COMBINATORIAL STRUCTURE OF PARAMETER SPACES OF POLYNOMIALS

by

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Abstract

In this thesis, we will explore the combinatorics of the Mandelbrot set.

Firstly, I will discuss relates the ordering of the hyperbolic components along veins to a classical result of Sharkovsky of orbit forcing for maps of one real dimension. In this problem I will assume V is a principal vein of combinatorial rotation number p/k (and denote $i = 1$), or an i-th secondary vein for $2 \leq i \leq k-1$. Suppose $C_V(n)$ is the hyperbolic component of period n along V closest to the main cardioid. I will show that $C_V(n) \succ_V C_V(m)$ if $n >_k m$. Additionally, the hyperbolic components of $C_V(n)$ with $n \equiv i \mod k$ are minimal and form a dynamically simple family whose Hubbard trees are spiral graphs.

Secondly, I will discuss a characterization of a family of abstract Hubbard trees by their topology. For any $n \geq 3$, let $\mathcal{M}(n)$ denote the set of parameters c in the Mandelbrot set with Hubbard tree T_c homeomorphic to an n-star. An n-vein is a union of veins in the Mandelbrot set that is homeomorphic to an n-star. I will show that the set of parameters of $\mathcal{M}(n)$ is a exactly the union of $\phi(n)$ n-stars and the hyperbolic components that pass through them. Additionally, I will identify a family of dynamically simple Misiurewicz tips that define the n-veins.

Thirdly, I will extend Milnor-Thurston kneading theory to the dynamics of Hubbard trees of postcritically finite maps. Using the first return map of a suitable interval, I will give an algorithm to find the kneading determinant for the Hubbard trees of along an arbitrary vein to a Misiurewicz tip, by identifying the kneading determinant of the tip and perturbing it for on the portion that shares the same branching.

Lastly, I will give another description of the contributions to pseudomonodromy of the Mandelbrot set. Given an oriented arc in the exterior of the Mandelbrot set that starts and ends at the root of the same hyperbolic component, it induces an automorphism on the space of kneading sequences. Ishii-Richards showed that the nontrivial contributions of this automorphism of kneading sequences arise from conspicuous components. I will present a natural description of conspicuous components using internal addresses.

To all the animals that sacrificed their lives for science.

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Introduction

The study of one dimensional complex dynamics is concerned with the iteration of analytic maps on a Riemann surface. The simplest non-trivial family maps of this form for which dynamically interesting properties emerge is the family of quadratic polynomials. These maps have a superattracting fixed point at ∞ of order 2, and one free critical point that maps to a finite value. Maps in this family have a unique representation up to conjugation by a Möbius map as a map of the form $f_c(z) = z^2 + c$, where $c \in \mathbb{C}$ is a parameter.

For a given map f_c acting on the dynamical plane $\hat{\mathbb{C}}$, we can partition the points in the dynamical plane by whether a neighbourhood around a given point behaves chaotically under iteration. More precisely, we define the family of forward iterates of the map $\{f_c^n\}_{n\in\mathbb{N}}$. For a given point $z \in \hat{\mathbb{C}}$, if there exists a open and connected neighbourhood $U(z)$ around z for which the forward iterates of f_c forms a normal family, then z is a point in the *Fatou set* $F(f_c)$. A *Fatou component* is a connected component of the Fatou set. The complement of the Fatou set is the *Julia set* $J(f_c)$, which consists of all the chaotic points where interesting dynamics happens. The *filled Julia set* $K(f_c)$ is the set of all points that are bounded under iteration, that is, union of the Julia set and all bounded Fatou components.

The *Mandelbrot set* consists of all parameters c such that the critical orbit is bounded. A parameter is hyperbolic if there is a neighbourhood around it in the parameter space for which all parameters in the neighbourhood have the same attracting periodic cycle. Given a prescribed orbit, the largest open set for which all parameters have that orbit as their attracting periodic cycle is a hyperbolic component corresponding to that orbit. Given a hyperbolic component, its centre is the unique parameter whose critical orbit is the periodic orbit associated to the hyperbolic component.

We are interested in the special case where the map is *postcritically finite*, that is, the forward orbits of the critical point is finite. These parameters occur as centres of hyperbolic components and

Misiurewicz points in the Mandelbrot set. They form a tree-like structure in the Mandelbrot set.

The smallest connected forward invariant set containing the critical orbit is the *Hubbard tree* T_c of f_c, and is a topological tree consisting of the union of regulated arcs between the points of the critical orbit. For a postcritically finite parameter, it has a Hubbard tree with finitely many vertices given by the critical orbit and branch points[7]. This subset of one real dimension captures many of the dynamically interesting invariants of f_c on the complex plane. For instance, it contains all periodic orbits with nontrivial portraits. Moreover, the Markov map given by the action of f_c on the edges of T_c allows for the development of a symbolic understanding of the dynamics on the plane.

A *vein* in the Mandelbrot set is an embedded arc from the centre of the main cardioid to a Misiurewicz tip parameter. The simplest such family is the family of real parameters, where the analysis of its Hubbard tree is the dynamics of a real quadratic polynomial on a real line. Using the Measurable Riemann Mapping Theorem, Branner-Douady identified a family of *principal veins*, which are a family of maps given by quasiconformal surgery of the real vein [4].

In this thesis, I will discuss other families of veins that generalize the principal vein, and relate various combinatorial structures that arise in the Mandelbrot set to a number of natural constructions in the theory of dynamical systems in one real dimension.

In Chapter 2 I will state relevant background material as ingredients in my results.

In Chapter 3 I will describe my first result, where I relate the periods of hyperbolic components to a classical result by Sharkovsky for dynamics of interval maps. For any vein V, let $C_V(n)$ be the hyperbolic component of period n along V closest to the main cardioid, if it exists. If C and C' are hyperbolic components that lie along V, denote by $C \succ_V C'$ to mean C is further than C' from the main cardioid. In any p/k limb, there exists the principal vein V_1, and veins V_i for each $i = 2, \ldots, k-1$. For $V = V_i$, I will show that $C_V(n) \succ_V C_V(m)$ if $n \geq_k m$ in the k-Sharkovsky ordering. Moreover, for $n \equiv i \mod k$, we have that the set of periods of the Hubbard tree of $C_V(n)$ is exactly $\{m | m \leq_k n\}$, and the dynamics of the Hubbard tree of $C_V(n)$ are spiral graphs, which is a dynamically straightforward family of parameters on these complex veins.

In Chapter 4 I will give a characterization of the topology of homeomorphic Hubbard trees in the Mandelbrot set. I will show the set of all parameters with Hubbard trees homeomorphic to an n-star is a union of $\phi(n)$ topological n-stars in the Mandelbrot set. Moreover, I will identify the Misiurewicz tip parameters that describe these veins in the Mandelbrot set.

Finally, in Chapter 5 I will describe a new characterization of the nontrivial contributions to pseudomonodromy. Given an oriented arc in the exterior of the Mandelbrot set with endpoints at the root of a hyperbolic component, we consider the automorphism it induces on space of kneading sequences. It is know by Ishii-Richards [8] that the non-trivial contributions to this automorphism induced by the monodromy at the root of a hyperbolic component H consist of all H' that are conspicuous to H. I will give a natural description of conspicuous components and the set of components they are conspicuous to, using internal addresses of hyperbolic components.

2

Background

2.1 Complex Dynamics of Polynomial Maps

We restrict our attention to polynomial maps with degree $d \geq 2$ on $\hat{\mathbb{C}}$.

2.1.1 Fatou and Julia Sets

Let f be a polynomial map. The *Fatou set*, $F(f)$, is the set of all points $z \in \hat{\mathbb{C}}$ such that there is a neighbourhood $U \ni z$ for which $\{f^n|_U\}_{n \geq 1}$ is a normal family. The *Julia set*, $J(f)$, is the complement of the Fatou set. Both $F(f)$ and $J(f)$ are f-invariant subsets of $\hat{\mathbb{C}}$.

A *Fatou component* is a connected component of the Fatou set. By Sullivan's No Wandering Domains Theorem, every Fatou component is eventually periodic. Moreover, by the Schwarz Lemma, every Fatou component is simply connected. Each Fatou component contains at most one point in the orbit of a periodic orbit.

The Julia set must be nonempty by the fact that the Fatou set cannot be the Riemann sphere. By a consequence of Montel's Theorem the Julia set must contain infinitely many points. The Julia set is contained in the closure of periodic points.

The *filled Julia set*, $K(f)$, is the union over the set of all bounded components of the Fatou set and the Julia set, and is also an f-invariant set.

2.1.2 Hubbard Trees

Suppose additionally f has the property that all critical points have finite orbit, i.e. f is a *post-critically finite* map. The *Hubbard tree* of f, T_f, is the smallest connected f-forward-invariant subset of $\hat{\mathbb{C}}$ that contains all critical points. Then T_f is a subset of $K(f)$ and is a topological tree with finitely many branch points and finitely many leaves, where the leaves are a subset of the critical orbits, and branch points are eventually periodic.

The *abstract Hubbard tree* of f, $T(f)$, is defined as the graph with vertex set $V = \mathcal{C} \cup \mathcal{B}$, with $\mathcal{C}$ the orbit of the critical points, and $\mathcal{B}$ the orbit of the branch points. Then f maps V to V with edges in between mapped homeomorphically to paths between vertices.

Associated to a Hubbard tree is its *Markov graph*, with vertex set given by the edge set of $T(f)$ and k arcs from e_i to e_j if $f(e_i)$ covers e_j with multiplicity k. The *Markov matrix*, A_f, is the incidence matrix of the Markov graph.

By Perron-Frobenius Theorem, A_f has a unique largest eigenvalue λ of algebraic multiplicity 1, such that $\log(\lambda)$ is the core entropy of f, i.e. the topological entropy of the restriction of f to its Hubbard tree.

2.1.3 Dynamical Rays

Let U_∞ be the immediate Fatou component of ∞. Then U_f can be uniformized to the set $D = \{z \in \hat{\mathbb{C}} | |z| > 1\}$ via a map $\phi : U_f \to D$, such that $\phi \circ f \circ \phi^{-1}$ is the map $z \mapsto z^d$. Then rays $R_D(\theta) := \{re^{i2\pi\theta} | r > 1\}$ in D pull back to arcs R_θ from ∞ under ϕ. We will call R_θ the *external ray* of angle θ.

A ray R_θ *lands* at a point z if $z = \lim_{r\to 1+} \phi^{-1}(re^{2\pi i\theta})$. If $\theta \in \mathbb{Q}$, then R_θ is eventually periodic. If θ is rational and f is postcritically finite, then R_θ lands at some point $z \in J(f)$, such that z has finite orbit [7].

2.1.4 Mandelbrot Set

A degree 2 polynomial is conjugate to a unique map of the form $z \mapsto z^2 + c$, $c \in \mathbb{C}$. The *Mandelbrot set*, $\mathcal{M}$, is the subset of the parameter plane $\mathbb{C}$ for which the unique critical point is bounded. It is known that $\mathcal{M}$ is connected [7], compact, and simply connected.

The exterior of the Mandelbrot set, $\mathbb{C} \setminus \mathcal{M}$, can be uniformized to the exterior of the disk $\mathbb{C} \setminus \overline{\mathbb{D}}$ via a map $\phi : \mathbb{C} \setminus \mathbb{C} \to D$ such that $\phi^{-1}(R_D(0))$ is the subset $(1/4, \infty)$ of the real line. A *parameter ray* of angle θ, $R_\mathcal{M}(\theta)$, is $\phi^{-1}(R_D(\theta))$. For the definition of *orbit portrait* see [7, 11].

A *hyperbolic component* $H \subseteq \mathcal{M}$ is a subset of $\mathcal{M}$ associated to an orbit portrait P_H for which every $c \in H$ has an attracting periodic orbit which has portrait P_H. The *main cardioid*, M_0 is the hyperbolic component where every $c \in M_0$ has an attracting fixed point. The *centre* of H is the unique parameter $c_H \in H$ with the critical orbit having the orbit portrait of P_H. The *root* of H is the unique parameter $r_H \in \partial H$ such that f_{r_H} has a parabolic critical orbit of portrait P_H.

If θ_1 and θ_2 are the angles of the external rays that bound the critical sector of P_H, then $R_\mathcal{M}(\theta_1), R_\mathcal{M}(\theta_2)$ land at r_H. For a hyperbolic component $H \neq M_0$, its *wake* W_H is the connected component of $\mathbb{C} \setminus (R_\mathcal{M}(\theta_1) \cup R_\mathcal{M}(\theta_2) \cup \{r_H\})$ that does not contain M_0.

A *Misiurewicz point* is a parameter $c \in \mathcal{M}$ such that the critical point is strictly preperiodic. Every Misiurewicz point is the landing point of a finite number of parameter rays. A *tip* is a Misiurewicz point that is the landing point exactly one one ray.

3

Sharkovsky Ordering on the Mandelbrot Set

3.1 Introduction

Orbit forcing in dynamical systems is the phenomenon where the existence of orbits of a certain combinatorial type forces the existence of orbits of another combinatorial type. The original problem of orbit forcing on continuous maps of the interval was solved by Sharkovsky in the 1960s. In his celebrated theorem, he defined a total ordering $<_2$ on the natural numbers such that any interval map $f : I \to I$ with a periodic point of period m also must have a periodic point of period n for all $n <_2 m$. That is, the set of periods $Per(f)$ of periodic orbits of f is a tail of this ordering. For a natural number $k \geq 2$, the k-Sharkovsky ordering is a partial ordering on $\mathbb{N}$ that describes the periods of periodic points of a continuous map on a tree $f : T \to T$. That is, if T is a tree with k leaves, then the set of periods $Per(f)$ of periodic orbits of f is a union of tails of i-Sharkovsky orderings, for $2 \leq i \leq k$.

The 2-Sharkovsky ordering manifests itself in the parameter space of real quadratic polynomials $z \mapsto z^2 + c$ where $c \in \mathbb{R}$. That is, for each $n \in \mathbb{N}$, let c_n be the largest parameter such that the critical point of c_n is of period n. Then $c_n < c_m$ iff $n >_2 m$. The goal of this paper is to generalize this ordering to the veins of the complex quadratic family.

A *vein* on the Mandelbrot set is an embedded arc from the parameter 0 in the main cardioid to a tip. We may assume that the vein intersects the boundary of every hyperbolic component in at most two points.

Suppose V is a vein of the Mandelbrot set $\mathcal{M}$. For hyperbolic components C_1, C_2 that intersect V, we say C_1 is *closer* to the main cardioid than C_2 along V if the connected component of $V \setminus C_2$ intersecting C_1 also intersects the main cardioid. For C_1, C_2 hyperbolic components on V, I will use the notation $C_1 \prec_V C_2$ if C_1 is closer to the main cardioid than C_2 along V. Since there are finitely

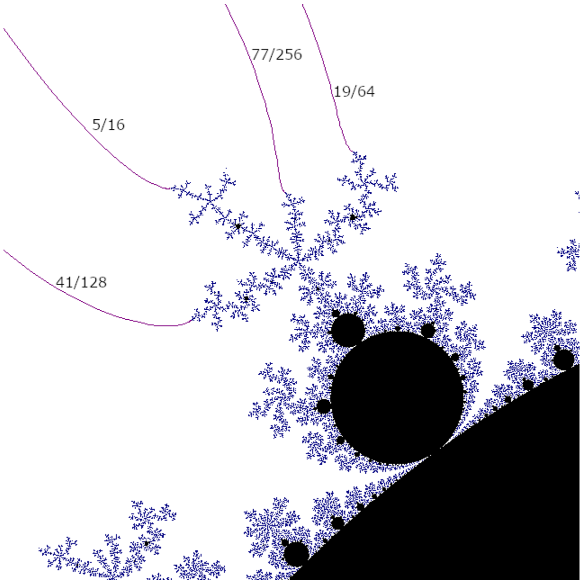

Figure 3.1: Principal and Secondary Veins in the 2/5-Limb.

many hyperbolic components of a given period, if there exist components of period n which intersect V nontrivially, there is a unique component of period n closest to the main cardioid along V, which we will denote by $C_V(n)$. My results describe the ordering $\succ_V$ and dynamics of the parameters in the hyperbolic components $C_V(n)$ along V.

Every quadratic map $f_c(z) = z^2 + c$ has two fixed points, α and β, where β is the landing point of the unique fixed external ray of angle 0. For natural numbers $k > p > 0$ where $\gcd(p, k) = 1$, the p/k limb of the Mandelbrot set is the set of parameters with combinatorial rotation number p/k at α [10].

The principal Misiurewicz point on this limb is the unique parameter $m_{p/k}$ that maps the critical point to the α fixed point with minimal preperiod $k-1$. Then $\mathcal{M}\setminus\{m_{p/k}\}$ is the union of k connected components. Denote as $\mathcal{M}_{p/k,0}$ the connected component that contains the main cardioid. For each of the other connected components, there is a parameter $c_l = c_{p/k,l}$ that maps the critical point to the β fixed point with minimal preperiod. Let c_1 be this parameter that maps to β with minimal preperiod across all of $\mathcal{M} \setminus \mathcal{M}_{p/k,0}$. Then c_1 has preperiod $k-1$, and let $\mathcal{M}_{p/k,1}$ be the component that contains c_1. For each of the $k-2$ other connected components, consider the parameter that maps the critical point to the β fixed point with minimal preperiod in that connected component. If the preperiod is $k+l-1$, then we will denote the parameter by c_l, with $l = 2, \ldots, k-1$, and call the connected component that contains it $\mathcal{M}_{p/k,l}$. A (k,l)-*vein* is a vein from the main cardioid to the c_l parameter on the p/k limb.

When $l = 1$, this is a principal vein. We will call the vein V a secondary vein when $2 \leq l \leq k-1$. For example, in the 2/5-limb, the principal vein is the vein from 0 to c_1 the landing point of the ray with external angle of 5/16 of preperiod 4. There are three secondary veins, V_l is the vein from 0 to c_l the landing point of the ray with external angle of θ_l, where $\theta_2 = 19/64$ of preperiod 6, $\theta_3 = 41/128$ of preperiod 7, and $\theta_4 = 77/256$ of preperiod 8.

Theorem 3.1 (Ordering of Hyperbolic Components along a Vein)**.** *Let V be a (k,l)-vein and let $\mathbb{N}(k,l) := \{n \ : \ n \geq k+l\} \cup \{k, 1\}$. Then:*

1. *$C_V(n)$ exists exactly for all $n \in \mathbb{N}(k,l)$.*
2. *If $l = 1$ and $n_1, n_2 \in \mathbb{N}(k,1)$ satisfy $n_1 >_k n_2$, then $C_V(n_1) \succ_V C_V(n_2)$.*
3. *If $l \neq 1$ and $n_1, n_2 \in \mathbb{N}(k,l)$ satisfy $n_1 >_k n_2$ and $n_2 \leq_k k+l$, then $C_V(n_1) \succ_V C_V(n_2)$.*

Theorem 3.2 (Dynamics of Hyperbolic Components along a Vein)**.** *Let V be a (k,l)-vein. Then:*

1. *If $i \geq 1$ then the dynamics of f_c for the centre c of $C_V(ik+l)$ is a spiral graph and $Per(f_c) = \{m \ : \ m \leq_k ik+l\}$.*
2. *For any $0 < l' < k$ with $l' \neq l$, the dynamics of f_c for the centre c of $C_V(ik+l')$ is not a spiral graph.*

When it is clear from context, I will use the notation $C(n) = C_V(n)$.

Theorem 3.3 (Explicit Description of the Vein Ordering)**.** *Let V be a (k,l) vein. We can obtain an explicit description of the vein ordering $\succ_V$ in some cases.*

1. *If $k = 2l$, then the ordering of $C_V(n)$ for $n \leq_k k+l$ is explicitly determined as follows:*

$$C(k+l) \succ C(2k+l) \ldots C(jk+l) \succ C((j+1)k+l) \ldots$$

$$C(3k) \succ C(5k) \succ C(7k) \cdots \succ C(4k) \succ C(2k) \succ C(k) \succ C(1)$$

where the above line is given by the k times the usual Sharkovsky ordering.

2. *If $k = 3l$ or $2k = 3l$, then the ordering of $C_V(n)$ for $n \leq_k k+l$ is explicitly determined as follows:*

$$C(k+l) \succ C(3k+2l) \succ C(4k+2l) \succ$$

$$C(2k+l) \succ C(5k+2l) \succ C(6k+2l) \succ$$

$$C(3k+l) \succ C(7k+2l) \succ C(8k+2l) \succ$$

$$\ldots$$

$$C(jk+l) \succ C((2j+1)k+2l) \succ C((2j+2)k+2l) \succ$$

$$C((j+1)k+l) \succ C((2(j+1)+1)k+2l) \succ C((2(j+1)+2)k+2l) \succ$$

$$\ldots$$

$$C(3k) \succ C(5k) \succ C(7k) \cdots \succ C(4k) \succ C(2k) \succ C(k) \succ C(1)$$

where the above line is given by k times the usual Sharkovsky ordering.

3.1.1 Background

The problem of orbit forcing on trees has been described by Baldwin in the 1980s [3]. In his setting, he considers a continuous map on a tree, and what can be said about the set of periods of periodic orbits of such a map. In his result, he describes the correspondence between maps on k_0-stars with a union of tails of k-Sharkovsky orderings, where $2 \leq k \leq k_0$.

In order to understand complex dynamics of quadratic maps using combinatorial descriptions of such dynamics on trees, Bruin and Schleicher have developed a correspondence between the dynamics on Hubbard trees and combinatorial objects such as internal addresses and kneading sequences [5].

This paper is a novel application of Baldwin's orbit forcing results to the complex dynamics setting. We will not use the internal address approach in this result.

3.2 Definitions and Notation

3.2.1 Complex Dynamics

Consider the family of quadratic maps $f_c(z) = z^2 + c$, where $c \in \mathbb{C}$. The Mandelbrot set $\mathcal{M} \subseteq \mathbb{C}$ consists of all parameters c such that the orbit of the critical point is bounded.

The external of the Mandelbrot set $\mathbb{C} \setminus \mathcal{M}$ can be uniformized to the external of the closed disk $\mathbb{C} \setminus \overline{\mathbb{D}}$ so that $\frac{\Phi(c)}{c} = 1$ as $c \to \infty$ by the unique conformal isomorphism $\Phi : \mathbb{C} \setminus \mathcal{M} \to \mathbb{C} \setminus \overline{\mathbb{D}}$. Then the pull back of rays on the external of the disk are the external rays of the Mandelbrot set, or parameter rays, and we denote as $R^\theta = \Phi^{-1}(\{re^{i\theta} | r > 1\})$.

We say that a parameter ray R^θ lands at a point c on the boundary of the Mandelbrot set if $\lim_{r\to 1^+} \Phi^{-1}(re^{i\theta}) = c$. Let $R^{\theta+}$ and $R^{\theta-}$ be external rays on the Mandelbrot set such that they land at a common point $c \in \partial\mathcal{M}$. Then $R^{\theta+} \cup R^{\theta-} \cup c$ divides $\mathbb{C}$ into two components.

Given a parameter $c \in \mathbb{C}$ we can define the filled Julia set K_c as the set of all points z in the dynamical plane such that the orbit of z under f_c remains bounded. For parameters $c \in \mathcal{M}$, K_c is connected and simply connected, so its external $\widehat{\mathbb{C}} \setminus K_c$ can be uniformized to the external of the closed disk $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and there is a unique conformal isomorphism $\Psi_c : \widehat{\mathbb{C}} \setminus K_c \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ such that the dynamics of f_c on the $\mathbb{C} \setminus K_c$ commutes with the $z \mapsto z^2$ on the external of the disk. Then the pull back of rays on the external of the disk are the external rays of the filled Julia set, or dynamical rays, and we denote as $R_c^\theta = \Psi^{-1}(\{re^{i\theta} | r > 1\})$.

For a parameter c, the Julia set $J_c = \partial K_c$ is the boundary of the filled Julia set. We say that a dynamical ray R_c^θ lands at a point z on the Julia set J_c if $\lim_{r\to 1^+} \Psi_c^{-1}(re^{i\theta}) = z$.

Orbit Portrait

An orbit portrait of a periodic orbit $\{z_1, \ldots, z_n\}$ for f_c, is a set of sets of angles $A_1, \ldots, A_n$, where A_i are all the angles $\theta_1, \ldots, \theta_k$ such that the dynamical rays $R_c^{\theta_1}, \ldots R_c^{\theta_k}$ land at z. The valence of a point z_i in the orbit is the number of dynamical rays landing at z_i, and since only points with valence ≥ 2 can be on the Hubbard tree H_c, we are only interested in orbit portraits with valence ≥ 2.

A hyperbolic component C of the Mandelbrot set is a connected component of the interior of the Mandelbrot set, such that every parameter in C has an attracting periodic cycle.

Suppose that the parameter rays $R^{\theta+}$ and $R^{\theta-}$ land at a root $r \neq \frac{1}{4}$ of a hyperbolic component C. Moreover, suppose that $R^{\theta+} \cup R^{\theta-} \cup r$ divides $\mathbb{C}$ into two connected open components, W, and $\mathbb{C} \setminus \overline{W}$ such that $C \supseteq W$ and $\mathbb{C} \setminus \widehat{W} \supseteq C(1)$. Then we say $W = W_C$ is the wake of the hyperbolic component C.

A parameter $c \in \mathbb{C}$ has a periodic orbit with orbit portrait P if and only if $c \in W \cup \{r\}$. If P and Q are distinct orbit portraits, then the wakes W_P and W_Q are either disjoint or strictly nested (ie, either $W_P \cap W_Q = \emptyset, \overline{W}_P \subseteq W_Q$, or $\overline{W}_Q \subseteq W_P$). If c_1 and c_2 are parameters for which V is a vein between them, then the wakes of roots of hyperbolic components along V are all nested.

Suppose f_c has an orbit with portrait P. Then there is hyperbolic component C with root r_P such that f_{r_P} has a parabolic orbit with portrait P, and there are parameter rays $R^{\theta+}, R^{\theta-}$ that land at r_P. Moreover these rays divide $\mathbb{C}$ into W_P and $\mathbb{C} \setminus W_P$, where $W_P \supseteq C$ is the wake of the orbit portrait P.

A hyperbolic component C is visible from C' if $V = [C, C']$ is a vein, $C \succ_V C'$, and all components $C'' \in (C, C')$ have the property that $per(C'') > per(C)$. At each hyperbolic component C with internal angle p/q, there is a tree $T_{p/q}$ consisting of all hyperbolic components visible from C in the p/q sub-wake of C. There are finitely many such hyperbolic components on each $T_{p/q}$,

Quasiconformal Surgery of Sectors

A vein is a continuous embedding of an arc inside $\mathcal{M}$. Denote the real vein $V_R = \mathcal{M} \cap \mathbb{R}$.

Let $k \geq 3$ and let $1 \leq p \leq k-1$ be relatively prime to k.

Let $c \in V_R$ be a parameter, and consider the dynamical plane of f_c. We will cut the dynamical plane along the rays at the α fixed point. The sector that contains the critical point will be denoted as the critical sector, $S_{c,R}$ and the other sector will be the non-critical sector $S_{n,R}$.

Consider the plane divided into k sectors labelled $S_0, S_1, \ldots S_{k-1}$ at angles $0, p/k, 2p/k, \ldots, (k-1)p/k$ respectively. Embed the critical sector in S_0 via the map $\phi_0 : S_{c,R} \to S_0$.

For each $1 \leq i \leq k-1$, embed the non-critical sector in S_i via the map $\phi_i : S_{n,R} \to S_i$.

We will duplicate the dynamics of the non-critical sector $k-2$ times, at the sectors $S_1, S_2, \ldots, S_{k-1}$.

The new map under the surgery $f_{\Phi_{p,k}(c)}$ will be defined by

$$f_{\Phi_{p,k}(c)} = \phi_{i+1} \circ f \circ \phi_i^{-1}$$

on the i-th sector for $i = 1, \ldots, k-1$ and we take $\phi_k = \phi_0$. For $i = 0$, define $f_{\Phi_{p,k}(c)}$ to be $\phi_0 \circ f \circ \phi_0^{-1}|_{f^{-1}(S_{c,R})}$ and $\phi_1 \circ f \circ \phi_0^{-1}|_{f^{-1}(S_{n,R})}$.

That is, $f_{\Phi_{p,k}(c)}$ duplicates the dynamics of the non critical sector $k-1$ times and maps $S_1 \to S_2 \to \cdots \to S_{k-1} \to S_0$.

Define $\Phi_{p,k}$ to me the map from V_R to its image, which is a principal p/k-vein. This is an embedding of V_R [4].

3.2.2 Sharkovsky Ordering

Interval Maps

An interval map is a continuous function $f : I \to I$ on a closed interval. The set of periods of f is $Per(f) := \{n | f \text{ has a periodic point of exact period } n\}$.

The *Sharkovsky ordering* $>_2$ is an ordering on $\mathbb{N}$ that describes orbit forcing and realization of interval maps. It is given by $3 >_2 5 >_2 7 >_2 9 >_2 11 >_2 \ldots 2 \times 3 >_2 2 \times 5 >_2 \ldots 2^n \times 3 >_2 2^n \times 5 >_2 \cdots >_2 2^n >_2 2^{n-1} >_2 \cdots >_2 2 >_2 1$.

A tail of the $>_2$ ordering is a set $S \subseteq \mathbb{N}$ such that whenever $s \in S$ and $t \in \mathbb{N} \setminus S$ then $t >_2 s$. Every interval map f has its set of periods $Per(f)$ given by a tail of $>_2$, and every tail of this ordering is realized as the set of periods of some interval map.

Suppose n is odd and $p_0 \mapsto p_1 \mapsto \cdots \mapsto p_n = p_0$ is a period n cycle of $f : I \to I$. Then $\{p_1, \ldots, p_n\}$ is a *Štefan cycle* if the points in the cycle occur in the following configuration along I or in the reverse orientation: $p_{n-1} < p_{n-3} < \cdots < p_4 < p_2 < p_0 = p_n < p_1 < p_3 < \cdots < p_{n-4} < p_{n-2}$. If f has a period n orbit, then it has a period n orbit that is a Štefan cycle.

Star Maps

For $k \geq 3$, a k-star T is a topological tree with k leaves joined to one vertex of degree k.

The k-Sharkovsky ordering is a partial ordering on $\mathbb{N}$ that describes the orbit forcing and realization of continuous maps on stars. The k-Sharkovsky ordering is a partial ordering defined by:

$\forall n, minS_k$

1. if $n > 1$ then $n >_k 1$,
2. if $n, m \equiv 0 \mod k$ and $n/k >_2 m/k$, then $n >_k m$,
3. if $n \not\equiv 0 \mod k$ and $m = in + jk$ for some $i \geq 0, j \geq 1$ then $n >_k m$.

A tail of the $>_k$ ordering is a set $S \subseteq \mathbb{N}$ such that whenever $s \in S$ and $t \in \mathbb{N} \setminus S$ then $t \not<_k s$. Every continuous map on a k-star has its set of periods given by a union of tails of $>_i$ for $i = 2, \ldots, k$, and every such union of tails is realized as the set of periods of a continuous map on a k-star.

Suppose T is a k-star with a continuous map $f : T \to T$ that fixes the central degree k vertex α of T, such that $p_0 \mapsto p_1 \mapsto \cdots \mapsto p_n = p_0$ is a period n cycle of f, where $n = ik + j$ for some $i \geq 1$ and $1 \leq j \leq n - 1$. Then $\{p_1, \ldots, p_n\}$ is a *spiral cycle* if the points in the cycle occur in an outward spiral on all k arms of T as evenly as possible in the following way: Denote the leaves of T as $L_1, \ldots, L_k$, with $L_i = [\alpha, v_i]$. Then the vertices $p_j \in L_i$ if $j \equiv i \mod k$. Moreover if $j \leq n - k$ then $p_j \in [\alpha, p_{j+k})$.

Suppose T is tree with vertices$\{p_1, \ldots, p_n\} \cup \{\alpha\}$. Then $f : T \to T$ is a *spiral graph* if f maps $\{p_1, \ldots, p_n\}$ in a spiral cycle, f fixes α, and f maps every edge $[v, w]$ of T homeomorphically to the path $[f(v), f(w)]$.

Markov Graphs

Suppose $T = (V, E)$ is a k-star with a fixed central vertex α and $f : T \to T$ maps vertices to vertices and maps edges to unions of edges homeomorphically. We can define the Markov graph of f (or

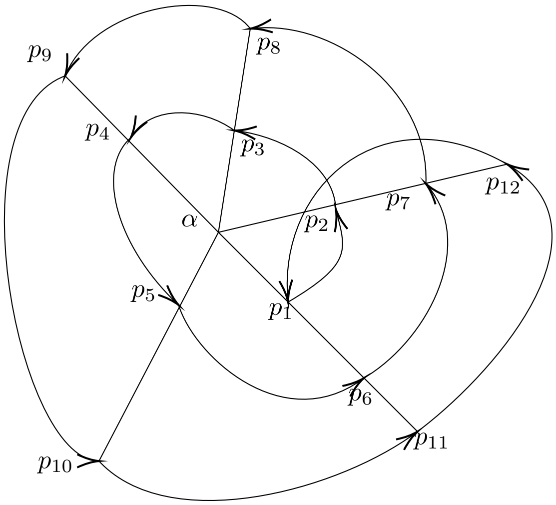

Figure 3.2: Schematic of a 5-star with a spiral cycle of period 12.

equivalently of T) as the directed graph $G_T = (E, A)$, where there is an arc $A_{i,j} = (E_i, E_j)$ from E_i to E_j if $f(E_i) \supseteq E_j$.

A nonrepetitive loop of length m is a loop in the Markov graph $L = E_{1,0}, E_{i_1}, E_{i,2}, \dots, E_{i,m} = E_{i,0}$ such that is not the union of l repeats of a smaller loop.

Suppose moreover that $f : T \to T$ maps p_i to p_{i+1} for all i, and fixes α is a spiral graph with k leaves and p vertices, where $p \equiv i \mod k$ for some $0 < i \leq p - 1$. Denote the spiral cycle that defines f to be $p_1, p_2, p_3, \dots, p_n = p_0$ where $f(p_i) = p_{i+1}$ with $p_1, \dots p_k$ the vertices closest to α on each branch.

Define the edge I_j to be the unique edge through vertex p_j that is closest to α. Then the Markov graph of f consists of the following p-cycle: $I_0 \to I_1 \to I_2 \to \cdots \to I_{p-1} \to I_p = I_0$ and arcs $I_p \to I_i, I_p \to I_{k+i}, \dots, I_p \to I_{p-k}$, $I_p \to I_1$, and $I_k \to I_1$.

Suppose $k \neq m \neq 1$. Then f has a point of period m if G_T has a nonrepetitive loop of length m that does not consist purely of edges adjacent to α. G_T has a loop of length m if f has a point of period m.

Suppose T is a k-star with vertices α of degree k and $p_0, \dots, p_{n-1}$ of degree at most 2. Moreover suppose $f : T \to T$ has a spiral cycle $p_0 \mapsto p_1 \mapsto \cdots \mapsto p_n = p_0$ of period $n > 1$, $n \not\equiv 0 \mod k$, and fixed point α of degree k, and f maps any edge between these vertices homeomorphically. Then $Per(f) = \{m | m \leq_k n\}$.

3.3 Results

First, we will show that the ordering of $C_V(n)$ along the real vein is the 2-Sharkovsky ordering. In this section, when referring to the dynamics of f_c where c is the centre of a hyperbolic component C, I will sometimes use C instead of f_c, such as in the context of $Per(C)$.

Proposition 3.4 (The Real Vein)**.** *Let* $V = V_R = [-2, 0]$.

1. $C_V(n)$ *exists for all* $n \in \mathbb{N}$.

2. *$C_V(n) \succ_V C_V(m)$ iff $n >_2 m$.*

3. *The set of periods of any $T_{C_V(n)}$ is $\{m|m \leq_2 n\}$.*

4. *If n is odd, then $C_V(n)$ is a Štefan cycle.*

Proof. 1. If $c \in V$ then the Hubbard tree of c is a line segment. Since the airplane c is a parameter of period 3 in V_R, T_c contains periodic orbits of all periods. Since periodic orbits in T_c correspond to orbit portraits of distinct wakes nested along V. For every n there is a unique parameter of period n that is the root of the wake of the orbit portrait of period n closest to the main cardioid along V. So $C_V(n)$ exists for every n.

2. By definition, $T_{C_V(n)}$ contains a periodic point of period n, so it contains a periodic point of period m for all $m \leq_2 n$. Corresponding to each of these periodic orbits of period m is its wake. Note that for the wake $W_{C_V(n)}$ of $C_V(n)$ is contained in the wake $W_{C_V(m)}$ of $C_V(m)$ iff $C_V(n) \succ_V C_V(m)$. Since these wakes are nested along V, let W_C be the wake of period m along V that contains all others, where C is the hyperbolic component corresponding to the root of W_C. Note that there is no other wake containing W_C that corresponds to a period m orbit, so $C = C_V(m)$. Since $m <_2 n$ implies $C_V(m) \prec_V C_V(n)$, and $<_2$ is a total ordering, the implication goes both ways.

3. Every periodic orbit of $T_{C_V(n)}$ corresponds to a wake of a hyperbolic component of that period that contains $C_V(n)$. For any $m >_2 n$, since $C_V(m) \succ_V C_V(n)$, all of the wakes of period m orbits are contained in $W_{C_V(n)}$, thus there are no wakes of period $m >_2 n$ that contain $C_V(n)$. So the set of periods of $T_{C_V(n)}$ is $\{m|m \leq_2 n\}$.

4. By a theorem of Burns and Hasselblatt [6], if f is an interval map with $Per(f) = \{m|m \leq_2 2i+1\}$ where $i \geq 0$ then f every orbit of period $2i+1$ is a Štefan cycle. Consider $T_{C_V(2i+1)}$, since $Per(T_{C_V(2i+1)}) = \{m|m \leq_2 2i+1\}$ its critical orbit is a Štefan cycle of this period.

□

Next, we will use a lemma to derive the statement for principal veins of p/k-limbs. We will use quasiconformal surgery of sectors to obtain an embedding of V_R into any p/k-principal vein.

Lemma 3.5 (From Real to Principal Vein). *Let V be a $(k,1)$-vein on the p/k-limb. Then $T(c)$ is a k-star for all $c \in V \setminus C_V(1)$. Moreover $C_V(ik+1)$ are spiral graphs for all $i \geq 0$.*

Proof. Consider the tip $c = -2$ of V_R. Its critical orbit is preperiodic to the β fixed point with preperiod 1, that is $\{0, c = -2, \beta = 2\}$. Then the α fixed point exists withing the edge $(-2, 0)$ of T_c. Let $\Phi_{p/k} : V_R \to V_{p/k,1}$ be the quasiconformal surgery of sectors that maps V_R onto the principal vein $V_{p/k,1}$ in the p/k-limb. Then $c_{p/k,1} = \Phi_{p/k}(c)$ is a parameter at the tip of $V_{p/k,1}$ such that the critical orbit of $c_{p/k,1}$ maps to its beta fixed point with preperiod $k-1$. This is the Misiurewicz point in the p/k-limb that maps to β with minimal preperiod [12].

Note that $T_{c_{p/k,1}}$ is a k-star, with critical orbit $\{c_0 = 0, c_1, \ldots, c_k = \beta\}$ such that $c_i \mapsto c_{i+1}$ for all $0 \leq i \leq k-1$ and $c_k \mapsto c_k$, so it is the unique parameter of minimal preperiod $k-1$ that maps to β.

For any $i \geq 1$, and any $c \in C_{V_R}(2i+1)$, T_c has a Štefan cycle as its critical orbit, say $\{c_0 = 0, c_1, c_2, \ldots, c_{2i+1}\}$ with $c_j \mapsto c_{j+1}$ for all $j \geq 0$. Then their relative position along the interval

$T_{C_{V_R}(2i+1)} = [c, f_c(c)]$ is given by $c_1 < c_{2i} < \cdots < c_6 < c_4 < c_3 < c_5 < \cdots < c_{2i-1} < c_0 = c_{2i+1} < c_2$. Then the α fixed point occurs between c_3 and c_4 (or in the case $i = 1$, between c_0 and c_1). If we denote $I_0 = [\alpha, c_2]$ the critical sector of the interval, and $I_1 = [c_1, \alpha]$ the non-critical sector, then there are i points of the critical orbit in I_1 and $i + 1$ in I_0.

The dynamics of $\Phi_{p/k}(c)$ can be obtained by duplicating the non-critical sector $k - 2$ with rotation number p/k about α. Denote by $I_2, \ldots, I_{k-1}$ copies of I_1, attached at α, and embedded into the plane at a combinatorial rotation number of p/k about α. For each $1 \leq j \leq k - 2$ let $\phi_{j,j+1} : I_j \to I_{j+1}$ be a homemorphjsm onto jts jmage, wjth $\phi_{j,j+1}(\alpha) = \alpha$. Then he dynamics of $f_{\phi p/k(c)}$ is given by $\phi_{j,j+1} : I_j \to I_{j+1}$ for $0 \leq j \leq k - 2$ and $f_c|_{I_1} \circ \phi_{1,2}^{-1} \circ \phi_{2,3}^{-}1 \circ \cdots \circ \phi_{k-2,k-1}^{-1} : I_{k-1} \to I_0$, $f_c|_{T_c} : I_0 \to I_0 \cup I_1$. Thus $T_{\Phi_{p/k}(c)}$ is a spiral graph with a critical orbit of period $jk + 1$.

□

Lemma 3.6 (From Principal to Secondary Vein)**.** *For each $1 \leq l \leq k - 2$ let V_l be a (k, l) vein on the p/k-limb. Then there is a homeomorphism $\Psi_l = \Psi_{p/k,l}$ of V_l to the $(k, l + 1)$-vein V_{l+1} of this limb, such that for all $i \geq 1$, $\Psi_l(C_{V_l}(ik + l)) = C_{V_{l+1}}(ik + l + 1)$ and this is a spiral graph.*

Proof. We will use the homeomorphism Ψ_l from V_l to V_{l+1} as described by Riedl in his thesis. If we remove $m_{p/k}$ the principal Misiurewicz point from $\mathcal{M}$, and remove the connected component of the main cardioid, we obtain $k - 1$ connected components. In each of these connected components there is a Misiurewicz point that maps to β with minimal preperiod. Then Ψ_l maps the vein V_l of the the l-th connected component to the vein V_{l+1} of the $l + 1$-th connected component and sends c_l to c_{l+1}, the Misiurewicz points of their respective sector of minimal preperiods prefixed to β.

I will briefly describe Ψ_l as a transformation of $c \in C_V(ik + l)$ to a parameter $\Psi_l(c)$ hybrid equivalent to $C_{V_{i+1}}(ik + l + 1)$. Let T_c be the Hubbard tree of f_c embedded in the plane, and suppose it is a k-star, with the critical orbit $c_0 = 0, c_1 = c, c_2, \ldots, c_{ik+l-1}$ and suppose it is a spiral cycle. Denote the edges of T_c as $I_1, I_2, \ldots, I_ik + l - 1$, where I_j is the edge closest to α incident to the vertex c_j. Consider the Markov graph, since $I_0 \to I_1 \to I_2 \to \cdots \to I_{k-1}$, and I_{k-1} maps to the union of edges which make up the segment $[c_1, c_k]$. Since c_1 and c_k are on different sectors of T_c, $\alpha \in [c_1, c_k]$, thus there is a point $\alpha_{-k} \in I_1$ such that $f^k(\alpha_{-k}) = \alpha$, where α_{-k} is a preperiod $k - 1$ point that maps to α. Let $\alpha_{-k}, \alpha_{-k+1}, \alpha_{-k+2}, \ldots \alpha_{-}1$ be the preperiodic points of α_{-k}.

Consider T_c embedded in the plane as a subset of K_c. Since I_{2k} maps to the union of edges I_{2k+1}, I_{k+1} (or if $i = 1$, reduce all indices mod $k+1$ so we have I_{k-1} maps to I_k, I_0), there is a point $c'_{k+1} \in I_{2k}$ that maps to c_{k+1}. Let $I' = [\alpha, c'_{k+1}]$, and let $J_{k-1+j} = f^{-j}(I')$ where we take the j-th preimage from α_{-j} for $1 \leq j \leq k$. Define $\overline{T_c} = T_c \bigcup \cup_{1 \leq j \leq k J_j}$, then each $\alpha_{-j} \in \overline{T_c}$ is of degree 3. Let $c' = f_c^{-k}(c'_{k+1})$, the newly created leaf in J_1.

Define $T'_c = \overline{T_c} \setminus \cup_{1 \leq j \leq k}(\alpha_{-j}, c_j]$. Define $f_{c'} : T_{c'} \to T_{c'}$ by $c_0 \mapsto c'$, $f_{c'}|J_j = f_c$ for $1 \leq j \leq k$. For all other vertices in $T_c \cap T_{c'}$, $f_c = f_{c'}$, and map the edges homeomorphically. This results in the following period $ik + l + 1$ critical orbit: $\{c' = c'_1, f_c(c') = c'_2, \ldots, f_c^{k+1}(c') = c_{k+1} = c'_{k+2}, \ldots, f_c^{k+l-1}(c') = c_{k+l-1} = c'_{k+l}\}$. Thus this $T_{c'}$ is a spiral graph.

□

Proposition 3.7. *Let V be a (k, l)-vein. Then $C_V(k + l)$ is a narrow component in the tree of visible hyperbolic components in the $1/2$-wake of $C_V(k)$.*

Proof. All hyperbolic components adjacent to the main cardioid are narrow. Since $C_V(k)$ is narrow, its tree of visible components in its $1/2$-wake consist of exactly k hyperbolic components of period

$k+1, k+2, \dots, 2k$, by Lau-Schleicher. Moreover the tree of visible components from $C_V(k)$ is a tree with $k-1$ branches, with $k-1$ leaves, and the component of period $2k$ occurs at internal angle $1/2$ from $C_V(k)$.

Let us denote V_l by the (k,l)-vein in the p/k-limb. Putting together all $k-1$ visible components of period $k+1, \dots, 2k-1$ along each of $V_1, \dots, V_{k-1}$, along with the component $C_{V_1}(2k)$ we obtain the tree of visible components. This tree consist of $k-1$ leaves, one for each of the $C_{V_l}(k+l)$, with the root component $C_{V_1}(2k)$.

Suppose it is not the case that $C_{V_l}(k+l)$ is narrow. Then $C_{V_l}(k+l)$ must contain in its wake another component of period less than $k+l$. Suppose m is the minimal period $< k+l$ for which there exist components of period m in the wake of $C_{V_l}(k+l)$. Let C' be a component of period m in the wake of $C_{V_l}(k+l)$ closest to $C_{V_l}(k+l)$ along the vein $[C', C_{V_l}(k+l)]$. Then C' must be visible from $C_{V_l}(k)$, so C' must be in the tree of visible components of $C_{V_l}(k)$, contradicting the fact that $C_{V_l}(k+l)$ is a leaf. $\square$

Corollary 3.8 (Visible Narrow Component)**.** *Suppose V is a (k,l)-vein. Then $C_V(k+l)$ is a narrow component whose Hubbard tree is a spiral graph. Moreover the set periods of hyperbolic components of V are exactly the set of $n \geq (k/m-1)(k+l), n \in m\mathbb{N}$, where $m = \gcd(k,l)$. Moreover, for all $n \equiv l \mod k$, $C'_V(n) = C_V(n)$ is minimal.*

Proof. This follows from the fact that since $C_V(k+l)$ is a spiral graph, the set of periods of $C_V(k+l)$ is exactly the set $\{n | n \leq_k k+l\}$, which is exactly the set of $\{n | n \geq (k/m-1)(k+l), n \in m\mathbb{N}\}$. $\square$

Proposition 3.9 (Orbit Forcing)**.** *If $n_1 >_k n_2$ and both $C_V(n_1), C_V(n_2)$ exist such that $n_2 \leq_k n+l$ then $C_V(n_1) \succ_V C_V(n_2)$.*

Proof. First suppose both $1 < n_1, n_2 \leq_k k+l$. Then $T_V(n_1)$ and $T_V(n_2)$ are k-stars. Label the vertex set of $T_V(n_1)$ as $\{p_1, \dots, p_{n_1}\} \cup \{\alpha\}$ with $p_i \mapsto p_{i+1}$ for all i the critical orbit, and $p_1, \dots, p_k$ the vertices closest to α on each of the k segments from α to a leaf of $T_V(n_1)$. Denote the edge I_i as the edge incident with p_i closest to α. Then the Markov graph of $T_V(n_1)$ contains the two cycles $I_1 \to \dots \to I_{k-1} \to I_1$ of length k, and $I_1 \to \dots \to I_{n_1} \to I_1$ of length n_1. So the Markov graph contains cycles of length $ik + j(n_1)$ for all $i \geq 1, j \geq 0$. So $Per(C_V(n_1)) \supseteq \{m | m \leq_k n_2\} \ni n_2$, so $C_V(n_1) \succ_V C_V(n_2)$.

Suppose $C_V(n_1) \not\leq_k k+l$ and $C_V(n_2) \leq_k k+l$. Then $C_V(n_l) \succ_V C_V(k+l)$ and $C_V(k+l) \succeq_V C_V(n_2)$. $\square$

We now prove Theorem 3.1.

Proof.
1. First note that $C_V(n)$ cannot exist for $1 < n < k$ because V is on a p/k limb. Additionally, $C_V(n)$ cannot exist for $k < n < n+l$, since $C_V(k+l)$ is a narrow component so $C_V(n) \not\succ_V C_V(k+l)$, and $C_V(k+l)$ is a spiral graph so $C_V(k+l) \not\succ_V C_V(n)$.

   First consider $l = 1$, and label the critical orbit of f_{c_l} by $\{c_0, c_1, c_2, \dots, c_{k-1}\}$ where $c_0 = 0$, and $c_i \mapsto c_{i+1}$ for $i \leq k-1$, and $c_k \mapsto c_k$. Then for c_l, its Hubbard tree is a k-star with vertex set $\{c_0, c_1, \dots, c_k\} \cup \{\alpha\}$, where $c_1, \dots, c_k$ are leaves, α is of degree k, $c_0 \in (\alpha, c_k)$. Denote $I_1, \dots, I_k$ the edges incident to $c_1, \dots, c_k$ respectively, I_0 the edge $[\alpha, 0]$. The Markov graph consists of the following cycle of length $k+1$ $I_k \mapsto I_0 \mapsto I_1 \mapsto \dots \mapsto I_{k-1} \mapsto I_k$, and the

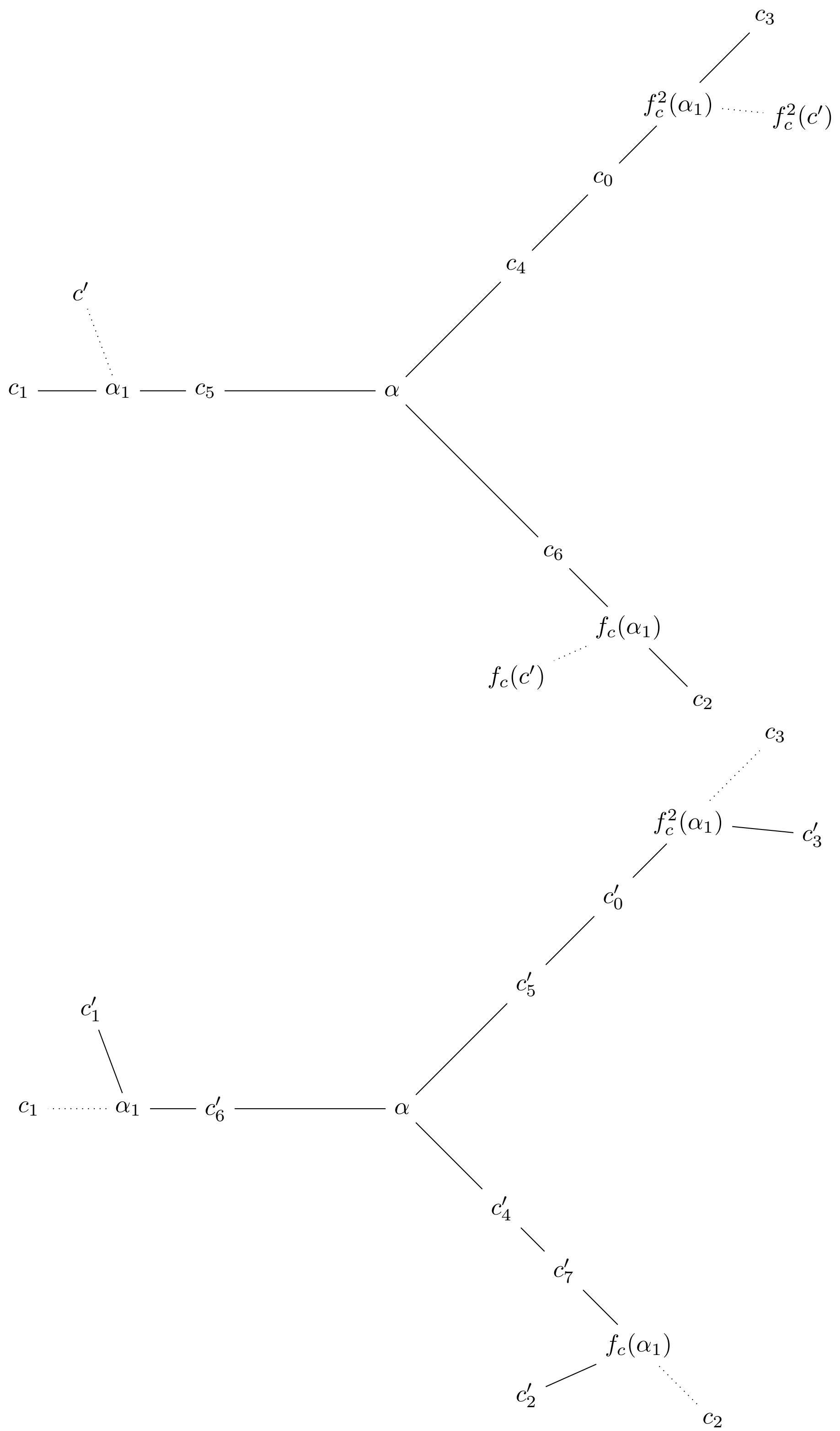

Figure 3.3: Schematic diagram of T_c and $T_{c'}$. Top: T_c in solid lines, with dotted lines indicating the edges $J_1, \ldots, J_k$. Bottom: T'_c in solid lines, with dotted lines indicating pruned edges of T_c.

additional arcs $I_{k-1} \mapsto I_0$, $I_k \mapsto I_k$, and $I_k \mapsto I_1$. Then by repeatedly concatenating the loop $I_k \mapsto I_k$ to the cycle of length $k+1$, we can obtain non-elementary cycles of any length greater than k. The fact that the only periods less than $k+1$ are k and 1 is a consequence of the fact that $C_V(k+1)$ is a narrow component visible from $C_V(k)$.

We now suppose $l > 1$. Consider c_l the tip of the (k,l)-vein. We can label critical orbit by $\{c_0, c_1, c_2, \ldots, c_{k+l-1}\}$ where $c_0 = 0$, and $c_i \mapsto c_{i+1}$ for $i \leq k-1$, and $c_{k+l-1} \mapsto c_{k+l-1}$. Then $c_{k+l-1} \mapsto c_{k+l}$. Since c_l is a tip, each of the points $\{c_1, c_2, \ldots, c_{k+l-1}\}$ is a leaf in the Hubbard tree T_{c_l} of f_{c_l}. Consider T_{c_l} as an abstract tree, with vertex set given by the critical orbit and all points of degree > 2 in T_{c_l}, as well as α_1, the other preimage of α. The vertex set of T_{c_l} consists of $\{c_0, \ldots, c_{k+l-1}\} \cup B$, where B is a finite set consisting of the branch points of T_{c_l} and are preimages of α. In particular α_{-1}, the preimage of α, is in the vertex set, and is located on the sector of T_{c_l} containing 0.

The edge set of T_{c_l} consists of edges between vertices in T_{c_l}. Denote the edges incident to each of the vertices of $c_1, \ldots, c_{k+l}$ by $I_1, \ldots, I_{k+l}$ respectively, and denote the edge I_0 by the edge incident to c_0 and α.

I claim that the other endpoint of the edge I_{k+l} is α_{-1}. Suppose there is some $\alpha_i \in (0, \beta)$ with degree > 2 for T_{c_l}, such that α_i does not map to α. Since α has rotation number p/k the sectors around α are permuted via this rotation. So $\alpha_i \mapsto \alpha_{i+1} \mapsto \ldots, \mapsto \alpha_{i+k-1}$, with each α_{i+j} in a distinct sector. Consider a local tripod T centered at α_i, T is mapped homeomorphically under f_{c_l}, so each of $\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+k-1}$ consist of at two children, and thus the sector containing each of these points consists of at least two leaves for a total of at least $2k$ leaves. This contradicts the fact that the critical orbit has preperiod $k+l-1$ for some $l \leq k-1$.

The Markov graph of T_{c_l} consists of the path $I_0 \to I_1 \to I_2 \to I_3 \to \cdots \to I_{k+l-2} \to I_{k+l-1}$. Since $\alpha_{-1} \mapsto \alpha$, $I_{k+l} \to I_0$ and $I_{k+l} \to I_{k+l}$ are arcs in the Markov graph. Thus $I_0 \to I_1 \to I_2 \to I_3 \to \cdots \to I_{k+l-1} \to I_{k+l} \to I_0$ is a $k+l+1$-cycle in the Markov graph. By concatenating with arbitrary repeats of the arc $I_{k+l} \to I_{k+l}$, we can obtain cycles of length $k+l+i$ for all $i \geq 1$ in the Markov graph, hence we have periodic orbits of periods of all periods $\geq k+l+1$.

2. By Proposition 3.4 and Lemma 3.5, $T_{C_V(n_1)}$ is a k-star. If $p_1, \ldots, p_{n_1}$ are the critical orbit with $p_1, \ldots, p_k$ the vertices of the critical orbit closest to α along each of the k arms of $T_{C_V(n_1)}$, and $p_i \mapsto p_{i+1}$ for all i, then vertex set of $T_{C_V(n_1)}$ is $\{p_1, \ldots, p_{n_1}\} \cup \{\alpha\}$. If $I_1, \ldots, I_{n_1}$ are the edges incident with $p_1, \ldots, p_{n_1}$ closest to α where $I_i \to I_{i+1}$ for all i, then the Markov graph contains the k-cycle $I_1 \to \cdots \to I_k \to I_1$ and the n_1-cycle $I_1 \to \cdots \to I_{n_1} \to I_1$, so there exist periodic orbits of all $m \leq_k n_1$.

3. The result follows from Lemma 3.6 and the same argument as the previous part.

□

We now prove Theorem 3.2.

Proof. 1. The result follows from Proposition 3.4, Lemma 3.5, and Lemma 3.6.

2. If for some other $l' \neq l$ and $i \geq 1$ such that $C_{V_l}(ik+l')$ is also a spiral graph, then its set of periods is $Per(C_{V_l})(ik+l') = \{m | m \leq_k ik+l'\} = \{sk + t(ik+l) | s \geq 1, t \geq 0\} \cup \{1\}$. In particular, $ik+l \notin Per(C_{V_l}(ik+l'))$, so this means $W_{C_{V_l}(ik+l')} \not\subseteq W_{C_{V_l}(ik+l)}$. But $Per(C_{V_l}(ik+l)) = \{sk+t(ik+l) | s \geq 1, t \geq 0\} \cup \{1\}$, and $ik+l' \notin Per(C_{V_l}(ik+l))$, so $W_{C_{V_l}(ik+l)} \not\subseteq W_{C_{V_l}(ik+l')}$. Thus the two wakes are disjoint, but this contradicts the fact that any two wakes along the vein must be nested.

□

We now prove Theorem 3.1.

Proof. 1. This is because for $k = 2l$ elements of $Per(C_V(k+l))$ form a total ordering of $\leq_k$.

2. Since $Per(C_V(ik+l))$ are all spiral graphs, $Per(C_V(jk+l)) \setminus Per(C_V((j+1)k+l)) = \{jk+l, (2j+1)k+l, (2j+2)k+l\}$, so $C_V(jk+l) \succ_V \{C_V((2j+1)k+l), C_V((2j+2)k+l)\} \succ_V C_V((2j+2)k+l)$. Since $(2j+1)k+l >_k (2j+2)k+l$ we must have $C_V((2j+1)k+l) \succ_V C_V((2j+2)k+l)$.

□

4

Isomorphic Hubbard Trees in the Mandelbrot Set

4.1 Introduction

A *graph* $G = (V, E)$ is a 2-tuple of a finite set of vertices V and edges E, such that E consists of 2 element subsets of V. Given $G = (V, E)$ and $G' = (V', E')$, a *graph isomorphism* between G and G' is a map $f : G \to G'$ such that f is a bijection between V and V' that induces a bijection between E and E'. That is, for all $\{u, v\} \in E$, $f(\{u, v\}) = \{f(u), f(v)\} \in E'$ is a bijection from E to E'. The *valence* of a vertex v is the number of edges $e \in E$ such that $v \in e$.

Given an embedded graph X in the plane, we can define $G(X)$ to be the *abstract graph* of X with vertex set given by the set of all points $c \in X$ such that $X \setminus \{c\}$ does not consist of exactly 2 connected components. The edge set of $G(X)$ is the set of all paths $[u, v] \in X$ between vertices $u \neq v$ such that there is no other vertex in the interior of this path.

Let $c \in \mathcal{M}$, and its Hubbard tree T_c is a subset of the dynamical plane. The *abstract Hubbard tree* $T(c)$, is the abstract graph of the Hubbard tree T_c. That is, $T(c) = (V_c, E_c)$ where V_c consists of the set of all points $v \in T_c$ such that the number of connected components of $T_c \setminus \{v\}$ is not equal to 2, and $\{u, v\} \in E_c$ if and only if the path between u, v in T_c does not contain any other vertices in V_c. That is, the topological Hubbard tree is the abstract tree of T_c with vertices consisting of all points of degree not equal to 2.

One could ask the following question.

Question[1]: Given a topological tree T, what is the set of parameters $c \in \mathcal{M}$, such that $T(c)$ is isomorphic to T?

In this chapter, I will describe an answer to the simplest case of this question.

The simplest family of graphs to consider in answering this question is the star graphs. If

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[1] Thanks to Caroline Davis for asking this question.

$n \geq 3$, an *n-star* is a graph isomorphic to the graph with vertex set $\{0, 1, \ldots, n\}$, and edge set $\{\{0, i\} | 1 \leq i \leq n\}$.

Given a topological tree T, let $\mathcal{M}(T)$ be the set of parameters $c \in \mathcal{M}$ such that T_c is isomorphic to T. An *n-vein* in $\mathcal{M}$ is a union of segments of veins in $\mathcal{M}$ such that as an abstract topological tree, it is an n-star. For example, in the 1/5-limb, the union of the principal and the i-th secondary veins for $i = 2, 3, 4$ is a 5-vein.

Theorem 4.1. *Let $n \geq 3$. Suppose T is an n-star. Then the set of postcritically finite parameters of $\mathcal{M}(T)$ is the union of $\phi(n)$ disjoint n-veins. Moreover the set of all such parameters of $\mathcal{M}(T)$ are either found on these n-veins, or on hyperbolic components passing through the centres of these veins.*

Note that, for instance, the 5-vein satisfying the result of Theorem 4.1 is not the union of the principal and secondary veins discussed in the example above. In general I have identified a family of n-veins for which the topology is constant that is different from the union of the principal and secondary veins.

Additionally, I will show that if T is a topological tree that is homeomorphic to a Hubbard tree of f_c for some $c \in \mathcal{M}$, then we can always find a tip parameter in a wake or sector of c that has a Hubbard tree homeomorphic to T.

Theorem 4.2. *Suppose $c \in \mathcal{M}$ is a postcritically finite parameter. Then there is some $c' \in \mathcal{M}$ a Misiurewicz tip with vein V from c' to the main cardioid, with $c \in V$ and $c' \succ_V c$, such that $T(c)$ is isomorphic to $T(c')$. Moreover, there are at most finitely many tip parameters c' for which which is true.*

4.2 Results

4.2.1 Star-Shaped Subsets of Star-Shaped Hubbard Trees

We will first show that for each m/n-limb, there is an n-vein $V_{m/n}$ consisting of postcritically finite parameters c with $T(c)$ isomorphic to an n-star.

Suppose V is a vein from the main cardioid to a Misiurewicz tip. Let c_n be a sequence of critically periodic parameters in $\mathcal{M}$ and suppose $c_n, c \in V$ for all n, and $c_n \succ_V c_m$ whenever $n > m$. Suppose the critical angles of c_n, given by θ_n^-, θ_n^+ satisfy that $\lim_{n\to\infty} \theta_n^- = \theta^-$ and $\lim_{n\to\infty} \theta_n^+ = \theta^+$. If θ_n^-, θ_n^+ are external rays landing at a postcritically finite parameter c, then we say that c_n *converges* to c.

Lemma 4.3. *Let c be a Misiurewicz point, and suppose c_m is a sequence of parameters in $\mathcal{M}$ such that $c \succ c_m$ for all m and $\lim_{m\to\infty} c_m = c$ in terms of their external angles. Suppose that $T(c_m)$ is isomorphic to T for all large enough m.*

Then $T(c)$ is isomorphic to T.

Proof. Suppose c has preperiod k and period i. If c is a branch Misiurewicz point let θ^-, θ^+ be the external rays landing at c that bound the component contain the main cardioid, and denote $\theta = \theta^-$. If c is a tip Misiurewicz point, let θ be its external ray.

We may assume c_m are increasine, ie if $m > m'$ then $c_m \succ c'_m$. Suppose c_m has period n_m. For each c_m, consider the portrait of c_m in $T(c)$, given by $\{c_m^1, c_m^2, \dots, c_m^{n_m}\}$. Suppose the characteristic rays of c_m are θ_m^-, θ_m^+, and denote $\theta_m = \theta_m^-$.

Denote the ray R_m^j as the external ray of angle $2^j\theta_m$ in the dynamical plane of c, and let z_m^j its landing point if $J(f_c)$. Denote the ray R^j as the external ray of angle $2^j\theta$ in the dynamical plane of c. I will show that for each $j = 1, \dots, k+i$, $\lim_m z_m^j = z^j$.

Since R_m^j is a ray of the periodic orbit of c_m, it lands at a point in T_c. Moreover, for any ϵ we can find m such that $|\theta_m - \theta| < 2^{-(k+i)}\epsilon$. So for large m, (R_m^j, R^j) are not separated by 0 for all $j = 1, \dots k+i$ and $[z_m^j, z^j]$ maps homeomorphically to $[z_m^{j+1}, z^{j+1}]$ as arcs of T_c. Moreover $[z_{m'}^j, z^j] \subseteq [z_m^j, z^j]$ for $m' > m$. Then $\lim_{m\to\infty} z_m^j$ exist.

Denote $\lim_{m\to\infty} z_m^j = z_\infty^j$ and suppose z_∞^j is not equal to z^j. Since $J_c = K_c$, $[z_\infty^j, z^j] \subseteqq J_c$. Since periodic points are dense in J_c, we can find periodic points $a \neq b$ in $[z_\infty^j, z^j]$. Since repelling periodic points on J_c are the landing points of external rays, let R^a, R^b be two external rays at a, b between $(R_{z_m}^j, R^j)$. Then we have $R_m^j < R^a < R^b < R^j)$ for every m so $\lim_{m\to\infty} R_m^j \neq R^j$, a contradiction.

Thus for each $1 \leq j \leq k+i$, z_m^j are points in the compact set T_c such that $\lim_{m\to\infty} z_m^j = z^j$. Thus $\mathit{conv.hull}(\{z_m^j\}_{1\leq j\leq k+i}) \subseteq \mathit{conv.hull}(\{z_m^j\}_{1\leq j\leq k+i}) \subseteq T_c$, and $\lim_m \mathit{conv.hull}(\{z_m^j\}_{1\leq j\leq k+i}) = \mathit{conv.hull}(\lim_m\{z_m^j\}_{1\leq j\leq k+1}) = T_c$.

□

Proposition 4.4. *Suppose c is the principal Misiurewicz point on the m/n-limb. Then for each $i = 1, \dots, n-1$ there is a tip parameter c_i in the i-th sector at c such that $T(c_i)$ is isomorphic to $T(c)$.*

Proof. We will assume c is in a $1/n$-limb as abstract Hubbard trees of parameters on conjugate p/n-limbs are isomorphic. By my previous result, this is true up to parameter $C_{V_i}(n+i)$ for all i.

The wake of $C_{V_i}(n+i)$ is defined by external rays of angle θ^- and θ^+ where $\theta^- = \overline{0\dots010\dots01}$ where the first string of 0's is of length $n-1$ and the second is of length $i-1$. Since this is a narrow hyperbolic component, $\theta^+ = \overline{0\dots010\dots010}$ where the first string of 0's is of length $n-1$ and the second is of length $i-2$.

Then there is a parameter c_i of preperiod $n-i$ and period i that is the landing point of the ray of angle $\theta = 0\dots0\overline{0\dots01}$, where $\theta \in (\theta^-, \theta^+)$.

Suppose c_i has another ray θ' that lands at it, then it must have the same preperiod and period. But for any such $\theta' \in (\theta^-, \theta^+)$, it must have the first n digits given by $0\dots01$, so $\theta' = \theta$. Thus c_i is a tip Misiurewicz point.

The kneading sequence of θ is given by $\overline{1\dots101\dots1\star}$ where the first sequence of 1's is of length $n-1$ and the second is of length $i-1$.

Denote $\theta_1 = \theta$, and for each m, define $\theta_m = \overline{(0\dots01)(0\dots01)\dots(0\dots01)}$, where it is the periodic repeat of the length $n+mi$ sequence given by concatenating the length i sequence $0\dots01$ to the end of the length $n+(m-1)i$ sequence of θ_{m-1}.

Using the kneading sequence of θ, the internal address of c_i given by $1 \to n \to n+i \to n+2i \to n+3i \to \dots$ [5]. By induction, the Hubbard tree corresponding to the internal address $1 \to n \to n+i \to \dots \to n+ki$ is topologically an n-star. Let z_m be the point in T_{c_i} with internal address $1 \to n \to n+i \to \dots \to n+mi$.

Let T_{z_m} be the subset of T_c given by the convex hull of the orbit of z_m. Then the abstract Hubbard tree of T_{z_m} is isomorphic to an n-star for all m, so by Lemma 4.3 $T(c_i)$ is also isomorphic to an n-star.

□

We will argue that every postcritically finite parameter c that is not on an m/n-limb cannot possibly have a Hubbard tree isomorphic to an n-star. First note that every postcritically finite parameter c in any m'/n'-limb for $n \neq n'$, $n' > 2$ has a branch point of valence exactly n' at the α fixed point. Then we will show that every postcritically finite parameter in the $1/2$-limb that has a point of valence n must have at least two vertices of valence n.

Lemma 4.5. *Let c be a postcritically finite parameter on the $1/2$-limb. Suppose $T(c)$ has a point p of valence n. Then $T(c)$ has at least two points of valence $k > 2$.*

Proof. Consider the α fixed point of T_c. Then α has valence 2 on T_c. Let T_c^1 be the component of $T_c \setminus \{\alpha\}$ containing c and T_c^0 be the component containing 0. Then $f_c(T_c^1) \subseteq T_c^0$ homoeomorphically, so if $p \in T_c^1$ then there is a point of valence n in T_c^0.

Suppose $p \in T_c^0$. Note that $p \neq 0$ since c has valence 1 this would imply f_c is a n to 1 mapping at 0. Since $p \neq 0$ there is a small neighbourhood $N_c(p)$ of p that is mapped homeomorphically under f_c. Since $p \neq \alpha$ and $p \neq \beta$, $f_c(p) \neq p$, so $f_c(p)$ is a point also has valence at least n. □

We now show that every hyperbolic component that is on the m/n-limb that is not on the candidate n-vein $V_{m/n}$ cannot be isomorphic to T_c. First, note that for any postcritically finite parameter c not on $V_{m/n}$, there is a first point of bifurcation, c'. That is, if V is the vein from the main cardioid to c, then $V_{m/n} \cap V$ is the vein from the main cardioid to some post critically finite parameter c'. Moreover this point of bifurcation c' must occur at either the centre of a hyperbolic component or a branch Misiurewicz point.

If c' is the centre of a hyperbolic component, then V must bifurcate from $V_{m/n}$ at a satellite of the hyperbolic component C'' of the hyperbolic component of c' with internal angle $m'/n' \neq 1/2$. This implies that T_c contains points with orbit portraits given by c'' the centre of C'' with $T_{c''}$ not isomorphic to a sub tree of an n-star.

If c' is a branch Misiurewicz point, it cannot be the principal Misiurewicz point. We will show that $V_{m/n} \setminus \{c'\}$ must be contained in two sectors at c', U_0, U_1, and c is contained in another sector U_j with $j \neq 0, 1$. We will show the Hubbard tree of $T_{c'}$ must contain additional branch points.

Lemma 4.6. *Let C be any critically periodic parameter and C' be any postcritically finite parameter. Then $T(C \rhd C')$ is isomorphic to the abstract tree of T_C after replacing a finite set of intervals of with copies of $T_{C'}$.*

Proof. The argument is exactly given by the procedure of tuning. Briefly, let $c_1, \ldots, c_n$ be the attracting periodic orbit of f_C, and let $U_i, \dot{;}U_n$ be their Fatou components. Then for each i, let $v_{i,0}$ and $v_{i,1/2}$ be the points on the boundary of U_i at internal angle 0 and $1/2$ respectively. Identify the internal angles of U_0 by the external angles of $K(f_{C'})$, and identify all bounded Fatou components of $K(f_C)$ in this way via pullbacks to obtain $K(f_{C \rhd C'})$.

The dynamics of the critical orbit of $C \rhd C'$ is given by $\{c_{i,j}\}_{0 \leq i \leq n-1, 0 \leq j}$ where $c_{i,j}$ is the copy of $f_{C'}^j(0)$ in the copy of $K(f_{C'})$ in U_i. We will use an abuse of notation and call the i-th copy of $K(f_{C'})$ in the former Fatou component U_i by U_i. The dynamics of $f_{C'}$ is such that $f_{C'}$ maps the copy of $K(f_{C'})$ in U_i to the copy in U_{i+1} such that $f_{C'}(c_{i,j}) = (c_{i+1,j})$ for $0 \leq i \leq n-2$ and $f_{C'}(c_{n-1,j}) = c_{0,j+1}$. Then $f^n_{C'}$ is the first return map of U_i to U_i, such that $f^n_{C'}(c_{i,j}) = c_{i,j+1}$.

Then the abstract Hubbard tree of the first return map $f^n|_{U_i}$ is isomorphic to $T(C')$. If $H = C \rhd C'$, then we can replace a neighbourhood of each point in the critical orbit of T_C with a small copy of $T_{C'}$ in the following way. For each c_i in the critical orbit of T_C, where c_i is in the centre of its Fatou component U_i. Let $v_{i,0}$ and $v_{i,1/2}$ be points on the boundary of U_i at internal angle 0 and 1/2 respectively, and both are found in T_C unless c_i is a leaf, in which case one of these points is found.

In order to create $T_{C \rhd C'}$, first define T'_C by taking T_C and extending the regulated arcs along the internal rays of angle 0 and 1/2 in U_i. Next, define the extended Hubbard tree $T'_{C'}$ by extending the arcs at the endpoints of $T_{C(1/2)}$ to the β fixed point and $-\beta$ via a union of regulated arcs. Then $T'_{C \rhd C'}$ is obtained by replacing the edge $\{v_{i,0}, v_{i,1/2}\}$ with the graph $T'_{C'}$ by identifying $v_{i,0}$ with β and $v_{i,1/2}$ with $-\beta$. The abstract graph $T(C \rhd C'$ is isomorphic to the abstract graph of $T'_{C \rhd C'}$.

□

Lemma 4.7. *Suppose C is a critically periodic parameter that is not* 0*, and suppose H is a postcritically finite parameter in the real vein of the small copy of the Mandelbrot set at C. Then $T(C)$ and $T(H)$ are isomorphic.*

Proof. H is the tuning $C \rhd C'$, where C' is some parameter on the real vein of the small copy of the Mandelbrot set at C.

The graphs $T(C)$ and $T(C \rhd C')$ are isomorphic because the tuning replaced the intervals $\{v_{i,0}, v_{i,1/2}\}$ with intervals. □

Lemma 4.8. *Suppose C is any critically periodic parameter, and C' is a postcritically finite parameter in the m/n limb for some $m/n \neq 1/2$. Suppose H is the tuning of $C \rhd C(m/n)$. Then $T(H)$ has more vertices with degree > 2 than $T(C)$.*

Proof. $T(C)$ is a tree with finitely many vertices and edges.

Since C' is in the m/n-limb, the α fixed point of $T_{C(m/n)}$ has rotation m/n with $n \geq 3$. Thus $T(C')$ is a tree with a vertex of degree ≥ 3.

$T(C \rhd C')$ is obtained by replacing a local neighbourhood at the attracting periodic orbit of T_C by $T_{C'}$. Thus $T(C \rhd C')$ has more vertices of degree > 2 than $T(C)$, so they cannot be isomorphic. □

Proposition 4.9. *Suppose c is a Misiurewicz branch point along the n-vein $V_{m/n}$ in the m/n-limb that is not the principal Misiureiwcz point and is not a tip. Suppose $U_0, U_1, \ldots U_{k-1}$ are the sectors at c such that U_0 and U_1 are the two sectors that contain $V_{m/n} \setminus \{c\}$ with U_0 containing the main cardioid and U_1 containing a tip of $V_{m/n}$. Suppose $j \neq 0, 1$ with $c'' \in U_j$, then $T(c'')$ is not isomorphic to $T(c)$.*

Proof. Suppose U_ϵ is a small neighbourhood of c in $\mathcal{M}$, such that $U_\epsilon \setminus \{c\}$ consists of k connected components, where U_0 and U_1 are the sectors containing the principal vein, with U_0 the sector whose external rays bound the main cardioid. It suffices to show that if $c'' \in U_j$ for any other j-th sector, then $T(c'')$ is not isomorphic to $T(c)$.

Consider the orbit portrait of c. If the parameter rays at c rays are not preimages of the parameter rays of the principal Misiureiwcz point of the m/n-limb under the doubling map, denote the portrait of c by $\{\{\phi_{1,1}, \phi_{1,2}, \dots, \phi_{1,k}\}\{\phi_{2,1}, \dots, \phi_{2,k}\}, \dots, \{\phi_{l,1}, \dots, \phi_{l,k}\}\}$, where $\phi_{1,1}, \dots, \phi_{1,k}$ are the angles of external rays landing at c. Otherwise, if the angles of the rays landing at c are a preimage of the angles of the rays landing at the principal Misiurewicz point, denote the portrait of c by $\{\{\phi_{1,1}, \phi_{1,2}, \dots, \phi_{1,n}\}\{\phi_{2,1}, \dots, \phi_{2,n}\}, \dots, \{\phi_{l+1,1}, \dots, \phi_{l+1,n}\}\}$, where $\phi_{1,1}, \dots, \phi_{1,k}$ are angles of external rays landing at c, and $\phi_{l+1,1}, \dots, \phi_{l+1,n}$ are angles of external rays at the α fixed point.

Moreover, suppose the parameter rays with angles $\phi_{1,1}$ and $\phi_{1,2}$ bound U_1 in the parameter plane, and the parameter rays with angles $\phi_{1,j}$ and $\phi_{1,j+1}$ bound U_j in the parameter plane.

Let c' be a postcritically finite parameter on the principal vein of the m/n-limb such that $c' \succ_V c$. That is, $c' \in U_1$. By Lemma 4.10 or Lemma 4.11, $f^i_{c'}(N_{c'}(c) \cap T_{c'})$ is contained in at most two sectors of the external rays landing at $f^i_{c'}(c) \in J(f_{c'})$ for each $1 \leq i \leq l$.

Let c'' be a postcritically finite point in U_i for $i \neq 0, 1$. Then c'' has rays landing in $S_j(c'')$. Now consider the sectors $S_0(c), S_1(c), \dots, S_k(c)$ in the dynamical plane of f_c, with $S_0(c), S_1(c), S_j(c)$ all different sectors.

Let $N_c(c)$ be a small neighbourhood of c in the dynamical plane of f_c, that maps homeomorphically under f^i_c for $1 \leq i \leq l$. Moreover, consider the first time f^i_c maps c to an interior point of T_c. This must happen for $i_0 \leq l$ as c is not a Misiurewicz point at a tip, and $f^{i_0}_c(c)$ must have valence 2, so $f^{i_0}_c$ maps exactly two of the local sectors in the neighbourhood $N(c)$ to local sectors containing the two local edges in $(f^{i_0}_c(N_c(c)) \cap T_c) \setminus \{f^{i_0}_c(c)\}$.

Suppose it was the case that $T_{c''}$ is also a topological star. Consider a small neighbourhood $N_{c''}(c)$ of c in the dynamical plane of $f_{c''}$ such that $N_{c''}(c)$ maps homeomorphically under $f^i_{c''}$ for $1 \leq i \leq l$. Then one local edge of c in $N_{c''}(c) \cap T_{c''}$ is contained in two local sectors at c, one of which is $S_i(c'')$. But $S_0(c'')$ and $S_1(c'')$ are also sectors at c and they map homeomorphically by $f^i_{c''}$ for $1 \leq i \leq l$.

Suppose $f^{i_0}_c(c)$ is the first iterate $1 \leq i_0 \leq l$ where c is mapped to a point of valence > 1. Then c must be valence 2 since it is a Misiureiwcz point that is not the principal Misiurewicz point. The two local edges I_0, I_1 at $f^{i_0}_c(c)$ must be contained in $f^{i_0}_c(S_0(c) \cap N(c))$ and $f^{i_0}_c(S_1(c) \cap N(c))$ respectively. Thus $f^{i_0}_c(S_j(c) \cap N(c))$ must be mapped to a local sector that does not intersect T_c.

Since I_0 and I_1 are contained in $J(f_c)$, they are the closure of a dense set of repelling periodic points. Since $J(f_c) = K(f_c)$, these points have nontrivial orbit portraits and these orbit portraits exist in $T_{c''}$. Now consider the three sectors in the dynamical plane of $f_{c''}$ at this point $f^{i_0}_{c''}(c)$, $f^{i_0}_{c''}(S_0(c'')), f^{i_0}_{c''}(S_1(c'')), f^{i_0}_{c''}(S_j(c''))$. All three of these sectors contain points with nontrivial orbit portraits, so all three sectors must contain points in the Hubbard tree. Thus the point $f^{i_0}_{c''}(c)$ must have valence at least 3. But since it is not the α fixed point, $T(c'')$ cannot be isomorphic to $T(c)$.

□

Lemma 4.10. *Suppose c is a Misiurewicz point on the n-vein $V_{m/n}$ of the m/n-limb of the Mandelbrot set, such that the external rays landing at c are not preimages of the external ray landing at the principal Misiurewicz point of the m/n-limb under the doubling map.*

Denote the orbit portrait of c by $\{\{\phi_{1,1}, \phi_{1,2}, \dots, \phi_{1,k}\}\{\phi_{2,1}, \dots, \phi_{2,k}\}, \dots, \{\phi_{l,1}, \dots, \phi_{l,k}\}\}$, where $\phi_{1,1}, \dots, \phi_{1,k}$ are the angles of external rays landing at c in the dynamical plane.

Suppose V is any vein in the m/n-limb of $\mathcal{M}$ passing through c. If c' is a postcritically finite parameter with $c' \succ_V c$, and $T(c')$ is a topological n-star, then for every $1 \leq i \leq l$, $T_{c'}$ is contained *in at most two sectors of the external rays* $\phi_{i,1}, \dots, \phi_{i,k}$.

Proof. If $T(c')$ is an n-star, then the α fixed point of $f_{c'}$ must be at the unique branch point of c'. By assumption c is on $T(c')$ and the orbit of c does not contain the α fixed point.

Suppose it was not the case that $T_{c'}$ is contained in at most two sectors of the external rays of some $f^j(c)$. Then $f^j(c)$ is a branch point of T_c hence it must be the α fixed point, so the external rays of c must be a preimage of the external rays of the principal Misiureiwcz point. $\square$

Lemma 4.11. *Suppose c is a Misiurewicz point on the n-vein $V_{m/n}$ of the m/n-limb, such that the external rays landing at c are the primages of the external rays landing at the principal Misiureiwcz point of the m/n-limb under the doubling map.*

Denote the orbit portrait of c by $\{\{\phi_{1,1}, \phi_{1,2}, \dots, \phi_{1,k}\}, \{\phi_{2,1}, \dots, \phi_{2,k}\}, \dots, \{\phi_{l+1,1}, \dots, \phi_{l+1,k}\}\}$, where $\phi_{1,1}, \dots, \phi_{1,k}$ are the angles of external rays landing at c in the dynamical plane.

Suppose V is any vein in the m/n-limb passing through c. If c' is a postcritically finite parameter with $c' \succ_V c$ and $T(c')$ is a topological n-star, then for every $1 \leq i \leq l$, $T_{c'}$ is contained in at most two sectors of the external rays $\phi_{i,1}, \dots, \phi_{i,k}$.

Proof. If $T(c')$ is an n-star, then the α fixed point of $f_{c'}$ must be at the unique branch point of c'. By assumption c is on $T(c')$ and the orbit of c does not contain the α fixed point.

Suppose it was not the case that $T_{c'}$ is contained in at most two sectors of the external rays of some $f^j(c)$. Then $f^j(c)$ is a branch point of T_c hence it must be the α fixed point, so the external rays at $f^j(c)$ are given by angles $\phi_{l,1}, \dots, \phi_{l,k}$. $\square$

Proof. Proof of Theorem 4.1.

There are $\phi(n)$ limbs of rotations number m/n, where $1 \leq m \leq n-1$ is relatively prime to n. By Proposition 4.4, the set of parameters c with $T(c)$ isomorphic to an n-star contains a union of $\phi(n)$ n-veins. Let $V_{m/n}$ be the n-vein of the m/n-limb, given by the union of veins from the main cardioid to the tips c_i for $1 \leq i \leq n-1$ in Proposition 4.4.

We now show there are no other parameters with Hubbard tree isomorphic to an n-star. First note that if $n' \neq n$, then for any c' on an m/n' limb, $T_{c'}$ must contain a branch point of valence n' at its α fixed point, thus cannot be isomorphic to an n-star if $n' > 2$. By Lemma 4.5 there is no parameter in the 1/2-limb isomorphic to an n-star.

We now show there are no other parameters with Hubbard tree isomorphic to an n-star on any m/n-limb, that is not on $V_{m/n}$. Suppose c' is a postcritically finite parameter on this limb not on $V_{m/n}$. Let $V_{c'}$ be the vein from c' to the main cardioid. Then $V_{m/n} \cap V_{c'}$ is the vein from the main cardioid to some c, where c is either the centre of a hyperbolic component or a Misiurewicz point.

Suppose c is the centre of a hyperbolic component C, then $V_{m/n}$ bifurcates from $V_{m/n}$ along some p/q sublimb of C where $p/q \neq 1/2$. Then $T(c)$ cannot be isomorphic to $T(c')$ by Lemma 4.8.

If c is a Misiurewicz point, then c cannot be the principal Misiurewicz point nor a tip. Denote the sectors in the Mandelbrot set bounded by adjacent parameter rays landing at c by $U_0, U_1, \dots, U_{k-1}$, with U_0 the sector containing the main cardioid, and U_1 the sector containing tip c_t of $V_{m/n}$ not in U_0. Then c' is in another sector U_j, $j \neq 0, 1$. By Proposition 4.9, $T(c)$ cannot be isomorphic to $T_{c'}$. $\square$

4.2.2 Not All Abstract Trees are Realizable

In this section, we will construct an example of a tree that is not realizable as the abstract Hubbard tree of a postcritically finite quadratic polynomial. In general, we expect that a topological tree that is realizable as an abstract Hubbard tree should have a balanced distribution of branch points.

Proposition 4.12. *Let $N > n > 2$. Consider the abstract tree T with vertex set given by $\{u, u_1, u_2, \ldots, u_N, v, v_1, v_2, \ldots, v_n\}$ and edge set consisting of $\{\{u, u_i\} | 1 \leq i \leq N-1\} \cup \{\{v, v_i\} | 1 \leq i \leq n-1\} \cup \{\{u, v\}\}$. That is, the tree consists of two vertices u and v of valence N and n respectively, and their neighbours. Then this tree is not realized as a Hubbard tree of a postcritically finite quadratic map.*

Proof. Suppose f is a postcritically finite quadratic map that realizes this Hubbard tree with critical point 0. Then u and v must have finite forward orbits. Since the critical point must map to a leaf, if either u or v was the critical point the local edges incident to it must map to the local edge incident to the leaf, which would be more than 2 to 1. Moreover we cannot have $f(u) = u$ and $f(v) = v$ because u and v would be the α and β fixed points, but the β fixed point can only occur on a leaf.

Since these u and v cannot map to points of lower valence and also cannot both be fixed, we must have that one is mapped to another. If u with valence N mapped to a point of lower valence that would imply the higher valence is a critical point, so we can only have that $f(v) = u$ and $f(u) = u$. Thus the candidate parameter c of f must occur in a $1/N$ limb as with the sectors of the α fixed point separated by N external rays landing at u.

Now consider the critical point 0. Since $f(v) = u$ and $f(u) = u$, 0 must occur on the arc between u and v. Moreover, $f^i(0)$ must be a leaf for all $1 \leq i \leq N+n-2$, since all leaves are in the critical orbit, and if the preimage of a leaf is not valence 1, it must be a critical point.

Since $f(u) = u$, u must be the α fixed point, so the critical value $f(0)$ must occur in the critical sector of u, which is distinct from the sector containing the critical point. So $f(0)$ must be a leaf of u, call it u_1.

Now we consider how the leaves u_i and v_j are mapped. Consider the set of indices i such that $f(u_i) = v_j$ for some j. For each such i, the edge between u_i and u is mapped to the path between v_j and u. So there is a preimage of 0 on the path $[u_i, u]$. By Zakeri there should only be one preimage of 0 in a sector of α not containing 0, so there should only be one such u_i that maps to v_i. Hence the only way we can traverse all the leaves is for the orbit of 0 to be $u_1, u_2, \ldots, u_{N-1}$ and then $v_1, v_2, \ldots, v_{n-1}$.

Consider each of the paths $[v, v_i]$, since $f(v) = u$, $f([v, v_i])$ covers $[u, v_{i+1}]$ so there is a preimage of 0 in $[v, v_i]$. But we can only have one preimage of 0, since there was already one preimage of 0 on $[u_{N-1}, u]$, a contradiction.

□

4.2.3 Topology of Hubbard Trees Extends to Misiurewicz Tips

In this section, we show that if $T(c)$ is the abstract Hubbard tree of some postcritically finite parameter c, then there is some Misiurewicz tip c' with $T(c')$ isomorphic to $T(c)$. More specifically, if c is a centre of a hyperbolic component C, then c' is in the wake of C, and if c is a Misiurewicz point, then c' is a tip in a sector of c.

Let $c \in \mathcal{M}$ be a non-tip Misiurewicz point. Let f_c be the map associated to the parameter c and T_c be the Hubbard tree of c. Suppose $i \geq 1$ is the smallest positive number such that $f_c^i(c)$ is not a leaf in T_c. We say that c is a *simple Misiureiwcz point* if the valence of $f_c^i(c)$ is equal to the number of external rays landing at $f_c^i(c)$. This definition generalizes principal Misiurewicz points.

Proposition 4.13. *Suppose c is a simple Misiurewicz point. Let $U_0, U_1, \ldots, U_{k-1}$ be the sectors in the parameter space at c, such that the main cardioid is contained in U_0. Then for some $1 \leq j \leq k-1$ there is a postcritically finite parameter $c_j \in U_j$ with $T(c_j)$ isomorphic to $T(c)$.*

Proof. Let $i \geq 1$ be the first iterate $f_c^i(c)$ such that it is not a tip. Since $f_c^i(c)$ has valence k in T_c, its iterates must all have valence k since the critical point is not periodic. Suppose l is the largest number such that $f_c^i(c)$ are disjoint for all $0 \leq i \leq l$.

Consider T_c, and let $S_0(c), \ldots, S_{k-1}(c)$ be the sectors in the dynamical plane at c bounded by the external rays landing at c, where $S_0(c)$ is the sector containing the edge in T_c incident to c. Let $N_c(c)$ be a small neighbourhood of c, such that $N_c(c)$ maps homeomorphically up to the l-th iterate. Let $e_0, \ldots, e_{k-1}$ be edges incident to $f_c^i(c)$ in T_c, with e_0 the edge in the local sector $f_c^l(N_c(c) \cap S_0(c))$. Then there is a local edge $e'_0 = f_c^{-l}(e_0 \cap f_c^l(N_c(c))) \in S_0(c)$ and preimages of local edges $e'_j = f_c^{-l}(e_j \cap f_c^l(N_c(c))) \in S_j(c)$ for $1 \leq j \leq k-1$ in each of the other sectors.

Since $J(f_c) = K(f_c)$ and by the density of repelling periodic points, we can find a dense set of repelling periodic points in each of these local edges. In each sector $S_j(c)$, the dynamical rays landing at the periodic points in the local edge e'_j have angles in the dynamical sector U_j in the parameter plane. Moreover, every such point c' is mapped into the interior of T_c, their Hubbard tree must be ismorphic to T_c. Thus the abstract Hubbard tree of any parameter on the vein from the main cardioid to c' must be isomorphic to $T(c)$. □

Using the same idea, we can show that the persistence of topology along Misiurewicz points in general.

Proposition 4.14. *Let c be any branch Misiurewicz point. Let $U_0, U_1, \ldots, U_{k-1}$ be sectors at c in the parameter space. Suppose $i > 0$ is the smallest positive iterate such that $f_c^i(c)$ has valence $d > 1$. Then there are d sectors at c in the dynamical plane, $U_{i_1}, \ldots, U_{i_d}$ such that there exists postcritically finite $c_j \in U_{i_j}$ with $T(c_i)$ isomorphic to $T(c)$.*

Proof. The idea is the same as Proposition 4.13. Let $i \geq 1$ be the first iternate $f_c^i(c)$ such that it is not a tip. Since $f_c^i(c)$ has valence d in T_c, its iterates must all have valence $\geq d$ since they the critical point is not periodic. Suppose l is the largest number such that $f_c^i(c)$ are disjoint for all $0 \leq i \leq l$.

Consider T_c, and let $S_0(c), \ldots, S_{k-1}(c)$ be the sectors in the dynamical plane at c bounded by the external rays landing at c, where $S_0(c)$ is the sector containing the edge in T_c incident to c. Let $N_c(c)$ be a small neighbourhood of c, such that $N_c(c)$ maps homeomorphically up to the l-th iterate. Let $e_0, \ldots, e_{d-1}$ be edges incident to $f_c^i(c)$ in T_c, with e_0 the edge in the local sector $f_c^l(N_c(c) \cap S_0(c))$. Then there is a local edge $e'_0 = f_c^{-l}(e_0 \cap f_c^l(N_c(c))) \in S_0(c)$ and preimages of local edges $e'_j = f_c^{-l}(e_j \cap f_c^l(N_c(c))) \in S_j(c)$ for $1 \leq j \leq d-1$ in each of the other sectors.

Since $J(f_c) = K(f_c)$ and by the density of repelling periodic points, we can find a dense set of repelling periodic points in each of these local edges. In each sector $S_j(c)$, the dynamical rays landing at the periodic points in the local edge e'_j have angles in the dynamical sector U_j in the parameter plane. Moreover, every such point c' is mapped into the interior of T_c, their Hubbard tree must be ismorphic to T_c. Thus the abstract Hubbard tree of any parameter on the vein from the main cardioid to c' must be isomorphic to $T(c)$. $\square$

Let c be a postcritically finite tip parameter in $\mathcal{M}$. Then T_c has finitely many branch points, each of which must be periodic or preperiodic since T_c maps locally homeomorphically away from the critical point. Since there can be at most finitely many simple Misiureiwcz points associated to each periodic or preperiodic cycle of branch points, there can be at most finitely many simple Misiureiwcz points.

We will now show that given any postcritically finite parameter c, we can find a tip parameter in the wake or sector of c that has an abstract Hubbard tree isomorphic to $T(c)$. We will extend the definition of principal veins in light of this result. Let c be a postcritically finite parameter. We say a connected subset of the arc V from c to a Misiurewicz tip is a *topological principal sub-vein* for c, if $T(c)$ is isomorphic $T(c')$ for any $c' \in V$.

For a postcritically finite parameter c, if c_t is a Misiurewicz tip in the wake or sector of c such that $T(c)$ is isomorphic to $T(c_t)$, a *topological principal vein* containg c is $V = \cup_{c''}[c'', c_t]$, where the union is taken over all $c'' \in [0, c_t]$ with $T(c')$ isomorphic to $T(c)$.

Proof. 4.2 Let $c \in \mathcal{M}$ be a postcritically finite parameter.

By Lemma 4.7 at any hyperbolic component, we can extend any topological principal sub-vein the the 1/2 vein of small copies of Mandelbrot sets. By Proposition **??**, we can extend any topological principal sub-vein at a Misiurewicz point. By Lemma 4.3, the topology of the abstract Hubbard tree does not change at cusps of hyperbolic components, nor at the limit Misiurewicz point.

To prove there are finitely many of these Misiurewicz tips with the same topological tree, note that there are finitely many leaves of T_c, and every Misiureiwcz tip c_t has the property that $f^i_{c_t}(c_t)$ is a leaf for all $i \geq 0$. Since there are finitely preperiodic maps on the leaves and finitely many orderings of vertices up to embedding, there are finitely many such Misiureiwcz tips. $\square$

4.2.4 Visible Hyperbolic Components

We say that a hyperbolic component C is *narrow* if C has period n and for $R^-, R+$ the parameter rays landing at the root of C, $R^+ - R^- = \frac{1}{2^n - 1}$. For hyperbolic components C and C' where C is in the wake of C', we say C is *visible* C' if there is no hyperbolic component C'' with $C \succ C'' \succ C'$ such that $per(C'') < per(C)$.

Let $\{R_{\phi_1}, R_{\phi_2}, \ldots, R_{\phi_k}\}$ be the set of dynamical rays that land at some point $z \in J(f_c)$. A *sector* of these rays is a connected component of $\mathbb{C} \setminus (R_{\phi_i} \cup R_{\phi_j} \cup \{z\})$ that does not contain any other of the rays that land at z. For a given point $z \in J(f_c)$, denote $S_0(c), S_1(c), \ldots S_{k-1}(c)$ as sectors at z in the dynamical plane of f_c. For a neighbourhood $N(z)$ of z, the *local sectors* of z are the sets $N(z) \cap S_j(c)$ for $0 \leq j \leq k-1$. If $z \in T_c$, the *local edges* at z are $T_c \cap N(z) \cap S_j(c)$.

Proposition 4.15. *Let C and C' be narrow hyperbolic components such that $C' \succ C$, C' is visible from C in the 1/2 sub-limb of C. Then there exists a Misiurewicz tip c in the wake of C' such that $T(c)$ is isomorphic to $T(C')$.*

Proof. Suppose C has period n. Since C is narrow, C' has period $n+i$. Since C' is in the 1/2 sub-limb of C, we can suppose C has address $1 \to k_1 \to k_2 \to \cdots \to k_l \to n$ and C' has internal address $1 \to k_1 \to k_2 \to \cdots \to k_l \to n \to n+i$.

By the translation principal [**LS**], the tree of visible hyperbolic components of the $1/2$ sub-limb of C' contains a visible hyperbolic component C'' of period $n+2i$. Moreover suppose the angles of the external rays landing at the root of C' and C are given by $\theta^{\pm}_{C} = \overline{\theta^{\pm}_{C,1}, \ldots, \theta^{\pm}_{C,n}}$ and $\theta^{\pm}_{C'} = \overline{\theta^{\pm}_{C',1}, \ldots, \theta^{\pm}_{C',n+i}}$.

Then $\theta^{\pm}_{C''} = \overline{\theta^{\pm}_{C',1}, \ldots, \theta^{\pm}_{C',n+i}\theta^{\pm}_{C',n+1} \cdots \theta^{\pm}_{C',n+i}}$, where the $n+2i$ digits of $\theta^{\pm}_{C''}$ is obtained by repeating the last i digits of $\theta^{\pm}_{C'}$ after the $n+i$ digits of $\theta C'^{\pm}$. This works because we can verify that C'' has the correct internal address, thus it is visible from C.

By the criterion for narrow components (cite Sc), C'' is narrow iff there does not exist $1 \leq k < i$ such that $\rho(k) = i$.

Denote C_m to be the component of period $n+mi$ that has the above prescribed dynamics. Then $C^{\pm}_{\infty} = \lim_m \theta^{\pm}_{C_m} = \theta^{\pm}_{C,1} \ldots \theta^{\pm}_{C,n}\overline{\theta^{\pm}_{C',n+1} \cdots \theta^{\pm}_{C',n+i}}$. So $C^{-}_{\infty} = C^{+}_{\infty}$ because it is in the intersection of arbitrarily small wakes.

□

5

Conspicuous Components and Pseudomonodromy on the Mandelbrot Set

The motivation for understanding the pseudomonodromy of the Mandelbrot set comes from Lipa [9] where it is used as a tool to understand the parameter space of Hénon maps. Given an oriented arc γ in the exterior of the Mandelbrot set with its endpoints at the root of a Hyperbolic component, the pseudomonodromy of γ is the automorphism on the space of kneading sequences induced by traversing along γ.

It is known by Ishii-Richards [8] that the non-trivial contributions to this automorphism induced by the monodromy at the root of a hyperbolic component H consist of all H' that are conspicuous to H. Other results in understanding the pseudomonodromy of the Mandelbrot set comes from Baik-Baik [2] and Atela [1].

In this Chapter, I will give a natural description of conspicuous components and the set of components they are conspicuous to, using internal addresses of hyperbolic components.

Definition 5.1 (Lipa)**.** *Let H and H' be two hyperbolic components of $\mathcal{M}$. We say that H is* conspicuous *to H' if:*

- $H' \succ H$,
- $per(H') \leq per(H)$*, and*
- *There is no hyperbolic component* $H'' \in [H, H']$ *with* $per(H'') < per(H')$.

By this original definition in [9], a hyperbolic component H is always conspicuous to itself.

I call a hyperbolic component H' *conspicuous* if there is another component $H \neq H'$ such that H is conspicuous to H'.

Proposition 5.2. *The set of conspicuous components is exactly the set of primitive components except for the main cardioid* H_0.

Proof. This is because for any H' primitive component, there are at most finitely many components with period $< per(H')$ along the vein $[H', H_0]$, so you can always find a segment of this vein, $[H', K]$ that does not contain any components of period $< per(H')$. Moreover for any satellite component H' has another component K immediately before it in the vein $[H, H_0]$ with $per(K) < per(H')$, so H' cannot be conspicuous. ☐

Lemma 5.3. *For a conspicuous component* H'*, the set of components* H *such that* H' *is conspicuous to* H *is a sub vein* $[H', K]$ *of* $[H', H_0]$*. Moreover* K *is a satellite component.*

Proof. This is because the only thing preventing this sub vein from being extended further is another component of period $< per(H')$.

So I can associate each conspicuous component H' with the satellite component K that corresponds to the furthest component in the sub vein of conspicuous components of H'. ☐

Theorem 5.4. *Suppose* H' *is a conspicuous component with angled internal address given by* $(1)_{p_0/q_0} \to (n_1)_{p_1/q_1} \to (n_2)_{p_2/q_2} \to \cdots \to (n_k)_{p_k/q_k} \to (n)$*. Suppose* K' *is a hyperbolic component at the angled internal address* $(1)_{p_0/q_0} \to (n_1)_{p_1/q_1} \to (n_2)_{p_2/q_2} \to \cdots \to (n_k)$*. Suppose* K *is the immediate satellite of* K' *with angled internal address* $(1)_{p_0/q_0} \to (n_1)_{p_1/q_1} \to (n_2)_{p_2/q_2} \to \cdots \to (n_k)_{p_k/q_k} \to (q_k n_k)$.

Then H' *is conspicuous to all hyperbolic components on the sub vein* $(H', K]$.

Proof. Denote the previous component to K in the vein $[H', H_0]$ as K' (ie K is an immediate satellite of K'). Then $per(K') = n' < n$, but $n < n_k$. If n' is not equal to any of the n_i's, then $n_i < n' < n_{i+1}$ for some $i < k$, but that means the internal address of K would be $1 \to n_1 \to \cdots \to n_i \to n' \to n_k$, contradiction.

Denote the component in the vein $[H', H_0]$ corresponding to $1 \to n_1 \to \cdots \to n_i$ as H_{n_i}. Now suppose K' is not H_{n_k}, so there is some component $H_{n''}$ of smaller period $n'' < n_k$ separating H_{n_k} and K'. If this n'' is not any of the n_i's it would be in the internal address of K, a contradiction. So n'' is one of the n_i's, but since $H_{n''} > H_{n_k} > H_{n_i}$ we must have that there is another component between $H_{n''}$ and H_{n_i} with smaller period. We can repeat this argument and obtain a contradiction via infinite descent on the natural numbers. ☐

For example, as a corollary, this answers the following question raised by Thomas Richards:

Corollary 5.5. *Suppose* H *is conspicuous to* H'*, then* H' *must be in the* $1/2$*-sublimb of* H.

Works Cited

[1] P. Atela. “The Mandelbrot set and σ-automorphisms of quotients of the shift”. In: *Transactions of the American Mathematical Society* 335.2 (1993), pp. 683–703.

[2] H. Baik and J. Baik. *Monodromy through bifurcation locus of the Mandelbrot set.* URL: `https://arxiv.org/abs/2305.04218`.

[3] S. Baldwin. “An extension of Šarkovskiĭ’s theorem to the n-od”. In: *Ergodic Theory and Dynamical Systems* 11.2 (1991), pp. 249–271. DOI: `10.1017/S0143385700006131`.

[4] B. Branner and A. Douady. “Surgery on complex polynomials”. In: *Lecture Notes in Mathematics* 1345.1 (1988), pp. 11–72. DOI: `10.1007/BFb0081395`.

[5] H. Bruin, A. Kaffl, and D. Schleicher. *Symbolic dynamics of quadratic polynomials.* URL: `https://www.mat.univie.ac.at/~bruin/talks/TreesBook.pdf`.

[6] K. Burns and Hasselblatt. B. *Sharkovsky’s theorem.* URL: `https://sites.math.northwestern.edu/~burns/papers/boris1/SharkovskyISubmitted.pdf`.

[7] A. Douady and J. Hubbard. *Exploring the Mandelbrot set. The Orsay Notes.* URL: `https://pi.math.cornell.edu/~hubbard/OrsayEnglish.pdf`.

[8] Y. Ishii and T Richards. “Pseudo-monodromy and the Mandelbrot set”. In: *Transactions of the American Mathematical Society* (2025). DOI: `10.1090/tran/9580`.

[9] C Lipa. “Monodromy and Hénon mappings”. PhD thesis. Cornell University, 2009. 117 pp. URL: `https://www.math.stonybrook.edu/~ebedford/PapersForM655/Lipa_Thesis.pdf`.

[10] C. L. Petersen and P. Roesch. “The Yoccoz combinatorial analytic invariant”. In: *Holomorphic Dynamics and Renormalization* 53 (2000), pp. 145–176.

[11] A. Poirer. *On post critically finite polynomials part two: Hubbard trees.* URL: `https://arxiv.org/pdf/math/9307235`.

[12] G. Tiozzo. “Entropy, dimension and combinatorial moduli for one-dimensional dynamical systems”. PhD thesis. Harvard University, 2013. 103 pp. URL: `https://www.math.utoronto.ca/tiozzo/docs/Thesis%5C_Tiozzo%5C_web.pdf`.