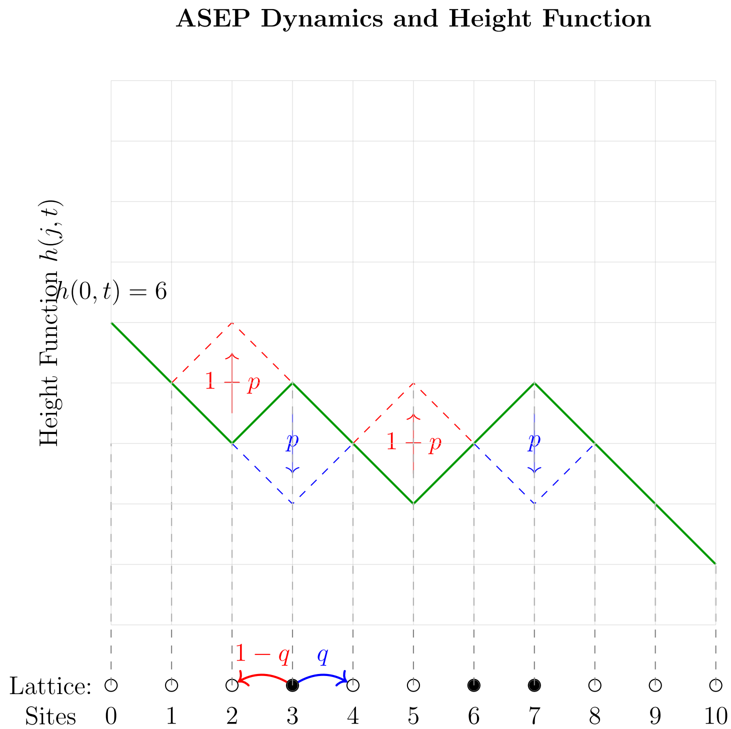
where f_1, f_2 are smooth functions with compact support.

Making sense of this SPDE is a difficult task, see [HQ18]. However, taking a formal Cole-Hopf transformation, which is $h = \nu\lambda^{-1}\log(z)$, $z(t,x)$ satisfies the stochastic heat equation with multiplicative chaos.

$$\partial_t z = \nu\partial_x^2 z + \lambda\nu^{-1}\xi z.$$

KPZ equation can be approximated by the following discrete particle system called the asymmetric exclusion process. This is a continuous time Markov process on $\{0,1\}^{\mathbb{Z}}$, with parameter $0 \le p \le 1$. One should think of it as an interacting particle system on $\mathbb{Z}$ such that 1 represents a particle occupying the sites. The dynamics are the following: there is an exponential clock with rate 1 associated with each particle; when the clock rings, the particle jumps to the right with probability p and jumps to the left with probability $(1-p)$. When the direction is determined, the particle checks whether its target site is empty. If the site is not empty, then the jump is blocked and the clock starts to count again; if the site is empty, then the particle performs the jump. This model is easier in the sense that some observables of the model can be directly computed. It turns out such an interacting particle system has a natural view as surface growth models. Let $\eta(x), x \in \mathbb{Z}$ be the occupation variable, i.e. $\eta(x) = 1$ if there is a particle at site x, and $\eta(x) = 0$ if there is no particle at site x. Let $h(0,0) = 0$, and $h(x,0) - h(x-1,0) = 1$ if there is a particle at x and is -1 if there is no particle at x. Thus, there is a simple random walk-type path associated with each particle configuration (given $h(0,0) = 0$). The dynamics reflected on the height function are local maxima flipped to local minima and local minima flipped to local maxima.

In [TW08],[TW10], a certain type of probability distribution can even be computed: Let $Y = \{y_1, \cdot, y_N\}$ with $y_1 < \cdots < y_N$ be the initial configuration of the



particles, let $X = \{x_1, \cdots, x_N\}$ be a possible configuration of the system at time t. Tracy and Widom give the probability distribution for

$$\mathbb{P}_Y(X;t),$$

which is the probability that the system starting with N particles at Y and being at position X at time t. The formula is only suitable for asymptotic analysis for certain types of initial configurations.

Now we yield one more step and consider a particular case of ASEP, that is, the case $p = 0$. In this case, particles can only jump to the right, which is called the totally asymmetric exclusion process (TASEP), which was first introduced by Spitzer [Spi70]. For a precise definition and properties, see [Lig85, Lig99]. In [Sch97], Schutz gave an explicit formula for TASEP that is of the same type as above $\mathbb{P}_Y(X;t)$, using the coordinate Bethe ansatz method. The date comes earlier since we are going to a simpler model. The formula that Schutz derived at the time is also not quite for the scaling limit.

Near 2000, Johansson solved the TASEP starting from the narrow wedge condition, i.e. initially, all the particles are at negative integer sites and the positive integer sites are empty. Johansson relates the model to the probability of last passage percolation and further to random matrix probabilities. The result is

Theorem 1.1.1. *[Joh00] We start TASEP with all negative integer sites being occupied by particles. Let $Y(k,t)$ be the number of particles to the right of k at time t. For each $u \in [0,1)$,*

$$\lim_{t\to\infty} \mathbb{P}(Y([ut],t)) \leq \frac{t}{4}(1-u)^2 + \frac{(1-u)^{2/3}}{(1+u)^{1/3}}\xi t^{1/3} = 1 - F(-\xi),$$

where $F(x)$ is the Tracy-Widom GUE distribution.

The model is solved by relating it to exponential last passage percolation (LPP), which is the limit of geometric LPP, whose value can be derived algebraically through the RSK mapping. We will introduce some of the models later; for now, we continue with the development of TASEP. After [Joh00], people managed to solve TASEP with other initial conditions.

Theorem 1.1.2. *[Sas05, BFPS07] We start TASEP with the initial condition that all the even sites are occupied and all the odd sites are empty.Let $x_i(t)$ be the position of i-th particle at time t. i-th particle is placed at $-2i$ initially. Then*

$$\lim_{t\to\infty} \mathbb{P}(x_{[t/4](t)\leq -st^{1/3}}) = F_1(2s)$$

where F_1 is the Tracy-Widom GOE distribution.

The method is by rewriting Schutz's formula as a non-intersecting line ensemble [Sas05] [BFPS07]. The probability is given as the Fredholm determinant implicitly, in the sense that the kernel is the solution of a bi-orthogonalization, from which the transition probability can be derived. Under some special initial conditions, the biorthogonalization problem is solved [BFP07][BFS08][BFPS07].

Another type of initial condition is also solved with a method similar to [Joh00], which is the stationary initial condition.

Theorem 1.1.3. *[BFP10] We start TASEP with the following random initial condition. Independently, each site is occupied by a particle with probability $1/2$ and empty with probability $1/2$, then*

$$\lim_{t\to\infty} \mathbb{P}(x_{[t/4]}(t) \geq t/4 - t^{1/3}s/2^{1/3}) = F_{\text{Baik-Rains}}(s)$$

where $F_{\text{Baik-Rains}}(s)$ is the Baik-Rains distribution.

It is worth mentioning that both theorems above are proved with general density cases. Also, both theorems proved that the TASEP height function converges as a random process in the large-time limit, i.e. they proved the convergence of the multi-point height probability distribution:

$$\mathbb{P}(h(t,x_1) \leq r_1, \cdots, h(t,x_n) \leq r_n)$$

after proper rescaling.

Eventually, TASEP is solved with general deterministic initial conditions in [MQR21].

Theorem 1.1.4. *Assume that the TASEP initial condition X_0 satisfies $X_0(j) = \infty$ for all $j \leq 0$. Then for any distinct positive integers $n_1, \cdots, n_m$ and $t \geq 0$,*

$$\mathbb{P}(X_t(n_j) > a_j, i = 1, \cdots, m) = \det(I - K_t^{\text{TASEP}})_{l^2(\{n_1,\cdots,n_m\}\times\mathbb{Z})},$$

where K_t^{TASEP} is some explicit kernel.

The importance of the theorem is that solving the model with enough initial conditions allows us to access the limiting universal process. It is believed that for surface growth models with similar types of mechanisms (smoothing, sticking effect, local dynamics), they will all converge to the same random process in the large time limit, independent of the exact local randomness and dynamics. In order to access this limiting process, the most straightforward method is to solve the discrete model and take the scaling limit, just like De Moivre and Laplace solved the sum of i.i.d. Bernoulli trials and derived the central limit theorem. In the same paper [MQR21], the existence of the KPZ fixed point $\mathfrak{h}(t,x)$, the conjecturally universal process, is established, and its one-time distribution

$$\mathbb{P}(\mathfrak{h}(t,x_1) \leq r_1, \cdots, \mathfrak{h}(t,x_n) \leq r_n)$$

is derived. As a space-time random process, it is natural to ask what the multi-time, multi-point distribution

$$\mathbb{P}(\mathfrak{h}(t_1,x_1) \leq r_1, \cdots, \mathfrak{h}(t_n,x_n) \leq r_n)$$

is, which is given in Liu [Liu22], and Johansson and Rahman [JR21].

Furthermore, TASEP can be thought of as evolving according to some random exponential field. Here we introduce the exponential last passage value problem. We first establish the following relation between TASEP and last passage percolation. We stack the space of the TASEP height function with boxes; see figure (1.2). Each box is identified by its center coordinates and is associated with an exponential random variable with mean 1. Now we can define the last passage value between two points:

$$L(x, r; y, s) = \max_{\Pi} \sum_i \omega_{\Pi(i)}$$

where $\Pi(i)$ is a Southwest-Southeast path with step size 1, connecting $(x, r-1)$ and $(y, s+1)$. Now we can state the connection between the TASEP and LPP problems:

Lemma 1.1.5. *Assume we start TASEP from the initial configuration h_0 with a local maximum at $(x_1, r_1), \cdots (x_n, r_n)$, then*

$$\mathbb{P}(h(t, y_1) \leq r_1, \cdots, h(t, y_m) \leq r_m) = \mathbb{P}(\max_{i,j}\{L(x_i, r_i; y_j, s_j) \leq t)\}$$

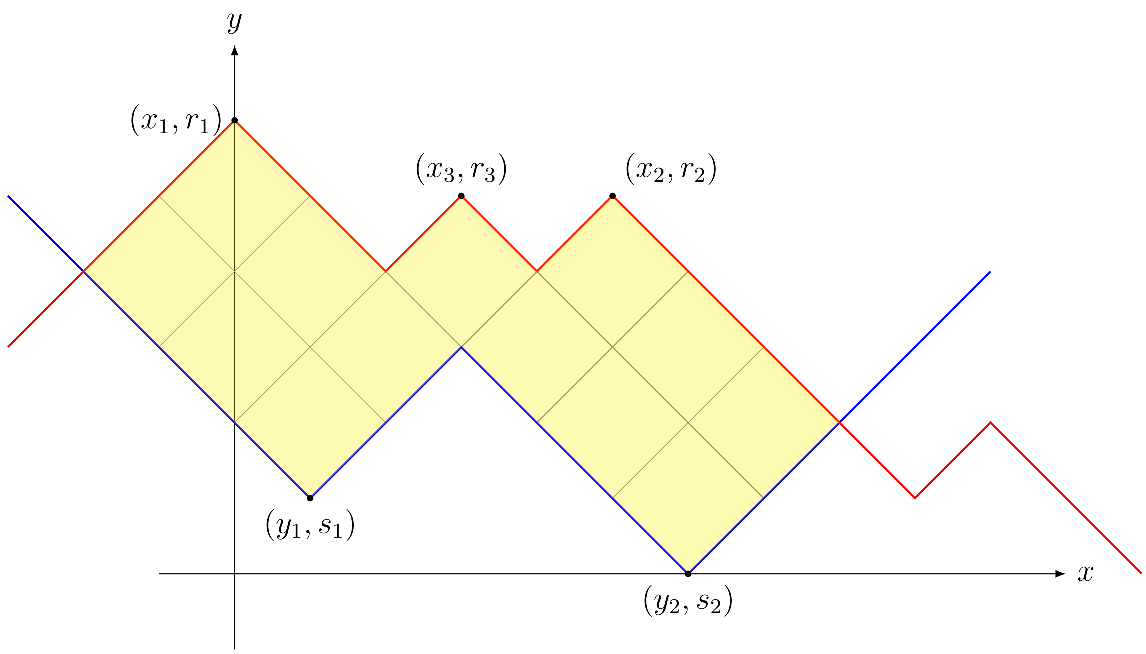


Figure 1.1: TASEP and LPP

With this point of view of TASEP, we can think of TASEP as evolving in these random exponential fields. Then one could ask what the probability is of different TASEP height functions running through the same random fields. Although there is no explicit formula for these objects, Dauvergne, Ortmann, and Virag [DOV22] constructed *the directed landscape* which characterize the limit of last passage values of the random field.

Theorem 1.1.6. *The directed landscape is a random continuous function $\mathcal{L}$ from*

$\{(x, s; y, t) \in \mathbb{R}^4 : s < t\} \to \mathbb{R}$ *satisfying the metric composition law*

$$\mathcal{L}(x, r; y, t) = \max_{z \in \mathbb{R}} \mathcal{L}(x, r; z, s) + \mathcal{L}(z, s; y, t) \text{ for } s \in (r, t)$$

with the property that $\mathcal{L}(\cdot, t_i; \cdot, t_i + s_i^3)$ *are independent Airy sheets of scale* s_i *for any set of disjoint time intervals* $(t_i, t_i + s_i^3)$*.*

For now, the directed landscape contains most information about the limiting process. Now we want to bring the same story to the half-space domain. Before we continue, we introduce another solvable model that also plays an important role in accessing the limiting process.

Polynuclear growth model (PNG) is a continuous-time Markov process whose state space is the set of upper semi-continuous functions $h : \mathbb{R} \to \mathbb{Z} \cup \{-\infty\}$. The dynamics have two parts. First, the height function spreads in two directions, i.e., $h(x) = \sup_{|y-x| \leq t} h(y)$. Second, we think that there is a space-time Poisson point process with rate 2. If (t_0, x_0) is a point in the process, then the height function h increases by one at time t. If we start the PNG model with the initial condition that $h(0, x) = 0$ if $x = 0$ and $-\infty$ if $x \neq 0$, then the problem is equivalent to the famous Ulam problems [Ula61]. The problem is: given a permutation of π_N of $1, 2, \cdots, N$, let $l(\pi_N)$ be the length of the longest increasing subsequence in $\pi_N(1), \cdot, \pi_N(N)$. The question is what is the behavior of $l(\pi_N)$ as $N \to \infty$ for a uniformly chosen π_N? In [BDJ99], they gave a firm answer that

Theorem 1.1.7.

$$\lim_{N \to \infty} \mathbb{P}\left(\frac{l(\pi_N) - 2\sqrt{N}}{N^{1/6}} \leq x\right) = F(x)$$

where $F(x)$ *is the Tracy-Widom distribution.*

Later, [PS02] we show the convergence of the height function to the Airy process and Airy line ensemble, which are the central objects in the theory, see [CH14, AH23].

1.2 Half-space models

We first describe the three models: TASEP, Exponential LPP, and the PNG model in the half-space. The half-space TASEP with rate α is a continuous-time Markov process on $\mathbb{Z}^+$. Particles jump to the right in continuous time at rate 1 with exclusion. There is a reservoir of an infinite number of particles at the origin, and the particles jump to site 1 at rate α if the site 1 is empty. Let $\eta : \mathbb{N} \to \{0, 1\}$ be the occupation variables. $\eta_t(x)$ is 1 if there is a particle at position x at time t and 0 otherwise. For finite range $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$, the generator is given by:

$$\mathcal{L}f(\eta) = \alpha(f(1,\eta_2,\eta_3,\cdots) - f(\eta_1,\eta_2,\cdots)) + \sum_{x\in\mathbb{Z}^+} \eta_x(1-\eta_{x+1})(f(\eta_{x,x+1}) - f(\eta))$$

where $\eta_{x,x+1}$ is obtained by switching the occupation variables η at sites x and $x+1$. There is also the height representation $h(t,x), x > 0$, where

$$h(t,x) = \begin{cases} -2J_{0,t} - \sum_{i=1}^{x}(1-2\eta_t(i)), & x \geq 1 \\ -2J_{0,t}, & x = 0 \end{cases}$$

where $J_{0,t}$ is the number of new particles that have entered the positive real line up to time t. Similarly, the half-space PNG model is also restricted to the positive real line, and then along the timeline at $x = 0$, there is a one-dimensional Poisson point process with rate α which represents the nucleation happening at the origin. For the exponential LPP problem, let $(w_{n,m})_{n\geq m\geq 0}$ be a sequence of independent exponential random variables with rate 1 when $n \geq m+1$ and with rate α when $n = m$. The exponential last passage percolation time on the half-quadrant, denoted by $H(n,m)$, is defined by

$$H(n,m) = w_{n,m} + \begin{cases} \max\{H(n-1,m), H(n,m-1)\} & \text{if } n \geq m+1 \\ H(n,m-1) & \text{if } n = m \end{cases}$$

with $H(n,0) = 0$. The correspondence between the half-space TASEP and exponential LPP also exists in the half-space: We stack the space of the TASEP height function with boxes; see figure (1.2). Each box is identified with its center coordinates and is associated with some exponential random variables. We define

$$\omega_{i,j} \sim \begin{cases} \text{Exp}(1), & \text{if } i \neq 0, h_{\text{final}}(i) < j < h_{\text{init}}(i) \\ \text{Exp}(\alpha), & \text{if } i = 0, h_{\text{final}}(i) < j < h_{\text{init}}(i) \\ 0, & \text{otherwise .} \end{cases}$$

Now we can define the last passage value between two points:

$$L(x,r;y,s) = \max_{\Pi} \sum_i \omega_{\Pi(i)}$$

where $\Pi(i)$ is a Southwest-Southeast path with step size 1, connecting $(x, r-1)$ and $(y, s+1)$. Now we can state the connection between the TASEP and LPP problem:

Lemma 1.2.1.

$$\mathbb{P}((x_1, r_1; \cdots; x_n, r_n)_t \leq \{y_1, s_1; \cdots; y_m, s_m\}) = \mathbb{P}(\max_{i,j}\{L(x_i, r_i; y_j, s_j)\} \leq t).$$

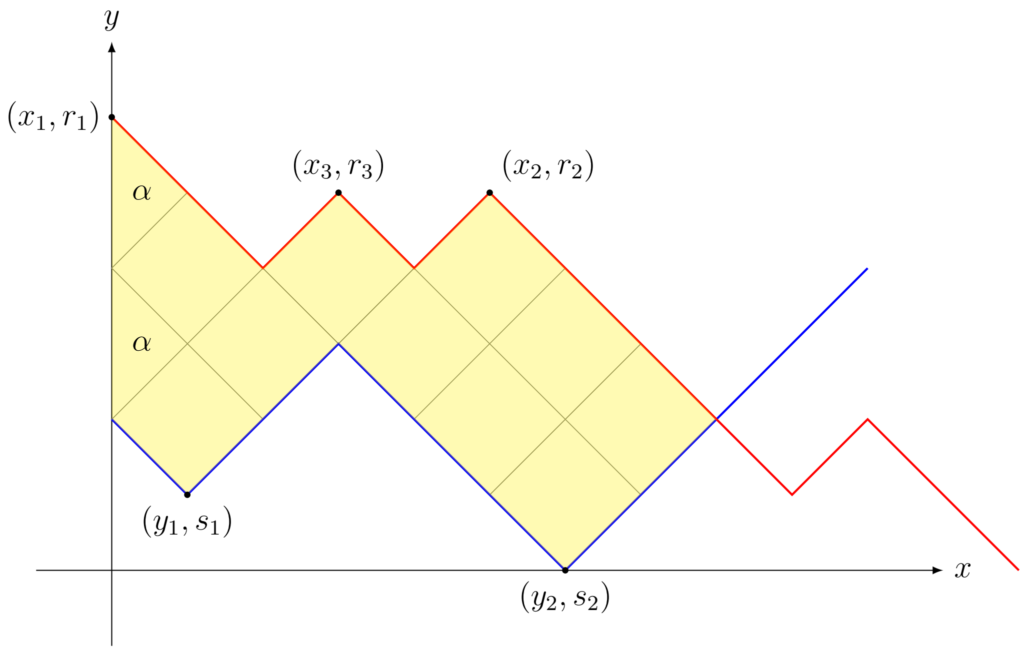


Figure 1.2: Half-space TASEP and LPP

The earliest studied half-space models in the KPZ universality class are symmetrized LPP with geometric weights, in [BR01c, BR01b, BR01a], where the phase transition of the one-point distribution at the diagonal has been established. The large time behavior of the model depends on the size of the parameter. There is a critical value of the parameter in different models. In the Poisson LPP problem which Baik and Rains consider, it is 1. If we properly scale the height function, in the case that $0 \leq \alpha < 1$ (sub-critical case), the fluctuation is of order $N^{1/3}$ and has Tracy-Widom GSE distribution; if $\alpha = 1$, the fluctuation is of order $N^{1/3}$ and has Tracy-Widom GOE distribution; if $\alpha > 1$ (super-critical case), the fluctuation is of order $N^{1/2}$ and has Gaussian distribution. The $N^{1/2}$ is non-KPZ fluctuations. This is purely due to the effect of the central limit theorem: the effect of the diagonal is too large that the last passage value behaves like the sum of i.i.d. random variables on the diagonal. Furthermore, depending on whether the position is away from the origin or not, one can see all different types of limiting distributions. It is interesting that the Tracy-Widom GSE distribution is not present in the full space models.

Further, [SI04] study the PNG model in half-space, in which they studied the multipoint distribution of the model, with or without nucleation at 0. The fluctuation of the process near the origin gives the symplectic-unitary transition in random matrix theory in the subcritical case and gives the orthogonal-unitary transition in the critical case [FNH99].

Later, the exponential last passage percolation in the first quadrant is studied in [BBCS18b, BBCS18a]. They start from the geometric LPP problem, scale it to the exponential LPP, then adopt asymptotic analysis. They derive the multi-point distribution of a process that interpolates between the symplectic-unitary regime and orthogonal-unitary regime. That requires a weak scaling of the parameter α around its critical value, so that in the limit, one still sees the effect of boundary injection.

It is worth mentioning that, from the point of view of lateral growth, all three models are equivalent to solving the model from the narrow wedge initial condition. In terms of half-space TASEP, that is to say, the model starts with all sites being empty initially.

For TASEP starting from the product Bernoulli initial condition, which is equivalent to the half-space stationary LPP model, which is studied in [BFO20, BFO22]. In these two cases, the formula is derived using that the half-space LPP with geometric weights is a marginal of the Pfaffian Schur process; for more about the Pfaffian Schur process, see [BR05, BBNV18, SI04].

There are a lot of other models in the KPZ universality class that are studied in the half-space. For example, polymer model with wall [BLD21, Sep12, BBC16, BCD23, DZ24], stochastic six-vertex model in the half quadrant [BBCW18, GdGMW24], the half-space ASEP [BC24, He24, Par19], the half-space KPZ equation [BKLD20, DNKLDT20, BKLD22, KL20], the half-space MacDonald process [BBC20], and many algebraic structures related to half-space models [IMS22, IMS23, Ass23]. For more studies related to the properties of models in the half-space, see [He22, FO24, Che24].

The invariant measure of the half-space TASEP is an interesting problem. The product of Bernoulli(α) measure is invariant for the half-space TASEP with origin rate α, the proof is essentially in [Lig77][Lig75]. Invariant measures are not unique; see [Gro04]. Other invariant measures can be derived using the invariant measure for open ASEP models, which is studied in [BCY24][WWoY24]. For more studies on invariant measures for half-space models, see [BD22][BC23][CK24][Cor22].

TASEP under other geometries is also studied; for example, TASEP in the periodic domain is studied in [BL21, BLS22, Lia22].

From the point of view of the full space development, one wants to investigate the conjectural limiting process in half-space, which should be the scaling of all the KPZ universality class models in half-space. In order to define the process using explicit transition probability, one wants to solve the model with more initial conditions, which is the main problem that is going to be addressed in this thesis. We solve the half-space TASEP starting from a general deterministic initial condition. Thus, one can take the scaling limit of the model and access the transition probability of the limit process, which is **the half-space KPZ fixed point**. Various aspects of the half-space KPZ fixed point are already known from the previous work. The probability distribution of the half-space fixed point starting from a narrow wedge initial condition is in [BBCS18b, BBCS18a], which is a Fredholm Pfaffian with kernel $\mathsf{K}^{\text{cross}}$, which should be thought of as the half-space Airy$_2$ process, and the kernel in [BFO22] is the half-space Airy stat process.

1.3 Method and organization of the thesis

We will solve the half-space TASEP with a general deterministic initial condition. Consider a continuous-time Markov process X_t in measurable state space $(S, \mathcal{S})$ with the generator $\mathcal{L}$. For any $A \in \mathcal{S}, x \in S$, the Markov transition function $P_t(x, A)$ satisfies the Kolmogorov backward equation:

$$\begin{cases} (\partial_t - \mathcal{L})P_t(x, A) = 0, \\ \lim_{t\to 0} P_t(x, A) = 1_{x\in A}. \end{cases} \tag{1.1}$$

If we can find a function $\tilde{P}_t(x, A)$ that satisfies (1.1) and show that the solution is unique, then the function $\tilde{P}_t(x, A)$ must be the transition density of the Markov process. This enables a guess-and-check approach to find the transition probability.

We will use this general scheme to prove a new formula for full space TASEP and solve the one-time, multi-point Half-space TASEP with a general deterministic initial condition. The formula is largely inspired by [MQR21][NQR20][BBCS18b][BBCS18a]. In the first paper, it reveals the key philosophy that "initial conditions should come in as hitting probability." From the second paper, the Kolmogorov equation is verified for full space TASEP. In the last two papers, many important ingredients of the half-space formula are present. The formula we give can be characterized loosely by "path integral formula from [BBCS18b] with hitting probability."

In Chapter 2, we present a new formula for the full-space TASEP. In Chapter 3, we solve the half-space TASEP. In both cases, we present a one-point formula first; this is mainly because going from a general one-point to a multi-point formula is normally not difficult for KPZ universality models, but the notations and indices become more complicated. We present the full-space formula first because almost all the mechanisms in full space are used in half-space, and the full-space formula is simpler and easier to understand. Thus, one does not waste much time even if they are only interested in half-space formulas.

In Chapter 4, we take the scaling limit of the half-space TASEP and derive the formula and existence of the half-space KPZ fixed point.

1.4 List of symbols

There are many small variables and notations in this thesis. We summarize them in this section for quick reference.

- Peaks positions (2.2):

$$(\vec{x},\vec{h}) = (x_1,h_1;x_2,h_2;\cdots x_n,h_n)_t, \quad x_1 < \cdots < x_n.$$

- Trough positions (2.3):

$$\{\vec{y},\vec{s}\} = \{y_1,s_1;y_2,s_2;\cdots y_m,s_m\}, \quad y_1 < \cdots < y_m.$$

- Primordial peak (2.5):

$$x_{\mathrm{prim}} = \frac{h_n - h_1 + x_n + x_1}{2}, \quad h_{\mathrm{prim}} = \frac{h_1 + h_n + x_n - x_1}{2}.$$

- u_i (number of up wedges from peak x_i to x_{i+1}); d_i (number of down wedges from peak x_i to x_{i+1}); u is the distance from *the primordial peak* to the first peak; d is the distance from *the primordial peak* to the last peak (2.6):

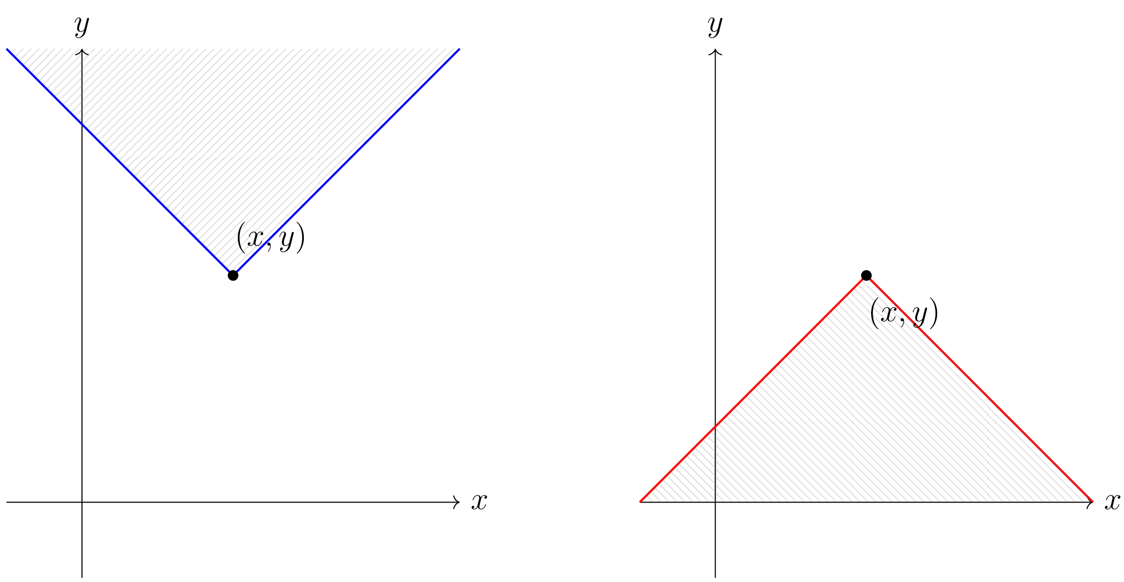
$$u_i = (x_{i+1} - x_i + h_{i+1} - h_i)/2, \quad d_i = (x_{i+1} - x_i - h_{i+1} + h_i)/2,$$
$$u = \sum_{i=1}^{n-1} u_i, \quad d = \sum_{i=1}^{n-1} d_i, \quad u_{ij} = \sum_{k=i}^{j-1} u_k, \quad d_{ij} = \sum_{k=i}^{j-1} d_k.$$

- Given a trough configuration $\{\vec{y}, \vec{s}\}$, u'_i, d'_i are u, d parameterized by $(\vec{y}, -\vec{s})$ (2.63). More precisely,

$$u'_i = (y_{i+1} - y_i - s_{i+1} + s_i)/2, \quad d'_i = (y_{i+1} - y_i + s_{i+1} - s_i)/2,$$
$$u' = \sum_{i=1}^{n-1} u'_i, \quad d' = \sum_{i=1}^{n-1} d'_i, \quad u'_{ij} = \sum_{k=i}^{j-1} u'_k, \quad d'_{ij} = \sum_{k=i}^{j-1} d'_k.$$

- Cone $C_{x,y}, C^{x,y}$ (3.4),(3.3):

$$C_{x,y} = \{(a,b) \in \mathbb{Z}^2 : b \geq |a - x| + y\},$$
$$C^{x,y} = \{(a,b) \in \mathbb{Z}^2 : b \leq -|a - x| + y\}.$$



(a) Cone $C_{x,y}$ (b) Cone $C^{x,y}$

Figure 1.3: Cone graph

- $l_{p,q}(\vec{x}, \vec{h}), r_{p,q}(\vec{x}, \vec{h})$ are the distances from *the primordial peak* of $(\vec{x}, \vec{h})$ to the left and right sides of the cone $C_{p,q}$, respectively (3.5):

$$l_{p,q}(\vec{x}, \vec{h}) := (h_n - q + x_n - p)/2, \quad r_{p,q}(\vec{x}, \vec{h}) := (h_1 - q - x_1 + p)/2.$$

- **Full space one-point.** $l = l_{0,0}(\vec{x}, \vec{h}), r = h_{0,0}(\vec{x}, \vec{h})$ (2.7):

$$l = (x_{\text{prim}} + h_{\text{prim}})/2, \quad r = (h_{\text{prim}} - x_{\text{prim}})/2.$$

- **Full space multi-point.** $l_i = l_{y_i,s_i}(\vec{x},\vec{h}), r_i = r_{y_i,s_i}(\vec{x},\vec{h})$ (2.66):

$$l_i = (x_{\text{prim}} + h_{\text{prim}} - y_i - s_i)/2, \quad r_i = (h_{\text{prim}} - s_i - x_{\text{prim}} + y_i)/2$$

- **Half space one-point.** $l = l_{0,s-y}(\vec{x},\vec{h}), r = r_{0,s-y}(\vec{x},\vec{h}), l' = l_{0,-x_n-h_n}(y,-s)$, $r' = r_{0,-x_n-h_n}(y,-s)$ in half-space (3.10):

$$l = (h_n + x_n - s + y)/2, \quad r = (h_1 - x_1 - s + y)/2,$$
$$l' = (-s + y + x_n + h_n), \quad r' = (-s - y + x_r + h_n)/2.$$

  Notice $l = l'$, thus we actually never use l'.

  Abuse of notation: We use both l, l_i, r, r_i in full-space and half-space, but they have different definitions. However, there should be no confusion since one is only used in the full-space chapter, and the other is only used in the half-space chapter.

- **Half space multi-point.** $l_i = l_{0,s_i-y_i}(\vec{x},\vec{h}), r_i = r_{0,s_i-y_i}(\vec{x},\vec{h}), l'_i = l_{0,-x_n-h_n}(y_i,-s_i), r'_i = r_{0,-x_n-h_n}(y_i,-s_i)$ in half-space (3.10):

$$l_i = (h_n + x_n - s + y)/2, \quad r_i = (h_1 - x_1 - s + y)/2,$$
$$l'_i = (-s + y + x_n + h_n), \quad r'_i = (-s - y + x_r + h_n)/2.$$

- 'Hitting operator' (2.8):

$$W = (I - W_0 \bar{1}^t W_{1,2} \bar{1}^t W_{2,3} \cdots \bar{1}^t W_{n-1,n} \bar{1}^t W_{n+1}), \quad n > 1;$$
$$W = 1_t, \quad n = 1;$$
$$W_{i,i+1} = \mathtt{a}^{-u_i} \mathtt{a}_*^{-d_i}, \quad W_0 = \mathtt{a}^u, \quad W_{n+1} = \mathtt{a}_*^d.$$

- Some auxiliary 'W' type operator (2.17):

$$W_{i,j} = W_{i,i+1} \bar{1}^t W_{i+1,i+2} \cdots W_{j-2,j-1} \bar{1}^t W_{j-1,j}, \quad i + 1 < j.$$

- 'Differential operator' $\mathtt{a}, \mathtt{a}_*$ (2.9):

$$\mathtt{a} = 1 - 2D, \quad \mathtt{a}_* = 1 + 2D.$$

- 'Integral operator' $\mathtt{a}^{-1}, \mathtt{a}_*^{-1}$ (2.10):

$$\mathtt{a}^{-1}(x,y) = \frac{1}{2} e^{(x-y)/2} 1_{x \le y}, \quad \mathtt{a}_*^{-1}(x,y) = \frac{1}{2} e^{(y-x)/2} 1_{y \le x}.$$

- Indicator function 1_b^a (2.11):

$$1_b^a(x) = 1_{b<x<a}.$$

If a is ∞ or b is $-\infty$, it will be omitted; if the endpoint is included, it will be $\overline{1}, \underline{1}$ or $\overline{\underline{1}}$.

- Differential and integral operators appear in half-space (3.6):

$$\mathtt{b} = 2\alpha - 1 - 2D, \quad \mathtt{b}_* = 2\alpha - 1 + 2D.$$

For $\alpha \neq \frac{1}{2}$,

$$\mathtt{b}^{-1}(x,y) = \tfrac{1}{2} e^{(2\alpha-1)(x-y)/2} 1_{x \le y}, \quad \mathtt{b}_*^{-1}(x,y) = \tfrac{1}{2} e^{(2\alpha-1)(y-x)/2} 1_{y \le x}.$$

For $\alpha = \frac{1}{2}$,

$$\mathtt{b}_*^{-1}(x,y) = -1_{x<y} + 1_{x \ge y}, \quad \mathtt{b}^{-1}(x,y) = 1_{x<y} - 1_{x \ge y}.$$

- Composition of $\mathtt{b}^{-1}$ and $\mathtt{b}_*^{-1}$. (3.13):

For $0 < \alpha < \frac{1}{2}$,

$$\overline{\mathtt{b}^{-1}\mathtt{b}_*^{-1}}(x,z) = 1_{x \ge z} \tfrac{1}{4(2\alpha-1)} e^{(1-2\alpha)(x-z)/2} + 1_{x<z} \tfrac{1}{4(2\alpha-1)} e^{(1-2\alpha)(z-x)/2}.$$

For $\alpha = \frac{1}{2}$,

$$\overline{\mathtt{b}^{-1}\mathtt{b}_*^{-1}}(x,z) = \tfrac{1}{4}((x-z)1_{x \ge z} + (z-x)1_{x<z}).$$

- $\overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}}$ (3.14):

$$\overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} := D\overline{\mathtt{b}^{-1}\mathtt{b}_*^{-1}} = \overline{\mathtt{b}^{-1}\mathtt{b}_*^{-1}} D.$$

- Modification of W used in half-space (3.11):

$$V = \mathtt{a}_*^{r-l} \mathtt{b}_*^{-1} W \mathtt{a}_*^{l-r} \mathtt{b}_*,$$
$$V' = \mathtt{b}_*^{-1} \mathtt{a}_*^{r'-l} 1_0 \mathtt{a}_*^{l-r'} \mathtt{b}_*.$$

Some combination of V and W (3.23):

$$V_0 = \mathtt{a}_*^{r-l}\mathtt{b}_*^{-1}W_0, \quad V_{0,i} = V_0\overline{1}^t W_{1,i} \text{ for } 1 < i \leq n,$$
$$V_{n+1} = W_{n+1}\mathtt{a}_*^{l-r}\mathtt{b}_*, \quad V_{i,n+1} = W_{i,n}\overline{1}^t V_{n+1}, \quad \text{for } 1 \leq i < n.$$

- A notation for parametrization of matrix (3.22):

$$\begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix}_{y,s} := \begin{pmatrix} -S_{0,0}^{r,r'} & DS_{1,-1}^{l,r+r'-l} \\ D^{-1}S_{-1,1}^{r+r'-l,l} & -S_{0,0}^{r',r} \end{pmatrix},$$
$$\begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix}_{y,s} := \begin{pmatrix} S_{0,0}^{r',r} & DS_{1,-1}^{l,r+r'-l} \\ D^{-1}S_{-1,1}^{r+r'-l,l} & S_{0,0}^{r,r'} \end{pmatrix}.$$

- $S^{i,j}$ kernel in full space (2.13):

$$S^{i,j}(x,y) = 1_{x+y\geq 0} \cdot s^{i,j}(x,y),$$
$$s^{i,j}(x,y) = \int_\Gamma e^{-(x+y)w} \frac{(1+2w)^j}{(1-2w)^i} \frac{dw}{2\pi i}.$$

- $S_{a,b}^{i,j}$ kernel in half space (3.9):

$$s_{a,b}^{i,j}(x,y) = \int_\Gamma e^{-(x+y)w} \frac{(1+2w)^j(2\alpha-1+2w)^b}{(1-2w)^i(2\alpha-1-2w)^a} \frac{dw}{2\pi i},$$
$$S_{a,b}^{i,j}(x,y) = s_{a,b}^{i,j}(x,y) 1_{x+y\geq 0}.$$

- S kernel with Dirac delta function (2.2.13):

$$B^{n,m}(x,y) = 2s^{n,m}(x,y) \cdot \delta_0(x+y).$$

- $\tilde{S}$ space (2.44):

$$\tilde{S} = \{S^{i,j}f \mid f \in L^2([0,\infty))\}.$$

Chapter 2

Full-space TASEP with a general initial condition

2.1 Models and notation

TASEP is a continuous-time Markov process on state space $\Omega = \{0,1\}^{\mathbb{Z}}$. One notation for the configuration is the occupation variable $\eta = \{\eta_j, j \in \mathbb{Z} | \eta_j \in \{0,1\}\}$. The dynamics of TASEP is that each particle jumps to the right site after an exponential 1 amount of waiting time, provided that the right site is empty. More precisely, let $f : \Omega \to \mathbb{R}$ be a function that only depends on a finite number of coordinates, the backward generator of the TASEP is given by

$$\mathcal{L}f(\eta) = \sum_{j \in \mathbb{Z}} \eta_j (1 - \eta_{j+1})(f(\eta^{j,j+1}) - f(\eta)), \tag{2.1}$$

where $\eta^{j,j+1}$ is the configuration that η_j and η_{j+1} values are switched.

We will use another set of observables to record TASEP configurations. We are interested in the following probability distribution:

$$\mathbb{P}(h((t, x; h_{\text{init}}) \le h_{\text{final}}(x)).$$

There are slightly different assumptions on the types of functions h_{init} and h_{final} that are allowed. We will always assume that the configuration $h_{\text{init}}(t,x)$ that is evolving has a finite number of peaks (local maxima) and $h_{\text{init}}(t,x) \to -\infty$ as $x \to \infty$. $h_{\text{final}}(x)$ has a finite number of troughs (local minima) and $h_{\text{final}}(x) \to \infty$ as $x \to \infty$. Under these assumptions, $h_{\text{init}}(t,x)$ is uniquely determined by the positions of the peaks x_i

and the heights of the peaks h_i at time t. We use notation

$$(\vec{x}, \vec{h}) = (x_1, h_1; x_2, r_2; \cdots x_n, h_n)_t, \quad x_1 < \cdots < x_n \tag{2.2}$$

to denote it. If a tuple $(\vec{x}, \vec{h})$ represents a configuration of TASEP, it satisfies the following parity constraints: $x_i + h_i$ all have the same parity; $|h_{i+1} - h_i| < x_{i+1} - x_i$. Similarly, $h_{\text{final}}(x)$ is uniquely determined by the position of the troughs y_i and the heights of the troughs s_i. We use notation

$$\{\vec{y}, \vec{s}\} = \{y_1, s_1; y_2, s_2; \cdots y_m, s_m\}, \quad y_1 < \cdots < y_m. \tag{2.3}$$

Here $y_i + s_i$ all have the same parity and $|s_{i+1} - s_i| < y_{i+1} - y_i$. The final condition does not have an index t as it does not depend on time. Given this notation, we can write the generator $\mathcal{L}$ as the sum of n pieces:

$$\mathcal{L} f(\vec{x}, \vec{h}) = \sum_{i=1}^{n} \mathcal{L}_{x_i} f(\vec{x}, \vec{h}) \tag{2.4}$$

with the obvious meaning that $\mathcal{L}_{x_i} f$ is the f evaluated at the configuration obtained from a flip at x_i, subtracting the f evaluated at the original configuration.

We now develop some notations for further discussion. Each such initial configuration can be thought of as having been obtained through a sequence of downward flips from *the primordial peak* configuration, which we denote as $(x_{\text{prim}}, h_{\text{prim}})_t$, where

$$x_{\text{prim}} = \frac{h_n - h_1 + x_n + x_1}{2}, \quad h_{\text{prim}} = \frac{h_1 + h_n + x_n - x_1}{2}. \tag{2.5}$$

We will refer to this configuration as *the primordial peak* that corresponds to $(x_1, h_1; \cdots x_n, h_n)_t$. See Figure (2.1), the point $(3, 11)$ is *the primordial peak* for the configuration $(0, 8; 2, 8; 5, 9)$. Now we want to introduce another set of variables that record the relative position of peaks with respect to *the primordial peak*. Let u_i, d_i be the number of wedges that go upward and downward from the peak x_i to x_{i+1}, respectively. More precisely,

$$u_i = (x_{i+1} - x_i + h_{i+1} - h_i)/2, \quad d_i = (x_{i+1} - x_i - h_{i+1} + h_i)/2. \tag{2.6}$$

Now, the configuration $(x_1, h_1; \cdots; x_n, h_n)$ is equivalently parameterized by *the pri-*

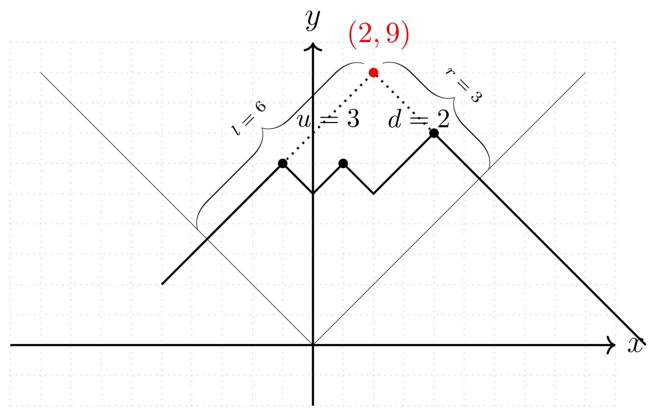


Figure 2.1: Configuration $(-1, 6; 4, 7; 1, 96)$ with *the primordial peak* $(2, 9)$

mordial peak and all u_i, d_i. We define

$$u = u_1 + \cdots u_{n-1}, \quad d = d_1 + \cdots d_{n-1}.$$

It is easy to see that u is the distance from *the primordial peak* to the first peak, and d is the distance from *the primordial peak* to the last peak; see Figure (2.1).

We also want to define

$$l = (x_{\text{prim}} + h_{\text{prim}})/2, \quad r = (h_{\text{prim}} - x_{\text{prim}})/2. \tag{2.7}$$

l is the signed distance from the *primordial peak* to the line $y = -x$, and r, which is the signed distance from the *primordial peak* to the line $y = x$.

We want to define the following "operator product ansatz" kernel associated with configuration $(x_1, h_1; \cdots; x_n, h_n)$. When $n > 1$, we define operator $W : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$,

$$\begin{aligned}
&W = (I - W_0 \overline{1}^t W_{1,2} \overline{1}^t W_{2,3} \cdots \overline{1}^t W_{n-1,n} \overline{1}^t W_{n+1}), \quad n > 1 \\
&W = 1_t, \quad n = 1 \\
&W_{i,i+1} = \mathtt{a}^{-u_i} \mathtt{a}_*^{-d_i}, \quad W_0 = \mathtt{a}^u, \quad W_{n+1} = \mathtt{a}_*^d,
\end{aligned} \tag{2.8}$$

where

$$\mathtt{a} = 1 - 2D, \quad \mathtt{a}_* = 1 + 2D. \tag{2.9}$$

D is the differential operator, in the distributional sense. $\mathtt{a}^{-1}, \mathtt{a}_*^{-1}$ are integral operators with the kernels

$$\mathtt{a}^{-1}(x, y) = \frac{1}{2} e^{(x-y)/2} 1_{x \le y}, \quad \mathtt{a}_*^{-1}(x, y) = \frac{1}{2} e^{(y-x)/2} 1_{y \le x}. \tag{2.10}$$

Notice that $\mathtt{aa}^{-1}(x,y) = \mathtt{a}_*\mathtt{a}_*^{-1}(x,y) = \delta(x-y)$, in the distribution sense.

$\overline{1}^t$ is the projection operator with the following multiplication kernel,

$$1_b^a(x) = 1_{b<x<a}. \tag{2.11}$$

We use $\overline{1}$, $\underline{1}$, and $\overline{\underline{1}}$ whenever the endpoints are included, and a or b are omitted if they are ∞ or $-\infty$, respectively. The index in W_0, W_{n+1} does not have meaning; it is simply for notational convenience.

Let us also introduce our notation for the Dirac delta function and Bra-Ket notation. $\delta_t(x)$ is the Dirac delta function at t. We will often omit the subscript when it is 0. $|\delta_t\rangle\langle\delta_t|$ is the operator with integral kernel $\delta_0(x-t)\delta_0(y-t)$. $\langle\delta_t| f$ just means $f(t)$. For an integral operator K with kernel $K(x,y)$, $\langle\delta_t| K |\delta_t\rangle = K(t,t)$. We also use the notation $\langle f|$ to denote an operator that acts by taking the inner product with f.

We will be more precise about what is W, i.e. on which space the operator acts. We will discuss it in detail in later sections.

2.2 One-point distribution

Theorem 2.2.1. *Assume that we start the full-space TASEP with the initial configuration having peaks at $(x_1, h_1; \ldots, x_n, h_n)$. The probability that at time t it is below the configuration $\{0, 0\}$ is given by:*

$$\mathbb{P}((x_1, h_1; \ldots; x_n, h_n)_t \le \{0,0\}) = \det(I - S^{l,r} W S^{r,l})_{L^2([0,\infty))}, \tag{2.12}$$

where W is defined in (2.8) . $S^{i,j}$ is an integral operator from $L^2([0,\infty)) \to L^2(\mathbb{R})$ with the following kernel:

$$\begin{aligned} S^{i,j}(x,y) &= 1_{x+y\ge 0} \cdot s^{i,j}(x,y), \\ s^{i,j}(x,y) &= \int_\Gamma e^{-(x+y)w} \frac{(1+2w)^j}{(1-2w)^i} \frac{dw}{2\pi i}, \end{aligned} \tag{2.13}$$

where Γ is a simple, positively oriented loop that includes $w = 1/2$.

Remark 2.2.2. For complete mathematical rigor, the $1_{x+y\ge 0}$ in the $S^{i,j}(x,y)$ is interpreted as the limit of a sequence of smooth approximations of the indicator function $\phi_n(x+y)$ such that for any $\varepsilon > 0$, for large enough n, $\phi_n(x) = 1$ for $x > \varepsilon$ and $\phi(x) = 0$ for $x < -\varepsilon$, and $\phi'(x) \to \delta(x)$ in the distributional sense.

We will prove in the next section that:

Proposition 2.2.3. *The kernel in* (2.12) *is well-defined and is a trace-class operator on* $L^2([0,\infty))$.

We will prove the theorem after we study some properties of the kernel in the next section.

Given Theorem (2.2.1), it is easy to state and prove the general one-point probability distribution, due to the fact that TASEP is translation invariant.

Corollary 2.2.4. *If we start the TASEP from the initial configuration* $(x_1, h_1 \cdots x_n, h_n)$*, the probability that at time* t *it is below the configuration* $\{y, s\}$ *is given by:*

$$\begin{aligned}&\mathbb{P}((x_1, h_1 \cdots x_n, h_n)_t) \leq \{y, s\}) = \\ &\det(I - S^{l_y, r_y}(I - W_0 \overline{1}^t W_{1,2} \overline{1}^t W_{2,3} \cdots \overline{1}^t W_{n-1,n} \overline{1}^t W_{n+1}) S^{r_y, l_y})_{L^2([0,\infty))},\end{aligned} \tag{2.14}$$

where $l_y = (x_{\text{prim}} + h_{\text{prim}} - y - s)/2, r_y = (h_{\text{prim}} - s - x_{\text{prim}} + y)/2$*, and* W *is defined in* (2.8)*, parameterized by* $(x_1, h_1; \cdots ; x_n, h_n)$.

Proof. Since

$$\mathbb{P}((x_1, h_1 \cdots x_n, h_n)_t) \leq (y, s)) = \mathbb{P}((x_1 - y, h_1 - s \cdots x_n - y, h_n - s)_t) \leq (0, 0)),$$

we plug the new parameter into Theorem (2.2.1); notice that $W_{i,j}$ only records the relative position and relative height between peaks, so they remain unchanged. □

2.2.1 Properties of the kernel

As discussed in Remark (2.2.2), the kernel in (2.12) actually given by

$$\lim_{n \to \infty} S^{l,r}_{\phi_n} W S^{r,l}_{\phi_n} \tag{2.15}$$

where $S^{l,r}_{\phi_n}$ is the operator with kernel $s^{l,r}(x, y) \cdot \phi_n(x + y)$. The main point of this section is to show two transformations on the kernel, which are useful to deduce that the kernel (2.15) is well-defined and trace-class from $L^2([0,\infty)) \to L^2([0,\infty))$.

Before we begin to prove the theorem, we establish certain properties about the kernel. There are a few simple facts about the kernel. Recall

$$\begin{aligned}
&W = I - W_0\overline{1}^t W_{1,2}\overline{1}^t W_{2,3}\cdots\overline{1}^t W_{n-1,n}\overline{1}^t W_{n+1}.\\
&W_{i,i+1} = \mathtt{a}^{-u_i}\mathtt{a}_*^{-d_i}, \qquad W_0 = \mathtt{a}^{u}, \qquad W_{n+1} = \mathtt{a}_*^{d}\\
&s^{i,j}(x,y) = \int_\Gamma e^{-(x+y)w}\frac{(1+2w)^j}{(1-2w)^i}\\
&S^{i,j}(x,y) = s^{i,j}(x,y)1_{x+y\geq 0}.
\end{aligned} \tag{2.16}$$

1. There are the same numbers of $\mathtt{a}_*$ and $\mathtt{a}_*^{-1}$; $\mathtt{a}$ and $\mathtt{a}^{-1}$ in W, and they all commute. Thus, if all indicator functions $\overline{1}^t$ are not present, $W = 0$.
2. All differential operators ($\mathtt{a}, \mathtt{a}_*$ with positive powers) are present in W_0 and W_{n+1}. All integral operators ($\mathtt{a}, \mathtt{a}_*$ with negative powers) are in $W_{i,i+1}$.
3. $S^{i,j} = 0$ if $i \leq 0$, because the integrand is analytic in w and contains no poles inside the contour.
4. $S^{i,j}(x,y)$ is an operator that depends only on $x+y$. Thus, we have

$$DS^{i,j} = -S^{i,j}D.$$

where the equality is taken in the distributional sense. We define some notions for sequences of W operators for the convenience of the discussion.

$$\begin{aligned}
&W_{i,j} = W_{i,i+1}\overline{1}^t W_{i+1,i+2}\cdots W_{j-2,j-1}\overline{1}^t W_{j-1,j}, \quad 0 < i+1 < j < n+1;\\
&W_{0,i} = W_0\overline{1}^t W_{1,2}\cdots W_{i-1,i}, \quad i < n;\\
&W_{j,n+1} = W_{j,j+1}\overline{1}^t\cdots W_{n-1,n}\overline{1}^t W_{n+1}, 0 < j.
\end{aligned} \tag{2.17}$$

For $\mathtt{a}^{-1}, \mathtt{a}_*^{-1}$, we have the following simple but useful lemma:

Lemma 2.2.5. As an operator from $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, for $n \in \mathbb{Z}^+$,

$$\begin{aligned}
\overline{1}^t\mathtt{a}^{-n}\overline{1}^t = \mathtt{a}^{-n}\overline{1}^t, \quad \overline{1}^t\mathtt{a}_*^{-n}\overline{1}^t = \overline{1}^t\mathtt{a}_*^{-n},\\
\overline{1}^t\mathtt{a}^{-n}\left|\delta_t\right\rangle = \mathtt{a}^{-n}\left|\delta_t\right\rangle, \quad \left\langle\delta_t\right|\mathtt{a}_*^{-n}\overline{1}^t = \left\langle\delta_t\right|\mathtt{a}_*^{-n}.
\end{aligned} \tag{2.18}$$

Proof. From the integral kernel representation for $\mathtt{a}^{-1}$ in (2.10), $\mathtt{a}^{-1}(x,y) = \frac{1}{2}e^{(x-y)/2}1_{x\leq y}$. Thus for each $n \in \mathbb{Z}^+$, $\mathtt{a}^{-n}$ is also an integral operators, and its kernel $\mathtt{a}^{-n}(x,y)$ is supported on $x \leq y$, i.e. $\mathtt{a}^{-n}(x,y) = f(x,y)\cdot 1_{x\leq y}$, where $f(x,y)$ is a smooth function. Similarly, $\mathtt{a}_*^{-n}(x,y)$ is supported on $x \geq y$. Thus, (2.18) followed by the support of the operators. □

For differential operators $\mathtt{a}, \mathtt{a}_*$, they are local operators in the sense that their support is at single point as a distribution. Formally, that is equivalent of saying $\delta(x-y)$ is supported on $|x-y| < \varepsilon$ for arbitrary small ε. Thus, we have the following relations.

Lemma 2.2.6. For any $t_1 \neq t_2$, let $\phi_n(x)$ be the smooth approximation to the indicator function defined in (2.2.2). The following objects:

$$\langle \delta_{t_1} | \, \mathtt{a} 1_{t_2}, \quad \langle \delta_{t_1} | \, \mathtt{a}_* 1_{t_2} \tag{2.19}$$

make sense as an functional on differentiable functions, defined by:

$$\langle \delta_{t_1} | \, \mathtt{a} 1_{t_2} := \lim_{n\to\infty} \langle \delta_{t_1} | \, \mathtt{a} \phi_n^{t_2}, \quad \langle \delta_{t_1} | \, \mathtt{a}_* 1_{t_2} := \lim_{n\to\infty} \langle \delta_{t_1} | \, \mathtt{a}_* \phi_n^{t_2}. \tag{2.20}$$

For $t_1 < t_2$, we have

$$\langle \delta_{t_1} | \, \mathtt{a} 1_{t_2} = 0, \quad \langle \delta_{t_1} | \, \mathtt{a}_* 1_{t_2} = 0.$$

Proof. Let f be a smooth function.

$$\langle \delta_{t_1} | \, \mathtt{a} 1_{t_2} f = \lim_{n\to\infty} \langle \delta_{t_1} | \, \mathtt{a}(\phi_n^{t_2} f) = \begin{cases} (\mathtt{a} f)(t_1) \text{ if } t_2 < t_1, \\ 0 \text{ if } t_2 > t_1. \end{cases}$$

If $t_2 < t_1$, the Dirac delta function δ_{t_1} does not see the jump in $(\phi_n^{t_2} f)$. If $t_2 > t_1$, $(\phi_n^{t_2} f)$ will be 0 for large enough n. □

Remark 2.2.7. *By our definition of $\phi_n(x)$, $\phi_n'(x)$ has compact support. Thus, the limit in* (2.20) *is eventually constant. Thus, it is a formalism to define 'how to differentiate the indicator function when essentially you do not care about the jump', rather than an approximation.*

Throughout the paper, when there is a distribution acting on a cut-off of a smooth function, it is interpreted as the limit of the approximated sequence. Notice that for differential operators, we only use the property (2.2.6) when the two endpoints t_1 and t_2 are separated; thus, the fact is independent of how the smooth functions ϕ_n are chosen. We never encounter $1_t \mathtt{a}_* \, |\delta_t\rangle \, , 1_t \mathtt{a} \, |\delta_t\rangle$. Lemma (2.2.5) and Lemma (2.2.6) are the two key facts that we will use frequently. Using these two facts, we also have numerous facts of the same type:

Lemma 2.2.8. *(Support of the operator) Assume* $t_1 < t_2$*,* $k, u, d > 0$*, we have*

i. $\overline{1}^{t_1}\mathtt{a}_*^{-k}\mathtt{a}^u\overline{1}^{t_2} = \overline{1}^{t_1}\mathtt{a}_*^{-k}\mathtt{a}^u, \quad \langle\delta_{t_1}|\,\mathtt{a}_*^{-k}\mathtt{a}^u\overline{1}^{t_2} = \langle\delta_{t_1}|\,\mathtt{a}_*^{-k}\mathtt{a}^u$

ii. $\overline{1}^{t_2}\mathtt{a}_*^d\overline{1}^{t_1} = \mathtt{a}_*^d\overline{1}^{t_1}, \quad \overline{1}^{t_2}\mathtt{a}_*^d\,|\delta_{t_1}\rangle = \mathtt{a}_*^d\,|\delta_{t_1}\rangle$

iii. $\overline{1}^{t_1}\mathtt{a}_*^{-k}\mathtt{a}^u 1_{t_2} = 0, \quad \langle\delta_{t_1}|\,\mathtt{a}_*^{-k}\mathtt{a}^u 1_{t_2} = 0$

iv. $1_{t_2}\mathtt{a}_*^d\overline{1}^{t_1} = 0, \quad 1_{t_2}\mathtt{a}_*^d\,|\delta_{t_1}\rangle = 0$

There are numerous properties for the operators involving $\mathtt{a}^{-1}(x, y)$, which is supported on $x \le y$. Lemma (2.2.8) is used either for

- Drop an indicator on one side when it is inherited from the operator itself; or
- Reduce the operator to zero when the restriction contradicts the operator's support.

Whenever we use such a fact, we say it is 'because of the support of the operator' rather than referring to this lemma.

Proof. The proof is the same for all of them, which is about the domain of the operators. If you have an operator $A(x, y)$ that is supported on $x \ge y$, if $x \le t$, then the variable y is also supported on $y \le t$. Notice that $\mathtt{a}_*^{-1}(x, y)$ is supported on $x \ge y$; $\mathtt{a}(x, y), \mathtt{a}_*(x, y)$ is supported on $|x - y| < \varepsilon$ for some arbitrarily small ε, because differential operators are local operators. All the above statements follow by observing the domain of the operator. □

2.2.2 Switching differential operators with indicator functions

The manipulation of the kernel involves switching the order of the differential operators in W_0 and W_{n+1} and the indicator functions, which brings commutators. There are two types of movement. The operators in W_0, W_{n+1} can switch with $\overline{1}^t$ and cancel with the terms in $\overline{1}^t W_{1,2}\overline{1}^t \cdots \overline{1}^t W_{n-1,n}\overline{1}^t$; the operators in W_0, W_{n+1} can act on the operator S. In this subsection, we discuss the first type. The commutators for switching $\mathtt{a}$ and $\overline{1}^t$ are

$$[\mathtt{a}, \overline{1}^t] = 2\,|\delta_t\rangle\,\langle\delta_t|\,, \quad [\overline{1}^t, \mathtt{a}_*] = 2\,|\delta_t\rangle\,\langle\delta_t|\,,$$
$$[\mathtt{a}^{-1}, \overline{1}^t] = 2\mathtt{a}^{-1}\,|\delta_t\rangle\,\langle\delta_t|\,\mathtt{a}^{-1}, \quad [\overline{1}^t, \mathtt{a}_*^{-1}] = 2\mathtt{a}_*^{-1}\,|\delta_t\rangle\,\langle\delta_t|\,\mathtt{a}_*^{-1}.$$

We start with two concrete examples to illustrate what will be proven in this section. The idea is really simple; however, the notation is complex.

Example 2.2.9. We take the kernel in (2.12) with $l = 3, r = 3, u = 2, d = 2, u_1 = u_2 = 1, d_1 = d_2 = 1$.

$$S^{3,3}WS^{3,3} = S^{3,3}(I - \mathtt{a}^2\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_*^2)S^{3,3} \tag{2.21}$$

We start by switching $\mathtt{a}, \mathtt{a}_*$ with the indicator function $\overline{1}^t$, which will generate $|\delta_t\rangle\langle\delta_t|$ terms.

$$\begin{aligned}(2.21) &= S^{3,3}(I - \mathtt{a}\overline{1}^t\mathtt{a}\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_*^2 - 2\mathtt{a}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_*^2)S^{3,3}\\ &= S^{3,3}(I - \mathtt{a}\overline{1}^t\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_*^2 - 2\mathtt{a}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_*^2)S^{3,3}\\ &= S^{3,3}(I - \mathtt{a}\overline{1}^t\mathtt{a}_*^{-1}\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_*^2 - 2\mathtt{a}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_*^2)S^{3,3}.\end{aligned} \tag{2.22}$$

The third equality is by the support of $\mathtt{a}_*^{-1}$. Now move the first $\mathtt{a}$ again, we have

$$\begin{aligned}(2.22) = S^{3,3}\Big(&I - \overline{1}^t\mathtt{a}_*^{-2}\overline{1}^t\mathtt{a}_*^2 - 2\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-2}\overline{1}^t\mathtt{a}_*^2\\ &- 2\mathtt{a}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_*^2\Big)S^{3,3}.\end{aligned} \tag{2.23}$$

The term $I - \overline{1}^t\mathtt{a}_*^{-2}\overline{1}^t\mathtt{a}_*^2 = I - \overline{1}^t\mathtt{a}_*^{-2}\mathtt{a}_*^2 = 1_t$. Now we examine the last two terms with $|\delta_t\rangle\langle\delta_t|$, we move $\mathtt{a}_*^*$ across $\overline{1}^t$:

$$\begin{aligned}&S^{3,3}\Big(-2\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-2}\overline{1}^t\mathtt{a}_*^2 - 2\mathtt{a}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_*^2\Big)S^{3,3}\\ &= S^{3,3}\Big(-2\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_* - 4\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-2}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}_*\\ &- 2\mathtt{a}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\overline{1}^t\mathtt{a}_* - 4\mathtt{a}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}_*\Big)S^{3,3}.\end{aligned} \tag{2.24}$$

The third term can be simplified to $-2\mathtt{a}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-2}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_*$ due to the support of $\mathtt{a}^{-1}$.

And we move the $\mathtt{a}_*$ in the first and third term across $\overline{1}^t$ again,

$$\begin{aligned}(2.24) = S^{3,3}\Big(& -2\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\overline{1}^t - 4\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-1}\,|\delta_t\rangle\langle\delta_t| \\ & -4\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-2}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}_* - 2\mathtt{a}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-2}\overline{1}^t \\ & -4\mathtt{a}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-2}\mathtt{a}_*^{-1}\,|\delta_t\rangle\langle\delta_t| - 4\mathtt{a}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}_*\Big)S^{3,3}.\end{aligned} \tag{2.25}$$

Now notice the fact that the terms with only one $|\delta_t\rangle\langle\delta_t|$ term, i.e. the first and fourth terms are 0, because of the support of $\mathtt{a}^{-1}$ (Lemma (2.2.8)). All the terms left are rank one operators. This example illustrates that all W can be written as 1_t plus finite rank terms.

Now we make it rigorous in the following proposition. We define the object $W_{1,n}^{i,j}, i \le u, j \le d$ to simplify the notation. In words, this is the notation when all i numbers of $\mathtt{a}$ and j numbers of $\mathtt{a}_*$ on two sides of $W_{1,n}$ go through all the indicator functions. To define it rigorously, let p be the smallest natural number such that $i \le u_1 + u_2 + \cdots u_p$, and let q be the largest natural number such that $j \le d_q + d_{q+1} + \cdots d_n$, then

If $p+1 < q-1$,

$$W_{1,n}^{i,j} = \mathtt{a}^{-(u_1+\cdots u_p)+i}\mathtt{a}_*^{-(u_1+\cdots u_p)}\overline{1}^t W_{p+1,q-1}\overline{1}^t\mathtt{a}^{-(d_q+d_{q+1}+\cdots d_n)}\mathtt{a}_*^{-(d_q+d_{q+1}+\cdots d_n)+j}. \tag{2.26}$$

If $p = q-1$,

$$W_{1,n}^{i,j} = \mathtt{a}^{-(u_1+\cdots u_p)+i}\mathtt{a}_*^{-(u_1+\cdots u_p)}\overline{1}^t\mathtt{a}^{-(d_q+d_{q+1}+\cdots d_n)}\mathtt{a}_*^{-(d_q+d_{q+1}+\cdots d_n)+j}.$$

If $p > q-1$,

$$W_{1,n}^{i,j} = \mathtt{a}^{-u+i}\mathtt{a}_*^{-d+j}.$$

Here are some examples to help understand. Let $W_{1,3} = \mathtt{a}^{-2}\mathtt{a}_*^{-2}\overline{1}^t\mathtt{a}^{-2}\mathtt{a}_*^{-2}\overline{1}^t\mathtt{a}^{-2}\mathtt{a}_*^{-2}$.

$W_{1,3}^{1,2} = \mathtt{a}^{-1}\mathtt{a}_*^{-2}\overline{1}^t\mathtt{a}^{-2}\mathtt{a}_*^{-2}\overline{1}^t\mathtt{a}^{-2}$(one $\mathtt{a}$ in the front cancels, two $\mathtt{a}_*$ in the end cancels)
$W_{1,3}^{2,3} = \mathtt{a}_*^{-2}\overline{1}^t\mathtt{a}^{-4}\mathtt{a}_*^{-1}$(two $\mathtt{a}$ in the front cancels, three $\mathtt{a}_*$ in the end cancels)
$W_{1,3}^{3,3} = \mathtt{a}^{-3}\mathtt{a}_*^{-3}$(three $\mathtt{a}$ in the front cancels, three $\mathtt{a}_*$ in the end cancels)

The only important fact about $W_{1,n}^{i,j}$ is that they are all integral operators from $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Proposition 2.2.10. For $n > 1$,

$$W = 1_t + \sum_{i=1}^{u}\sum_{j=1}^{d} 4\mathtt{a}^{u-i} |\delta_t\rangle \langle\delta_t| W_{1,n}^{i-1,j-1} |\delta_t\rangle \langle\delta_t| \mathtt{a}_*^{d-j}. \tag{2.27}$$

they are equal as an operator from $\mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$.

Proof. We want to bring all the differential operators across the indicator functions. Using the relation that

$$[\mathtt{a}, \overline{1}^t] = 2|\delta_t\rangle\langle\delta_t|, \quad [\overline{1}^t, \mathtt{a}_*] = 2|\delta_t\rangle\langle\delta_t|.$$

We have

$$W = I - 2\sum_{i=1}^{u} \mathtt{a}^{u-i} |\delta_t\rangle \langle\delta_t| W_{1,n}^{i-1,0}\overline{1}^t \mathtt{a}_*^d - \overline{1}^t W_{1,n}^{u,0}\overline{1}^t \mathtt{a}_*^d \tag{2.28}$$

The third term is what one gets after switching W_0 and $\overline{1}^t$, and the second term is where all the commutators appear during the switching.

Now we switch all the $\mathtt{a}_*^d$ to the left of the $\overline{1}^t$,

$$\begin{aligned}(2.28) = I &- \sum_{i=1}^{u}\sum_{j=1}^{d} 4\mathtt{a}^{u-i} |\delta_t\rangle \langle\delta_t| W_{1,n}^{i-1,j-1} |\delta_t\rangle \langle\delta_t| \mathtt{a}_*^{d-j} - 2\sum_{i=1}^{u} \mathtt{a}^{u-i} |\delta_t\rangle \langle\delta_t| W_{1,n}^{i-1,d}\overline{1}^t \\ &- 2\sum_{j=1}^{d} \overline{1}^t W_{1,n}^{u,j-1} |\delta_t\rangle \langle\delta_t| \mathtt{a}_*^{d-j} - \overline{1}^t W_{1,n}^{u,d}\overline{1}^t =: \textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4} + \textcircled{5}\end{aligned} \tag{2.29}$$

Notice that ② is the second term we want in (2.27). ① + ⑤ is 1_t, since $W_{1,n}^{u,d}$ is the term that all the $\mathtt{a}^{-1}, \mathtt{a}_*^{-1}$ in $W_{1,n}$ are canceled; what is left is $\overline{1}^t$. Lastly, we want to show ③, ④ are 0. Notice d is the total number of $\mathtt{a}_*^{-1}$ in $W_{1,n}$, thus by (2.26) $W_{1,n}^{i-1,d} = \mathtt{a}^{u-i+1}$, thus each term in the summation ③ is $2\mathtt{a}^{u-i}|\delta_t\rangle\langle\delta_t|\mathtt{a}^{u-i+1}\overline{1}^t$, which is 0 due to the support of $\mathtt{a}$ (Lemma (2.2.8)). Similarly, in ④, $\overline{1}^t W_{1,n}^{u,j-1}|\delta_t\rangle\langle\delta_t|\mathtt{a}_*^{d-j} = \overline{1}^t \mathtt{a}_*^{d-j+1}|\delta_t\rangle\langle\delta_t|\mathtt{a}_*^{d-j} = 0$. Thus, the statement is proved.

□

Corollary 2.2.11. Let $\tilde{S} = \{S^{i,j}f | f \in L^2[0,\infty)\}$, the equality

$$W = 1_t + \sum_{i=1}^{u}\sum_{j=1}^{d} 4\mathtt{a}^{u-i} |\delta_t\rangle \langle\delta_t| W_{1,n}^{i-1,j-1} |\delta_t\rangle \langle\delta_t| \mathtt{a}_*^{d-j} \tag{2.30}$$

is true in the sense of an operator from $\tilde{S} \to \tilde{S}'$. Here $\tilde{S}'$ means a linear functional on the space $\tilde{S}$. In particular, this shows that the kernel we have in (2.2.1) is well-defined.

Proof. Look at the summation term on the right-hand side. $\langle\delta_t| W_{1,n}^{i-1,j-1} |\delta_t\rangle$ is a scalar value; thus, the terms in the summation are in the form of $c_{ij}\mathtt{a}^{u-i} |\delta_t\rangle \langle\delta_t| \mathtt{a}_*^{d-j}$, which are rank-one operators.

Let $f \in L^2[0,\infty)$, $S^{i,j}f = \lim_{n\to\infty} S^{i,j}_{\phi_n}$, where ϕ_n is the smooth approximation we define in (2.2.2).

$$\langle\delta_t| \mathtt{a}_*^{d-j} S^{i,j} f = \lim_{n\to\infty} \langle\delta_t| \mathtt{a}_*^{d-j} S^{i,j}_{\phi_n} f \tag{2.31}$$

Recall $S^{i,j}_{\phi_n} = \int_0^\infty dy S^{i,j}(x,y)\phi_n(x+y)f(y)$, which is the smooth approximation to

$$1_{x\geq 0}\int_0^\infty S^{i,j}(x,y)f(y)dy + 1_{x<0}\int_{-x}^\infty S^{i,j}(x,y)f(y)dy$$

Same as the discussion in Lemma (2.18), since $t > 0$

$$\lim_{n\to\infty} \langle\delta_t| \mathtt{a}_*^{d-j} S^{i,j}_{\phi_n} f = \left(\mathtt{a}_*^{d-j}\int_0^\infty S^{i,j}(x,y)f(y)dy\right)\Bigg|_{x=t} \tag{2.32}$$

Notice that the limit is eventually constant due to the compact support of $\phi_n'(x)$. Similarly, $1_t S^{i,j} f = \lim_{n\to\infty} 1_t S^{i,j}_{\phi_n} f$, thus the equality in (2.27) makes sense as an operator on $\tilde{S}$. □

Proposition 2.2.12. *The kernel $S^{l,r}WS^{r,l}$ in* (2.12) *is a trace-class operator.*

Proof. When $n > 1$, by Prop (2.2.10), we have

$$W = 1_t + \sum_{i=1}^{u}\sum_{j=1}^{d} 4\mathtt{a}^{u-i} |\delta_t\rangle \langle\delta_t| W_{1,n}^{i-1,j-1} |\delta_t\rangle \langle\delta_t| \mathtt{a}_*^{d-j} =: \tilde{W}. \tag{2.33}$$

Now we calculate what is $s^{n,m}$

$$\begin{aligned} s^{n,m}(x,y) &= \mathrm{Res}(e^{-(x+y)w}\frac{(1+2w)^m}{(1-2w)^n}, 1/2) \\ &= (-1)^n \frac{2^{m-n}}{(n-1)!}\sum_{i=0}^{n-1\wedge m}\binom{n-1}{i}\frac{m!}{(m-i)!}(-x-y)^{n-1-i}e^{-\frac{1}{2}(x+y)}. \end{aligned} \tag{2.34}$$

which has exponential decay at $+\infty$, thus $\left\|S^{r,l}(x,y)1_{x\geq t}1_{y\geq 0}\right\|_2 < \infty$ and $\left\|S^{l,r}(x,y)1_{x\geq 0}1_{y\geq t}\right\|_2 < \infty$, thus both $1_0 S^{l,r}1_t$ and $1_t S^{r-s,l-y}1_0$ are Hilbert-Schmidt operators. Thus, $1_0 S^{l,r} 1_t S^{r,l} 1_0$ is a trace-class operator. For all i,j,

$$1_0 S^{l,r} 4\mathtt{a}^{u-i}\left|\delta_t\right\rangle\left\langle\delta_t\right| W_{1,n}^{i-1,j-1}\left|\delta_t\right\rangle\left\langle\delta_t\right| \mathtt{a}_*^{d-j} S^{r,l} 1_0$$

are all rank-one operators. Thus, $S^{l,r}WS^{r,l}$ is a finite rank perturbation of a trace-class operator on $L^2([0,\infty))$, which is trace-class.

□

2.2.3 Differential operators acting on S operator

The differential operators in W_0, W_{n+1} can also act on the operator $S^{n,m}$. Due to the indicator function in $S^{n,m}$, it will also generate delta functions when acted upon by a differential operator.

Lemma 2.2.13. *Let $B^{n,m}(x,y) = 2s^{n,m}(x,y)\cdot\delta_0(x+y)$. For $n,m>0$*

$$\mathtt{a}_* S^{n,m} = S^{n,m}\mathtt{a} = S^{n-1,m} + B^{n,m}. \tag{2.35}$$

Proof. Recall

$$\begin{aligned} s^{n,m}(x,y) &= \int_\Gamma e^{-(x+y)w}\frac{(1+2w)^m}{(1-2w)^n}\frac{dw}{2\pi i}, \\ S^{n,m}(x,y) &= s^{n,m}(x,y)1_{x+y\geq 0}. \end{aligned}$$

$B^{n,m}$ comes when the operator hits the indicator function. When the differential operator hits the contour integral part, we can bring the operator into the integral, and the relation follows.

□

Lemma 2.2.14. For $0<k<n$,

$$\mathtt{a}_*^{-k}(\mathtt{a}_*^k S^{n,m}) = S^{n,m}, \quad (S^{n,m}\mathtt{a}^k)\mathtt{a}^{-k} = S^{n,m} \tag{2.36}$$

Proof. We will prove the first relation, and it suffices to prove the formula when $k = 1$. From the previous lemma, we have $\mathtt{a}_* S^{n,m} = S^{n-1,m} + B^{n,m}$. So

$$\begin{aligned}
&\mathtt{a}_*^{-1}(\mathtt{a}_* S^{n,m}) = \int_{-z}^{x} dy \frac{1}{2} e^{-x/2+y/2} \cdot \int_\Gamma e^{-(y+z)w} \frac{(1+2w)^m}{(1-2w)^{n-1}} \frac{dw}{2\pi i} \\
&+ 2\int_{-\infty}^{x} dy \frac{1}{2} e^{-x/2+y/2} \cdot \delta_0(y+z) \int_\Gamma e^{-(y+z)w} \frac{(1+2w)^m}{(1-2w)^n} \frac{dw}{2\pi i} \\
&= \int_\Gamma e^{-(x+z)w} \frac{(1+2w)^m}{(1-2w)^n} \frac{dw}{2\pi i} - 1_{x+z\geq 0} e^{-x/2-z/2} \int_\Gamma e^{-((-z)+z)w} \frac{(1+2w)^m}{(1-2w)^n} \frac{dw}{2\pi i} \\
&+ 1_{x+z\geq 0} e^{-x/2-z/2} \int_\Gamma e^{-((-z)+z)w} \frac{(1+2w)^m}{(1-2w)^n} \frac{dw}{2\pi i}.
\end{aligned} \tag{2.37}$$

The second term cancels exactly the third term, what is left is $S^{n,m}$.

□

Now we want to show another manipulation on the kernel. The differential operators in W_0, W_{n+1} can hit S, which will generate a delta function. Once there is a delta function on S, the rest of the differential operators can get through the indicator functions. In precise language, let f be a differentiable function,

$$\begin{aligned}
f\overline{1}^t \mathtt{a}_*(S^{i,j}(x,y) \cdot \delta(x+y))1_0 &= \lim_{n\to\infty} f\overline{1}^t \mathtt{a}_*(S^{i,j}(x,y) \cdot \phi_n'(x+y))1_0 \\
&= \lim_{n\to\infty} f\mathtt{a}_*(S^{i,j}(x,y) \cdot \phi_n'(x+y))1_0 \\
&= (f\mathtt{a}_*)(-y)S^{i,j}(-y,y), \quad y > 0.
\end{aligned} \tag{2.38}$$

The second equality is due to the fact that $y > 0$ and ϕ_n' have compact support. The limit is also eventually constant.

We will also give an explicit example.

Example 2.2.15. We take the kernel in (2.12) $l = 3, r = 3, u = 2, d = 2, u_1 = u_2 = 1, d_1 = d_2 = 1$.

$$\begin{aligned}
1_0 S^{3,3} W S^{3,3} 1_0 &= 1_0 S^{3,3}(I - \mathtt{a}^2 \overline{1}^t \mathtt{a}^{-1} \mathtt{a}_*^{-1} \overline{1}^t \mathtt{a}^{-1} \mathtt{a}_*^{-1} \overline{1}^t \mathtt{a}_*^2) S^{3,3} 1_0 \\
&= 1_0 S^{3,3}(\mathtt{a}_*^{-1} - \mathtt{a}^2 \overline{1}^t \mathtt{a}^{-1} \mathtt{a}_*^{-1} \overline{1}^t \mathtt{a}^{-1} \mathtt{a}_*^{-1} \overline{1}^t \mathtt{a}_*) \mathtt{a}_* S^{3,3} 1_0
\end{aligned}$$

We add the indicator 1_0 at the beginning and end to emphasize that the kernel is from $L^2([0,\infty)) \to L^2([0,\infty))$. By Lemma (2.2.14), we have

$$\begin{aligned}
&1_0 S^{3,3}(\mathtt{a}_*^{-1} - \mathtt{a}^2 \overline{1}^t \mathtt{a}^{-1} \mathtt{a}_*^{-1} \overline{1}^t \mathtt{a}^{-1} \mathtt{a}_*^{-1} \overline{1}^t \mathtt{a}_*) \mathtt{a}_* S^{3,3} 1_0 \\
&= 1_0 S^{3,3}(\mathtt{a}_*^{-1} - \mathtt{a}^2 \overline{1}^t \mathtt{a}^{-1} \mathtt{a}_*^{-1} \overline{1}^t \mathtt{a}^{-1} \mathtt{a}_*^{-1} \overline{1}^t \mathtt{a}_*)(S^{2,3} + B^{3,3}) 1_0
\end{aligned} \tag{2.39}$$

Now $\mathtt{a}_* B^{3,3}$ can act nicely on the left-hand side, since $\mathtt{a}_* s^{n,m}(x,y)\delta_0(x+y)$ is a distribution supported at $-y$, and since $y>0$, thus the indicator $\overline{1}^t, t>0$ won't affect. So the term in (2.39) with B is

$$\begin{aligned}1_0 S^{3,3}(\mathtt{a}_*^{-1} - \mathtt{a}^2\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\mathtt{a}_*)B^{3,3}1_0 &= 1_0 S^{3,3}(\mathtt{a}_*^{-1} - \mathtt{a}^2\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1})B^{3,3}1_0 \\ &= 1_0 S^{3,3}(\mathtt{a}_*^{-1} - \mathtt{a}^2\overline{1}^t\mathtt{a}^{-2}\mathtt{a}_*^{-1})B^{3,3}1_0.\end{aligned} \tag{2.40}$$

The third equality is due to the support of $\mathtt{a}^{-1}$. Now we let $\mathtt{a}$ act on the left $S^{3,3}$,

$$\begin{aligned}(2.40) &= 1_0(S^{2,3}+B^{3,3})(\mathtt{a}^{-1}\mathtt{a}_*^{-1} - \mathtt{a}\overline{1}^t\mathtt{a}^{-2}\mathtt{a}_*^{-1})B^{3,3}1_0 \\ &= 1_0 S^{2,3}(\mathtt{a}^{-1}\mathtt{a}_*^{-1} - \mathtt{a}\overline{1}^t\mathtt{a}^{-2}\mathtt{a}_*^{-1})B^{3,3}1_0 + 1_0 B^{3,3}(\mathtt{a}^{-1}\mathtt{a}_*^{-1} - \mathtt{a}^{-1}\mathtt{a}_*^{-1})B^{3,3}1_0.\end{aligned} \tag{2.41}$$

The second equality is due to the support of $\mathtt{a}^{-1}$. Continuing to do this,

$$\begin{aligned}(2.41) &= 1_0 S^{1,3}(\mathtt{a}^{-2}\mathtt{a}_*^{-1} - \overline{1}^t\mathtt{a}^{-2}\mathtt{a}_*^{-1})B^{3,3}1_0 + 1_0 B^{2,3}(\mathtt{a}^{-2}\mathtt{a}_*^{-1} - \overline{1}^t\mathtt{a}^{-2}\mathtt{a}_*^{-1})B^{3,3}1_0 \\ &= 1_0 S^{1,3}1_t\mathtt{a}^{-2}\mathtt{a}_*^{-1}B^{3,3}1_0.\end{aligned}$$

There is one term remaining in (2.39) with $S^{2,3}$ we need to consider,

$$\begin{aligned}&1_0 S^{3,3}(\mathtt{a}_*^{-1} - \mathtt{a}^2\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_*)S^{2,3}1_0 \\ &\quad = 1_0 S^{3,3}(\mathtt{a}_*^{-2} - \mathtt{a}^2\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t)(S^{1,3}+B^{2,3})1_0.\end{aligned} \tag{2.42}$$

Then move $\mathtt{a}$,

$$\begin{aligned}(2.42) &= 1_0 S^{2,3}(\mathtt{a}^{-1}\mathtt{a}_*^{-2} - \mathtt{a}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t)S^{1,3}1_0 \\ &+ 1_0 B^{3,3}(\mathtt{a}^{-1}\mathtt{a}_*^{-2} - \mathtt{a}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t)S^{1,3}1_0 \\ &+ 1_0 S^{2,3}(\mathtt{a}^{-1}\mathtt{a}_*^{-2} - \mathtt{a}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t)B^{2,3}1_0 \\ &+ 1_0 B^{3,3}(\mathtt{a}_*^{-2} - \mathtt{a}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t)B^{2,3}1_0 \\ &= 1_0 S^{1,3}(\mathtt{a}^{-2}\mathtt{a}_*^{-2} - \overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t)S^{1,3}1_0 \\ &+ 1_0 B^{2,3}(\mathtt{a}^{-2}\mathtt{a}_*^{-2} - \overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t)S^{1,3}1_0 \\ &+ 1_0 B^{3,3}(\mathtt{a}^{-1}\mathtt{a}_*^{-2} - \mathtt{a}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t)S^{1,3}1_0 \\ &+ 1_0 S^{1,3}(\mathtt{a}^{-2}\mathtt{a}_*^{-2} - \overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t)B^{2,3}1_0 \\ &+ 1_0 B^{2,3}(\mathtt{a}^{-2}\mathtt{a}_*^{-2} - \overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t)B^{2,3}1_0 \\ &+ 1_0 B^{3,3}(\mathtt{a}_*^{-2} - \mathtt{a}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t)B^{2,3}1_0\end{aligned}$$

The main point of this example is that the differential operator W_0, W_{n+1} can be moved to act on S and the structure of W is maintained. The term without any Dirac delta function, which is

$$1_0 S^{1,3}(\mathtt{a}^{-2}\mathtt{a}_*^{-2} - \overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t)S^{1,3}1_0 \tag{2.43}$$

is important and has good probabilistic meaning. We will illustrate that further later. Now we state the previous example as a proposition:

Proposition 2.2.16. Let

$$\tilde{S} = \{S^{i,j}f \mid f \in L^2([0,\infty))\}. \tag{2.44}$$

Operator W is well-defined as an operator from $\tilde{S} \to \tilde{S}'$. For any $f, g \in L^2([0,\infty))$,

$$\begin{aligned} gS^{l,r}WS^{r,l}f = {} & gS^{l-u,r}(\mathtt{a}^{-u}\mathtt{a}_*^{-d} - \overline{1}^tW_{1,n}\overline{1}^t)S^{r-d,l}f \\ & + \sum_{j=1}^{d} gS^{l-u,r}(\mathtt{a}^{-u}\mathtt{a}_*^{-j} - \overline{1}^tW_{1,n}^{0,d-j})B^{r-j+1,l}f) \\ & + \sum_{i=1}^{u}\sum_{j=1}^{d} gB^{l-i+1,r}(\mathtt{a}^{-i}\mathtt{a}_*^{-d} - W_{1,n}^{u-i,0}\overline{1}^t)S^{r-d,l}f \\ & + \sum_{i=1}^{u}\sum_{j=1}^{d} gB^{l-i+1,r}(\mathtt{a}^{-i}\mathtt{a}_*^{-j} - W_{1,n}^{u-i,d-j})B^{r-j+1,l}f. \end{aligned} \tag{2.45}$$

where $W_{1,n}$ is defined in (2.17), $W_{i,j}^{a,b}$ is defined in (2.26).

Remark 2.2.17. *The point is that all the terms in* (2.45) *are well-defined. The actual representation will not be used later.*

Proof. By the integral operator definition,

$$S^{i,j}f(x) = 1_{x\geq 0}\int_0^\infty S^{i,j}(x,y)f(y)dy + 1_{x<0}\int_{-x}^\infty S^{i,j}(x,y)f(y)dy. \tag{2.46}$$

We start from the left-hand side,

$$
\begin{aligned}
gS^{l,r}WS^{r,l}f &= gS^{l-u,r}(\mathtt{a}^{-u} - \overline{1}^t\tilde{W}_{1,n}\overline{1}^t W_{n+1})S^{r,l}f \\
&+ \sum_{i=1}^{u} gB^{l-i+1,r}(\mathtt{a}^{-i} - \mathtt{a}^{u-i}\overline{1}^t\tilde{W}_{1,n}\overline{1}^t W_{n+1})S^{r,l}f \\
&= gS^{l-u,r}(\mathtt{a}^{-u} - \overline{1}^t\tilde{W}_{1,n}\overline{1}^t W_{n+1})S^{r,l}f \\
&+ \sum_{i=1}^{u} gB^{l-i+1,r}(\mathtt{a}^{-i} - \tilde{W}_{1,n}^{u-i,0}\overline{1}^t W_{n+1})S^{r,l}f.
\end{aligned}
\tag{2.47}
$$

The second equality is because $gB^{l-i+1,r}$ is a distribution supported at 0. Now we move $\mathtt{a}_*^d$ in W_{n+1} on the right-hand side,

$$
\begin{aligned}
(2.47) &= gS^{l-u,r}(\mathtt{a}^{-u}\mathtt{a}_*^{-d} - \overline{1}^t W_{1,n}\overline{1}^t)S^{r-d,l}f \\
&+ \sum_{j=1}^{d} gS^{l-u,r}(\mathtt{a}^{-u}\mathtt{a}_*^{-j} - \overline{1}^t W_{1,n}\overline{1}^t\mathtt{a}_*^{d-j})B^{r-j+1,l}f \\
&+ \sum_{i=1}^{u}\sum_{j=1}^{d} gB^{l-i+1,r}(\mathtt{a}^{-i}\mathtt{a}_*^{-d} - W_{1,n}^{u-i,0}\overline{1}^t)S^{r-d,l}f \\
&+ \sum_{i=1}^{u}\sum_{j=1}^{d} gB^{l-i+1,r}(\mathtt{a}^{-i}\mathtt{a}_*^{-j} - W_{1,n}^{u-i,0}\overline{1}^t\mathtt{a}_*^{d-j})B^{r-j+1,l}f.
\end{aligned}
$$

The second term can be simplified to

$$
\sum_{j=1}^{d} gS^{l-u,r}(\mathtt{a}^{-u}\mathtt{a}_*^{-j} - \overline{1}^t W_{1,n}^{0,d-j})B^{r-j+1,l}f.
$$

The fourth term can be simplified to

$$
\sum_{i=1}^{u}\sum_{j=1}^{d} gB^{l-i+1,r}(\mathtt{a}^{-i}\mathtt{a}_*^{-j} - W_{1,n}^{u-i,d-j})B^{r-j+1,l}f.
$$

Thus, we derive the desired result. □

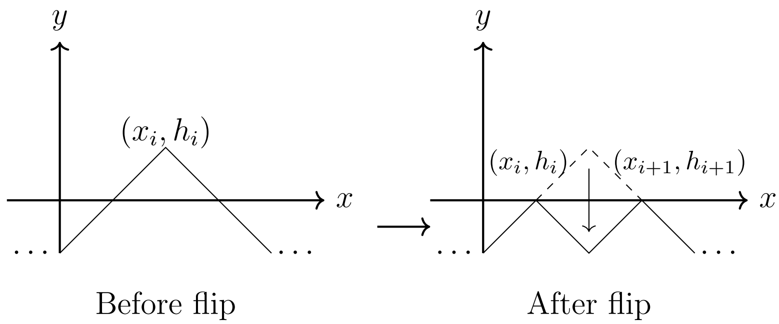
Now we are going to prove Theorem (2.2.1).

2.2.4 Kolmogorov equation

Lemma 2.2.18. *Let $W^{\downarrow x_i}$ be the kernel parameterized by the configuration that is obtained by flipping at x_i from $(x_1, h_1; \dots; x_n, h_n)$, then $W^{\downarrow x_i} - W = W_{0,i}\left|\delta_t\right\rangle\left\langle\delta_t\right| W_{i,n+1}$.*

Proof. Depending on the type of peak, the kernel changes in four ways. Notice that the type of change in peaks depends on whether there is more than 1 down step on the right-hand side of the peak and whether there is more than 1 up step on the left-hand side of the peak. Let us see what that means.

Type-1:(more than one down step on the right and more than one up step on the left)



Assume that the configuration before the flip has a kernel: $W_{0,i}\overline{1}^t W_{i,n+1}$. Then the configuration after the type-1 flip at x_i has a kernel:

$$W_{0,i}\mathtt{a}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_* W_{i,n+1}.$$

Taking the difference, we have

$$\begin{aligned}&W_{0,i}(\mathtt{a}\overline{1}^t\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_* - \overline{1}^t)W_{i,n+1}\\&= W_{0,i}\big(2\overline{1}^t\mathtt{a}_*^{-1}\left|\delta_t\right\rangle\left\langle\delta_t\right| + 2\left|\delta_t\right\rangle\left\langle\delta_t\right|\mathtt{a}^{-1}\overline{1}^t + 4\left|\delta_t\right\rangle\left\langle\delta_t\right|\mathtt{a}^{-1}\mathtt{a}_*^{-1}\left|\delta_t\right\rangle\left\langle\delta_t\right|\big)W_{i,n+1}.\end{aligned}$$

The three terms in the bracket are the commutator terms from switching $\mathtt{a}, \mathtt{a}_*$ with the indicators. Notice that $\mathtt{a}_*^{-1}$ and $\mathtt{a}^{-1}$ both have integral kernels.

$$\mathtt{a}^{-1}(x,y) = \frac{1}{2}e^{\frac{1}{2}(x-y)}1_{y\geq x}, \quad \mathtt{a}_*^{-1}(x,y) = \frac{1}{2}e^{\frac{1}{2}(y-x)}1_{x\geq y},$$

and

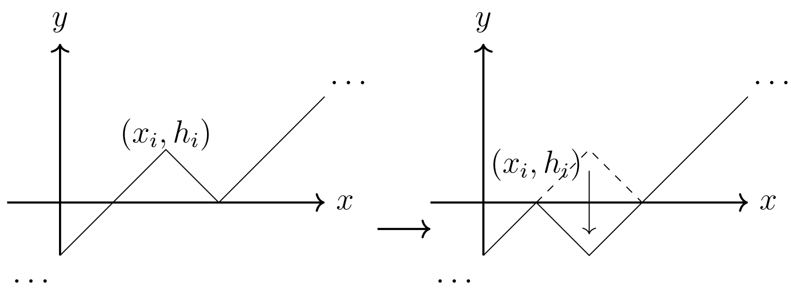
$$(\mathtt{a}_*^{-1}\mathtt{a}^{-1})(x,y) = \frac{1}{4}e^{-\frac{1}{2}|x-y|}. \tag{2.48}$$

Due to the support of the operator $\mathtt{a}_*, \mathtt{a}$ (Lemma (2.2.8)), we see $\overline{1}^t\mathtt{a}_*^{-1}\left|\delta_t\right\rangle = 0, \left\langle\delta_t\right|\mathtt{a}^{-1}\overline{1}^t = 0.$. Then what is left is

$$W_{0,i}\big(4\left|\delta_t\right\rangle\left\langle\delta_t\right|\mathtt{a}^{-1}\mathtt{a}_*^{-1}\left|\delta_t\right\rangle\left\langle\delta_t\right|\big)W_{i,n+1} = W_{0,i}\left|\delta_t\right\rangle\left\langle\delta_t\right|W_{i,n+1}.$$

The equality is due to the fact that the middle part of the kernel is a real number: $\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-1}\,|\delta_t\rangle$ means evaluating the integral kernel (2.48) at $x = t, y = t$, which is 1/4.

Type-2:(One down step on the right and more than one up step on the left)



This corresponds to

$$W_{0,i}\overline{1}^t W_{i,n+1} \to W_{0,i}\mathtt{a}\overline{1}^t\mathtt{a}^{-1}W_{i,n+1}.$$

We take the difference and get

$$W_{0,i}\big(\mathtt{a}\overline{1}^t\mathtt{a}^{-1} - \overline{1}^t\big)W_{i,n+1} = W_{0,i}\big(2\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\big)W_{i,n+1}. \tag{2.49}$$

$2\,|\delta_t\rangle\langle\delta_t|$ is the commutators from switching $\mathtt{a}$ with $\overline{1}^t$. We show the following special fact

$$2\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}W_{i,i+1}\overline{1}^t = |\delta_t\rangle\langle\delta_t|\,W_{i,i+1}\overline{1}^t. \tag{2.50}$$

Note that in the graph on the left, there is only one down step from peak x_i to x_{i+1}, which means $d_i = 1$, so $W_{i,i+1}\overline{1}^t$ can be written as

$$W_{i,i+1}\overline{1}^t = \mathtt{a}_*^{-1}\mathtt{a}^{-u_i}\overline{1}^t = \mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-u_i}\overline{1}^t.$$

The second equality is true since $\mathtt{a}^{-u_i}(x,y)$ is supported on $x \le y$, so $y \le t$ implies that the support of x must be in $x \le t$. Then take the difference of the terms in (2.50), we have

$$\begin{aligned} &2\,|\delta_t\rangle\langle\delta_t|\,\mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-u_i}\overline{1}^t - |\delta_t\rangle\langle\delta_t|\,\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-u_i}\overline{1}^t \\ &= |\delta_t\rangle\langle\delta_t|\,(2\mathtt{a}^{-1} - 1)\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}^{-u_i}\overline{1}^t. \end{aligned} \tag{2.51}$$

Using the definition of $\mathtt{a}^{-1}$ and $\mathtt{a}_*^{-1}$, we have

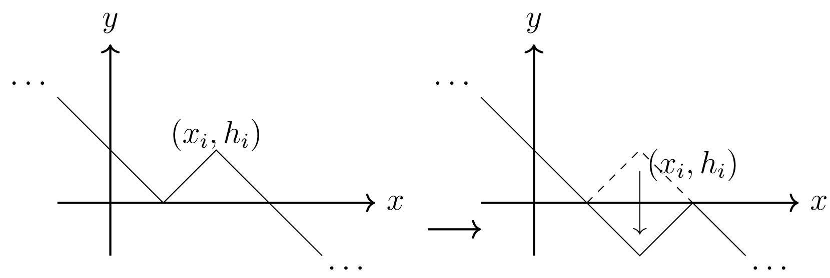
$$(2\mathtt{a}^{-1} - 1)\mathtt{a}_*^{-1} = \big(\tfrac{2}{1-2D} - 1\big)\cdot\tfrac{1}{1+2D} = \tfrac{1}{1-2D}.$$

So

$$(2.51) = |\delta_t\rangle \langle\delta_t| \, \mathtt{a}^{-1}\overline{1}^t \mathtt{a}^{-u_i}\overline{1}^t = 0$$

The second equality is due to the support of $\mathtt{a}^{-1}$. Thus, (2.49) is $W_{0,i} |\delta_t\rangle \langle\delta_t| W_{i,n+1}$, which is what we want.

Type-3:(More than one down step on the right and one up step on the left.)



This corresponds to

$$W_{0,i}\overline{1}^t W_{i,n+1} \to W_{0,i}\mathtt{a}_*^{-1}\overline{1}^t \mathtt{a}_* W_{i,n+1}.$$

The difference is

$$W_{0,i}\big(\mathtt{a}_*^{-1}\overline{1}^t\mathtt{a}_* - \overline{1}^t\big)W_{i,n+1} = W_{0,i}\big(2\mathtt{a}_*^{-1} |\delta_t\rangle \langle\delta_t| \big)W_{i,n+1}. \tag{2.52}$$

Similarly, we will show that it is equal to $W_{0,i} |\delta_t\rangle \langle\delta_t| W_{i,n+1}$ when it acts on $W_{i-1,i}$. Note that in the graph on the left, there is only one up step from peak x_{i-1} to x_i, which means $u_{i-1} = 1$, so $\overline{1}^t W_{i-1,i}$ can be written as

$$\overline{1}^t W_{i-1,i} = \overline{1}^t \mathtt{a}_*^{-d_{i-1}}\mathtt{a}^{-1} = \overline{1}^t \mathtt{a}_*^{-d_{i-1}}\overline{1}^t\mathtt{a}^{-1}.$$

The second equality is true since $\mathtt{a}_*^{-d_{i-1}}(x,y)$ is supported on $x \geq y$, so $x \leq t$ implies that the support of y must be in $y \leq t$. Then taking the difference of the term in (2.52) with $W_{0,i} |\delta_t\rangle \langle\delta_t| W_{i,n+1}$, we have

$$W_{0,i-1}\left(\overline{1}^t\mathtt{a}_*^{-d_{i-1}}\overline{1}^t\mathtt{a}^{-1}(2\mathtt{a}_*^{-1} - 1) |\delta_t\rangle \langle\delta_t| \right)W_{i,n+1}. \tag{2.53}$$

Using the definition of $\mathtt{a}^{-1}$ and $\mathtt{a}_*^{-1}$, we have

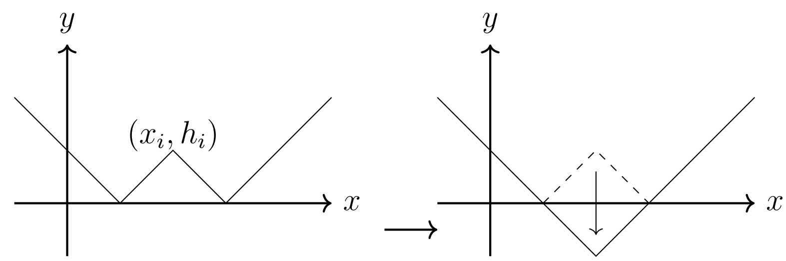
$$\mathtt{a}^{-1}(2\mathtt{a}_*^{-1} - 1) = \tfrac{1}{1-2D} \cdot \big(\tfrac{2}{1+2D} - 1\big) = \tfrac{1}{1+2D}.$$

So

$$(2.53) = W_{0,i-1}\overline{1}^t \mathtt{a}_*^{-d_{i-1}} \overline{1}^t \mathtt{a}_*^{-1} \left|\delta_t\right\rangle \left\langle\delta_t\right| \overline{1}^t W_{i,n+1} = 0$$

The second equality is due to the fact that $\overline{1}^t \mathtt{a}_*^{-1} \left|\delta_t\right\rangle$ in the middle of the equation is 0.

Type-4:(Single up step on the left and single down step on the right.)



In this case,

$$W_{0,i}\overline{1}^t W_{i,n+1} \to W_{0,i} W_{i,n+1}.$$

The difference is $W_{0,i}1_t W_{i,n+1}$. We will show that it is equal to $W_{0,i}\left|\delta_t\right\rangle \left\langle\delta_t\right| W_{i,n+1}$. Due to the special structure that there is only one up step on the left and one down step on the right, we have $W_{i-1,i} = \mathtt{a}^{-1}\mathtt{a}_*^{-d_{i-1}}$ and $W_{i,i+1} = \mathtt{a}_*^{-1}\mathtt{a}^{-u_i}$. We can write

$$\overline{1}^t W_{i-1,i} = \overline{1}^t \mathtt{a}_*^{-d_{i-1}} \overline{1}^t \mathtt{a}^{-1}, \quad W_{i,i+1}\overline{1}^t = \mathtt{a}_*^{-1}\overline{1}^t \mathtt{a}^{-u_i}\overline{1}^t.$$

Then $\overline{1}^t W_{i-1,i} 1_t W_{i,i+1}\overline{1}^t = \overline{1}^t \mathtt{a}_*^{-d_{i-1}} \overline{1}^t \mathtt{a}^{-1} 1_t \mathtt{a}_*^{-1} \overline{1}^t \mathtt{a}^{-u_i}\overline{1}^t$. Using the integral kernel definition of $\mathtt{a}^{-1}, \mathtt{a}_*^{-1}$, we have

$$\begin{aligned}(\overline{1}^t \mathtt{a}^{-1} 1_t \mathtt{a}_*^{-1} \overline{1}^t)(x,z) &= 1_{x\le t, z\le t}\int_t^\infty dy \tfrac{1}{4} 1_{x\le y} e^{\frac{1}{2}(x-y)} 1_{z\le y} e^{\frac{1}{2}(z-y)} \\ &= 1_{x\le t, z\le t}\tfrac{1}{4} e^{-t+\frac{1}{2}x+\frac{1}{2}z}.\end{aligned} \tag{2.54}$$

The indicator function $1_{x\le y}$ and $1_{z\le y}$ in the integration can be dropped since we know $y > t$ and $x \le t, z \le t$. Now notice that (2.54) is the integral kernel of the rank one operator $\mathtt{a}^{-1}\left|\delta_t\right\rangle \left\langle\delta_t\right| \mathtt{a}_*^{-1}$, because $(\mathtt{a}^{-1}\left|\delta_t\right\rangle)(x) = 1_{x\le t}\frac{1}{2}e^{-t/2+x/2}$ and $(\left\langle\delta_t\right| \mathtt{a}_*^{-1})(z) = 1_{z\le t}\frac{1}{2}e^{-t/2+z/2}$.

The last thing we need to check is the flip at x_1 and x_n. For a flip at the last peak x_n, it can only be Type-1 and Type-3. In Type-1, the kernel change is:

$$W \to W_{0,n-1}\overline{1}^t W_{n-1,n}\mathtt{a}\overline{1}^t \mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t \mathtt{a}_* W_{n+1} \tag{2.55}$$

It is worth explaining the presence of the $\mathtt{a}_*$ term. Recall $W_{n+1} = \mathtt{a}_*^d$. d is the distance from *the primordial peak* to the last peak. When there is a flip in the last peak, the distance will increase by one, which is why we have $\mathtt{a}_* W_{n+1}$ in the end. Then the proof that the difference of the two terms in (2.55) is $W_{0,n-1}\overline{1}^t W_{n-1,n} |\delta_t\rangle \langle\delta_t| W_{n+1}$ is the same as in the previous paragraph.

In Type-3, the kernel change is:

$$W \to W_{0,n-1}\overline{1}^t W_{n-1,n}\mathtt{a}_*^{-1}\overline{1}^t \mathtt{a}_* W_{n+1}$$

The proof is exactly the same as before.

For a flip at the first peak x_1. It can only be Type-1 or Type-2. In Type-1, the kernel change is

$$W \to W_0 \mathtt{a}\overline{1}^t \mathtt{a}^{-1}\mathtt{a}_*^{-1}\overline{1}^t \mathtt{a}_* W_{1,2}\overline{1}^t W_{2,n+1}.$$

In Type-2, the kernel change is

$$W \to W_0 \mathtt{a}\overline{1}^t \mathtt{a}^{-1} W_{1,2}\overline{1}^t W_{2,n+1}.$$

The reason we have $W_0\mathtt{a}$ in the front is that the distance from *the primordial peak* to the first peak increases by one. Then the proof of showing the differences is $W_0 |\delta_t\rangle \langle\delta_t| W_{1,2}\overline{1}^t W_{2,n+1}$ is the same.

□

Lemma 2.2.19. *Let $K = S^{l,r} W S^{r,l}$ defined in* (2.12)*, then $(\partial_t - \mathcal{L})K = 0$, where $\mathcal{L}$ is the generator for the full-space TASEP.*

Proof. Recall

$$W = I - \mathtt{a}^u \overline{1}^t W_{1,2} \dots W_{n-1,n}\overline{1}^t \mathtt{a}_*^d.$$

$\partial_t = \sum_{i=1}^n \partial_{t,i}$, where $\partial_{t,i}$ is when ∂ hits the i-th $\overline{1}^t$ and becomes $|\delta_t\rangle \langle\delta_t|$. Recall the generator $\mathcal{L}$ is also written as $\sum_{i=1}^n \mathcal{L}_{x_i}$ (2.4), where $\mathcal{L}_{x_i}$ corresponds to the kernel change when the i-th peak flips. The previous proof shows that $\mathcal{L}_{x_i}$ also changes $\overline{1}^t$ to $|\delta_t\rangle \langle\delta_t|$. Thus, we have $(\partial_t - \mathcal{L})W = 0$, which implies that $(\partial_t - \mathcal{L})K = 0$. □

Lastly, we check how $\partial_t - \mathcal{L}$ acts on the determinant.

Proposition 2.2.20. *Let $K = S^{l,r} W S^{r,l}$, then*

$$(\partial_t - \mathcal{L}) \det(I - K)_{L^2([0,\infty))} = 0. \tag{2.56}$$

Proof. Using the known equality,

$$\partial_t \det(I-K) = \det(I-K)\,\mathrm{tr}\left((I-K)^{-1}\partial_t K\right).$$

For the generator $\mathcal{L}$, notice that each $\mathcal{L}_{x_i}$ acts on the kernel, giving a rank-one operator, which we denote as $|h_i\rangle\langle g_i|$. Then

$$\mathcal{L}_{x_i}\det(I-K) = \det(I-K-|h_i\rangle\langle g_i|) - \det(I-K).$$

Using the known equality for the Fredholm determinant of a rank-one perturbation,

$$\begin{aligned}\det(I-K-|h_i\rangle\langle g_i|) - \det(I-K) &= \det(I-K)\,\mathrm{tr}\left((I-K)^{-1}|h_i\rangle\langle g_i|\right)\\ &= \det(I-K)\,\mathrm{tr}\left((I-K)^{-1}\mathcal{L}_i K\right).\end{aligned}$$

Thus, we have that

$$(\partial_t-\mathcal{L})\det(I-K) = \det(I-K)\,\mathrm{tr}\left((I-K)^{-1}(\partial_t-\mathcal{L})K\right) = 0.$$

□

2.2.5 Initial condition

To prove the initial condition, we first need to develop some properties regarding the operator $S^{n,m}$ and the kernel.

The following absorbing lemma states that in operator $\mathtt{a}_*^{-k}W\mathtt{a}_*^{k}$ (W is surrounded by $\mathtt{a}_*$ and $\mathtt{a}_*^{-1}$), they can be absorbed into S.

Lemma 2.2.21 (Absorbing Lemma I)**.** *For $k<l$, $0<t$,*

$$1_0 S^{l,r}(I-\mathtt{a}_*^{-k}W\mathtt{a}_*^{k})S^{r,l}1_0 = 1_0 S^{l,r-k}(I-\mathtt{a}^{u}W\mathtt{a}_*^{d})S^{r-k,l}1_0.$$

Proof. Use equation (2.27), we have

$$(I-\mathtt{a}_*^{-k}\mathtt{a}^{u}\overline{1}^{t}\tilde{W}_{1,n}\overline{1}^{t}\mathtt{a}_*^{d}\mathtt{a}_*^{k}) = \mathtt{a}_*^{-k}1_t\mathtt{a}_*^{k} - \sum_{i=1}^{u}\sum_{j=1}^{d}4\mathtt{a}_*^{-k}\mathtt{a}^{u-i}|\delta_t\rangle\langle\delta_t|W_{1,n}^{i-1,j-1}|\delta_t\rangle\langle\delta_t|\mathtt{a}_*^{d-j}\mathtt{a}_*^{k}. \tag{2.57}$$

Using Lemma (2.2.25),

$$\langle\delta_t|\mathtt{a}_*^{d-j}\mathtt{a}_*^{k}S^{r,l}1_0 = \langle\delta_t|\mathtt{a}_*^{d-j}S^{r-k,l}1_0, \quad \mathtt{a}_*^{-k}1_t\mathtt{a}_*^{k}S^{r,l}1_0 = \mathtt{a}_*^{-k}1_tS^{r-k,l}1_0.$$

Using Lemma (2.2.26) , we have

$$1_0 S^{l,r} \mathtt{a}_*^{-k} \mathtt{a}^{u-i} \left|\delta_t\right\rangle = 1_0 S^{l,r-k} \mathtt{a}^{u-i} \left|\delta_t\right\rangle, \quad 1_0 S^{l,r} \mathtt{a}_*^{-k} 1_t S^{r-k,l} 1_0 = 1_0 S^{l,r-k} 1_t S^{r-k,l} 1_0.$$

Plugging this into (2.57), we get the desired relation. □

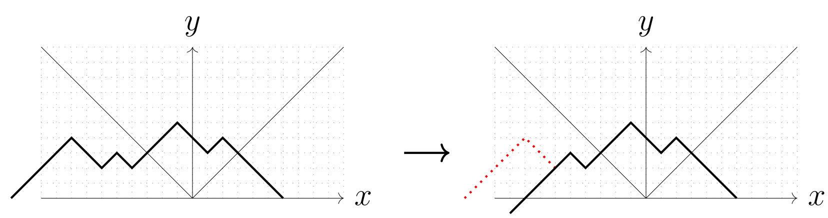
Lemma 2.2.22 (Absorbing Lemma II)**.** *Let* $0 < t$. *If* $l \le u$, *recall that* $W_{1,2} = \mathtt{a}^{-u_1} \mathtt{a}_*^{-d_1}$*; we have*

$$1_0 S^{l,r} (I - \mathtt{a}^u \overline{1}^t W_{1,n} \overline{1}^t \mathtt{a}_*^d) S^{r,l} 1_0 = 1_0 S^{l,r-d_1} (I - \mathtt{a}^{u-u_1} \overline{1}^t W_{2,n} \overline{1}^t \mathtt{a}_*^{d-d_1}) S^{r-d_1,l} 1_0.$$

If $r \le d$, *we have the similar formula on the other side. Recall* $W_{n-1,n} = \mathtt{a}^{-u_{n-1}} \mathtt{a}_*^{-d_{n-1}}$. *We have*

$$1_0 S^{l,r} (I - \mathtt{a}^u \overline{1}^t W_{1,n} \overline{1}^t \mathtt{a}_*^d) S^{r,l} 1_0 = 1_0 S^{l-u_n,r} (I - \mathtt{a}^{u-u_n} \overline{1}^t W_{1,n-1} \overline{1}^t \mathtt{a}_*^{d-d_n}) S^{r,l-u_n} 1_0.$$

Remark 2.2.23 (Graphical View)**.** $l \le u$ *means that the leftmost peak is outside of the cone* $C_{0,0}$, *then the kernel should be equivalent to the kernel with the leftmost peak not present, since it does not affect the probability.*



Proof. We will only prove the first equality; the second is the same. Note

$$1_0 S^{l,r} (I - \mathtt{a}^u \overline{1}^t W_{1,n} \overline{1}^t \mathtt{a}_*^d) S^{r,l} 1_0 = 1_0 S^{l,r} (I - \mathtt{a}^u (I - 1_t) W_{1,n} \overline{1}^t \mathtt{a}_*^d) S^{r,l} 1_0.$$

Notice the term involved 1_t vanishes since

$$1_0 S^{l,r} \mathtt{a}^u 1_t W_{1,2} = \sum_{i=0}^{l-1} 1_0 B^{l-i,r} \mathtt{a}^{u-i} 1_t W_{1,2} + S^{l-u,r} 1_t W_{1,2} = 0.$$

The summation is 0 due to the support of $\mathtt{a}^{u-i}$ (Lemma (2.2.8)). The second term is 0 since $S^{l-u,r}$ is 0 when $l \le u$. This means that when $l \le u$, the first indicator function before $W_{1,2}$ disappears and $\mathtt{a}^u$ can cancel the terms in $W_{1,2}$, which is $\mathtt{a}^u W_{1,2} =$

$\mathtt{a}^{u-u_1}\mathtt{a}_*^{-d_1}$, so the formula is

$$1_0 S^{l,r}(I-\mathtt{a}^{u}\overline{1}^{t}W_{1,n}\overline{1}^{t}\mathtt{a}_*^{d})S^{r,l}1_0 = 1_0 S^{l,r}(I-\mathtt{a}^{u-u_1}\mathtt{a}_*^{-d_1}\overline{1}^{t}W_{2,n}\overline{1}^{t}\mathtt{a}_*^{d})S^{r,l}1_0.$$

Then apply Lemma (2.2.21) to bring $\mathtt{a}_*^{-d_1}$ to the first S and $\mathtt{a}_*^{d_1}$ to the second S, we get the desired result. □

Lemma 2.2.24. *$S^{n,m}(x,-x)=0$ if $n-2\geq m>0$; $S^{n,m}(x,-x)=\frac{(-1)^n}{2}$ if $m=n-1$;*

Proof. Let $g(x,y)=e^{-(x+y)w}\frac{(1+2w)^m}{(1-2w)^n}$. We have

$$\begin{aligned} S^{n,m}(x,y) &= \operatorname{Res}(g,1/2) \\ &= (-1)^n\frac{2^{m-n}}{(n-1)!}\sum_{i=0}^{n-1\wedge m}\binom{n-1}{i}\frac{m!}{(m-i)!}(-x-y)^{n-1-i}e^{-\frac{1}{2}(x+y)}. \end{aligned} \tag{2.58}$$

When $x=-y$ and $n\geq m-2$, the degree of $(-x-y)$ is always positive; thus, it is 0.

For $S^{n,n-1}(x,-x)$, in equation (2.58), $i=m$ in the summation is not 0 and is easily seen to be $\frac{(-1)^n}{2}$ when $x=-y$.

□

Lemma 2.2.25. *For any $-t_1<t_2$,*

$$1_{t_1}S^{n,m}\mathtt{a}\left|\delta_{t_2}\right\rangle = 1_{t_1}S^{n-1,m}\left|\delta_{t_2}\right\rangle, \quad \left\langle\delta_{t_2}\right|\mathtt{a}_*S^{n,m}1_{t_1} = \left\langle\delta_{t_2}\right|S^{n-1,m}1_{t_1}.$$

For any $-t_1\neq t_2$,

$$\left\langle\delta_{t_1}\right|S^{n,m}\mathtt{a}\left|\delta_{t_2}\right\rangle = \left\langle\delta_{t_1}\right|S^{n-1,m}\left|\delta_{t_2}\right\rangle, \quad \left\langle\delta_{t_2}\right|\mathtt{a}_*S^{n,m}\left|\delta_{t_1}\right\rangle = \left\langle\delta_{t_2}\right|S^{n-1,m}\left|\delta_{t_1}\right\rangle.$$

Proof. The proof is to note that when there is a Dirac delta function present on the other side, the commutator term $B_{a,b}^{n,m}$ in (2.35) will be 0 due to support. □

Lemma 2.2.26.
1. $(S^{n,m}\mathtt{a}_*^{-1})(x,z)1_{z\geq -x}=S^{n,m-1}(x,z)$ for all $n,m\in\mathbb{Z}$.
2. $S^{n,m}\mathtt{a}^{-1}=S^{n+1,m}$ for $n\geq m+1\geq 1$.
3. $(\mathtt{a}^{-1}S^{n,m})(x,z)1_{x\geq -z}=S_2^{n,m-1}(x,z)$ for all $n,m\in\mathbb{Z}$.
4. $\mathtt{a}_*^{-1}S^{n,m}=S^{n+1,m}$ for $n\geq m+1\geq 1$.

Proof. 1. By definition,

$$\begin{aligned}(S^{n,m}\mathtt{a}_*^{-1})(x,z)1_{z\geq -x} &= \int_z^\infty dy \int dw e^{-(x+y)w}\frac{(1+2w)^m}{(1-2w)^n}\frac{1}{2}e^{(z-y)/2} \\ &= \int dw e^{-(x+z)w}\frac{(1+2w)^{m-1}}{(1-2w)^n} = S^{n,m-1}(x,z).\end{aligned} \tag{2.59}$$

2. For $n \geq m+1$,

$$\begin{aligned}S^{n,m}\mathtt{a}^{-1} &= \int_{-x}^z dy \int dw e^{-(x+y)w}\frac{(1+2w)^m}{(1-2w)^n}\frac{1}{2}e^{(y-z)/2} \\ &= \int dw e^{-(x+z)w}\frac{(1+2w)^m}{(1-2w)^{n+1}} - e^{-(x+z)/2}\int dw\frac{(1+2w)^m}{(1-2w)^{n+1}}.\end{aligned} \tag{2.60}$$

When $n \geq m+1$, the second term is 0 by Lemma (2.2.24).

The proof of (3), (4) is completely the same. □

Lemma 2.2.27. *For $t \geq 0$, if $a \geq c+2, c \geq 0$,*

$$\langle \delta_t | S^{a,b}1_{-t}S^{b,c} | \delta_t \rangle = 0.$$

Also, if $a > b \geq c$ or $c > d \geq a$,

$$\langle \delta_t | S^{a,b}1_{-t}S^{c,d} | \delta_t \rangle = 0.$$

Proof. Looking at the variable range, we can drop the indicator function in S, thus

$$\langle \delta_t | S^{a,b}1_{-t}S^{b,c} | \delta_t \rangle = \int_{-t}^\infty dy s^{a,b}(t,y)s^{b,c}(y,t).$$

Recall $s^{n,m}$ defined in (2.58). Continue to integrate by parts to take $(1+2w)$ from left s to right s we have

$$\begin{aligned}&\int_{-t}^\infty dy s^{a,b}(t,y)s^{b,c}(y,t) \\ &\qquad = 2\sum_{i=0}^{b-1} s^{a,b-1-i}(t,-t)s^{b-i,c}(-t,t) + \int_{-t}^\infty dy s^{a,0}(t,y)s^{0,c}(y,t).\end{aligned}$$

The last integral is zero. All boundary terms are also 0 since $s^{b-i,c}(0,0) = 0$ when $b-i \geq c+2$ and $s^{a,b-1-i}(0,0) = 0$ when $b-i < c+2 \leq a$.

For $a > b \geq c$,

$$\begin{aligned}\langle\delta_t| S^{a,b} 1_{-t} S^{c,d} |\delta_t\rangle &= \int_{-t}^{\infty} dy s^{a,b}(t,y) s^{c,d}(y,t) \\ &= 2\sum_{i=0}^{c-1} s^{a,b-1-i}(t,-t) s^{c-i,d}(-t,t) + \int_{-t}^{\infty} dy s^{a,b-c}(t,y) s^{0,d}(y,t),\end{aligned}$$

which is 0 for the same reason above. The proof for the case $c > d \geq a$ is the same. □

Lemma 2.2.28. *(Eigenfunction lemma for S) When $n, m > 0$, the function $S^{n,0}|\delta_0\rangle$ is an eigenfunction of the operator $S^{n,m}1_0 S^{m,n}$ with eigenvalue 1, i.e.,*

$$1_0 S^{n,m} 1_0 S^{m,n} S^{n,0} |\delta_0\rangle = S^{n,0} |\delta_0\rangle . \tag{2.61}$$

Proof.

$$\begin{aligned}S^{n,m} 1_0 S^{m,n} S^{n,0} |\delta_0\rangle &= \int_0^{\infty} dy \int_0^{\infty} dz s^{n,m}(x,y) s^{m,n}(y,z) s^{n,0}(z,0) \\ &= \sum_{i=0}^{m-1} 2 \int_0^{\infty} dz s^{n,m-1-i}(x,0) s^{m-i,n}(0,z) s^{n,0}(z,0) \\ &+ \int_0^{\infty} dy \int_0^{\infty} dz s^{n,0}(x,y) s^{0,n}(y,z) s^{n,0}(z,0).\end{aligned}$$

From Lemma (2.2.27), all of the terms are zero except when $i = m - 1$. Thus, we have

$$2\int_0^{\infty} dz s^{n,0}(x,0) s^{1,n}(0,z) s^{n,0}(z,0). \tag{2.62}$$

Following the proof of the last lemma, it is easy to see that

$$\int_0^{\infty} dz 2 s^{1,n}(0,z) s^{n,0}(z,0) = 4 s^{1,0}(0,0) s^{1,0}(0,0) = 1$$

Thus, equation (2.62) is $s^{n,0}|\delta_0\rangle$. Since $x > 0$ from the first indicator function in (2.61), it is equal to $S^{n,0}|\delta_0\rangle$. □

Proposition 2.2.29. *Given the initial condition $(\vec{x}, \vec{h})$. Let $H(x)$ be the height function of TASEP associated with $(\vec{x}, \vec{h})$. Let $F(t, H) = \mathbb{P}((\vec{x}, \vec{h})_t \leq \{0, 0\})$. Then*

$$\lim_{t\to 0} F(t, H) = 1_{H(0)\leq 0}.$$

Proof. Now we check the initial condition. The kernel is

$$1_0 S^{l,r}(I - \mathtt{a}^u \overline{1}^t W_{1,2} \dots W_{n-1,n} \overline{1}^t \mathtt{a}_*^d) S^{r,l} 1_0$$

If either $l \leq u$ or $r \leq d$, we can use Lemma (2.2.22) to reduce the term. There are two cases: either we reduce to the case that ($l > u$ and $r > d$) or there is only a peak left. First, we discuss the case that $l > u$ and $r > d$. That means there are some peaks in the cone $C_{0,0}$, and the kernel after reduction represents the configuration in the cone $C_{0,0}$, so we want to show that the initial probability is 0. Using (2.27), the kernel is

$$1_0 S^{\tilde{l},\tilde{r}}(1_0 + \sum_{i=1}^{\tilde{u}} \sum_{j=1}^{\tilde{d}} 4\mathtt{a}^{\tilde{u}-i} |\delta_0\rangle \langle\delta_0| W_{1,n}^{i-1,j-1} |\delta_0\rangle \langle\delta_0| \mathtt{a}_*^{\tilde{d}-j}) S^{\tilde{r},\tilde{l}} 1_0$$

Here, all variables $\tilde{l}, \tilde{r}, \tilde{u}, \tilde{d}$ are parametrized by the black configuration in the right graph.

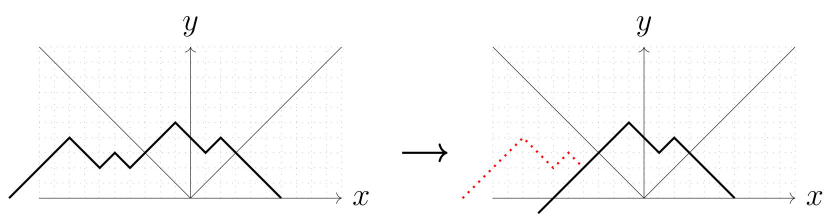


Figure 2.2: Figure: peak reduction

We will show that $S^{l,0} |\delta_0\rangle$ is an eigenfunction of the kernel with eigenvalue 1. Using Lemma (2.2.27), we can see that all the terms in the summation that act on $S^{l,0} |\delta_0\rangle$ will be 0 since

$$\langle\delta_0| \mathtt{a}_*^{\tilde{d}-j} S^{\tilde{r},\tilde{l}} 1_0 S^{\tilde{l},0} |\delta_0\rangle = \langle\delta_0| S^{\tilde{r}-\tilde{d}+j,\tilde{l}} 1_0 S^{\tilde{l},0} |\delta_0\rangle$$

Since $\tilde{l} \geq 1, \tilde{r} > \tilde{d}$, so $\tilde{r} - \tilde{d} + j \geq 2$, the condition of Lemma (2.2.27) is satisfied. Then using Lemma (2.2.28), we have

$$1_0 S^{\tilde{l},\tilde{r}} 1_0 S^{\tilde{r},\tilde{l}} 1_0 S^{\tilde{l},0} |\delta_0\rangle = S^{\tilde{l},0} |\delta_0\rangle .$$

Thus, we get the desired result.

Now we consider the case that there is only one peak left after reduction. In this case, W in the kernel reduces to 1_0 and the kernel reduces to

$$S^{\tilde{l},\tilde{r}} 1_0 S^{\tilde{r},\tilde{l}}.$$

If the peak is not in the cone $C_{0,0}$, that means the whole configuration is completely below the $(0,0)$, thus we want to show that the probability is 1. In the formula, that means either $\tilde{r} \le 0$ or $\tilde{l} \le 0$, which means either $S^{\tilde{l},\tilde{r}} = 0$ or $S^{\tilde{r},\tilde{l}} = 0$, thus the determinant is 1. If the peak is in the cone $C_{0,0}$, that means both $\tilde{l} > 0$ and $\tilde{r} > 0$. Using the Lemma (2.2.28), we can show that there is an eigenfunction with eigenvalue 1, thus the probability is 0, as desired.

□

In principle, one needs to prove the uniqueness of the Kolmogorov equation. Since the model is already solved and the method is exactly the same as the half-space case, we thus omit it here.

2.3 Multi-point distribution

Now we give the multi-point distribution formula for the full-space TASEP starting from a general deterministic initial condition.

Recall some notations from previous sections. Kernel W is defined in (2.8),

$$\begin{aligned} &W = (I - W_0 \overline{1}^t W_{1,2} \overline{1}^t W_{2,3} \cdots \overline{1}^t W_{n-1,n} \overline{1}^t W_{n+1}), \\ &W_{i,i+1} = \mathtt{a}^{-u_i} \mathtt{a}_*^{-d_i}, \quad W_0 = \mathtt{a}^{u}, \quad W_{n+1} = \mathtt{a}_*^{d}. \end{aligned}$$

Recall W is parameterized by $(\vec{x}, \vec{h}) = (x_1, h_1; \cdots; x_n, h_n)$, and all the variables u_i, d_i, u, d are recording the relative positions between different peaks.

If we have a trough configuration $\{y_1, s_1; \cdots; y_m, s_m\}$, we can flip the configuration with respect to the x-axis so that $(y_1, -s_1; \cdots; y_m, -s_m)$ is a peak configuration. We define u_i', d_i', u', d' to be the variable parameterized by $(y_1, -s_1; \cdots; y_m, -s_m)$, i.e.

$$u_i' = (y_{i+1} - y_i - s_{i+1} + s_i)/2, \quad d_i' = (y_{i+1} - y_i + s_{i+1} - s_i)/2. \tag{2.63}$$

and $u' = \sum u_i'$, $d' = \sum d_i'$.

Theorem 2.3.1. *Assume that we start the full-space TASEP with the initial configuration having peaks at $(x_1, h_1; \ldots, x_n, h_n)$. The probability that at time t it is below* *the configuration* $\{y_1, s_1; \cdots; y_m, s_m\}$ *is given by:*

$$\mathbb{P}((x_1, h_1; \ldots; x_n, h_n)_t \leq \{y_1, s_1; \cdots; y_m, s_m\}) = \det(I - K)_{(L^2([0,\infty))^m}, \tag{2.64}$$

where K *is a matrix-valued kernel on* m *copies of* $L^2([0, \infty))$.

$$K(i, \cdot; j, \cdot) = 1_{i<j}(\mathtt{a})^{-u'_{ij}}(\mathtt{a}_*)^{-d'_{ij}} + S^{l_i, r_i} W S^{r_j, l_j} \tag{2.65}$$

where

$$\begin{aligned} u'_{ij} &= \sum_{k=i}^{j-1} u'_k, \quad d'_{ij} = \sum_{k=i}^{j-1} d'_k \\ l_i &= (x_{\text{prim}} + h_{\text{prim}} - y_i - s_i)/2, \quad r_i = (h_{\text{prim}} - s_i - x_{\text{prim}} + y_i)/2 \end{aligned} \tag{2.66}$$

where $x_{\text{prim}}, h_{\text{prim}}$ *are defined in* (2.5). W *is defined in* (2.8)*, parameterized by* $(x_1, h_1; \cdots; x_n, h_n)$*, which is the same object defined in the one-point formula.*

Remark 2.3.2. *We want to emphasize how you should think of this formula as built from the one-point kernel. This point of view will also be applied to the half-space case. In the operator* SWS*,* W *purely depends on* $(\vec{x}, \vec{h})$*, thus it is unchanged in the multi-point case. Both* S, S *depend on the relative position between* $(x_{\text{prim}}, h_{\text{prim}})$ *and* $\{\vec{y}, \vec{s}\}$*. In the multi-point case, in the* $K(i, \cdot; j, \cdot)$*, the left piece* S *is parametrized by* $\{y_i, s_i\}$*, and the right piece* S *is parametrized by* $\{y_j, s_j\}$*.*

We will prove that the formula satisfies the Kolmogorov equation with the proper initial condition.

2.3.1 Kolmogorov equation

Proposition 2.3.3. *Let* K *be the kernel defined in Theorem* (2.3.1)*. We have* $(\partial_t - \mathcal{L}) \det(I - K) = 0$

Proof. Let us first check $(\partial_t - \mathcal{L})K = 0$. Since K is a matrix kernel, the operator acts entry-wise; thus, we need to check $(\partial_t - \mathcal{L})K(i, \cdot; j, \cdot) = 0$. This is the same as Lemma (2.2.19), since all the variables x_i, h_i, t are in the operator W.

Next, we check that both operators can go through the determinant. For the derivative in t, we have

$$\partial_t \det(I - K) = \det(I - K) \operatorname{tr}\big((I - K)^{-1} \partial_t K\big).$$

which is the same as the one-point case.

For the generator $\mathcal{L}$, notice that each $\mathcal{L}_k$ acts on the kernel $K(i,\cdot;j,\cdot)$, giving a rank-one operator, which we denote as $|h_k^i\rangle\langle g_k^j|$. Define the following row vector and column vector:

$$|h_k\rangle = \Big(|h_k^1\rangle, |h_k^2\rangle, \cdots, |h_k^m\rangle\Big), \quad \langle g_k| = \begin{pmatrix} \langle g_k^1| \\ \langle g_k^2| \\ \cdots \\ \langle g_k^m| \end{pmatrix}.$$

Then $\mathcal{L}_{x_k}K$ is also a rank-one operator on $(L^2([0,\infty)))^m$, which is $|h_k\rangle\langle g_k|$.

Then

$$\mathcal{L}_{x_k}\det(I-K) = \det(I-K-|h_k\rangle\langle g_k|) - \det(I-K).$$

Using the known equality for the Fredholm determinant of a rank-one perturbation,

$$\begin{aligned}\det(I-K-|h_k\rangle\langle g_k|) - \det(I-K) &= \det(I-K)\,\mathrm{tr}\,\big((I-K)^{-1}|h_k\rangle\langle g_k|\big) \\ &= \det(I-K)\,\mathrm{tr}\,\big((I-K)^{-1}\mathcal{L}_k K\big).\end{aligned}$$

Thus, we have that

$$(\partial_t - \mathcal{L})\det(I-K) = \det(I-K)\,\mathrm{tr}\left((I-K)^{-1}(\partial_t-\mathcal{L})K\right) = 0.$$

□

2.3.2 Initial condition

Proposition 2.3.4. *Given the initial condition $(\vec{x},\vec{h})$. Let $H(x)$ be the height function of the TASEP associated with $(\vec{x},\vec{h})$. Let $F(t,H) = \mathbb{P}((\vec{x},\vec{h})_t \le \{\vec{y},\vec{s}\})$. Then*

$$\lim_{t\to 0} F(t,H) = 1_{(\vec{x},\vec{h})\le\{\vec{y},\vec{s}\}} = \Pi_{i=1}^m 1_{H(y_i)\le s_i}.$$

Proof. Notice that if there is a trough (y_k,s_k) outside the cone $C^{x_{\mathrm{prim}},h_{\mathrm{prim}}}$, then there are two cases.

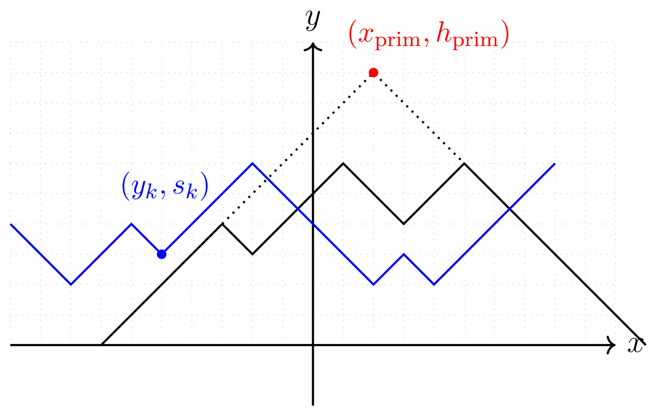


Figure 2.3: Tough outside the cone from the left

Case1: $y_k \le x_{\text{prim}}$ and $s_k \ge h_{\text{prim}} - x_{\text{prim}} + y_k$. In this case, it is clear that all the troughs (y_j, s_j) for $j \le k$ are also outside the cone $C^{x_{\text{prim}}, h_{\text{prim}}}$.

In this case, $r_j = (h_{\text{prim}} - s_j - x_{\text{prim}} + y_j)/2 \le 0$ for all $j \le k$. By definition of S, $S^{r_j, l_j} = 0$, thus in every j-th column $(j \le k)$, $K(i, \cdot; j, \cdot) = 1_{i<j}(\mathtt{a})^{-u'_{ij}}(\mathtt{a}_*)^{-d'_{ij}}$.

Thus, the first $k \times k$ diagonal block of $I - K$ is an upper triangular matrix with the identity operator along the first k diagonal position, so the determinant reduces:

$$\det((I-K)^m_{i=1,j=1}) = \det((I-K)^m_{i=k+1,j=k+1}).$$

Case2:

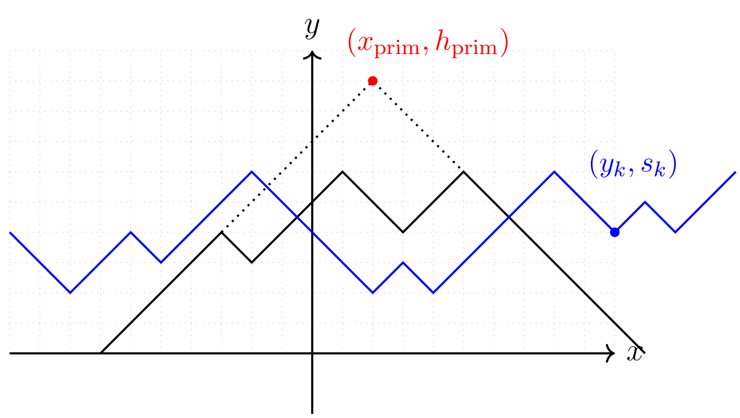


Figure 2.4: Tough outside the cone from the right

This is the case that $y_k \ge x_{\text{prim}}$ and $s_k \ge h_{\text{prim}} + x_{\text{prim}} - y_k$. In this case, it is clear that all the troughs (y_i, s_i) for $i \ge k$ are also outside the cone $C^{x_{\text{prim}}, h_{\text{prim}}}$.

In this case, $l_i = (h_{\text{prim}} + s_j - x_{\text{prim}} - y_j)/2 \le 0$ for all $i \ge k$. By the definition of S, $S^{l_i, r_i} = 0$, thus in every i-th row, $K(i, \cdot; j, \cdot) = 1_{i<j}(\mathtt{a})^{-u'_{ij}}(\mathtt{a}_*)^{-d'_{ij}}$ for $i \ge k$.

Thus, the last $k \times k$ diagonal block of $I - K$ is an upper triangular matrix with the identity operator along the first k diagonal position, so the determinant reduces:

$$\det((I-K)^m_{i=1,j=1}) = \det((I-K)^k_{i=1,j=1}).$$

Now we are ready to discuss the proof.

If all the troughs are outside the cone $C^{x_{\text{prim}},h_{\text{prim}}}$, then the configuration is already less than or equal to $\{\vec{y}, \vec{s}\}$; we want to show $F(0, H) = 1$. In this case, the whole kernel $I - K$ reduces to an upper triangular matrix with the identity operator along the diagonal; thus, its determinant is 1.

Now assume there exist some troughs in the cone. Since what is outside the cone does not affect the determinant, WLOG, we can assume that all the troughs are in the cone. If there exists $l_i \geq u$ or $r_i \geq d$, we apply the absorbing lemma (2.2.22) to reduce the kernel. The following figure illustrates when that is needed:

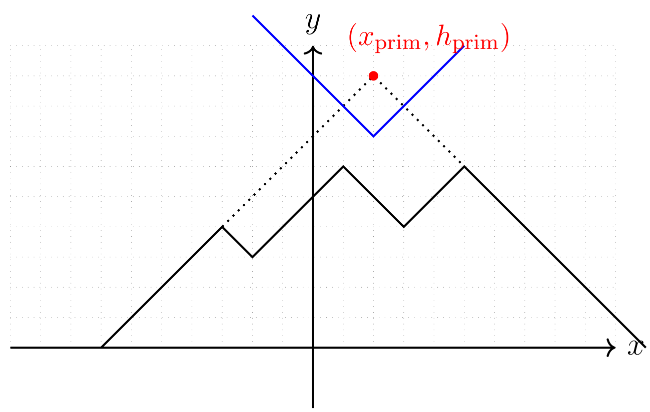


Figure 2.5: Tough inside cone but reduction needed

As we have shown in the one-point case, if the trough is completely above the initial configuration, the kernel $S^{l_i,r_i}WS^{r_i,l_i}$ reduces to 0. Thus, if all the troughs inside the cone $C^{x_{\text{prim}},h_{\text{prim}}}$ are of this type, the kernel again reduces to an upper triangular matrix with an identity along the diagonal. Thus, we have $F(0, H) = 1$, which is what we want.

Now assume that there exists a trough (y_k, s_k) that is below the initial configuration. Now we want to show the $F(0, H) = 0$. To do that, we present an eigenfunction for the kernel K with eigenvalue 1. Recall

$$K(i,\cdot;j,\cdot) = -1_{i<j}(\mathtt{a})^{-d'_{ij}}(\mathtt{a}_*)^{-u'_{ij}} + S^{l_i,r_i}WS^{r_j,l_j}.$$

WLOG we can assume y_k, s_k is the last trough in the configuration. From Lemma

(2.2.28), we know that $S^{l_m,0}(x,0)$ is an eigenfunction of $S^{l_m,r_m}WS^{r_m,l_m}$ with eigenvalue 1. Now we show that

$$S = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ S^{l_m,0}\left|\delta_0\right\rangle \end{pmatrix}$$

is the eigenfunction we want.

$$KS = \begin{pmatrix} -(\mathtt{a})^{-u'_{1m}}(\mathtt{a}_*)^{-d'_{1m}}S^{l_m,0}\left|\delta_0\right\rangle + S^{l_1,r_1}WS^{r_m,l_m}S^{l_m,0}\left|\delta_0\right\rangle \\ -(\mathtt{a})^{-u'_{2m}}(\mathtt{a}_*)^{-d'_{2m}}S^{l_m,0}\left|\delta_0\right\rangle + S^{l_2,r_2}WS^{r_m,l_m}S^{l_m,0}\left|\delta_0\right\rangle \\ \cdots \\ S^{l_m,r_m}WS^{r_m,l_m}S^{l_m,0}\left|\delta_0\right\rangle \end{pmatrix} \tag{2.67}$$

The last entry is $S^{l_m,0}\left|\delta_0\right\rangle$, using the proof from the one-point case. Now we need to show that

$$-(\mathtt{a})^{-u'_{im}}(\mathtt{a}_*)^{-d'_{im}}S^{l_m,0}\left|\delta_0\right\rangle + S^{l_i,r_i}WS^{r_m,l_m}S^{l_m,0}\left|\delta_0\right\rangle = 0 \tag{2.68}$$

First, notice that $l_i = l_m + u'_{im}, r_i = r_m - d'_{im}$ by definition. From the proof of the one-point case, we know that all the finite rank parts act $S^{l_m,0}$ will be 0, thus all we need to show is that

$$-(\mathtt{a})^{-u'_{im}}(\mathtt{a}_*)^{-d'_{im}}S^{l_m,0}\left|\delta_0\right\rangle + S^{l_i,r_i}1_0S^{r_m,l_m}S^{l_m,0}\left|\delta_0\right\rangle = 0 \tag{2.69}$$

The second term is

$$\begin{aligned} S^{l_i,r_i}1_0S^{r_m,l_m}S^{l_m,0}\left|\delta_0\right\rangle &= \int_0^\infty dy\int_0^\infty dz s^{l_i,r_i}(x,y)s^{r_m,l_m}(y,z)s^{l_m,0}(z,0) \\ &= \sum_{k=0}^{r_m-1} 2\int_0^\infty dz s^{l_i,r_i-k-1}(x,0)s^{r_m-k,l_m}(0,z)s^{l_m,0}(z,0) \\ &+ \int_0^\infty dy\int_0^\infty dz s^{l_i,r_i-r_m}(x,y)s^{0,l_m}(y,z)s^{l_m,0}(z,0). \end{aligned}$$

From Lemma (2.2.27), all of the terms are zero except when $k = r_m - 1$. Thus, we have

$$2\int_0^\infty dz s^{l_i,r_i-r_m}(x,0)s^{1,n}(0,z)s^{n,0}(z,0) \tag{2.70}$$

Following the proof of Lemma (2.2.27), it is easy to see that

$$\int_0^\infty dz 2s^{1,n}(0,z)s^{n,0}(z,0) = 4s^{1,0}(0,0)s^{1,0}(0,0) = 1.$$

Thus, equation (2.62) is $S^{l_i,-d'_{im}} |\delta_0\rangle$. On the other hand, using Lemma (2.2.26), we have

$$(\mathtt{a})^{-d'_{im}}(\mathtt{a}_*)^{-u'_{im}} S^{l_m,0} |\delta_0\rangle = S^{l_i,-d'_{im}} |\delta_0\rangle \tag{2.71}$$

Thus, we get the desired result. □

Chapter 3

Half-space TASEP with a general initial condition

3.1 One-point distribution

3.1.1 Notation for half-space

The half-space TASEP with rate α is a continuous-time Markov process on $\mathbb{Z}^+ \cup \{0\}$. Particles jump to the right in continuous time at rate 1 with exclusion. There is a reservoir of an infinite number of particles at the origin, and the particles jump to site 1 at rate α if the site 1 is empty. Let $\eta : \mathbb{N} \to \{0, 1\}$ be the occupation variables. $\eta_t(x)$ is 1 if there is a particle at position x at time t and 0 otherwise. For finite range $f : \{0, 1\}^{\mathbb{N}} \to \mathbb{R}$, the generator is given by:

$$\mathcal{L}f(\eta) = \alpha(f(1, \eta_2, \eta_3, \cdots) - f(\eta_1, \eta_2, \cdots)) + \sum_{x \in \mathbb{Z}^+} \eta_x(1 - \eta_{x+1})(f(\eta_{x,x+1}) - f(\eta))$$

where $\eta_{x,x+1}$ is obtained by switching the occupation variables η at sites x and $x + 1$.

Similar to the full space case, we are interested in the following probability distribution:

$$\mathbb{P}(h((t, x; h_{\text{init}}) \leq h_{\text{final}}).$$

Now both h_{init} and h_{final} are functions on non-negative integer points, representing the height function of TASEP. We have similar assumptions on the types of functions h_{init} and h_{final} that are allowed. We will always assume that the configuration $h_{\text{init}}(t, x)$ that is evolving has a finite number of peaks (local maxima) and $h_{\text{init}}(t, x) \to -\infty$ as $x \to \infty$. $h_{\text{final}}(x)$ has a finite number of troughs (local minima) and $h_{\text{final}}(x) \to \infty$ as

$x \to \infty$. Under these assumptions, we can also use

$$(\vec{x}, \vec{h}) = (x_1, h_1; x_2, r_2; \cdots x_n, h_n)_t, \quad 0 \le x_1 < \cdots < x_n \tag{3.1}$$

to denote the initial configurations and use

$$\{\vec{y}, \vec{s}\} = \{y_1, s_1; y_2, s_2; \cdots y_m, s_m\}, \quad 0 \le y_1 < \cdots < y_m. \tag{3.2}$$

To denote the final configuration, the only extra requirements compared to the full space case are that all x_i, y_i are non-negative. The notion of *the primordial peak* can be defined the same way in (2.5). All the variables u_i, d_i, u, d are defined the same as in (2.6). The kernel W, which records the configuration of $(\vec{x}, \vec{h})$, is defined the same way in (2.8).

Now we are going to define a generalization of the variable l, r that is defined in the full space case. Recall that $C_{x,y}$ is the cone starting at (x, y), open to the top; that is,

$$C_{x,y} = \{(a, b) \in \mathbb{Z}^2 : b \ge |a - x| + y\}. \tag{3.3}$$

and $C^{x,y}$ is the cone starting at (x, y) and open to the bottom; that is,

$$C^{x,y} = \{(a, b) \in \mathbb{Z}^2 : b \le -|a - x| + y\}. \tag{3.4}$$

Now we define

$$l_{p,q}(\vec{x}, \vec{h}) := (h_n - q + x_n - p)/2, \quad r_{p,q}(\vec{x}, \vec{h}) := (h_1 - q - x_1 + p)/2. \tag{3.5}$$

$l_{p,q}, r_{p,q}$ are the distances from *the primordial peak* of $(\vec{x}, \vec{h})$ to the left and right sides of the cone $C_{p,q}$, respectively. $l_{p,q}, r_{p,q}$ is basically a change in the coordinate system for $(x_{\text{prim}}, h_{\text{prim}})$. Whenever there is no confusion on p, q, the subscript will be dropped. See Figure(3.1) for all the geometric meanings of variables. Notice that the l, r defined in (2.7) is $l_{0,0}, r_{0,0}$. The reason for this generalization is that in the full space TASEP, all that matters is the relative position between $(x_{\text{prim}}, h_{\text{prim}})$ and $\{y_{\text{prim}}, s_{\text{prim}}\}$, since the model is translation invariant. However, in the half-space TASEP, the model would be different if there is a horizontal shift.

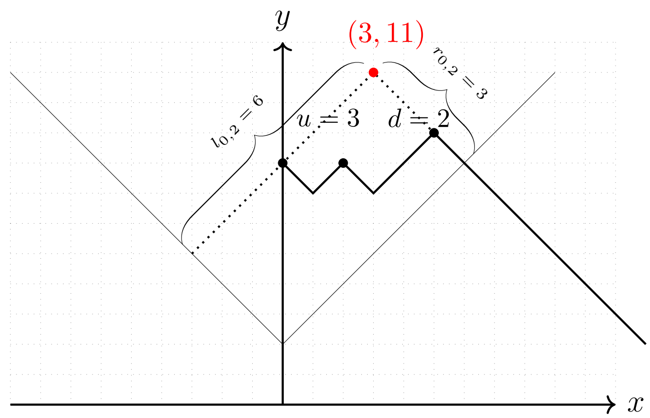


Figure 3.1: Configuration $(0, 8; 2, 8; 5, 9)$ with *the primordial peak* $(3, 11)$

We also need to define an operator similarly to $\mathtt{a}, \mathtt{a}_*$. For $\alpha > 0$,

$$\mathtt{b} = 2\alpha - 1 - 2D, \quad \mathtt{b}_* = 2\alpha - 1 + 2D \tag{3.6}$$

Now we are ready to state the main theorem.

Theorem 3.1.1. *Assume that we start the half-space TASEP with rate* $0 < \alpha < 1$ *with the initial configuration having peaks at* $(x_1, h_1; \ldots, x_n, h_n), x_1 \geq 0$. *The probability that at time t, it being below the configuration* $\{y, s\}, y \geq 0$ *is given by:*

$$\begin{aligned} \mathbb{P}((x_1, h_1 \cdots x_n, h_n)_t \leq \{y, s\}) &= \mathrm{Pf}(J + JK_{\mathrm{form1}})_{L^2(\mathbb{R})} \\ &= \sqrt{\det(I + K_{\mathrm{form1}})_{L^2(\mathbb{R}) \times L^2(\mathbb{R})}} \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} K_{\mathrm{form1}} = &\begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix} \begin{pmatrix} \mathtt{a}^{l-r}\mathtt{b}W^*\mathtt{a}^{r-l}\mathtt{b}^{-1} & -\mathtt{a}^{l-r}\mathtt{b}W^*\mathtt{a}^{r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r-l}W\mathtt{a}_*^{l-r}\mathtt{b}_* \\ 0 & \mathtt{a}_*^{r-l}\mathtt{b}_*^{-1}W\mathtt{a}_*^{l-r}\mathtt{b}_* \end{pmatrix} \\ &\cdot \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} \begin{pmatrix} \mathtt{a}^{l-r'}\mathtt{b}1_0\mathtt{a}^{r'-l}\mathtt{b}^{-1} & \mathtt{a}^{l-r'}\mathtt{b}1_0\mathtt{a}^{r'-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r'-l}1_0\mathtt{a}_*^{l-r'}\mathtt{b}_* \\ 0 & \mathtt{b}_*^{-1}\mathtt{a}_*^{r'-l}1_0\mathtt{a}_*^{l-r'}\mathtt{b}_* \end{pmatrix}. \end{aligned} \tag{3.8}$$

and $J(x, y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta_{xy}$. *In* K_{form1}, *every* S *is* $S_{1,1}^{l,l}$ *which is defined below:*

$$\begin{aligned} s_{a,b}^{i,j}(x, y) &= \int_\Gamma e^{-(x+y)w} \frac{(1 + 2w)^j (2\alpha - 1 + 2w)^b}{(1 - 2w)^i (2\alpha - 1 - 2w)^a} \frac{dw}{2\pi i} \\ S_{a,b}^{i,j}(x, y) &= s_{a,b}^{i,j}(x, y) 1_{x+y \geq 0}. \end{aligned} \tag{3.9}$$

Γ is a simple, positively oriented loop that includes $w = 1/2$ and $w = \pm(2\alpha - 1)/2$. W is the kernel defined in (2.8) *parameterized by the configuration* $(x_1, h_1; \ldots, x_n, h_n)$. *Furthermore,*

$$\begin{aligned} l &= l_{0,s-y}(\vec{x}, \vec{h}) := (h_n + x_n - s + y)/2, \\ r &= r_{0,s-y}(\vec{x}, \vec{h}) : (h_1 - x_1 - s + y)/2, \\ r' &= r_{0,-x_n-h_n}(\vec{y}, \vec{-s}) := (-s - y + x_n + h_n)/2. \end{aligned} \tag{3.10}$$

The operator $\overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}}$ *will be defined precisely in the remark* (3.1.3).

Remark 3.1.2. *We will name*

$$\begin{aligned} V &= \mathtt{b}_*^{-1} \mathtt{a}_*^{r-l} W \mathtt{a}_*^{l-r} \mathtt{b}_* \\ V' &= \mathtt{b}_*^{-1} \mathtt{a}_*^{r'-l} 1_0 \mathtt{a}_*^{l-r'} \mathtt{b}_*, \end{aligned} \tag{3.11}$$

then the kernel is K_{form1} is

$$K_{\text{form1}} = \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix} \begin{pmatrix} V^* & -V^* D V \\ 0 & V \end{pmatrix} \cdot \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} \begin{pmatrix} V'^* & V'^* D V' \\ 0 & V' \end{pmatrix}. \tag{3.12}$$

We write the kernel out in order to emphasize that there is a potential problem in $V^* D V$ when $0 < \alpha < \frac{1}{2}$, which will be illustrated in the following remark.

Remark 3.1.3. *We state precisely what we mean by $D^{-1}, \mathtt{b}^{-1}, \mathtt{b}_*^{-1}$ in the kernel. $D^{-1} f(x) = \int_{-\infty}^{x} f(t) dt$. For $\alpha > 0, \alpha \neq 1/2$, $\mathtt{b}^{-1}, \mathtt{b}_*^{-1}$ are the notation for the following two integral kernels:*

$$\mathtt{b}^{-1}(x, y) = \tfrac{1}{2} e^{(2\alpha-1)(x-y)/2} 1_{x \le y}, \quad \mathtt{b}_*^{-1}(x, y) = \tfrac{1}{2} e^{(2\alpha-1)(y-x)/2} 1_{y \le x}.$$

When $\alpha = 1/2$,

$$\mathtt{b}_*^{-1}(x, y) = 2D^{-1}(x, y) = -1_{x<y} + 1_{x \ge y}, \quad \mathtt{b}^{-1}(x, y) = -2D^{-1}(x, y) = 1_{x<y} - 1_{x \ge y}.$$

When $\alpha > 1/2$, the kernels have an exponential decay at infinity. When $\alpha < 1/2$, the kernels go to infinity at infinity, which is the non-physical Green's function. They can only act on functions with faster decay. $\mathtt{a}_*^{-1}, \mathtt{a}^{-1}$ decay fast enough since $\frac{1}{2} > \frac{1-2\alpha}{2}$ when $0 < \alpha < 1$. Thus, there is no problem when $\mathtt{b}^{-1}, \mathtt{b}_*^{-1}$ is followed by $\mathtt{a}^{-1}, \mathtt{a}_*^{-1}$.

When $\mathtt{b}^{-1}$ is followed by $\mathtt{b}_*^{-1}$, we explain what does it mean. "$\mathtt{b}^{-1}\mathtt{b}_*^{-1}$" should be thought of as the notation for one integral kernel, which we denote $\overline{\mathtt{b}^{-1}\mathtt{b}_*^{-1}}$, defined as *the following. When* $0<\alpha<1/2$*,*

$$\overline{\mathtt{b}^{-1}\mathtt{b}_*^{-1}}(x,z) = 1_{x\geq z}\tfrac{1}{4(2\alpha-1)}e^{(1-2\alpha)(x-z)/2} + 1_{x<z}\tfrac{1}{4(2\alpha-1)}e^{(1-2\alpha)(z-x)/2}. \tag{3.13}$$

and when $\alpha=1/2$*,*

$$\overline{\mathtt{b}^{-1}\mathtt{b}_*^{-1}}(x,z) = \tfrac{1}{4}((x-z)1_{x\geq z} + (z-x)1_{x<z}).$$

The reader might ask: What about the D *operator between them? One can think that* D *commutes with* $\mathtt{b}_*^{-1},\mathtt{b}^{-1}$*. Precisely, we define:*

$$\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}} := D\overline{\mathtt{b}^{-1}\mathtt{b}_*^{-1}} = \overline{\mathtt{b}^{-1}\mathtt{b}_*^{-1}}D. \tag{3.14}$$

When $0<\alpha<1/2$*, using definition (3.14), the two delta functions from taking derivatives on the indicator function in* (3.13) *cancels, and we get*

$$\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}(x,z) = -1_{x\geq z}\tfrac{1}{8}e^{(1-2\alpha)(x-z)/2} + 1_{x<z}\tfrac{1}{8}e^{(1-2\alpha)(z-x)/2}.$$

When $\alpha=1/2$*,*

$$\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}(x,z) = \tfrac{1}{4}(1_{x\geq z} - 1_{x<z}).$$

Similarly for $S_{1,1}^{l,l}\mathtt{b}_*^{-1},\mathtt{b}^{-1}S_{1,1}^{l,l}$ *(they are not well-defined in the normal operator composition since one of residue from* $S_{1,1}^{l,l}$ *does not decay at* ∞*), they are defined in the following way: when* $0<\alpha\leq 1/2$*,*

$$S_{1,1}^{l,l}\mathtt{b}_*^{-1} := S_{0,1}^{l,l}\overline{\mathtt{b}^{-1}\mathtt{b}_*^{-1}}, \quad \mathtt{b}^{-1}S_{1,1}^{l,l} := \overline{\mathtt{b}^{-1}\mathtt{b}_*^{-1}}S_{0,1}^{l,l}. \tag{3.15}$$

Notice that $\overline{\mathtt{b}^{-1}\mathtt{b}_*^{-1}}$ still behaves like the composition of $\mathtt{b}^{-1}$ and $\mathtt{b}_*^{-1}$, since it satisfies the relations

Lemma 3.1.4.

$$\begin{aligned} &\mathtt{b}_*\overline{(\mathtt{b}_*\mathtt{b})^{-1}} = \mathtt{b}^{-1}, \quad \overline{(\mathtt{b}_*\mathtt{b})^{-1}}\mathtt{b} = \mathtt{b}_*^{-1}. \\ &(S_{1,1}^{l,l}\mathtt{b}_*^{-1})\mathtt{b}_* := S_{1,1}^{l,l}, \quad \mathtt{b}(\mathtt{b}^{-1}S_{1,1}^{l,l}) := S_{1,1}^{l,l} \end{aligned} \tag{3.16}$$

Lemma 3.1.5. $\overline{(\mathtt{b}_*\mathtt{b})^{-1}}$ *commutes with* $\mathtt{a}$ *and* $\mathtt{a}_*$*.*

Both lemmas are straightforward calculus calculations.

Now we are going to prove Theorem (3.1.1).

3.1.2 Properties of operators

We will discuss the properties of objects in half-space kernels. Notice that $\mathtt{b}, \mathtt{b}_*$ is qualitatively the same as $\mathtt{a}, \mathtt{a}_*$. All the properties are similar to the full space, except that the notation becomes more complicated.

Lemma 3.1.6. *(Support of the operator) Let* $\mathtt{x}$ *be* $\mathtt{a}$ *or* $\mathtt{b}$*. Assume* $t_1 < t_2$*,* $k, u, d > 0$*; we have*

i. $\bar{1}^{t_1}\mathtt{x}_*^{-k}\mathtt{x}^u\bar{1}^{t_2} = \bar{1}^{t_1}\mathtt{x}_*^{-k}\mathtt{x}^u, \quad \langle\delta_{t_1}|\,\mathtt{x}_*^{-k}\mathtt{x}^u\bar{1}^{t_2} = \langle\delta_{t_1}|\,\mathtt{x}_*^{-k}\mathtt{x}^u$

ii. $\bar{1}^{t_2}\mathtt{x}_*^d\bar{1}^{t_1} = \mathtt{x}_*^d\bar{1}^{t_1}, \quad \bar{1}^{t_2}\mathtt{x}_*^d\,|\delta_{t_1}\rangle = \mathtt{x}_*^d\,|\delta_{t_1}\rangle$

iii. $\bar{1}^{t_1}\mathtt{x}_*^{-k}\mathtt{x}^u 1_{t_2} = 0, \quad \langle\delta_{t_1}|\,\mathtt{x}_*^{-k}\mathtt{x}^u 1_{t_2} = 0$

iv. $1_{t_2}\mathtt{x}_*^d\bar{1}^{t_1} = 0, \quad 1_{t_2}\mathtt{x}_*^d\,|\delta_{t_1}\rangle = 0$

The proof is omitted since it is the same as the full space case.

We now state the absorbing lemma for half-space. Recall the Lemma (2.2.21),

Lemma 3.1.7 (Absorbing Lemma I)**.** *For* $k < l$*,* $0 < t$*,*

$$1_0 S^{l,r}(I - \mathtt{a}_*^{-k}\mathtt{a}^u\bar{1}^t\tilde{W}_{1,n}\bar{1}^t\mathtt{a}_*^d\mathtt{a}_*^k)S^{r,l}1_0 = 1_0 S^{l,r-k}(I - \mathtt{a}^u\bar{1}^t\tilde{W}_{1,n}\bar{1}^t\mathtt{a}_*^d)S^{r-k,l}1_0. \quad (3.17)$$

Although we in the Lemma (2.2.21), there are only $\mathtt{a}, \mathtt{a}_*$ operators. It is easy to see that statements with one of the $\mathtt{a}, \mathtt{a}_*$ replaced by $\mathtt{b}, \mathtt{b}_*$ will still be true. The following version is what we will actually apply on $S_{1,1}^{l,l}WS_{1,1}^{l,l}$.

Corollary 3.1.8. *For* $k < l$*,* $0 < t$

$$1_0 S_{1,1}^{l,l}(I - \mathtt{a}_*^{-k}\mathtt{b}_*^{-1}\mathtt{a}^u\bar{1}^t W_{1,n}\bar{1}^t\mathtt{a}_*^d\mathtt{a}_*^k\mathtt{b}_*)S_{1,1}^{l,l}1_0 = 1_0 S_{1,0}^{l,l-k}(I - \mathtt{a}^u\bar{1}^t W_{1,n}\bar{1}^t\mathtt{a}_*^d)S_{0,1}^{l-k,l}1_0.$$

For the kernel $S_{1,1}^{l,l}\partial^{-1}W_I^*\partial S_{1,1}^{l,l}$, we have the similar equation:

Corollary 3.1.9. *For* $k < l$*,* $0 < t$*,*

$$\begin{aligned} &1_0 D^{-1}S_{1,1}^{l,l}(I - \mathtt{a}^k\mathtt{b}\mathtt{a}^d\bar{1}^t W_{1,n}^*\bar{1}^t\mathtt{a}_*^u\mathtt{a}^{-k}\mathtt{b}^{-1})DS_{1,1}^{l,l}1_0 \\ &\qquad = 1_0 D^{-1}S_{0,1}^{l-k,l}(I - \mathtt{a}^d\bar{1}^t W_{1,n}^*\bar{1}^t\mathtt{a}_*^u)DS_{1,0}^{l,l-k}1_0. \quad (3.18)\end{aligned}$$

Proof. The proof of Lemma (2.2.21) consists of two steps: move $\mathtt{a}^{-k}\mathtt{b}^{-1}$ to the second $S_{1,1}^{l,l}$ and move $\mathtt{a}^k\mathtt{b}$ into the first $S_{1,1}^{l,l}$. Since D^{-1}, D commutes with $\mathtt{a}^{-1}$ and $\mathtt{b}^{-1}$, and

D is a local operator, the first step of the proof is the same. For the second step, we want to conclude

$$\langle \delta_{-x}|\, D^{-1}\mathtt{a}^{k+d-i}1_t = 0 \text{ for } -x < 0 < t,$$

which is true by Lemma (3.1.6) if D^{-1} is not present. However, it is easy to check that since $D^{-1}(x,y)$ is supported on $x \geq y$, it is still correct. □

Lastly, for the kernel $D^{-1}S^{l,l}_{1,1}V^*DVS^{l,l}_{1,1}$, we have

Corollary 3.1.10. *For $k < l$, $0 < t$,*

$$\begin{aligned} &1_0 S^{l,l}_{1,1} D^{-1}\mathtt{a}^k\mathtt{b}(I - \mathtt{a}^d\bar{1}^t W^*_{1,n}\bar{1}^t\mathtt{a}^u_*)\mathtt{a}^{-k}\overline{\mathtt{b}^{-1}D\mathtt{b}^{-1}_*}\mathtt{a}^{-k}_*(I - \mathtt{a}^u\bar{1}^t W_{1,n}\bar{1}^t\mathtt{a}^d_*)\mathtt{a}^k_*\mathtt{b}_* S^{l,l}_{1,1}1_0 \\ &= 1_0 S^{l-k,l}_{0,1} D^{-1}(I - \mathtt{a}^d\bar{1}^t W^*_{1,n}\bar{1}^t\mathtt{a}^u_*)\mathtt{a}^{-k}\overline{\mathtt{b}^{-1}D\mathtt{b}^{-1}_*}\mathtt{a}^{-k}_*(I - \mathtt{a}^u\bar{1}^t W_{1,n}\bar{1}^t\mathtt{a}^d_*) S^{l-k,l}_{0,1}1_0. \end{aligned} \tag{3.19}$$

The proof is just that we apply step two twice on both sides of the equation. The middle of the operator remains unchanged.

Remark 3.1.11. *With the absorbing lemmas in hand, we present another form of the kernel in* (3.8)*. Recall from* (3.11) *that $V = \mathtt{a}^{r-l}_*\mathtt{b}^{-1}_* W \mathtt{a}^{l-r}_*\mathtt{b}_*$, what surrounded W are inverses of each other, thus by applying Corollary* (3.1.8),(3.1.9) *and* (3.1.10)*, all the operators $\mathtt{a}, \mathtt{a}_*, \mathtt{b}, \mathtt{b}_*$ surrounding $W^*, W, 1_0$ can be brought into $S^{l,l}_{1,1}$. Meanwhile, the last matrix in* (3.8) $\begin{pmatrix} (V')^* & (V')^*DV' \\ 0 & V' \end{pmatrix}$ *can be split into three matrices. If we call $Y = \mathtt{b}^{-1}_*\mathtt{a}^{r'-l}_*$, then it is*

$$\begin{aligned} &\begin{pmatrix} \mathtt{a}^{l-r'}\mathtt{b}1_0 & 0 \\ 0 & \mathtt{b}^{-1}_*\mathtt{a}^{r'-l}_*1_0 \end{pmatrix} \begin{pmatrix} 1_0 & 1_0\mathtt{a}^{r'-l}\overline{\mathtt{b}^{-1}D\mathtt{b}^{-1}_*}\mathtt{a}^{r'-l}_*1_0 \\ 0 & 1_0 \end{pmatrix} \begin{pmatrix} 1_0\mathtt{b}^{-1}\mathtt{a}^{r'-l} & 0 \\ 0 & 1_0\mathtt{a}^{l-r'}_*\mathtt{b}_* \end{pmatrix} \\ &= \begin{pmatrix} (Y^*)^{-1}1_0 & 0 \\ 0 & Y1_0 \end{pmatrix} \begin{pmatrix} I & 1_0 Y^* D Y 1_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 1_0 Y^* & 0 \\ 0 & 1_0 Y^{-1} \end{pmatrix} \end{aligned} \tag{3.20}$$

The first matrix can be brought into S using the absorbing lemma, and the last matrix can be brought into the first S using $\det(I - AB) = \det(I - BA)$*. Thus, the kernel also has the following form*

$$\begin{aligned} K_{\text{form2}} = &\begin{pmatrix} -S^{r,r'}_{0,0} & DS^{l,r+r'-l}_{1,-1} \\ D^{-1}S^{r+r'-l,l}_{-1,1} & -S^{r',r}_{0,0} \end{pmatrix} \begin{pmatrix} W^* & -W^*\mathtt{a}^{r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}^{-1}_*}\mathtt{a}^{r-l}_*W \\ 0 & W \end{pmatrix} \\ &\cdot \begin{pmatrix} S^{r',r}_{0,0} & DS^{l,r+r'-l}_{1,-1} \\ D^{-1}S^{r+r'-l,l}_{-1,1} & S^{r,r'}_{0,0} \end{pmatrix} \begin{pmatrix} 1_0 & 1_0\mathtt{a}^{r'-l}\overline{\mathtt{b}^{-1}D\mathtt{b}^{-1}_*}\mathtt{a}^{r'-l}_*1_0 \\ 0 & 1_0 \end{pmatrix}. \end{aligned} \tag{3.21}$$

We want to emphasize one important feature of this form. Here the second matrix only depends on the initial configuration $(\vec{x}, \vec{h})$. *All the parameters in the first and third matrix depend on both* $(x_{\text{prim}}, h_{\text{prim}})$ *and* $\{y, s\}$. *We will use the following notation for the first and third matrix:*

$$\begin{aligned} \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix}_{y,s} &:= \begin{pmatrix} -S_{0,0}^{r,r'} & DS_{1,-1}^{l,r+r'-l} \\ D^{-1}S_{-1,1}^{r+r'-l,l} & -S_{0,0}^{r',r} \end{pmatrix} \\ \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix}_{y,s} &:= \begin{pmatrix} S_{0,0}^{r',r} & DS_{1,-1}^{l,r+r'-l} \\ D^{-1}S_{-1,1}^{r+r'-l,l} & S_{0,0}^{r,r'} \end{pmatrix} \end{aligned} \tag{3.22}$$

This will be useful when discussing the initial condition and the multi-point distribution.

3.1.3 Kolmogorov equation

We first study how the kernel changes if there is a flip from the initial configuration. For checking the Kolmogorov equation, we will use the kernel in the form 1:

$$K_{\text{form1}} = \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix} \begin{pmatrix} V^* & -V^*DV \\ 0 & V \end{pmatrix} \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} \begin{pmatrix} V'^* & V'^*DV' \\ 0 & V' \end{pmatrix}$$

Notice that the variable l does not change during the dynamics of the TASEP; thus, all the variables t and initial configuration information are in the second kernel in (3.8), which is

$$\begin{pmatrix} V^* & -V^*DV \\ 0 & V \end{pmatrix} = \begin{pmatrix} \mathtt{a}^{l-r}\mathtt{b}W^*\mathtt{a}^{r-l}\mathtt{b}^{-1} & -\mathtt{a}^{l-r}\mathtt{b}W^*\mathtt{a}^{r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r-l}W \\ 0 & \mathtt{a}_*^{r-l}\mathtt{b}_*^{-1}W\mathtt{a}_*^{l-r}\mathtt{b}_* \end{pmatrix}$$

In order for notation convenience, we define $V_0 = \mathtt{a}_*^{r-l}\mathtt{b}_*^{-1}W_0, V_{n+1} = W_{n+1}\mathtt{a}_*^{l-r}\mathtt{b}_*$. Similar to (2.17), we define

$$\begin{aligned} V_{0,i} &= V_0\overline{1}^t W_{1,i} \text{ for } 1 < i \leq n, \\ V_{i,n+1} &= W_{i,n}\overline{1}^t V_{n+1}, \quad \text{for } 1 \leq i < n. \end{aligned} \tag{3.23}$$

Proposition 3.1.12. *Recall the definition of* K_{form1} *in* (3.8),

$$(\partial_t - \mathcal{L})K_{\text{form1}} = 0$$

We need the following two lemmas to prove the proposition.

Lemma 3.1.13. *Let $\partial_{t,i}V$ be the result that the partial operator hits i-th indicator function $\overline{1}^t$. Recall $\mathcal{L}_{x_i}V = V^{\downarrow x_i} - V$ if $x_i \neq 0$ and $\mathcal{L}_{x_i}V = \alpha(V^{\downarrow x_i} - V)$ if $x_i = 0$, where $V^{\downarrow x_i}$ is the kernel V parametrized by the configuration obtained from a flip at x_i from $(x_1, h_1; \dots, x_n, h_n)$.*

$$(\partial_{t,i} - \mathcal{L}_{x_i})V = \begin{cases} 0, & x_i \neq 0 \\ -\mathtt{ba}_*^{-1} V_0 \left|\delta_t\right\rangle \left\langle\delta_t\right| V_{1,n+1}, & x_i = 0. \end{cases} \tag{3.24}$$

Lemma 3.1.14. *For $x_i \geq 0$,*

$$\mathcal{L}_{x_i}(V^* D V) = (\mathcal{L}_{x_i}V^*)DV + V^* D \mathcal{L}_{x_i} V. \tag{3.25}$$

Lemma 3.1.15. *For any $\alpha > 0$, when $x_1 = 0$,*

$$((\partial_{t,1} - \mathcal{L}_{x_1})V^*)DV + V^* D(\partial_{t,1} - \mathcal{L}_{x_1})V = ((\partial_{t,1} - \mathcal{L}_{x_1})V^*)D + D(\partial_{t,1} - \mathcal{L}_{x_1})V \tag{3.26}$$

We first comment on these lemmas. V, V^* is essentially the same as the full-space kernel. Lemma (3.1.13) says that all are the same except for a flip at 0, which is expected. One then expects that the term V^*DV will cancel the non-zero term from (3.24). Lemma (3.1.14) says the operator $\mathcal{L}_{x_i}$ is actually Leibniz, which is due to the special structure of this operator. Lastly, Lemma (3.1.15) says that the operator almost becomes "$V^*D + DV$" after calculating $\partial_{t,1} - \mathcal{L}_{x_1}$. This is the lemma that illustrates what the half-space mechanism is. Lastly, we need to show that $\partial_t - \mathcal{L}$ acting on the determinant is 0, which is the following proposition.

Proposition 3.1.16. *Let*

$$F(t; (\vec{x}, \vec{h})) = \mathbb{P}((x_1, h_1 \cdots x_n, h_n)_t) \leq \{y, s\})$$

then

$$(\partial_t - \mathcal{L})F(t; (\vec{x}, \vec{h})) = 0 \tag{3.27}$$

This is the general structure for the Kolmogorov equation in the half-space. Now we will prove all the lemmas and propositions. We will first prove the Proposition (3.1.12) with the help of the two lemmas.

Proof. (Proof of Proposition (3.1.12)) We will show

$$\begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix} \begin{pmatrix} (\partial_t - \mathcal{L})V^* & (\partial_t - \mathcal{L})(-V^*DV) \\ 0 & (\partial_t - \mathcal{L})V \end{pmatrix} \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} = 0. \tag{3.28}$$

By Lemma (3.1.14), the middle matrix is

$$\begin{pmatrix} (\partial_t - \mathcal{L})V^* & -(\partial_t - \mathcal{L})V^*DV - V^*D(\partial_t - \mathcal{L})V \\ 0 & (\partial_t - \mathcal{L})V \end{pmatrix}.$$

By Lemma (3.1.13), we only need to consider the case that $x_1 = 0$. Using Lemma (3.1.15), we can represent the matrix as $\begin{pmatrix} M_{11} & -M_{11}D - DM_{22} \\ 0 & M_{22} \end{pmatrix}$ where $M_{11} = (\partial_{t,1} - \mathcal{L}_{x_1})V^*, M_{22} = (\partial_{t,1} - \mathcal{L}_{x_1})V$.

$$\begin{aligned}
(3.28) &= \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix} \begin{pmatrix} M_{11} & -M_{11}D - DM_{22} \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} \\
&= \begin{pmatrix} -SM_{11} & SM_{11}D + SDM_{22} + DSM_{22} \\ D^{-1}SM_{11} & -D^{-1}SM_{11}D - D^{-1}SDM_{22} - SM_{22} \end{pmatrix} \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} \\
&= \begin{pmatrix} -SM_{11}S + SM_{11}S + SDM_{22}D^{-1}S + DSM_{22}D^{-1}S & -SM_{11}DS + SM_{11}DS + SDM_{22}S + DSM_{22}S \\ D^{-1}SM_{11}S - D^{-1}SM_{11}S - D^{-1}SDM_{22}D^{-1}S - SM_{22}D^{-1}S & D^{-1}SM_{11}DS - D^{-1}SM_{11}DS - D^{-1}SDM_{22}S - SM_{22}S \end{pmatrix}
\end{aligned} \tag{3.29}$$

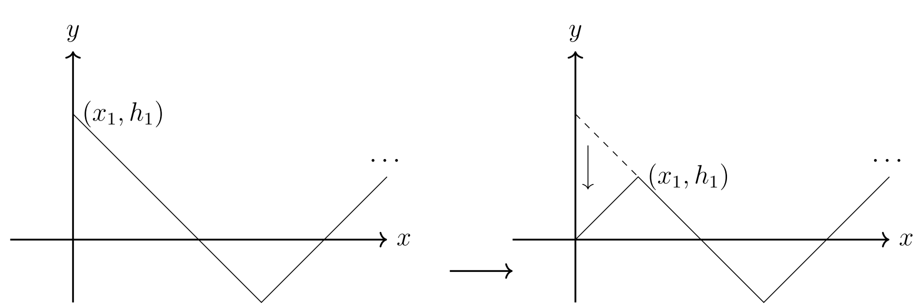
Using the relation that $DS = -SD$, we can see that the result is 0, which is what we want to prove.

□

Proof. (Proof of Lemma ((3.1.13))) If the flip is at a place that is not 0, the proof is completely the same as the full-space case. Thus, we focus on flips that happen at 0.

For a flip at 0, there are two types.

Type-1: This corresponds to the kernel change:



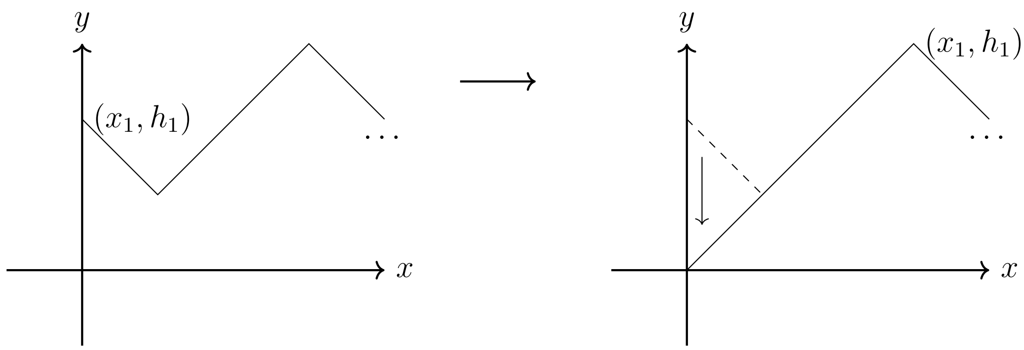
$$V_0\overline{1}^t W_{1,2}\cdots\overline{1}^t V_{n+1} \to \mathtt{a}_*^{-1} V_0 \overline{1}^t \mathtt{a}_* W_{1,2}\cdots V_{n+1}.$$

By switching $\mathtt{a}_*$ with $\overline{1}^t$ and commuting with V_0, it cancels $\mathtt{a}_*^{-1}$ in the front, so their difference is the commutator term

$$2\mathtt{a}_*^{-1} V_0 \left|\delta_t\right\rangle \left\langle\delta_t\right| W_{1,2}\cdots V_{n+1},$$

which is what we want.

Type-2: In this case, we have the following kernel change:



$$V_0\overline{1}^t W_{1,2}\overline{1}^t V_{2,n+1} \to V_0 W_{1,2}\overline{1}^t V_{2,n+1}.$$

This kernel transformation is not that straightforward. Recall $W_{1,2} = \mathtt{a}_*^{-1}\mathtt{a}^{-u_1}, V_0 = \mathtt{a}_*^{r-l}\mathtt{b}_*\mathtt{a}^u$. After the flip at the origin, note that the distance from *the primordial peak* to the first peak decreases by u_1. For any cone $C_{p,q}$, the distance from *the primordial peak* to the left side of the cone $C_{p,q}$ remains unchanged; the distance from *the primordial peak* to the right side of the cone $C_{p,q}$ decreases by 1. By removing the indicator function $\overline{1}^t$ between V_0 and $W_{1,2}$, we have $V_0 W_{1,2} = \mathtt{a}_*^{r-l-1}\mathtt{a}^{u-u_1}\mathtt{b}_*$. This is exactly the new "V_0" term that corresponds to the configuration on the right graph.

Now we compute

$$V_0 W_{1,2}\overline{1}^t V_{2,n+1} - V_0\overline{1}^t W_{1,2}\overline{1}^t V_{2,n+1} = V_0 1_t W_{1,2}\overline{1}^t V_{2,n+1}.$$

Notice that there is only one $\mathtt{a}_*^{-1}$ in $W_{1,2}$. By switching the order of $\mathtt{a}_*^{-1}$ in $W_{1,2}$ with 1_t, we have

$$V_0 1_t W_{1,2}\overline{1}^t V_{2,n+1} = 2\mathtt{a}_*^{-1} V_0 \left|\delta_t\right\rangle \left\langle\delta_t\right| W_{1,2}\overline{1}^t V_{2,n+1} + V_0\mathtt{a}_*^{-1} 1_t \mathtt{a}^{-u_1}\overline{1}^t V_{2,n+1}.$$

The second term is 0 since $1_t\mathtt{a}^{-u_1}\overline{1}^t = 0$, thus we have the desired result. □

Proof. (Proof of Lemma (3.1.14)) By definition,

$$\begin{aligned}\mathcal{L}_{x_i}(V^*DV) =&(V^*DV)^{\downarrow x_i} - (V^*DV)\\ &= (V^*)^{\downarrow x_i} DV^{\downarrow x_i} - (V^*)^{\downarrow x_i} DV\\ &+ (V^*)^{\downarrow x_i} DV - V^*DV.\end{aligned} \tag{3.30}$$

The third line is $(\mathcal{L}_{x_i}V^*)DV$. The second line is $(V^*)^{\downarrow x_i} D(\mathcal{L}_{x_i}V)$. So the difference between equation (3.25) and equation (3.30) is

$$V^*D(\mathcal{L}_{x_i}V) - (V^*)^{\downarrow x_i} D(\mathcal{L}_{x_i}V) = (\mathcal{L}_{x_i}V^*)D(\mathcal{L}_{x_i}V). \tag{3.31}$$

Recall from Lemma (2.2.18), (3.1.13) that $\mathcal{L}_{x_i}V = V^{\downarrow x_i} - V$ is a rank-one operator for all x_i, so we write it as $|h\rangle\langle g|$ for some h and g. Then we have $(\mathcal{L}_{x_i}V^*) = (\mathcal{L}_{x_i}V)^* = |g\rangle\langle h|$. So we conclude that

$$(3.31) = |g\rangle\langle h| D |h\rangle\langle g| = 0.$$

since D is an antisymmetric operator, which shows that $\mathcal{L}_{x_i}$ is also Leibniz. □

Proof. (Proof of Lemma (3.1.15)) We first discuss the case $\alpha > 1/2$. Look at the term $V^*_{1,n+1}|\delta_t\rangle\langle\delta_t| V^*_0 \mathtt{b}_*\mathtt{a}^{-1}DV$, write out what is V, we have

$$V^*_{1,n+1}|\delta_t\rangle\langle\delta_t| V^*_0 \mathtt{b}_*\mathtt{a}^{-1}DV = V^*_{0,n}|\delta_t\rangle\langle\delta_t| V^*_0 \mathtt{b}_*\mathtt{a}^{-1}D(I - V_0\overline{1}^t V_{1,n+1}) \tag{3.32}$$

Notice one key fact: when the first peak is at 0*, that means* $l - r = u$. So we have

$$V^*_0 = \mathtt{a}^{-u}\mathtt{a}^u_*\mathtt{b}^{-1}, \quad V_0 = \mathtt{a}^{-u}_*\mathtt{a}^u\mathtt{b}^{-1}_*.$$

Plug into equation (3.32), we claim that the part containing $V_0\overline{1}^t V_{1,n+1}$ is 0, because

$$\begin{aligned}&V^*_{0,n}|\delta_t\rangle\langle\delta_t| V^*_0\mathtt{b}_*\mathtt{a}^{-1}DV_0\overline{1}^t V_{1,n+1}\\ =&V^*_{0,n}|\delta_t\rangle\langle\delta_t| \mathtt{a}^{-u}\mathtt{a}^u_*\mathtt{b}^{-1}\mathtt{b}_*\mathtt{a}^{-1}D\mathtt{a}^{-u}_*\mathtt{a}^u\mathtt{b}^{-1}_*\overline{1}^t V_{1,n+1}\\ =&V^*_{0,n}|\delta_t\rangle\langle\delta_t| \mathtt{b}^{-1}\mathtt{a}^{-1}D\overline{1}^t V_{1,n+1}.\end{aligned}$$

The second equality is true because all operators in the middle commute. By Lemma (3.1.6), the integral operator $(\mathtt{b}^{-1}\mathtt{a}^{-1}D)(x,y)$ is supported on $y \geq x$, thus

$\langle\delta_t|\,\mathtt{b}^{-1}\mathtt{a}^{-1}D\overline{1}^t = 0$. So equation (3.32) equals

$$V_{0,n}^*\,|\delta_t\rangle\,\langle\delta_t|\,V_0^*\mathtt{b}_*\mathtt{a}^{-1}D.$$

which is exactly the same as $(\partial_{t,1} - \mathcal{L}_{x_1})V^*D$ when $x_1 = 0$. Similarly, the term $V^*D\mathtt{b}\mathtt{a}_*^{-1}V_0\,|\delta_t\rangle\,\langle\delta_t|\,V_{1,n+1}$ is

$$\begin{aligned} V^*D\mathtt{b}\mathtt{a}_*^{-1}V_0\,|\delta_t\rangle\,\langle\delta_t|\,V_{1,n+1} &= (I - V_{1,n+1}^*\overline{1}^tV_0^*)D\mathtt{b}\mathtt{a}_*^{-1}V_0\,|\delta_t\rangle\,\langle\delta_t|\,V_{1,n+1} \\ &= -D\mathtt{b}\mathtt{a}_*^{-1}V_0\,|\delta_t\rangle\,\langle\delta_t|\,V_{1,n+1} \end{aligned} \tag{3.33}$$

because the part containing $V_{1,n+1}^*\overline{1}^tV_0^*$ is

$$\begin{aligned} &V_{1,n+1}^*\overline{1}^tV_0^*D\mathtt{b}\mathtt{a}_*^{-1}V_0\,|\delta_t\rangle \\ &= V_{1,n+1}^*\overline{1}^t\mathtt{a}^{-u}\mathtt{a}_*^{u}\mathtt{b}^{-1}D\mathtt{b}\mathtt{a}_*^{-1}\mathtt{a}_*^{-u}\mathtt{a}^{u}\mathtt{b}_*^{-1}\,|\delta_t\rangle \\ &= V_{1,n+1}^*\overline{1}^tD\mathtt{a}_*^{-1}\mathtt{b}_*^{-1}\,|\delta_t\rangle = 0. \end{aligned} \tag{3.34}$$

The last term is 0 for the same reason that $(D\mathtt{a}_*^{-1}\mathtt{b}_*^{-1})(x,y)$ is supported on $x \geq y$.

For $1/2 > \alpha > 0$, we need to check the term $\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}$ carefully.

$$\begin{aligned} &(\partial_{t,1} - \mathcal{L}_{x_1})(V^*\partial V) = \\ &-\mathtt{b}(V_{1,n+1}^*)^\circ\,|\delta_t\rangle\,\langle\delta_t|\,(V_0^*)^\circ\mathtt{b}_*\mathtt{a}^{-1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}(V)^\circ\mathtt{b}_* \\ &-\mathtt{b}(V^*)^\circ\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{b}\mathtt{a}_*^{-1}(V_0)^\circ\,|\delta_t\rangle\,\langle\delta_t|\,(V_{1,n+1})^\circ\mathtt{b}_*. \end{aligned} \tag{3.35}$$

$V^\circ, (V^*)^\circ, V_0^\circ, (V_0^*)^\circ, (V_{1,n+1})^\circ, (V_{1,n+1}^*)^\circ$ all means that operator $\mathtt{b}_*, \mathtt{b}$ and their inverses are pulled out. In particular,

$$(V_0^*)^\circ = \mathtt{a}^{-u}\mathtt{a}_*^{u}, \quad V_0^\circ = \mathtt{a}_*^{-u}\mathtt{a}^{u}.$$

Now we write V° and $(V^*)^\circ$ as in the case $\alpha > 1/2$. $V^\circ = (I - V_0^\circ\overline{1}^tV_{1,n+1}), (V^*)^\circ = (I - V_{1,n+1}^*\overline{1}^t(V_0^*)^\circ)$. Look at the first term in (3.35),

$$-\mathtt{b}(V_{1,n+1}^*)^\circ\,|\delta_t\rangle\,\langle\delta_t|\,(V_0^*)^\circ\mathtt{b}_*\mathtt{a}^{-1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}(I - V_0^\circ\overline{1}^tV_{1,n+1}) \tag{3.36}$$

We first show the term has $V_0^\circ\overline{1}^tV_{1,n+1}$ is 0, that is to show

$$-\mathtt{b}(V_{1,n+1}^*)^\circ\,|\delta_t\rangle\,\langle\delta_t|\,(V_0^*)^\circ\mathtt{b}_*\mathtt{a}^{-1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}V_0^\circ\overline{1}^tV_{1,n+1} = 0.$$

Notice that $(V_0^*)^\circ$ cancels the term V_0° since they all commute with $\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}$, what is left is

$$-\mathtt{b}(V_{1,n+1}^*)^\circ |\delta_t\rangle \langle\delta_t| \mathtt{b}_*\mathtt{a}^{-1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\overline{1}^t V_{1,n+1}.$$

Looking at $\mathtt{b}_*\mathtt{a}^{-1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}$,

$$\mathtt{b}_*\mathtt{a}^{-1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}} = \mathtt{b}_*\mathtt{a}^{-1}D\overline{(\mathtt{b}_*\mathtt{b})^{-1}} = \mathtt{a}^{-1}D\mathtt{b}_*\overline{(\mathtt{b}_*\mathtt{b})^{-1}} = \mathtt{a}^{-1}D\mathtt{b}^{-1} \tag{3.37}$$

The first equality is according to our definition, the second equality is according to commutativity, and the third equality is according to the lemma (3.1.4). Then the term is 0 because of the support of the operator. Notice that this is the reason we want to define $\mathtt{b}_*^{-1}, \mathtt{b}^{-1}$ as the non-physical Green's function. It maintains the same domain as in $\alpha > 1/2$.

Thus,

$$\begin{aligned}(3.36) &= -\mathtt{b}(V_{1,n+1}^*)^\circ |\delta_t\rangle \langle\delta_t| (V_0^*)^\circ \mathtt{b}_*\mathtt{a}^{-1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}} \\ &= -\mathtt{b}(V_{1,n+1}^*)^\circ |\delta_t\rangle \langle\delta_t| (V_0^*)^\circ \mathtt{a}^{-1}\mathtt{b}^{-1}D \\ &= (\partial_{t,1} - \mathcal{L}_{x_1})V^*D, \quad x_1 = 0\end{aligned} \tag{3.38}$$

For the second line in (3.35), using the same proof, we can see that it is $-D(\partial_{t,1} - \mathcal{L}_{x_1})V$. This finishes the proof of the lemma in the case that $0 < \alpha < 1/2$.

Lastly, we check the case $\alpha = 1/2$. All the properties used in (3.37) are still valid for $\alpha = 1/2$. Notice that the definition of $\mathtt{b}^{-1}$ is that it is the integral operator with kernel $\mathtt{b}^{-1}(x,z) = 1_{x<z} - 1_{x\geq z}$, so $D\mathtt{b}^{-1} = -2I$. Thus, $\mathtt{a}^{-1}D\mathtt{b}^{-1} = -2\mathtt{a}^{-1}$, so $\langle\delta_t| \mathtt{a}^{-1}D\mathtt{b}^{-1}\overline{1}^t = 0$. This shows that the proof in the case $0 < \alpha < 1/2$ will all go through; thus, the lemma also works in the case $\alpha = 1/2$.

□

Proof. (Proof of Proposition (3.1.16)) It is a well-known identity that if the kernel depends smoothly on a parameter t, then the partial derivative of the Fredholm determinant is

$$\partial_t \sqrt{\det(I-K)} = \frac{1}{2}\sqrt{\det(I-K)}\,\mathrm{tr}(I-K)^{-1}\partial_t K.$$

Now we check how $\mathcal{L}_{x_i}$ acts on the square root of the determinant. By Lemma (2.2.18) and Lemma (3.1.13), $\mathcal{L}_{x_i}V$ is a rank-1 operator for all $x_i \geq 0$. Let us denote

$$\mathcal{L}_{x_i}V = |h\rangle\langle g|, \quad \mathcal{L}_{x_i}V^* = |g\rangle\langle h|$$

for some function h, g. Recall our kernel in (3.8) is defined by

$$K = \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix} \begin{pmatrix} V^* & -V^*DV \\ 0 & V \end{pmatrix} \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} \begin{pmatrix} (V')^* & -(V')^*DV' \\ 0 & V' \end{pmatrix}. \quad (3.39)$$

In order to write things in matrix form, we name the first matrix in (3.39) by $\mathcal{S}_1$, the second matrix by $\mathcal{V}$, the third matrix by $\mathcal{S}_2$, and the last matrix $\mathcal{V}'$.

So

$$K^{\downarrow x_i} = K + \mathcal{S}_1 \begin{pmatrix} |g\rangle \langle h| & -|g\rangle \langle h| DV - V^*D |h\rangle \langle g| \\ 0 & |h\rangle \langle g| \end{pmatrix} \mathcal{S}_2\mathcal{V}'$$

Define the rank one operator

$$G_1 = \begin{pmatrix} |g\rangle \\ 0 \end{pmatrix}, \quad H_1 = \begin{pmatrix} \langle h| & -\langle h| DV \end{pmatrix},$$

$$G_2 = \begin{pmatrix} -V^*D |h\rangle \\ |h\rangle \end{pmatrix} \quad H_2 = \begin{pmatrix} 0 & \langle g| \end{pmatrix}.$$

Let $\mathcal{G} = \begin{pmatrix} G_1 & G_2 \end{pmatrix}, \mathcal{H} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$,

$$\begin{pmatrix} |g\rangle \langle h| & -|g\rangle \langle h| DV - V^*D |h\rangle \langle g| \\ 0 & |h\rangle \langle g| \end{pmatrix} = \mathcal{G}\mathcal{H}.$$

Thus, generator $\mathcal{L}_{x_i}$ gives a rank two perturbation. Thus, by Proposition (5.1.3),

$$\det(I - K^{\downarrow x_i}) = \det(I - K) \det(I - (I - K)^{-1}\mathcal{S}_1\mathcal{G}\mathcal{H}\mathcal{S}_2\mathcal{V}') \quad (3.40)$$

Using Proposition (5.1.2),

$$(3.40) = \det(I - K) \det(I - \mathcal{H}\mathcal{S}_2\mathcal{V}'(I - K)^{-1}\mathcal{S}_1\mathcal{G}). \quad (3.41)$$

The second determinant on the right-hand side is a 2×2 determinant. We denote this matrix as $\begin{pmatrix} 1 - M_{11} & M_{12} \\ M_{21} & 1 - M_{22} \end{pmatrix}$ and write out each component.

$$M_{12} = H_1\mathcal{S}_2\mathcal{V}'(I - K)^{-1}\mathcal{S}_1G_2.$$

We will show that M_{12} is anti-symmetric, i.e. $M_{12} = M_{12}^*$. Since M_{12} is a scalar, thus it can only be 0.

$$M_{12}^* = G_2^* \mathcal{S}_1^* ((I-K)^{-1})^* (\mathcal{V}')^* \mathcal{S}_2^* H_1^*.$$

Notice the relation that

$$\begin{aligned} &G_2^* = H_1 J, \quad H_1^* = JG_2, \\ &\mathcal{S}_1^* = J\mathcal{S}_2 J, \quad \mathcal{S}_2^* = J\mathcal{S}_1^* J, \\ &\mathcal{V}^* = -J\mathcal{V}J, \quad (\mathcal{V}')^* = -J\mathcal{V}'J. \end{aligned} \tag{3.42}$$

J is the matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, thus $J^2 = -I$.

$$\begin{aligned} M_{12}^* &= H_1 J J \mathcal{S}_2 J ((I-K)^{-1})^* (\mathcal{V}')^* J \mathcal{S}_1^* J J G_2 \\ &= H_1 \mathcal{S}_2 J ((I-K)^{-1})^* (\mathcal{V}')^* J \mathcal{S}_1^* G_2. \end{aligned}$$

For $((I-K)^{-1})^*$, we have

$$((I-K)^{-1})^* = \sum_{n=0}^{\infty} ((\mathcal{S}_1 \mathcal{V} \mathcal{S}_2 \mathcal{V}')^n)^* = \sum_{n=0}^{\infty} -J(\mathcal{V}' \mathcal{S}_1 \mathcal{V} \mathcal{S}_2)^n J$$

Thus,

$$J((I-K)^{-1})^* (\mathcal{V}')^* J = \sum_{n=0}^{\infty} -(\mathcal{V}' \mathcal{S}_1 \mathcal{V} \mathcal{S}_2)^n \mathcal{V}' = \sum_{n=0}^{\infty} -\mathcal{V}' (\mathcal{S}_1 \mathcal{V} \mathcal{S}_2 \mathcal{V}')^n = -\mathcal{V}'(I-K)^{-1}.$$

This finishes the proof that M_{12} is antisymmetric, thus $M_{12} = 0$. $M_{21} = H_2 \mathcal{S}_2 \mathcal{V}'(I - K)^{-1} \mathcal{S}_1 G_1$, since we also have the relation that

$$H_2^* = -JG_1, G_1^* = -H_2 J, \tag{3.43}$$

Thus M_{21} is also antisymmetric. With the same proof, thus $M_{21} = 0$.

Now we want to prove $M_{11}^* = M_{22}$, from which we can derive $M_{11} = M_{22}$ since they are scalars.

$$\begin{aligned} M_{11} &= H_1 \mathcal{S}_2 \mathcal{V}' (I-K)^{-1} \mathcal{S}_1 G_1, \\ M_{22} &= H_2 \mathcal{S}_2 \mathcal{V}' (I-K)^{-1} \mathcal{S}_1 G_2. \end{aligned}$$

Using the relations in (3.42) and (3.43), it is easy to see that $M_{11}^* = M_{22}$. Thus we have

$$\det \begin{pmatrix} 1 - M_{11} & M_{12} \\ M_{21} & 1 - M_{22} \end{pmatrix} = (1 - M_{11})(1 - M_{22}) = (1 - \tfrac{1}{2}(M_{11} + M_{22}))^2.$$

Now we have

$$\begin{aligned} &\sqrt{\det(I - K^{\downarrow x_i})} - \sqrt{\det(I - K)} \\ &= \sqrt{\det(I - K)(1 - \tfrac{1}{2}(M_{11} + M_{22}))^2} - \sqrt{\det(I - K)} \\ &= \tfrac{1}{2}\sqrt{\det(I - K)}(M_{11} + M_{22}) = \tfrac{1}{2}\sqrt{\det(I - K)}\,\mathrm{tr}(I - K)^{-1}\mathcal{L}_{x_i}K. \end{aligned}$$

The last equality is true because for the rank one operators $|h\rangle\langle g|$, we have

$$\mathrm{tr}(I - K)^{-1}|h\rangle\langle g| = \langle g|(I - K)^{-1}|h\rangle$$

By summing all x_i, the proposition is proved. □

3.1.4 Initial condition

To check the initial condition, similar to the full space case, we need an absorbing lemma to reduce unuseful peaks and require some properties about $S^{m,n}_{a,b}$.

We want to emphasize that the lemma in the half-space version is not that different; the reason is that $\mathtt{b}, \mathtt{b}_*$ are exactly the same type of operator as $\mathtt{a}, \mathtt{a}_*$.

Now we introduce the absorbing lemma for half-space.

Lemma 3.1.17. *Let K_{form1} be defined as the kernel defined in Theorem* (3.1.1)*.*

$$K_{\mathrm{form1}} = \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix} \begin{pmatrix} V^* & -V^*DV \\ 0 & V \end{pmatrix} \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} \begin{pmatrix} V'^* & V'^*DV' \\ 0 & V' \end{pmatrix}$$

Recall that the first three matrices are parameterized by $C_{0,s-y}(x_1, h_1; \ldots; x_n, h_n)$. If $r \le d$, then

$$K_{\mathrm{form1}} = K^{\backslash x_n}_{\mathrm{form1}} \tag{3.44}$$

The notation $K^{\backslash x_n}_{\mathrm{form1}}$ means that it is the kernel parameterized by the configuration with the peak x_n removed:

$$C_{0,s_m-y_m}(x_1, h_1; \ldots, x_{n-1}, h_{n-1}).$$

Proof. (Proof of Lemma (3.1.17). Let us multiply out the first three matrices to make it clearer.

$$\begin{pmatrix} -SV^*S+SV^*DVD^{-1}S+DSVD^{-1}S & -SV^*DS+SV^*DVS+DSVS \\ D^{-1}SV^*S-D^{-1}SV^*DVD^{-1}S-SVD^{-1}S & D^{-1}SV^*DS-D^{-1}SV^*DVS-SVS \end{pmatrix} \tag{3.45}$$

There are three types of operators: SVS, SV^*S, SV^*DVS if we ignore the D, D^{-1} in between them. It turns out that D, D^{-1} will not affect the arguments in this proof.

We look at SVS first. Applying Corollary (3.1.8) on $1_0 S^{l,l}_{1,1} V S^{l,l}_{1,1} 1_0$, it equals

$$1_0 S^{l,r}_{1,0}(I - \mathtt{a}^u \overline{1}^t W_{1,n} \overline{1}^t \mathtt{a}^d_*) S^{r,l}_{0,1} 1_0.$$

Since $r \leq d$, we can apply Lemma (2.2.22) on the right-hand side of the equation, it becomes

$$1_0 S^{l-u_n,r}_{1,0}(I - \mathtt{a}^{u-u_n} \overline{1}^t W_{1,n-1} \overline{1}^t \mathtt{a}^{d-d_n}_*) S^{r,l-u_n}_{0,1} 1_0.$$

Since $l - u_n \geq r$, we apply Corollary (3.1.8) again, it becomes

$$1_0 S^{l-u_n,l-u_n}_{1,1}(I - \mathtt{a}^{-l+u_n+r}_* \mathtt{b}^{-1}_* \mathtt{a}^{u-u_n} \overline{1}^t W_{1,n-1} \overline{1}^t \mathtt{a}^{d-d_n}_* \mathtt{a}^{l-u_n-r}_* \mathtt{b}_*) S^{l-u_n,l-u_n}_{1,1} 1_0.$$

This is exactly the kernel SVS parameterized by $C_{0,s_m-y_m}(x_1, h_1; \ldots; x_{n-1}, r_{n-1})$.

For $1_0 S^{l,l}_{1,1} V^* S^{l,l}_{1,1} 1_0$, it is similar. Applying Corollary (3.1.9), we have it equal to

$$1_0 S^{r,l}_{0,1}(I - \mathtt{a}^d \overline{1}^t W^*_{1,n} \overline{1}^t \mathtt{a}^u_*) S^{l,r}_{1,0} 1_0.$$

Now we can apply Lemma (2.2.22) on the left side of the equation; it becomes

$$1_0 S^{r,l-u_n}_{0,1}(I - \mathtt{a}^{d-d_n} \overline{1}^t W^*_{1,n-1} \overline{1}^t \mathtt{a}^{u-u_n}_*) S^{l-u_n,r}_{1,0} 1_0.$$

Applying Corollary (3.1.9) again, it becomes

$$1_0 S^{l-u_n,l-u_n}_{1,1}(I - \mathtt{a}^{l-u_n-r} \mathtt{b} \mathtt{a}^{d-d_n} \overline{1}^t W^*_{1,n-1} \overline{1}^t \mathtt{a}^{u-u_n}_* \mathtt{a}^{-l+u_n+r}_* \mathtt{b}^{-1}_*) S^{l-u_n,l-u_n}_{1,1} 1_0,$$

which is the kernel SVS parameterized by $C_{0,s_m-y_m}(x_1, h_1; \ldots; x_{n-1}, h_{n-1})$. For kernel $S^{l,l}_{1,1} V^* DV S^{l,l}_{1,1}$, it is slightly different. After we apply Corollary (3.1.10), we get

$$S^{r,l}_{0,1}(I - \mathtt{a}^d \overline{1}^t W^*_{1,n} \overline{1}^t \mathtt{a}^u_*) \mathtt{a}^{r-l} \overline{\mathtt{b}^{-1} D \mathtt{b}^{-1}_*} \mathtt{a}^{r-l}_* (I - \mathtt{a}^u \overline{1}^t W_{1,n} \overline{1}^t \mathtt{a}^d_*) S^{r,l}_{0,1}.$$

On both sides of the equation, the same manipulation is still true; we have

$$S_{0,1}^{r,l-u_n}(\mathtt{a}_*^{u_n} - \mathtt{a}^{d-d_n}\overline{1}^t W_{1,n-1}^* \overline{1}^t \mathtt{a}_*^{u})\mathtt{a}^{r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r-l}(\mathtt{a}^{u_n} - \mathtt{a}^{u}\overline{1}^t W_{1,n-1}\overline{1}^t \mathtt{a}_*^{d-d_n})S_{0,1}^{r,l-u_n}.$$

Pull out $\mathtt{a}_*^{u_n}$ in the first bracket and $\mathtt{a}^{u_n}$ in the second bracket, using the fact that they both commute with $\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}$, i.e.

$$\mathtt{a}_*^{u_n}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}^{u_n} = \mathtt{a}^{u_n}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{u_n},$$

we can see that $\mathtt{a}_*^{u_n}, \mathtt{a}^{u_n}$ adds to $\mathtt{a}_*^{r-l}, \mathtt{a}^{r-l}$ on the other side of $\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}$. The formula becomes

$$\begin{aligned} S_{0,1}^{r,l-u_n}(I - \mathtt{a}^{d-d_n}\overline{1}^t W_{1,n-1}^* \overline{1}^t \mathtt{a}_*^{u-u_n})\mathtt{a}^{u_n+r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}} \\ \circ\, \mathtt{a}_*^{u_n+r-l}(I - \mathtt{a}^{u-u_n}\overline{1}^t W_{1,n-1}\overline{1}^t \mathtt{a}_*^{d-d_n})S_{0,1}^{r,l-u_n}. \end{aligned}$$

Last step, apply Corollary (3.1.10), we have it equal to

$$\begin{aligned} S_{1,1}^{l-u_n,l-u_n}\mathtt{a}^{l-u_n-r}\mathtt{b}(I - \mathtt{a}^{d-d_n}\overline{1}^t W_{1,n-1}^* \overline{1}^t \mathtt{a}_*^{u-u_n})\mathtt{a}^{u_n+r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}} \\ \circ\, \mathtt{a}_*^{u_n+r-l}(I - \mathtt{a}^{u-u_n}\overline{1}^t W_{1,n-1}\overline{1}^t \mathtt{a}_*^{d-d_n})\mathtt{a}^{l-u_n-r}\mathtt{b}S_{0,1}^{r,l-u_n}. \end{aligned}$$

which is the kernel $S_{1,1}^{l,l}V^*DVS_{1,1}^{l,l}$ parameterized by $C_{0,s_m-y_m}(x_1, h_1; \cdots; x_{n-1}, h_{n-1})$. After the simplification, the kernel can be again factored into

$$\begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix}\begin{pmatrix} V^* & -V^*DV \\ 0 & V \end{pmatrix}\begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix}$$

with the new parameters. □

Recall Lemma (2.2.25) illustrates when there is a correct Dirac delta function on one side, $\mathtt{a}_*, \mathtt{a}$ can act on S nicely. We have the same lemma for $\mathtt{b}, \mathtt{b}_*$,

Lemma 3.1.18. *For any* $-t_1 < t_2$,

$$1_{t_1}S_{a,b}^{n,m}\mathtt{b}\,|\delta_{t_2}\rangle = 1_{t_1}S_{a-1,b}^{n,m}\,|\delta_{t_2}\rangle\,, \quad \langle\delta_{t_2}|\,\mathtt{b}_*S_{a,b}^{n,m}1_{t_1} = \langle\delta_{t_2}|\,S_{a-1,b}^{n,m}1_{t_1}. \tag{3.46}$$

For any $-t_1 \neq t_2$,

$$\langle\delta_{t_1}|\,S_{a,b}^{n,m}\mathtt{b}\,|\delta_{t_2}\rangle = \langle\delta_{t_1}|\,S_{a-1,b}^{n,m}\,|\delta_{t_2}\rangle\,, \quad \langle\delta_{t_2}|\,\mathtt{b}_*S_{a,b}^{n,m}\,|\delta_{t_1}\rangle = \langle\delta_{t_2}|\,S_{a-1,b}^{n,m}\,|\delta_{t_1}\rangle\,. \tag{3.47}$$

Lemma 3.1.19. *Recall the definition of $s^{n,m}_{a,b}$ in (3.7).*

$$s^{n,m}_{0,0}(x,-x) = 0 \text{ if } n-2 \geq m > 0, \quad s^{n,n-1}_{0,0}(x,-x) = -1/2$$
$$s^{n,m}_{1,0}(x,-x) = 0 \text{ if } n-1 \geq m > 0$$

Proof. Evaluating the residue of the integrand,

$$s^{n,m}_{0,0}(x,y) = (-1)^n \frac{2^{m-n}}{(n-1)!} \sum_{i=0}^{n-1\wedge m} \binom{n-1}{i} \frac{m!}{(m-i)!} (-x-y)^{n-1-i} e^{-\frac{1}{2}(x+y)}. \quad (3.48)$$

When $x = -y$ and $n-2 \geq m$, the degree of $(-x-y)$ is always positive; thus, it is 0. When $m = n-1$, it can easily be seen that the value is $-1/2$.

Now we prove for $s_{1,0}$, which has residue at both $w = 1/2$ and $w = (2\alpha-1)/2$.

$$s^{n,m}_{1,0}(x,y) = 2^{m-n-1}(-1)^{n+1}\Bigg\{ e^{-\frac{2\alpha-1}{2}(x+y)} \frac{\alpha^m}{(\alpha-1)^n} + \frac{e^{-\frac{1}{2}(x+y)}}{(n-1)!} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i\wedge m} \binom{n-1}{i}\binom{n-1-i}{j} \frac{(-x-y)^i m!(-(n-1-i-j))!}{(m-j)!(1-\alpha)^{n-1-i-j}} \Bigg\} \quad (3.49)$$

Plug in $y = -x$, it equals

$$\begin{aligned} &= 2^{m-n-1}(-1)^{n+1}\left\{ \frac{\alpha^m}{(\alpha-1)^n} + \frac{1}{(n-1)!} \sum_{j=0}^{n-1\wedge m} \binom{n-1}{j} \frac{m!(-(n-1-j))!}{(m-j)!(1-\alpha)^{n-j}} \right\} \\ &= 2^{m-n-1}(-1)^{n+1}\left\{ \frac{\alpha^m}{(\alpha-1)^n} - \sum_{j=0}^{n-1\wedge m} \frac{1}{j!} \frac{m!}{(m-j)!} (\frac{-1}{1-\alpha})^{n-j} \right\}. \end{aligned} \quad (3.50)$$

When $n > m$, the last term is

$$\sum_{j=0}^{m} \frac{1}{j!} \frac{m!}{(m-j)!} (\frac{-1}{1-\alpha})^{n-j}) = (1-(1-\alpha))^m (\alpha-1)^{-n} = \frac{\alpha^m}{(\alpha-1)^n},$$

which cancels the first term in the parenthesis. □

The next lemma prepares the eigenfunction for the kernel.

Lemma 3.1.20. Recall the V' defined to be $V' = \mathtt{b}_*^{-1}\mathtt{a}_*^{r'-l}1_0\mathtt{a}_*^{l-r'}\mathtt{b}_*$ in (3.11), then

$$V'S_{1,0}^{l,0}\left|\delta_0\right\rangle = S_{1,0}^{l,0}\left|\delta_0\right\rangle,$$
$$(V')^*DV'S_{1,0}^{l,0}\left|\delta_0\right\rangle = (V')^*DS_{1,0}^{l,0}\left|\delta_0\right\rangle.$$

Proof. The first equation is obtained by switching $\mathtt{a}_*, \mathtt{b}_*$ with the indicator function 1_0,

$$\begin{aligned}&\mathtt{b}_*^{-1}\mathtt{a}_*^{r'-l}1_0\mathtt{a}_*^{l-r'}\mathtt{b}_*S_{1,0}^{l,0}\left|\delta_0\right\rangle\\ &= 1_0S_{1,0}^{l,0}\left|\delta_0\right\rangle + \mathtt{b}_*^{-1}\mathtt{a}_*^{r'-l}\left|\delta_0\right\rangle\left\langle\delta_0\right|\mathtt{a}_*^{l-r}S_{1,0}^{l,0}\left|\delta_0\right\rangle\\ &+ \mathtt{a}_*^{l-r'}\sum_{i=1}^{l-r'}\mathtt{a}_*^{r'-l+i-1}\left|\delta_0\right\rangle\left\langle\delta_0\right|\mathtt{a}_*^{l-r'-i}S_{1,0}^{l,0}\left|\delta_0\right\rangle.\end{aligned}$$

By Lemma (3.1.18) and Lemma (3.1.19), we have

$$\left\langle\delta_0\right|\mathtt{a}_*^{l-r'-i}S_{1,0}^{l,0}\left|\delta_0\right\rangle = S_{1,1}^{r'+i} \text{ for all } i = 0,\cdots l-r'$$

(Here Lemma (3.1.18) is used in the sense that we take $S_{1,0}^{l,0}\left|\delta_0\right\rangle = \lim_{\varepsilon\to 0} S_{1,0}^{l,0}\left|\delta_\varepsilon\right\rangle$). What is left is $1_0S_{1,0}^{l,0}\left|\delta_0\right\rangle = S_{1,0}^{l,0}\left|\delta_0\right\rangle$.

For the second statement, it is obvious in the case $\alpha > 1/2$, via the first equality. When $0 < \alpha \le 1/2$, it is not obvious since there is $\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}$ in $(V')^*DV'$. Recall

When $0 < \alpha \le 1/2$, using the same argument above, by switching $\mathtt{b}_*, \mathtt{a}^{l-r'}$ across 1_0, we get

$$\begin{aligned}&\mathtt{a}^{l-r'}\mathtt{b}1_0\mathtt{a}^{r'-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r'-l}1_0\mathtt{a}_*^{l-r'}\mathtt{b}_*S_{1,0}^{l,0}\left|\delta_0\right\rangle\\ &= \mathtt{a}^{l-r'}\mathtt{b}1_0\mathtt{a}^{r'-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{b}_*1_0S_{1,0}^{l,0}\left|\delta_0\right\rangle\end{aligned} \tag{3.51}$$

Since all the commutator terms are still 0. Then using (3.1.4),

$$(3.51) = \mathtt{a}^{l-r'}\mathtt{b}1_0\mathtt{a}^{l-r'}\mathtt{b}^{-1}D1_0S_{1,0}^{l,0}\left|\delta_0\right\rangle, \tag{3.52}$$

which is what we want. □

Similar to Lemma (2.2.27), we have

Lemma 3.1.21. *For $t \ge 0$, if $a \ge c+2, c \ge 0$,*

$$\left\langle\delta_t\right|S_{0,1}^{a,b}1_{-t}S_{1,0}^{b,c}\left|\delta_t\right\rangle = 0, \quad \left\langle\delta_t\right|D^{-1}S_{0,1}^{a,b}1_{-t}DS_{1,0}^{b,c}\left|\delta_t\right\rangle = 0.$$

Proof. For the first equality,

$$\begin{aligned}\langle\delta_t| S^{a,b}_{0,1} 1_{-t} S^{b,c}_{1,0} |\delta_t\rangle &= \int_{-t}^{\infty} dy s^{a,b}_{0,1}(t,y) s^{b,c}_{1,0}(y,t) \\ &= 2s^{a,b}(t,-t) s^{b,c}_{1,0}(-t,t) + \langle\delta_t| S^{a,b} 1_t S^{b,c} |\delta_t\rangle\end{aligned}$$

In the first term, either $s^{a,b}(t,-t)$ is 0 or $s^{b,c}_{1,0}(-t,t)$ is 0 due to Lemma (3.1.19); the second term is 0 by Lemma (2.2.27). For the second equality,

$$\langle\delta_t| D^{-1} S^{a,b}_{0,1} 1_{-t} D S^{b,c}_{1,0} |\delta_t\rangle = D^{-1} s^{a,b}_{0,1}(t,-t) s^{b,c}_{1,0}(-t,t) + \langle\delta_t| S^{a,b}_{0,1} 1_{-t} S^{b,c}_{1,0} |\delta_t\rangle$$

The first boundary term is 0 due to Lemma (3.1.19); the second term reduces to the first case. ☐

Lemma 3.1.22. *(Eigenfunction lemma for $S_{0,1}, S_{1,0}$)*

$$\begin{aligned}&1_0 S^{n,m}_{1,0} 1_0 S^{m,n}_{0,1} S^{n,0}_{1,0} |\delta_0\rangle = S^{n,0}_{1,0} |\delta_0\rangle \\ &1_0 D S^{n,m}_{1,0} 1_0 D^{-1} S^{m,n}_{0,1} 1_0 D S^{n,0}_{1,0} |\delta_0\rangle = -D S^{n,0}_{1,0} |\delta_0\rangle\end{aligned} \tag{3.53}$$

when $n > 0$.

Proof.

$$\begin{aligned}S^{n,m}_{1,0} 1_0 S^{m,n}_{0,1} S^{n,0}_{1,0} |\delta_0\rangle &= \int_0^\infty dy \int_0^\infty dz s^{n,m}_{1,0}(x,y) s^{m,n}_{0,1}(y,z) s^{n,0}_{1,0}(z,0) \\ &= \sum_{i=0}^{m-1} 2\int_0^\infty dz s^{n,m-1-i}_{1,0}(x,0) s^{m-i,n}_{0,1}(0,z) s^{n,0}_{1,0}(z,0) \\ &+ \int_0^\infty dy \int_0^\infty dz s^{n,0}_{1,0}(x,y) s^{0,n}_{0,1}(y,z) s^{n,0}_{1,0}(z,0).\end{aligned} \tag{3.54}$$

From Lemma (3.1.21), all terms are zero except when $i = m-1$. Thus, we have

$$S^{n,m}_{1,0} 1_0 S^{m,n}_{0,1} S^{n,0}_{1,0} |\delta_0\rangle = 2\int_0^\infty dz s^{n,0}_{1,0}(x,0) s^{1,n}_{0,1}(0,z) s^{n,0}_{1,0}(z,0). \tag{3.55}$$

Following the proof of Lemma (3.1.21), it is easy to see that

$$\int_0^\infty dz 2 s^{1,n}_{0,1}(0,z) s^{n,0}_{1,0}(z,0) = 4s^{1,0}(0,0) s^{1,0}(0,0) = 1$$

Thus, equation (3.55) is $s^{n,0}_{1,0} |\delta_0\rangle$

To prove the second statement, first, one needs to carefully check that when D

and D^{-1} hit the indicator function in $S_{1,0}$, it generates nothing due to the indicator functions 1_0 and 1_0. Then we have

$$
\begin{aligned}
&1_0 D S_{1,0}^{n,m} 1_0 D^{-1} S_{0,1}^{m,n} 1_0 D S_{1,0}^{n,0} \left|\delta_0\right\rangle \\
&= \int_0^\infty dy \int_0^\infty dz D s_{1,0}^{n,m}(x,y) D^{-1} s_{0,1}^{m,n}(y,z) D s_{1,0}^{n,0}(z,0) \\
&= -\int_0^\infty dy D s_{1,0}^{n,m}(x,y) D^{-1} s_{0,1}^{m,n}(y,0) s_{1,0}^{n,0}(0,0) \\
&- \int_0^\infty dy \int_0^\infty dz D s_{1,0}^{n,m}(x,y) s_{0,1}^{m,n}(y,z) s_{1,0}^{n,0}(z,0)
\end{aligned}
$$

The first term is 0 due to $s_{1,0}^{n,0}(0,0)$. Continuing with the second term, we have

$$
\begin{aligned}
&- \int_0^\infty dy \int_0^\infty dz D s_{1,0}^{n,m}(x,y) s_{0,1}^{m,n}(y,z) s_{1,0}^{n,0}(z,0) \\
&= -\int_0^\infty dy D s_{1,0}^{n,m}(x,y) s^{m,n}(y,0) s_{1,0}^{n,0}(0,0) \\
&- \int_0^\infty dy \int_0^\infty dz D s_{1,0}^{n,m}(x,y) s^{m,n}(y,z) s^{n,0}(z,0) \\
&= -\int_0^\infty dy D s_{1,0}^{n,m}(x,y) s^{m,n}(y,0) s_{1,0}^{n,0}(0,0) \\
&- \sum_{i=0}^{n-1} 2 \int_0^\infty dy D s_{1,0}^{n,m}(x,y) s^{m,n-i-1}(y,0) s^{n-i,0}(z,0).
\end{aligned}
$$

Similarly, the first term is 0, and in the summation, the only term that is not 0 is $i = n-1$, we have

$$
-\int_0^\infty D 2 s_{1,0}^{n,m}(x,y) s^{m,0}(y,0) s^{1,0}(0,0) = \int_0^\infty D s_{1,0}^{n,m}(x,y) s^{m,0}(y,0).
$$

Now, keep doing integration by parts again, we have

$$
\begin{aligned}
&\int_0^\infty D s_{1,0}^{n,m}(x,y) s^{m,0}(y,0) = \sum_{i=0}^{m-1} 2 D s_{1,0}^{n,m-i-1}(x,0) s^{m-i,0}(0,0) \\
&= -D s_{1,0}^{n,0}(x,0).
\end{aligned}
$$

Recall there is 1_0 at the beginning, thus it is $-D S_{1,0}^{n,0} \left|\delta_0\right\rangle$. □

Lemma 3.1.23. For $n > 0$,

$$\begin{aligned} &S_{0,1}^{m,n} W^* \mathtt{a}^{r-l} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{r-l} W S_{0,1}^{m,n} S_{1,0}^{n,0} |\delta_0\rangle \\ &= S_{0,1}^{n,m} W^* \mathtt{a}^{r-l} \mathtt{b}^{-1} D S_{1,0}^{n,0} |\delta_0\rangle \end{aligned} \tag{3.56}$$

Proof. From explicit expressions for $S_{1,0}^{n,m}, S_{0,1}^{n,m}$ in Lemma (3.1.19), we see that when $l > r$,

$$S_{0,0}^{l,r} = \varrho \frac{(1-2D)^r}{(1+2D)^l}, \text{ or } S_{0,0}^{l,r} = \frac{(1+2D)^r}{(1-2D)^l} \varrho$$

Using the same proof as in Lemma (3.1.22),

$$S_{0,0}^{n,m} W S_{0,1}^{m,n} S_{1,0}^{n,0} |\delta_0\rangle = S_{0,0}^{n,0}.$$

Then using definition (3.15) and (3.1.4), we have

$$S_{0,1}^{m,n} W^* \mathtt{a}^{r-l} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} S_{0,0}^{n,0} |\delta_0\rangle = S_{0,1}^{m,n} W^* \mathtt{a}^{r-l} \mathtt{b}^{-1} D S_{1,0}^{n,0} |\delta_0\rangle,$$

which is what we want to prove. □

Proposition 3.1.24. *Given the initial condition $(\vec{x}, \vec{h})$. Let $H(x)$ be the height function of the TASEP associated with $(\vec{x}, \vec{h})$. Let $F(t, H) = \mathbb{P}((\vec{x}, \vec{h})_t \leq \{y, s\})$ be defined in* (3.1.1). *Then*

$$\lim_{t \to 0} F(t, H) = 1_{(\vec{x}, \vec{h}) \leq \{y,s\}} = 1_{H(y) \leq s}.$$

Proof. If the trough $\{y, s\}$ is outside the cone $C^{x_{\text{prim}}, h_{\text{prim}}}$, we have either $r \leq 0$ or $r' \leq 0$ in the kernel (3.21). Recall the kernel is

$$\begin{aligned} K_{\text{form2}} = &\begin{pmatrix} -S_{0,0}^{r,r'} & D S_{1,-1}^{l,r+r'-l} \\ D^{-1} S_{-1,1}^{r+r'-l,l} & -S_{0,0}^{r',r} \end{pmatrix} \begin{pmatrix} W^* & -W^* \mathtt{a}^{r-l} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{r-l} W \\ 0 & W \end{pmatrix} \\ &\cdot \begin{pmatrix} S_{0,0}^{r',r} & D S_{1,-1}^{l,r+r'-l} \\ D^{-1} S_{-1,1}^{r+r'-l,l} & S_{0,0}^{r,r'} \end{pmatrix} \begin{pmatrix} 1_0 & 1_0 \mathtt{a}^{r'-l} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{r'-l} 1_0 \\ 0 & 1_0 \end{pmatrix}. \end{aligned} \tag{3.57}$$

If $r \leq 0$, then $r + r' - l \leq 0$ since $r' \leq l$ is always true by definition. In that case, both $S_{0,0}^{r,r'}, S_{-1,1}^{r+r'-l,l}$ are 0, so the whole kernel is 0. If $r \leq 0$, then $r + r' - l \leq 0$ since $r \leq l$ is always true by definition. In that case, both $S_{0,0}^{r',r}, S_{-1,1}^{r+r'-l,l}$ are 0. Thus we have that the probability is 1, which is what we want to show.

Now assume the trough $\{y, s\}$ is inside the cone. If $r \leq d$, then we can apply the absorbing Lemma (3.1.17) to reduce the kernel. We keep applying the kernel until either there is only one peak or $r > d$ after some steps. In the case that $r > d$, we want to show that the probability is 0, by giving an explicit eigenfunction. We show that $\begin{pmatrix} DS^{l,0}_{1,0}\,|\delta_0\rangle \\ -S^{l,0}_{1,0}\,|\delta_0\rangle \end{pmatrix}$ is an eigenfunction with eigenvalue -1. To check the eigenvalue, we use the kernel in the form in (3.8), which is

$$K_{\text{form1}} = \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix} \begin{pmatrix} \mathtt{a}^{l-r}\mathtt{b}W^*\mathtt{a}^{r-l}\mathtt{b}^{-1} & -\mathtt{a}^{l-r}\mathtt{b}W^*\mathtt{a}^{r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r-l}W \\ 0 & \mathtt{a}_*^{r-l}\mathtt{b}_*^{-1}W\mathtt{a}_*^{l-r}\mathtt{b}_* \end{pmatrix}$$
$$\cdot \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} \begin{pmatrix} \mathtt{a}^{l-r'}\mathtt{b}1_0\mathtt{a}^{r'-l}\mathtt{b}^{-1} & \mathtt{a}^{l-r'}\mathtt{b}1_0\mathtt{a}^{r'-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r'-l}1_0\mathtt{a}_*^{l-r'}\mathtt{b}_* \\ 0 & \mathtt{b}_*^{-1}\mathtt{a}_*^{r'-l}1_0\mathtt{a}_*^{l-r'}\mathtt{b}_* \end{pmatrix}. \tag{3.58}$$

Using Lemma (3.1.20),

$$\begin{pmatrix} \mathtt{a}^{l-r'}\mathtt{b}1_0\mathtt{a}^{r'-l}\mathtt{b}^{-1} & \mathtt{a}^{l-r'}\mathtt{b}1_0\mathtt{a}^{r'-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r'-l}1_0\mathtt{a}_*^{l-r'}\mathtt{b}_* \\ 0 & \mathtt{b}_*^{-1}\mathtt{a}_*^{r'-l}1_0\mathtt{a}_*^{l-r'}\mathtt{b}_* \end{pmatrix} \begin{pmatrix} DS^{l,0}_{1,0}\,|\delta_0\rangle \\ -S^{l,0}_{1,0}\,|\delta_0\rangle \end{pmatrix} \tag{3.59}$$

which is $\begin{pmatrix} 0 \\ -S^{l,0}_{1,0}\,|\delta_0\rangle \end{pmatrix}$.

Now write out the result of the multiplication of the first three matrices, set $H := \mathtt{a}^{r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r-l}$, $S_2 = S^{l,r}_{1,0}$, $S_1 = S^{r,l}_{0,1}$ to save some space; it is

$$\begin{pmatrix} -S_1W^*S_2 + DS_2WD^{-1}S_1 - S_1W^*HWD^{-1}S_1 & -S_1W^*DS_2 + DS_2WS_1 - S_1W^*HWS_1 \\ D^{-1}S_1W^*S_2 - S_2WD^{-1}S_1 + D^{-1}S_1W^*HWD^{-1}S_1 & D^{-1}S_1W^*DS_2 - S_2WS_1 + D^{-1}S_1W^*HWS_1 \end{pmatrix}$$

By Lemma (3.1.22),(3.1.23),

$$\begin{aligned} (-S_1W^*DS_2 + DS_2WS_1 - S_1W^*HWS_1)(-S^{l,0}_{1,0}\,|\delta_0\rangle) &= -DS^{l,0}_{1,0}\,|\delta_0\rangle \\ (D^{-1}S_1W^*DS_2 - S_2WS_1 + D^{-1}S_1W^*HWS_1)(-S^{l,0}_{1,0}\,|\delta_0\rangle) &= S^{l,0}_{1,0}\,|\delta_0\rangle \end{aligned} \tag{3.60}$$

which finishes the proof that the eigenvalue is -1.

□

3.1.5 Uniqueness

In the previous sections, we showed that the equation we proposed in (3.1.1) satisfies the Kolmogorov equation, with the correct initial condition. Now we want to show that this solution is unique, which ensures that the equation in (3.1.1) is the probability distribution for the half-space TASEP.

Up to now, we have been using the TASEP peak function representations. We want to switch to the particle occupation variable notation. Notice that there is a one-to-one correspondence between the peak functions and the occupation variable plus the initial height $h(0)$.

$$(\vec{x}, \vec{h}) \longleftrightarrow (h(0), \eta).$$

Assume u is a solution to the following equation

$$\begin{cases} (\partial_t - \mathcal{L})u(t, h(0), \eta) = 0, \\ u(0, h(0), \eta) = 0. \end{cases}$$

Notice that $\mathcal{L}$ acts on $(h(0), \eta)$. We want to show that $u(T, h(0), \eta) = 0$ for all $T > 0$, which implies the uniqueness of the solution. Assume μ is the product of product Bernoulli(α) measures on η, and the counting measure on $2\mathbb{Z}$, which is invariant for $\mathcal{L}$. Let $T > 0$, for any $v(t, \eta)$, we have

$$\int_0^T \int (\partial_t - \mathcal{L})u(t, \eta)v(t, \eta)dtd\mu = 0. \tag{3.61}$$

By integration by parts,

$$\int u(T, \eta)v(T, \eta) - u(0, \eta)v(0, \eta)d\mu - \int_0^T \int u(t, x)(\partial_t + \mathcal{L}^*)v(t, \eta)dtd\mu = 0 \tag{3.62}$$

If we have a set of $v(t, h(0), \eta)$ which satisfies the following condition:

$$\begin{cases} (\partial_t + \mathcal{L}^*)v(t, h(0), \eta) = 0 \\ v(T, h(0), \eta) = 1_{h(0)=k, \eta_1 = a_1, \cdots, \eta_n = a_n} \text{ where } a_i \in \{0, 1\}, k \in 2\mathbb{Z}. \end{cases} \tag{3.63}$$

then we have $\int u(T, h(0), \eta)1_{\eta_1 = a_1, \cdots, \eta_n = a_n} d\mu = 0$, for all finite particle configurations with a prefixed height at 0, which implies that $u(T, h(0), \eta) = 0$.

Now we only need to solve equation (3.63). It turns out that $\mathcal{L}^*$ is the generator of TASEP with particles jumping to the left with rate 1, and with rate $(1 - \alpha)$, the particle at site 1 will be annihilated. See Proposition (3.1.25). This is also the model that the half-space TASEP is running backward; thus, we solved it while solving the half-space TASEP. It is just the equation in (3.98) with t replaced by $-t$. This finishes the proof that the solution is unique for $0 < \alpha < 1$.

Proposition 3.1.25. *The adjoint of the half-space TASEP generator $\mathcal{L}_\alpha$ under the* *stationary measure μ_α is*

$$\mathcal{L}^* f(h(0),\eta) = \sum_{x\geq 2} \eta_x(f(\eta_{x,x-1}) - f(\eta)) + (1-\alpha)(f(h(0)+2,\eta_1=0) - f(h(0),\eta)). \tag{3.64}$$

Proof.

$$\begin{aligned}\int \mathcal{L}f(\eta)g(\eta)d\mu_\alpha = \int \Big(& \sum_{x\geq 1} \eta_x(f(\eta_{x,x+1}) - f(\eta)) \\ & + (1-\alpha)\big(f(h(0)-2;\eta_1=1) - f(h(0);\eta)\big)\Big) g(\eta)d\mu_\alpha.\end{aligned} \tag{3.65}$$

By renaming $\eta_{x,x+1}$ to be η, we have it equal to

$$\begin{aligned}&= \int \sum_{x\geq 2} \eta_x f(\eta)(g(\eta_{x,x-1}) - g(\eta))d\mu_\alpha + \int \eta_N f(\eta)(g(\eta)) - \int \eta_1 f(\eta)g(\eta)d\mu_\alpha \\ &+ \int (1-\alpha)(f(h(0)-2;\eta_1=1) - f(h(0);\eta))g(\eta)d\mu_\alpha.\end{aligned} \tag{3.66}$$

η_N, η_1 are independent, have a probability α of being 1, and a probability $1-\alpha$ of being 0, thus it equals

$$\begin{aligned}&= \int d\mu_\alpha \sum_{x\geq 2} \eta_x f(\eta)(g(\eta_{x,x-1}) - g(\eta)) + \int d\mu_\alpha^{\eta_1} \alpha^2 f(\eta_1=1)g(\eta_1=1) \\ &+ \int d\mu_\alpha^{\eta_1} \alpha(1-\alpha) f(\eta_1=0)g(\eta_1=0) - \int d\mu_\alpha^{\eta_1} \alpha f(\eta_1=1)g(\eta_1=1) \\ &+ \int d\mu_\alpha^{\eta_1} \alpha(1-\alpha)\big(f(h(0)-2;\eta_1=1) - f(h(0);\eta_1=0)\big)g(\eta_1=0).\end{aligned} \tag{3.67}$$

$d\mu_\alpha^{\eta_1}$ means the product measure excluding η_1. Observe that $f(\eta_1 = 0)g(\eta_1 = 0)$ cancels. It equals

$$\begin{aligned}&\int \sum_{x\geq 2} \eta_x f(\eta)(g(\eta_{x,x-1}) - g(\eta))d\mu_\alpha \\ &+ \int d\mu_\alpha^{\eta_1} \alpha(1-\alpha) f(h(0)-2,\eta_1=1)g(\eta_1=0) - f(\eta_1=1)g(\eta_1=1)).\end{aligned} \tag{3.68}$$

For the second line, since we have the counting measure on $h(0)$, by a change of variable, it equals

$$\begin{aligned}&\int d\mu_\alpha^{\eta_1}\alpha(1-\alpha)f(h(0),\eta_1=1)g(h(0)+2,\eta_1=0)-f(\eta_1=1)g(\eta_1=1))\\&=\int d\mu_\alpha(1-\alpha)\big(g(h(0)+2,\eta_1=0)-g(\eta_1=1)\big)f(\eta_1=1)\end{aligned}\tag{3.69}$$

which is what we want. □

3.2 Multi-point distribution

In this section, we will present the multi-point distribution formula for the half-space TASEP. We will show it satisfies the Kolmogorov equation and has the correct initial condition. The uniqueness argument is the same as in the previous section. Before we state the theorem, recall the kernel in the one-point case in (3.8),

$$K_{\text{form1}}=\begin{pmatrix}-S & DS\\ D^{-1}S & -S\end{pmatrix}\begin{pmatrix}V^* & -V^*DV\\ 0 & V\end{pmatrix}\begin{pmatrix}S & DS\\ D^{-1}S & S\end{pmatrix}\begin{pmatrix}(V')^* & -(V')^*DV'\\ 0 & V'\end{pmatrix}\tag{3.70}$$

, where V is parametrized by $\{\vec{x},\vec{h}\}$; V' is parametrized by $\{y,s\}$. We need to transform the kernel into another form. Recall $V'=\mathtt{b}_*^{-1}\mathtt{a}_*^{r'-l}1_0\mathtt{a}_*^{l-r'}\mathtt{b}_*$. Let's denote $Y=\mathtt{b}_*^{-1}\mathtt{a}_*^{r'-l}$; thus, $V'=Y1_0Y^{-1}$. Apply Proposition (5.1.2),

$$\begin{aligned}&\det(I-K_{\text{form1}})\\&=\det\left(I-\begin{pmatrix}1_0(Y^*) & 0\\ 0 & 1_0Y^{-1}\end{pmatrix}\begin{pmatrix}-S & DS\\ D^{-1}S & -S\end{pmatrix}\begin{pmatrix}V^* & -V^*DV\\ 0 & V\end{pmatrix}\right.\\&\left.\cdot\begin{pmatrix}S & DS\\ D^{-1}S & S\end{pmatrix}\begin{pmatrix}(Y^*)^{-1}1_0 & 0\\ 0 & Y1_0\end{pmatrix}\begin{pmatrix}I & 1_0Y^*DY1_0\\ 0 & I\end{pmatrix}\right)\end{aligned}\tag{3.71}$$

Call $\tilde{K}$ to be the composition of the first 5 matrices in the kernel; then it becomes

$$\begin{aligned}\det(I-K_{\text{form1}})&=\det(I-\tilde{K}\begin{pmatrix}I & 1_0Y^*DY1_0\\ 0 & I\end{pmatrix})\\&=\det(I-\tilde{K}\begin{pmatrix}I & 1_0Y^*DY1_0\\ 0 & I\end{pmatrix})\det(I-\begin{pmatrix}0 & 1_0Y^*DY1_0\\ 0 & 0\end{pmatrix})\\&=\det(I-\begin{pmatrix}0 & 1_0Y^*DY1_0\\ 0 & 0\end{pmatrix}-\tilde{K})\end{aligned}\tag{3.72}$$

The second equality is true since the second determinant is 1; the third equality is true since $\begin{pmatrix} I & Y^*DY \\ 0 & I \end{pmatrix}\begin{pmatrix} I & -Y^*DY \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Notice that both $\tilde{K}$ and $1_0 Y^* DY 1_0$ are on $L^2([0,\infty)$, thus the Fredholm Pfaffian can be thought of as defined on $L^2([0,\infty)) \times L^2([0,\infty))$. We will use this new kernel in defining the multipoint formula. To prepare for the multipoint formula, we need to slightly modify Y^*DY. Let $\tilde{Y}^* = \mathtt{b}\mathtt{a}^{-l}\mathtt{a}_*^r, \tilde{Y} = \mathtt{b}_*\mathtt{a}_*^{-l}\mathtt{a}^r$. In the one-point case, $\tilde{Y}^*D\tilde{Y} = Y^*DY$ since all these operators commute; however, in the multi-point case, they will be indexed by different variables.

The next thing is that we want to use the kernel in the form in (3.21), thus

$$\tilde{K} = \begin{pmatrix} 1_0(Y^*) & 0 \\ 0 & 1_0 Y^{-1} \end{pmatrix}\begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix}\begin{pmatrix} V^* & -V^*DV \\ 0 & V \end{pmatrix} \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix}\begin{pmatrix} (Y^*)^{-1}1_0 & 0 \\ 0 & Y1_0 \end{pmatrix} \quad (3.73)$$

Now invoke the general philosophy from the full-space case, see remark (2.3.2): In the ij entry, the left piece is parametrized by $\{y_i, s_i\}$; the right piece is parametrized by $\{y_j, s_j\}$. Now we can state the theorem.

Theorem 3.2.1. *Assume that we start the half-space TASEP with the initial configuration having peaks at $(x_1, h_1; \ldots, x_n, h_n)$. The probability that at time t it is below the configuration $\{y_1, s_1; \cdots; y_m, s_m\}$ is given by:*

$$\mathbb{P}((x_1, h_1; \ldots; x_n, h_n)_t \leq \{y_1, s_1; \cdots; y_m, s_m\}) = \mathrm{Pf}(J + JK)_{L^2([0,\infty))^m}, \quad (3.74)$$

where K is a matrix-valued kernel on m copies of $L^2([0,\infty)) \times L^2([0,\infty))$. That means the kernel K maps $\{1, \cdots, m\} \times \mathbb{R}$ to a 2×2 antisymmetric matrix.

$$K(i, \cdot; j, \cdot) = \begin{pmatrix} 1_{i<j}(\mathtt{a})^{-d'_{ij}}(\mathtt{a}_*)^{-u'_{ij}} & 0 \\ 0 & 1_{i<j}(\mathtt{a}_*)^{-d'_{ij}}(\mathtt{a})^{-u'_{ij}} \end{pmatrix} + \tilde{K}_{ij} + \begin{pmatrix} 0 & 1_0 Y_i^* \mathtt{a}_*^{l_j - l_i} D \mathtt{a}^{r_i - r_j} Y_j 1_0 \\ 0 & 0 \end{pmatrix} \quad (3.75)$$

where

$$\tilde{K}_{ij} = \begin{pmatrix} 1_0 & 0 \\ 0 & 1_0 \end{pmatrix} \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix}_{y_i,s_i} \begin{pmatrix} W^* & -W^* \mathtt{a}^{r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r-l}W \\ 0 & W \end{pmatrix} \cdot \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix}_{y_j,s_j} \begin{pmatrix} 1_0 & 0 \\ 0 & 1_0 \end{pmatrix} \tag{3.76}$$

The second and fourth matrices are defined in (4.11)*, with parameters being*

$$\begin{aligned} l_i &= l_{0,s_i-y_i}(\vec{x},\vec{h}) := (h_n + x_n - s_i + y_i)/2 \\ r_i &= r_{0,s_i-y_i}(\vec{x},\vec{h}) : (h_1 - x_1 - s_i + y_i)/2 \\ r_i' &= r_{0,-x_n-h_n}(\{y_i,s_i\}) := (-s_i - y_i + x_r + h_n)/2 \end{aligned} \tag{3.77}$$

The third matrix is the same object as in the one-point case, which only depends on $(\vec{x},\vec{h})$ *(Notice that $l-r$ is x_{prim})).* $Y_i = \mathtt{b}_*^{-1}\mathtt{a}_*^{r_i'-l_i}$*, and*

$$\begin{aligned} &u_{ij}' = \sum_{k=i}^{j-1} u_k', \quad d_{ij}' = \sum_{k=i}^{j-1} d_k' \\ &u_i' = (y_{i+1} - y_i - s_{i+1} + s_i)/2, \quad d_i' = (y_{i+1} - y_i + s_{i+1} - s_i)/2. \end{aligned} \tag{3.78}$$

Recall from the full space case that u_i', d_i' *records information from* $\{y_i, s_i\}$ *to* $\{y_{i+1}, s_{i+1}\}$.

3.2.1 Kolmogorov equation

Proposition 3.2.2. $(\partial_t - \mathcal{L})\sqrt{\det(I+K)} = 0$

Proof. For each $K(i,\cdot;j,\cdot)$, we have $(\partial_t - \mathcal{L})K(i,\cdot;j,\cdot) = 0$, using the same proof as in the one-point case.

Now we need to show that $(\partial_t - \mathcal{L})$ can go through the determinant. For the derivative in t, we have

$$\partial_t \sqrt{\det(I-K)} = \frac{1}{2}\sqrt{\det(I-K)}\,\mathrm{tr}(I-K)^{-1}\partial_t K.$$

From the proof of the one-point case, we know each $\mathcal{L}_{x_i}K(i,\cdot;j,\cdot)$ gives a rank two perturbation of a 2×2 matrix kernel, thus we denote it as $\mathcal{G}^j\mathcal{H}^j$. Define the following row vector and column vector:

$$\mathcal{H}_k = \left(\mathcal{H}_k^1, \mathcal{H}_k^2, \cdots, \mathcal{H}_k^m\right), \quad \mathcal{G}_k = \begin{pmatrix} \mathcal{G}_k^1 \\ \mathcal{G}_k^2 \\ \cdots \\ \mathcal{G}_k^m \end{pmatrix} \tag{3.79}$$

Then

$$\begin{aligned} \mathcal{L}_{x_k}\sqrt{\det(I-K)} &= \sqrt{\det(I-K-\mathcal{G}_k\mathcal{H}_k)} - \sqrt{\det(I-K)} \\ &= \sqrt{\det(I-K)}(\sqrt{\det(I-\mathcal{H}_k(I-K)^{-1}\mathcal{G}_k)} - 1) \end{aligned}$$

Again $\det(I-\mathcal{H}_k(I-K)^{-1}\mathcal{G}_k)$ is a 2×2 matrix, with the same structure as in the one-point case, thus

$$(\sqrt{\det(I-\mathcal{H}_k(I-K)^{-1}\mathcal{G}_k)} - 1) = \frac{1}{2}\operatorname{tr}(I-K)^{-1}\partial_t K. \tag{3.80}$$

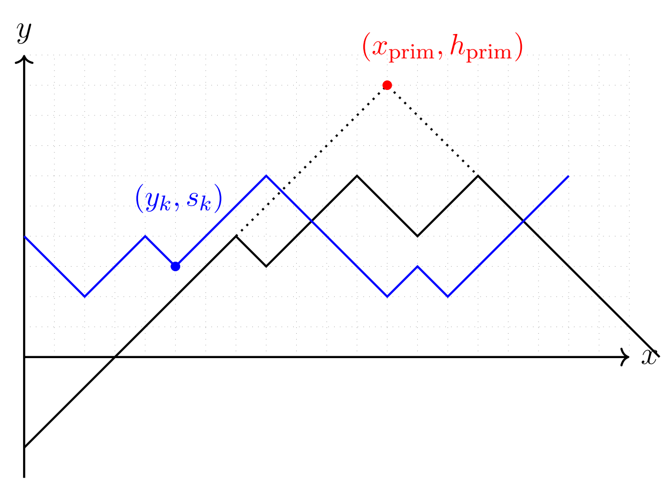
□

3.2.2 Initial condition

Proposition 3.2.3. *Given the initial condition $(\vec{x}, \vec{h})$. Let $h(x)$ be the height function of the half-space TASEP associated with $(\vec{x}, \vec{h})$. Let $F(t,h) = \mathbb{P}((\vec{x}, \vec{h})_t \le \{\vec{y}, \vec{s}\})$. Then*

$$\lim_{t\to 0} F(t,h) = 1_{(\vec{x},\vec{h})\le\{\vec{y},\vec{s}\}} = \Pi_{i=1}^m 1_{h(y_i)\le s_i}.$$

Proof. We analyze similarly to the full-space case. Notice that if there is a trough (y_k, s_k) outside the cone $C^{x_{\text{prim}}, h_{\text{prim}}}$, there are two cases.



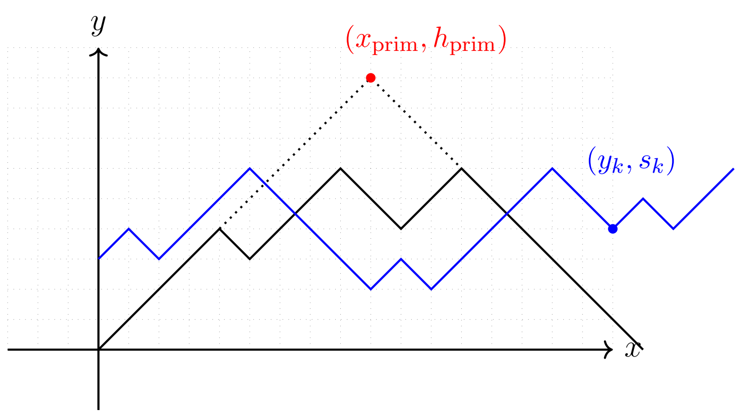
Case1: $y_k \le x_{\text{prim}}$ and $s_k \ge h_{\text{prim}} - x_{\text{prim}} + y_k$. In this case, it is clear that all the troughs (y_j, s_j) for $j \le k$ are also outside the cone $C^{x_{\text{prim}}, h_{\text{prim}}}$.

In this case, $r_j = (r_1 - s_j - x_1 + y_j)/2 \le 0$ for all $j \le k$. By definition, $S_{0,0}^{r_j, r'_j} = 0$, $D^{-1} S_{-1,1}^{r_j + r'_j - l_j, l_j} = 0$ for the same reason as in the one-point case. Thus, in every j-th column $(j \le k)$, $K(i, \cdot; j, \cdot) = 1_{i<j} (\mathtt{a})^{-u'_{ij}} (\mathtt{a}_*)^{-d'_{ij}}$.

Thus, the first $k \times k$ diagonal block of $I - K$ is an upper triangular matrix of 2×2 matrices with matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ operators along the first k diagonal position, so the determinant reduces:

$$\det((I-K)_{i=1,j=1}^{m}) = \det((I-K)_{i=k+1,j=k+1}^{m}) \tag{3.81}$$

Case2:



$y_k \ge x_{\text{prim}}$ and $s_k \ge h_{\text{prim}} + x_{\text{prim}} - y_k$. In this case, it is clear that all the troughs (y_i, s_i) for $i \ge k$ are also outside the cone $C^{x_{\text{prim}}, h_{\text{prim}}}$.

In this case, $r'_i = (r_n - s_i + x_n - y_i)/2 \le 0$ for all $i \ge k$. By definition, $S_{0,0}^{r'_i, r_i} = 0$, $D^{-1} S_{-1,1}^{r_i + r'_i - l_i, l_i} = 0$, thus in every i-th row, $K(i, \cdot; j, \cdot) = 1_{i<j} (\mathtt{a})^{-u'_{ij}} (\mathtt{a}_*)^{-d'_{ij}}$ for $i \ge k$.

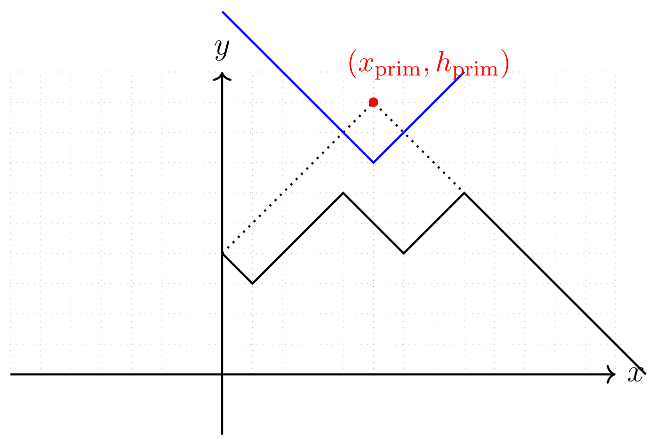
Thus, the last $k \times k$ diagonal block of $I - K$ is an upper triangular matrix of 2×2 matrices with matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ operator along the last $m - k$ diagonal position, so the determinant reduces:

$$\det((I-K)_{i=1,j=1}^{m}) = \det((I-K)_{i=1,j=1}^{k}) \tag{3.82}$$

Now we are ready to discuss the proof.

If all the troughs are outside the cone $C^{x_{\text{prim}}, h_{\text{prim}}}$, then the configuration is already less than or equal to $\{\vec{y}, \vec{s}\}$. We want to show $F(0, h) = 1$. In this case, the whole kernel $I - K$ reduces to an upper triangular matrix with the identity operator along the diagonal; thus, its determinant is 1.

Now assume there exist some troughs in the cone. Since what is outside the cone does not affect the determinant, WLOG, we can assume that all the troughs are in the cone. If there exists $r_i' \geq u$ or $r_i \geq d$, we apply the absorbing lemma (2.2.22) to reduce the kernel. The following figure illustrates when that is needed:



As we have shown in the one-point case, if the trough is completely above the initial configuration, the kernel $S^{l_i,r_i}WS^{r_i,l_i}$ reduces to 0. Thus, if all the troughs inside the cone $C^{x_{\text{prim}},h_{\text{prim}}}$ are of this type, the kernel again reduces to an upper triangular matrix with an identity along the diagonal. Thus, we have $F(0,h) = 1$, which is what we want.

Now assume that there exists a trough (y_k, s_k) that is below the initial configuration. Now we want to show the $F(0,h) = 0$. To do that, we present an eigenfunction for the kernel K with eigenvalue 1. Recall

$$K(i,\cdot;j,\cdot) = \begin{pmatrix} 1_{i<j}(\mathtt{a})^{-d'_{ij}}(\mathtt{a}_*)^{-u'_{ij}} & 0 \\ 0 & 1_{i<j}(\mathtt{a})^{-u'_{ij}}(\mathtt{a}_*)^{-d'_{ij}} \end{pmatrix} + \tilde{K}_{ij} + \begin{pmatrix} 0 & 1_0 Y_i^* \mathtt{a}_*^{l_j-l_i} D \mathtt{a}^{r_i-r_j} Y_j 1_0 \\ 0 & 0 \end{pmatrix} \quad (3.83)$$

where

$$\tilde{K}_{ij} = \begin{pmatrix} 1_0 & 0 \\ 0 & 1_0 \end{pmatrix} \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix}_{y_i,s_i} \begin{pmatrix} W^* & -W^*\mathtt{a}^{r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r-l}W \\ 0 & W \end{pmatrix} \cdot \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix}_{y_j,s_j} \begin{pmatrix} 1_0 & 0 \\ 0 & 1_0 \end{pmatrix}$$

Recall that in the one-point case we find the eigenfunction for the kernel in the form

$$K_{\text{matrix}} = \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix} \begin{pmatrix} V^* & -V^*DV \\ 0 & V \end{pmatrix} \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} \begin{pmatrix} (V')^* & -(V')^*DV' \\ 0 & V' \end{pmatrix} \tag{3.84}$$

We write the last matrix as the product of three matrices.

$$\begin{pmatrix} (Y^*)^{-1}1_0 & 0 \\ 0 & Y1_0 \end{pmatrix} \begin{pmatrix} I & 1_0Y^*DY1_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 1_0(Y^*) & 0 \\ 0 & 1_0Y^{-1} \end{pmatrix} \tag{3.85}$$

and move the last matrix to the front using $\det(I - AB) = \det(I - BA)$. Thus, the eigenfunction of $K(k, \cdot; k, \cdot)$ would be

$$\begin{aligned} f_k \left|\delta_0\right\rangle &:= \begin{pmatrix} I & 1_0Y_k^*DY_k1_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 1_0(Y^*)_k & 0 \\ 0 & 1_0Y_k^{-1} \end{pmatrix} \begin{pmatrix} DS_{1,0}^{l_k,0}\left|\delta_0\right\rangle \\ -S_{1,0}^{l_k,0}\left|\delta_0\right\rangle \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1_0Y_k^{-1}S_{1,0}^{l_k,0}\left|\delta_0\right\rangle \end{pmatrix}. \end{aligned} \tag{3.86}$$

WLOG we can assume y_k, s_k is the last trough in the configuration. From () we know that is an eigenfunction of $K(k, \cdot; k, \cdot)$ with eigenvalue 1. Now we show that

$$f = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ f_k\left|\delta_0\right\rangle \end{pmatrix} \tag{3.87}$$

is the eigenfunction we want. 0 in f is a 2×1 column vector with 0 entries.

$$Kf = \begin{pmatrix} K(1, \cdot; k, \cdot)f_k\left|\delta_0\right\rangle \\ K(2, \cdot; k, \cdot)f_k\left|\delta_0\right\rangle \\ \cdots \\ K(k, \cdot; k, \cdot)f_k\left|\delta_0\right\rangle \end{pmatrix}. \tag{3.88}$$

Thus, we just need to show that for $1 \le i < k$, $K(i, \cdot; k\cdot)f = 0$

$$\tilde{K}(i,\cdot;k\cdot)f_k = \begin{pmatrix} 1_0(Y^*)_i & 0 \\ 0 & 1_0 Y_i^{-1} \end{pmatrix} \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix}_{l_i} \begin{pmatrix} V^* & -V^*DV \\ 0 & V \end{pmatrix} \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix}_{l_k}$$
$$\begin{pmatrix} 1_0(Y^*) & 0 \\ 0 & 1_0 Y_k \end{pmatrix} \begin{pmatrix} I & 1_0 Y_k^* D Y_k 1_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 1_0(Y^*)_k & 0 \\ 0 & 1_0 Y_k^{-1} \end{pmatrix} \begin{pmatrix} DS_{1,0}^{l_k,0}\,|\delta_0\rangle \\ -S_{1,0}^{l_k,0}\,|\delta_0\rangle \end{pmatrix} \tag{3.89}$$

From calculations in the one-point case, the result of the last four matrices becomes $\begin{pmatrix} 0 \\ -S_{1,0}^{l_k,0}\,|\delta_0\rangle \end{pmatrix}$ Let $H := \mathtt{a}^{r_i-l_i}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r_k-l_k}$,multiply out

$$\begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix}_{l_i} \begin{pmatrix} V^* & -V^*DV \\ 0 & V \end{pmatrix} \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix}_{l_k}.$$

we have

$$\begin{pmatrix} -S_{0,1}^{r_i,l_i}W^*S_{1,0}^{l_k,r_k} + DS_{1,0}^{l_i,r_i}WD^{-1}S_{0,1}^{r_k,l_k} - S_{0,1}^{r_i,l_i}W^*HWD^{-1}S_{0,1}^{r_k,l_k} & -S_{0,1}^{r_i,l_i}W^*DS_{1,0}^{l_k,r_k} + DS_{1,0}^{l_i,r_i}WS_{0,1}^{r_k,l_k} - S_{0,1}^{r_i,l_i}W^*HWS_{0,1}^{r_k,l_k} \\ D^{-1}S_{0,1}^{r_i,l_i}W^*S_{1,0}^{l_k,r_k} - S_{1,0}^{l_i,r_i}WD^{-1}S_{0,1}^{r_k,l_k} + D^{-1}S_{0,1}^{r_i,l_i}W^*HWD^{-1}S_{0,1}^{r_k,l_k} & D^{-1}S_{0,1}^{r_i,l_i}W^*DS_{1,0}^{l_k,r_k} - S_{1,0}^{l_i,r_i}WS_{0,1}^{r_k,l_k} + D^{-1}S_{0,1}^{r_i,l_i}W^*HWS_{0,1}^{r_k,l_k} \end{pmatrix}$$

Now compute

$$\begin{aligned} &(-S_{0,1}^{r_i,l_i}W^*DS_{1,0}^{l_k,r_k} + DS_{1,0}^{l_i,r_i}WS_{0,1}^{r_k,l_k} - S_{0,1}^{r_i,l_i}W^*HWS_{0,1}^{r_k,l_k})(-S_{1,0}^{l_k,0}\,|\delta_0\rangle) \\ &(D^{-1}S_{0,1}^{r_i,l_i}W^*DS_{1,0}^{l_k,r_k} - S_{1,0}^{l_i,r_i}WS_{0,1}^{r_k,l_k} + D^{-1}S_{0,1}^{r_i,l_i}W^*HWS_{0,1}^{r_k,l_k})(-S_{1,0}^{l_k,0}\,|\delta_0\rangle) \end{aligned} \tag{3.90}$$

First compute

$$DS_{1,0}^{l_i,r_i}WS_{0,1}^{r_k,l_k}(-S_{1,0}^{l_k,0}\,|\delta_0\rangle) \tag{3.91}$$

From the proof of the one-point case, we know that all the finite rank parts act $S^{l_m,0}$ will be 0, thus we just need to consider $DS_{1,0}^{l_i,r_i}1_0S_{0,1}^{r_k,l_k}S_{1,0}^{l_k,0}\,|\delta_0\rangle$

$$\begin{aligned} DS_{1,0}^{l_i,r_i}1_0S_{0,1}^{r_k,l_k}S_{1,0}^{l_k,0}\,|\delta_0\rangle &= \int_0^\infty dy \int_0^\infty dz D s_{1,0}^{l_i,r_i}(x,y) s_{0,1}^{r_k,l_k}(y,z) s_{1,0}^{l_k,0}(z,0) \\ &= \sum_{k=0}^{r_k-1} 2\int_0^\infty dz D s_{1,0}^{l_i,r_i-k-1}(x,0) s_{0,1}^{r_k-k,l_k}(0,z) s_{1,0}^{l_k,0}(z,0) \end{aligned} \tag{3.92}$$

From Lemma (2.2.27), all of the terms are zero except when $k = r_k - 1$. Thus we have it equal to

$$2\int_0^\infty dz D s_{1,0}^{l_i,r_i-r_m}(x,0) s_{0,1}^{1,n}(0,z) s_{1,0}^{n,0}(z,0) \tag{3.93}$$

Since

$$\int_0^\infty dz 2 s_{0,1}^{1,n}(0,z) s_{1,0}^{n,0}(z,0) = 4 s_{0,1}^{1,0}(0,0) s_{1,0}^{1,0}(0,0) = 1$$

So $DS_{1,0}^{l_i,r_i} 1_0 S_{0,1}^{r_k,l_k} S_{1,0}^{l_k,0} |\delta_0\rangle = DS_{1,0}^{l_i,r_i-r_k} |\delta_0\rangle$. On the other hand,

$$(-S_{0,1}^{r_i,l_i} W^* D S_{1,0}^{l_i,r_i} - S_{0,1}^{r_i,l_i} W^* H W S_{0,1}^{r_k,l_k})(-S_{1,0}^{l_k,0} |\delta_0\rangle)$$

is still 0 since $S_{0,1}^{r_i,l_i} W^* H W S_{0,1}^{r_k,l_k})(-S_{1,0}^{l_k,0} |\delta_0\rangle) = (-S_{0,1}^{r_i,l_i} W^* D S_{1,0}^{l_i,r_i})(-S_{1,0}^{l_k,0} |\delta_0\rangle)$ for the same reason as the one-point case. For a similar calculation, the second line in (3.2.2) gives $S_{1,0}^{l_i,r_i-r_k} |\delta_0\rangle$. Lastly, appending the first matrix, we have

$$\begin{aligned} \tilde{K}(i,\cdot;k,\cdot) f_k |\delta_0\rangle &= \begin{pmatrix} 1_0 (Y^*)_i & 0 \\ 0 & 1_0 Y_i^{-1} \end{pmatrix} \begin{pmatrix} -DS_{1,0}^{l_i,r_i-r_k} |\delta_0\rangle \\ S_{1,0}^{l_i,r_i-r_k} |\delta_0\rangle \end{pmatrix} \\ &= \begin{pmatrix} -DS_{1,-1}^{l_i,r_i-r_k+r_i'-l_i} |\delta_0\rangle \\ S_{0,0}^{r_i',r_i-r_k} |\delta_0\rangle \end{pmatrix}. \end{aligned} \tag{3.94}$$

Now we compute

$$\begin{aligned} &\begin{pmatrix} 0 & -1_0 Y_i^* \mathtt{a}_*^{l_k-l_i} D \mathtt{a}^{r_i-r_k} Y_k 1_0 \\ 0 & 0 \end{pmatrix} f_k |\delta_0\rangle \\ &= \begin{pmatrix} 0 & -1_0 Y_i^* \mathtt{a}_*^{l_k-l_i} D \mathtt{a}^{r_i-r_k} Y_k 1_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1_0 Y_k^{-1} S_{1,0}^{l_k,0} |\delta_0\rangle \end{pmatrix} \\ &= \begin{pmatrix} 1_0 Y_i^* \mathtt{a}_*^{l_k-l_i} D \mathtt{a}^{r_i-r_k} S_{1,0}^{l_k,0} |\delta_0\rangle \\ 0 \end{pmatrix} = \begin{pmatrix} 1_0 Y_i^* D S_{1,0}^{l_i,r_i-r_k} |\delta_0\rangle \\ 0 \end{pmatrix}, \end{aligned} \tag{3.95}$$

which cancels the first entry in (3.94). Lastly, check

$$\begin{aligned} &\begin{pmatrix} (\mathtt{a})^{-d_{ik}'} (\mathtt{a}_*)^{-u_{ik}'} & 0 \\ 0 & (\mathtt{a})^{-d_{ik}'} (\mathtt{a}_*)^{-u_{ik}'} \end{pmatrix} \begin{pmatrix} I & 1_0 Y_k^* D Y_k 1_0 \\ 0 & I \end{pmatrix} \\ &\cdot \begin{pmatrix} 1_0 (Y^*)_k & 0 \\ 0 & 1_0 Y_k^{-1} \end{pmatrix} \begin{pmatrix} DS_{1,0}^{l_k,0} |\delta_0\rangle \\ -S_{1,0}^{l_k,0} |\delta_0\rangle \end{pmatrix} \\ &= \begin{pmatrix} (\mathtt{a})^{-d_{ik}'} (\mathtt{a}_*)^{-u_{ik}'} & 0 \\ 0 & (\mathtt{a})^{-u_{ik}'} (\mathtt{a}_*)^{-d_{ik}'} \end{pmatrix} \begin{pmatrix} 0 \\ -1_0 Y_k^{-1} S_{1,0}^{l_k,0} |\delta_0\rangle \end{pmatrix} \end{aligned} \tag{3.96}$$

Recall that

$$\begin{aligned} &l_m - l_i = u'_{im}, r_m - r_i = u'_{im} \\ &r'_i - r'_m = d'_{im}; \end{aligned} \tag{3.97}$$

Thus, the last expression is $\begin{pmatrix} 0 \\ -S_{0,0}^{r'_i, r_i - r_k} \, |\delta_0\rangle \end{pmatrix}$, which cancels the second entry in (3.94). Thus, the proof is finished. □

3.3 Path integral version of the kernel

Here we want to give another version of the kernel, which is called the path integral formula:

Theorem 3.3.1. *Assume that we start the half-space TASEP with rate* $1 > \alpha > 0$ *with the initial configuration having peaks at* $(x_1, h_1; \ldots, x_n, h_n), x_1 \geq 0$*. The probability that at time* t*, it being below the configuration* $\{y_1, s_1; \ldots; y_m, s_m\}, y_1 \geq 0$ *is given by:*

$$\mathbb{P}((x_1, h_1 \cdots x_n, h_n)_t \leq \{y_1, s_1; \ldots; y_m, s_m\}) = \sqrt{\det(I - K_{\text{PI-s}})_{L^2(\mathbb{R})}} \tag{3.98}$$

where

$$K_{\text{PI-s}} = S_{1,1}^{l,l} K_{\text{init}} S_{1,1}^{l,l} K_{\text{final}}, \tag{3.99}$$

where $l = (h_n + x_n - s_m + y_m)/2$*, and*

$$\begin{aligned} K_{\text{init}} &= V_I + D^{-1} V_I^* D - D^{-1} V_I^* D V_I, \\ K_{\text{final}} &= V_F + D^{-1} V_F^* D - D^{-1} V_F^* D V_F, \end{aligned} \tag{3.100}$$

Γ *is a simple, positively oriented loop that includes* $w = 1/2$ *and* $w = (2\alpha - 1)/2$*.* V_I *is the kernel* V *that we introduced before the theorem (see 3.11), parameterized by the configuration* $C_{0,s_m - y_m}(x_1, h_1; \cdots, x_n, h_n)$*;* V_F *is the kernel* V *parameterized by* $C_{0,-x_n-h_n}(y_1, -s_1; \ldots; y_m, -s_m)$*, with all indicator functions* $\overline{1}^t$ *replaced by* $\overline{1}^0$*.*

This is called the path integral kernel since the final configuration information is also represented in a symmetric way as the information for the initial configuration. Also, the space of the kernel is on $\mathbb{R}$. The following is another form of the path integral version, where the kernel is a 2×2 matrix kernel.

Corollary 3.3.2. *Under the same notation and assumptions as Theorem* (3.3.1)*,*

$$\begin{aligned}\mathbb{P}((x_1,h_1\cdots x_n,h_n)_t)\le\{y_1,s_1;\dots;y_m,s_m\}) &= \mathrm{Pf}(J+JK_{\mathrm{PI-m}})_{L^2(\mathbb{R})}\\ &= \sqrt{\det(I+K_{\mathrm{PI-m}})_{L^2(\mathbb{R})\times L^2(\mathbb{R})}}\end{aligned} \tag{3.101}$$

where

$$K_{\mathrm{PI-m}}=\begin{pmatrix}-S & DS\\ D^{-1}S & -S\end{pmatrix}\begin{pmatrix}V_I^* & -V_I^*DV_I\\ 0 & V_I\end{pmatrix}\begin{pmatrix}S & DS\\ D^{-1}S & S\end{pmatrix}\begin{pmatrix}V_F^* & V_F^*DV_F\\ 0 & V_F\end{pmatrix}. \tag{3.102}$$

and $J(x,y)=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\delta_{xy}$*. All* S *in the kernel are* $S_{1,1}^{l,l}$*,*

There is a method in [BCR15] which is about transforming the kernel on m copies of L^2 space to a kernel on L^2 space. However, the method can only be applied formally in our scenario. Surprisingly, the kernel we derived in the end makes sense and can be proven using the Kolmogorov equation checking method. Since the proof is very similar, we will not redo it here. Rather, we show the kernel $K_{\mathrm{PI-m}}$ can be transformed to $K_{\mathrm{PI-s}}$. We write

$$\begin{pmatrix}-S & DS\\ D^{-1}S & -S\end{pmatrix}=\begin{pmatrix}D & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}I\\ -I\end{pmatrix}\begin{pmatrix}S & S\end{pmatrix}\begin{pmatrix}D^{-1} & 0\\ 0 & I\end{pmatrix},$$

$$\begin{pmatrix}S & DS\\ D^{-1}S & S\end{pmatrix}=\begin{pmatrix}D & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}I\\ I\end{pmatrix}\begin{pmatrix}-S & S\end{pmatrix}\begin{pmatrix}D^{-1} & 0\\ 0 & I\end{pmatrix},$$

and applying the identity $\det(I-AB)=\det(I-BA)$, we get the scalar version of the kernel.

To see the kernel JK is an antisymmetric kernel, we write

$$\begin{pmatrix}V_F^* & V_F^*DV_F\\ 0 & V_F\end{pmatrix}=\begin{pmatrix}V_F^* & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}I & D/2\\ 0 & I\end{pmatrix}\begin{pmatrix}I & D/2\\ 0 & I\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & V_F\end{pmatrix}$$

and bring $\begin{pmatrix}I & D/2\\ 0 & I\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & V_F\end{pmatrix}$ to the front using $\det(I-AB)=\det(I-BA)$, we have

$$\begin{pmatrix} I & D/2 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & V_F \end{pmatrix} \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix} \begin{pmatrix} V_I^* & -V_I^* D V_I \\ 0 & V_I \end{pmatrix} \cdot \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} \begin{pmatrix} V_F^* & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & D/2 \\ 0 & I \end{pmatrix}, \quad (3.103)$$

then it is easy to check that multiplying this kernel by the matrix J is antisymmetric.

Chapter 4

Scaling limit and the half-space KPZ fixed point

4.1 Transformation of the kernel

In the kernel (3.74), all the S operators are explicit, and a standard steepest descent method can be applied. The W operator, which records the initial condition, is harder to compute a limit. However, it has a nice probabilistic interpretation as a Brownian bridge hitting some curves. We will first do some transformations on the kernel. Let $\mathfrak{d}_{\vec{x}}^{\vec{h}}(y) = \begin{cases} y = r_i \text{ if } y = x_i \\ -\infty \text{ if } y \neq x_i \text{ for all } i. \end{cases}$ The strict epigraph of a function f is defined to be

$$\text{epi}(f) = \{(m, y) : m \in \mathbb{R}, y > f(m)\} \tag{4.1}$$

Recall W is defined by

$$\begin{aligned} &W = (I - W_1 \bar{1}^t W_{1,2} \bar{1}^t W_{2,3} \cdots \bar{1}^t W_{n-1,n} \bar{1}^t W_{n+1}), \\ &W_{i,i+1} = \mathtt{a}^{-u_i} \mathtt{a}_*^{-d_i}, \quad W_1 = \mathtt{a}^u, \quad W_{n+1} = \mathtt{a}_*^d, \end{aligned} \tag{4.2}$$

. Let ExpWalk(W) be a random walk such that it has u_1 steps with Exp(1/2) − 2 jumps, then followed by d_1 steps with 2 − Exp(1/2), then followed by u_2 steps with Exp(1/2) − 2 jumps, etc., and it ends with d_{n-1} steps with 2 − Exp(1/2). Let $\text{ExpWalk(W)}_{i,j}(x, y)$ be the transition density of the walk restricted to i-th to j-th steps, starting from x, ending at y. Let $\tau = \min\{i : \text{ExmWalk(W)}_i > -2r_i\}$. Define the following hit operator:

$$\mathrm{ExpWalk(W)}^{\mathrm{hit}_i}(x,z) = \int_{-r_i}^{\infty} dy \mathbb{P}_x(\tau = i, \mathrm{ExpWalk(W)}_i = y)\mathrm{ExpWalk(W)}_{i,u+d}(y,z)) \tag{4.3}$$

Proposition 4.1.1.

$$W = \mathtt{a}^u e^{-(t+2r_1)D} \sum_{i=1}^{n} \mathrm{ExpWalk(W)}^{\mathrm{hit}_i} e^{(t+2h_n)D} \mathtt{a}_*^d \tag{4.4}$$

Proof. Pull out $\mathtt{a}^u, \mathtt{a}_*^d$ outside the bracket and we focus on

$$\mathtt{a}^{-u}\mathtt{a}_*^{-d} - \bar{1}^t W_{1,2} \bar{1}^t \cdots \bar{1}^t W_{n-1,n} \bar{1}^t.$$

Notice that this operator is a combination of $\mathtt{a}_*^{-1}, \mathtt{a}^{-1}$ and the projection operator. We write out their integral form:

$$\mathtt{a}_*^{-1}(x,y) = 1_{x \geq y} \tfrac{1}{2} e^{(y-x)/2}, \quad \mathtt{a}^{-1}(x,y) = 1_{y \geq x} \tfrac{1}{2} e^{(x-y)/2} \tag{4.5}$$

Notice $\mathtt{a}_*^{-1}(x,y)$ is the transition density of a random walk with Exp(1/2) jump to the left, with mean -2; $\mathtt{a}^{-1}(x,y)$ is the transition density of a random walk with Exp(1/2) jump to the right, with mean 2. By composing them, we see that $W_{i,i+1}$ is also a transition density function for a random walk, with a drift $2r_{i+1} - 2r_i$. Since all $W_{i,i+1}(x,y)$ only depend on the difference of x, y, we can do a shift of t of the whole operator, and get:

$$e^{-tD}(\mathtt{a}^{-u}\mathtt{a}_*^{-d} - \bar{1}^0 W_{1,2} \bar{1}^0 \cdots \bar{1}^0 W_{n-1,n} \bar{1}^0) e^{tD}.$$

Then we want to shift the starting and endpoint of $W_{i,j}$ to make it mean 0, i.e., write

$$\begin{aligned}
&(\mathtt{a}^{-u}\mathtt{a}_*^{-d} - \bar{1}^0 W_{1,2} \bar{1}^0 W \cdots \bar{1}^0 W_{n-1,n} 1^0) \\
&= (\mathtt{a}^{-u}\mathtt{a}_*^{-d} - e^{-2r_1 D} \bar{1}^{-2r_1} e^{2r_1 D} W_{1,2} e^{-2r_2 D} \bar{1}^{-2r_2} \cdots \\
&\bar{1}^{-2r_{n-1}} e^{2r_{n-1} D} W_{n-1,n} e^{-2r_n D} \bar{1}^{-2h_n} e^{2r_n D}) \\
&= e^{-2r_1 D}(e^{2r_1 D} \mathtt{a}^{-u}\mathtt{a}_*^{-d} e^{-2r_n D} - \bar{1}^{-2r_1} e^{2r_1 D} W_{1,2} e^{-2r_2 D} \bar{1}^{-2r_2} \\
&\cdots \bar{1}^{-2r_{n-1}} e^{2r_{n-1} D} W_{n-1,n} e^{-2r_n D} \bar{1}^{-2h_n}) e^{2r_n D}
\end{aligned} \tag{4.6}$$

Denote

$$P^{\text{hit}}(x,y) = (e^{2r_1 D}\mathtt{a}^{-u}\mathtt{a}_*^{-d}e^{-2r_n D} - \overline{1}^{-2r_1}e^{2r_1 D}W_{1,2}e^{2r_2 D}\overline{1}^{-2r_2}\cdots\overline{1}^{-2r_{n-1}}e^{2r_{n-1}D}W_{n-1,n}e^{-2r_n D}\overline{1}^{-2h_n})(x,y) \tag{4.7}$$

All the translation operators will make all $W_{1,n}$ a mean 0 walk; we denote it as $W^\circ_{i,i+1}$. The (4.7) is the probability density that a random walk $W^\circ_{1,2}, W^\circ_{2,3}, \cdots, W^\circ_{n-1,n}$, starting from x, ends at y, being greater than $-2r_i$ at x_i for some i; we denote it as $P^{\text{hit}}(x,y)$.

We want to further rewrite the probability.

$$P^{\text{hit}} = 1_{-2r_1}\mathtt{a}^{-u}\mathtt{a}_*^{-d} + \sum_{i=2}^{n}\overline{1}^{-2r_1}W_{1,i}1_{-2r_i}W_{i,i+1}W_{i+1,i+2}\cdots W_{n-1,n} \tag{4.8}$$

This formula means that the probability is summing over the probability that the walk first hits the curve at the i-th wedge. Each term in the summation reads: the walk does not hit in the first $i-1$ wedges, then hits at the i-th wedge, then the walk can go to the endpoint freely. Using the notation we defined before the proposition, we have

$$P^{\text{hit}} = \sum_{i=1}^{n}\text{ExpWalk}(\text{W})^{\text{hit}_i}.$$

Thus the statement is proved. □

Lastly, we want to write out the kernel coefficient explicitly in (3.74). Recall

$$\tilde{K}_{ij} = \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix}_{y_i,s_i} \begin{pmatrix} W^* & -W^*\mathtt{a}^{r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r-l}W \\ 0 & W \end{pmatrix} \cdot \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix}_{y_j,s_j} \tag{4.9}$$

The second and fourth matrices are defined in (4.11), with parameters being

$$\begin{aligned} l_i &= l_{0,s_i-y_i}(\vec{x},\vec{h}) := (h_n + x_n - s_i + y_i)/2 \\ r_i &= r_{0,s_i-y_i}(\vec{x},\vec{h}) : (h_1 - x_1 - s_i + y_i)/2 \\ r_i' &= r_{0,-x_n-h_n}(\{y_i,s_i\}) := (-s_i - y_i + x_n + h_n)/2 \end{aligned} \tag{4.10}$$

$$\begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix}_{y,s} := \begin{pmatrix} -S_{0,0}^{r,r'} & DS_{1,-1}^{l,r+r'-l} \\ D^{-1}S_{-1,1}^{r+r'-l,l} & -S_{0,0}^{r',r} \end{pmatrix}$$
$$\begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix}_{y,s} := \begin{pmatrix} S_{0,0}^{r',r} & DS_{1,-1}^{l,r+r'-l} \\ D^{-1}S_{-1,1}^{r+r'-l,l} & S_{0,0}^{r,r'} \end{pmatrix} \tag{4.11}$$

Multiply the matrices out, we have $\tilde{K}_{ij} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$, where

$$\begin{aligned}
M_{11} &= -S_{0,0}^{r_i,r_i'} W^* S_{0,0}^{r_j',r_j} + S_{0,0}^{r_i,r_i'} W^* HWD^{-1} S_{-1,1}^{r_j+r_j'-l_j,l_j} + DS_{1,-1}^{l_i,r_i+r_i'-l_i} WD^{-1} S_{-1,1}^{r_j+r_j'-l_j,l_j}, \\
M_{12} &= -S_{0,0}^{r_i,r_i'} W^* DS_{1,-1}^{l_j,r_j+r_j'-l_j} + S_{0,0}^{r_i,r_i'} W^* HWS_{0,0}^{r_j,r_j'} + DS_{1,-1}^{l_i,r_i+r_i'-l_i} WS_{0,0}^{r_j,r_j'}, \\
M_{21} &= D^{-1} S_{-1,1}^{r_i+r_i'-l_i,l_i} W^* S_{0,0}^{r_j',r_j} - D^{-1} S_{-1,1}^{r_i+r_i'-l_i,l_i} W^* HWD^{-1} S_{-1,1}^{r_j+r_j'-l_j,l_j} \\
&- S_{0,0}^{r_j',r_j} WD^{-1} S_{-1,1}^{r_j+r_j'-l_j,l_j}, \\
M_{22} &= D^{-1} S_{-1,1}^{r_i+r_i'-l_i,l_i} W^* DS_{1,-1}^{l_j,r_j+r_j'-l_j} - D^{-1} S_{-1,1}^{r_i+r_i'-l_i,l_i} W^* HWS_{0,0}^{r_j,r_j'} - S_{0,0}^{r_j',r_j} WS_{0,0}^{r_j,r_j'}.
\end{aligned} \tag{4.12}$$

We analyze M_{11} term and the rest are the same. Using Proposition (4.1.1), we plug in

$$\begin{aligned}
W &= \mathtt{a}^u e^{-(t+2r_1)D} (\sum_{k=1} \mathrm{ExpWalk(W)}^{\mathrm{hit}_k}) e^{(t+2h_n)D} \mathtt{a}_*^d \\
&\mathrm{ExpWalk(W)}^{\mathrm{hit}_k}(x,z) = \\
&\int_{-2r_k}^{\infty} dy \mathbb{P}_x(\tau = k, \mathrm{ExpWalk(W)}_k = y) \mathrm{ExpWalk(W)}_{k,u+d}(y,z))
\end{aligned}$$

into the formula. The $\mathrm{ExpWalk(W)}_{k,u+d}(y,z) e^{(t+2h_n)D} \mathtt{a}_*^d$ can be absorbed into S since $y \geq -r_i$, i.e. we have

$$\begin{aligned}
&1_{-2r_k} \mathrm{ExpWalk(W)}_{k,u+d}(y,\cdot) e^{(t+2h_n)D} \mathtt{a}_*^d D^{-1} S_{-1,1}^{r_j+r_j'-l_j,l_j} \\
&= 1_{-2r_k} e^{(t+2r_i)D} D^{-1} S_{-1,1}^{\frac{r_k - x_k - s_j - y_j}{2}, \frac{r_k + x_k - s_j + y_j}{2}}
\end{aligned} \tag{4.13}$$

The reason for the change of shift operator from $e^{(t+2h_n)D}$ to $e^{(t+2r_k)D}$ is that in order to absorb random walk transition density into S, you need to change it back to the original walk that is not mean 0.

To use a simpler notation for the indices, define

$$S_{a,b}^{(-x_k,-y_k)} := S_{a,b}^{r_j+r_j'-l_j,l_j} = S_{a,b}^{\frac{r_k-x_k-s_j-y_j}{2},\frac{r_k+x_k-s_j+y_j}{2}} \tag{4.14}$$

We would like to mention this is a "posteriori" notation. All the terms that come into the coefficient in S will be in the form $(r_k - x_k - s_j - y_j)/2$ with the restriction that r, x comes in as a pair, s, j comes in as a pair; the sign for r is always positive; the sign for s is negative; the sign for x and y in the first and second superscripts are different; i.e. as long as we record the x and y in the first superscript, we know the whole S.

Further, to use a similar notation as in [MQR21], we define

$$\begin{aligned}&D^{-1}S_{-1,1}^{\mathrm{epi},-y_j}(x,z)\\&:=\sum_{k=1}\int_{-2r_k}^{\infty}\mathbb{P}_x(\tau=k,\mathrm{ExpWalk(W)}_k=y)e^{(t+2r_i)D}D^{-1}S_{-1,1}^{r_k-x_k-s_j-y_j,r_k+x_k-s_j+y_j}(y,z)\end{aligned} \tag{4.15}$$

Now the last term in M_{11} reduces to $DS_{1,-1}^{l_i,r_i+r_i'-l_i}\mathsf{a}^u e^{-(t+h_1)D}D^{-1}S_{-1,1}^{\mathrm{epi},-y_j}$, we want to bring all the a^u into S. When a hits S, it will generate boundary terms $B_{1,-1}^{l_i,r_i+r_i'-l_i}$,which is equivalent to having $S_{1,-1}^{l_i,r_i+r_i'-l_i}(x,-x)$. Since $l_i - r_i \geq u$ and $r_i' \leq l_i$, thus by Lemma (3.1.19), all the boundary terms are 0, and we have

$$\begin{aligned}DS_{1,-1}^{l_i,r_i+r_i'-l_i}\mathsf{a}^u e^{-(t+2r_1)D}S_{-1,1}^{\mathrm{epi},j} &= DS_{1,-1}^{\frac{x_1+h_1-s_i+y_i}{2},\frac{h_1-x_1-s_i-y_i}{2}}e^{-(t+h_1)D}D^{-1}S_{-1,1}^{\mathrm{epi},-y_j}\\&= DS_{1,-1}^{(x_1,y_i)}e^{-(t+2r_1)D}D^{-1}S_{-1,1}^{\mathrm{epi},-y_j}\end{aligned} \tag{4.16}$$

Now we look at the first term in M_{11},

$$W^* = \mathsf{a}^d e^{-(t+2h_n)D}(\sum_{k=1}^{n}\mathrm{ExpWalk(W)}^{\mathrm{hit}_k})^* e^{(t+2r_1)D}\mathsf{a}_*^u.$$

All $(\sum_{k=1}^n \mathrm{ExpWalk(W)}^{\mathrm{hit}_k})^*$ can be brought into $S_{0,0}^{r_i,r_i'}$, and it becomes $(S_{0,0}^{\mathrm{epi},-y_i})^*$. To bring a_*^u into $S_{0,0}^{r_j',r_j}$, there will also be boundary terms, $S_{0,0}^{r_j'-i,r_j}(x,-x)$ where $i < u$. Notice

$$\begin{aligned}&r_j' - r_j - u\\&= (-s_j - y_j + x_n + h_n)/2 + (h_1 - x_1 - s_j + y_j)/2 - (x_n - x_1 + h_n - h_1)/2\\&= 2r_1 - 2y_i\end{aligned} \tag{4.17}$$

which is not necessarily positive. However, in the scaling limit we are going to consider, $y_i \sim \varepsilon^{-1}$ and $h_1 \sim \varepsilon^{-3/2}$, thus we can assume that $h_1 - y_i > 0$. So, we can bring $\mathtt{a}_*^u$ into $S_{0,0}^{r_j',r_j}$ and

$$\mathtt{a}_*^u S_{0,0}^{r_j',r_j} = S_{0,0}^{\frac{-s_j-y_j+x_1+h_1}{2},\frac{h_1-x_1-s_j+y_j}{2}} = S_{0,0}^{(x_1,-y_j)}$$

Lastly, the second term in M_{11}, which is $S_{0,0}^{r_i,r_i'} W^* H W D^{-1} S_{-1,1}^{r_j+r_j'-l_j,l_j}$. Using the same arguments above,

$$\begin{aligned} WD^{-1}S_{-1,1}^{r_j+r_j'-l_j,l_j} &= \mathtt{a}^u e^{-(t+h_1)D} D^{-1} S_{-1,1}^{\text{epi},-y_j} \\ S_{0,0}^{r_i,r_i'} W^* &= (S_{0,0}^{\text{epi},-y_i})^* e^{(t+2r_1)D} \mathtt{a}_*^u \end{aligned} \tag{4.18}$$

Recall $H = \mathtt{a}^{r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r-l}$, so putting them together, it is

$$\begin{aligned} &(S_{0,0}^{\text{epi},-y_i})^* e^{(t+2r_1)D} \mathtt{a}_*^u \mathtt{a}^{r-l}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r-l} \mathtt{a}^u e^{-(t+h_1)D} D^{-1} S_{-1,1}^{\text{epi},-y_j} \\ &= (S_{0,0}^{\text{epi},-y_i})^* e^{(t+2r_1)D} \mathtt{a}^{r-l+u}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{r+u-l} e^{-(t+h_1)D} D^{-1} S_{-1,1}^{\text{epi},-y_j} \\ &= (S_{0,0}^{\text{epi},-y_i})^* e^{(t+2r_1)D} \mathtt{a}^{-x_1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-x_1} e^{-(t+h_1)D} D^{-1} S_{-1,1}^{\text{epi},-y_j} \end{aligned} \tag{4.19}$$

The second equality is due to the $\mathtt{a}, \mathtt{a}_*$ commuting with $\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}$. The third equality is by definition that $r+u-l = -x_1$. Up to now, we have transformed the kernel M_{11} into the form in which we will apply the asymptotic analysis. For M_{12}, M_{21}, M_{22}, the analysis is the same; we will just write them out in the form we want.

$$\begin{aligned} M_{11} &= -(S_{0,0}^{\text{epi},y_i})^* e^{(t+2r_1)D} S_{0,0}^{(x_1,-y_j)} + D S_{1,-1}^{(x_1,y_i)} e^{-(t+2r_1)D} D^{-1} S_{-1,1}^{\text{epi},-y_j} \\ &\quad + (S_{0,0}^{\text{epi},y_i})^* e^{(t+2r_1)D} \mathtt{a}^{-x_1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-x_1} e^{-(t+h_1)D} D^{-1} S_{-1,1}^{\text{epi},-y_j} \\ M_{12} &= -(S_{0,0}^{\text{epi},y_i})^* e^{(t+2r_1)D} D S_{1,-1}^{(x_1,y_j)} + D S_{1,-1}^{(x_1,y_i)} e^{-(t+2r_1)D} S_{0,0}^{\text{epi},y_j} \\ &\quad + (S_{0,0}^{\text{epi},y_i})^* e^{(t+2r_1)D} \mathtt{a}^{-x_1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-x_1} e^{-(t+h_1)D} S_{0,0}^{\text{epi},y_j} \\ M_{21} &= (D^{-1}S_{-1,1}^{\text{epi},-y_i})^* e^{(t+2r_1)D} S_{0,0}^{(x_1,-y_j)} - S_{0,0}^{(x_1,-y_i)} e^{-(t+2r_1)D} D^{-1} S_{-1,1}^{\text{epi},-y_j} \\ &\quad - (D^{-1}S_{-1,1}^{\text{epi},-y_i})^* e^{(t+2r_1)D} \mathtt{a}^{-x_1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-x_1} e^{-(t+h_1)D} D^{-1} S_{-1,1}^{\text{epi},-y_j} \\ M_{22} &= D^{-1}(S_{-1,1}^{\text{epi},-y_i})^* e^{(t+2r_1)D} D S_{1,-1}^{(x_1,y_j)} - S_{0,0}^{(x_1,-y_i)} e^{-(t+2r_1)D} S_{0,0}^{\text{epi},y_j} \\ &\quad - (D^{-1}S_{-1,1}^{\text{epi},-y_i})^* e^{(t+2r_1)D} \mathtt{a}^{-x_1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-x_1} e^{-(t+h_1)D} S_{0,0}^{\text{epi},y_j}. \end{aligned} \tag{4.20}$$

4.2 Point-wise limit of the kernel

Now we are ready to consider the scaling limit of the TASEP height function.

For $\varepsilon > 0$, the $1:2:3$ rescaled TASEP height function is

$$\mathbf{h}^{\varepsilon}(\mathbf{t},\mathbf{x}) := \varepsilon^{1/2}[h(2\varepsilon\mathbf{t}^{-3/2}, 2\varepsilon^{-1}\mathbf{x}) + \varepsilon^{-3/2}\mathbf{t}]. \tag{4.21}$$

This corresponds to scaling the initial condition diffusively,

$$\mathbf{h}^{\varepsilon}(0,\mathbf{x}) := \varepsilon^{1/2}h(0, 2\varepsilon^{-1}\mathbf{x}). \tag{4.22}$$

This scaling corresponds to studying the scaling limit to perturbations of density $1/2$. General density ρ could also be analyzed with the same method.

We have the following scaling on all the variables:

$$\begin{gathered} t^{\varepsilon} = 2\varepsilon^{-3/2}\mathbf{t}, \quad r_i^{\varepsilon} = \varepsilon^{-1/2}\mathbf{r}_i + \varepsilon^{-3/2}\mathbf{t}, \quad x_i^{\varepsilon} = 2\varepsilon^{-1}\mathbf{x}_i, \\ s_i^{\varepsilon} = \varepsilon^{-1/2}\mathbf{s}_i, \quad y_i^{\varepsilon} = 2\varepsilon^{-1}\mathbf{y}_i \end{gathered} \tag{4.23}$$

For the injection parameter α, we can either fix a $\alpha > 1/2$, in which case one can derive the formula in the symplectic-unitary transition scheme, or one can weakly scale the parameter around $1/2$, which is the case we will consider in the following. We scale

$$\alpha^{\varepsilon} = \frac{1 + \boldsymbol{\rho}\varepsilon^{1/2}}{2}. \tag{4.24}$$

The state space we will work with is

$$\begin{gathered} \mathrm{UC} := \{ \text{ upper semicontinuous functions } \mathfrak{f} : [0,\infty) \text{ with} \\ \mathfrak{f}(\mathbf{x}) \le C(1+|\mathbf{x}|) \text{ for some } C < \infty \text{ and } \mathfrak{h}(x) > -\infty \text{ for some } \mathbf{x}.\} \end{gathered} \tag{4.25}$$

For any function $f : [0,\infty) \to [-\infty,\infty)$, we define the *hypograph* of f as

$$\mathrm{hypo}(f) = \{(x,y) : y \le f(x)\} \tag{4.26}$$

A function $\mathfrak{f} \in \mathrm{UC}$ if and only if $\mathrm{hypo}(f)$ is closed. We define the metric on $[-\infty,\infty)$ to be $|x-y| = |e^x - e^y|$. Let d_H be the Hausdorff metric: for X, Y be non-empty subsets in a metric space (M,d)

$$d_H(X,Y) = \max\{\sup_{x\in X} d(x,Y), \sup_{y \in Y} d(X,y)\}.$$

$h_n \to h$ in *local Hausdorff* topology if for any $M \ge 1$,

$$d_H(\mathrm{hypo}(h_n)|_{[0,M]}, \mathrm{hypo}(h)|_{[0,M]}) \to 0$$

. UC is a Polish space under this topology. The Borel sigma algebra can be generated by the finite-dimensional set:

$$\{\mathfrak{f} \in \mathrm{UC} : \mathfrak{f}(\mathbf{x}_i) \leq \mathbf{r}_i, i = 1, \cdots, n\}.$$

We use LC to denote the set of functions f such that $-f \in \mathrm{UC}$.

Before we state the convergence results, we need to develop some notation for the limiting objects. Let us recall some operators from [MQR21]. For $\mathbf{t} > 0$,

$$\begin{aligned} \mathbf{S}_{\mathbf{t},\mathbf{x}}(\mathbf{z}_1, \mathbf{z}_2) :&= \tfrac{1}{2\pi i} \int_{C_1^{\pi/3}} dw e^{\mathbf{t}w^3/3 + \mathbf{x}w^2 + (\mathbf{z}_1 - \mathbf{z}_2)w} \\ &= \mathbf{t}^{-1/3} e^{\frac{2\mathbf{x}^3}{3\mathbf{t}^2} - \frac{(\mathbf{z}_1-\mathbf{z}_2)\mathbf{x}}{t}} \mathrm{Ai}(-\mathbf{t}^{-1/3}(\mathbf{z}_1 - \mathbf{z}_2) + \mathbf{t}^{-4/3}\mathbf{x}^2). \end{aligned} \tag{4.27}$$

where $C_a^{\pi/3} = \{a + re^{i\pi/3} : r \in [0,\infty)\} \cup \{a + re^{-i\pi/3} : r \in [0,\infty)\}$. This is the integral kernel for the operator $e^{\mathbf{x}\partial^2 + \mathbf{t}\partial^3/3}$. For $\mathbf{t} = 0$, the operator is still well defined for $\mathbf{x} > 0$. For $\mathbf{t}_1, \mathbf{t}_2 > 0$, it behaves like a group, i.e. $\mathbf{S}_{\mathbf{t}_1,\mathbf{x}_1}\mathbf{S}_{\mathbf{t}_2,\mathbf{x}_2} = \mathbf{S}_{\mathbf{t}_1+\mathbf{t}_2,\mathbf{x}_1+\mathbf{x}_2}$. One useful property of the operator is: $\mathbf{S}_{-\mathbf{t},\mathbf{x}} = (\mathbf{S}_{\mathbf{t},\mathbf{x}})^*$. We are going to use some variation of this operator.

$$\mathbf{S}_{a,b}^{\mathbf{t},\mathbf{x}}(\mathbf{z}_1, \mathbf{z}_2) = \int_{C_{a_w}^{\pi/3}} \frac{(w + \boldsymbol{\rho})^b}{(-w + \boldsymbol{\rho})^a} e^{\mathbf{t}w^3/3 + \mathbf{x}w^2 + (\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2)w} dw. \tag{4.28}$$

where $a_w < -|\boldsymbol{\rho}|$. Define $\mathfrak{p} = \boldsymbol{\rho} - D, \mathfrak{p}_* = \boldsymbol{\rho} + D$ For $\mathfrak{h} \in \mathrm{UC}$, define

$$\mathbf{S}_{a,b}^{\mathrm{hypo}(\mathfrak{h}),\mathbf{t},\mathbf{x}}(\mathbf{z}_1, \mathbf{z}_2) = \mathbb{E}_{\mathbf{B}(0)=\mathbf{z}_1}[\mathbf{S}_{a,b}^{\mathbf{t},\mathbf{x}-\boldsymbol{\tau}}(\mathbf{B}(\boldsymbol{\tau}, \mathbf{z}_2))1_{\boldsymbol{\tau}<\infty}]. \tag{4.29}$$

where $\mathbf{B}(x)$ is a Brownian motion with diffusion coefficient 2 and $\boldsymbol{\tau}$ is the hitting time of the hypograph of $\mathfrak{h}$. When $\mathfrak{h}$ is clear from the context, we will omit it from the superscript.

Now we are ready to state our main convergence theorem.

Theorem 4.2.1. *Let* $\mathfrak{h}_0 \in \mathrm{UC}$. *Let* $\mathbf{h}^\varepsilon(\mathbf{t}, \mathbf{x})$ *be the rescaled TASEP height function defined in* (4.21). *Assume* $\mathbf{h}^\varepsilon(0, \mathbf{x}) \to \mathfrak{h}_0$ *in UC in distribution. Then for any* $\mathbf{y}_1 < \cdots < \mathbf{y}_m \in [0, \infty), \mathbf{s}_1, \cdots, \mathbf{s}_m \in \mathbb{R}$,

$$\lim_{\varepsilon \to 0} \mathbb{P}(\mathbf{h}^\varepsilon(\mathbf{t}, \mathbf{y}_1) \leq \mathbf{s}_1, \cdots, \mathbf{h}^\varepsilon(\mathbf{t}, \mathbf{y}_m) \leq \mathbf{s}_m) = \mathrm{Pf}(J + JK^{\mathrm{fp}})_{(L^2[0,\infty) \times L^2[0,\infty))^m} \tag{4.30}$$

where K^{fp} *is a matrix-valued kernel on* m *copies of* $L^2[0,\infty)\times L^2[0,\infty)$*, given by*

$$\begin{aligned}K^{\mathrm{fp}}(i,\cdot;j,\cdot) &= 1_{i<j}\begin{pmatrix} e^{\mathbf{s}_i\mathbf{D}}e^{(\mathbf{y}_j-\mathbf{y}_i)\mathbf{D}^2}e^{-\mathbf{s}_j\mathbf{D}} & 0 \\ 0 & e^{-\mathbf{s}_i\mathbf{D}}e^{(\mathbf{y}_j-\mathbf{y}_i)\mathbf{D}^2}e^{\mathbf{s}_j\mathbf{D}}\end{pmatrix} + \tilde{K}^{\mathrm{fp}}(i,\cdot;j,\cdot) \\ &+ \begin{pmatrix} 0 & e^{\mathbf{s}_i\mathbf{D}+\mathbf{y}_i\mathbf{D}^2}\overline{\mathfrak{p}^{-1}\mathbf{D}\mathfrak{p}_*^{-1}}e^{-\mathbf{s}_j\mathbf{D}+\mathbf{y}_j\mathbf{D}^2} \\ 0 & 0\end{pmatrix}\end{aligned} \tag{4.31}$$

where $\tilde{K}^{\mathrm{fp}}$ *is*

$$\begin{aligned}&\begin{pmatrix} e^{\mathbf{s}_i\mathbf{D}} & 0 \\ 0 & e^{-\mathbf{s}_i\mathbf{D}}\end{pmatrix}\begin{pmatrix} -(\mathbf{S}_{0,0}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_i})^* & \mathbf{D}\mathbf{S}_{1,-1}^{-\mathbf{t},\mathbf{x}_1+\mathbf{y}_i} \\ (\mathbf{D}^{-1}\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_i})^* & \mathbf{S}_{0,0}^{-\mathbf{t},\mathbf{x}_1-\mathbf{y}_i}\end{pmatrix}\begin{pmatrix} I & e^{2\mathbf{x}_1\mathbf{D}^2}\overline{\mathfrak{p}^{-1}\mathbf{D}\mathfrak{p}_*^{-1}}e^{2\mathbf{x}_1\mathbf{D}^2} \\ 0 & I\end{pmatrix} \\ &\qquad\begin{pmatrix} \mathbf{S}_{0,0}^{\mathbf{t},\mathbf{x}_1-\mathbf{y}_j} & \mathbf{D}\mathbf{S}_{1,-1}^{\mathbf{t},\mathbf{x}_1+\mathbf{y}_j} \\ \mathbf{D}^{-1}\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_j} & \mathbf{S}_{0,0}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_j}\end{pmatrix}\begin{pmatrix} e^{-\mathbf{s}_j\mathbf{D}} & 0 \\ 0 & e^{\mathbf{s}_j\mathbf{D}}\end{pmatrix}\end{aligned} \tag{4.32}$$

Now we state the convergence theorem for each of the components. We will add one more subscript in S to denote that all the variables in S are under the scaling we are discussing.

Proposition 4.2.2. Recall $S_{a,b}^{x_k,y_i} = S_{a,b}^{\frac{x_k+r_k-s_i+y_i}{2},\frac{r_k-x_k-s_i-y_i}{2}}$. Let $z_1^\epsilon = 2\varepsilon^{-1/2}\boldsymbol{z}_1, z_2^\varepsilon = 2\varepsilon^{-1/2}\mathbf{z}_2$.

$$\begin{aligned}&2(\varepsilon^{-1/2})^{a-b}\varepsilon^{-1/2}S_{a,b,\varepsilon}^{x_k,y_i}e^{-(2r_k^\varepsilon)D}\varrho(z_1^\varepsilon,z_2^\varepsilon) \\ &\to \int_{C_{a_w}^{\pi/3}}\frac{(w+\boldsymbol{\rho})^b}{(-w+\boldsymbol{\rho})^a}e^{\mathbf{t}w^3/3+(\mathbf{x}_k+\mathbf{y}_i)z^2+(\mathbf{z}_2-\mathbf{z}_1-\mathbf{s}_i)w}dw =: (\mathbf{S}_{a,b}^{\mathbf{t},\mathbf{x}_k+\mathbf{y}_i})^*(\mathbf{z}_1,\mathbf{z}_2-\mathbf{s}_i)\end{aligned} \tag{4.33}$$

Where $a_w < -|\boldsymbol{\rho}|$. For $S_{a,b,\varepsilon}^{\mathrm{epi},y_j}$,

$$2(\varepsilon^{-1/2})^{a-b}\varepsilon^{-1/2}\varrho S_{a,b,\varepsilon}^{\mathrm{epi},y_j}(z_1^\varepsilon,z_2^\varepsilon)\to\mathbf{S}_{a,b}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_i}(\mathbf{z}_1,\mathbf{z}_2) \tag{4.34}$$

Proof. Recall the definition of $S_{a,b}^{i,j}$ in equation (3.7). Plugging in all the scaled variables, we have

$$\begin{aligned}&S_{a,b,\varepsilon}^{x_k,y_i}e^{-2r_1^\varepsilon D}\varrho(z_1^\varepsilon,z_2^\varepsilon) \\ &= \int_\Gamma dw\frac{(2w+\boldsymbol{\rho}\varepsilon^{1/2})^b}{(-2w+\boldsymbol{\rho}\varepsilon^{1/2})^a}\exp\{\varepsilon^{-3/2}f_1(w)+\varepsilon^{-1}f_2(w)+\varepsilon^{-1/2}f_3(w)\}\end{aligned}$$

where

$$\begin{aligned} f_1(w) &= -2\mathbf{t}w + \mathbf{t}(\log(1+2w) - \log(1-2w))/2, \\ f_2(w) &= -2\mathbf{x}_k(\log(1+2w) + \log(1-2w))/2, \\ f_3(w) &= -(2\mathbf{z}_1 - 2\mathbf{z}_2 + 2\mathbf{r}_k)w + (\mathbf{r}_k - \mathbf{s}_i)(\log(1+2w) - \log(1-2w))/2 \end{aligned} \tag{4.35}$$

we have $f_1'(0) = f_1''(0) = 0$. We want to move the contour to $C_0^{\pi/3}$ since this is a path on which the real part of f_1 is decreasing ([BBCS18b] Lemma 5.9). We also check here for completeness. $\mathrm{Re}[f_1(re^{\pm i\pi/3})]$ is $-\frac{\mathbf{t}}{4}(4r + \log(1-2r+4r^2) - \log(1+2r+4r^2))$.

$$\frac{d\mathrm{Re}[f_1(re^{\pm i\pi/3})]}{dr} = -\frac{8(r^2+2r^4)\mathbf{t}}{1+4r^2+16r^4} < 0.$$

And clearly, for any $\kappa_1 > 0$, there exists $c_1(\kappa_1) > 0$ such that $\mathrm{Re}[f_1(re^{\pm i\pi/3})] < -c_1$ for $r > \kappa_1$. But we cannot directly move the contour to $C_0^{\pi/3}$ since there can exist poles at $\pm\varepsilon^{1/2}|\boldsymbol{\rho}|/2$. We need to make a small blip at 0 to include the pole. We use the same contour and notation as in [BBCS18b]. The contour $C[\rho], \rho > 0$ is defined as in figure (4.1).

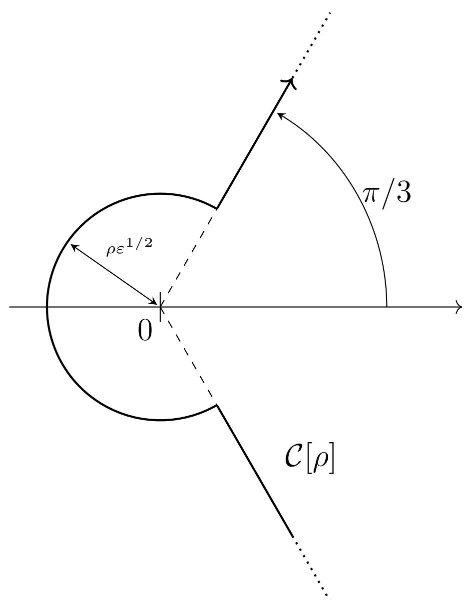


Figure 4.1: The contour $\mathcal{C}[\rho]$.

Fix $N > 0$. We will first cut off the contour outside the ball $B_0(N)$. The error would be

$$\int_{C_0^{\pi/3}\cap B_0(N)^c} dw \frac{(2w+\boldsymbol{\rho}\varepsilon^{1/2})^b}{(-2w+\boldsymbol{\rho}\varepsilon^{1/2})^a} \exp\{\varepsilon^{-3/2}f_1(w) + \varepsilon^{-1}f_2(w) + \varepsilon^{-1/2}f_3(w)\}$$

Parametrizing the curve by $\{re^{\pm i\pi/3} : r > N\}$. For ε small enough, there exists c such that

$$\mathrm{Re}[\varepsilon^{-3/2}f_1(w) + \varepsilon^{-1}f_2(w) + \varepsilon^{-1/2}f_3(w)\}] < -c\varepsilon^{-3/2}r,$$

and the term not in the exponent is bounded by $c|r|^{|a|+|b|}$, thus the integral would be $O(e^{-c\varepsilon^{-3/2}N})$, which goes to 0 as $\varepsilon \to 0$.

Now we focus on the contour that is $C[\boldsymbol{\rho}] \cap B_0(N)$. We take the Taylor expansion of the exponent and do the change of variable $w \to \varepsilon^{1/2}w/2$, and derive that

$$\begin{aligned}&(\varepsilon^{-3/2}f_1(w) + \varepsilon^{-1}f_2(w) + \varepsilon^{-1/2}f_3(w)) - (\frac{w^3}{3}\mathbf{t} + w^2\mathbf{x} - w(\mathbf{z}_1 + \mathbf{s} - \mathbf{z}_2)\\ &= \varepsilon^{1/2}O(w^4\mathbf{t} + w^3\mathbf{x} + w^2(\mathbf{r}_1 - \mathbf{s}_i))\end{aligned} \tag{4.36}$$

Denote $O(\mathbf{t}, \mathbf{x}, \mathbf{r}_1 - \mathbf{s}_i) = O(w^4\mathbf{t} + w^3\mathbf{x} + w^2(\mathbf{r}_1 - \mathbf{s}_i)$. All the extra $(\varepsilon^{1/2})^{b-a}$ cancels the one from (4.33). What we have is

$$\int dw \frac{(w+\boldsymbol{\rho})^b}{(-w+\boldsymbol{\rho})^a} \exp\{(\frac{w^3}{3}\mathbf{t} + w^2\mathbf{x} - w(\mathbf{z}_1 + \mathbf{s} - \mathbf{z}_2) + \varepsilon^{1/2}O(\mathbf{t}, \mathbf{x}, \mathbf{r}_1 - \mathbf{s}_i)\},$$

where the contour is $2\varepsilon^{-1/2}(C[\boldsymbol{\rho}] \cap B_0(N))$. Using the bound that $|e^x - 1| \le e^{|x|}|x|$, if we want to eliminate the error term in the exponent, we pick up an error

$$e^{\varepsilon^{1/2}O(w^4\mathbf{t}+w^3\mathbf{x}+w^2(\mathbf{r}_1-\mathbf{s}_i)}\varepsilon^{1/2}O(w^4\mathbf{t} + w^3\mathbf{x} + w^2(\mathbf{r}_1 - \mathbf{s}_i)$$

which is less than

$$e^{N\cdot O(w^3\mathbf{t}+w^2\mathbf{x}+w^1(\mathbf{r}_1-\mathbf{s}_i))}N \cdot O(w^3\mathbf{t} + w^2\mathbf{x} + w^1(\mathbf{r}_1 - \mathbf{s}_i))$$

Since the contour is in the ball $B_0(N)$. The integral over the circular region in $2\varepsilon^{-1/2}(C[\boldsymbol{\rho}] \cap B_0(N))$ is bounded which in ε, for the part on $C_0^{\pi/3}$, by choosing N small enough, we can ensure the coefficient of w^3 in the exponent is positive, thus w^3 will have exponential decay along $C_0^{\pi/3}$. By the dominated convergence theorem, the error would be $O(\varepsilon^{1/2})$ which goes to 0.

Lastly, we append the contour $2\varepsilon^{-1/2}(C[\boldsymbol{\rho}] \cap B_0(N))$ to infinity. Similar to the cutoff in the first step, due to the exponential decay of the exponent, as $\varepsilon \to 0$, the error of appending the contour goes to 0. Thus we get the desired result. For $S^{\mathrm{epi},y_j}_{a,b,\varepsilon}$

, recall the definition:

$$S^{\mathrm{epi},y_j}_{a,b,\varepsilon}(z_1,z_2)=\sum_{k=1}^{n}\int_{-2r_k}^{\infty}\mathbb{P}_{z_1}(\tau=k,\mathrm{ExpWalk(W)}_k=dz_3)e^{(t+2r_k)D}S^{-x_k,-y_j}_{a,b}(z_3,z_2)$$

Now we plug in the scaling, also scale $z_3^\varepsilon=2\varepsilon^{-1/2}\mathbf{z}_3$, we have

$$(\varepsilon^{-1/2})^{a-b}2\varepsilon^{-1/2}\varrho e^{(2r_k^\varepsilon)D}S^{-x_k,-y_j}_{a,b,\varepsilon}\to\mathbf{S}^{-\mathbf{x}_k,-\mathbf{y}_j}_{a,b}(\mathbf{z}_3,\mathbf{z}_2+\mathbf{s}_j)$$

The reason that the t is not present in the scaled shift operator is because $r_k^\varepsilon=\varepsilon^{-3/2}\mathbf{t}+\varepsilon^{-1/2}\mathbf{r}_k$, thus we need to re-shift by t^ε to place the random walk in the correct scale, thus it does not appear in the scaling. Now the probability becomes

$$\mathbb{P}_{-2\varepsilon^{-1/2}\mathbf{z}_1}(\tau^\varepsilon=k\varepsilon^{-1},\mathrm{ExpWalk}(\mathrm{W}_\varepsilon)=dz_3^\varepsilon).$$

The walk now takes steps $\mathrm{Exp}(1/2)-2$ and $2-\mathrm{Exp}(1/2)$, which has variance 8. Since we are diffusively scaling the random walk, with an extra factor 2 on the space, thus the walk can be thought of as a walk with steps $(\mathrm{Exp}(1/2)-2)/2$ and $(2-\mathrm{Exp}(1/2))/2$, which has variance 2. Thus, by Donsker's theorem, $\mathrm{ExpWalk}(\mathrm{W}_\varepsilon)$ converges locally uniformly to a Brownian motion with coefficient 2. Moreover, since we reflected the start and endpoint, now τ^ε is the hitting time of the hypograph of $\mathfrak{d}^{\vec{h}}_{\vec{x}}$ rather than hitting the epigraph of $\mathfrak{d}^{-2\vec{h}}_{\vec{x}}$. Using Proposition 3.2 in [MQR21], we have $\tau^\varepsilon\to\boldsymbol{\tau}$ in distribution, where τ is the time of Brownian motion $\mathbf{B}$ hitting the hypograph of $\mathfrak{h}_0=\lim_{\varepsilon\to0}\mathfrak{d}^{\vec{h}^\varepsilon}_{\vec{x}^\varepsilon}$. Thus

$$\begin{aligned}2(\varepsilon^{-1/2})^{a-b}\varepsilon^{-1/2}\varrho S^{\mathrm{epi},y_j}_{a,b,\varepsilon}(z_1^\varepsilon,z_2^\varepsilon)\to\int d\mathbf{k}\int_{-\infty}^{\mathbf{r}_k}d\mathbf{z}_3\mathbb{P}_{\mathbf{z}_1}(\boldsymbol{\tau}=\mathbf{k},\mathbf{B}=\mathbf{z}_3)\mathbf{S}^{-\mathbf{k}-\mathbf{y}_j}_{a,b}(\mathbf{z}_3,\mathbf{z}_2+\mathbf{s}_j)\\=\mathbf{S}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_j}_{a,b}(\mathbf{z}_1,\mathbf{z}_2+\mathbf{s}_j).\end{aligned}\tag{4.37}$$

□

This is the main structure of the kernel. Now we look closely at the exact kernels in (4.20).

$$
\begin{aligned}
M_{11} &= -(S_{0,0}^{\mathrm{epi},y_i})^* e^{(t+2r_1)D} S_{0,0}^{(x_1,-y_j)} + DS_{1,-1}^{(x_1,y_i)} e^{-(t+2r_1)D} D^{-1} S_{-1,1}^{\mathrm{epi},-y_j} \\
&\quad + (S_{0,0}^{\mathrm{epi},y_i})^* e^{(t+2r_1)D} \mathtt{a}^{-x_1} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{-x_1} e^{-(t+2r_1)D} D^{-1} S_{-1,1}^{\mathrm{epi},-y_j} \\
M_{12} &= -(S_{0,0}^{\mathrm{epi},y_i})^* e^{(t+2r_1)D} DS_{1,-1}^{(x_1,y_j)} + DS_{1,-1}^{(x_1,y_i)} e^{-(t+2r_1)D} S_{0,0}^{\mathrm{epi},y_j} \\
&\quad + (S_{0,0}^{\mathrm{epi},y_i})^* e^{(t+2r_1)D} \mathtt{a}^{-x_1} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{-x_1} e^{-(t+2r_1)D} S_{0,0}^{\mathrm{epi},y_j} \\
M_{21} &= (D^{-1} S_{-1,1}^{\mathrm{epi},-y_i})^* e^{(t+2r_1)D} S_{0,0}^{(x_1,-y_j)} - S_{0,0}^{(x_1,-y_i)} e^{-(t+2r_1)D} D^{-1} S_{-1,1}^{\mathrm{epi},-y_j} \\
&\quad - (D^{-1} S_{-1,1}^{\mathrm{epi},-y_i})^* e^{(t+2r_1)D} \mathtt{a}^{-x_1} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{-x_1} e^{-(t+2r_1)D} D^{-1} S_{-1,1}^{\mathrm{epi},-y_j} \\
M_{22} &= D^{-1} (S_{-1,1}^{\mathrm{epi},-y_i})^* e^{(t+2r_1)D} DS_{1,-1}^{(x_1,y_j)} - S_{0,0}^{(x_1,-y_i)} e^{-(t+2r_1)D} S_{0,0}^{\mathrm{epi},y_j} \\
&\quad - (D^{-1} S_{-1,1}^{\mathrm{epi},-y_i})^* e^{(t+2r_1)D} \mathtt{a}^{-x_1} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{-x_1} e^{-(t+2r_1)D} S_{0,0}^{\mathrm{epi},y_j}.
\end{aligned}
$$

There are multiple D, D^{-1} appearing in the kernel. Notice that since we scale the space by $2\varepsilon^{-1/2}$, each D in the new space becomes $\varepsilon^{1/2}\mathbf{D}/2$, and D^{-1} becomes $2\varepsilon^{-1/2}\mathbf{D}^{-1}$. Then looking at $\mathtt{a}^{-x_1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-x_1}$, if x_1 is scaled diffusively, i.e. $x_1^\varepsilon = 2\varepsilon^{-1}\mathbf{x}_1$, then by the central limit theorem,

$$
\frac{1}{\mathtt{a}^{x_1^\varepsilon}} e^{2\varepsilon^{-1/2} x_1 D} \to e^{\mathbf{x}_1 \mathbf{D}^2}, \quad e^{-2\varepsilon^{-1/2} x_1 D} \frac{1}{\mathtt{a}_*^{x_1^\varepsilon}} \to e^{\mathbf{x}_1 \mathbf{D}^2}
$$

The drift terms will cancel each other since $\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}(x,y)$ only depends on $x-y$. If x_1^ε is not scaled diffusively, i.e., if x_1 always has a fixed distance to the origin, then $x_1^\varepsilon \to 0$, and what is left is just $\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}$. Lastly, using the explicit formula in (3.1.3), we directly have the limit of the operator $\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}$:

$$
2\varepsilon^{1/2}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}} \to \overline{\mathfrak{p}^{-1}\mathbf{D}\mathfrak{p}_*^{-1}}
$$

Now we can combine all the ingredients to write out the limit for the kernels above

$$
\begin{aligned}
2\varepsilon^{-1/2}M_{11} \to e^{\mathbf{s}_i D}\Big[&-(\mathbf{S}_{0,0}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_i})^*\mathbf{S}_{0,0}^{\mathbf{t},\mathbf{x}_1-\mathbf{y}_j} + \mathbf{D}\mathbf{S}_{1,-1}^{-\mathbf{t},\mathbf{x}_1+\mathbf{y}_i}\mathbf{D}^{-1}\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_j} \\
&+ (\mathbf{S}_{0,0}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_i})^* e^{\mathbf{x}_1\mathbf{D}^2}\overline{\mathfrak{p}^{-1}\mathbf{D}\mathfrak{p}_*^{-1}}e^{\mathbf{x}_1\mathbf{D}^2}\mathbf{D}^{-1}\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_j}\Big]e^{-\mathbf{s}_j D} \\
4\varepsilon^{-1}M_{12} \to e^{\mathbf{s}_i D}\Big[&-(\mathbf{S}_{0,0}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_i})^*\mathbf{D}\mathbf{S}_{1,-1}^{\mathbf{t},\mathbf{x}_1+\mathbf{y}_j} + \mathbf{D}\mathbf{S}_{1,-1}^{-\mathbf{t},\mathbf{x}_1+\mathbf{y}_i}\mathbf{S}_{0,0}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_j} \\
&+ (\mathbf{S}_{0,0}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_i})^* e^{\mathbf{x}_1\mathbf{D}^2}\overline{\mathfrak{p}^{-1}\mathbf{D}\mathfrak{p}_*^{-1}}e^{\mathbf{x}_1\mathbf{D}^2}\mathbf{S}_{0,0}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_j}\Big]e^{-\mathbf{s}_j D} \\
M_{21} \to e^{-\mathbf{s}_i D}\Big[&(\mathbf{D}^{-1}\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_i})^*\mathbf{S}_{0,0}^{\mathbf{t},\mathbf{x}_1-\mathbf{y}_j} - \mathbf{S}_{0,0}^{-\mathbf{t},\mathbf{x}_1-\mathbf{y}_i}\mathbf{D}^{-1}\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_j} \\
&- (\mathbf{D}^{-1}\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_i})^* e^{\mathbf{x}_1\mathbf{D}^2}\overline{\mathfrak{p}^{-1}\mathbf{D}\mathfrak{p}_*^{-1}}e^{\mathbf{x}_1\mathbf{D}^2}\mathbf{D}^{-1}\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_j}\Big]e^{\mathbf{s}_j D} \\
2\varepsilon^{-1/2}M_{22} \to e^{-\mathbf{s}_i D}\Big[&\mathbf{D}^{-1}(\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_i})^*\mathbf{D}\mathbf{S}_{1,-1}^{\mathbf{t},\mathbf{x}_1+\mathbf{y}_j} - \mathbf{S}_{0,0}^{-\mathbf{t},\mathbf{x}_1-\mathbf{y}_i}\mathbf{S}_{0,0}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_j} \\
&- (\mathbf{D}^{-1}\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_i})^* e^{\mathbf{x}_1\mathbf{D}^2}\overline{\mathfrak{p}^{-1}\mathbf{D}\mathfrak{p}_*^{-1}}e^{\mathbf{x}_1\mathbf{D}^2}\mathbf{S}_{0,0}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_j}\Big]e^{\mathbf{s}_j D}.
\end{aligned}
$$

We write this in matrix product form.

$$
\begin{pmatrix} e^{\mathbf{s}_i\mathbf{D}} & 0 \\ 0 & e^{-\mathbf{s}_i\mathbf{D}} \end{pmatrix}
\begin{pmatrix} -(\mathbf{S}_{0,0}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_i})^* & \mathbf{D}\mathbf{S}_{1,-1}^{-\mathbf{t},\mathbf{x}_1+\mathbf{y}_i} \\ (\mathbf{D}^{-1}\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_i})^* & \mathbf{S}_{0,0}^{-\mathbf{t},\mathbf{x}_1-\mathbf{y}_i} \end{pmatrix}
\begin{pmatrix} I & e^{2\mathbf{x}_1\mathbf{D}^2}\overline{\mathfrak{p}^{-1}\mathbf{D}\mathfrak{p}_*^{-1}}e^{2\mathbf{x}_1\mathbf{D}^2} \\ 0 & I \end{pmatrix}
$$
$$
\begin{pmatrix} \mathbf{S}_{0,0}^{\mathbf{t},\mathbf{x}_1-\mathbf{y}_j} & \mathbf{D}\mathbf{S}_{1,-1}^{\mathbf{t},\mathbf{x}_1+\mathbf{y}_j} \\ \mathbf{D}^{-1}\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_j} & \mathbf{S}_{0,0}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},\mathbf{y}_j} \end{pmatrix}
\begin{pmatrix} e^{-\mathbf{s}_j\mathbf{D}} & 0 \\ 0 & e^{\mathbf{s}_j\mathbf{D}} \end{pmatrix}
\tag{4.38}
$$

This completes the pointwise asymptotic analysis for $\tilde{K}(i,\cdot;j,\cdot)$ in (3.74). There are two other terms required in (3.74) that require analysis.

$$
(\mathtt{a})^{-u'_{ij}}(\mathtt{a}_*)^{-d'_{ij}} \tag{4.39}
$$

where $u'_{ij} = (y_j - y_i - s_j + s_i)/2, d'_{ij} = (y_j - y_i + s_j - s_i)/2$. This is the diffusive scaling of the transition density of a random walk; thus, by the central limit theorem,

$$
\begin{aligned}
2\varepsilon^{-1/2}(\mathtt{a})^{-u'_{ij}}(\mathtt{a}_*)^{-d'_{ij}} &\to e^{\mathbf{s}_i\mathbf{D}}e^{(\mathbf{y}_j-\mathbf{y}_i)\mathbf{D}^2}e^{-\mathbf{s}_j\mathbf{D}} \\
2\varepsilon^{-1/2}(\mathtt{a}_*)^{-u'_{ij}}(\mathtt{a})^{-d'_{ij}} &\to e^{\mathbf{s}_j\mathbf{D}}e^{(\mathbf{y}_j-\mathbf{y}_i)\mathbf{D}^2}e^{-\mathbf{s}_i\mathbf{D}}.
\end{aligned}
\tag{4.40}
$$

Lastly, for the element $Y_i^* \mathtt{a}_*^{l_j-l_i} D \mathtt{a}^{r_i-r_j} Y_j$,

$$\begin{aligned} 4\varepsilon^{-1} Y_i^* D Y_j &= \mathtt{a}^{r_i-r_j+r_i'-l_i} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{l_j-l_i+r_j'-l_j} \\ &= \mathtt{a}^{(-s_i^\varepsilon-y_i^\varepsilon+s_j^\varepsilon-y_j^\varepsilon)/2} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{(-y_i^\varepsilon-y_j^\varepsilon-s_j^\varepsilon+s_i^\varepsilon)/2} \\ &\to e^{\mathbf{s}_i \mathbf{D}+\mathbf{y}_i \mathbf{D}^2} \overline{\mathfrak{p}^{-1} \mathbf{D} \mathfrak{p}_*^{-1}} e^{-\mathbf{s}_j \mathbf{D}+\mathbf{y}_j \mathbf{D}^2} \end{aligned} \tag{4.41}$$

4.3 Trace Norm bounds

Up to now, we have shown the pointwise convergence of the kernel. In order to show the Fredholm determinant convergence, we need to show that the kernel is convergent in trace norm; thus, we now want to give a uniform bound of all the kernels above in trace norm.

Proposition 4.3.1. *Define M_k as the multiplication operator such that*

$$M_k f(x) = e^{kx} f(x).$$

For any $0 < \delta < 1/2$, the operator $M_{-|\boldsymbol{\rho}|} D S_{1,-1,\varepsilon}^{(x_1,y_i)} e^{-(2r_1^\varepsilon)D} D^{-1} S_{-1,1,\varepsilon}^{\mathrm{epi},-y_j} M_{|\boldsymbol{\rho}|}$ is bounded in trace norm, uniformly in ε.

Proof. In this proof, it should be understood that all the intermediate space variables $z_1, z_2 \cdots$ are scaled versions, which is $2\varepsilon^{-1/2}\mathbf{z}_1, 2\varepsilon^{-1/2}\mathbf{z}_2, \cdots$. We start with the operator $D S_{1,-1,\varepsilon}^{(x_1,y_i)} e^{-(2r_1^\varepsilon)D} D^{-1} S_{-1,1,\varepsilon}^{\mathrm{epi},-y_j}(z_1, z_4)$, which is

$$\int d\mathbf{z}_2 \int_0^\infty d\mathbf{k} \int d\mathbf{z}_3 \mathbb{P}_{\mathbf{z}_2}(\tau^\varepsilon = \mathbf{k}, \mathrm{ExpWalk(W)}_\varepsilon = d\mathbf{z}_4) \cdot S_{1,-1,\varepsilon}^{(x_1,y_i)}(\mathbf{z}_1, \mathbf{z}_2 + 2\mathbf{r}_1) S_{-1,1,\varepsilon}^{(k^\varepsilon,-y_i)}(\mathbf{z}_3, \mathbf{z}_4). \tag{4.42}$$

Using the fact that

$$\begin{aligned} \|(4.42)\|_1 \le \int d\mathbf{z}_2 \int_0^\infty d\mathbf{k} \int d\mathbf{z}_3 \mathbb{P}_{\mathbf{z}_2}(\tau^\varepsilon = \mathbf{k}, \mathrm{ExpWalk(W)}_\varepsilon = d\mathbf{z}_4) \\ \cdot \| S_{1,-1,\varepsilon}^{(x_1,y_i)}(\mathbf{z}_1, \mathbf{z}_2 + 2\mathbf{r}_1) S_{-1,1,\varepsilon}^{(\mathbf{k},-y_i)}(\mathbf{z}_3, \mathbf{z}_4) \|_1. \end{aligned} \tag{4.43}$$

Notice that the last operator is a rank-one operator in variable $\mathbf{z}_1, \mathbf{z}_4$. Using the fact that the trace norm of a rank-one operator is the product of its L^2 norm, i.e.

$$\| |f\rangle \langle g| \|_1 = \|f\|_{L^2} \|g\|_{L^2}$$

Also, appending the multiplication operator $M_{|\boldsymbol{\rho}|}, M_{-|\boldsymbol{\rho}|}$, the trace norm becomes

$$[\int d\mathbf{z}_1 e^{-2|\boldsymbol{\rho}|\mathbf{z}_1}(S^{(x_1,y_i)}_{1,-1,\varepsilon}(\mathbf{z}_1,\mathbf{z}_2+2\mathbf{r}_1))^2]^{1/2}\int d\mathbf{z}_4 e^{2|\boldsymbol{\rho}|\mathbf{z}_4}(S^{(k^\varepsilon,-y_j)}_{-1,1,\varepsilon}(\mathbf{z}_3,\mathbf{z}_4))^2]^{1/2}.$$

The probability term is well understood from classical theory. We cite the following result in [MQR21]: there exist $\kappa > 0$ such that

$$\mathbb{P}_{\mathbf{z}_2}(\tau^\varepsilon \leq \mathbf{k}) \leq \exp\{-\kappa\frac{(\mathbf{z}_2+C(1+\mathbf{k})^2}{\mathbf{k}}\} \tag{4.44}$$

From (4.35), it is easy to see that there exist $c_1, c_2 > 0$ such that

$$\|S^{(x_1,y_i)}_{1,-1,\varepsilon}(\mathbf{z}_1,\mathbf{z}_2)\|_{L^2} \leq c_1 e^{c_2\mathbf{z}_2}$$

since the error in the convergence does not depend on variable $\mathbf{z}_2$. The $e^{-2|\boldsymbol{\rho}|\mathbf{z}_1}$ is required since otherwise the residue from $-|\varepsilon^{1/2}\boldsymbol{\rho}|$ will not have decay at ∞. Thus the $d\mathbf{z}_2$ integral is convergent. For $\mathbf{z}_3$ and $\mathbf{k}$, first $\mathbf{z}_3$ is the place where the $\mathrm{ExpWalk}(W)_\varepsilon$ hits the initial condition. There is a natural bound on the place it hits; by our assumption on the initial condition, we have $\mathbf{z}_3 \geq -C(1+\mathbf{s})$. On the other hand, a mean 0 random walk with finite variance almost surely cannot grow linearly; thus, $\mathbf{z}_3 \leq \mathbf{z}_2 + \varepsilon^{-1/2}\mathbf{s}$. For the bound on $\|S^{(\mathbf{k},-\mathbf{y}_i)}_{-1,1,\varepsilon}(\mathbf{z}_3,\mathbf{z}_4)\|_{L^2}$, it is

$$\left(\int_{C_1} dw_1 \int_{C_2} dw_2 (\varepsilon\boldsymbol{\rho}^2 - 4w_1^2)(\varepsilon\boldsymbol{\rho}^2 - 4w_2^2)\frac{e^{F(w_1)+F(w_2)}}{2\varepsilon^{-1/2}(w_1+w_2)-2|\boldsymbol{\rho}|}\right)^{1/2} \tag{4.45}$$

where F is the expression in (4.35). Here we define $\tilde{F}(w_1,\mathbf{k}) = \varepsilon^{-3/2}f_1 + \varepsilon^{-1}f_2$ (notice that $\mathbf{x}$ in (4.35) becomes $-\mathbf{k}$).We do not need to add terms involving $\mathbf{r}_k$ since we do not consider the regime that $\mathbf{r}_k$ is large; we do not need to add $\mathbf{z}_3$ since it is not involved in the expansion of ε.

Solving $\partial_{w_1}\tilde{F}(w_1,\mathbf{k}) = 0$, we see that two roots are 0 and $\varepsilon^{1/2}\mathbf{k}/\mathbf{t}$. Now we want to move the contour to the critical point $\varepsilon^{1/2}\mathbf{k}/\mathbf{t}$. WLOG we can assume that $\mathbf{k}$ is large enough (since we want to investigate the integrability in $\mathbf{k}$) so that we do not encounter the pole at $2\varepsilon^{-1/2}(w_1+w_2) - 2\varepsilon^{1/2}|\boldsymbol{\rho}|$. On the other hand, we also will not cross the other pole at $1/2$ since if $\varepsilon^{1/2}\mathbf{k}/\mathbf{t} \geq 1/2$, the integrand in (4.45) is analytic and the whole integral reduces to 0. Thus, we simply take the contour to be $C^{\pi/3}_{\varepsilon^{1/2}\mathbf{k}/\mathbf{t}}$, i.e. $\{\varepsilon^{1/2}\mathbf{k}/\mathbf{t} + re^{\pm\pi/3} : r > 0\}$. We show that the real part is strictly decreasing:

$$\begin{aligned}&\frac{d(\varepsilon^{-3/2}f_1 + \varepsilon^{-1}f_2)(\varepsilon^{1/2}\mathbf{k}/\mathbf{t} + re^{\pm\pi/3})}{dr}\\ &= -\frac{(4r\mathbf{t}^2(2r\mathbf{t}^3 + 4r^3\mathbf{t}^3 + \varepsilon^{1/2}\mathbf{k}(\mathbf{t}^2 - 4\varepsilon\mathbf{k}^2))}{(\varepsilon^{3/2}((3r^2\mathbf{t}^2 + (2\varepsilon^{1/2}\mathbf{k} + (-1+r)\mathbf{t})^2))((3r^2\mathbf{t}^2 + (2\varepsilon^{1/2}\mathbf{k} + (1+r)\mathbf{t})^2)))} < 0\end{aligned} \tag{4.46}$$

Notice that $(\mathbf{t}^2 - 4\varepsilon\mathbf{k}^2) > 0$ exactly because of our restriction $\varepsilon^{1/2}\mathbf{k}/\mathbf{t} < 1/2$. Now we can use the value of the integrand in (4.45) at $\varepsilon^{1/2}\mathbf{k}/\mathbf{t}$. We can write $\tilde{F}(w, x)$ in the following form

$$\begin{aligned}&\tilde{F}(w, x)\\ &= (\frac{8\mathbf{t}w^3}{3} - 4w^2x)\sum_{n\geq 0}\frac{3}{(2n+1)n}(\varepsilon^{1/2}w)^{2n} + (8\mathbf{t}w^3 - 8w^2x)\sum_{n\geq 1}\frac{n-1}{(2n+1)n}(\varepsilon^{1/2}w)^{2n}\\ &= (\frac{w^3}{3} + w^2x)\nu_1(\varepsilon^{1/2}w) + (w^3 + 2w^2x)\nu_2(\varepsilon^{1/2}w)\end{aligned} \tag{4.47}$$

where both ν_1, ν_2 are uniformly bounded in absolute value and non-negative. So plugging $w = \varepsilon^{1/2}\mathbf{k}/\mathbf{t}$, we get that there exists $\delta > 0$ such that

$$\tilde{F}(\varepsilon^{1/2}\mathbf{k}/\mathbf{t}, \mathbf{k}) \leq -(\frac{4}{3} - \delta)\frac{\mathbf{k}^3}{\mathbf{t}^2}.$$

So (4.45) $\in O(e^{-(\frac{4}{3}-\delta)\frac{\mathbf{k}^3}{\mathbf{t}^2}})$, which clearly makes the integral in (4.43) convergent.

Thus, the trace norm is uniformly bounded in ε. □

Next, we investigate the other type of kernel in (4.20), the term

$$(S_{0,0}^{\mathrm{epi},y_i})^* e^{2r_1^\varepsilon D}\mathtt{a}^{-x_1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-x_1}e^{-2r_1^\varepsilon D}D^{-1}S_{-1,1}^{\mathrm{epi},-y_j}.$$

Write out the integration:

$$\begin{aligned}&\int_{\mathbf{z}_2,\mathbf{z}_3,k_1,k_2,\mathbf{z}_4,\mathbf{z}_5} S_{0,0,\varepsilon}^{(-\mathbf{k}_1-\mathbf{y}_i)}(\mathbf{z}_2,\mathbf{z}_1)\mathbb{P}_{\mathbf{z}_2}(\tau^\varepsilon = \mathbf{k}_1, \mathrm{ExpWalk(W)}_\varepsilon = d\mathbf{z}_3)\\ &(\mathtt{a}^{r_i+u-l_i}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}^{r_j+u-l_j})(\mathbf{z}_3,\mathbf{z}_4)\cdot\mathbb{P}_{z_4}(\tau^\varepsilon = \mathbf{k}_2, \mathrm{ExpWalk(W)}_\varepsilon = d\mathbf{z}_5)\\ &\cdot S_{0,0,\varepsilon}^{(-\mathbf{k}_2,-\mathbf{y}_j)}(\mathbf{z}_5,\mathbf{z}_6)\end{aligned} \tag{4.48}$$

Using the same procedure,

$$\begin{aligned}\|(4.48)\|_1 \leq \int_{\mathbf{z}_2,\mathbf{z}_3,k_1,k_2,\mathbf{z}_4,\mathbf{z}_5} & \mathbb{P}_{\mathbf{z}_2}(\tau^{\varepsilon} = \mathbf{k}_1, \mathrm{ExpWalk}(\mathrm{W})_{\varepsilon} = d\mathbf{z}_3) \\ & (\mathtt{a}^{r_i+u-l_i}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}^{r_j+u-l_j})(\mathbf{z}_3,\mathbf{z}_4)\cdot \mathbb{P}_{\mathbf{z}_4}(\tau^{\varepsilon} = \mathbf{k}_2, \mathrm{ExpWalk}(\mathrm{W})_{\varepsilon} = d\mathbf{z}_5) \\ & \|S_{0,0,\varepsilon}^{(-\mathbf{k}_1-\mathbf{y}_i)}(\mathbf{z}_2,\mathbf{z}_1)S_{0,0,\varepsilon}^{(-\mathbf{k}_2,-\mathbf{y}_j)}(\mathbf{z}_5,\mathbf{z}_6)\|_1\end{aligned} \tag{4.49}$$

The middle operator is the differential of a transition probability; there exists $c_1, c_2 > 0$ such that

$$(\mathtt{a}^{r_i+u-l_i}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}^{r_j+u-l_j})(\mathbf{z}_2,\mathbf{z}_3) \leq e^{c_1\mathbf{z}_2+c_2\mathbf{z}_3}.$$

From previous calculations,

$$\begin{aligned}\mathbb{P}_{\mathbf{z}_2}(\tau \leq \mathbf{k}_1) \leq \exp\{-\kappa\frac{(\mathbf{z}_2 + C(1+\mathbf{k}_1))^2}{\mathbf{k}_1}\} \\ \mathbb{P}_{\mathbf{z}_3}(\tau \leq \mathbf{k}_2) \leq \exp\{-\kappa\frac{(\mathbf{z}_3 + C(1+\mathbf{k}_2))^2}{\mathbf{k}_2}\}\end{aligned} \tag{4.50}$$

Also with a bound on $\mathbf{z}_3, \mathbf{z}_5$ that

$$\begin{aligned}-(C+1)\mathbf{k}_1 \leq \mathbf{z}_3 \leq \mathbf{z}_2 + \varepsilon^{-1}\mathbf{k}_1 \\ -(C+1)\mathbf{k}_2 \leq \mathbf{z}_5 \leq \mathbf{z}_3 + \varepsilon^{-1}\mathbf{k}_2\end{aligned} \tag{4.51}$$

Together with the bound for $S_{0,0,\varepsilon}^{(-\mathbf{k}_1,-\mathbf{y}_i)}, S_{0,0,\varepsilon}^{(-\mathbf{k}_2,-\mathbf{y}_j)}$ in $\mathbf{k}_1, \mathbf{k}_2$,

$$\|S_{0,0,\varepsilon}^{(-\mathbf{k}_1,-\mathbf{y}_i)} \leq e^{-c_3\mathbf{k}_2^3/\mathbf{t}^2}, \quad \|S_{0,0,\varepsilon}^{(-\mathbf{k}_2,-\mathbf{y}_j)}\|_{L^2} \leq e^{-c_4\mathbf{k}_1^3/\mathbf{t}^2} \tag{4.52}$$

Combining these together,we can see that (4.49) is finite. Analogously, one can show that all the components in the kernel are uniformly bounded in the trace norm.

Now we can show that the kernel converges in trace norm, following the argument in [MQR21].

Theorem 4.3.2. *The operator* $DS_{1,-1,\varepsilon}^{(x_1,y_i)}e^{-(2r_1^{\varepsilon})D}D^{-1}S_{-1,1,\varepsilon}^{\mathrm{epi},-y_j}$ *converges to* $\mathbf{D}\mathbf{S}_{1,-1}^{-\mathbf{t},\mathbf{x}_1+\mathbf{y}_i}\mathbf{D}^{-1}\mathbf{S}_{-1,1}^{\mathrm{hypo}(\mathfrak{h}_0),\mathbf{t},-\mathbf{y}_j}$ *in trace norm.*

Proof. By Donsker's theorem $\mathrm{ExpWalk}(\mathrm{W})_{\varepsilon} \to \mathbf{B}$ uniformly on compact sets, where $\mathbf{B}$ is a Brownian motion with diffusion coefficient 2. By [MQR21] Proposition 3.2,

$$\mathbb{P}_{\mathbf{z}_2}(\tau^{\varepsilon} = \mathbf{k}, \mathrm{ExpWalk}(\mathrm{W})_{\varepsilon} = d\mathbf{z}_4) \to \mathbb{P}_{\mathbf{B}(0)=\mathbf{z}_2}(\boldsymbol{\tau} \in d\mathbf{k}, \mathbf{B}(\tau) \in d\mathbf{z}_4)$$

as measures. Since it is uniformly bounded in ε, we can restrict $\mathbf{k}, \mathbf{z}_2, \mathbf{z}_4$ to compact intervals, on which the measure is finite. Lastly, using $\|\int f_\varepsilon d\mu_\varepsilon - \int f d\mu\|_1 \le \|\int f_\varepsilon d\mu_\varepsilon - \int f d\mu_\varepsilon\|_1 + \|\int f d\mu_\varepsilon - \int f d\mu\|_1$, the first term goes to 0 by the dominated convergence theorem (recall the norm becomes L^2 norm on a rank one operator), and the second term goes to 0 since $\mu_\varepsilon \to \mu$ weakly as a finite measure and f is bounded. □

The proof of the kernel

$$(S_{0,0}^{\mathrm{epi},y_i})^* e^{2r_1^\varepsilon D} \mathtt{a}^{-x_1} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{-x_1} e^{-2r_1^\varepsilon D} D^{-1} S_{-1,1}^{\mathrm{epi},-y_j}.$$

is the same. Thus, we showed that our kernel in (4.20) converges in the trace norm.

4.4 Tightness and Markov property

To show that the limiting probability function is the transition probability of a Markov process, we follow the same scheme in [MQR21]. Since the argument is very similar, we just point out the method and what is different and needs to be checked in the half-space case. First, we want to show the tightness, which is a result of Hölder regularity. We want to show

Proposition 4.4.1. *Fix $t > 0$, assume the initial condition of TASEP $\mathfrak{h}^\varepsilon(0,\cdot) \to \mathfrak{h}_0$ in distribution, in UC. Then for each $\beta \in (0, 1/2)$ and $M < \infty$*

$$\lim_{A\to\infty} \limsup_{\varepsilon\to 0} \mathbb{P}(\|\mathfrak{h}^\varepsilon(\mathbf{t})\|_{\beta,[0,M]} \ge A) = 0 \tag{4.53}$$

From the regularity, we can get the tightness; see the section on tightness and the Markov property [MQR21] .

Tightness gives us that $\mathbb{P}(\mathfrak{h}(\mathbf{t}, \mathbf{x}_1) \le \mathbf{r}_1, \cdots, \mathfrak{h}(\mathbf{t}, \mathbf{x}_n) \le \mathbf{h}_n)$ is a probability distribution for each fixed t. We want to show that as a process t, it is the transition probability of a Markov process. Using the fact that the convergence we proved is uniform over initial conditions $\mathfrak{h}^\varepsilon(0,\cdot)$ in sets of locally bounded Hölder β norm for $0 < \beta < 1/2$, using Lemma 3.10 in [MQR21], it finishes the proof of the existence of a Markov process with transition probability given by (3.2.1), which is the *the half-space KPZ fixed point*.

Thus, in the following section, we just need to show (4.53). To prove this, we use a version of the Kolmogorov continuity theorem:

Theorem 4.4.2. *Let $\mathfrak{h}(x)$ be a stochastic process defined for x in an interval $[0, M]$, such that for some $p > 1$ and $\alpha > 0$,*

$$\mathbb{E}[|\mathfrak{h}(x) - \mathfrak{h}(y)|^p] \le C_1|x - y|^{1+\alpha}.$$

Then for every $\beta < \alpha/p$, there is a constant $C_2 = C_2(p, \alpha, \beta, C_1)$ such that

$$\mathbb{P}(\|\mathfrak{h}\|_{\beta,[0,M]} \ge R) \le C_2 R^{-p} \tag{4.54}$$

We will fix $\mathbf{t} = 1$ afterwards since the bound is only in the spatial variable. Let $\mathbf{h}^\varepsilon$ to be the rescaled TASEP height function. Let $\mathbf{h}^\varepsilon_N$ to be the cut-off:

$$\mathbf{h}^\varepsilon_N(\mathbf{y}) := (\mathbf{h}^\varepsilon(\mathbf{y}) \wedge N) \vee (-N).$$

The reason we can have this cut-off is that

$$\limsup_{N\to\infty} \limsup_{\varepsilon\to 0} \mathbb{P}(\sup_{\mathbf{y}\in[0,M]} |\mathbf{h}^\varepsilon(\mathbf{y})| \ge N) = 0.$$

Let F^ε be the cumulative distribution function, i.e. $F^\varepsilon(\mathbf{y}_1, \mathbf{s}_1; \mathbf{y}_2, \mathbf{s}_2) = \mathbb{P}(\mathbf{h}^\varepsilon(\mathbf{y}_1) \le \mathbf{s}_1, \mathbf{h}^\varepsilon(\mathbf{y}_2) \le \mathbf{s}_2)$. Use [MQR21] Lemma $C.3$, for $p \ge 2$,

$$\begin{aligned}&\mathbb{E}[|\mathbf{h}^\varepsilon_N(\mathbf{y}_1) - \mathbf{h}^\varepsilon_N(\mathbf{y}_2)|] = p(p-1) \int_{-N}^{N} \int_{-N}^{N} d\mathbf{s}_1 d\mathbf{s}_2 |\mathbf{s}_1 - \mathbf{s}_2|^{p-2} \\ &[F^\varepsilon(\mathbf{y}, \mathbf{s}_1) 1_{\mathbf{s}_1 < \mathbf{s}_2} + F^\varepsilon(\mathbf{y}_1, \mathbf{s}_2) 1_{\mathbf{s}_1 \ge \mathbf{s}_2} - F^\varepsilon(\mathbf{y}_1, \mathbf{s}_1; \mathbf{y}_2, \mathbf{s}_2)]\end{aligned} \tag{4.55}$$

We analyze the case that $\mathbf{s}_1 < \mathbf{s}_2$ first; the other case is the same. We want to bound

$$\int_{-N \le \mathbf{s}_1 < \mathbf{s}_2 \le N} d\mathbf{s}_1 d\mathbf{s}_2 |\mathbf{s}_1 - \mathbf{s}_2|^{p-2} [F^\varepsilon(\mathbf{y}, \mathbf{s}_1) - F^\varepsilon(\mathbf{y}_1, \mathbf{s}_1; \mathbf{y}_2, \mathbf{s}_2)] \tag{4.56}$$

by $C(N)|\mathbf{y}_1 - \mathbf{y}_2|^{1+\alpha}$ for some constant $C(N)$. The difference of the two square roots of the determinant can be bounded by

$$\sqrt{\det(I-A)} - \sqrt{\det(I-B)} \le \frac{1}{\sqrt{\det(I-A)} + \sqrt{\det(I-B)}} \|A - B\|_1 e^{\|A\|_1 + \|B\|_1 + 1}$$

The fact that the denominator is bounded in both $\mathbf{y}_1, \mathbf{y}_2$ is because we have the cut-off at N and both square roots of determinants represent a probability CDF. Since from (4.3.1), we know the terms on the exponential are bounded, it remains to bound $\|A - B\|_1$, where A, B are the corresponding kernels for $F^\varepsilon(\mathbf{y}_1, \mathbf{s}_1), F^\varepsilon(\mathbf{y}_1, \mathbf{s}_1; \mathbf{y}_2, \mathbf{s}_2)$.

Here we need to use the path integral formula for the multi-point distribution (3.3.2). Recall from (3.102)

$$K_{\mathrm{PI-m}} = \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix} \begin{pmatrix} V_I^* & -V_I^* D V_I \\ 0 & V_I \end{pmatrix} \begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} \begin{pmatrix} V_F^* & V_F^* D V_F \\ 0 & V_F \end{pmatrix}. \tag{4.57}$$

Using $\det(I-AB) = \det(I-BA)$, we bring the first matrix to the end. Since the second matrix does not contain any information about the final configuration, we focus on

$$\begin{pmatrix} S & DS \\ D^{-1}S & S \end{pmatrix} \begin{pmatrix} V_F^* & V_F^* D V_F \\ 0 & V_F \end{pmatrix} \begin{pmatrix} -S & DS \\ D^{-1}S & -S \end{pmatrix}$$

In the one-point $\mathbf{y}_1, \mathbf{s}_1$ case, the kernel is

$$\begin{aligned} &\begin{pmatrix} S^{l_1,l_1} & DS^{l_1,l_1} \\ D^{-1}S^{l_1,l_1} & S^{l_1,l_1} \end{pmatrix}_\varepsilon \begin{pmatrix} \mathtt{a}^{y_1^\varepsilon} \mathtt{b} 1_0 \mathtt{a}^{-y_1^\varepsilon} \mathtt{b}^{-1} & \mathtt{a}^{y_1^\varepsilon} \mathtt{b} 1_0 \mathtt{a}^{-y_1^\varepsilon} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{-y_1^\varepsilon} 1_0 \mathtt{a}_*^{y_1^\varepsilon} \mathtt{b}_* \\ 0 & \mathtt{a}_*^{-y_1^\varepsilon} \mathtt{b}_*^{-1} 1_0 \mathtt{a}_*^{y_1^\varepsilon} \mathtt{b}_* \end{pmatrix} \\ &\cdot \begin{pmatrix} -S^{l_1,l_1} & DS^{l_1,l_1} \\ D^{-1}S^{l_1,l_1} & -S^{l_1,l_1} \end{pmatrix}_\varepsilon \\ &= \begin{pmatrix} S_{0,1}^{r_1,l_1} & DS_{1,0}^{l_1,r_1} \\ D^{-1}S_{0,1}^{r_1,l_1} & S_{1,0}^{l_1,r_1} \end{pmatrix}_\varepsilon \begin{pmatrix} 1_0 & 1_0 \mathtt{a}^{-y_1^\varepsilon} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{-y_1^\varepsilon} 1_0 \\ 0 & 1_0 \end{pmatrix} \begin{pmatrix} -S_{1,0}^{l_1,r_1} & DS_{1,0}^{l_1,r_1} \\ D^{-1}S_{0,1}^{r_1,l_1} & -S_{0,1}^{r_1,l_1} \end{pmatrix}_\varepsilon \end{aligned} \tag{4.58}$$

where $l_1 = (h_n + x_n - s_1 + y_1)/2, r_1 = (h_n + x_n - s_1 - y_1)/2$. In the two-point $\mathbf{y}_1, \mathbf{s}_1; \mathbf{y}_2, \mathbf{s}_2$ case, let $W' = (I - \mathtt{a}^{u'} \overline{1}^0 \mathtt{a}^{-u'} \mathtt{a}_*^{-d'} \overline{1}^0 \mathtt{a}_*^{d'})$ the kernel be

$$\begin{aligned} &\begin{pmatrix} S_{0,1}^{r_2,l_2} & DS_{1,0}^{l_2,r_2} \\ D^{-1}S_{0,1}^{r_2,l_2} & S_{1,0}^{l_2,r_2} \end{pmatrix} \begin{pmatrix} (W')^* & (W')^* \mathtt{a}^{-\frac{y_1+y_2}{2}} \overline{\mathtt{b}^{-1} D \mathtt{b}_*^{-1}} \mathtt{a}_*^{-\frac{y_1+y_2}{2}} W' \\ 0 & W' \end{pmatrix} \\ &\qquad \begin{pmatrix} -S_{1,0}^{l_2,r_2} & DS_{1,0}^{l_2,r_2} \\ D^{-1}S_{0,1}^{r_2,l_2} & -S_{0,1}^{r_2,l_2} \end{pmatrix} \end{aligned} \tag{4.59}$$

Thus, we have where $l_2 = (-h_n + x_n - s_2 + y_2)/2, r_2 = (h_n + x_n - s_1 - y_1)/2$. Notice that $r_2 = r_1$. We define W' as the summation of two parts,

$$W' = \mathtt{a}^{u'}(1_0 \mathtt{a}^{-u'} \mathtt{a}_*^{-d'} + \overline{1}^0 \mathtt{a}^{-u'} \mathtt{a}_*^{-d'} 1_0) \mathtt{a}_*^{d'} =: W_1 + W_2.$$

. The part only involves W_1 reduces to the one-point case,

$$\begin{pmatrix} S_{0,1}^{r_2,l_2} & DS_{1,0}^{l_2,r_2} \\ D^{-1}S_{0,1}^{r_2,l_2} & S_{1,0}^{l_2,r_2} \end{pmatrix} \begin{pmatrix} W_1^* & (W_1)^*\mathtt{a}^{-\frac{y_1+y_2}{2}}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-\frac{y_1+y_2}{2}}W_1 \\ 0 & W_1 \end{pmatrix}$$
$$\cdot \begin{pmatrix} -S_{1,0}^{l_2,r_2} & DS_{1,0}^{l_2,r_2} \\ D^{-1}S_{0,1}^{r_2,l_2} & -S_{0,1}^{r_2,l_2} \end{pmatrix}$$
$$= \begin{pmatrix} S_{0,1}^{r_1,l_1} & DS_{1,0}^{l_1,r_1} \\ D^{-1}S_{0,1}^{r_1,l_1} & S_{1,0}^{l_1,r_1} \end{pmatrix} \begin{pmatrix} 1_0 & 1_0\mathtt{a}^{-y_1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-y_1}1_0 \\ 0 & 1_0 \end{pmatrix} \begin{pmatrix} -S_{1,0}^{l_1,r_1} & DS_{1,0}^{l_1,r_1} \\ D^{-1}S_{0,1}^{r_1,l_1} & -S_{0,1}^{r_1,l_1} \end{pmatrix}. \tag{4.60}$$

The rest is $\begin{pmatrix} W_2^* & M_{21} \\ 0 & W_2 \end{pmatrix}$ where

$$\begin{aligned} M_{21} &= (W_2)^*\mathtt{a}^{-\frac{y_1+y_2}{2}}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-\frac{y_1+y_2}{2}}W_2 \\ &+ (W_1)^*\mathtt{a}^{-\frac{y_1+y_2}{2}}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-\frac{y_1+y_2}{2}}W_2 \\ &+ (W_2)^*\mathtt{a}^{-\frac{y_1+y_2}{2}}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-\frac{y_1+y_2}{2}}W_1. \end{aligned} \tag{4.61}$$

Thus

$$\|A-B\|_1 \le \|A\|_1 \| \begin{pmatrix} W_2^* & M_{21} \\ 0 & W_2 \end{pmatrix} \|_{\mathrm{op}} \tag{4.62}$$

From the previous section, we know that $\|A\|_1$ is bounded. For the operator norm of the matrix, notice that

$$\|W_2\|_{\mathrm{op}} = \|W_2^*\|_{\mathrm{op}} = \|\bar{1}^0\mathtt{a}^{-u'}\mathtt{a}_*^{-d'}1_0\|_{\mathrm{op}}.$$

Notice that the term $\mathtt{a}^{u'}$ and $\mathtt{a}_*^{d'}$ can be absorbed into S, thus not appearing in the operator norm. Let $B_{y_2-y_1}^{\varepsilon} = \mathrm{ExpWalk}_{u'+d'}^{\varepsilon}$ be the scaled mean 0 version of $\mathtt{a}^{-u'}\mathtt{a}_*^{-d'}$, thus

$$\|\bar{1}^0\mathtt{a}^{-u'}\mathtt{a}_*^{-d'}1_0\|_{\mathrm{op}} = \|\bar{1}^{\mathtt{s}_1}B_{y_2-y_1}^{\varepsilon}1_{\mathtt{s}_2}\|_{\mathrm{op}} \le \int_{|\mathtt{s}_2-\mathtt{s}_1|}^{\infty} B_{y_2-y_1}^{\varepsilon}(y)dy.$$

For the three terms in M_{21}, notice that in the middle of the terms they have the common operator

$$\mathtt{a}_*^{u'}\mathtt{a}^{-\frac{y_1+y_2}{2}}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-\frac{y_1+y_2}{2}}\mathtt{a}^{u'} = \mathtt{a}^{-y_1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-y_1},$$

which is the derivative of the transition density of a mean 0 random walk, thus

$$\|\mathtt{a}^{-y_1}\overline{\mathtt{b}^{-1}D\mathtt{b}_*^{-1}}\mathtt{a}_*^{-y_1}\|_{\text{op}} \leq C$$

for some constant C. What is left in W_1 is 1_0, whose operator norm is bounded; what is left in W_2 is $\overline{1}^0\mathtt{a}^{-u'}\mathtt{a}_*^{-d'}1_0$, which can be bounded in the same way above. Thus in total,

$$\|\begin{pmatrix} W_2^* & M_{21} \\ 0 & W_2 \end{pmatrix}\|_{\text{op}} \leq C\int_{|\mathbf{s}_2-\mathbf{s}_1|}^{\infty} B^{\varepsilon}_{y_2-y_1}(y)dy.$$

Plug this bound into (4.56),

$$\begin{aligned}(4.56) &\leq C\int_{-N\leq \mathbf{s}_1<\mathbf{s}_2\leq N} d\mathbf{s}_1 d\mathbf{s}_2 |\mathbf{s}_2-\mathbf{s}_1|^{p-2}\int_{|\mathbf{s}_2-\mathbf{s}_1|}^{\infty} B^{\varepsilon}_{y_2-y_1}(y)dy \\ &\leq CN\mathbb{E}[B^{\varepsilon}_{y_2-y_1}]^{p-1} \leq C(N,p)|\mathbf{y}_2-\mathbf{y}_1|^{\frac{p-1}{2}}\end{aligned} \tag{4.63}$$

Using the Kolmogorov continuity theorem (4.4.2) with $\alpha = \frac{p-3}{2}$, we get that the Hölder continuity with $\beta = \frac{1}{2} - \frac{3}{2p}$ for any $p \geq 2$, which is the bound in (4.54), which further implies the equation in (4.4.1). Thus, we proved the process is local Hölder $1/2^-$ in space.

Chapter 5

Appendix

5.1 Fredholm determinant and Fredholm Pfaffian

In this section, we review the definition of the Fredholm determinant and some of the important properties. All the properties are proved either in some textbooks, see [Sim15, Sim79, Lax02]. For properties regarding the Pfaffian, see [OQR17]. We will only cite the properties.

Let X be a compact metric space with μ a finite measure on X. Let $K : X \times X \to \mathbb{C}$ be a continuous function. The K can be thought of as an operator on $f \in L^2(X, \mu)$ such that

$$(Kf)(x) = \int K(x, y) f(y) d\mu(y). \tag{5.1}$$

Definition 1. *The Fredholm determinant is defined by*

$$\det(I + \lambda K)_{L^2(X,d\mu)} = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int \dots \int \det(K(x_i, y_j))_{1 \le i,j \le n} d\mu(x_1) \dots d\mu(x_n) \tag{5.2}$$

Definition 2. *Assume K is a 2 matrix-valued skew-symmetric kernel,*

$$K(x, y) = \begin{pmatrix} K_{11}(x, y) & K_{12}(x, y) \\ K_{21}(x, y) & K_{22}(x, y) \end{pmatrix}; \quad x, y \in X$$

its Fredholm Pfaffian is defined by

$$\mathrm{Pf}(J + \lambda K)_{L^2(X,d\mu)} = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int \dots \int \mathrm{Pf}(K(x_i, y_j))_{1 \le i,j \le n} d\mu(x_1) \dots d\mu(x_n), \tag{5.3}$$

where the kernel J is defined by

$$J(x,y) = 1_{x=y}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For a skew-symmetric $2k \times 2k$ matrix A, its Pfaffian is defined by

$$\mathrm{Pf}((a_{i,j})_{1\le i,j\le 2k}) = \frac{1}{2^k k!}\sum_{\sigma\in S_{2n}} \mathrm{sign}(\sigma) a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2k-1)\sigma(2k)}. \tag{5.4}$$

For $2k \times 2k$ skew-symmetric matrix A, it is known that $\mathrm{Pf}(A)^2 = \det(A)$; we also have the same relation between the Fredholm determinant and the Fredholm Pfaffian.

Proposition 5.1.1. *For any skew-symmetric 2×2 matrix kernel K and $\lambda \in \mathbb{C}$, we have*

$$\mathrm{Pf}(J + \lambda K)^2_{L^2(X,\mu)} = \det(I - \lambda JK)_{L^2(X,\mu)\times L^2(X,\mu)} \tag{5.5}$$

given that both sides are convergent.

Proposition 5.1.2. *(Cyclic property) If $K_1 : L^2(X_1) \to L^2(X_2)$ and $K_1 : L^2(X_2) \to L^2(X_1)$, then*

$$\det(I + K_1K_2)_{L^2(X_2)} = \det(I + K_2K_1)_{L^2(X_1)} \tag{5.6}$$

Proposition 5.1.3.

$$\det((I - A)(I - B)) = \det(I - A)\det(I - B) \tag{5.7}$$

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