On topological and Hodge theoretic invariants of curves and families of curves

by

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Abstract

In this thesis, we study topological and Hodge theoretic invariants associated to smooth complex algebraic varieties with particular focuses on algebraic curves and smooth projective families of curves. The central question we would like to understand is to what extent these invariants capture morphisms between the corresponding varieties.

More precisely, inspired by Grothendieck's section conjecture in anabelian geometry, we formulated and studied a topological and a Hodge theoretic section question for smooth projective family of curves and made progress towards answering those questions. In the topological setting, many of our results are analogous to existing results in anabelian geometry. In the Hodge setting, much less is known, and so we also studied the connection between the Hodge theoretic section question and classical results in non-abelian Hodge theory. In a different direction, motivated by the famous theorem of Torelli, we studied if the Torelli's theorem can be made functorial. We construct interesting examples of morphisms of Hodge structures which do not arise from morphism between curves, and connect our construction to the study of isotrivial isogeny factors in family of curves.

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Contents

Chapter 1

Introduction

One of the most useful ideas in mathematics is that one may study complicated mathematical objects (e.g. geometry of algebraic varieties) by finding and studying suitable functorial invariants attached to these mathematical objects. These functorial invariants tend to be very useful when one wants to distinguish different mathematical objects: if two mathematical objects have different invariants, then they must be different to start with. An elementary but also illustrative example of this principle is that one may distinguish smooth compact orientable closed surfaces by looking at their topological fundamental groups and/or first homology groups (which are abelianizations of their topological fundamental groups).

It's then very natural to ask when the converse of this principle holds, i.e., if we have two mathematical objects with isomorphic invariants, when can we say that these two objects are the same? In other words, we would like to know to what extent our mathematical objects are determined by these invariants. In the case of topological surfaces, our invariants turn out to extremely good: not only can we say that two surfaces with isomorphic topological fundamental groups are homeomorphic, but we can also say that continuous maps between topological surfaces up to homotopy correspond exactly to maps between topological fundamental groups. In other words, these topological surfaces are examples of $k(\pi, 1)$ spaces.

Now let's turn to the world of complex algebraic geometry. What are some invariants in algebraic geometry that are powerful enough to determine the geometry of algebraic varieties and morphisms between them? One classical approach to find such invariants is via Hodge theory. The idea is the following: given a smooth connected complex algebraic variety X, we can consider its (singular) cohomology groups $H^i(X, \mathbb{Z})$. By a classical theorem of Deligne [12], we know that these cohomology groups carry extra linear algebraic data known as mixed Hodge structures. Therefore, one may ask to what extent the mixed Hodge structures on the cohomology groups of algebraic varieties capture the geometry of the corresponding algebraic varieties. Perhaps the most celebrated theorem in this direction is the Torelli theorem for smooth projective curves [3, page 245]: if X and Y are two smooth projective curves of genus g such that $H^1(X, \mathbb{Z})$ and $H^1(Y, \mathbb{Z})$ are isomorphic as polarized integral Hodge structures, then X and Y must have isomorphic algebraic structures.

There's a long and beautiful line of work extending the classical Torelli's theorem to varieties of dimension greater than 1 (e.g. K3 surfaces [45], cubic fourfolds [57], etc.). In a different direction, one may try to extend the classical Torelli's theorem by asking if Hodge structures can be used to capture morphisms between algebraic varieties. One very successful theorem in this direction is the pointed Torelli theorem of Hain [23] and Pulte [46]. In this extension, cohomology groups are replaced by fundamental groups: given a complex algebraic variety X and a point $x \in X$, Hain functorially defines a mixed Hodge structure on the truncated integral group ring $\mathbb{Z}[\pi_1(X,x)]/J^{r+1}$, where J is the augmentation ideal (see [23, Theorem 5.1]; Morgan [42] proved an equivalent result using a different method). The pointed Torelli theorem of Hain and Pulte says the following: Let (X,x) and (Y,y) be two smooth projective pointed curves of genus g. If there's a ring isomorphism

$$\theta : \mathbb{Z}[\pi_1(X,x)]/J^3 \to \mathbb{Z}[\pi_1(Y,y)]/J^3$$

which induces an isomorphism of mixed Hodge structures, then there exists an isomorphism $f : X \to Y$ such that, with the possible exception of two points, $f(x) = y$.

In a different area of algebraic geometry, people have also been studying the role of fundamental groups and how they might capture morphisms of the corresponding algebraic variety. This is the anabelian geometry program proposed by Grothendieck [19]. Here the fundamental groups in consideration are étale fundamental groups defined by Grothendieck. Recall that given a quasi-compact and geometrically integral scheme over some number field k, fixing a separable closure $\overline{k}$ of k, we have a short exact sequence of étale fundamental groups

$$1 \to \pi_1^{\text{ét}}(X_{\bar{k}}) \to \pi_1^{\text{ét}}(X) \to \mathrm{Gal}(\bar{k}/k) \to 1 \tag{1.0.1}$$

where $X_{\overline{k}}$ is the base-change of X to the separable closure (see e.g. [52, Theorem 5.6.1]). Grothendieck then proposed that there is a "natural" class of schemes, known as anabelian schemes, such that morphism between anabelian schemes are entirely determined by maps of extensions of (1.0.1). Motivated by this philosophy, Grothendieck proposed his famous section conjecture: if X is a smooth projective curve of genus $g \geq 2$, then the set of k-rational points of X is in bijection with the set of equivalence classes of splittings of (1.0.1).

This thesis aims to continue these traditions and ideas. More precisely, we are interested in understanding to what extent invariants like fundamental groups and cohomology capture morphisms between smooth connected *complex* algebraic varieties. Given the prominence of curves and more generally, family of curves in these lines of research, we will also focus on them. In particular, we mainly study the following two types of morphisms:

1. algebraic sections to smooth projective family of curves (Chapter 3 and Chapter 4);
2. morphisms between curves of different genus (Chapter 5).

For algebraic sections to smooth projective family of curves, we formulated a topological and a Hodge theoretic analogue of Grothendieck's section conjecture and made partial progress towards understanding these analogous questions. In particular, in the topological setting, we proved the following collection of results

Proposition 1.0.1.

1. *The topological section map is injective for Kodaira fibrations (i.e. non-isotrivial smooth projective*

family of curves over smooth projective curves) if the monodromy representation has no invariants (see Corollary 3.1.4);

2. *The topological section map is in general not surjective (see Corollary 3.2.4).*

In the Hodge theoretic setting, we replace topological fundamental group of a complex variety X with the category of graded-polarizable, admissible variation of mixed Hodge structures over X. We proved the following theorems:

Proposition 1.0.2.

1. *If we consider the category of $\mathbb{Z}$-variation of mixed Hodge structures, then the Hodge theoretic section map is injective (see Proposition 4.2.3);*

2. *If we consider the category of $\mathbb{Q}$-variation of mixed Hodge structure, we may apply Tannakian duality and obtain an exact sequence of Tannakian fundamental groups for any smooth projective map $f : X \to B$ with connected fibers (see Proposition 4.3.1)*

$$\pi_1(\mathrm{LS}^{hdg}_{\mathbb{Q}}(X_b)) \to \pi_1(\mathrm{VMHS}_{\mathbb{Q}}(X)) \to \pi_1(\mathrm{VMHS}_{\mathbb{Q}}(B)) \to 1. \tag{1.0.2}$$

3. *Fixing some $g \geq 2$, and working with $\mathbb{Q}$-variation of mixed Hodge structures, then we can show that the Hodge theoretic section map is a bijection for universal family of curves $f : \mathcal{C}_g \to \mathcal{M}_g$ and for the moduli space of degree 1 line bundles on the universal curve $p : \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g} \to \mathcal{M}_g$ (see Corollary 4.6.6).*

We defer the precise definition of these objects and the formulation of the topological and Hodge theoretic section conjecture to later chapters. To motivate the discussion of these results, we also include a brief recap of some of the results appearing in the study of Grothendieck's section conjecture (Chapter 2). One will see that many of the results proven in this thesis have counterparts in the study of Grothendieck's section conjecture.

It's also worth pointing out that the Hodge theoretic formulation is relatively new (but see also the thesis work of Ferrario [16]) and we investigated many natural questions one may ask in this Hodge theoretic setting (see Chapter 4). In particular, we are able to relate the study of the exact sequence (1.0.2) to Simpson's work on non-abelian Hodge locus [47]. The majority of the content of Chapter 3 and Chapter 4 are contained in the arXiv preprints [59] and [58].

For morphisms between curves, we are interested in the question of whether Torelli's theorem can be made functorial for curves of different genus. We give examples to show that this is not true, and also relate the constructions to the study of isotrivial isogeny factors in family of curves (Chapter 5). More precisely, we prove that

Proposition 1.0.3.

1. *Let R be any curve of genus 2. Then for any $g > 2$, there exists a topological cover C of R of genus g such that the covering map is not homotopic to an algebraic map but induces maps of Hodge structure on cohomology (see Proposition 5.1.1);*

2. *The construction above can be modified to give subvarieties of $\mathcal{M}_g$ whose associated families of curves have isotrivial isogeny factor and the dimension of these subvarieties is as big as possible (see Proposition 5.2.1).*

Chapter 2

A brief review of Grothendieck's section conjecture

This section does not aim to be a comprehensive overview of Grothendieck's section conjecture. A good reference for that is Stix's book on the section conjecture [51]. Instead we only mention those results that will be relevant to our discussion later on. We first give a brief overview of the anabelian philosophy.

Let X be a quasi-compact and geometrically integral scheme over some number field k. Fix some separable closure $\bar{k}$ of k as before and fix some geometric point of X. We will however omit the geometric point in our notation as it will not be very relevant for us. Consider the functor from the category of k-schemes to the category of short exact sequence of (profinite) groups which sends X to the extension

$$1 \to \pi_1^{\text{ét}}(X_{\bar{k}}) \to \pi_1^{\text{ét}}(X) \to \mathrm{Gal}(\bar{k}/k) \to 1.$$

Given a morphism of short exact sequence of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & K_1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H_2 & \longrightarrow & G_2 & \longrightarrow & K_2 & \longrightarrow & 1 \end{array}$$

we may define an equivalence relation on the set of morphism between these two sequences by the conjugation action of H_2. Then morphisms in the category of extensions are defined to be equivalence classes of morphisms of short exact sequences under this conjugation action.

Remark 2.0.1. The equivalence relation is needed to account for the fact that we've fixed a base point and the conjugation action amounts to changing the base points. One may remove this equivalence relation by working with pointed varieties.

Grothendieck then conjectured that there's a class of schemes, known as the anabelian schemes, for which this functor is fully faithful. Furthermore, he then conjectured that this class of schemes should at least have the following properties:

1. smooth projective curves of genus $g \geq 2$ should be anabelian;
2. the moduli stack $\mathcal{M}_{g,n}$ should be anabelian;
3. this class of schemes should be closed under taking fibrations: if $f : X \to Y$ is a smooth projective map where Y is anabelian and so are the fibers X_b, then X should be anabelian as well.

If one in addition believes that $\operatorname{Spec} k$ is anabelian, then one arrives at Grothendieck's section conjecture: Let X be a smooth projective curve of genus $g \geq 2$ over some number field k. Any k-rational point induces a splitting of (1.0.1) and so we have a section map

$$\mathrm{sec} : X(k) \to \{\text{splittings of } (1.0.1)\}/\text{conjugation by } \pi_1^{\text{ét}}(X_{\overline{k}}).$$

Then the section conjecture says that

Conjecture 2.0.2 (Grothendieck's section conjecture)**.** The map sec is a bijection.

One direction of this conjecture is already known to Grothendieck.

Proposition 2.0.3 (Grothendieck; see section 7 of [51] for a proof)**.** *The section map is injective.*

The question of surjectivity is much more difficult and not much is known. The only examples where we can verify the section conjecture involve curves with no rational points (see for example the constructions in section 7 of [50]). On the other hand, Stix showed that the case of curves with no rational points are in fact the most interesting case, as in fact the question of surjectivity, and hence the section conjecture itself, may be reduced to this case entirely. To state this result, we first state the weak section conjecture.

Conjecture 2.0.4 (Weak section conjecture)**.** Let X be a smooth projective curve of genus $g \geq 2$ over some number field k. Then X has a k-rational point if and only if the sequence (1.0.1) splits.

It's clear that the weak section conjecture is implied by the Grothendieck's section conjecture. What's less obvious is that if weak section conjecture holds for all curves, then the Grothendieck's section conjecture also holds for all curves. More precisely, Stix proved the following:

Proposition 2.0.5 (Theorem 31 in [50])**.** *Let X be a smooth projective curve of genus $g \geq 2$ over some number field k. Then the section map is surjective for X if and only if the weak section conjecture is true for every finite étale cover of X which is geometrically connected over k.*

One may also consider other fields. Of particular interest to us is the section conjectures on generic curves over the generic point of $\mathcal{M}_g$ first proven by Hain. More precisely, we have the following proposition:

Proposition 2.0.6 ([20], [37])**.** *Let $f : \mathcal{C}_g \to \mathcal{M}_g$ be the universal family of genus $g > 2$ curves, and K the function field of $\mathcal{M}_g$. Suppose $\overline{K}$ is a separable closure of K. Let $\mathcal{C}_{g,K}$ be the pullback of the universal family to the function field (i.e. the generic curve), and $\mathcal{C}_{g,\overline{K}}$ the base change to the separable closure. Then the sequence*

$$1 \to \pi_1^{ét}(\mathcal{C}_{g,\overline{K}}) \to \pi_1^{ét}(\mathcal{C}_{g,K}) \to \mathrm{Gal}(\overline{K}/K) \to 1$$

do not split, and hence the section conjecture is trivially true for the generic curve.

Remark 2.0.7. Both the work of Hain [20] and the work of Li-Litt-Salter-Srinivasan [37] proved much more than this result. For example, Hain proved analogous statements for moduli space of curves with level structure and unipotent version of the section conjecture for moduli space of curves with marked points. The work of Li-Litt-Salter-Srinivasan proved analogous statement for the moduli space of degree 1 line bundles on the universal curve $p : \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g} \to \mathcal{M}_g$. They also formulated and proved many cases of a tropical version of the Grothendieck's section conjecture, which concerns generic curves of a given reduction type.

Chapter 3

Topological section conjecture for family of curves

Let $f : X \to B$ be a smooth projective family of smooth projective curves of genus g. By the universal property of $\mathcal{M}_g$, this is equivalent to the data of a map from B into $\mathcal{M}_g$. Associated to such a family is a long exact sequence of topological fundamental groups

$$\cdots \to \pi_2(B) \to \pi_1(X_b) \to \pi_1(X) \to \pi_1(B) \to 1$$

where X_b is the fiber of $f : X \to B$ over some base point $b \in B$.

Lemma 3.0.1. *The map $\pi_1(X_b) \to \pi_1(X)$ is always injective and hence we get a short exact sequence*

$$1 \to \pi_1(X_b) \to \pi_1(X) \to \pi_1(B) \to 1. \tag{3.0.1}$$

Proof. Since this family is pulled back from the universal family $\mathcal{C}_g \to \mathcal{M}_g$ via the map $B \to \mathcal{M}_g$, we have the following commutative diagram

$$\begin{tikzcd}
\pi_1(X_b) \arrow[r] & \pi_1(\mathcal{C}_g) \arrow[r] & \pi_1(\mathcal{M}_g) \arrow[r] & 1 \\
\pi_1(X_b) \arrow[r] \arrow[u, "="'] & \pi_1(X) \arrow[r] \arrow[u] & \pi_1(B) \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

Then if the map $\pi_1(X_b) \to \pi_1(X)$ has a non-trivial kernel, so does the map $\pi_1(X_b) \to \pi_1(\mathcal{C}_g)$. However, the map $\pi_1(X_b) \to \pi_1(\mathcal{C}_g)$ is necessarily injective because $\mathcal{M}_g$ is a $k(\pi, 1)$-space, and so $\pi_2(\mathcal{M}_g) = 0$. $\square$

Now any algebraic section s of the family $f : X \to B$ induces a splitting of the short exact sequence (3.0.1). Hence, we get a map

$$\mathrm{sec}_{\mathrm{top}} : \{\text{algebraic sections to } f : X \to B\} \to \{\text{splittings of } (3.0.1)\}/\sim$$

where the equivalence relation on the set of splittings is defined in the following way: we say two sections $s_1, s_2 : \pi_1(B) \to \pi_1(X)$ are isomorphic if and only if they are conjugate via some elements in $\pi_1(X_b)$. We are interested in the following question:

Question 3.0.2 (Topological section question)**.** Is the map $\mathrm{sec}_{\mathrm{top}}$ a bijection?

This question has a very simple answer in the case where $f : X \to B$ has trivial monodromy (i.e. $X = B \times X_b$ for some curve X_b and f is the projection map onto B). In this case, there are infinitely many algebraic sections but only one group theoretic section, and so the map $\mathrm{sec}_{\mathrm{top}}$ is surjective but never injective.[1] Therefore, to make this question interesting, we would like to add the additional assumption that $f : X \to B$ is non-isotrivial. Of special interest to us is the case where B is another smooth projective curve.

Definition 3.0.3. A Kodaira fibration is a smooth projective, non-isotrival family X of curves over some smooth projective curve B whose fibers are all smooth projective of genus g.

Remark 3.0.4.

1. Some constructions of Kodaira fibrations are recorded in Appendix A.
2. It's known that if $f : X \to B$ is a Kodaira fibration, then $g(B) \geq 2$ and $g(X_b) \geq 3$ [28, Theorem 1.1], and so by Grothendieck's anabelian philosophy, Kodaira fibrations should be considered as an anabelian scheme as well.

In this chapter, our goal is to provide partial answers to Question 3.0.2 for Kodaira fibrations. The main idea is to replace sequence (3.0.1) with an abelian version:

$$0 \to H_1(X_b, \mathbb{Z}) \to \pi_1(X)/[\pi_1(X_b), \pi_1(X_b)] \to \pi_1(B) \to 1. \tag{3.0.2}$$

This is obtained from the original sequence (3.0.1) by pushing out along the abelianization map $\pi_1(X_b) \twoheadrightarrow H_1(X_b, \mathbb{Z})$. Furthermore, this sequence also has a geometric interpretation: this is the short exact sequence of topological fundamental groups associated to the family of Jacobians $p : \mathrm{Pic}^0_{X/B} \to B$. Therefore, one can similarly define a section map

$$\mathrm{sec}_{\mathrm{ab}} : \{\text{algebraic sections to } p : \mathrm{Pic}^0_{X/B} \to B\} \to \{\text{splittings of } (3.0.2)\}/\sim$$

and ask if this map $\mathrm{sec}_{\mathrm{ab}}$ is a bijection. The first two sections are devoted to understanding this abelian version of the topological section question, and we will use the results to deduce consequences for the original topological section question (i.e. Question 3.0.2). In the next section, we consider the general case where the base is no longer assumed to be a curve, and relate the question of surjectivity of $\mathrm{sec}_{\mathrm{top}}$ to a certain weak topological section question, and in the last section, we give examples of Kodaira fibrations for whom $\mathrm{sec}_{\mathrm{top}}$ is not surjective.

[1] One may remedy this issue by replacing the space of algebraic sections by π_0 of that space and this is the approach used in the formulation of real section conjectures; see for example [51, Section 16.1].

3.1 Family of Jacobians and injectivity

Results in this section holds more generally for any abelian schemes over a curve, so we first work in that generality. Let $p : \mathcal{A} \to B$ be an abelian scheme over a smooth projective curve B. Associated to this map is a short exact sequence of topological fundamental groups

$$1 \to H_1(\mathcal{A}_b, \mathbb{Z}) \to \pi_1(\mathcal{A}) \to \pi_1(B) \to 1.$$

Given an algebraic section $s : B \to \mathcal{A}$, we get a group theoretic splitting of this short exact sequence. It's well known that isomorphism classes of group theoretic splittings are parametrized by the cohomology group $H^1(\pi_1(B), H_1(\mathcal{A}_b, \mathbb{Z}))$ [34, Chapter 8, Theorem 1.3], and so the map $\mathrm{sec_{ab}}$ may be rewritten as

$$\mathrm{sec_{ab}} : H^0(B, \mathcal{A}) \to H^1(\pi_1(B), H_1(\mathcal{A}_b, \mathbb{Z})),$$

where we view $\mathcal{A}$ as a sheaf over B and $H^0(B, \mathcal{A})$ is the group of algebraic sections of $p : \mathcal{A} \to B$. We would like to relate $\mathrm{sec_{ab}}$ with a boundary map in some long exact sequence of cohomology groups which we now explain.

Consider the short exact sequence $0 \to R^1p_*\mathbb{Z}(1) \to R^1p_*\mathcal{O} \to \mathcal{A} \to 0$. Over each $b \in B$, this short exact sequence becomes the universal covering map $H^1(\mathcal{A}_b, \mathbb{Z}) \to \mathcal{A}_b$ with kernel $H_1(\mathcal{A}_b, \mathbb{Z})$. Taking the long exact sequence in cohomology, we get

$$\cdots \to H^0(B, R^1p_*\mathcal{O}_{\mathcal{A}}) \to H^0(B, \mathcal{A}) \xrightarrow{\psi} H^1(B, R^1p_*\mathbb{Z}(1)) \to H^1(B, R^1p_*\mathcal{O}_{\mathcal{A}}) \to \dots$$

We would like to construct an isomorphism $F : H^1(\pi_1(B), H_1(\mathcal{A}_b, \mathbb{Z})) \to H^1(B, R^1p_*\mathbb{Z}(1))$ such that the following diagram commutes

$$\begin{tikzcd}
H^0(B,\mathcal{A}) \arrow[r, "\mathrm{sec_{ab}}"'] \arrow[d, "\psi"] & H^1(\pi_1(B), H_1(\mathcal{A}_b,\mathbb{Z})) \arrow[dl, dashed, "F"] \\
H^1(B, R^1p_*\mathbb{Z}(1)) &
\end{tikzcd}$$

To do so, first observe that there's a natural inclusion map $\iota : H^0(B, \mathcal{A}) \to H^0(B, (\mathcal{A})^{\mathrm{cont}})$, where $\mathcal{A}^{\mathrm{cont}}$ is the sheaf of continuous sections to $\mathcal{A}$, and the map $\mathrm{sec_{ab}}$ evidently factors through this natural inclusion. Now we claim that ψ also factors through ι. Consider the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & R^1p_*\mathbb{Z}(1) \arrow[r] \arrow[d, "="] & R^1p_*\mathcal{O}_{\mathcal{A}} \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^1p_*\mathbb{Z}(1) \arrow[r] & (R^1p_*\mathcal{O}_{\mathcal{A}})^{\mathrm{cont}} \arrow[r] & (\mathcal{A})^{\mathrm{cont}} \arrow[r] & 0
\end{tikzcd}$$

Taking long exact sequence in cohomology, we get

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H^0(B,\mathcal{A}) & \xrightarrow{\psi} & H^1(B,R^1p_*\mathbb{Z}(1)) & \longrightarrow & H^1(B,R^1p_*\mathcal{O}) & \longrightarrow & \dots \\
 & & \downarrow\iota & & \downarrow= & & \downarrow & & \\
\dots & \rightarrow & H^0(B,(\mathcal{A})^{\mathrm{cont}}) & \xrightarrow{\psi_{\mathrm{cont}}} & H^1(B,R^1p_*\mathbb{Z}(1)) & \rightarrow & H^1(B,(R^1p_*\mathcal{O})^{\mathrm{cont}}) = 0 & &
\end{array}$$

This shows that ψ factors through ι. In particular, we see that it's enough to construct an isomorphism $F : H^1(\pi_1(B), H_1(\mathcal{A}_b,\mathbb{Z})) \to H^1(B, R^1p_*\mathbb{Z}(1))$ so that the following diagram commutes:

$$\begin{tikzcd}
H^0(B,(\mathcal{A})^{\mathrm{cont}}) \arrow[r, "(\mathrm{sec}_{\mathrm{ab}})_{\mathrm{cont}}"'] \arrow[d, "\psi_{\mathrm{cont}}"] & H^1(\pi_1(B),H_1(\mathcal{A}_b,\mathbb{Z})) \arrow[dl, dashed, "F"] \\
H^1(B,R^1p_*\mathbb{Z}(1)) &
\end{tikzcd}$$

Note that because $(R^1p_*\mathcal{O})^{\mathrm{cont}}$ is a fine sheaf, $H^1(B,(R^1p_*\mathcal{O})^{\mathrm{cont}}) = 0$ and the map ψ_{cont} is surjective. On the other hand, because $p : \mathcal{A} \to B$ is a map of $k(\pi, 1)$ spaces, it follows that the map $(\mathrm{sec}_{\mathrm{ab}})_{\mathrm{cont}}$ is also surjective. This suggests the following definition of F: for any $[s] \in H^1(\pi_1(B), H_1(\mathcal{A}_b,\mathbb{Z}))$, represent it by some continuous section $s : B \to \mathcal{A}$, and define $F([s])$ to be $\psi_{\mathrm{cont}}(s)$.

Lemma 3.1.1. *The map F constructed as above is well-defined, and is an isomorphism that makes the following diagram commutes*

$$\begin{tikzcd}
H^0(B,\mathcal{A}) \arrow[r, "\mathrm{sec}_{ab}"'] \arrow[d, "\psi"] & H^1(\pi_1(B),H_1(\mathcal{A}_b,\mathbb{Z})) \arrow[dl, dashed, "F"] \\
H^1(B,R^1p_*\mathbb{Z}(1)) &
\end{tikzcd}$$

Proof. Note that by construction, this map F, if well-defined, automatically makes the desired diagram commutative and is also automatically surjective.

To show that F is well-defined, we need to show that given any pair of continuous section s_1 and s_2, if they are homotopic, then $\psi_{\mathrm{cont}}(s_1) = \psi_{\mathrm{cont}}(s_2)$ and if they induce conjugate group theoretic splittings, then we also have $\psi_{\mathrm{cont}}(s_1) = \psi_{\mathrm{cont}}(s_2)$. To this end, we need to use an explicit description of the map ψ_{cont}. It's well-known that $H^1(B, R^1p_*\mathbb{Z})$ parametrizes $R^1p_*\mathbb{Z}(1)$ torsors on B [49, Lemma 21.4.3], and by unwinding the proof (e.g. the one given in [49]), we see that ψ_{cont} admits the following description: let $s \in H^0(B,(\mathcal{A})^{\mathrm{cont}})$ be a continuous section. Then $\psi_{\mathrm{cont}}(s)$ is the isomorphism class of $R^1p_*\mathbb{Z}(1)$-torsor $\mathcal{F} \subset (R^1p_*\mathcal{O})^{\mathrm{cont}}$ defined by the sections which maps to s.

Now suppose s_1 and s_2 are homotopic continuous sections and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the torsor they induce respectively. Explicitly, over some trivializing open subset $\mathcal{U}$, $\mathcal{F}_i(U)$ is the set of sections $s : U \to U \times H^1(\mathcal{A}_b, \mathcal{O}_{\mathcal{A}_b})$ which, after compositing with the projection $U \times H^1(\mathcal{A}_b, \mathcal{O}_{\mathcal{A}_b}) \to U \times \mathcal{A}_b$ becomes $s_i|_U$, $i = 1, 2$. Note that $U \times H^1(\mathcal{A}_b, \mathcal{O}_{\mathcal{A}_b}) \to U \times \mathcal{A}_b$ is a covering map and hence we may lift the homotopy between s_1 and s_2 to homotopies between sections of $\mathcal{F}(U_1)$ and $\mathcal{F}(U_2)$. Hence we get a bijection $\mathcal{F}_1(U) \cong \mathcal{F}_2(U)$ and an isomorphism $\mathcal{F}_1 \cong \mathcal{F}_2$.

Finally, suppose that s_1 and s_2 induce conjugate group theoretic splittings. This means that they are related via the Deck transformation coming from $R^1p_*\mathbb{Z}(1)$ and so by definition, the two torsors $\mathcal{F}_1$ and $\mathcal{F}_2$ agree. Furthermore, this also shows that the map F is injective, and hence it's an isomorphism as desired. $\square$

Therefore, if we would like to understand if $\mathrm{sec}_{\mathrm{ab}}$ is injective/surjective, it's enough to understand if ψ is injective/surjective.

Proposition 3.1.2. *Let $p : \mathcal{A} \to B$ be an abelian scheme. Then the section map $\phi : H^0(B, \mathcal{A}) \to H^1(\pi_1(B), H_1(\mathcal{A}_b, \mathbb{Z}))$ is a injective if and only if the monodromy action of $\pi_1(B)$ on $H^1(\mathcal{A}_b, \mathbb{Z})$ has no invariant factors.*

Proof. First suppose that the monodromy action has no invariant factors. By Lemma 3.1.1, it's enough to show that in this case, $H^0(B, R^1p_*\mathcal{O}_{\mathcal{A}}) = 0$. Consider the Higgs bundle associated to the variation of Hodge structure $R^1p_*\mathbb{Z}$:

$$\mathcal{E} := p_*\omega_{\mathcal{A}/B} \oplus R^1p_*\mathcal{O} \xrightarrow{\theta} p_*\omega_{\mathcal{A}/B} \oplus R^1p_*\mathcal{O} \otimes \omega_B,$$

where the Higgs field θ is defined by the following two maps

$$p_*\omega_{\mathcal{A}/B} \xrightarrow{\nabla} R^1p_*\mathcal{O} \otimes \omega_B$$
$$R^1p * \mathcal{O} \xrightarrow{\text{zero map}} p_*\omega_{\mathcal{A}/B} \oplus R^1p_*\mathcal{O} \otimes \omega_B$$

Here ∇ is the flat connection associated to the vector bundle $\pi_*\omega_{\mathcal{A}/B}$.

Now if $H^0(B, R^1p_*\mathcal{O}_{\mathcal{A}}) \neq 0$, then $\mathcal{O}_B$ maps into $R^1p_*\mathcal{O}_A$, and hence $(\mathcal{O}, 0)$ is a sub-Higgs bundle of $(R^1p_*\mathcal{O}_A, 0)$ and hence a sub-Higgs bundle of $(\mathcal{E}, \theta)$. On the other hand, by a theorem of Simpson [48, Theorem 1], Higgs bundles associated to a variation of Hodge structure are polystable, i.e., it's a direct sum of stable Higgs bundles of the same slope. Since $\mathcal{O}$ is a line bundle, we see that $(\mathcal{O}, 0)$ must be one of the irreducible factors of $(\mathcal{E}, \theta)$. In particular, by the non-abelian Hodge correspondence, the trivial representation should appear as a sub-representation of the monodromy representation associated to $R^1p_*\mathbb{Z}$. This contradicts the fact that the monodromy representation no invariant factors.

Now suppose the section map ϕ is injective. Again by Lemma 3.1.1, we know that $H^0(B, R^1p_*\mathbb{Z}(1)) \cong H^0(B, R^1p_*\mathcal{O})$. However, the former is a discrete group whereas the latter is a vector space over $\mathbb{C}$ and hence they are isomorphic if and only if both are 0. In particular, the invariants $H^0(B, R^1p_*\mathbb{Z}(1)) = H^0(B, R^1p_*\mathbb{Z})(1)$ of the local system $R^1p_*\mathbb{Z}(1)$ is trivial. It follows that the action of $\pi_1(B)$ on $H^1(\mathcal{A}_b, \mathbb{Z})$ has no invariants, as desired. $\square$

We can immediately deduce the following corollary for family of Jacobians associated to a Kodaira fibration:

Corollary 3.1.3. *Let $f : X \to B$ be a Kodaira fibration and $p : \mathrm{Pic}^0_{X/B} \to B$ be the corresponding family of Jacobians. Then*

$$\phi : H^0(X, \mathrm{Pic}^0_{X/B}) \to H^1(\pi_1(B), H_1(X_b, \mathbb{Z}))$$

is injective if and only if the associated monodromy action on $H^1(X_b, \mathbb{Z})$ has no invariants.

Furthermore, the abelian version of the section question is also related to the original topological section question for Kodaira fibrations:

Corollary 3.1.4. *Let $f : X \to B$ be a Kodaira fibration whose monodromy action on $H^1(X_b, \mathbb{Z})$ has no invariants, then the corresponding map*

$$\{\textit{algebraic sections to } f : X \to B\} \to \{\textit{sections of } (3.0.1)\}\,/\,\textit{conjugation}$$

is injective.

Proof. If $f : X \to B$ has no sections, then the statement is trivially true so let's assume that we have a fixed section $s_0 : B \to X$. Then we may define a B-morphism $h : C \to \mathrm{Pic}^0_{X/B}$ which maps $x \in X_b$ to the divisor class $[s_0(f(x)) - x]$. Note that h is injective, as it's just the Abel-Jacobi map on each fiber.

Now recall that we have the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_b) \arrow[r] \arrow[d] & \pi_1(X) \arrow[r] \arrow[d] & \pi_1(B) \arrow[r] \arrow[d, "="] & 1 \\
0 \arrow[r] & H_1(X_b, \mathbb{Z}) = \pi_1(X_b)^{\mathrm{ab}} \arrow[r] & \pi_1(\mathrm{Pic}^0_{X/B}) \arrow[r] & \pi_1(B) \arrow[r] & 1
\end{tikzcd}$$

Let s and s' be two distinct algebraic sections of $f : X \to B$. By post-composing with h and using the injectivity of h, we get two distinct sections $\tilde{s}$ and $\tilde{s}'$ of $\pi : \mathrm{Pic}^0_{X/B} \to B$. If s and s' are conjugate via some element $g \in \pi_1(X_b)$, then $\tilde{s}$ and $\tilde{s}'$ must be conjugate via the image of g in $H_1(X_b, \mathbb{Z})$, contradicting Corollary 3.1.3. Hence, the map from algebraic sections to group theoretic splittings modulo conjugation is injective as desired. □

Remark 3.1.5. Note that by Lemma A.0.1, we see that the Kodaira fibration constructed using the moduli construction will satisfy the assumption of Corollary 3.1.4. On the other hand, many classical constructions of Kodaira fibration (including Kodaira's original construction [30]) involves taking branched covers of a product of curves and so typically the monodromy action will have invariants. Bregman gave a partial converse to this observation in [5] when the dimension of the invariants is small.

3.2 Family of Jacobians and non-surjectivity

In this section, we come back to the special case of family of Jacobians $p : \mathrm{Pic}^0_{X/B} \to B$ associated to a Kodaira fibration. The goal is to show that under the assumption that the Kodaira fibration $f : X \to B$ admits an algebraic section, the map $\mathrm{sec}_{\mathrm{ab}}$ is never surjective. Note that the existence of such Kodaira fibrations is guaranteed by Proposition A.0.2.

In view of Lemma 3.1.1, it's enough to show that the connecting homomorphism $\psi : H^0(C, \mathrm{Pic}^0_{X/B}) \to H^1(C, R^1p_*\mathbb{Z}(1))$ is not surjective. Since we have an algebraic section, we get a map $h : X \to \mathrm{Pic}^0_{X/B}$, which induces canonical isomorphisms $R^1p_*\mathbb{Z}(1) \cong R^1f_*\mathbb{Z}(1)$ and $R^1p_*\mathcal{O} \cong R^1f_*\mathcal{O}$. Therefore, we may work within the relative curve setting and instead study the map $H^1(B, R^1f_*\mathbb{Z}(1)) \to$

$H^1(B, R^1 f_* \mathcal{O}_X)$. To understand this map, we first compute the degree of the vector bundle $R^1 f_* \mathcal{O}_X$. This computation is probably known to experts but we could not find a reference:

Lemma 3.2.1. *The vector bundle $R^1 f_* \mathcal{O}_X$ is of negative degree.*

Proof. Let $f : X \to B$ be a Kodaira fibration such that the fiber X_b has genus g and the base B has genus h. By Grothendieck-Riemann-Roch, we know that

$$\mathrm{ch}(f_!(\mathcal{O}_X)) = f_*(\mathrm{ch}(\mathcal{O}_X) \cdot \mathrm{td}_{X/B}),$$

where ch denotes the Chern character, and $\mathrm{td}_{X/B}$ is the Todd class of the relative tangent bundle $\mathcal{T}_{X/B}$. Since we are in the relative curve setting, we know that the higher derived pushforward $R^i f_*(\mathcal{O}_X)$ vanishes for all $i \geq 2$, so we can rewrite the the left hand side of the equation to get

$$\begin{aligned}
\mathrm{ch}(f_!(\mathcal{O}_X)) &= \mathrm{ch}(f_* \mathcal{O}_X) - \mathrm{ch}(R^1 f_*(\mathcal{O}_X)) \\
&= \mathrm{ch}(\mathcal{O}_B) - \mathrm{ch}(R^1 f_*(\mathcal{O}_X)) \\
&= 1 - (\mathrm{rk}(R^1 f_*(\mathcal{O}_X)) + c_1(R^1 f_*(\mathcal{O}_X)) + \dots).
\end{aligned}$$

On the other hand, since $\mathrm{ch}(\mathcal{O}_X) = 1$, we see that the right hand side is simply

$$\begin{aligned}
f_* \,\mathrm{td}_{X/B} &= f_*(\mathrm{td}(\mathcal{T}_{X/B})) \\
&= f_* \left(1 + \frac{c_1(\mathcal{T}_{X/B})}{2} + \frac{c_1^2(\mathcal{T}_{X/B}) + c_2(\mathcal{T}_{X/B})}{12} + \dots \right).
\end{aligned}$$

As $\mathcal{T}_{X/B}$ is a line bundle, we know that it has no higher Chern classes. It follows then

$$\deg R^1 f_*(\mathcal{O}_X) = c_1(R^1 f_*(\mathcal{O}_X)) = -f_* \left(\frac{c_1^2(\mathcal{T}_{X/B})}{12} \right).$$

Thus, it's enough to compute $f_*(c_1^2(\mathcal{T}_{X/B}))$. Since $\mathcal{T}_{X/B} = \Omega_{X/B}^\vee$, we know that $c_1^2(\mathcal{T}_{X/B}) = c_1^2(\Omega_{X/B})$ so we can work with the relative differential. Consider the following short exact sequence

$$0 \to f^* \Omega_B \to \Omega_X \to \Omega_{X/B} \to 0.$$

By taking the wedge power, we get the following isomorphism

$$\wedge^2 \Omega_X \cong f_* \Omega_B \otimes \Omega_{X/B}.$$

Since c_1 is a group homomorphism, we know that $c_1(\wedge^2 \Omega_X) = c_1(f_* \Omega_B) + c_1(\Omega_{X/B})$. Then

$$\begin{aligned}
c_1(\Omega_{X/B})^2 &= (c_1(\wedge^2 \Omega_X))^2 - 2c_1(\wedge^2 \Omega_X) \cdot c_1(f^* \Omega_B) + (c_1(f^* \Omega_B))^2 \\
&= K_X^2 - 2K_X \cdot c_1(f^*(K_B)) + (c_1(f^*(K_B))^2)
\end{aligned}$$

where K_X is a canonical divisor on X, and K_B is a canonical divisor on B. Now since c_1 is functorial and f^* is a ring homomorphism, we know that

$$(c_1(f^*(K_B)))^2 = f^*(c_1(K_B)^2).$$

Because B is a curve, this has to vanish. It follows then

$$(c_1(\Omega_{X/B}))^2 = K_X^2 - 2K_X \cdot f^*K_B$$

and hence

$$f_*((c_1(\Omega_{X/B}))^2) = f_*(K_X^2) - 2f_*(K_X \cdot f^*K_B).$$

Since we may compute degrees both before and after pushing-forward, we know that $f_*(K_B^2) = K_B^2$. To understand the last term, we first use projection formula to write it as

$$2f_*(K_X \cdot f^*K_B) = 2f_*(K_X) \cdot K_B.$$

Now $\deg f_*(K_X) = K_X \cdot X_b$, and because $K_X \cdot X_b = \deg K_X|_{X_b} = \deg K_{X/B}|_{X_b}$ for any generic fiber X_b, we see that $K_X \cdot X_b = 2g - 2$,. Now since K_B has degree $2h - 2$, we see that

$$f_*((c_1(\Omega_{X/B}))^2) = K_X^2 - 8(g-1)(h-1).$$

Hence, we have

$$\deg R^1 f_* \mathcal{O}_X = c_1(R^1 f_* \mathcal{O}_X) = \frac{-K_X^2 + 8(g-1)(h-1)}{12} = \frac{-K_X^2 + 2\chi_X}{12},$$

where χ_X is the Euler characteristics of X. Finally, by the signature formula of Hirzebruch, Atiyah and Singer, we know that the signature $\sigma(X)$ of X is precisely given by

$$\sigma(X) = \frac{1}{3}(K_X^2 - 2\chi_X).$$

Since Kodaira fibrations necessarily have positive signatures [8, Corollary 42], $\deg R^1 f_* \mathcal{O}_X < 0$ as desired. □

Corollary 3.2.2. *$\dim H^1(B, R^1 f_* \mathcal{O}_X) > 3$.*

Proof. By Riemann-Roch, we know that

$$\dim H^1(B, R^1 f_* \mathcal{O}_X) = -\left(\deg R^1 f_* \mathcal{O}_X + \mathrm{rk}(R^1 f_* \mathcal{O}_X)(1-h) - \dim H^0(B, R^1 f_* \mathcal{O}_X)\right).$$

Lemma 3.2.1 says that $-\deg R^1 f_* \mathcal{O}_X > 0$. Furthermore, by [28, Theorem 1.1], the base curve of a Kodaira fibration has genus at least 2 and the fiber has genus at least 3, so we know that

$$\dim H^1(B, R^1 f_* \mathcal{O}_X) > \mathrm{rk}(R^1 f_* \mathcal{O}_X)(h-1) \geq 3 \cdot 1 = 3,$$

as desired. □

We will use this corollary to show that $H^1(B, R^1 f_* \mathbb{Z}) \to H^1(B, R^1 f_* \mathcal{O}_X)$ is non-zero.

Proposition 3.2.3. *The map $\theta : H^1(B, R^1 f_*\mathbb{Z}(1)) \to H^1(B, R^1 f_*\mathcal{O}_X)$ is non-zero.*

Proof. Note that there's a map from $\Theta : H^2(X, \mathbb{Z}(1)) \to H^2(X, \mathcal{O}_X)$ and by Leray spectral sequence, both cohomology groups are equipped with Leray filtrations. Furthermore, the map $\theta : H^1(B, R^1 f_*\mathbb{Z}(1)) \to H^1(B, R^1 f_*\mathcal{O}_X)$ may be viewed as the map induced by Θ between the associated graded pieces of the Leray filtration. Because the map $f : X \to B$ admits a section, the Leray filtration splits, and so we may view $H^1(B, R^1 f_*\mathbb{Z}(1))$ (resp. $H^1(B, R^1 f_*\mathcal{O}_X)$) as a subgroup of $H^2(B, \mathbb{Z}(1))$ (as a subspace of $H^2(S, \mathcal{O}_X)$). As the map Θ factors through $H^2(X, \mathbb{C}(1))$, we have the following commutative diagram:

$$\begin{tikzcd}
H^2(X,\mathbb{Z}(1)) \arrow[r] & H^2(X,\mathbb{C}(1)) \arrow[r] & H^2(X,\mathcal{O}_X) \\
H^1(B,R^1f_*\mathbb{Z}(1)) \arrow[u] \arrow[rr, "\theta"'] & & H^1(B,R^1f_*\mathcal{O}_X) \arrow[u]
\end{tikzcd}$$

Notice that the map from $H^2(X, \mathbb{C}(1)) \to H^2(X, \mathcal{O}_X)$ is induced by the exponential map (twisted by $\mathbb{Z}(1)$) and hence agrees with the projection map coming from Hodge decomposition and in particular is surjective.

Since $f : X \to B$ is a relative curve over another curve, we see that $H^2(X, \mathcal{O}_X) \cong H^1(B, R^1 f_*\mathcal{O}_X)$ and hence by Corollary 3.2.2, dim $H^2(X, \mathcal{O}_X)$ has dimension at least 4. On the other hand, as both $H^2(B, \mathbb{Z}(1))$ and $H^0(B, R^2 f_*\mathbb{Z}(1))$ are of dimension 2, we see that $H^2(B, R^1 f_*\mathbb{Z}(1))$ is a lattice that generates a subspace of codimension 2 inside $H^2(X, \mathbb{C}(1))$. In particular, for the projection map to be surjective, the restriction onto the lattice $H^1(B, R^1 f_*\mathbb{Z}(1))$ has to be non-zero. Hence, θ is non-zero, as desired. □

Finally, using Lemma 3.1.1, we arrive at the following conclusion.

Corollary 3.2.4. *Let $f : X \to B$ be a Kodaira fibration with an algebraic section, and $p : \mathrm{Pic}^0_{X/B} \to B$. The map sec_{ab} is never surjective for these families of Jacobians.*

Remark 3.2.5. In the case where $f : X \to B$ has an algebraic section, one may identify $\mathrm{Pic}^0_{X/B}$ with $\mathrm{Pic}^1_{X/B}$, and hence our results shows that the topological section question has a negative answer for $\mathrm{Pic}^1_{X/B} \to B$ in the case where the associated Kodaira fibration has a section. This in fact differs from the universal case: the universal family of moduli space of degree 1 line bundles $p' : \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g} \to \mathcal{M}_g$ does satisfies the topological section question when $g \geq 3$ in a trivial way, i.e., there's no topological section to p' (this is first proven by Morita when $g \geq 9$; see [43, Corollary 3, Theorem 4]. The strengthened result is proven in [37]).

3.3 A note on weak topological section question

In this section, we work with the general case of a smooth projective non-isotrivial family of curves $f : X \to B$. We are interested in understanding the surjectivity of the map $\mathrm{sec}_{\mathrm{top}}$ and we claim that, similar to the arithmetic setting described in section 2, the surjectivity of $\mathrm{sec}_{\mathrm{top}}$ is related to a "weak topological section conjecture". More precisely, we have the following two statements

Statement 3.3.1 (Weak topological section conjecture)**.** The map $X \to B$ admits an algebraic section if and only if the associated short exact sequence of topological fundamental groups split.

Statement 3.3.2 (Surjectivity of top. section question)**.** The topological section map

$$\mathrm{sec}_{\mathrm{top}} : \{\text{algebraic sections to } f : X \to B\} \to \{\text{splittings of } (3.0.1)\}/\sim$$

is surjective.

We prove the following proposition:

Proposition 3.3.3. *Let $f : X \to B$ be a smooth projective non-isotrivial family of curves whose fibers are of genus at least 2. Then Statement 3.3.2 being true for $X \to B$ is equivalent to Statement 3.3.1 being true for all finite étale connected covers X' of X such that the composed map $X' \to B$ has connected fibers.*

Our proof is very much inspired by the proof in the arithmetic setting. To distinguish an algebraic section of $f : X \to B$ and a group theoretic section of (3.0.1), we will denote the former s and the latter x.

Definition 3.3.4 (Neighbourhood of a section x)**.** Let $x : \pi_1(B) \to \pi_1(X)$ be a group theoretic splitting of the short exact sequence of topological fundamental groups (3.0.1). Then a *neighbourhood* of x is a finite étale connected cover S' of S such that $S' \to C$ has connected fibers and the finite index subgroup $\pi_1(S') \subset \pi_1(S)$ contains the image $x(\pi_1(C))$ of the section x.

Note that given a neighbourhood of x, we get a lift of x to a group theoretic section $x' : \pi_1(B) \to \pi_1(X')$ of the short exact sequence of fundamental groups associated to $X' \to B$. Furthermore, if we post-compose x' with the natural inclusion map $\pi_1(X') \to \pi_1(X)$, we recover the section x so one may alternatively define a neighbourhood of x as a pair (X', x') of finite étale covers S' with connected fibers over B and a group theoretic section x' which descends to x.

Lemma 3.3.5. *Let $x = x_s$ be a geometric section, i.e., it's induced by some algebraic section $s : B \to X$, then a neighbourhood of x is the same as pair $(X', x_{s'})$, where X' is a finite étale connected cover of X with connected fibers over B, and $x_{s'}$ is a group theoretic section induced by some algebraic section $s' : B \to X$ that is a lift of s.*

Proof. Recall that a finite étale connected covers is the same as a finite set with a transitive $\pi_1(X)$ action. In this case, the finite set is given by the set $\pi_1(X)/\pi_1(X')$ of cosets of $\pi_1(X')$. Using the section $x_s : \pi_1(B) \to \pi_1(X)$, we get an induced action of $\pi_1(B)$ on this set. Since $\pi_1(X')$ contains $\pi_1(B)$, it follows that this action has a fixed point. Therefore, the cover that corresponds to this action of $\pi_1(B)$ is disconnected and has a copy of B. Hence, we may lift the section s to an algebraic section $s' : B \to X$. □

Given a group theoretic splitting $x : \pi_1(B) \to \pi_1(X)$, let X_x be the pro-étale cover of X defined by the the projective system $(X' \to X)$, where X' runs over all neighbourhoods of x.

Lemma 3.3.6. *Let x_1 and x_2 be two group theoretic sections. Then $X_{x_1} = X_{x_2}$ if and only if they are conjugate to each other.*

Proof. It's enough to show that $\pi_1(X_x) = x(\pi_1(C))$. This is the case since $X_{x_1} = X_{x_2}$ is equivalent to $\pi_1(X_{x_1})$ being conjugate to $\pi_1(X_{x_2})$. Hence, if $\pi_1(X_x) = x(\pi_1(B))$, then $\pi_1(X_{x_1})$ being conjugate to $\pi_1(X_{x_2})$ is equivalent to x_1 being conjugate to x_2.

Now to see that $\pi_1(X_x) = s(\pi_1(B))$, consider all finite index subgroups H of $\pi_1(X)$ containing $x(\pi_1(B))$. Observe that $\pi_1(X_x) = \bigcap H$ so it's enough to show that $x(\pi_1(B)) = \bigcap H$. Since we have a section, $\pi_1(X)$ can be written as a semi-direct product $\pi_1(X) \cong \pi_1(X_b) \rtimes \pi_1(B)$. Let $N_i := \bigcap_{[\pi_1(X_b):H]=i} H \subset \pi_1(X_b)$. Note that because $\pi_1(X_b)$ is finitely generated, it admits finitely many maps into the symmetric group S_i and hence there are only finitely many index i subgroups of $\pi_1(X_b)$. In particular, this intersection is finite and N_i is again of finite index. Since every automorphism of $\pi_1(X_b)$ preserves the index of a subgroup, we see that N_i is also characteristics. It follows that $N_i x(\pi_1(B))$ are finite index subgroups of $\pi_1(X)$. Since fundamental group of a surface is residually finite [25], we know that $\bigcap N_i$ is trivial. It follows that $N_i x(\pi_1(B)) = x(\pi_1(B))$ and hence the intersection of all finite index subgroups of $\pi_1(X)$ containing $x(\pi_1(B))$ is $x(\pi_1(B))$ as desired. $\square$

Combining these two lemmas, we may give a characterization of group theoretic sections that come from algebraic geometry:

Corollary 3.3.7. *A group theoretic section x is conjugate to x_s for some algebraic section $s : B \to X$ if and only if s belongs to the image of the natural map $X_x(B) \to X(B)$, where $X_x(B)$ is the set of algebraic sections of $X_x \to B$ and $X(B)$ is the set of algebraic sections of $X \to B$.*

Proof. If $x : B \to X$ is an algebraic section, by Lemma 3.3.5, it lifts to a compatible system of algebraic sections and hence it lifts to an algebraic section of $X_x \to B$.

Conversely, if s is in the image of $X_x(B) \to X(B)$, the section x and x_s have the same collection of neighbourhood and so are conjugate to each other by Lemma 3.3.6. $\square$

Finally, we need the following lemma to prove the main proposition.

Lemma 3.3.8. *Let $f : X \to B$ be any non-isotrivial smooth projective family of curves of genus at least* 1. *Then the set of algebraic sections is finite.*

Proof. When B is a curve, this is the content of the geometric Mordell conjecture [39]. Now suppose $\dim B > 1$. If $X \to B$ has infinitely many algebraic sections, note that the locus where these infinitely many algebraic sections agree is a countable union of closed subvarieties of B, and hence we may find a smooth proper curve $C \subset B$ such that the restriction of all these sections are distinct. Furthermore, we may also choose C such that the map from B to $\mathcal{M}_g$ restricts to a non-constant map on C. This then gives us a Kodaira fibration with infinitely many algebraic sections, contradicting the geometric Mordell conjecture. $\square$

Now we are ready to prove Proposition 3.3.3.

Proof of Prop. 3.3.3. First let's deduce the weak topological section conjecture for connected finite étale covers with connected fibers over B from the surjectivity of the section map for $f : X \to B$. Suppose there exists a connected finite étale cover $X' \to X$ with connected fibers over B and a group theoretic section x'. Then x' descends to a group theoretic section from $\pi_1(B) \to \pi_1(X)$ and hence by the surjectivity, there exists an algebraic section from $B \to X$. By Lemma 3.3.5, we may lift this to an algebraic section from B to X', and hence the weak topological section question holds for $X' \to B$.

For the other direction, by Corollary 3.3.7, it's enough to show that $X_x(B) = \varprojlim X'(B)$ is non-empty, where $X'(B)$ is the set of algebraic sections of $X' \to B$. Since every neighbourhood X' has a topological section by definition, it follows from the weak topological section conjecture that $X'(B)$ is non-empty. It's finite by Lemma 3.3.8, and therefore it's a non-empty set with a compact Hausdorff topology. Then such projective limit is always non-empty as desired. □

3.4 Kodaira fibrations and non-surjectivity after S. Lee and C. Serván

In this section, we record and slightly generalize an example due to Seraphina Lee and Carlos Serván of Kodaira fibrations with infinitely many π_1-sections [35]. We thank Seraphina and Carlos for generously allowing us to include their construction in this thesis. The construction is similar to the one used in their recent paper [36]. In view of the geometric Mordell conjecture, $\mathrm{sec}_{\mathrm{top}}$ is not surjective for such Kodaira fibrations. More precisely, we prove the following result:

Proposition 3.4.1. *Let $f : X \to B$ be a Kodaira fibration. Then there exists a branched cover $B' \to B$ such that the base change $X' := X \times_B B' \to B'$ is a Kodaira fibration with infinitely many π_1-sections up to conjugation.*

Let $f : X \to B$ be a Kodaira fibration whose fiber X_b has genus g. Since we can always replace B by a finite cover, by Proposition A.0.2, we may without loss of generality assume that $f : X \to B$ has an algebraic section. Let $B' \to B$ be a double cover of B branched at 2 points and consider the base change $f' : X' := B' \times_B X \to B'$. We would like to give a topological description of this fiber bundle. First, we introduce some notations.

Let Σ_g be a closed smooth compact orientable genus g surface. Then X_b is homeomorphic to Σ_g for all $b \in B$. Let $B_1 = B_2 = B - D^2$ for some open disk $D^2 \subset B$. Then B' can be constructed by gluing together B_1 and B_2 along the boundaries $\partial B_1 = \partial B_2 = S^1$ via the reflection map r. Similarly, set $X_i := f^{-1}(B_i)$. Notice that since D^2 is contractible, the family $f|_{X_i} : X_i \to B_i$ must have trivial monodromy along the boundary $\partial B_i = S^1$ and hence the fiber bundle restricts to the trivial one $S^1 \times \Sigma_g \to S^1$ along the boundary. Then, X' can be constructed by gluing together X_1 and X_2 along the boundary $\partial X_1 = \partial X_2 = S^1 \times \Sigma_g$ via the map (r, id).

It follows from Van Kampen theorem that

$$\pi_1(B') = \pi_1(B_1) *_{\pi_1(S^1)} \pi_1(B_2) \text{ and } \pi_1(X') = \pi_1(X_1) *_{\pi_1(S^1 \times \Sigma_g)} \pi_1(X_2).$$

Let s_i be the restriction of s to B_i. They certainly induce group theoretic sections $(s_i)_* : \pi_1(B_i) \to \pi_1(X_i)$.

Definition 3.4.2. Let $\gamma \in \pi_1(\Sigma_g)$. Define a group theoretic $s_\gamma : \pi_1(B') \to \pi_1(X')$ by

$$s_\gamma(l) = \begin{cases} (s_1)_*(l) & \text{if } l \in \pi_1(X_1) \\ \gamma(s_2)_*(l)\gamma^{-1} & \text{if } l \in \pi_1(X_2) \end{cases}$$

Lemma 3.4.3. *The sections s_γ is well-defined.*

Proof. Let $l \in \pi_1(S^1) = \pi_1(\partial B_1) = \pi_1(\partial B_2)$. Note that the restriction of the section to the boundary

$$((s_i)_*)|_{\pi_1(\partial B_i)} : \pi_1(S^1) \to \pi_1(S^1 \times \Sigma_g) = \pi_1(S^1) \times \pi_1(\Sigma_g)$$

is simply $(\mathrm{id}, 1)$. In particular, the action of $\pi_1(\Sigma_g)$ commutes with the section and so $\gamma(s_2)_*(l)\gamma^{-1} = (s_2)_*(l) = (s_1)_*(l)$ as desired. □

Now Proposition 3.4.1 follows from the following lemma:

Lemma 3.4.4. *If two group theoretic sections s_γ is conjugate to s_δ, then $[\gamma] - [\delta] \in W$, where W is the invariants of the monodromy action of $\pi_1(B)$ on $H_1(\Sigma_g, \mathbb{Z})$.*

Proof. Suppose that $s_\gamma = m s_\delta m^{-1}$ for some $m \in \pi_1(\Sigma_g)$. More explicitly, this means that

$$\begin{cases} (s_1)_*(l) = m(s_1)_*(l)m^{-1} & \text{if } l \in \pi_1(B_1) \\ \gamma(s_2)_*(l)\gamma^{-1} = m\delta(s_2)_*(l)\delta^{-1}m^{-1} & \text{if } l \in \pi_1(B_2) \end{cases}$$

We know that $\pi_1(B_1) = \pi_1(B_2)$ and $\pi_1(X_1) = \pi_1(X_2)$. Furthermore, under this identification, $(s_1)_*(l) = (s_2)_*(l) \in \pi_1(X_1) = \pi_1(X_2)$ for every $l \in \pi_1(B_1) = \pi_1(B_2)$. On the other hand, the conjugation relation says that in $\pi_1(X_i)$ we have

$$m(s_1)_*(l)m^{-1} = (s_1)_*(l) = (s_2)_*(l) = \gamma^{-1}m\delta(s_2)_*(l)\delta^{-1}m^{-1}\gamma.$$

This means that

$$(s_2)_*(l) = (s_1)_*(l) = (m^{-1}\gamma^{-1}m\delta)(s_2)_*(l)(\delta^{-1}m^{-1}\gamma m)$$

In particular, we see that $(s_2)_*(l)$ commutes with $m^{-1}\gamma^{-1}m\delta$ for all $l \in \pi_1(B_2)$. Therefore the monodromy action of $\pi_1(B_2)$ on $H_1(\Sigma_g, \mathbb{Z})$ must fixes $[m^{-1}\gamma^{-1}m\delta]$. In other words, $[m^{-1}\gamma^{-1}m\delta] = [-\gamma] + [\delta] \in W$, as desired. □

Corollary 3.4.5. *There exists a Kodaira fibration with a continuous section but no algebraic sections*

Proof. This follows directly from Proposition 3.3.3. □

Chapter 4

Hodge theoretic fundamental groups and section conjecture

In this section, we formulate and study a Hodge theoretic analogue of Grothendieck's section conjecture for families of curves and study many natural questions one might ask in this Hodge theoretic setting. The main player here is the category of variation of mixed Hodge structure. We first introduce some notations

4.1 Notation

Let X be a smooth connected variety over $\mathbb{C}$. Let $\mathrm{VMHS}_R(X)$ be the category of admissible, graded-polarizable, R-variation of mixed Hodge structures (VMHS) over X, where R is either $\mathbb{Z}$ or $\mathbb{Q}$. For a precise definition of such an object, see [29]. Note that this is an abelian tensor category. The unit object in this category is the constant variation of mixed Hodge structure $\underline{R(0)}$ and by trivial objects, we mean direct sums of the unit objects. Finally, let $\mathrm{LS}^{\mathrm{hdg}}(X)$ be the category of R-local systems on X which are subquotients of local systems underlying a variation of mixed Hodge structures.

Suppose X and Y are two smooth complex varieties. Any functor $F : \mathrm{VMHS}_R(X) \to \mathrm{VMHS}_R(Y)$ is assumed to be exact, additive $\otimes$-functor. Note that if we have a map $f : Y \to X$, then the pull-back functor f^* satisfies these assumptions.

Remark 4.1.1. Admissibility is a technical condition on the behavior of a variation of mixed Hodge structure at infinity that will not play a big role in this thesis. The main point is that it ensures the variation of mixed Hodge structure to have some nice properties, which all variations of mixed Hodge structure of geometric origin have. Any VMHS that comes from geometry is admissible. If one wants to ignore this technical point, one may assume that all the spaces in this chapter are compact, in which case all graded-polarizable variation of R-mixed Hodge structures are automatically admissible.

4.2 Formulation of the question and injectivity for families of curves

For this section, assume $R = \mathbb{Z}$. Let $f : X \to B$ be a smooth projective morphism with connected fibers between two smooth connected varieties. Let X_b be the fiber over $b \in B$, and $\iota : X_b \to X$ the natural inclusion map. We have the following sequence

$$\mathrm{VMHS}_{\mathbb{Z}}(B) \xrightarrow{f^*} \mathrm{VMHS}_{\mathbb{Z}}(X) \xrightarrow{\iota^*} \mathrm{LS}^{\mathrm{hdg}}(X_b)$$

where f^* is the pullback functor induced by f, and ι^* is defined by first pulling back a variation of mixed Hodge structure on X to a fiber and then taking the underlying local system. Furthermore, if $s : B \to X$ is an algebraic section to f, we get a functor $s^* : \mathrm{VMHS}_{\mathbb{Z}}(X) \to \mathrm{VMHS}_{\mathbb{Z}}(B)$ such that $f^* \circ s^* = \mathrm{id}_{\mathrm{VMHS}_{\mathbb{Z}}(Y)}$. We can make a formal definition:

Definition 4.2.1. A functor $F : \mathrm{VMHS}_{\mathbb{Z}}(X) \to \mathrm{VMHS}_{\mathbb{Z}}(B)$ is a section to f^* if $f^* \circ F$ is isomorphic to the identity functor on $\mathrm{VMHS}_{\mathbb{Z}}(B)$.

Now we formulate the question we are interested in:

Question 4.2.2 (Hodge theoretic section question)**.**

1. (injectivity) If s_1, s_2 are two distinct algebraic sections to $f : X \to B$, can the functors s_1^* and s_2^* be isomorphic?
2. (surjectivity) Suppose that $F : \mathrm{VMHS}_{\mathbb{Z}}(X) \to \mathrm{VMHS}_{\mathbb{Z}}(B)$ is a functor which is a section to f^*. Then can we find an algebraic section $s : B \to X$ such that F is isomorphic to s^*?

In this section, we study Question 4.2.2 for families of curves. Let $f : X \to B$ be a smooth projective family of curves of genus ≥ 1. We would like to prove the following proposition:

Proposition 4.2.3. *For any pair of algebraic sections $s_1, s_2 : B \to X$, if s_1^* is isomorphic to s_2^* as functors from $\mathrm{VMHS}_{\mathbb{Z}}(X) \to \mathrm{VMHS}_{\mathbb{Z}}(B)$, then $s_1 = s_2$.*

To prove this proposition, we need to find a graded-polarizable, admissible $\mathbb{Z}$-variation of mixed Hodge structure on X whose associated period map is injective (or at least injective on each fiber). We do so by using the canonical variation of mixed Hodge structure of Hain and Zucker. We first recall some definitions and facts.

Let X be a smooth algebraic variety over $\mathbb{C}$, and let PX be the space of piecewise-smooth paths in X endowed with the compact open topology. The free path fibration $p : PX \to X \times X$ is defined as

$$\begin{aligned} p : PX &\to X \times X \\ \gamma &\mapsto (\gamma(0), \gamma(1)) \end{aligned}$$

Denote the $P_{x,y}$ the fiber of $p : PX \to X \times X$ over the point (x, y). Now there's a isomorphism $H_0(P_{x,x}, \mathbb{Z}) \cong \mathbb{Z}[\pi_1(X, x)]$. Let J_x be the augmentation ideal of the group ring $\mathbb{Z}[\pi_1(X, x)]$. Note that $H_0(P_{x,y})$ carries a canonical left $\mathbb{Z}[\pi_1(X, x)]$-module structure, so we get an induced filtration $J^\bullet$ by the augmentation ideal J_x.

Proposition-Definition 4.2.4 (r-th canonical VMHS, Prop. 4.20 + Def. 4.21 of [24])**.** *Let X be a smooth algebraic variety over $\mathbb{C}$ and $x \in X$ a fixed point. Then there exists a graded-polarizable variation $\mathcal{J}_x$ of mixed Hodge structure on X such that for any $y \in X$,*

$$\mathcal{J}_{x,y} := (\mathcal{J}_x)_y = H_0(P_{x,y}, \mathbb{Z})/J^{r+1}.$$

We will in particular be interested in the case where $r = 1$. In this case, we have an extension of mixed Hodge structures [24, Prop. 5.39]

$$0 \to H_1(X, \mathbb{Z}) \to H_1(X, \{x, y\}) \to \mathbb{Z}(0) \to 0$$

In particular, when $x \neq y$, we just have $H_0(P_{x,y}, \mathbb{Z})/J^2 \cong H_1(X, \{x, y\})$. We have the following proposition which classifies such extensions:

Proposition 4.2.5. *[7] Extensions of this form is classified by the Albanese $\mathrm{Alb}(X)$ of X, and the map $y \mapsto H_1(X, \{x, y\})$ agrees with the Albanese mapping with basepoint x:*

$$\begin{aligned} \alpha_x : X &\to \mathrm{Alb}(X) := F^1 H^1(X)^\vee / H_1(X, \mathbb{Z}) \\ y &\mapsto \left(\omega \mapsto \int_\gamma \omega \right) \end{aligned}$$

where γ is any path from x to y.

Theorem 4.2.6. *[24, Cor. 5.40] This period map α_x agrees with the period map for the 1-st canonical VMHS.*

Corollary 4.2.7. *When (X, x) is a curve, the 1-st canonical VMHS on X with base point x has injective period map.*

Now we may proceed to the proof of Proposition 4.2.3.

Proof of Prop. 4.2.3. If $f : X \to B$ has no section, then the claim is trivially true, so we may without loss of generality assume that we have an algebraic section $s_0 : B \to X$. Then as before we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"] & \mathrm{Pic}^0_{X/B} \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

The fibers of $\pi : \mathrm{Pic}^0_{X/B} \to B$ is $\mathrm{Jac}(X_b) = \mathrm{Alb}(X_b)$, which we may view as a mixed period domain. In particular, $\mathrm{Pic}^0_{X/B}$ carries a universal variation of mixed Hodge structure $\mathcal{U}$ such that for a given point $p \in \mathrm{Jac}(X_b)$, $\mathcal{U}|_p$ is the extension class in $\mathrm{Ext}^1(\mathbb{Z}(0), H_1(X_b))$ corresponding to p. Now pulling back $\mathcal{U}$ along h, we get a variation of mixed Hodge structure $\mathcal{J} := h^*\mathcal{U}$ on $\mathcal{X}$, whose period map factors through h, and which, when restricting to the fiber X_b, agrees with $J_{s_0(b),y}$. By Corollary 4.2.7, we know that $\mathcal{V}$ is injective on each fiber.

Therefore, if s_1^* is isomorphic to s_2^* as functors, then for all $b \in B$, $s_1^*\mathcal{V}|_b \cong s_2^*\mathcal{V}|_b$ or equivalently, $\mathcal{V}|_{s_1(b)} \cong \mathcal{V}|_{s_2(b)}$. By the injectivity of the period map, we see that $s_1(b) = s_2(b)$. Therefore, the desired proposition follows immediately if one can verify that $\mathcal{J}$ is admissible. $\square$

Lemma 4.2.8. *The $\mathbb{Z}$-VMHS $\mathcal{J}$ constructed in the proof above is admissible.*

This lemma, again, is automatically true when $\mathcal{X}$ is proper. It was proven in [22] and we provide a sketch of a proof that's motivated by Beilinson-Deligne-Goncharov's construction of mixed Hodge structures on truncated fundamental groups [13]. For a detailed description of this construction, see section 3.6 of [6].

Sketch of a proof. We claim that this variation of mixed Hodge structures comes from geometry as it's the cohomology of a family of cosimplicial schemes. Since any variation of mixed Hodge structure that comes from geometry is admissible, this proves the desired claim. The main idea is to run the Beilinson-Deligne-Goncharov's construction of the mixed Hodge structure on $\mathbb{Z}[\pi_1(X,x,y)]/J^{r+1}$ in families.

Consider the fiber product

$$\begin{tikzcd} X \times_B X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

We get a fibration $\varphi : X \times_B X \to B$, where over each point $b \in B$, the fiber is given by the product of curves $\varphi^{-1}(b) = X_b \times X_b$. Let $\mathcal{Z}_0$ be the image of the diagonal map $\Delta : X \to X \times_B X$. Let D be the image of the fixed section $s_0 : B \to X$, and let $\mathcal{Z}_1$ be the preimage of D in $X \times_B X$ under the second projection map p_2. Note that $\mathcal{Z}_0 \cap \pi^{-1}(b)$ is the closed subset $Z_0 \subset X_b \times X_b$ defined by $\{x_1 = x_2\}$, where the x_i are coordinates of $X_b \times X_b$ and $\mathcal{Z}_1 \cap \pi^{-1}(b)$ is the closed subset $Z_1 \subset X_b \times X_b$ defined by $\{x_1 = s_0(b)\}$

Let $\underline{\mathbb{Z}}_{\mathcal{Z}_i}$ be the extension by zero of the constant sheaf on $\mathcal{Z}_i$ along the natural inclusion map. We can define the following complex

$$\mathcal{K}_s : 0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}_{\mathcal{Z}_0} \oplus \underline{\mathbb{Z}}_{\mathcal{Z}_1} \to 0,$$

where the map $\underline{\mathbb{Z}} \to \underline{\mathbb{Z}}_{\mathcal{Z}_0} \oplus \underline{\mathbb{Z}}_{\mathcal{Z}_1}$ is given by the alternating sum of the natural restriction map. Note that if we restrict this complex to $\pi^{-1}(b)$, we recover the complex of sheaves on $X_b \times X_b$ used in Beilinson-Deligne-Goncharov's construction

$$\bullet\mathcal{K}_{s(b)}\langle 1\rangle : 0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}_{Z_0} \oplus \underline{\mathbb{Z}}_{Z_1} \to 0.$$

Now the desired variation of mixed Hodge structure agrees with the variation of mixed Hodge structure defined on the local system $R^1(p_1)_*(\mathcal{K}_s)$ on X, whose fiber at $y \in S_b$ is given by the hypercohomology $\mathbb{H}^1(X_{b,y}\mathcal{K}_{s(b)}\langle 1\rangle)$, which agrees with $H^1(X_b, \{s(b), y\})$ when $s(b) \neq y$. When $s(b) = y$, this hypercohomology becomes the split extension of $H^1(X_b, s(b))$ by $\underline{\mathbb{Z}(0)}$. $\square$

4.3 Exact sequence of Hodge theoretic fundamental groups

In this section, we set $R = \mathbb{Q}$. Note that, after fixing a base point $x \in X$ and thus a fiber functor, both $\mathrm{VMHS}_{\mathbb{Q}}(X)$ and $\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(X)$ are Tannakian categories over $\mathbb{Q}$, and thus we may talk about

Tannakian fundamental groups. We will again omit the choice of base point in our notation. This is then more analogous to Grothendieck's anabelian philosophy as we may now use group theoretic information to understand the geometry of algebraic varieties and maps between them. In addition, for technical reasons we shall explain later, we will henceforth require all variations of mixed Hodge structures and local systems to have underlying *integral* structures. For some preliminaries on Tannakian categories, see Appendix B. Now given a Serre fibration of $k(\pi, 1)$-spaces $f : X \to B$, we should get a (long) exact sequence of topological fundamental groups, so it's natural to ask that if we have a smooth projective family $f : X \to B$ with connected fibers between two smooth connected varieties, do we have an analogous exact sequence of Tannakian fundamental groups?

If B is a point, this question is answered in [10], where the authors showed that for any smooth connected complex variety X, we have a short exact sequence of groups

$$1 \to \pi_1^{\mathrm{Tann}}(\mathrm{LS}_{\mathbb{Q}}^{\mathrm{hdg}}(X)) \to \pi_1^{\mathrm{Tann}}(\mathrm{VMHS}_{\mathbb{Q}}(X)) \to \pi_1^{\mathrm{Tann}}(\mathrm{MHS}_{\mathbb{Q}}) \to 1.$$

In this section, we generalize this to the case where B is not a single point. The key theorems we need from the theory of Tannakian categories are summarized in Appendix B.

Proposition 4.3.1. *Given a smooth projective morphism $f : X \to B$ with connected fibers between two smooth connected varieties, we have an exact sequence*

$$\pi_1^{Tann}(\mathrm{LS}_{\mathbb{Q}}^{hdg}(X_b)) \xrightarrow{\pi(\iota)} \pi_1^{Tann}(\mathrm{VMHS}_{\mathbb{Q}}(X)) \xrightarrow{\pi(f)} \pi_1^{Tann}(\mathrm{VMHS}_{\mathbb{Q}}(B)) \to 1 \tag{4.3.1}$$

Proof. To see that $f^*(\mathrm{VMHS}_{\mathbb{Q}}(B)) \subset \mathrm{VMHS}_{\mathbb{Q}}(X)$ is a full subcategory, note that since f is smooth projective and $\mathcal{V}$ is admissible, $f_*\mathcal{V}$ is naturally a variation of mixed Hodge structure. Furthermore, since f has connected fibers, we know that $f_* \circ f^* = \mathrm{id}$. In particular, by adjunction, we know that

$$\mathrm{Hom}(f^*\mathcal{V}, f^*\mathcal{W}) = \mathrm{Hom}(\mathcal{V}, f_*f^*\mathcal{W}) = \mathrm{Hom}(\mathcal{V}, \mathcal{W}).$$

It follows that $f^*(\mathrm{VMHS}_{\mathbb{Q}}(B))$ is a full subcategory. It's closed under taking subobjects because if $\mathcal{W}$ is a subobject of $f^*\mathcal{V}$, then it must come from $f_*\mathcal{W} \subset f_*f^*\mathcal{V} = \mathcal{V}$. By part (1) of Proposition B.0.3, $\pi(f)$ is faithfully flat.

Now $\iota^* \circ f^*$ is certainly trivial, as the objects in the image are local systems underlying a variation of mixed Hodge structure on X_b that are pulled back from a point. Furthermore, the semisimple objects in $\mathrm{VMHS}_{\mathbb{Q}}(X)$ are exactly the pure variation of $\mathbb{Q}$-Hodge structures, and by Deligne's semisimplicity theorem, the underlying local system is also semisimple. Finally, we claim that the maximal trivial subobjects of $\iota^*(\mathcal{V})$ comes from $f_*\mathcal{V}$. It's enough to check this at the level of local systems. Let $\mathcal{U}$ be the maximal trivial part of $\iota^*\mathcal{V}$; it's a constant sheaf with values in $H^0(X_b, \iota^*\mathcal{V})$. Since $\iota^*f^*f_*\mathcal{V}$ is also trivial, there's certainly a map $\iota^*f^*f_*\mathcal{V} \to \mathcal{U}$. It's an isomorphism because the induced maps on stalks are the isomorphism $(f_*\mathcal{V})_b \cong H^0(X_b, \iota_*\mathcal{V})$. Therefore, by Proposition B.0.4, the sequence is exact in the middle. □

Furthermore, we have the following proposition:

Proposition 4.3.2. *Let $f : X \to B$ be as above. Then the map*

$$\pi_1^{Tann}(\mathrm{LS}_{\mathbb{Q}}^{hdg}(X_b)) \to \pi_1^{Tann}(\mathrm{VMHS}_{\mathbb{Q}}(X))$$

is injective if and only if every local system $\mathbb{V} \in \mathrm{LS}_{\mathbb{Q}}^{hdg}(X_b)$ is a subquotient of a local system which extends to a local system $\mathbb{W}$ on all of X with $\mathbb{W}$ underlying a (graded-polarizable, admissible) $\mathbb{Q}$-variation of mixed Hodge structure on X.

Proof. This is a direct consequence of part 2 of Proposition B.0.3. □

Using this proposition and building upon recent work of Landesman-Litt [33] and Lam [31], we prove the following corollary:

Corollary 4.3.3. *Fix $g \geq 2$, and consider the universal family of curves $\mathcal{C}_g \to \mathcal{M}_g$. Then for some $b = [X_b] \in \mathcal{M}_g$, the induced map*

$$\pi_1^{Tann}(\mathrm{LS}_{\mathbb{Q}}^{hdg}(X_b)) \to \pi_1^{Tann}(\mathrm{VMHS}_{\mathbb{Q}}(\mathcal{C}_g))$$

is not injective.

The key to apply results from [33] and [31] is the following lemma.

Lemma 4.3.4. *Let $f : X \to B$ be a smooth projective family of algebraic varieties and let $\mathbb{V}$ be an irreducible $\mathbb{Q}$-local system on X_{b_0} for some $b_0 \in B$. If $\mathbb{V}$ is a subquotient of a local system $\mathbb{W}$ which extends to a local system underlying some $\mathbb{Q}$-variation of mixed Hodge structure in $\mathrm{VMHS}_{\mathbb{Q}}(X)$, then $\mathbb{V}$ underlies a pure $\mathbb{Q}$-variation of Hodge structure on X_b for every $b \in B$.*

Proof. Suppose that $\mathbb{W}$ extends to the local system $\mathbb{W}'$ on all of X, and suppose $\mathcal{W}$ is some variation of mixed Hodge structure in $\mathrm{VMHS}(X)$ whose underlying local system is $\mathbb{W}'$. Then the parallel transport of $\mathbb{V}$ (which we will again call $\mathbb{V}$) must be a subquotient of $\mathbb{W}'|_{X_b}$ for any $b \in B$. It follows that $\mathbb{V}$ is a sub-local system of $\mathrm{Gr}_\bullet\, \mathbb{W}'|_{X_b}$. Since $\mathcal{W}'|_{X_b}$ underlies a $\mathbb{Q}$-variation of mixed Hodge structure, then $\mathrm{Gr}_\bullet\, \mathcal{W}'|_{X_b}$ underlies a $\mathbb{Q}$, and hence $\mathbb{C}$-variation of pure Hodge structures. Since $\mathbb{V}$ is irreducible, non-abelian Hodge correspondence then tells us that $\mathbb{V}$ must underlies a pure $\mathbb{C}$-variation of Hodge structure on X_b for every $b \in B$. Since $\mathbb{V}$ is a $\mathbb{Q}$-local system, we see that it must underlies a $\mathbb{Q}$-variation of pure Hodge structure. □

Therefore to prove that the map $\pi_1^{\mathrm{Tann}}(\mathrm{LS}_{\mathbb{Q}}^{\mathrm{hdg}}(X_b)) \to \pi_1^{\mathrm{Tann}}(\mathrm{VMHS}_{\mathbb{Q}}(\mathcal{C}_g))$ is not injective, it's enough to produce an irreducible $\mathbb{Q}$-local system $\mathbb{V}$ on a curve X_b such that after we deform the algebraic structure on X_b, $\mathbb{V}$ no longer underlies a variation of Hodge structure. This can be done by applying one of the main theorems in Landesman-Litt [33, Theorem 1.2.12] which says that this is possible if the rank of the local system is relatively small compared to the genus. More precisely, suppose X is a curve and $\mathbb{V}$ is a local system on X with rank $\leq 2\sqrt{g+1}$. Then if the isomonodromy deformation of $\mathbb{V}$ to an analytic general nearby curve still underlies a variation of Hodge structure, then $\mathbb{V}$ must be unitary. Therefore, any non-unitary irreducible local systems of small rank on curves of large genera can be used to prove Corollary 4.3.3.

To make things more explicit, and to make our result holds for all $g \geq 2$, we instead opted to slightly generalize the proof of [31, Theorem 1.1]. The main idea is to use any motivic uniformizing Higgs bundles in the sense of Hitchin [26]. Recall that a uniformizing Higgs bundle $(\mathcal{E}, \theta)$ on a smooth projective curve X is defined by $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^\vee$, where $\mathcal{L}$ is some square root of the canonical bundle, $\theta|_{\mathcal{L}}$ is the tautological isomorphism between $\mathcal{L}$ and $\mathcal{L} \otimes \Omega_S$, and $\theta|_{\mathcal{L}^\vee} = 0$. Note that this defines a system of Hodge bundles, with $\mathcal{H}^{1,0} = \mathcal{L}$ and $\mathcal{H}^{0,1} = \mathcal{L}^\vee$ and hence is the Higgs bundle associated to an irreducible local system $\mathbb{V}$ on X which underlies a $\mathbb{C}$ (in fact, $\mathbb{R}$)-variation of Hodge structure. Furthermore, consider the classifying map

$$\phi : \tilde{S} \to \mathcal{H},$$

where $\tilde{X}$ is the universal cover of X, and $\mathcal{H}$ is the upper half plane, viewed as the classifying space of rank 2 weight 1 polarized $\mathbb{R}$-Hodge structure. Since the derivative of ϕ agrees with the Higgs field, we see that ϕ must also be an isomorphism, and thus the local system is uniformizing and $\rho(\pi_1(X)) \subset \mathrm{PSL}_2(\mathbb{R})$ determines X.

Lemma 4.3.5. *If $\rho(\pi_1(X))$ is an arithmetic subgroup, then there are only finitely many curves of genus g for which the local system $\mathbb{V}$ underlies a $\mathbb{Q}$-variation of Hodge structure.*

Proof. This follows from the finiteness of conjugacy classes of arithmetic Fuchsian groups of bounded genus due to Takeuchi [53]. □

Proof of Corollary 4.3.3. In view of the two lemmas above, we see that it's enough to show that for every $g \geq 2$, there exists of a curve C of genus g with a uniformizing local system whose associated monodromy representation $\rho(\pi_1(X_b)) \subset \mathrm{PSL}_2(\mathbb{R})$ is arithmetic. If $g = 2$, we may take a genus 2 Shimura curve (see e.g. [54]), as any Shimura curve carries a motivic SL_2-local system $\mathbb{V}$. If $g \geq 3$, then there's a curve C' which is a degree $g-1$ cover of a genus 2 Shimura curve C, and we can simply pull the local system $\mathbb{V}$ back to C'. Again for degree reasons, the local system will be uniformizing, and the image of the monodromy representation will be arithmetic, as the monodromy representation factors through the monodromy representation associated to C and $\mathbb{V}$. □

Remark 4.3.6. The same argument also shows that any positive dimensional family of smooth projective curves whose fiber contains a curve with such a uniformizing local system gives example for which the sequence (4.3.1) is not exact. In particular, any positive dimensional subvariety of $\mathcal{M}_g$ passing through the point corresponding such a curve would work.

4.4 Injectivity and non-abelian Hodge locus

We continue to work in the setting as in the previous section: let $f : X \to B$ be a smooth projective map between smooth connected complex varieties with connected fibers. We have just seen that the map

$$\pi_1^{\mathrm{Tann}}(\mathrm{LS}_{\mathbb{Q}}^{\mathrm{Hdg}}(X_b)) \to \pi_1^{\mathrm{Tann}}(\mathrm{VMHS}_{\mathbb{Q}}(X))$$

is not always injective. In this section, by using the work of Simpson on non-abelian Hodge locus [47], we show that in many cases, this map is in fact usually injective if we restrict to the sub-category generated by the semi-simple objects. We explain what that means: semi-simple objects in $\mathrm{VMHS}_{\mathbb{Q}}(X)$ are precisely the direct sums of polarizable $\mathbb{Q}$-variations of pure Hodge structures and the semi-simple objects in $\mathrm{LS}^{\mathrm{Hdg}}_{\mathbb{Q}}(X_b)$ are exactly local systems that are sub-objects of local systems underlying polarizable $\mathbb{Q}$-variation of pure Hodge structures (see [1, Remark on page 8]). Call these two categories $\mathrm{VHS}_{Q}(X)$ and $\mathrm{LS}^{\mathrm{hdg,ss}}_{\mathbb{Q}}(X_b)$. On the group theory side, their corresponding Tannakian fundamental groups are the pro-reductive quotients of the original groups we've worked with:

$$\pi_1^{\mathrm{Tann}}(\mathrm{VHS}_{\mathbb{Q}}(X)) = \left(\pi_1^{\mathrm{Tann}}(\mathrm{VMHS}_{\mathbb{Q}}(X))\right)^{\mathrm{red}}, \quad \pi_1^{\mathrm{Tann}}(\mathrm{LS}^{\mathrm{hdg,\,ss}}_{\mathbb{Q}}(X_b)) = \left(\pi_1^{\mathrm{Tann}}(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(X_b))\right)^{\mathrm{red}}.$$

The same argument that proves that sequence (4.3.1) is exact now shows that the following sequence is exact

$$\pi_1(\mathrm{LS}^{\mathrm{hdg,ss}}_{\mathbb{Q}}(X_b)) \xrightarrow{\iota_b^{ss}} \pi_1(\mathrm{VHS}_{\mathbb{Q}}(X)) \to \pi_1(\mathrm{VHS}_{\mathbb{Q}}(B)) \to 1.$$

Definition 4.4.1. Let $f : X \to B$ be given as above. Define the

$$NG(X/B) := \{b \in B : \iota_b : \pi_1^{\mathrm{Tann}}(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(X_b)) \to \pi_1^{\mathrm{Tann}}(\mathrm{VMHS}_{\mathbb{Q}}(X)) \text{ is not injective}\}$$
$$NG(X/B)^{\mathrm{red}} := \{b \in B : \iota_b^{ss} : \pi_1^{\mathrm{Tann}}(\mathrm{LS}^{\mathrm{hdg,ss}}_{\mathbb{Q}}(X_b)) \to \pi_1^{\mathrm{Tann}}(\mathrm{VHS}_{\mathbb{Q}}(X)) \text{ is not injective}\}$$

In view of Proposition B.0.4 part (2), we see that the points $b \in NG(X/B)$ can be thought of as the set of points over which the fibers X_b have "extra" local systems that underlies $\mathbb{Q}$-variation of Hodge structures. This can be viewed as a non-abelian analogue of the Hodge exceptional locus studied in Cattani-Deligne-Kaplan [9], which is the locus of points over which the fibers X_b have "extra" Hodge classes. Because Hodge filtrations vary holomorphically in a polarizable $\mathbb{Q}$-variation of Hodge structure, it's not hard to see that the Hodge exceptional locus is a countable union of closed analytic subsets (see e.g. [55, section 3.1] for an detailed exposition). We prove an analogous result in the non-abelian setting:

Theorem 4.4.2. *Suppose $f : X \to B$ is a smooth projective family of algebraic varieties such that $\pi_1^{\acute{e}t}(X_b)$ injects into $\pi_1^{\acute{e}t}(X)$. Then $NG(X/B)^{red}$ is a countable union of closed analytic subsets of B.*

We prove this theorem by relating this set $NG(X/B)^{\mathrm{red}}$ to Simpson's non-abelian Hodge locus [47, section 12] (also known as the non-abelian Noether-Lefschetz locus). We briefly recall the definition of non-abelian Hodge locus and some well-known results about it. The definition and all the relevant facts can be found in [47]. Let $f : X \to B$ be a smooth projective family of algebraic varieties. Let $\mathcal{M}_{\mathrm{Dol}}(X/B)$ be the relative moduli space of semistable Higgs bundles $(\mathcal{E}, \phi)$ with vanishing rational Chern classes and $\mathcal{M}_{\mathrm{dR}}(X/B)$ the relative moduli space of vector bundles with flat connections. From non-abelian Hodge theory, we get two things:

1. an homeomorphism between $\mathcal{M}_{\mathrm{Dol}}(X/B)$ and $\mathcal{M}_{\mathrm{dR}}(X/B)$ known as the non-abelian Hodge correspondence;

2. a $\mathbb{G}_m$-action on $\mathcal{M}_{\mathrm{Dol}}(X/B)$ whose fixed points are exactly those Higgs bundles that correspond to systems of Hodge bundles associated to a $\mathbb{C}$-variation of Hodge structure.

Let V be the set of fixed points of the action of $\mathbb{G}_m$ on $\mathcal{M}_{\mathrm{Dol}}(X/B)$ and let V_{dR} be its image in $\mathcal{M}_{\mathrm{dR}}(X/B)$ under the non-abelian Hodge correspondence. Now let $\mathcal{M}_{\mathrm{dR}}(X/B,\mathbb{Z})$ be the subset of $\mathcal{M}_{\mathrm{dR}}(X/B)$ which over each fiber X_b correspond to a flat bundle whose associated monodromy representation has an integral structure. Simpson's non-abelian Hodge locus $NAHL(X/B)$ is then defined as the intersection of these two sets

$$NAHL(X/B) := V_{\mathrm{dR}} \cap \mathcal{M}_{\mathrm{dR}}(X/B,\mathbb{Z})$$

The important fact we need is the following:

Proposition 4.4.3 (Theorem 12.1 in [47])**.**

1. *Simpson's non-abelian Hodge locus $NAHL(X/B)$ has a unique structure of a reduced analytic variety such that the inclusion $NAHL(X/B) \hookrightarrow \mathcal{M}_{dR}(X/B)$ is complex analytic.*
2. *The canonical map $\mathcal{M}_{dR}(X/B) \to B$ restricts to a proper map from $NAHL(X/B) \to B$.*

The key lemma that relates Simpson's non-abelian Hodge locus and $NG(X/B)^{\mathrm{red}}$ is the following:

Lemma 4.4.4. *If $f : X \to B$ is a smooth projective family of algebraic varieties such that $\pi_1^{\acute{e}t}(X_b)$ injects into $\pi_1^{\acute{e}t}(X)$. Then the set $NG(X/B)^{red}$ is the image of the components of $NAHL(X/B) \subset \mathcal{M}_{dR}(X/B)$ which do not surject onto B under the canonical map $\mathcal{M}_{dR}(X/B) \to B$.*

Proof. It's clear that the image of the components which do not surject onto B is contained in $NG(X/B)^{\mathrm{red}}$, so it's enough to show that if $\mathbb{V}$ is a local system on X_b which is contained in a connected component that does surjects onto all of B, then $\mathbb{V}$ is a subquotient (equivalently subobjects) of some local system which does extend to a variation of Hodge structure on all of X. Since $\mathbb{V}$ is contained in such a component, we know that the isomonodromic deformation of $\mathbb{V}$ onto any fiber of $f : X \to B$ is a local system which underlies a variation of Hodge structure. By [14, Theorem 1.4], we know that the orbit of $\mathbb{V}$ under the action of $\pi_1(B)$ is of finite orbit, and therefore it extends to a local system $\mathbb{V}'$ on some finite cover $X' \to X$ that underlies a $\mathbb{Q}$-variation of Hodge structure on X'. Then $f_*\mathbb{V}'$ is a local system underlying some $\mathbb{Q}$-variation of Hodge structure on X whose restriction contains $\mathbb{V}$ as a sub-local system as desired. □

Proof of Theorem 4.4.2. The desired result follows immediately from Lemma 4.4.4 since the map $f : NAHL(X/B) \to B$ is proper and hence closed. □

Remark 4.4.5.

1. The assumption that $\pi_1^{\text{ét}}(X_b)$ injects into $\pi_1^{\text{ét}}(X)$ is needed to apply the results of Esnault-Kerz. It's very reasonable to conjecture that this description is still true without this assumption.

2. This assumption however is important from a purely group theoretic perspective: The main group theoretic result [14, Theorem 4.1] of Esnault-Kerz says that if H is a subgroup of G such that $\widehat{H}$ injects into $\widehat{G}$, then a semi-simple representation of H extends to a semi-simple representation of some finite index subgroup G' of G containing H if and only if this representation is of finite orbit under the action of G/H. This result is not true if we don't make any assumption. Indeed, this statement with no assumptions imposed would imply that any extension of residually finite group by residually finite group is residually finite, as we explain now: consider the extension of groups

$$1 \to H \to G \to G/H \to 1,$$

where both H and G/H are residually finite. To show that G is residually finite, it's enough to show that every non-zero element $g \in H \subset G$ is non-zero in some finite quotient of G. Since H is residually finite, we know that H has some finite quotient in which g remains non-zero. As every finite group is linear, we see that there's a representation ρ of H of finite image such that $\rho(g) \neq 0$. Since every representation of finite image is automatically of finite orbit under the action of G/H, then under the assumption, we may extend it to a representation ρ' of some finite index subgroup G' of G containing H. In particular $\rho'(g) \neq 0$. Let K' be the kernel of ρ'. It's also of finite index in G and hence the intersection of all subgroups conjugate to K' is a normal, finite index subgroup. Furthermore, g is again non-zero in the corresponding finite quotient, and so G is residually finite as claimed. On the other hand, Millson [40] has constructed an example of an extension of residually finite group by a finite group that's not residually finite. Hence, some assumption is needed for the main group theoretic result of Esnault-Kerz to hold.

Let's return to the set $NG(X/B)$. We know that in the abelian setting, the Hodge conjecture famously implies that the Hodge exceptional locus is in fact algebraic and this is eventually proven unconditionally in [9]. Similarly in the non-abelian setting, the non-abelian version of the Hodge conjecture [47, Conjecture 12.4] would also imply that Simpson's non-abelian Hodge locus, and therefore $NG(X/B)^{\mathrm{red}}$, is algebraic. In fact, it seems reasonable to ask if the same is true with no assumptions and without having to pass to pro-reductive quotient:

Question 4.4.6. Let $f : X \to B$ be a smooth projective family of algebraic varieties. Is $NG(X/B)$ always a countable union of algebraic subsets of B?

4.5 Case study: moduli space of degree 1 line bundles on universal curves, part 1

In this section, we study the example of moduli space of degree 1 line bundles on the universal curve $p : \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g} \to \mathcal{M}_g$. Let C be a smooth projective curve of genus g. We first give an explicit description of $\pi_1(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(\mathrm{Pic}^1(C)))$. In fact, this description works for any algebraic variety A whose topological fundamental group is a finitely generated free abelian group, so we first work in that generaltiy. More specifically, we prove that

Proposition 4.5.1. *Let A be smooth connected complex algebraic variety with $\pi_1(A) = \mathbb{Z}^k$. For any choice of base point, we have $\pi_1(\mathrm{LS}^{hdg}_{\mathbb{Q}}(A)) = \mathbb{G}_a^k \times \widehat{\mathbb{Z}}^k$.*

The following lemma is probably well-known to experts, and we only prove it for completeness.

Lemma 4.5.2. *Let A be a smooth connected complex algebraic variety with abelian fundamental groups and $\mathbb{V}$ a $\mathbb{Q}$-local system underlying a polarizable $\mathbb{Q}$-variation of Hodge structure with an underlying integral structure. Then $\mathbb{V}$ is of finite monodromy.*

Proof. Let $\mathbb{V}$ be such a local system. Then $\mathbb{V} \otimes \mathbb{C}$ underlies a polarizable $\mathbb{C}$-variation of Hodge structure and since $\pi_1(A)$ is abelian, it splits into irreducible rank 1 summands $\mathbb{V} \otimes \mathbb{C} = \bigoplus \mathbb{V}_i$. Now by non-abelian Hodge theory, we know that each $\mathbb{V}_i$ underlies a polarized $\mathbb{C}$-variation of Hodge structure. In particular, the polarization has to be positive-definite and hence $\mathbb{V}_i$ all have unitary monodromy. It follows that $\mathbb{V}$ has unitary monodromy as well. Finally, since it in addition has an associated integral structure, it's finite as desired. □

Remark 4.5.3. This lemma does use the fact that the local systems we are considering have integral structures, and Proposition 4.5.1 will not be correct without the integrality assumption.

Proof of Proposition 4.5.1. We need to show that the category of representation of $\mathbb{G}_a^k \times \widehat{\mathbb{Z}}^k$ is equivalent to $\mathrm{LS}^{\mathrm{Hdg}}(A)$ and the equivalence is compatible with tensor products and taking fiber functors. Let $\mathbb{V}$ be a local system in $\mathrm{LS}^{\mathrm{Hdg}}(A)$. Then it's given by k commuting linear operators L_i. Lemma 4.5.2 tells us that the semi-simplification $\mathrm{Gr}_\bullet\, \mathbb{V}$ of $\mathbb{V}$ has finite monodromy, and so by the Jordan-Chevalley decomposition, we may write $L_i = U_i N_i$, where U_i is of finite order and N_i is unipotent. Since U_i commutes with N_i, we see that we get a representation of $\mathbb{G}_a^k \times \widehat{\mathbb{Z}}^k$. This defines a functor

$$\Phi : \mathrm{LS}^{\mathrm{Hdg}}(A) \to \mathrm{Rep}(\mathbb{G}_a^k \times \widehat{\mathbb{Z}}^k).$$

Conversely, given a representation of $\mathbb{G}_a^k \times \widehat{\mathbb{Z}}^k$, we need to produce a local system in $\mathrm{LS}^{\mathrm{Hdg}}(A)$. Such a representation is the same as k linear operators U_i of finite order and k nilpotent operators N_i that all commute with each other. Hence we get a local system $\mathbb{V}$ of A by assigning each generators of $\mathbb{Z}^k$ to $U_i N_i$. We need to show that $\mathbb{V}$ is a subquotient of a local system which underlies a variation of mixed Hodge structure. Since the semi-simplification of $\mathrm{Gr}^\bullet\, \mathbb{V}$ is of finite monodromy, we know that there exists some positive integer m such that $[m]^*\mathbb{V}$ is a unipotent local system on A, where $[m] : A \to A$ is the multiplication-by-m map. It follows that $\mathbb{V}$ is a sub-local system of $[m]_*[m]^*\mathbb{V}$. Therefore, it's enough to show that every unipotent local system on $\mathbb{V}$ is a subquotient of some local system which underlies a variation of mixed Hodge structure.

Let V be the fiber of $\mathbb{V}$ over a point in A and let $\{v_1, \ldots, v_r\}$ be a basis for V. Then we may define a $\pi_1(A)$-equivariant surjection

$$\begin{aligned} \mathbb{Q}[\pi_1(A,x)]^r &\longrightarrow V \\ 1_i &\mapsto v_i \end{aligned}$$

where 1_i is the multiplicative identity in the i-th copy of $\mathbb{Q}[\pi_1(A,x)]$. As V is a unipotent representation, we know that there exists some N such that $(g-1)^N$ is in the kernel of this map for every $g \in \mathbb{Q}[\pi_1(A,x)]$. It follows that this map factors through $(\mathbb{Q}[\pi_1(A,x)]/J^N)^r$, where J is the augmentation ideal inside the group ring. The work of Hain and Zucker [24] shows that $\mathbb{Q}[\pi_1(A,x)]/J^N$ is a local system underlying an admissible, graded-polarizable $\mathbb{Q}$-variation of mixed Hodge structure, and hence so does $(\mathbb{Q}[\pi_1(A,x)]/J^N)^r$. It also has an integral structure coming from the integral group ring. It follows that every unipotent local system on is a subquotient of some local system which underlies a variation of mixed Hodge structure.

Finally, it's clear that two constructions are inverses of each other, and are compatible with taking tensor products and fiber functors, so the desired result follows from Tannakian duality. □

Remark 4.5.4. The fact that every $\mathbb{Q}$-unipotent local system (potentially with no integral structures) is a subquotient of some local system that underlies a graded-polarizable, admissible $\mathbb{Q}$-variation of mixed Hodge structure also follows from the work of D'Addezio-Esnault [10, Theorem 4.4] and the work of Jacobsen [27, Theorem 7.2]. In fact, they showed that the full subcategory of local systems that are subquotients of local systems underlying such variation of mixed Hodge structures is closed under taking extensions.

Now we return to the case of $p : \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g} \to \mathcal{M}_g$. The main theorem of this section is the following:

Theorem 4.5.5. *Fix some $g \geq 2$, and let C be a smooth projective curve of genus g. Then the following sequence is exact*

$$1 \to \mathbb{G}_a^{2g} \times \widehat{\mathbb{Z}}^{2g} = \pi_1(\mathrm{LS}^{hdg}_{\mathbb{Q}}(\mathrm{Pic}^1(C))) \to \pi_1(\mathrm{VMHS}_{\mathbb{Q}}(\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}) \to \pi_1(\mathrm{VMHS}_{\mathbb{Q}}(\mathcal{M}_g)) \to 1$$

Note that to show that the map

$$\mathbb{G}_a^{2g} \times \widehat{\mathbb{Z}}^{2g} = \pi_1(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(\mathrm{Pic}^1(C))) \to \pi_1(\mathrm{VMHS}_{\mathbb{Q}}(\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}))$$

is injective, it's enough to find representations of $\pi_1(\mathrm{VMHS}_{\mathbb{Q}}(\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}))$ whose restrictions to $\mathbb{G}_a^{2g} \times \widehat{\mathbb{Z}}^{2g}$ are jointly faithful.

By unpacking the identification of $\pi_1(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(\mathrm{Pic}^1(C))) = \mathbb{G}_a^{2g} \times \widehat{\mathbb{Z}}^{2g}$ in the proof of Proposition 4.5.1 and using Tannakian duality, we see that it's enough to find local systems $(\mathcal{E}_i)_{i\in\mathbb{N}}$ on $\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$ satisfying the following conditions:

1. All of the $\mathcal{E}_i$'s underlie some graded-polarizable, admissible $\mathbb{Q}$-variation of mixed Hodge structures $\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$;
2. the restrictions of $\mathcal{E}_0$ to the fiber $\mathrm{Pic}^1(C)$ is unipotent and faithful;
3. the restrictions of $(\mathcal{E}_i)_{i\geq 0}$ to the fiber $\mathrm{Pic}^1(C)$ are of finite monodromy and are jointly faithful.

We first explain how to construct the local system $\mathcal{E}_0$. This is inspired by the local systems studied by Hain-Matsumoto in [21] and the key tool is to use homologically trivial relative cycles.

Consider $\mathcal{X} := \mathcal{C}_g \times_{\mathcal{M}_g} \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$ together with two projection maps $p_1 : \mathcal{X} \to \mathcal{C}_g$ and $p_2 : \mathcal{X} \to \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$. By definition of $\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$, we know that there's a universal line bundle $\mathbb{L}^{\mathrm{univ}}$ on $\mathcal{X}$. On each fiber of p_2, the line bundle $\mathbb{L}^{\mathrm{univ}}$ restricts to a line bundle of degree 1. We would like to modify $\mathbb{L}^{\mathrm{univ}}$ so that the restriction is homologically trivial. Let $\omega := p_1^*\omega_{\mathcal{C}_g/\mathcal{M}_g}$ where $\omega_{\mathcal{C}_g/\mathcal{M}_g}$ is the relative canonical bundle on the universal curve $\mathcal{C}_g \to \mathcal{M}_g$. Then we see that the line bundle

$$\mathbb{L} := (\mathbb{L}^{\mathrm{univ}})^{2g-2} \otimes \omega^\vee$$

has the desired property, as the restriction of $\mathbb{L}$ to each fiber of p_2 is of degree 0. Let D be a divisor corresponding to $\mathbb{L}$ and let $|D|$ be the support of D together with the inclusion map $\iota : |D| \to \mathcal{X}$.

Now the relative cycle class map yields an exact sequence

$$0 \to R^1(p_2)_*\mathbb{Z}(1) \to R^1(p_2|_{\mathcal{X}-|D|})_*\mathbb{Z}(1) \to (p_2|_{|D|})_*\mathcal{H}^2_{|D|}(\mathbb{Z}(1)) \to R^2(p_2)_*\mathbb{Z}(1) \to \dots$$

where $\mathcal{H}^2_{|D|}(\mathbb{Z}(1)) := R^2\iota^!\mathbb{Z}(1)$ is the sheaf on $|D|$ whose pushforward gives the bundle of local cohomology groups with respect to the subset $|D|$ on $\mathcal{X}$. Since D is homologically trivial on each fiber, we see that the image of the natural inclusion map $\mathbb{Z} \to (p_2|_{|D|})_*\mathcal{H}^2_{|D|}(\mathbb{Z}(1))$ induced by D goes to 0 in $R^2(p_2)_*\mathbb{Z}(1)$ and hence we may pullback to get a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & R^1(p_2)_*\mathbb{Z}(1) \arrow[r] & R^1(p_2|_{\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}-|D|})_*\mathbb{Z}(1) \arrow[r] & (p_2|_{|D|})_*\mathcal{H}^2_{|D|}(\mathbb{Z}(1)) \arrow[r] & R^2(p_2)_*\mathbb{Z}(1) \\
0 \arrow[r] & R^1(p_2)_*\mathbb{Z}(1) \arrow[r] \arrow[u, "="] & \mathcal{E}_0 \arrow[r] \arrow[u] & \mathbb{Z}(0) \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Notice that all the maps involved are maps of variation of mixed Hodge structures, so $\mathcal{E}_0$ is a local system which underlies a graded-polarizable, admissible variation of mixed Hodge structure as well. Furthermore, since the fibers of p_2 over any point of $\mathrm{Pic}^1(C) \subset \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$ is simply C, if we restrict $\mathcal{E}_0$ to $\mathrm{Pic}^1(C)$, we get a unipotent local system

$$0 \to \underline{H^1(C,\mathbb{Z}(1))} \to (\mathcal{E}_0)|_{\mathrm{Pic}^1(C)} \to \underline{\mathbb{Z}(0)} \to 0$$

We would like to compute the monodromy of this unipotent local system.

Recall that monodromy is a topological invariant and does not depend on the base point we choose. Let $x \in C$. This corresponds to a degree 1 line bundle on C and hence gives us a point $[x] \in \mathrm{Pic}^1(C)$. We compute the monodromy with respect to this base point. Since the local system is unipotent, the monodromy reprensentation is determined by a map

$$H_1(C,\mathbb{Z}) = \pi_1(\mathrm{Pic}^1(C),[x]) \to \mathrm{Hom}(\mathbb{Z}(0), H^1(C,\mathbb{Z}(1))) = H^1(C,\mathbb{Z}(1)) = H_1(C,\mathbb{Z}).$$

In particular, the monodromy of the local system $(\mathcal{E}_0)|_{\mathrm{Pic}^1(C)}$ is faithful if $\rho_{\mathcal{E}_0}$ is injective.

Observe that $[x] \in \mathrm{Pic}^1(C)$ is in the image of the natural inclusion map $C \to \mathrm{Pic}^1(C)$ and we have the following commutative diagram

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, hook] & \mathrm{Pic}^1(C) \arrow[d, hook] \\ \mathcal{C}_g \arrow[r, hook, "j"] & \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g} \end{tikzcd}$$

The inclusion from C into $\mathrm{Pic}^1(C)$ induces a surjection of topological fundamental group $\pi_1(C,x) \twoheadrightarrow \pi_1(\mathrm{Pic}^1(C),[x])$ and hence to compute the monodromy of $(\mathcal{E}_0)|_{\mathrm{Pic}^1(C)}$, it's enough to compute the monodromy of $(j^*\mathcal{E}_0)|_C$.

Now if we view $\mathcal{C}_g$ as $\mathcal{M}_{g,1}$, when we pull the family $p_2 : \mathcal{C}_g \times_{\mathcal{M}_g} \to \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$ back along this embedding j, we get

$$(\mathcal{C}_g \times_{\mathcal{M}_g} \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}) \times_{\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}} \mathcal{C}_{g,1} = \mathcal{M}_{g,1} \times_{\mathcal{M}_g} \mathcal{M}_{g,1} = \mathcal{C}_{g,1} \to \mathcal{M}_g$$

which is the universal pointed curve over $\mathcal{M}_{g,1}$, and the divisor can be chosen to be $(2g-2)\xi - K$, where ξ is the unique section from $\mathcal{M}_{g,1}$ to $\mathcal{C}_{g,1}$, and K is a relative canonical divisor for the map $\mathcal{C}_{g,1} \to \mathcal{M}_{g,1}$. In particular, we can relate $j^*\mathcal{E}_0$ to a certain local system studied by Hain and Matsumoto in [21]. In particular, building on their monodromy computation [21, Proposition 6.4], we can prove the following:

Proposition 4.5.6. *The monodromy representation is given by*

$$\begin{array}{ccccc} \pi_1(C,x) & \twoheadrightarrow & \pi_1(\mathrm{Pic}^1(C),[x]) & \longrightarrow & H_1(C,\mathbb{Z}) \\ \gamma & \longmapsto & & & (g-1)[\gamma] \end{array}$$

In particular, $(\mathcal{E}_0)|_{\mathrm{Pic}^1(C)}$ *is faithful.*

The proof of Proposition 4.5.6 goes through the Johnson homomorphism and since it requires us to go on a tangent, we defer the proof to the Appendix C.

Proof of Theorem 4.5.5. It remains to write down local systems $(\mathcal{E}_i)_{i\geq 1}$ whose restrictions to $\mathrm{Pic}^1(C)$ are of finite monodromy and are jointly faithful. Let $\rho : \pi_1(\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}) \to \mathrm{GL}_r(\mathbb{C})$ be the monodromy representation associated to $\mathcal{E}_0$. Consider all finite quotients of the image of ρ. As every finite group is linear, we may pick faithful representations of these finite quotients and hence we get local systems $\mathcal{E}_i$ on $\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$ of finite monodromy, and hence underlies graded-polarizable, admissible $\mathbb{Q}$-variation of mixed Hodge structures on $\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$. Consider the restriction of $\mathcal{E}_i$ to the fiber $\mathrm{Pic}^1(C)$. The restrictions are certainly of finite monodromy, and it remains to show that they are jointly faithful. Since we showed that the restriction of ρ to $\pi_1(\mathrm{Pic}^1(C))$ is faithful, we may view it as a subgroup of the image of ρ. Since $\rho(\pi_1(\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}))$ is a finitely generated linear group, it's residually finite [38], and hence for any $g \in \pi_1(\mathrm{Pic}^1(C))$, there's some finite quotient of $\rho(\pi_1(\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}))$ for which g remains non-zero. Hence, the $\mathcal{E}_i|_{\mathrm{Pic}^1(C)}$'s are jointly faithful as desired. $\square$

Remark 4.5.7. Note that since $\pi_1(\mathcal{M}_g)$ is residually finite [15, Theorem 6.11], this argument also shows that the topological fundamental group $\pi_1(\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g})$ of $\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$ is residually finite. In fact, suppose $f : X \to B$ is any smooth projective family of curves over some base B with $\pi_1(B)$ residually finite. Then the same argument shows that the topological fundamental group $\pi_1(\mathrm{Pic}^1_{X/B})$ is residually finite.

Notice that in the example of $p : \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g} \to \mathcal{M}_g$, the group $\pi_1(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(\mathrm{Pic}^1(C)))$ does not depend on the choice of base point $[C]$ in $\mathcal{M}_g$. Recall that in the abelian setting, the Hodge exceptional locus also has a group-theoretic characterization: it's the locus where the Mumford-Tate group of the fiber becomes smaller than the Mumford-Tate group of a general fiber. Therefore, it's reasonable to ask the following question

Question 4.5.8. Given a smooth projective families of algebraic varieties $f : X \to B$, can $NG(X/B)$ be characterized as the points b of B such that $\pi_1^{\mathrm{Tann}}(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(X_b))$ differs from that of a general fiber?

If this question has an affirmative answer, then using Proposition 4.5.1, we can for example conclude that for any smooth projective family of abelian varieties $f : \mathcal{A} \to B$, the locus $NG(\mathcal{A}/B)$ is always empty.

4.6 Case study: moduli space of degree 1 line bundles on universal curves, part 2

We continue our study of $p : \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g} \to \mathcal{M}_g$. In this section, we would like to go back and study the Hodge theoretic section question for this family as well as for the universal family of curves $f : \mathcal{C}_g \to \mathcal{M}_g$.

Recall that in Grothendieck's original formulation of anabelian geometry, group theoretic maps are not just maps of étale fundamental groups but maps of extensions (1.0.1). Now the short exact sequence proven by D'Addezio and Esnault

$$1 \to \pi_1^{\mathrm{Tann}}(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(X)) \to \pi_1^{\mathrm{Tann}}(\mathrm{VMHS}_{\mathbb{Q}}(X)) \to \pi_1^{\mathrm{Tann}}(\mathrm{MHS}_{\mathbb{Q}})) \to 1$$

may be viewed as the analogue of the short exact sequence 1.0.1. This motivates the following definition:

Definition 4.6.1. Suppose X and Y are two smooth connected complex algebraic varieties. Then a map $f : \pi_1(\mathrm{VMHS}_{\mathbb{Q}}(X)) \to \pi_1(\mathrm{VMHS}_{\mathbb{Q}}(Y))$ is called *Hodge theoretic* if it induces a map of extensions

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(X)) \arrow[r] \arrow[d] & \pi_1(\mathrm{VMHS}_{\mathbb{Q}}(X)) \arrow[r] \arrow[d] & \pi_1(\mathrm{MHS}_{\mathbb{Q}}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(Y)) \arrow[r] & \pi_1(\mathrm{VMHS}_{\mathbb{Q}}(Y)) \arrow[r] & \pi_1(\mathrm{MHS}_{\mathbb{Q}}) \arrow[r] & 1
\end{tikzcd}$$

Note that any algebraic morphism between X and Y induces Hodge theoretic maps between their Hodge theoretic fundamental groups. Then instead of going into the set of group theoretic splittings of (4.3.1), we require that it lands in the set of Hodge theoretic splittings:

$$\mathrm{sec}_{\mathrm{Hdg}} : \{\text{algebraic sections to } f : X \to B\} \to \{\text{Hodge theoretic splittings of (4.3.1)}\}/\backsim,$$

where the equivalence relation is defined by the conjugation action of $\pi_1(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(B))$. With this new requirement, the Hodge theoretic section question becomes the following

Question 4.6.2. When is $\mathrm{sec}_{\mathrm{Hdg}}$ a bijection?

We show that this section map is a bijection in the case $p : \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g} \to \mathcal{M}_g$ (as well as the case $f : \mathcal{M}_g \to \mathcal{C}_g$). The main idea is to relate maps of Tannakian fundamental groups to the maps of étale fundamental groups. More precisely, we have the following theorem:

Theorem 4.6.3. *Let $f : X \to B$ be a smooth projective map between smooth connected complex quasi-projective varieties. If $\pi_1(\mathrm{VMHS}_{\mathbb{Q}}(X)) \to \pi_1(\mathrm{VMHS}_{\mathbb{Q}}(B))$ admits a Hodge theoretic splitting, then the induced map of étale fundamental groups $\pi_1^{\acute{e}t}(X) \to \pi_1^{\acute{e}t}(B)$ also splits.*

Proof. Suppose that we have a Hodge theoretic section $s : \pi_1(\mathrm{VMHS}_{\mathbb{Q}}(B)) \to \pi_1(\mathrm{VMHS}_{\mathbb{Q}}(X))$. By definition, s restricts to a splitting $s|_{\mathrm{LS}_{\mathbb{Q}}} : \pi_1(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(B)) \to \pi_1(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(X))$ of the map $\pi_1(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(X)) \to \pi_1(\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(B))$. Applying Tannakian duality, we see that at the categorical side, we get a functor $s^*|_{\mathrm{LS}} : \mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(X) \to \mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(B)$ such that $s^*|_{\mathrm{LS}} \circ f^*$ is isomorphic to the identity functor on $\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(B)$.

We claim that any additive tensor functor between $\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(X)$ and $\mathrm{LS}^{\mathrm{hdg}}_{\mathbb{Q}}(B)$ has to preserve local systems of finite monodromy. Assuming this claim is true, then $s^*|_{\mathrm{LS}}$ further restricts to a functor from the category $\mathrm{LS}_{\mathrm{fin}}(X)$ of local systems with finite monodromy on X to $\mathrm{LS}_{\mathrm{fin}}(B)$, such that precomposing with f^* is isomorphic to the identity functor on $\mathrm{LS}_{\mathrm{fin}}(B)$. This concludes the proof as the étale fundamental group of a smooth connected complex algebraic variety X can be identified with the Tannakian fundamental group $\pi_1^{\mathrm{Tann}}(\mathrm{LS}_{\mathrm{fin}}(X))$. □

Therefore, it remains to verify the following lemma:

Lemma 4.6.4. *Let X and Y be two smooth connected complex quasi-projective varieties. Then any additive tensor functors between (sub)categories of local systems have to send local systems of finite monodromy to local systems of finite monodromy.*

Proof. This follows from Nori's criterion for a local system to be of finite monodromy: a local system $\mathbb{V}$ on a smooth connected quasi-projective variety X is of finite monodromy if and only if there are two polynomials $P(x) \neq Q(x) \in \mathbb{N}[x]$ such that $P(\mathbb{V}) = Q(\mathbb{V})$, where multiplication of local system is interpreted as tensor products and addition is interpreted as direct sum. This is first proven by Nori in the case where X is a curve [44], and is later generalized to the case where X is a smooth connected quasi-projective variety by the work of Biswas-Holla-Schumacher [4]. □

Remark 4.6.5. This criterion also works for $\mathcal{M}_g$ and $\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$ as one can check if a local system is of finite monodromy by pulling back this local system to a finite étale cover, and both $\mathcal{M}_g$ and $\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$ admit finite étale covering maps by smooth connected quasi-projective varieties.

Corollary 4.6.6. *Let U be any open subsets of $\mathcal{M}_g$ and $f^{-1}(U)$ its preimage in $\mathcal{C}_g$ and $p^{-1}(U)$ its preimage in $\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}$. Then the maps*

$$\pi_1(\mathrm{VMHS}(p^{-1}(U))) \to \pi_1(\mathrm{VMHS}(U)), \qquad \pi_1(\mathrm{VMHS}(f^{-1}(U))) \to \pi_1(\mathrm{VMHS}(U))$$

do not admit any Hodge theoretic sections, and the section maps $\mathrm{sec}_{\mathrm{Hdg}}$ are bijections in these two cases. In particular, taking U to be $\mathcal{M}_g$ gives affirmative answers to Hodge theoretic section question for $p : \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g} \to \mathcal{M}_g$ and $f : \mathcal{C}_g \to \mathcal{M}_g$.

Proof. This follows from the fact that the associated maps of étale fundamental groups do not split when the base is the generic point of $\mathcal{M}_g$ (see [20] and [37]). □

Remark 4.6.7. André observed in [2] that given a smooth connected complex algebraic variety X, the étale fundamental group $\pi_1^{\text{ét}}(X)$ can be identified with $\pi_0(\pi_1(\mathrm{VMHS}(X))) = \pi_0(\pi_1(\mathrm{LS}^{\mathrm{Hdg}}(X)))$. In particular, Theorem 4.6.3 still holds even if we just work with group theoretic sections. It's then natural to ask if every group theoretic section is automatically Hodge theoretic.

Chapter 5

Maps between curves

In this section we switch gears and discuss algebraic maps between curves. We know that any algebraic maps between curves must preserve Hodge structures on cohomology. Furthermore, by Torelli's theorem [3, page 245], we know that the algebraic structure of a smooth projective curve X is determined by the polarized Hodge structure on $H^1(X, \mathbb{Z})$. We ask if maps between cohomology which preserves Hodge structures are in one-to-one correspondence to the set of algebraic maps.

Now it's in fact not difficult to see that the answer to this naïve question is no. For example, as an integral Hodge structure, $H^1(X, \mathbb{Z})$ always have non-trivial automorphisms, as the -1 maps is always an automorphism. On the other hand, there are many curves of genus at least 2 which do not admit any non-trivial automorphisms. However, in the case of automorphisms, a slightly modified version of the naïve indeed has a positive solution.

Proposition 5.0.1. *Let X be a smooth projective curve. Given any isomorphism f of Hodge structures $H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{Z})$ which preserves the polarization, there's always an automorphism ϕ of X whose induced map on cohomology is $\pm f$.*

Therefore, in some sense, it's well-understood how automorphism of polarized integral Hodge structure captures (or fails to capture) automorphisms of curves. In the next section, we produce more examples of morphisms between curves of different genera that do not come from algebraic geometry. These examples are in some sense more extreme and are related to finding families of curves with isotrivial isogeny factors. In particular, this construction gives examples of families of curves with isotrivial isogeny factors that are in some sense maximal. We explain this relation in section 5.2.

5.1 Hodge structure on cohomology is not enough

Let $\phi : C \to R$ be a non-constant map of algebraic curves where C is of genus g and R some curve of genus $h < g$. Note that we get an induced map of Jacobians $\mathrm{Jac}(R) \hookrightarrow \mathrm{Jac}(C)$ and by Poincaré complete reducibility theorem [41, Theorem 10.1], we know that $\mathrm{Jac}(C)$ is isogenous to $\mathrm{Jac}(R) \times A'$ for some abelian variety A' of dimension $g - h$. In particular, this means that $\mathcal{M}_g$ intersects nontrivially with some Hecke translate of $[\mathrm{Jac}(R)] \times \mathcal{A}_{g-h}$ inside $\mathcal{A}_g$. Now the codimension of $[\mathrm{Jac}(R)] \times \mathcal{A}_{g-h}$ inside $\mathcal{A}_g$ is given by

$$\binom{g+1}{2} - \binom{g-h+1}{2} = hg + \frac{h-h^2}{2}$$

and so the expected dimension of the intersection of $[\mathrm{Jac}(R)] \times \mathcal{A}_{g-h}$ with $\mathcal{M}_g$ inside $\mathcal{A}_g$ is given by $3g - 3 - hg - \frac{h-h^2}{2}$.

In particular, when $h = 2$, we see that the expected dimension of the intersection is $g - 2 > 0$. Therefore, since the intersection is non-empty, we get a positive dimensional family of curves $\{C_b\}$ of genus g, parametrized by some base B. Since varying the algebraic structure does not change the underlying topological space, we see that for all $b \in B$, C_b is still an unramified topological cover of R with covering map $\phi_b := \phi$. Note that ϕ_b need not to be algebraic with respect to the algebraic structure on X_b and R.

However, by construction, for every $b \in B$, the Jacobian $\mathrm{Jac}(C_b)$ of the fiber contains $\mathrm{Jac}(R)$ as a polarized subvariety. It follows then that the map between cohomology induced by ϕ_b is a morphism of polarized Hodge structure. Using this observation, we may construct examples of maps of Hodge structures which do not come from algebraic geometry:

Proposition 5.1.1. *Let R be any smooth projective curve of genus 2. Let $\phi : C \to R$ be an étale cover of genus g. As explained above, we get a family $\{C_b\}$ of curves with maps ϕ_b into R. Then all but finitely many of these maps ϕ_b are not homotopic to an algebraic map.*

Proof. This follows from the fact that there are only finitely many étale covers of a fixed degree over R and the fact that as continous maps, ϕ_b are all unramified covering maps. □

Remark 5.1.2.

1. Note that by Riemann-Hurwitz, every non-constant map from a curve of genus 3 to a curve of genus 2 is étale. In particular, there are examples where we have non-trivial maps of Hodge structures but no algebraic maps at all.

2. When $h = 1$, we know that the expected dimension of the intersection is $2g - 3 > 0$, but étale covers of a genus 1 curve is still of genus 1 so this argument does not work.

3. The expected dimension of the intersection can be rewritten as

$$3g - 3 - hg - \frac{h-h^2}{2} = 3g - 3 - \frac{hg}{2} - h\left(\frac{g-h+1}{2}\right).$$

   Note that when $h \geq 6$, this expression is clearly negative; the cases of $h = 3, 4,$ and 5 can be checked individually to be no greater than 0. Therefore, this argument does not work for any R of genus greater than 2.

4. The proof the pointed Torelli theorem of Hain and Pulte crucially uses Proposition 5.0.1. Therefore, if one would like to capture morphisms of pointed curves using the mixed Hodge structure on fundamental groups, one may need a different idea.

5.2 Family of curves and isotrivial isogeny factors

Note that the construction in the previous section produces a subvariety B inside $\mathcal{M}_g$, whose associated family of curves has an isotrivial isogeny factor. In this section, we show that the families constructed in the following way are in some sense maximal.

Proposition 5.2.1. *Let B be a subvariety of $\mathcal{M}_g$ whose associated family of curves has an isotrivial isogeny factor, then* $\dim B \leq 2g-3$*. This bound is sharp as the construction in the previous section produces such a family when $h = 1$.*

To prove this proposition, we first relate having an isotrivial isogeny factor to the derivative of the period maps. Suppose $f : X \to B$ is a family of curves corresponding to an embedding $\iota : B \hookrightarrow \mathcal{M}_g$ and suppose that the associated family of Jacobians $\pi : \mathcal{A} \to B$ has an isotrivial isogeny factor A_f. The derivative of the (nonpolarized) period map associated to $f : X \to B$ at some point $b = [C] \in B$ is given by the following:

$$dP : T_{B,b} \to T_{\mathcal{M}_g,b} \to \mathrm{Hom}(H^0(C, \omega_C), H^1(C, \mathcal{O}_C)).$$

Another way to think about it is that it comes from the Gauss-Manin connection

$$\nabla : R^1 f_* \mathbb{C} \otimes \mathcal{O}_B \to R^1 f_* \mathbb{C} \otimes \Omega_B$$

in the following way: by Griffiths transversality, if we take the associated graded piece of ∇, we get and $\mathcal{O}_B$-linear map

$$\overline{\nabla} : f_* \Omega_{\mathcal{X}/B} \to R^1 f_* \mathcal{O}_{\mathcal{X}} \otimes \Omega_B$$

and the derivative of the period map is obtained from first taking the fiber of $\overline{\nabla}$ at b

$$\overline{\nabla}_b : H^0(C, \omega_C) \to H^1(C, \mathcal{O}_C) \otimes (T^\vee_{B,b})$$

and then applying adjunction.

By definition, $\pi : \mathcal{A} \to B$ having isotrivial isogeny factor A_f is the same as saying that ∇ has a nontrivial kernel. Since the kernel is also a Hodge structure, it has to intersect $F^1(R^1 f_* \mathbb{C} \otimes \mathcal{O}_B)$ non-trivially and hence both $\overline{\nabla}$ and $\overline{\nabla}_b$ have non-trivial kernel.

Since the derivative of the period map factors through $T_{\mathcal{M}_g,b}$, we see that $\overline{\nabla}_b$ factors in the following way

$$\overline{\nabla}_b : H^0(C, \omega_C) \to H^1(C, \mathcal{O}_C) \otimes T^\vee_{\mathcal{M}_g,b} \to H^1(C, \mathcal{O}_C) \otimes (T^\vee_{B,b})$$

Recall that we have identification $H^1(C, \mathcal{O}_C)^\vee = H^0(C, \omega_C)$ coming from Hodge theory and $T_{\mathcal{M}_g,b} = H^1(C, T_C) = H^0(C, \omega_C^{\otimes 2})$ coming from Serre duality. Then using these identifications and the tensor-Hom adjunction, we can rewrite the map as

$$\overline{\nabla}_b : H^0(C, \omega_C) \to \mathrm{Hom}(H^0(C, \omega_C), H^0(C, \omega_C^{\otimes 2})) \to \mathrm{Hom}(H^0(C, \omega_C), (T^\vee_{B,b})).$$

By [56, Lemma 10.22], we know that the first map $H^0(C, \omega_C) \to \mathrm{Hom}(H^0(C, \omega_C), H^0(C, \omega_C^{\otimes 2}))$ is just given by multiplication of sections, i.e, for any $\varphi \in H^0(C, \omega_C)$, the homomorphism we get just sends ψ to $\psi \otimes \varphi$. In particular, for any $\varphi \in H^0(C, \omega_C)$, we have the following map

$$H^0(C, \omega_C) \xrightarrow{-\otimes\varphi} H^0(C, \omega_C^{\otimes 2}) = T^{\vee}_{\mathcal{M}_g, b} \xrightarrow{\iota^*} T^{\vee}_{B,b},$$

where ι^* is just the induced map on cotangent spaces.

Lemma 5.2.2. *For any $\varphi \neq 0 \in H^0(C, \omega_C)$, the map $\omega_C \to \omega_C^{\otimes 2}$ defined by $- \otimes \varphi$ is injective and so is the induced map on global sections.*

Proof. Given any non-zero global section ψ of ω_C, it's non-zero over some dense open subset of C. It follows that given two non-zero holomorphic 1-forms φ and ψ, both of them has to be non-zero at some point $p \in C$ and hence $\varphi \otimes \psi$ is also non-zero. □

Proof of Proposition 5.2.1. Let φ be a non-zero holomorphic 1-form in the kernel of $\overline{\nabla}$. Then by definition the composed map

$$H^0(C, \omega_C) \xrightarrow{-\otimes\varphi} H^0(C, \omega_C^{\otimes 2}) = T^{\vee}_{\mathcal{M}_g, b} \xrightarrow{\iota^*} T^{\vee}_{B,b}$$

is the zero map and hence the image of $- \otimes \varphi$ is contained in the kernel of the map on cotangent spaces $\iota^* : T^{\vee}_{\mathcal{M}_g, b} \to T^{\vee}_{B,b}$. By Lemma 5.2.2, $\ker \iota^*$ is at least g dimensional. Since B is a subvariety of $\mathcal{M}_g$, ι^* is surjective and so

$$\dim B = \dim T^{\vee}_{B,b} \leq \dim T^{\vee}_{\mathcal{M}_g, b} - \dim H^0(C, \omega_C) = 3g - 3 - g = 2g - 3$$

as claimed. □

If we additionally assume that the dimension of the isotrivial isogeny factor $\dim A_f$ is at least 2, then we get a linearly independent set of holomorphic 1-forms $\{\varphi_1, \varphi_2\} \subset \ker \overline{\nabla}_b$. Then as before we get two maps

$$m_i : H^0(C, \omega_C) \xrightarrow{-\otimes\varphi_i} H^0(C, \omega_C^{\otimes 2})$$

and the images of m_i are both contained in the kernel of ι^* so we would like to compute the dimension of the subspace spanned by the images of m_1 and m_2.

Lemma 5.2.3. *If the zero sets of φ_1 and φ_2 are disjoint, then the image of m_1 and the image of m_2 has exactly 1-dimensional intersection given by $Span(\varphi_1 \otimes \varphi_2) = Span(\varphi_2 \otimes \varphi_1)$.*

Proof. First note that the two quadratic differentials $\varphi_1 \otimes \varphi_2$ and $\varphi_2 \otimes \varphi_1$ are scalar multiples of each other, since they share the same zero sets and therefore differ by multiplying by some global section of $\mathcal{O}_C^*$. Hence, $\mathrm{Span}(\varphi_1 \otimes \varphi_2) = \mathrm{Span}(\varphi_2 \otimes \varphi_1)$ is contained in the intersection of the image of m_1 and m_2.

Now suppose that we have some non-zero quadratic differential $\eta = \psi_1 \otimes \varphi_1 = \psi_2 \otimes \varphi_2$ which is also contained in the intersection of the images of m_i's. In particular, η has to vanish on the zero sets of φ_1 and the zero sets of φ_2. Since $\deg \omega_C^{\otimes 2} = 4g - 4$, the zero sets of η has to be of size $4g - 4$ counting with multiplicity. On the other hand, since the zero sets of φ_1 and φ_2 are disjoint, and the union is of size $4g-4$ counting with multiplicity, it follows that η must share the same zero sets as $\varphi_1 \otimes \varphi_2$ and hence are non-zero scalar multiples of $\varphi_1 \otimes \varphi_2$ as claimed. □

Corollary 5.2.4. *Suppose that $B \subset \mathcal{M}_g$ is a subvariety of $\mathcal{M}_g$ whose associated family of Jacobians has a d-dimensional isotrivial isogeny factor for some $d > 1$, and suppose that there exists some point $b = [C] \in B$ with an unramified map $g : C \to A_f$ that factors through the Abel-Jacobi map of C into its own Jacobian. Then* $\dim B \leq g-2$.

Proof. It's enough to show that the dimension of the subspace spanned by the images of m_1 and m_2 is at least $2g-1$. By Lemma 5.2.3, it's enough to find two holomorphic 1-forms on C whose zero sets are disjoint and are contained in the kernel of $\overline{\nabla}$.

Note that if we start with a holomorphic 1-form u on the isotrivial isogeny factor A_f, then its pullback $g^*(u)$ will be a holomorphic 1-form on C which is in the kernel of $\overline{\nabla}$. Therefore, we just need to find two such 1-forms on A_f whose pullback will have disjoint zero sets. Now a point $x \in C$ is in the zero set of $g^*(u)$ if and only if $g(C)$ is tangent to the foliation induced by u at the point $g(x)$. Because $T_{A_f} \cong \Omega_{A_f} = \mathcal{O}^{\dim A_f}$, we know that $H^0(A_f, \Omega_{A_f})$ is dual to $T_{A_f,x}$ for every point $x \in A_f$, and $g(C)$ being tangent to the foliation induced by u at $g(x)$ is the same as saying that u gives zero when paired with the image of the line $T_{C,x}$ inside $T_{A_f,g(x)}$. Since g is unramified, the map on tangent spaces is injective at every point, so the image of $T_{C,x}$ is always a line.

Therefore, to find a second holomorphic 1-form u' so that the zero set of $g^*(u')$ is disjoint from $g^*(u)$, we need to look for such u' in the complement of the hyperplane of differential forms in $H^0(A_f, \Omega_{A_f})$ which paired with the image of $T_{C,x}$ in $T_{A_f,g(x)}$ to be zero. Given that the zero set of $g^*(u)$ is a finite set, we are looking at the complement of finite union of hyperplanes which is non-empty, as desired. □

Remark 5.2.5.

1. Note that this bound is also sharp by the construction from the previous section; to satisfy the assumption that $g : C \to A_f$ is unramified, we can just require the covering map to be unramified.

2. The assumption is also necessary; without this condition, we can take a family of degree 2 covers of a curve C of genus 2 branched over $2g-6$ points [32, Theorem C]. In this case, the base is a generically finite cover of $C^{(n)}$ the n-fold symmetric product of C; this is of dimension $2g-6$ and when g is large, this is greater than $g-2$.

Appendix A

Examples of Kodaira fibrations

In this appendix, we record some constructions of Kodaira fibrations. The goal is to give examples that satisfy the assumptions of the results presented in Chapter 3. First we would like to construct Kodaira fibrations whose monodromy representation has no invariant factors. The key is to use the fact that a Kodaira fibration $f : X \to B$ is the same as a non-constant map from B into $\mathcal{M}_g$. Therefore, to construct a Kodaira fibration, it's enough to construct complete curves inside $\mathcal{M}_g$. To avoid issues with stacks, we will work instead with the fine moduli space $\mathcal{M}_g[n]$ of genus g curves with fixed level $n \geq 3$ structure. The following construction, which we called the moduli construction, is fairly well-known (see for example [17, Prop. 2.1]).

Moduli construction: Suppose $g \geq 4$ and consider the Satake compactification $\mathcal{M}_g[n]^*$, i.e., the closure of $\mathcal{M}_g[n]$ inside the Satake compactification $\mathcal{A}_g[n]^*$ of $\mathcal{A}_g[n]$ via the Torelli map $J : \mathcal{M}_g[n] \to \mathcal{A}_g[n]$. Since $\mathcal{M}_g[n]^*$ is projective, we may embed it into some large projective space and cut it with hyperplane sections and produce a curve. Now that when $g \geq 4$, the boundary component $\mathcal{M}_g[n]^* - J(\mathcal{M}_g[n])$ is of codimension at least 2 and the hyperelliptic locus $\mathcal{H}_g[n]$, where the Torelli map fails to be an immersion, is of codimension at least 2. Hence, a curve obtained by cutting $\mathcal{M}_g[n]^*$ with hyperplane sections can avoid the boundary component as well as the hyperelliptic locus, and corresponds to a Kodaira fibration via the universal property of $\mathcal{M}_g[n]$.

Lemma A.0.1. *Let $f : X \to B$ be a Kodaira fibration constructed via the moduli construction explained above. Then the image of the monodromy representation*

$$\rho : \pi_1(C) \to \mathrm{Sp}_{2g}(\mathbb{Z})$$

is of finite index inside $\mathrm{Sp}_{2g}(\mathbb{Z})$ and hence the monodromy action on $H^1(X_b, \mathbb{Z})$ has no invariants (i.e. $H^0(C, R^1\pi_*\mathbb{Z}) = 0$).

Proof. By Lefschetz's hyerplane theorem for quasi-projective varieties [18, page 153], the monodromy representation ρ factors through $\pi_1(\mathcal{M}_g[n])$ via a surjection:

$$\pi_1(C) \twoheadrightarrow \pi_1(\mathcal{M}_g[n]) \to \mathrm{Sp}_{2g}(\mathbb{Z}),$$

and by definition, the last map, which corresponds to the monodromy representation for the universal family over $\mathcal{M}_g[n]$, surjects onto the kernel of $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z})$, which certainly acts on $H^1(X_b, \mathbb{Z})$ with no invariants. $\square$

Next, we need to give examples of Kodaira fibrations with algebraic sections

Proposition A.0.2. *[5, Prop. 4.2] For every Kodaira fibration $f : X \to B$, there exists a Kodaira fibration $\tilde{f} : \tilde{X} \to \tilde{B}$ with an algebraic section such that*

1. *The fibers of $f : X \to B$ and $\tilde{f} : \tilde{X} \to \tilde{B}$ are of same genus g*

2. *The monodromy homomorphisms $\pi_1(B) \to \mathrm{Mod}_g$ and $\pi_1(\tilde{B}) \to \mathrm{Mod}_g$ have the same image.*

Appendix B

Some Tannakian formalisms

The main references for this section are [11] and [10, Appendix A].

Definition B.0.1. A neutral Tannakian category over k is a rigid abelian k-linear tensor category $\mathcal{C}$ for which there exists an exact faithful k-linear tensor functor $\omega : \mathcal{C} \to \mathbf{Vec}_k$. Any such functor ω is said to be a fiber functor for $\mathcal{C}$.

Given a neutral Tannakian category over k together with a fiber functor ω, we may define a functor $\mathrm{Aut}^{\otimes}(\omega)$ of k-algebras by setting $\mathrm{Aut}^{\otimes}(\omega)(R)$ to be the families (λ_X), where X is an object of $\mathcal{C}$ and λ_X is an R-linear automorphism of $X \otimes R$ such that $\lambda_{X_1 \otimes X_2} = \lambda_{X_1} \otimes \lambda_{X_2}$, $\lambda_{\underline{1}}$ is the identity on R, and for any morphism α from X to Y in $\mathcal{C}$, we have

$$\lambda_Y \circ (\alpha \otimes 1) = (\alpha \otimes 1) \circ \lambda_X : X \otimes R \to Y \otimes R.$$

The main theorem of the theory of Tannakian categories is the following:

Theorem B.0.2 ([11], Theorem 2.11)**.** *Let $\mathcal{C}$ be a Tannakian category over k equipped with a fiber functor $\omega : \mathcal{C} \to \boldsymbol{Vec}_k$. Then*

1. *the functor $\mathrm{Aut}^{\otimes}(\omega)$ is representable by an affine group scheme G*
2. *ω defines an equivalence of tensor categories $\mathcal{C} \to \boldsymbol{Rep}_k(G)$.*

This group G is called the Tannakian fundamental group of $\mathcal{C}$. We need to relate properties of morphisms between Tannakian fundamental groups to properties of functors between Tannakian categories.

Proposition B.0.3 ([11], Proposition 2.21)**.** *Let $f : G \to G'$ be a homomorphism of affine group schemes over k, and let f^* be the corresponding functor $\boldsymbol{Rep}_k(G') \to \boldsymbol{Rep}_k(G)$. Then*

1. *f is faithfully flat if and only if f^* is fully faithful and for any object X' in $\boldsymbol{Rep}_k(G')$, every subobject of $f^*(X')$ is isomorphic to the image of a subobject of X';*
2. *f is a closed immersion if and only if every object of $\boldsymbol{Rep}_k(G)$ is isomorphic to a subquotient of an object of the form $f^*(X')$ for some object X' of $\boldsymbol{Rep}_k(G')$.*

We also have a criterion for exactness:

Proposition B.0.4 ([10], Proposition A. 12+Proposition A. 13)**.** *Let*

$$K \xrightarrow{f} G \xrightarrow{g} H$$

be a sequence of affine group schemes. Then this sequence is exact in the middle if and only if

1. $f^* : \boldsymbol{Rep}_k(G) \to \boldsymbol{Rep}_k(K)$ *sends semi-simple object to semi-simple objects (i.e. it's an observable functor);*

2. *for every* $M \in \boldsymbol{Rep}_k(G)$*, there exists* $U \in \boldsymbol{Rep}_k(H)$ *such that the maximal trivial subobject of* f^*M *comes from* $g^*(U) \subset M$.

Appendix C

Proof of Proposition 4.5.6

All the ingredients needed for a proof of Proposition 4.5.6 are contained in [21], although they did not do this monodromy computation explicitly. They were mostly concerned with computing the monodromy a related local system, which we will call $\mathcal{E}_{\mathrm{cer}}$, on $\mathcal{M}_{g,1}$ and we explain how to compute the monodromy of $j^*\mathcal{E}_0$ by using their results and some standard facts about the Johnson homomorphism (see [15, section 6.6]).

We first set up some notations. Fix some integer $g \geq 2$. Let $f : \mathrm{Pic}^0_{\mathcal{C}_{g,1}/\mathcal{M}_{g,1}} \to \mathcal{M}_{g,1}$ be the relative Jacobian of the universal curve $\mathcal{C}_{g,1}$ of genus g with 1 marked point and $[C,x] \in \mathcal{M}_{g,1}$ some base point. Let $\xi : \mathcal{M}_{g,1} \to \mathcal{C}_{g,1}$ be the unique section. Define two local systems on $\mathcal{M}_{g,1}$:

$$\mathbb{L}_{\mathbb{Z}} := R^{2g-3} f_* \mathbb{Z}(g-1) \qquad \mathbb{H}_{\mathbb{Z}} := R^{2g-1} f_* \mathbb{Z}(g).$$

Note that $\mathbb{H}_{\mathbb{Z}}$ is the local system corresponding to the action of $\pi_1(\mathcal{M}_{g,1}, [C,x])$ on $H := H^1(C, \mathbb{Z}(1))$ and $\mathbb{L}_{\mathbb{Z}}$ is the local system corresponding to the action of $\pi_1(\mathcal{M}_{g,1}, [C,x])$ on $L := [\wedge^3 H](-1)$. There's a natural projection map $c : L \to H$ defined by

$$c(x \wedge y \wedge z) = \omega(x,y)z + \omega(y,z)x + \omega(z,x)y$$

where ω is the intersection form.

The precise definition of the local system $\mathcal{E}_{\mathrm{cer}}$ is not relevant to our story. The key is that it's an extension of the form

$$0 \to \mathbb{L}_{\mathbb{Z}} \to \mathcal{E}_{\mathrm{cer}} \to \underline{\mathbb{Z}(0)} \to 0.$$

More importantly, by [46, Corollary 6.7], we know that the map c is equivariant with respect to the action of $\pi_1(\mathcal{M}_{g,1})$ and hence induces a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{L}_{\mathbb{Z}} \arrow[r] \arrow[d, two heads] & \mathcal{E}_{\mathrm{cer}} \arrow[r] \arrow[d] & \underline{\mathbb{Z}(0)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{H}_{\mathbb{Z}} \arrow[r] & j^*\mathcal{E}_0 \arrow[r] & \underline{\mathbb{Z}(0)} \arrow[r] & 0
\end{tikzcd}$$

In particular, the monodromy representation associated to $(j^*\mathcal{E}_0)|_C$ is determined by sending a loop

$\gamma \in \pi_1(C, x)$ to a map of the form

$$\mathbb{Z} \to L \xrightarrow{c} H.$$

Fix such a loop γ. Notice that $\pi_1(C, x)$ must be contained in the Torelli subgroup $T_{g,1}$ of $\pi_1(\mathcal{M}_{g,1})$ as it's already trivial under the projection onto $\pi_1(\mathcal{M}_g)$. The key result of Hain and Matsumoto is the following

Proposition C.0.1 (Proposition 6.4 in [21])**.** *The monodromy of $\mathcal{E}_{cer}$ around γ is given by the map $\gamma \mapsto \tau([\gamma])$, where τ is the Johnson homomorphism $\tau : T_{g,1} \to L$, and $[\gamma]$ is the homology class of γ in $H_1(C, \mathbb{Z})$.*

Proposition 4.5.6 now follows immediately:

Proof of Proposition 4.5.6. It's enough to show that the composition

$$\pi_1(C, x) \to T_{g,1} \xrightarrow{\tau} L \xrightarrow{c} H$$

is given by sending γ to $(g-1) \cdot [\gamma]$. Now we know that the map $\pi_1(C, x) \to T_{g,1} \xrightarrow{\tau} L$ has the following explicit description (see e.g. [15]): let $\{x_i, y_i\}_{i=1}^{g}$ be a symplectic basis for H with respect to the intersection form ω and then this map is given by

$$\begin{aligned} \pi_1(C, x) &\longrightarrow L \\ \gamma &\mapsto \left(\sum x_i \wedge y_i\right) \wedge [\gamma] \end{aligned}$$

Suppose $[\gamma] = \sum \alpha_i x_i + \beta_j y_i$. We may now compute that

$$\begin{aligned} c\left(\left(\sum x_i \wedge y_i\right) \wedge [\gamma]\right) &= \sum_{i=1}^{g} c(x_i \wedge y_i \wedge [\gamma]) \\ &= \sum_{i=1}^{g} [\gamma] + \omega(y_i, [\gamma]) x_i + \omega([\gamma], x_i) y_i \\ &= \sum_{i=1}^{g} [\gamma] - \alpha_i x_i - \beta_i y_i \\ &= (g-1)[\gamma] \end{aligned}$$

This concludes the proof of Proposition 4.5.6. $\square$

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