# PARTIAL DESINGULARIZATION PRESERVING NORMAL CROSSINGS AND MINIMAL SINGULARITIES IN LOW DIMENSION

by

RAMON RONZON LAVIE

A thesis submitted in conformity with
the requirements for the degree of

Doctor of Philosophy

Department of Mathematics
University of Toronto

# ABSTRACT


Partial Desingularization Preserving Normal Crossings and Minimal Singularities in Low Dimension

Ramon Ronzon Lavie

Doctor of Philosophy

Department of Mathematics

University of Toronto

2025

In this work we address the problem of partial desingularization while preserving normal crossings for algebraic varieties of dimension $\leqslant 4$ defined over an algebraically closed field of characteristic zero. The main result provides a sequence of smooth blowings-up where each blow-up preserves the normal crossings locus and such that the resulting variety has only singularities in a given *minimal finite list* $\mathscr{N}$, where each element in $\mathscr{N}$ is expressed in a precise local *normal form*. Each element of $\mathscr{N}$ can be described using determinants of *circulant matrices*, which are the $n \times n$ matrices spanned by the permutation matrices associated to *cyclic permutations* of $S_n$. This type of result is the best possible result that one may expect, as there is no resolution procedure — which works for all varieties — preserving the normal crossings locus and such that the resulting variety has only normal crossings singularities. The results in this thesis also apply for complex analytic varieties, but we do not provide the detailed proofs in this case.

The key tools used to obtain the main result are called *splitting lemmas*, *cleaning sequences* and an *algorithm* for eliminating all non-minimal *limit singularities* of the neighbours of *circulant normal forms*.

As a first step towards our main goal, we provide a description for non normal crossings singularities in the closure of the normal crossings locus of X, after suitable blowings-up. Let us provide a more precise statement. Let $a$ be a non-normal crossings singularity which is a limit point of a smooth curve $C$ of normal crossings points of $X$, that is, assume that for all $b \in C$ in a punctured neighbourhood $U \setminus \{a\}$ of $a$ we have that $X$ is normal crossings at $b$. Then, we find a proper birational morphism $\sigma : X' \to X$ consisting of a composition of *equimultiple* blowings-up so that, if $f = 0$ is a local equation defining $X'$ at $a' \in \sigma^{-1}(a)$, then $f = f_1, \ldots, f_n$ where each $f_i$ is a smooth element of a finite extension $S$ of the ring of functions of $X'$, given by formal power substitutions. This is what we call the *splitting theorem*.

We then perform another sequence of equimultiple blowings-up after which we can express the irreducible components of $X$ at $a$ as elements of a given finite family of singularities. We call these blow-up sequences *cleaning sequences*. The local equation defining $X$ at these singularities is expressed in terms of determinants of *circulant matrices*.

A *circulant singularity* is a simple example in this family that occurs as a limit of the normal crossings locus, and it is a generalization in arbitrary dimension of the *pinch-point* of the Whitney umbrella.

Circulant singularities cannot be eliminated by birational morphisms that preserve the normal crossings locus. Thus, we require subsequent blowings-up to preserve circulant singularities. Consequently, the locus of neighbouring singularities of a circulant point is also preserved; and so we need to eliminate the limits of neighbouring singularities that do not belong to $\mathscr{N}$. For this, we develop an *algorithm* that finds a finite sequence of equimultiple blowings-up using only *combinatorial centres*, such that we can cover the fibre of the circulant point with a finite cover of affine charts where the origin of each affine chart is in normal form. Moreover, when we apply this moving away algorithm to varieties of dimension $\leqslant 4$, we may express all limits of the neighbouring locus in a minimal finite family $\mathscr{N}$ of singularities.

*A mi abuelo,*

*Por sus incontables palabras de aliento y cariño.*

# ACKNOWLEDGEMENTS

I am very grateful to Prof. Edward Bierstone for the enormous amount of help over many years and at every stage of the project. I also thank Profs. Askold Khovanskii and Stephen Kudla who worked as members of the Supervisory Committee since 2019. Thanks to Prof. Tristan Collins for agreeing to serve as part of the Examination Committee. Thanks in advance to Prof. Mickaël Matusinski, from Université de Bordeaux, for accepting to function as external appraiser. Special thanks to Prof. Pierre Milman for his financial support in the final year of this project.

This work was created with the financial support provided by the CONACyT (now CONAHCyT) scholarship (Becas al Extranjero 2018-000009-01EXTF-00250).

Thanks to my friend Malors for encouraging me to resume my academic path. Thanks to my friends Ulises, Vincent and Adriano all of whom have helped me keep the impostor syndrome at bay, and provided some needed company during pandemic times. I need to point out how lucky I am for all the support Alice Rolf gave me. I also thank Alice for a couple of discussions on categorical notions, which gave me enough clarity to establish Remark 3.51. Thanks to Charlie Wu for listening to my ramblings about normalization. Apologies to Matthew Bolan as I was unable to deduce a reduction to normal form theorem for singularities which admit an action by a product of two cyclic groups. Otherwise, I would have included one of his ideas addressing part of this problem.

# CONTENTS

1

# INTRODUCTION

This work concerns partial desingularization, that is, a procedure that resolves all singularities except those belonging to a given family of singularities. More precisely, we present an algorithm that preserves the normal crossings locus of a variety $X$ over an algebraically closed field of characteristic zero, and we show that there is a *minimal finite family* $\mathscr{N}$ of singularities that remains after performing this procedure. Moreover, we provide an explicit description of each element of $\mathscr{N}$.

## 1.1 MAIN RESULTS

Consider a variety $X$ defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. One of the simplest models of singularities is a *simple normal crossings singularity*, which are points where $X$, in a neighbourhood $U$ of $a$, can be described as

$$\{x_1 \dots x_n = 0\}, \tag{1.1}$$

for some regular coordinate system $x_1, \dots, x_n, u_1, \dots, u_m$ defined in $U$. If there is a *formal* (or *étale*, or *analytic*) coordinate system defined on $U$ expressing $X$ as in (1.1), then we say that $a$ is a *normal crossings singularity* of $X$ of order $n$ ($\mathrm{nc}(n)$ for short).

*Circulant singularities* are singularities that can be expressed as the determinant of a circulant matrix. Given indeterminates $a_0, \dots, a_{n-1}$, we define the *circulant matrix* $C(a_0, \dots, a_{n-1})$ as the matrix

$$\begin{bmatrix} a_0 & a_{n-1} & \dots & a_1 \\ a_1 & a_0 & \dots & a_2 \\ \ddots & \ddots & \ddots & \ddots \\ a_{n-1} & a_{n-2} & \dots & a_0 \end{bmatrix}.$$

We say that $a \in X$ is a *circulant singularity of order* $n$, $\mathrm{cp}(n)$ for short[1], if there exists a formal (or étale, or analytic) coordinate system $w, x_1, \dots, x_{n-1}, z$ at $a$ such that $X$ is locally given by the vanishing locus of

$$\begin{aligned} \Delta_n(z, w^{1/n}x_1, \dots, w^{(n-1)/n}x_{n-1}) :=& \\ \det(C(z, w^{1/n}x_1, \dots, w^{(n-1)/n}x_{n-1})).& \end{aligned}$$

1 The initials cp come from "cyclic point", which was one of the previous terms used to refer to this type of singularities, see [BM12] and [BLM12].

An example of a circulant singularity is the *pinch-point*, that is, the singularity at the origin of the Whitney umbrella
$$\{z^2 - wx^2 = 0\}$$
(see [GH14], Chapter IV, Section 6). Circulant singularities occur as a limit of normal crossings singularities, and they cannot be eliminated while preserving the normal crossings locus.

A *resolution of singularities* of $X$ is a proper birational morphism $X' \xrightarrow{\sigma} X$ such that the singular locus of $X'$ is empty. In this work, we are mainly interested in resolutions given by compositions of blowings-up, that is, $\sigma = \sigma_t \circ \ldots \circ \sigma_1$ where,

$$X' := X_t \xrightarrow{\sigma_t} X_{t-1} \xrightarrow{\sigma_{t-1}} \ldots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_0} X_0 := X, \tag{1.2}$$

are blowings-up. Let us present some terms of the language of resolution of singularities that make the discussion of the desingularization techniques simpler.

Given a pair $(X, E)$ consisting of a variety $X$ embedded in a smooth variety $Z$ and a simple normal crossings divisor $E \subset Z$ (see Definition 2.44), we say that the blow-up $\sigma : X' \to X$ with centre $C$ is $(X, E)$-*admissible* if for every point $a \in C$ there exists a regular coordinate system around $a$ for which $C$ is a coordinate subspace, $E$ is the union of coordinate hyperplanes, and $C$ is not tangent to any component of $E$. Moreover, the morphisms in a sequence of blowings-up as in (1.2) are called $(X, E)$-*admissible* if for each $k \in \{1, \ldots, t\}$ we have that $\sigma_k$ is $(X_k, E_k)$-admissible, where $X_k$ is the strict transform of $X_0$ and $E_k$ denotes the strict transform of $E_0$ together with the exceptional divisors, that is, the sum of the exceptional divisors created by $\sigma_1, \ldots, \sigma_{k-1}, \sigma_k$. To simplify our notation, and when the context is clear, we simply say that $C$ is admissible. We say that $(X, E)$ is *simple normal crossings* at $a$ if $X \cup E$ is simple normal crossings at $a$, and the pair is *normal crossings* at $a$ if $X \cup E$ is normal crossings at $a$.

If every morphism $\sigma_k$ in (1.2) restricts to an isomorphism on the simple normal crossings locus of $X$, we say that (1.2) is a *partial resolution sequence* preserving simple normal crossings. If $X'$ is simple normal crossings at all points, then we say that the sequence is *strong*. In [BM97], [BM12], and [BdSMV14], Bierstone et al. show that for all $X$, there is a strong partial resolution sequence that preserves simple normal crossings. This leads to posing the question of existence of strong partial resolutions preserving normal crossings. In Example 8 of [Kol08], Kollár shows that any birational morphism that preserves the $w$-axis of the Whitney umbrella $\{z^2 - wx^2 = 0\}$ also preserves the singularity at the origin. As such, the Whitney umbrella is an example of a variety that cannot admit a *strong partial resolution preserving normal crossings*. Nonetheless, in [BM12] and [BLM12], Bierstone *et al.* show that if $\dim X \leqslant 3$, then there is a *minimal finite family* of singularities $\mathscr{N}$ together with a finite sequence of equimultiple blowings-up, such that every singularity of $X'$ is in $\mathscr{N}$. As such, we say that $X$ admits a *partial resolution preserving normal crossings* if there exists a finite family of singularities $\mathscr{N}$ and a finite sequence of blowings-up as in (1.2) such that all singularities of $X'$ belong to $\mathscr{N}$. In this case, we say that (1.2) is a *partial resolution sequence*.

The partial desingularization procedure we present in this work has extra desirable properties. Let us provide the relevant definitions. We say that $X$ has *vanishing order* $d$ at $a$, denoted by $\mathrm{ord}_X(a)$, if the stalk $\mathcal{O}_{X,a}$ satisfies that $\mathcal{O}_{X,a} \subset \mathfrak{m}_a^d$ but $\mathcal{O}_{X,a} \not\subset \mathfrak{m}_a^{d-1}$. If $(X, E)$ is normal crossings at $a$ and there exist $r$ irreducible components of $E$ passing through $a$, we say that $(X, E)$ is *normal crossings of order* $(n, r)$, or $a$ is $\mathrm{nc}(n, r)$ for short, where $n = \mathrm{ord}_X(a)$. We consider the class of nc singularities to be ordered with the lexicographic ordering on the pairs $(n, r)$. If $r = 0$ we simplify this to $\mathrm{nc}(n)$. Given a subset $C \subset X$, we say that $X$ is *equimultiple* along $C$ if for any two elements $a, b \in C$ we have that $\mathrm{ord}_X(a) = \mathrm{ord}_X(b)$. We say that a pair $(X, E)$ is *equimultiple* along $C$ if $X \cup E$ is equimultiple along $C$.

The ideas present in this work are built on those present in [BM12], [BLM12], and [BdSMV14]. Let us state the main results in this work. Throughout this thesis, $\mathbb{K}$ is an algebraically closed field of characteristic zero.

**Theorem 1.1** (Partial Desingularization Theorem)**.** *There exists a finite collection of singularities $\mathscr{N}$ such that, for any pair $(X, E)$ consisting of a $\mathbb{K}$-variety $X$ of dimension $\leqslant 4$ embedded into a smooth variety $Z$ and a simple normal crossings divisor $E \subset Z$, there is a proper birational morphism $\sigma : Z' \to Z$ given by a composition of blowings-up with smooth equimultiple centres preserving normal crossings, and such that all singularities of the strict transform $X'$ of $X$ are in $\mathscr{N}$. Moreover, we can describe the elements of $\mathscr{N}$ in local coordinates $w, x, z$ as in Table 1.1 below.*

In the case where $\dim X = 2$, we can find local coordinates $w, x, z$ to describe the elements of $\mathscr{N}$ in the following *normal forms*

1. $\mathrm{nc}(1) : z$,
2. $\mathrm{nc}(2) : xz$,
3. $\mathrm{nc}(3) : wxz$,
4. $\mathrm{cp}(2) : \Delta_2(z, w^{1/2}x) = z^2 - wx^2$.

In the case where $\dim X = 3$, there are local coordinates $w, x_1, x_2, z$ in which we can describe the elements of $\mathscr{N}$ using the following normal forms

1. $\mathrm{nc}(k)$, for $1 \leqslant k \leqslant 4$,
2. $\mathrm{cp}(2)$,
3. $\mathrm{cp}(3) : \Delta_3(z, w^{1/3}x_1, w^{2/3}x_2) = z^3 - 3wx_1x_2z + wx_1^3 + w^2x_2^3$,
4. Degenerate pinch-point: $\Delta_3(z, w^{1/3}x_1, w^{2/3})$,
5. $\mathrm{nc}(1) \times \mathrm{cp}(2) : x_2\Delta_2(z, w^{1/2}x_1)$.

The following is a table containing the normal forms for minimal singularities in dimension four.

1. $\mathrm{nc}(k)$, for some $k \leqslant 5$,
2. $\mathrm{cp}(k)$, for some $2 \leqslant k \leqslant 4$,
3. $\Delta_4(z, w^{1/4}x_1, w^{2/4}x_2, w^{3/4})$,
4. $\Delta_4(z, w^{1/4}x_1, w^{2/4}, w^{3/4}x_2x_3)$,
5. $\Delta_4(z, w^{1/4}x_1, w^{2/4}, w^{3/4}x_2)$,
6. $\Delta_4(z, w^{1/4}x_1, w^{2/4}, w^{3/4})$,
7. $x_2\Delta_3(z, w^{1/3}x_1, w^{2/3}))$,
8. Degenerate pinch-point,
9. $\mathrm{nc}(k) \times \mathrm{cp}(\ell)$, where $k + \ell \leqslant 4$ and $2 \leqslant \ell$.

Table 1.1: List of normal forms for minimal singularities in dimension 4.

**Theorem 1.2** (Partial Desingularization for Bounded Order)**.** *Let $(X, E)$ be a pair in arbitrary dimension. Then there is a finite sequence of blowings-up with smooth equimultiple centres preserving the normal crossings locus of $(X, E)$ of order at most $(3, 0)$, and such that all the singularities of the pair $(X', E')$ are in $\mathscr{N}$ (see Table 1.1 for the list of normal forms), where $X'$ is the strict transform of $X$ and $E'$ denotes the strict transform of $E$.*

The key tools used to prove Theorem 1.1 and Theorem 1.2 are *splitting theorems*, *cleaning theorems* and a *moving away algorithm*. The following are versions of these statements to illustrate the types of results they entail.

**Theorem 1.3** (Splitting Theorem)**.** *Let $n \geqslant 2$, and consider a hypersurface $X \hookrightarrow Z$ where $Z$ is a smooth variety over $\mathbb{K}$ with $\dim Z = n + 1$, that is, $\dim X = n$. Assume there is a smooth curve $C \subset X$ such that $X$ is $\mathrm{nc}(n, 0)$ everywhere in $C$, except at isolated points. Then, after a finite sequence of blowings-up*

$$X' := X_t \xrightarrow{\sigma_t} \dots \xrightarrow{\sigma_1} X_0 := X$$

*with discrete centres inside the successive strict transforms of $C$ such that, at each point $a' \in X'$ over $a \in C$ where $X$ is not $\mathrm{nc}$ at $a$, there exist étale coordinates (or analytic, or formal coordinates) $w, x_1, \dots, x_{n-1}, z$ satisfying that $C$ is locally defined by $\{z = x_1 = \dots = x_{n-1} = 0\}$ and, if $f(w, x, z) = 0$ is a local equation of the strict transform $X'$ of $X$, then there exists $p \in \mathbb{N}$ such that $f(w^{1/p}, x, z)$ splits into $n$ irreducible factors of vanishing order $1$.*

**Theorem 1.4** (Reduction to a Product of Circulants)**.** *Let $n \geqslant 2$, and let $X$ be a hypersurface of a smooth variety $Z$ over $\mathbb{K}$ with $\dim X = n$, together with a $\mathrm{snc}$ divisor $E \subset Z$. Assume that after a suitable sequence of admissible and equimultiple blowings-up for $(X, E)$, there is a curve $C$ in the strict transform $X'$ of $X$ such that $X'$ is generically $\mathrm{nc}(n)$ in $C$. Let $a \in C$ be such that $X$ is not $\mathrm{nc}(n)$ at $a$. Then, there are a sequence of admissible and equimultiple blowings-up for $(X, E)$ that restrict to an isomorphism on the $(X, E)$-normal crossings locus, together with étale coordinates (or analytic, or formal coordinates) $w, x_{1,0}, \dots, x_{1,d_1-1}, \dots, x_{s,0}, \dots, x_{x,d_s-1}, u_1, \dots, u_m$ defined at the point $a'$ in the final transform $X''$ of $X$ over $a$, such that $X''$ is a product of circulants at $a'$. More precisely, $X''$ can be locally described at $a'$ as the vanishing locus of*

$$\prod_{i=1}^{s} \Delta_{d_i}(x_{i,0}, w^{1/d_i} x_{i,1}, \dots, w^{(d_i-1)/d_i} x_{i,d_i-1}) = 0.$$

Notice that applying Theorem 1.4 at every non-nc point of (the respective transform of) $C$, we deduce that $X$ is a product of circulants at all non-nc points in $C$.

**Algorithm 1.5** (Moving Away Algorithm)**.** *Given an integer $n$, there is a finite list of singularities $\mathscr{N}$, such that for any pair $(X, E)$ where $\dim X = n$, we can construct a finite sequence of blowings-up*

$$X' := X_t \xrightarrow{\sigma_t} \dots \xrightarrow{\sigma_1} X_0 := X,$$

*with admissible and equimultiple centres for $(X, E)$ where each $\sigma_j$ preserves the $(X, E)$-$\mathrm{nc}$ locus and if we define $\sigma := \sigma_t \circ \dots \sigma_0$ then there is a smooth* distinguished divisor *$\mathscr{D}$, which is a union of components of $E_t$ containing the $\mathrm{cp}(n)$ locus of $(X_t, E_t)$, and a finite affine cover $\{U_\alpha \subset Z_t\}_{\alpha \in A}$ of $\mathscr{D}$ where the origin in each $U_\alpha$ is a singularity in $\mathscr{N}$ and $X_t$ contains only $(X_t, E_t)$-$\mathrm{nc}$ singularities in $\cup_\alpha U_\alpha \setminus \mathscr{D}$. Moreover, if $n \leqslant 4$, then all the singularities of $X_t$ inside $\mathscr{D}$ are in $\mathscr{N}$.*

*INPUT: The finite list of isolated singularities of* $\mathrm{cp}(n)$ *of* $X$.
*OUTPUT: A finite list of charts* $\{U_\alpha\}_{\alpha \in A}$, *where the origin is a singularity in* $\mathcal{N}$.

The main part of this work was published in [BdSBRL25], where we can find different versions of Theorem 1.1 and Theorem 1.2. The overall strategy for partial desingularization followed in [BdSBRL25] and in this thesis is the same, where the key differences lie in the moving away procedure. More precisely, [BdSBRL25] and this thesis, both present two versions of a splitting theorem and two versions of a cleaning theorem. The sequences of blowings-up used in splitting and cleaning theorems have admissible and equimultiple centres in both works. On the other hand, in [BdSBRL25], the moving away procedure is carried out explicitly for $\mathrm{cp}(4)$ singularities, where the sequence of blowings-up is admissible, but not necessarily equimultiple. In this thesis, we present a moving away procedure such that, after a suitable sequence of admissible and *equimultiple* blowings-up, all the limit singularities of the neighbouring locus of the $\mathrm{cp}(4)$ locus belong to a finite list of normal forms. Nonetheless, the *minimal list* of singularities in both works is the same. Notice that the moving away procedure in this thesis works for $\mathrm{cp}(n)$ singularities, for arbitrary values of $n$, but the problem of determining if *all* the remaining singularities can be put in a finite family of local expressions is still open for $n \geqslant 5$.

## 1.2 BACKGROUND OF THE PROBLEM AND MOTIVATION

One of the most important results in the area of resolution of singularities is that of [BM97]. Given a variety $X$, Bierstone and Milman find a proper birational morphism $\sigma : X' \to X$, which is explicitly described as a composition of blowing-up morphisms with smooth centres, and such that $X'$ is a smooth variety.

Even though this desingularization strategy allows us to find a smooth model for any variety, there are contexts in which we need to admit models with certain types of "mild" singularities.

For example, if we consider the total transform $\overline{X}$ of $X$ instead of the strict transform, we obtain that $\overline{X}$ is in simple normal crossings at all points. A similar phenomenon occurs when we consider the *log-resolution* of an ideal with respect to a divisor $D$: the variety resulting from a log-resolution cannot be assumed to be smooth but rather in simple normal crossings.

In [BM97], [BM12], and [BdSMV14], Bierstone *et al.* show that given any variety $X$, we can find an suitable sequence of blowings-up

$$X_t \xrightarrow{\sigma_t} X_{t-1} \xrightarrow{\sigma_{t-1}} \ldots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 := X,$$

where each $\sigma_j$ restricts to an isomorphism over the *simple normal crossings* locus and such that every point in $\mathrm{Sing}\,(X_t)$ is simple normal crossings.

In order to provide some insight on the difference between the notions of snc and nc, let us consider the *Whitney umbrella*, that is, the variety $X \subset \mathbb{A}^3_{\mathbb{C}}$ given by the locus $\{z^2 - wx^2 = 0\}$. Notice that at any point $a \neq 0$ in the $w$-axis we can consider the local formal factorization

$$z^2 - wx^2 = (z - \sqrt{w}x)(z + \sqrt{w}x).$$

Briefly, the singularities (that are not the pinch-point!) of the Whitney umbrella are normal crossings but not simple normal crossings because the factorization happens in étale coordinates.

Let us go back to the question of partial desingularization. We claim that there is no strong partial desingularization preserving normal crossings for all varieties. Consider again the variety $X$ given by the Whitney umbrella. We have established that $X$ is normal crossings at any non-zero point of the $w$-axis. On the other hand, $X$ is not normal crossings at the origin. Moreover, any blow-up whose centre only contains singularities of $X$ and that restricts to an isomorphism over the normal crossings locus can only have the origin as a centre of blow-up. But after blowing-up the pinch-point $\sigma : X' \to X$, we obtain that $X'$ also has a pinch-point singularity.

In short, any blow-up that preserves the normal crossings locus of $X$ also preserves the pinch-point. This phenomenon of the insolubility of the origin in the Whitney umbrella while preserving the normal crossings locus has been previously identified in [Kol08] and [BM12].

Given that the limit singularities of the normal crossings locus cannot be eliminated in general, we need to admit extra singularities. This leads us to the following.

**Question.** Is there a minimal family of singularities $\mathscr{N}$ such that, for any variety $X$, there is a partial desingularization sequence preserving normal crossings and if $X'$ denotes the strict transform of $X$, then all the singularities of $X'$ are in $\mathscr{N}$?

[BM12] and [BLM12] are a sequence of papers that give a positive answer to the question above, when $\dim X \leqslant 3$. In said papers, Bierstone *et al.* explain why it is not possible to obtain a strong partial desingularization result, and why we need to allow for extra singularities, e.g. *circulant points*, which are singularities that occur as limit points of a normal crossings locus. They also present a list $\mathscr{N}$ of *minimal singularities*, each described in local coordinates, that remain after a partial desingularization preserving the normal crossings locus of a variety of dimension $\leqslant 3$. The central idea in [BM12] and [BLM12] is to follow the algorithm for the classical resolution of singularities (see [BM97]) by blowing-up the stratum with maximal value of inv (see Section 1.3 for details) at points which are not limit points of the normal crossings locus. After this, for each limit point $a \in X$ of the normal crossings locus, the authors find a sequence of blowings-up that reduces $a$ to a *circulant normal form*.[2] More precisely, this sequence of blow-up morphisms is such that if $a'$ is a limit point of the strict transform of the normal crossings locus, then $a'$ is an element of $\mathscr{N}$.

In this thesis we can find the tools and techniques that allow to deduce a positive answer for the question above, for varieties $X$ with $\dim X \leqslant 4$.

## 1.3 GENERAL STRATEGY OF THE PARTIAL DESINGULARIZATION ALGORITHM

In order to properly address the strategy for partial desingularization we follow, let us briefly discuss the classical desingularization strategy in [BM97].

In said paper, Bierstone and Milman construct a desingularization invariant $\mathrm{inv} := \mathrm{inv}_X$ which is Zariski upper-semicontinuous function with respect to the lexicographic order,

2 This result combines the use of a *splitting lemma* together with a *cleaning lemma*.

and the locus of points where inv is constant is a locally closed smooth variety. Moreover, the loci of points with constant inv value form a finite partition of $X$ by locally closed smooth sets. Moreover, for any $I$ we have that the topological closure of $\{b \in X : \mathrm{inv}(b) = I\}$ is $\{b \in X : \mathrm{inv}(b) \geqslant I\}$. These properties allow us to define a stratification of $X$. If $X$ has singular points then, by blowing-up the collection of points with maximal value of inv, we obtain a morphism $X_1 \xrightarrow{\sigma_1} X_0 := X$, which restricts to an isomorphism over the collection of smooth points of $X$. Because all points in the centre of blow-up of $\sigma_1$ are inv-constant, we say that $\sigma_1$ is inv-*admissible*.

Moreover, given a sequence of inv-admissible blowings-up

$$X_r \xrightarrow{\sigma_r} \dots \xrightarrow{\sigma_1} X_0 := X, \tag{1.3}$$

where each $\sigma_i$ is inv-admissible, then the authors construct functions $\mathrm{inv} : X_k \to \Sigma$ for each $k \in \{1, \dots, r\}$. In particular, each $X_k \to \Sigma$ is dependent on the functions $X_i \to \Sigma$ for $i < k$. The main result of [BM97] is that blowing-up the stratum with maximal value of inv, after finitely many steps, gives us a resolution of singularities of $X$ preserving the locus of smooth points of $X$.

Let us go back to the description of the general strategy for partial desingularization present in this thesis. The algorithm we present can be described recursively, like we do in Chapter 6, or iteratively, where the iterates are pairs $(n, r) \in \mathbb{Z}^2_{\geqslant 0}$ ordered lexicographically. In the iterative description, a pair $(n, r)$ represents the locus of points $S_{n,r}$ with invariant equal to

$$\mathrm{inv}_{n,r} := (n, r, 1, 0, \dots, 1, 0, \infty),$$

where the right hand side of the equation has $2(n + r) + 1$ entries, and each iteration can be separated into 4 *stages*. Notice that $\mathrm{inv}_{n,r}$ is the value of inv at a normal crossings point of order $(n, r)$, and so $S_{n,r}$ contains all $\mathrm{nc}(n, r)$ singularities. Let $(X, E)$ be a pair where $X \hookrightarrow Z$ is embedded in a smooth variety $Z$. Let us describe the structure of the general step for a pair $(n, r)$.

In the *first stage*, we follow the classical desingularization procedure (see [BM97]) until we may assume that

- outside of a closed subvariety $\Sigma_{(n,r)^+}$ there are no singularities with invariant $> \mathrm{inv}_{n,r}$, where $(n, r)^+$ stands for the successor of $(n, r)$, which can be $(n, r + 1)$ or $(n + 1, 0)$ depending on $n, r$ and $\dim X$ (see Chapter 6 for more details on this). Let us assume that $(n, r)^+ = (n, r + 1)$.
- all singularities in $\Sigma_{n,r+1}$ are in normal form,
- the inv-admissible stratum $S_{n,r} \subset X \setminus \Sigma_{n,r}$ of points with invariant equal to $\mathrm{inv}_{n,r}$ is generically $\mathrm{nc}(n, r)$.

The variety $\Sigma_{n,r+1}$ consists of a finite union of *distinguished* components of $E$ and finitely many inv-constant strata (see (1.4) below to find how $\Sigma_{n,r+1}$ is iteratively built-up).

In the *second stage*, we follow a blow-up sequence given by a *splitting theorem*, after which we can split the local expression of the strict transform of $X$ in a finite formal extension of $\mathcal{O}_Z$ at any limit point of $S_{n,r}$.

In the *third stage*, we follow an *cleaning sequence* of blowings-up, after which we may reduce any limit point of $S_{n,r}$ to a circulant normal form, i.e. a product of circulants (see Definition 3.13.[3]

In the *fourth stage*, we follow a *move away* procedure; or more precisely, a version of Algorithm 1.5. After this blow-up sequence, all the limit points of the neighbouring locus of a singularity in the closure of $S_{n,r}$ lie inside a smooth divisor $\mathscr{D}_{n,r}$ and can be expressed in a finite list of normal forms.

We can now proceed to the next iterate $(n,r)^-$, which we define as $(n, r-1)$, if $r > 0$ (or $(n-1, r')$ for some $r'$, if $r = 0$). Let us assume that $(n,r)^- = (n, r-1)$. Then,

$$\Sigma_{n,r} := \Sigma_{(n,r+1)} \cup \mathscr{D}_{n,r} \cup S_{n,r} \tag{1.4}$$

After finitely many steps, we obtain a partial desingularization sequence after which, all singularities of $(X, E)$ admit a local expression in one of the normal forms of Table 1.1, for varieties with $\dim X \leqslant 4$.

## 1.4 STRUCTURE OF THE THESIS

Chapter 2 is a compilation of the elements of algebraic geometry, birational geometry, and resolution of singularities that are needed in the approach we follow for partial resolution.

Chapter 3 is dedicated to deducing the properties of circulant singularities that are relevant to the results in this thesis. For example, we show that if $X' \to X$ is the normalization of $X$, then $X'$ is smooth at the point of the fibre of the circulant points of $X$. We also show that the ring of functions defined around a circulant singularity admits an action by a *cyclic group*.

At the end of this chapter we present the notion of *group precirculant singularities*, which are singularities that are a limit of a normal crossings locus, and such that the ring of functions locally defined around them admits an action by an *abelian group*. This notion is not relevant to the main result of this thesis, but it is included as I believe it could prove useful when addressing the problem of partial desingularization in higher dimensions, but more work is needed.

In Chapter 4, we present two versions of a *splitting theorem*. One of them is Theorem 1.3. The other version is concerned with the splitting at limit points of a normal crossings locus of order 3, regardless of $\dim X$. The latter is the version needed for partial desingularization in dimension $\leqslant 4$, but the former is a result which gives partial progress addressing the problem of partial desingularization in arbitrary dimension.

In Chapter 5, we present two versions of a *reduction to normal form theorem*. The version given by Theorem 1.4 addresses the problem of finding a minimal family of normal forms for the limits of the one-dimensional stratum of $\mathrm{nc}(n)$ singularities of an $n$-dimensional variety, for arbitrary values of $n$. The other reduction theorem, establishes the existence of a minimal family of singularities that occur as limits of the $\mathrm{nc}(3)$ locus, up to a finite sequence of admissible and equimultiple blowings-up. This result is established for general varieties, regardless of dimension, but only applies to the limits of the $\mathrm{nc}(3)$ locus.

3 The situation described in the conclusion of Theorem 1.4 corresponds to the result of following both the second and third stages.

In Chapter 6, we present the details of the strategy we follow for partial desingularization, which we have briefly described in Section 1.3.

In Chapter 7 we present a moving away algorithm that can be applied to $cp(n)$ singularities, regardless of the dimension of $X$. This algorithm begins by blowing-up the $cp(n)$ locus of points, which introduces a *distinguished divisor* $D_0$. This algorithm finds a family $\mathscr{N}$ of local expressions of singularities together with a finite sequence of admissible and equimultiple blowings-up after which we can cover (the strict transform of) $D_0$ with finitely many affine charts $\{U_\alpha\}_{\alpha \in A}$ such that the origin $a = 0 \in U_\alpha$ belongs to $D_0$ and is an element of $\mathscr{N}$, for all $\alpha \in A$. This result can be improved when $n \leqslant 4$, in which case we show that all points in $D_0$ are in $\mathscr{N}$.

# 2 PRELIMINARIES

The purpose of this chapter is to provide the necessary definitions for the development of this text. Some key concepts presented in this chapter are:

- Concepts from algebraic geometry. For example: Varieties, birational morphisms, étale morphism, regular system of parameters, uniformizing parameters, and normalization.
- Concepts from singularity theory. For example: Singularities, smooth varieties, vanishing order, blow-up morphisms, normal crossings, equisingularity.
- Concepts from the calculus of marked ideals. For example: Controlled transform by a blow-up, test sequences, derivatives of marked ideals, coefficient ideals, maximal contact hypersurfaces.
- Concepts from resolution of singularities. For example: Hilbert-Samuel function, monomial part and residual part of a marked ideal, and the desingularization invariant as defined in [BM97].

This chapter is not meant to provide an exhaustive development of the theory behind these concepts. Its only goal is to gather together a sufficient collection of concepts that will be used throughout the text. The fundamental references in the writing of this chapter were the following books: [Har77], [Eis95], [Mum99], [Kol07], [Vak24], as well as the article [BM97].

In this whole text, all rings we consider are noetherian commutative rings with $1$. If $R$ is a ring, and $F \subset R$, we denote the ideal generated by $F$ with $\langle F \rangle$. If $\{f_1, \ldots, f_r\} \subset R$, we define $\langle f_1, \ldots, f_r \rangle := \langle \{f_1, \ldots, f_r\} \rangle$

## 2.1 ELEMENTS OF ALGEBRAIC GEOMETRY

**Definition 2.1** (Spectrum of a ring as a topological space)**.** Given $R$ a commutative ring with $1$, we define the *spectrum* of $R$ as

$$\mathrm{Spec}(R) := \{p \subset R : p \text{ is a prime ideal of } R\}.$$

We define the *Zariski topology* on $\mathrm{Spec}(R)$ by specifying its closed subsets. We declare all subsets of $\mathrm{Spec}(R)$ of the form

$$V(I) := \{q \in \mathrm{Spec}(R) : I \subset q\},$$

as closed sets for the Zariski topology.

**Remark 2.2.** From the definition, it follows that a singleton $\{p\} \subset \mathrm{Spec}(R)$ is closed if and only if $p$ is a maximal ideal, as any non-maximal ideal is contained inside some maximal ideal.

In this thesis, given a topological space $X$, when we use the expression *let* $a \in X$ *be a point*, we will mean *let* $a \in X$ *be a closed point*. When referring to non-closed points we explicitly indicate this.

**Definition 2.3** (Presheaf of rings, morphism of presheaves, presheaf of ideals)**.** Let $X$ be a topological space, let $\mathrm{Op}(X)$ be the category whose objects consist of open sets of $X$ and morphisms are the inclusion maps $\iota : U \to V$, where $U \subset V$, and let Ring denote the category of commutative rings with 1. We define a *presheaf of rings on* $X$ as a contravariant functor $\Gamma : \mathrm{Op}(X) \to \mathrm{Ring}$. In other words:

- to each $U \in \mathrm{Op}(X)$ we associate a ring $\Gamma(U)$,
- to each inclusion map $U \xrightarrow{\iota} V$ we associate a ring homomorphism $\Gamma(\iota) : \Gamma(V) \to \Gamma(U)$; we call this the *restriction morphism* ,
- for the identity map $U \xrightarrow{\mathrm{Id}_U} U$, we have that $\Gamma(\mathrm{Id}_U) = \mathrm{Id}_{\Gamma(U)}$,
- if $U \xrightarrow{\iota_1} V \xrightarrow{\iota_2} W$, then $\Gamma(\iota_1) \circ \Gamma(\iota_2) = \Gamma(\iota_2 \circ \iota_1)$.

If $U, V \in \mathrm{Op}(X)$ are as above, we define $\mathrm{res}_{V,U} : \Gamma(V) \to \Gamma(U)$ as $\Gamma(\iota)$.

Given two presheaves $\Gamma_1, \Gamma_2$ defined on $X$, we define a *morphism of presheaves* as a collection of homomorphisms $\phi_U : \Gamma_1(U) \to \Gamma_2(U)$ such that for all $U \subset V$ we have that

$$\phi_U \circ \mathrm{res}^{\Gamma_1}_{V,U} = \mathrm{res}^{\Gamma_2}_{V,U} \circ \phi_V.$$

We say that $\mathcal{I} : \mathrm{Op}(X) \to \mathrm{Ring}$ is a *presheaf of ideals of* $\Gamma$ if for each $U \in \mathrm{Op}(X)$, $\mathcal{I}(U)$ is an ideal of $\Gamma(U)$, and if $\mathcal{I}$ satisfies the identity and composition axioms.

**Example 2.4.** Let $X$ be a topological space. Given a ring $R$, we may define the *constant sheaf* as $\Gamma(U) = R$ for all $U \in \mathrm{Op}(X)$, and for all $\iota : U \to V$, as $\Gamma(\iota) := \mathrm{Id}_R$.

**Definition 2.5** (Sheaf of rings, morphism of sheaves, ideal sheaf)**.** Given a presheaf $\Gamma$ defined on a topological space $X$, we say that $\Gamma$ is a *sheaf of rings* if $\Gamma$ satisfies:

- *Locality*, that is, for any open set $U$, for any open cover $\{U_i\}_{i \in I}$ of $U$, and for any $s, t \in \Gamma(U)$, we have that $s = t$ if and only if, for all $i$
$$\mathrm{res}_{U,U_i}(s) = \mathrm{res}_{U,U_i}(t).$$
- *Glueing*, that is, for any open set $U$, for any open cover $\{U_i\}_{i \in I}$ of $U$, and for any set $\{s_i \in U_i\}_{i \in I}$ satisfying
$$\mathrm{res}_{U_i, U_i \cap U_j}(s_i) = \mathrm{res}_{U_j, U_i \cap U_j}(s_j)$$
there exists an element $s \in \Gamma(U)$ such that $\mathrm{res}_{U,U_i}(s) = s_i$, for each $i \in I$.

If $\Gamma_1, \Gamma_2$ are sheaves on $X$, we say that $\phi : \Gamma_1 \to \Gamma_2$ is a *sheaf morphism* (or *morphism of sheaves*) if $\phi$ is a morphism of presheaves.

An *sheaf of ideals* (or *ideal sheaf*) $\mathcal{I}$ of a sheaf $\Gamma$ is a presheaf of ideals of $\Gamma$.

**Example 2.6.** Given a differentiable manifold $M$, to each open subset $U \subset M$ we can associate the ring of smooth functions defined on $U$

$$\Gamma(U) := C^\infty(U) = \{f : U \to \mathbb{R} : f \text{ is a smooth map}\}.$$

If $\iota : U \to V$ is an inclusion of open subsets of $M$, then we define $\Gamma(\iota) : \Gamma(V) \to \Gamma(U)$ as $\Gamma(\iota)(f) = f|_U$.

**Example 2.7.** We can define a sheaf on spaces of the form $\mathrm{Spec}(R)$. Given an open set $U \subset \mathrm{Spec}(R)$, and given $p \in U$ we denote by $R_p$ the localization of $R$ at the multiplicative set given by the complement of $p$. We define $\Gamma(U)$ as the set of functions $s : U \to \sqcup_{p \in U} R_p$ satisfying that for any $q \in U$, there exist $V \in \mathrm{Op}(X)$, $f, g \in R$ with $V \subset U$ and such that for all $p' \in V$ we have that

$$s(p') = \frac{f}{g} \tag{2.1}$$

where $g \notin p'$. Addition and product are defined point-wise, and the constant function $1 \in \Gamma(U)$, for any $U$. Thus, $\Gamma(U)$ is a commutative ring with 1. Because addition and product are defined point-wise, we can define the restriction map $\mathrm{res}_{V,U} : \Gamma(V) \to \Gamma(U)$ and it is immediately verified that it is a ring homomorphism. Similarly, for functoriality and identity properties. By construction, we have that $\Gamma(X)$ satisfies glueing and uniqueness.

**Definition 2.8** (Ringed space)**.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ where $X$ is a topological space and $\mathcal{O}_X$ is a sheaf defined on $X$.

A *morphism of ringed spaces* is a pair $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ where $f : X \to Y$ is a continuous map and $f^\#$ is a collection of maps $f^\sharp_U : \mathcal{O}_Y(U) \to \mathcal{O}_Y(f^{-1}(U))$ satisfying that for all $U, V \in \mathrm{Op}(Y)$ with $U \subset V$, we have that

$$\mathrm{res}_{f^{-1}(V), f^{-1}(U)} \circ f^\sharp_V = f^\sharp_U \circ \mathrm{res}_{V,U}.$$

A morphism $(f, f^\#)$ of ringed spaces is called an *isomorphism* if there exists a morphism of ringed spaces $(g, g^\#)$ such that $f$, $f^\#$ and $g$, $g^\#$ are the respective inverses of each other.

From now on, given $U \in \mathrm{Op}(X)$, we denote by $\mathcal{O}_U$ the ring $\mathcal{O}_X(U)$.

**Definition 2.9** (Stalk at a point)**.** Given a ringed space $(X, \mathcal{O}_X)$ and given $x \in X$, we may notice that the collection of neighbourhoods $U$ containing $x$ forms a directed system with respect to reverse inclusion. As such, we can define the *stalk of* $X$ *at* $x$ as the ring given by the direct limit

$$\mathcal{O}_{X,x} := \varinjlim_{\substack{U \\ x \in U}} \mathcal{O}_{\mathcal{U}}.$$

**Definition 2.10** (Locally ringed space, morphism of locally ringed spaces)**.** Given a ringed space $(X, \mathcal{O}_X)$, we say that it is a *locally ringed space*, if for all $x \in X$, we have that $\mathcal{O}_{X,x}$ is a local ring.

Given a morphism of ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, we say that $(f, f^\#)$ is *morphism of locally ringed spaces* if for all $x \in X$ the induced map $f^\#_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a homomorphism of local rings.

Given a morphism of locally ringed spaces $(f, f^\#)$ we say that it is an *isomorphism* if there exists a morphism of locally ringed spaces $(g, g^\#)$ such that $f, f^\#$ and $g, g^\#$ are the respective inverses of each other.

**Example 2.11.** We can provide $Z := \mathrm{Spec}(R)$ with the structure of a locally ringed space with the sheaf $\mathcal{O}_Z$, where $\mathcal{O}_Z$ is the sheaf constructed in Example 2.7, as the stalk of $\mathcal{O}_Z$ at a point $p$ is isomorphic to the localization $R_p$, which is a local ring with maximal ideal equal to $\iota(p)$, where $\iota$ is the localization morphism $\iota : R \to R_p$.

**Definition 2.12** (Affine scheme, affine morphism)**.** We say that a locally ringed space $(X, \mathcal{O}_X)$ is an affine scheme if it is isomorphic to $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$.

An *affine morphism* is a morphism of locally ringed spaces between two affine schemes $(X, \mathcal{O}_X) \xrightarrow{(f,f^\sharp)} (Y, \mathcal{O}_Y)$.

Given a scheme $(X, \mathcal{O}_X)$, $a \in X$ and an open set $U \subset X$, we say that $U$ is an affine neighbourhood of $a$ if $(X, \mathcal{O}_U)$ is an affine scheme.

**Definition 2.13** (Scheme, morphism of schemes)**.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ where for every point $x \in X$ there exists $U \in \mathrm{Op}(X)$ such that $(U, \mathcal{O}_U)$ is an affine scheme. In this case, we call $X$ and $\mathcal{O}_X$ the *topological space* and *structure sheaf* of $(X, \mathcal{O}_U)$, respectively; and we say that $U$ is an affine chart.

If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are two schemes, and if

$$(X, \mathcal{O}_X) \xrightarrow{(f,f^\sharp)} (Y, \mathcal{O}_Y),$$

is a morphism of locally ringed spaces, we say that $(f, f^\sharp)$ is a *morphism of schemes*.

**Remark 2.14.** Given two schemes $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$, let $\hom(X, Y)$ denote the set of continuous maps $X \to Y$ coming from a morphism of schemes. Similarly, let $\hom(\mathcal{O}_Y, \mathcal{O}_X)$ denote the set of sheaf morphisms coming from a morphism of schemes.

Given an arbitrary scheme $(X, \mathcal{O}_X)$ and an affine scheme $(Y, \mathcal{O}_Y)$ there is a natural isomorphism

$$\hom(X, Y) \to \hom(\mathcal{O}_Y, \mathcal{O}_X).$$

(see Chapter 1, Section 3, Proposition 3.5, in [Har77]).

In particular, notice that there is a 1-1 correspondence between the ring of morphisms $R \to S$ and the continuous maps $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ coming from an affine morphism. From this point on, when we refer to a scheme we omit the structure sheaf, that is, we will have notation of the sort "let $X$ be a scheme".

**Example 2.15.** Given a field $K$, we define the $n$*-dimensional affine space* as the affine scheme associated to the ring of polynomials in $n$ variables. That is,

$$\mathbb{A}^n_K := \mathrm{Spec}(K[x]),$$

where $x$ denotes the vector of formal variables $x_1, \ldots, x_n$.

**Definition 2.16** (Reduced scheme)**.** Given a scheme $X$ we say that it is a *reduced scheme* if for all $a \in X$ the only nilpotent element of the local ring $\mathcal{O}_{X,a}$ is $0$.

**Definition 2.17** (Variety)**.** Given a reduced scheme $X$ and a field $K$, we say that $X$ is a *variety over* $K$, if there exists a morphism of schemes $X \xrightarrow{s} \mathrm{Spec}(K)$ such that $s$ is *separated* (see Chapter 10 of [Vak24]), and if there is a cover $\{U_\alpha\}_{\alpha \in A}$ of $X$ where each $U_\alpha$ is an affine scheme and such that $\mathcal{O}_U$ is finitely generated algebra over $K$. A *morphism of varieties* is a morphism $X \xrightarrow{\varphi} Y$ of the underlying schemes.

**Definition 2.18** (Affine variety)**.** Given a variety $X$, we say that $X$ is an affine variety if $X$ is a reduced subscheme of $\mathrm{Spec}(K[x_1, \dots, x_n])$, where $K$ is a field.

**Example 2.19.** Fix a field $K$, let $R$ be an integral $K$-algebra generated by $t_1, \dots, t_n$, and let $F := \{f_1, \dots, f_\ell\} \subset R$, and consider the localization $F^{-1}R$ of $R$ by the multiplicative set generated by $F$. Then, $F^{-1}R$ is a finitely generated $K$-algebra as

$$F^{-1}R \simeq R[y_1, \dots, y_\ell]/\langle y_1 f_1 - 1, \dots, y_\ell f_\ell - 1\rangle.$$

By the universal property of localization, and because $R$ is integral, there exists a homomorphism $\iota : R \to F^{-1}R$. Notice also that $\mathrm{Spec}(F^{-1}R)$ is a variety (in order to check for separatedness, see Corollary 4.2 of [Har77]).

If $X \xrightarrow{\varphi} Y$ is a morphism such that for every $b \in Y$ there is an affine neighbourhood $b \in U \subset Y$ where $\mathcal{O}_U \xrightarrow{\varphi^\sharp} \mathcal{O}_X$ is a localization morphism as above, we say that $X$ is an *open subvariety* of $Y$.

**Example 2.20.** Let $I$ be a proper radical ideal of a $K$-algebra $R$ generated by $x_1, \dots, x_n$. The morphism $\varphi$ associated to the quotient homomorphism $f^\sharp : K \to R/I$ is a morphism of affine varieties, thus separated (see Corollary 4.2 of [Har77]). Given that $I$ is radical, we have that $R/I$ does not have non-trivial nilpotent elements. Therefore, $\mathrm{Spec}(R/I)$ is a variety.

If $X \xrightarrow{\varphi} Y$ is a morphism such that for every $b \in Y$ there is an affine neighbourhood $b \in U \subset Y$ where $\mathcal{O}_U \xrightarrow{\varphi^\sharp} \mathcal{O}_X$ is a quotient morphism as above, we say that $X$ is a *closed subvariety* of $Y$.

**Definition 2.21** (Support, cosupport)**.** Given an ideal sheaf $\mathcal{I}$ of a locally ringed space $(X, \mathcal{O}_X)$, we define the *support* of $\mathcal{I}$ as

$$\mathrm{supp}(\mathcal{I}) := \{p \in X : \mathcal{I}_p \neq 0\}.$$

Similarly, we define the *cosupport* of $\mathcal{I}$ as

$$\mathrm{cosupp}(\mathcal{I}) := \mathrm{supp}(\mathcal{O}_X/\mathcal{I}).$$

**Example 2.22.** Given an ideal $I \subset K[x_1, \dots, x_n]$ we may construct an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{A}^n_K}$ as follows. Let $f_1, \dots, f_r$ be generators of $I$ and let $U \subset \mathbb{A}^n_K$ be an open subset. Notice that we can identify each $f_j$ with the function $p \in U \mapsto i_p(f_j)$, where $i_p : K[x_1, \dots, x_n] \to K[x_1, \dots, x_n]_p$ is the localization at the complement of the prime ideal $p$. Thus, the ideal generated by the elements $i_p(f_1), \dots, i_p(f_r)$ (which we denote by $\langle i_p(f_1), \dots, i_p(f_r)\rangle$) is an ideal of $\mathcal{O}_U$. Thus, this construction constitutes a sheaf of ideals.

Notice that, unless $I$ is the zero ideal, the support of $\mathcal{I}$ is $\mathbb{A}^n$, which makes the support a trivial concept when we discuss ideals defining affine varieties.

On the other hand, we may notice that $\mathrm{cosupp}(\mathcal{I})$ equals the underlying topological space of $V(I)$ as $\mathcal{O}_{X,p}/\mathcal{I}_p = 0$ if and only if $1 \in \mathcal{I}_p$, which in turn happens if and only if there is at least one $f_j$ such that the localization $i_p(f_j)$ is not an element of the maximal ideal $\mathfrak{m}_p$ of the local ring $K[x_1, \dots, x_n]_p$.

Thus, the cosupport of a sheaf of ideals is a closed subset. Moreover, if $X \subset Z$ is closed and $Z$ is a variety, then there is a unique ideal sheaf $\mathcal{I}_X$ such that $\operatorname{cosupp}(\mathcal{I}_X) = X$. We call $\mathcal{I}_X \subset \mathcal{O}_Z$ the *ideal sheaf associated to* $X$.

**Remark 2.23.** Let $Z$ be a variety over an algebraically closed field $\mathbb{K}$, let $a \in Z$ be a closed point and let $\mathfrak{m}_a$ be the maximal ideal of the local ring $\mathcal{O}_{Z,a}$. By definition, $Z$ comes equipped with a *structure homomorphism* $\mathbb{K} \to \mathcal{O}_{Z,a}$ which expresses $\mathcal{O}_{Z,a}$ as a finitely generated algebra over $\mathbb{K}$. On the other hand, we have that $\mathcal{O}_{Z,a}/\mathfrak{m}_a$ is a finite field extension of $\mathbb{K}$. Thus, the quotient morphism gives us a morphism

$$\begin{aligned} \mathcal{O}_{Z,a} &\to \mathbb{K} \\ f_a &\mapsto f(a), \end{aligned}$$

which we call the *evaluation map*. The composition of these morphisms is a ring homomorphism

$$\varepsilon_a : \mathcal{O}_{Z,a} \to \mathcal{O}_{Z,a},$$

whose image is the set of locally constant functions. Because $\mathcal{O}_{Z,a}$ is a $\mathbb{K}$-algebra, we use the notation $f(a)$ to denote both $f(a)$, and $\varepsilon_a(f_a)$.

**Definition 2.24** (Closed embedding)**.** Given two varieties $X, Z$ over a field $K$, a morphism $X \xrightarrow{i} Z$, is said to be a *closed embedding* if for every affine open subvariety $V \subset Z$ with $V \simeq \operatorname{Spec}(R)$ we have that $i^{-1}(V)$ is an affine open subvariety of $X$ with $i^{-1}(V) \simeq \operatorname{Spec}(S)$, and such that the morphism of rings $R \to S$ is surjective. In this case, we use the notation $X \overset{i}{\hookrightarrow} Z$.

**Example 2.25.** Let $Z$ be a variety over $K$, and let $\mathcal{I}$ be a principal ideal sheaf, that is, for every $a \in Z$ there exists $r \in \mathcal{O}_{Z,a}$ such that $\mathcal{I}_a = \langle r \rangle$. Then, the morphism of varieties associated to the quotient map $\mathcal{O}_{Z,a} \twoheadrightarrow \mathcal{O}_{Z,a}/\mathcal{I}_a$ is a closed embedding $V(\mathcal{I}) \hookrightarrow Z$.

In this context, we say that $V(\mathcal{I})$ is a *hypersurface* of $Z$.

Let us conclude this section with the Jacobian criterion for smoothness for varieties over an algebraically closed field of characteristic zero.

**Definition 2.26** (Smooth point, smooth variety, singularity)**.** Let $X$ be a variety over an algebraically closed field $\mathbb{K}$ of characteristic zero and let $a \in X$ be a closed point. We say that $X$ is *smooth at* $a$ if there exists an affine neighbourhood $U \subset X$ of $a$ with $U \simeq \operatorname{Spec}(\mathbb{K}[x_1, \dots, x_n]/I)$, together with a finite set of generators $f_1, \dots, f_r$ of $I$, such that the matrix $J(a) \in \operatorname{Mat}_{r \times n}(\mathbb{K})$ of partial derivatives

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(a) & \dots & \dfrac{\partial f_1}{\partial x_n}(a) \\ \vdots & \dots & \vdots \\ \dfrac{\partial f_r}{\partial x_1}(a) & \dots & \dfrac{\partial f_r}{\partial x_n}(a) \end{pmatrix}$$

evaluated at $a$ has rank $r$.

If $X$ is smooth at every closed point, we say that $X$ is a *smooth variety*.

If $X$ is not smooth at $a$ we say that $X$ is *singular* at $a$, or that $a$ is a singularity, for short.

**Definition 2.27** (Vanishing order)**.** Let $Z$ be a variety over a field $K$, let $\mathcal{O}_Z$ be its sheaf of functions, and let $a \in Z$ be a smooth point of $Z$. Given $f \in \mathcal{O}_{Z,a}$, we define the *vanishing order of* $f$ *at* $a$ , as

$$\mathrm{ord}_f(a) = \max\{d \in \mathbb{Z}_{\geqslant 0} : f \in \mathfrak{m}_a^d\},$$

where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_{Z,a}$, and where $\mathfrak{m}_a^0 := \mathcal{O}_{Z,a}$. We also define the vanishing order of $0$ as $\infty$.

Similarly, if $\mathcal{I} \neq 0$ is a sheaf of ideals, we define the *vanishing order* of $\mathcal{I}$ at a point $a$, as

$$\mathrm{ord}_{\mathcal{I}}(a) = \max\{d \in \mathbb{Z}_{\geqslant 0} : \mathcal{I}_a \subset \mathfrak{m}_a^d\}.$$

We also define the vanishing order of the zero ideal sheaf as $\infty$.

## 2.2 COORDINATE SYSTEMS AND ÉTALE MORPHISMS

Two key concepts in this work are those of *simple normal crossings* and *normal crossings* singularities. These concepts in turn are best stated in terms of what we call *regular* and *étale coordinate systems*. For convenience, we also present the notion of étale morphism, but we do not prove any of its properties. A good reference for the properties of étale morphisms is [AM99].

**Definition 2.28** (Regular system of parameters)**.** Let $a$ be a closed point of a variety $X$ over an algebraically closed field $\mathbb{K}$. We say that the functions $x_1, \dots, x_n \in \mathcal{O}_X$ are a *regular system of parameters at* $a$ if the localization of the ideal $\langle x_1, \dots, x_n\rangle$ at $a$ is the maximal ideal of the local ring $\mathcal{O}_{X,a}$.

If $U$ is an affine open neighbourhood of $a$, we say that $x_1, \dots, x_n \in \mathcal{O}_X$ are a *regular system of parameters on* $U$ if for all closed points $a \in U$, the functions

$$\{x_1 - x_1(a), \dots, x_n - x_n(a)\} \subset \mathcal{O}_U$$

form a regular system of parameters of $U$ at $a$.

Definition 2.28 was taken from [AM99], where it can be found under the name of *uniformizing parameters*. I decided to refer to this using the terminology of [AM69] as it fits better the way we use this notion in this text. Notice that in both definitions of regular system of parameters above, the domain of definition of the functions are a subset of $X$.

**Example 2.29.** If $X = \mathbb{A}^n_{\mathbb{K}} = \mathrm{Spec}(\mathbb{K}[t_1, \dots, t_n])$ then $t_1, \dots, t_n$ is a regular system of parameters of $X$.

**Example 2.30.** Let $X$ be a variety over an algebraically closed field $\mathbb{K}$ and assume that $X$ is smooth on an open neighbourhood $U$ of $a \in X$. By definition, there is an affine neighbourhood $U \in a$ such that $U$ is isomorphic to a closed subvariety of $\mathbb{A}^{n+r}_{\mathbb{K}}$ such that $\mathcal{O}_U \simeq \mathbb{K}[t_1, \dots, t_{n+r}]/\langle y_1, \dots, y_r\rangle$, where the Jacobian matrix has rank $r$. We claim that there is a regular system of parameters $x_1, \dots, x_{n+r}$ defined on an open subvariety $W$ of $\mathbb{A}^{n+r}_{\mathbb{K}}$ such that $U \cap W \simeq V(x_1, \dots, x_r)$. By rearranging the coordinates if necessary, we may assume that the $r \times r$ minor defined by the matrix $J$ with entries $J_{i,j} := \dfrac{\partial y_i}{\partial t_j}$ is invertible, when we evaluate at $a$. Let $W$ be an open subvariety of $\mathbb{A}^{n+r}_{\mathbb{K}}$ where all the entries of J and det J is a unit of $\mathcal{O}_W$. The adjugate of J given by $M \in \mathrm{Mat}_{r\times r}(\mathcal{O}_W)$ satisfies $MJ = \det(J)I$. Define $x_i$ as $y_i$ for $1 \leqslant i \leqslant r$ and $t_i$ for $r+1 \leqslant i \leqslant n+r$. We want to show that $x_1, \dots, x_{n+r}$ is a regular system of parameters on $W$. Fix $a \in W \cap U$ and let $\mathfrak{n}_a$ denote the ideal of $\mathcal{O}_{\mathbb{A}^{n+r}_{\mathbb{K}}}$ generated by $t_1, \dots, t_r$. The invertibility of J implies that $x_1, \dots, x_r$ generate the $\mathbb{K}$-vector space $\mathfrak{n}_a/\mathfrak{n}_a^2$. By Nakayama's lemma (see Corollary 4.8, part b of [Eis95]) we have that the localizations of $x_1, \dots, x_{n+r}$ generate the ideal $\langle t_1, \dots, t_{n+r}\rangle_a$, giving us what we wanted to show.

**Definition 2.31** (Finite type morphism)**.** Let $\varphi^\sharp : R \to S$ be a ring homomorphism. We say that $\varphi^\sharp$ is a *finite type morphism* (also referred to as *finite morphism*) if S is a finite $\varphi(R)$-module. We also say that S is a *finite ring extension* of R.

**Definition 2.32** (Étale morphism)**.** Given a morphism $\pi : X \to Y$ of varieties over an algebraically closed field $\mathbb{K}$ such that $\pi^\sharp : \mathcal{O}_Y \to calO_X$ is a morphism of rings of finite type, we say that $\pi$ is *étale at* $x \in X$ if there exist neighbourhoods $U \subset X$, $V \subset Y$ with $a \in U$, $\pi(U) \subset V$ such that $\mathcal{O}_U$ is isomorphic to some open subvariety of $\mathrm{Spec}(\mathcal{O}_V[t_1, \dots, t_n]/\langle f_1, \dots, f_n\rangle)$, where $\pi^* : \mathcal{O}_V \to \mathcal{O}_U$ is the inclusion map and, the Jacobian

$$\det\begin{pmatrix} \frac{\partial f_1}{\partial t_1} & \cdots & \frac{\partial f_1}{\partial t_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial t_1} & \cdots & \frac{\partial f_n}{\partial t_n} \end{pmatrix},$$

is non-vanishing at $x$. In this context, we say that $\pi$ is an *étale neighbourhood* of $\pi(a)$. We say that $\pi$ is *an étale morphism on* $U$ , if $\pi$ is étale at every point $a \in U$.

**Example 2.33.** Let $X \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$ be the curve $V(t^2 - s^2 - s^3)$. We want to find an étale map $W \xrightarrow{\pi} \mathbb{A}^2_{\mathbb{C}}$ such that for some open neighbourhood $U \subset X$ we have that $\pi^{-1}(U)$ is the union of two curves. For this, consider the ring $R := (s+1)^{-1}\mathbb{C}[s,t]$ given by the localization of $\mathbb{C}[s,t]$ by the multiplicative set generated by $s+1$ and the ring $S := r^{-1}\mathbb{C}[r,t]$ given by the localization of $\mathbb{C}[r,t]$ by the multiplicative set generated by $r$. Consider the homomorphism $R \xrightarrow{\rho^\sharp} S$ given by $t \mapsto t, s \mapsto r^2 - 1$. Notice that the respective inclusions of $t - r - r^3, t + r + r^3$ generate $\mathbb{K}[t,r]_a$ for all points $a$ outside $V(1 + 3r^2)$.

**Example 2.34.** Fix a positive integer $d$ and a non-negative integer $n$. Let R denote the ring $\mathbb{C}[w, x]$, where $x$ denotes the vector of variables $(x_1, \dots, x_n)$ and consider the R-algebra given by $S := R[v]/\langle v^d - w\rangle$. Notice that S is isomorphic to $\mathbb{C}[v, x]$, and consider the morphism $\phi : R \to S$ induced by the mappings

$$w \mapsto v^d, \quad x_k \mapsto x_k.$$

Thus, the map $\pi$ induced by $\phi$

$$\begin{aligned} \pi : \mathbb{A}^{n+1}_{\mathbb{C}} \cap D(v) &\to \mathbb{A}^{n+1}_{\mathbb{C}} \cap D(w) \\ (\alpha_0, \alpha_1, \dots, \alpha_n) &\mapsto (\alpha_0^d, \alpha_1, \dots, \alpha_n), \end{aligned}$$

is étale. We are interested in the particular case where $n = d = 2$.

**Definition 2.35** (Étale coordinate system)**.** Let $U$ be an open neighbourhood of a variety $X$ over an algebraically closed field $\mathbb{K}$. An *étale coordinate system of* $U$ is an étale neighbourhood $V \xrightarrow{\pi} U$ together with a regular system of parameters $(x_1, \dots, x_n)$ defined on $V$.

Notice in particular that an étale coordinate system of $U \subset X$ is not a set of functions defined on $U$, but rather a set of functions defined on some covering $V \xrightarrow{\pi} U$.

**Example 2.36.** This is a continuation of Example 2.33. Notice that

$$\begin{aligned} &r^{-1}\mathbb{C}[r,t]/\langle t^2 - r^2(r^2-1)^2\rangle \\ &\qquad \simeq r^{-1}\mathbb{C}[r,t]/\langle t - r(r^2-1)\rangle \oplus r^{-1}\mathbb{C}[r,t]/\langle t + r(r^2-1)\rangle. \end{aligned}$$

Thus, there is a surjective homomorphism

$$r^{-1}\mathbb{C}[r,t] \to r^{-1}\mathbb{C}[r][t]/\langle t - r(r^2-1)\rangle \oplus r^{-1}\mathbb{C}[r][t]/\langle t + r(r^2-1)\rangle,$$

which induces a map $U \subset \operatorname{Spec}(\mathbb{C}[r,t]_r/\langle t^2 - r^2(r^2-1)^2) \xhookrightarrow{F} W \subset \mathbb{A}^2_{\mathbb{C}}$, where the components of this map $x_1 := t - r - r^3, x_2 := t + r + r^3$ form a regular system of parameters on $F^{-1}(D(1+3r^2))$. Thus, we have found an étale system of coordinates $x_1, x_2$ of an open subvariety of $D(s+1)$ where $X \cap D(s+1)$ is given by the vanishing locus of $x_1x_2$.

**Example 2.37.** This is a continuation of Example 2.34, for the case $d = n = 2$. Notice that $v, z - vx_1, z + vx_1$ is a regular coordinate system defined on $D(v)$, which in turn gives us an étale system of coordinates $y_1, y_2, y_3$ of $D(w)$ which expresses $X \cap D(w)$ as the vanishing locus of $y_2y_3$.

We now need to establish the notion of partial derivatives w.r.t. a regular system of parameters. For this we need to briefly discuss some properties of the power series ring, and more generally, the completion of a ring.

Let us first discuss some concepts for general topological spaces. Given a topological space $X$, we can define the notion of a *filter* (see Definition A.12 of [AGM96]) and their *convergence* (see Definition 3.1.14 of [AGM96]). We can use filters to define the notion of *Cauchy filters* (see Definition 3.1.20 of [AGM96]). Just as in the construction of completion for metric spaces, we can define the *completion* of a filtered space $X$ as the set of filters on $X$ identified by the relation of their difference forming a Cauchy filter. A topological space $X$ is said to be *complete* if all Cauchy filters have a limit in $X$.

Let us now discuss how to apply these notions to rings. A ring $R$ is said to be a *topological ring* if $R$ is a topological space for which the operations $+, \cdot : R \times R \to R$ are continuous in the product topology of $R \times R$. Given an arbitrary ring $R$ and a maximal ideal $\mathfrak{m} \subset R$, the set $\{\mathfrak{m}^d\}_{d \in \mathbb{Z}_{\geqslant 0}}$ forms a basis of neighbourhoods of $0$ (see Proposition 1.2.1 and Proposition 1.2.2 in [AGM96] for the axioms of a basis of neighbourhoods). On the other hand, any system of neighbourhoods uniquely determines a topology for which the system of neighbourhoods is open (see Theorem 1.2.5 of [AGM96]). Thus, we may use this system of neighbourhoods to imbue $R$ with a topology, this is called the $\mathfrak{m}$*-adic topology* of $R$ (see Example 1.2.10 of [AGM96]) which realizes $R$ as a topological ring. This allows us to construct the *completion* $\hat{R}_{\mathfrak{m}}$ of a ring $R$ with respect to the $\mathfrak{m}$-adic topology defined on $R$. To simplify our notation, whenever the maximal ideal $\mathfrak{m}$ is understood we simply notate the completion of $R$ w.r.t. $\mathfrak{m}$ by $\hat{R}$. If $\hat{R}$ is a completion of $R$ then $\hat{R}$ is a complete topological ring and there is a ring homomorphism $\iota : R \to \hat{R}$ such that the image $\iota(R)$ is a dense subset of $\hat{R}$ (see Proposition 3.4 in [AGM96]). Moreover, for any complete topological ring $S$ and for any topological ring $R$, any continuous morphism of rings $f : R \to S$ admits a unique extension $\hat{f} : \hat{R} \to S$ satisfying that $\hat{f}(r) = f(r)$ for all $r \in R$ (see Theorem 3.2.27 in [AGM96]), we call this the *universal property of completions*.

**Example 2.38.** Let $R$ be a local ring, and let $\mathfrak{m}$ be a maximal ideal of $R$. Notice that the sequence of powers

$$R \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \dots$$

allows us to construct an inverse system $\{Q_d := R/\mathfrak{m}^d\}_{d \in \mathbb{Z}_{\geqslant 0}}$, with morphisms given by

$$\begin{aligned} q_{d_1,d_2} : Q_{d_2} &\to Q_{d_1} \\ f + \mathfrak{m}^{d_2} &\mapsto f + \mathfrak{m}^{d_1} \end{aligned}$$

when $d_1 \leqslant d_2$. Thus, we can construct the inverse limit $\varprojlim_d R/\mathfrak{m}^d$. Then the ring $\varprojlim_d R/\mathfrak{m}^d$ is isomorphic to the completion $\hat{R}$ w.r.t. $\mathfrak{m}$ (see p.174, Subsection 3.2.6 of [Bou98]).

In particular, if $R = K[t_1, \dots, t_n]$ where $K$ is a field, and if

$$\mathfrak{m} = \langle t_1, \dots, t_n \rangle,$$

then the completion $\hat{R}$ of $R$ w.r.t. $\mathfrak{m}$ is $K[\![t_1, \dots, t_n]\!]$ (see the Corollary at p. 175 of Subsection 3.2.6 of [Bou98]).

**Remark 2.39.** Let $\hat{\mathfrak{m}}$ denote the ideal

$$\hat{\mathfrak{m}} := \{s = (s_0, s_1, \dots) \in \hat{R} : s_0 = 0\}.$$

Notice that $\hat{R}/\hat{\mathfrak{m}} \simeq R/\mathfrak{m}$ which is a field, and so $\hat{\mathfrak{m}}$ is a maximal ideal.

**Remark 2.40.** Notice that for each $k$, $R/\mathfrak{m}^k$ is a local ring. This is because any maximal ideal $\mathfrak{n} \subset R/\mathfrak{m}^k$ is prime, and a prime ideal of $R/\mathfrak{m}^k$ corresponds to a prime ideal $P \subset R$ with $\mathfrak{m}^k \subset P$. Thus, $\mathfrak{m} \subset P$. This shows that $\mathfrak{n} = \mathfrak{m}$.

We now claim that $\hat{R}$ is a local ring. It suffices to show that if $s \in \hat{R} \setminus \hat{\mathfrak{m}}$ then $s$ is a unit. By definition, there exist representatives $\{s_d\} \subset R$ such that $s = (s_0, s_1, \dots)$ where $s_0 \neq 0$ (and so $s_d \notin \mathfrak{m} \cdot (R/\mathfrak{m}^d)$, for all $d$). Because each $R/\mathfrak{m}^d$ is a local ring with maximal ideal $\mathfrak{m} \cdot (R/\mathfrak{m}^d)$, we have that for each $d$, there exists $g_d \in R/\mathfrak{m}^d$ such that $g_d s_d = 1 + \mathfrak{m}^d$. Thus, $(g_0, g_1, \dots) \in \hat{R}$ is an inverse of $s$, which is what we wanted to prove.

**Proposition 2.41** (Definition 3.4 in [BM97])**.** *Let $X$ be a smooth variety over an algebraically closed field $\mathbb{K}$ of characteristic zero, let $a \in X$ be a closed point and let $x_1, \dots, x_n$ be a regular system of parameters defined on a neighbourhood $U$ of $a$. There is a homomorphism*

$$T_a : \mathcal{O}_{X,a} \to \mathbb{K}[\![Z_1, \dots, Z_n]\!]$$

*of $\mathbb{K}$-algebras satisfying that*

1. $T_a(x_i) = x_i(a) + Z_i$ *for all* $i$,

2. $T_a$ *admits a unique extension to the completion of* $\mathcal{O}_{X,a}$, $\hat{T}_a : \hat{\mathcal{O}}_{X,a} \to \mathbb{K}[\![Z_1, \dots, Z_n]\!]$, *and this extension is an isomorphism.*

*Proof.* By the argument followed in Example 2.30, there exists an open neighbourhood $U_1$ of $a$ such that

$$\mathcal{O}_{U_1} \simeq \mathbb{K}[z_1, \dots, z_n, y_1, \dots, y_r]/\langle p_1(z,y), \dots, p_r(z,y)\rangle \tag{2.2}$$

where each $x_k$ is represented by $z_k$, where the Jacobian with respect to $y_1, \dots, y_r$ $J = (\frac{\partial p_i}{\partial y_j})$ is invertible in some open subvariety $U_2 \ni a$ of $U_1$. Consider now the function $P := (p_1(Z,Y), \dots, p_r(Z,Y)) \in \mathbb{K}[\![Z,Y]\!]$ as a power series. By definition $P$ vanishes at $a$, and so, by the implicit function theorem for formal power series (see Proposition 3.1 part (a) in [Sok09]), we have that there is $\varphi \in \mathbb{K}[\![Z]\!]^r$ such that $P(z(a)+Z, y(a)+\varphi(Z)) = 0 \in \mathbb{K}[\![Z]\!]$. In other words, each $p_k(z(a)+Z, y(a)+Y) \in \mathbb{K}[\![Z,Y]\!]$ is an element of the ideal $\langle Y_1 - \varphi_1(Z), \dots, Y_r - \varphi_r(Z)\rangle$. By the invertibility of the Jacobian matrix we have that

$$\langle Y_1 - \varphi_1(Z), \dots, Y_r - \varphi_r(Z)\rangle = \langle p_1(Z,Y), \dots, p_r(Z,Y)\rangle. \tag{2.3}$$

Given $f \in \mathcal{O}_{X,a}$, by means of the isomorphism in (2.2), there exists $F \in \mathbb{K}[z,y]$ such that the class of $F$ in $\mathbb{K}[z,y]/\langle p_1, \dots, p_r\rangle$ corresponds uniquely to $f$. This allows us to define,

$$\mathcal{O}_{X,a} \to \mathbb{K}[\![Z,Y]\!]/\langle Y_1 - \varphi_1(Z), \dots, Y_r - \varphi_r(Z)\rangle \simeq \mathbb{K}[\![Z]\!],$$

which sends $f(z,y) \mapsto F(z(a)+Z, y(a)+\varphi(Z))$, which is well-defined by (2.3). In order to uniquely determine the morphism above, it suffices to define this function on each quotient ideal $\mathfrak{m}_a^k/\mathfrak{m}_a^{k+1}$, where $\mathfrak{m}_a$ is the maximal ideal corresponding to $a$. But these values are also uniquely determined by the values at the generators of $\mathfrak{m}_a/\mathfrak{m}_a^2$, that is, it suffices to define the morphism on $z_1 - z_1(a), \dots, z_n - z_n(a)$.

The existence of a unique extension to $\hat{\mathcal{O}}_{X,a}$ follows from the universal property of completion. On the other hand, this extension is an isomorphism because $\hat{\mathcal{O}}_{X,a}$ is complete and the generators of the maximal ideal of $\mathbb{K}[\![Z]\!]$ are in the image of $T_a$. $\square$

**Proposition 2.42** (Lemma 3.5 in [BM97]). *Assume the hypotheses of Proposition 2.41. For any* $j \in \{1, \dots, n\}$ *any for any* $f \in \mathcal{O}_U$ *there exists a unique function* $f_{(j)} \in \mathcal{O}_{X,a}$ *such that*

$$T_a(f_{(j)}) = \frac{\partial T_a(f)}{\partial Z_j}$$

*Proof.* Let us first provide an explicit way of computing $\dfrac{\partial T_a(f)}{\partial Z_j}$, expressed in terms of partial derivatives in $\mathcal{O}_U$. Given $f \in \mathcal{O}_U$ there exists a function $F \in \mathbb{K}[z,y]/\langle p_1, \dots, p_r\rangle$ that corresponds uniquely to $f$. By construction,

$$T_a(f) = F(Z, \varphi(Z)) \in \mathbb{K}[\![Z]\!].$$

Thus, by the chain rule, it suffices to compute partial derivatives of $\varphi$ w.r.t. $Z_j$. To describe this in terms of partial derivatives in $\mathcal{O}_U$, consider the function $P(Z,Y) = (p_1(Z,Y),\dots,p_r(Z,Y)) \in \mathbb{K}[\![Z,Y]\!]$ and the identity

$$P(Z,\varphi(Z)) = 0.$$

By considering the matrices of partial derivatives we obtain

$$J_z P + J_y P \cdot J_Z \varphi = 0.$$

Consequently, we have $J_Z\varphi = -J_y P^{-1} \cdot J_z P$. Notice that each of the entries of $J_y P^{-1}$ and $J_z P$ are rational functions in terms of elements of $\mathcal{O}_U$ (by the formula characterizing adjugate matrices). Thus,

$$\frac{\partial T_a(f)}{\partial Z_j} = \frac{\partial F}{\partial z_j} + \sum_{i=1}^{r} \frac{\partial F}{\partial y_i}\frac{\partial \varphi_i}{\partial Z_j},$$

corresponds to an element in $\mathcal{O}_V$, for some affine subvariety $V \subset U$. □

By successively applying the previous proposition we deduce the following.

**Theorem 2.43** (Lemma 3.5 in [BM97]). *Under the hypotheses of Proposition 2.41, given* $\alpha \in (\mathbb{Z}_{\geqslant 0})^n$*, and given* $f \in \mathcal{O}_U$*, there exists a unique function* $f_\alpha \in \mathcal{O}_{X,a}$ *such that*

$$T_a(f_\alpha) = \frac{\partial^{|\alpha|} T_a(f)}{\partial Z^\alpha}.$$

*In this case, we say that* $f_\alpha$ *is the* partial derivative of $f$ of order $\alpha$ *w.r.t.* $x_1,\dots,x_n$.

## 2.3 SINGULARITIES AND EXAMPLES

In general, a singularity can have many possible forms and behaviours. Nonetheless, there are some families of singularities that are mild enough to be described in simple ways, just like in the following definitions.

**Definition 2.44** (Simple Normal Crossings)**.** Let $X$ be a hypersurface of smooth variety $Z$ and let $a \in X$. We say that $X$ is *simple normal crossings of order* $d$ *at* $a$ if there exists a regular system of parameters $x_1,\dots,x_n$ defined on an open neighbourhood $a \in U \subset Z$ such that the ideal $\mathcal{I}_X$ that defines $X \hookrightarrow Z$ is a principal ideal generated by a monomial of the form

$$x_1 \cdot \ldots \cdot x_d.$$

Notice in particular that $d \leqslant n$. When $X$ is understood we say that $a$ is $\mathrm{snc}(d)$ for brevity.

Given a pair $(X, E)$ where $X$ is given as above and $E \subset Z$ is a reduced divisor, we say that the pair $(X, E)$ is snc at $a \in X$ if $X \cup E$ is $\mathrm{snc}(d)$ at $a$, for some $d$. For added specificity, we say that $(X,E)$ is $\mathrm{snc}(d,r)$ at $a \in X$ if $X, E$ are snc of order $d$, $r$, respectively.

More generally, given a pair $(X, E)$ where $X$ is an arbitrary variety embedded in a smooth variety $Z$, and $E \subset Z$ is a reduced divisor, we say that the pair $(X, E)$ is snc at $a \in X$ if there is a regular coordinate system $(x_1,\dots,x_n)$ on a neighbourhood $U \subset Z$ of $a$ such that $X \cap U$ is the vanishing locus of the ideal $\langle x_1,\dots,x_\ell\rangle$, and if for every irreducible component $E_j$

of $E$ passing through $a$ then $E_j \cap U = V(\langle x_{i_j} \rangle)$, for some $i_j \in \{\ell+1, \dots, n\}$. We say that the pair $(X, E)$ is snc if the pair is snc at every closed point $a \in X \cap E$.

Let us now present one of the key concepts in this work.

**Definition 2.45** (Normal Crossings)**.** Let $X$ be a hypersurface of a smooth variety $Z$ and let $a \in X$. We say that $X$ is *normal crossings of order* $d$ *at* $a$ if there exists an étale coordinate system $x_1, \dots, x_n$ of a neighbourhood $a \in U \subset Z$ such that the local expression in $V(\xrightarrow{\pi} U)$ of the ideal sheaf $\mathcal{I}_X$ defining $X$ at $a$ is a principal ideal generated by a monomial of the form

$$x_1 \cdot \ldots \cdot x_d.$$

When $X$ is understood, we say that $a$ is $\mathrm{nc}(d)$ for brevity.

Given a pair $(X, E)$ where $X$ is a hypersurface in a smooth variety $Z$, and $E \subset Z$ is a reduced divisor, we say that the pair $(X, E)$ is $\mathrm{nc}(d, r)$ at $a \in X$ if

- there is a regular coordinate system $(x_1, \dots, x_n)$ on a neighbourhood $U \subset Z$ of $a$ such that $X \cap U$ is the vanishing locus of the ideal $\langle x_1 \dots x_d \rangle$,
- for every irreducible component $E_j$ of $E$ passing through $a$ then $E_j \cap U = V(\langle x_{i_j} \rangle)$, for some $i_j \in \{d+1, \dots, n\}$,
- there are $r$ irreducible components of $E$ passing through $a$.

We define the normal crossings locus of $X$ as the set

$$S_{\mathrm{nc}(d,r)} := \{a \in X : (X, E) \text{ is normal crossings of order } (d, r) \text{ at } a\}.$$

We also define $S_{\mathrm{nc}(d)} := S_{\mathrm{nc}(d,0)}$.

**Remark 2.46.**

- Notice that if $X$ is snc at $a \in X$ then $X$ is nc at $a$, this is because the identity morphism is étale.
- Notice also that if $X$ is nc at $a$ then the embedding dimension of $X$ at $a$ is $\dim X + 1$.
- Notice that if either $X$ or $E$ are not reduced at $a$, then $X$ is not nc at $a$.

**Example 2.47.** This is a continuation of Example 2.33. Define $I := \langle t^2 - s^2(s+1) \rangle$. The remarks done in Example 2.36 allow us to verify that $V(I)$ is nc at the origin.

**Example 2.48.** This is a continuation of Example 2.34 in the case $d = n = 2$. Define $I := \langle z^2 - wx^2 \rangle$. The remarks done in Example 2.37 allow us to verify that $V(I)$ is nc at the origin.

## 2.4 ELEMENTS OF BIRATIONAL GEOMETRY

**Definition 2.49** (Rational map)**.** Given two varieties $X$ and $Y$, a *rational map* $\pi : X \dashrightarrow Y$ is an equivalence class of morphisms, where the equivalence relation is described as follows. Given $U, V$ open dense subvarieties of $X$ and given morphisms $U \xrightarrow{\varphi} Y$, $V \xrightarrow{\psi} Y$, we say that $(U, \varphi)$ and $(V, \psi)$ are equivalent if there exists an open dense subvariety $W \subset U, V$ such that $\varphi|_W = \psi|_W$. In particular, notice that a rational map does not need to be everywhere defined.

**Proposition 2.50** (Theorem 4.4 in [Har77])**.** *Any rational map of irreducible varieties* $X \dashrightarrow Y$ *induces a morphism* $\mathrm{Frac}(\mathcal{O}_Y) \to \mathrm{Frac}(\mathcal{O}_X)$*.*

*Proof.* Let $V$ be an affine open subvariety of $Y$ such that the rational map $\pi : X \dashrightarrow Y$ admits a representative $\varphi : U \to V$ where $\overline{\varphi(U)} = Y$. Let $\varphi^\sharp : \mathcal{O}_V \to \mathcal{O}_U$ be the morphism of algebras associated to $\pi$. We can then consider the morphism of stalks $\Psi : \mathcal{O}_{V,\pi(\chi)} \to \mathcal{O}_{U,\chi}$. Because $\mathcal{O}_V$ and $\mathcal{O}_U$ are both integral domains, we have that the generic points in each are represented by their respective zero ideal, and so

$$\Psi : \mathrm{Frac}(\mathcal{O}_V) \simeq \mathrm{Frac}(\mathcal{O}_Y) \to \mathrm{Frac}(\mathcal{O}_U) \simeq \mathrm{Frac}(\mathcal{O}_X),$$

is a field homomorphism. □

**Definition 2.51** (Dominant map)**.** Given a rational map $\pi : X \dashrightarrow Y$, we say that $\pi$ is a *dominant map* if there exists a representative $\varphi : U \to Y$ of $\pi$ such that the image $\varphi(U)$ is dense in $Y$.

**Remark 2.52.** Let $\pi : X \dashrightarrow Y$ be a rational map, and consider two representatives $\varphi : U \to Y$, $\psi : V \to Y$ of $\pi$ and assume that $\overline{\varphi(U)} = Y$. Let $W$ be an open subvariety of $U \cap V$ such that $\overline{W} = X$ and $\varphi|_W = \psi|_W$. Notice then

$$\begin{aligned} \overline{\psi(W)} &= \overline{\varphi(W)} \\ &= \overline{\varphi(\overline{W} \cap U)} \\ &= \overline{\varphi(U)} \\ &= Y. \end{aligned}$$

In particular, we obtain that $\overline{\psi(V)} = Y$.

**Proposition 2.53** (Ex. 7.5 in [Vak24])**.** *Let* $\pi : X \dashrightarrow Y$ *be a rational map of irreducible varieties. Then,* $\pi$ *is dominant if and only if* $\pi$ *maps the generic point of* $X$ *to the generic point of* $Y$*.*

*Proof.* Let us first assume that $\pi$ is dominant. Let $\varphi : U \to Y$ be a representative of $\pi$. Let $\chi \in X$ be its generic point. Because $\overline{\{\chi\}} = X$, we have that $\chi \in U$. On the other hand, we know that $\varphi(\overline{\{\chi\}}) = \overline{\varphi(\{\chi\})}$, and so $\varphi(\chi) \in Y$ is a point whose closure is $Y$, that is, $\varphi(\chi)$ is the generic point of $Y$.

Let us now assume that $\pi$ maps the generic point of $X$ to the generic point of $Y$, in other words, if $\varphi : U \to Y$ is a representative of $\pi$ then $\varphi(\chi)$ is the generic point of $Y$. Because $\varphi$ is continuous, we have that $\overline{\varphi(\overline{\{\chi\}} \cap U)} = \overline{\varphi(\{\chi\})}$. In other words,

$$\overline{\varphi(U)} = Y.$$

□

**Proposition 2.54** ([Vak24], Proposition 7.5.7)**.** *Let* $X$*,* $Y$ *be two irreducible varieties over* $K$ *such that there is an injective field morphism* $\overline{\varphi} : \mathrm{Frac}(\mathcal{O}_Y) \to \mathrm{Frac}(\mathcal{O}_X)$ *preserving the subfield* $K$*. Then, there exists a dominant rational map* $\varphi : X \dashrightarrow Y$ *such that the morphism on fields of rational functions induced by* $\varphi$ *is* $\overline{\varphi}$*.*

**Definition 2.55** (Birational equivalence, birational maps). Given two varieties $X, Y$ over a field $K$, we say that $X$ is *birationally equivalent to* $Y$ if there exist rational maps $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow X$ such that there are open subvarieties $U \subset X$ and $V \subset Y$ where $\psi \circ \varphi|_U = \mathrm{Id}_U$ and $\varphi \circ \psi|_V = \mathrm{Id}_V$. We say that $\psi$ and $\varphi$ are birational maps, and they are rational inverses of each other. A morphism of varieties is called birational, if it admits a rational inverse.

One desirable property of a resolution sequence is that, if $X$ is embedded in a smooth space $Z$, and if $\sigma : X' \to X$ is the composition of a resolution sequence, then we would like the strict transform $X'$ of $X$ under $\sigma$ to be embedded in the strict transform $Z'$ of $Z$. This can be done if $\sigma$ is the composition of blowings-up $\sigma_i$ whose centres $C_i$ of blow-up are smooth subvarieties of the strict transform $X_i$ of $X_{i-1}$. This leads us to the notion of *admissible blow-up*.

**Definition 2.56** (Admissible centre, admissible blow-up). Given a variety $X$ embedded into a smooth space $Z$, we say that the centre $C$ of a blow-up morphism $\sigma : Z' \to Z$ is *admissible* if $C$ is a closed smooth subspace of $X$. If $\sigma$ has an admissible centre, we say that $\sigma$ is *admissible*.

Given a pair $(X, E)$ where $X$ is embedded in a smooth space $Z$ and $E$ is a reduced divisor of $Z$, we say that a blow-up $\sigma$ with centre $C \subset Z$ is $(X, E)$-admissible if $C$ is a smooth subspace of $X$ and if $(C, E)$ is snc. When the pair is understood, we simply say that $C$ is admissible, for short.

**Definition 2.57** (Resolution of singularities, *cf.* Definition 3.3 in [Kol07]). A resolution of singularities of an embedded $K$ variety $X \hookrightarrow Z$, where $Z$ is a smooth $K$-variety is a sequence of admissible blowings-up

$$Z' := Z_t \xrightarrow{\sigma_t} Z_{t-1} \to \ldots \to Z_1 \xrightarrow{\sigma_1} Z_0 := Z$$

such that

- The strict transform $X'$ of $X$ at year $t$ is smooth.
- The composition $\sigma := \sigma_t \circ \ldots \circ \sigma_1$ restricts to an isomorphism outside the exceptional divisor $E \subset Z_t$ given by the strict transform of all exceptional divisors created in the blowing-up process.
- The pair $(X', E)$ is snc and for every irreducible component $X_j \not\subset E$.

One extra desirable condition of a resolution of singularities is to prevent unnecessary modifications to $X$. In other words, we would like that the set of points where $\sigma : X' \to X$ does not restrict to an isomorphism to be contained in $\mathrm{Sing}\,(X)$. One way of ensuring this, is by only blowing-up centres $C$ where all the points $a \in C$ satisfy that $\mathrm{ord}_X(a)$ is constant, and only consider centres consisting of points such that the ideal $\mathcal{I}_X$ has vanishing order $> 1$.

**Definition 2.58** (Equimultiple centre, equimultiple blow-up). We say that a centre $C$ of an admissible blow-up $\sigma : X' \to X$ is an *equimultiple centre* if there is a positive integer $d$ such that for all $a \in C$ we have that

$$\mathrm{ord}_{\mathcal{I}_X}(a) = d.$$

If $\sigma$ has an equimultiple centre, we say that $\sigma$ is an *equimultiple blow-up*.

## 2.5 NORMALIZATION

### 2.5.1 *Normalization of rings*

In this section, we provide the necessary definitions of integral elements, integral closure, normal rings, normal varieties, and normalization of a variety. The goal in this section is to show that if $X$ is the affine hypersurface associated to a circulant point then the normalization $\tilde{X}$ is smooth. A side goal is to provide an argument showing that partial derivatives of integral elements are integral.

**Definition 2.59** (Integral elements)**.** Let $R \xrightarrow{\varphi} S$ be a morphism of rings. We say that $f \in S$ is *integral over* $R$ if there exist elements $c_1, \dots, c_{n-1}$ such that $f^n + \varphi(c_1)f^{n-1} + \dots + \varphi(c_{n-1}) = 0$.

We say that $S$ is *integral over* $R$ if every element $s \in S$ is integral over $R$. In this section we will automatically assume that $R \hookrightarrow S$.

**Proposition 2.60** (Corollary 4.6 in [Eis95])**.** *Let $S$ be an $R$-algebra, and let $f \in S$. Then, $f$ is integral over $R$ if and only if $R[f]$ is a finitely generated $R$-module.*

*Proof.*
Notice that if $f$ is integral over $R$ then $1, f, \dots, f^{n-1}$ are generators of $R[f]$ as an $R$-module.

Assume now that $R[f]$ is a finitely generated $R$-module. By fixing a family of generators of $R[f]$ as an $R$-module, we can express $R$-module endomorphism $m_f : R[f] \to R[f]$ given by multiplication by $f$ as multiplication by a matrix $A$. By Cayley-Hamilton we have that there exists a monic polynomial $p$ such that $p(A) = 0$. Consequently, we have that $p(f) = 0$. □

**Corollary 2.61.** *If $S$ is an $R$-algebra which is finitely generated as an $R$-module then $S$ is integral over $R$.*

The proof of the corollary is the same as the second part of the proof of Proposition 2.60, as the proof is fundamentally the same when replacing $R[f]$ with any finitely generated $R$-module.

**Corollary 2.62.** *If $S$ is an $R$-algebra and $f_1, \dots, f_n$ are integral elements of $S$ over $R$, then $R[f_1, \dots, f_n]$ is integral over $R$.*

**Proposition 2.63** (Corollary 5.4 in [AM69])**.** *Let $T$ be an $S$-algebra and let $S$ be an $R$-algebra. Then, $T$ is integral over $R$ if and only if $T$ is integral over $S$ and $S$ is integral over $R$.*

*Proof.*
If $T$ is integral over $R$ then any element in $T$ satisfies a monic polynomial with coefficients over $R$. In particular, notice

- $S \subset T$, thus any element in $S$ satisfies a monic polynomial with coefficients in $R$.
- if $t \in T$ satisfies a monic polynomial with coefficients in $R$ in particular the coefficients are elements in $S$.

Thus obtaining that $T$ is integral over $S$ and $S$ is integral over $R$.

Assume now that T is integral over S and S is integral over R. Let $t \in T$, and consider $s_1, \dots, s_{n-1} \in S$ such that

$$t^n + s_1 t^{n-1} + \dots + s_{n-1} = 0.$$

Then $R[s_1, \dots, s_{n-1}][t]$ is a finitely generated $R[s_1, \dots, s_n]$-module, and so $R[s_1, \dots, s_{n-1}, t]$ is a finitely generated R-module. In particular, t is integral over R. $\square$

**Definition 2.64** (Integral Closure, Normalization, Normal Domain)**.** Let S be an R-algebra. We define the *integral closure of* R *in* S as the set $\tilde{R}_S$ of all elements in S which are integral over R.

In the case where R is an integral domain, we define the *normalization of* R as the integral closure $\tilde{R}$ of R in its fraction field.

Similarly, if R is an integral domain such that $\tilde{R} = R$ then we say that R is a *normal domain*.

**Remark 2.65.** Let R be an integral domain and let S be an R-algebra which is a normal domain. Assume that the morphism $R \xrightarrow{\varphi} S$ is injective. By the universal property of localizations, there exists a unique injective morphism $\psi : \mathrm{Frac}(R) \to \mathrm{Frac}(S)$ such that $\psi$ extends $\varphi$. Moreover, the image $\psi(\tilde{R})$ is a subset of $\tilde{S} = S$.

In short, any injective morphism $R \xrightarrow{\varphi} S$, factors uniquely as $R \xrightarrow{\iota} \tilde{R} \xrightarrow{\psi} S$. We call this the *universal property of normalization*.

**Proposition 2.66** (E. Noether, Corollary 13.13 in [Eis95])**.** *Let* R *be a noetherian normal domain and let* K *be its fraction field. Let* L|K *be a finite field extension. Then, the integral closure of* R *in* L *is a finite* R*-module.*

*Proof.*
Let S denote the integral closure of R in L. Our first observation is that S is integral over $\tilde{R}$ by Proposition 2.63. Notice also that $\mathrm{span}_K(S)$ is an R-algebra, by Corollary 2.62. Next we may notice that $\mathrm{span}_K(S)$ is a subfield of L as

$$K(S) = K[S] = \mathrm{span}_K(S).$$

Because L|K is a finite extension and $K(S) \subset L$, we have that $K(S)|K$ is a finite extension, and so there exist $\beta_1, \dots, \beta_r \in S$ such that

$$\mathrm{span}_K(S) = \mathrm{span}_K(\beta_1, \dots, \beta_r).$$

For each $a \in R$ let us consider the free R-module

$$M_a := \frac{1}{a}(R \cdot \beta_1 + \dots + R \cdot \beta_r),$$

and notice that if $a|b$ then $M_a \subset M_b$, thus $\{M_a\}_{a \in R}$ forms a directed system. Let M be the direct limit of this directed system, and notice that $M = \mathrm{span}_K(S)$. Then, notice that if P is a prime ideal of R and $\alpha_1, \dots, \alpha_s$ with $s > r$, then there must exist $D \in R$ such that $\alpha_1, \dots, \alpha_s \in M_D$ and so $[\alpha_1], \dots, [\alpha_s]$ can be generated with r elements in $M_D/PM_D$. By theorem 3.2 in [Vas70] we have that M is finitely generated. Notice then, that S is an R-submodule of M, thus S is finitely generated. $\square$

**Proposition 2.67** (Corollary 4.12 in [Eis95])**.** *If* $R$ *is a normal domain then* $R[x]$ *is a normal domain.*

*Proof.* Let $K$ denote the fraction field of $R$. Notice that $R[x] \subset K[x] \subset K(x)$. Thus, any element $f(x) \in K(x)$ integral over $R[x]$ is also integral over $K[x]$. On the other hand, because $K[x]$ is a Euclidean domain we have that $K[x]$ is normal. Thus, we may assume that the normalization of $R[x]$ is a subset of $K[x]$. But if $f(x) \in K[x]$ is such that

$$f(x)^n + r_1(x)f(x)^{n-1} + \ldots + r_n(x) = 0,$$

then the induced equation on degree $0$ gives us that the constant coefficient $c$ of $f$ is integral over $R$, thus an element of $R$. Thus, $f(x) - c$ is integral. Using a similar sequence of arguments we get that $f \in R[x]$. □

**Corollary 2.68.** *Let* $K$ *be a field and let* $R$ *denote the ring* $K[x_1, \ldots, x_n, y]$*. If* $f \in \mathrm{Frac}(R)$ *is integral over* $R$ *then* $\dfrac{\partial f}{\partial y}$ *is integral over* $R$*.*

*Proof.* First notice that $K[x_1]$ is normal as $K[x_1]$ is a Euclidean domain. By successively applying Proposition 2.67 we get that $K[x_1, \ldots, x_n, y]$ is normal. Let $f \in \mathrm{Frac}(R)$ be integral over $R$ and consider $r_1, \ldots, r_n \in R$ such that

$$f(x,y)^n + r_1(x,y)f(x,y)^{n-1} + \ldots + r_n(x,y) = 0.$$

By taking partial derivative with respect to $y$ in the previous equation we obtain elements $a(x,y), b(x,y) \in R$ such that

$$\frac{\partial f}{\partial y} \cdot a(x,y) + b(x,y) = 0,$$

and so $\partial f/\partial y \in K(f)$, and because $f$ is integral we have that $K(f)$ is a finite extension of $K$. Thus, by Proposition 2.66 we have that $R[\partial f/\partial y]$ is a finite $R$-module. □

### 2.5.2 *Normalization of varieties*

**Definition 2.69** (Normal variety)**.** Let $K$ be a field of characteristic zero. Let $X$ be a variety over $K$. We say that $X$ is *normal* if for every point $a \in X$ the stalk $\mathcal{O}_{X,a}$ is a normal domain.

**Proposition 2.70** (Ex. 5.4.A in [Vak24])**.** *If* $R$ *is a normal domain then* $\mathrm{Spec}(R)$ *is a normal variety.*

*Proof.* We first claim that if $R$ is an integral domain, and if we realize each stalk $R_{\mathfrak{m}}$ as a subset of $\mathrm{Frac}(R)$, where $\mathfrak{m}$ is a maximal ideal of $S$, then $R = \cap_{\mathfrak{m}} R_{\mathfrak{m}}$, where the intersection ranges over all maximal ideals of $R$.

It suffices to show that $\cap_{\mathfrak{m}} R_{\mathfrak{m}} \subset R$. Let $f \in \cap_{\mathfrak{m}} R_{\mathfrak{m}}$ and consider the ideal $D_f := \{g \in R : gf \in R\} \subset R$. Then $D_f$ is an ideal that is not contained in any maximal ideal, and so $D_f = R$, and so $f \in R$.

We now want to verify that if $R$ is a normal domain, and $S$ is a localization of $R$, that is, if there exists injective morphisms $R \to S \to \mathrm{Frac}(R)$, then $S$ is integrally closed in

$\mathrm{Frac}(S) = \mathrm{Frac}(R)$. Let $f \in \mathrm{Frac}(R)$ be such there exist elements $a_0, b_0, \ldots, a_{n-1}, b_{n-1} \in R$ such that

$$f^n + \frac{a_{n-1}}{b_{n-1}} f^{n-1} + \ldots + \frac{a_0}{b_0} = 0,$$

where $\frac{a_i}{b_i} \in S$. Notice then that $g := \left(\prod_{i=0}^{n-1} b_i\right) \cdot f$ is a solution of the equation

$$x^n + \frac{a_{n-1}}{b_{n-1}} \left(\prod_{i=0}^{n-1} b_i\right) x^{n-1} + \ldots + \frac{a_1}{b_1} \left(\prod_{i=0}^{n-1} b_i^{n-1}\right) x + \frac{a_0}{b_0} \left(\prod_{i=0}^{n-1} b_i^n\right) = 0.$$

That is, $g$ is integral over $R$, and so $g \in R \subset S$. On the other hand, $b_0, \ldots, b_{n-1}$ are all invertible in $S$, and so $f \in S$.

Finally, notice that a stalk $\mathcal{O}_{\mathrm{Spec}(R),\mathfrak{a}}$ at a point $\mathfrak{a} \in \mathrm{Spec}(R)$ is isomorphic to a localization of $R$, $R \to \mathcal{O}_{\mathrm{Spec}(R),\mathfrak{a}}$, and so we obtain that $\mathcal{O}_{\mathrm{Spec}(R),\mathfrak{a}}$ is a normal domain, for all (non-necessarily closed) points $\mathfrak{a}$. □

We now provide a definition of normalization.

**Definition 2.71** (Normalization of a variety)**.** Let $X$ be a variety over a field $K$. We say that a normal variety $\tilde{X}$ is a *normalization of* $X$ if there exists a dominant morphism $\tilde{X} \xrightarrow{\pi} X$ such that for any normal variety $Y$ and for any dominant morphism $q : Y \to X$ there exists a unique morphism $p : Y \to \tilde{X}$ satisfying $\pi \circ p = q$.

**Proposition 2.72** (Proposition 12.44 in [GW20])**.** *For any integral affine variety* $X$*, the normalization* $\tilde{X}$ *of* $X$ *exists.*

*Proof.* Let $R$ be such that $X = \mathrm{Spec}(R)$. Consider the inclusion morphism

$$R \xrightarrow{\pi^\sharp} \tilde{R}.$$

Given that this morphism is injective, we have that the morphism $\mathrm{Spec}(\tilde{R}) \xrightarrow{\pi} \mathrm{Spec}(R)$ maps the generic point of $\mathrm{Spec}(\tilde{R})$ to the generic point of $\mathrm{Spec}(R)$, and so this morphism is dominant as a rational map.

We now claim that if $Y$ is an integral normal variety, then $\mathcal{O}_Y$ is a normal domain. To deduce this, notice that

$$\mathcal{O}_Y = \cap_{\mathfrak{m}} \mathcal{O}_{Y,\mathfrak{m}} \subset \mathrm{Frac}(\mathcal{O}_Y).$$

Let $f \in \mathrm{Frac}(\mathcal{O}_Y)$ be integral over $\mathcal{O}_Y$, and let $a_0, \ldots, a_{n-1} \in \mathcal{O}_Y$ be such that

$$f^n + a_{n-1} f^{n-1} + \ldots + a_0 = 0.$$

Notice in particular that this identity still holds on the level of stalks and so $f$ is integral over $\mathcal{O}_{Y,\mathfrak{m}}$, in other words, $f \in \mathcal{O}_{Y,\mathfrak{m}}$. Thus, $f \in \cap_{\mathfrak{m}} \mathcal{O}_{Y,\mathfrak{m}} = \mathcal{O}_Y$. This shows that $\mathcal{O}_Y$ is a normal domain.

Because $X$ is affine, the morphism $q$ induces a ring morphism $q^\sharp : R \to \mathcal{O}_Y$, which in turn induces a morphism $\overline{q} : \mathrm{Frac}(R) \to \mathrm{Frac}(\mathcal{O}_Y)$. Notice that $\overline{q}$ maps integral elements of $\mathrm{Frac}(R)$ over $R$ to integral elements of $\mathrm{Frac}(\mathcal{O}_Y)$ over $\mathcal{O}_Y$, and because $\mathcal{O}_Y$ is normal, we have that $\overline{q}(\tilde{R}) \subset \mathcal{O}_Y$. Thus, there is a well-defined morphism $Y \to \mathrm{Spec}(R)$. Given that $\overline{q}$ is a map associated to a localization, it is the unique map satisfying $\overline{q} = p^\sharp \circ \pi^\sharp$. □

While normalizations exist for arbitrary (locally noetherian) varieties (see Exercises 10.7.B and 10.7.D of [Vak24]), the proof of this is beyond the scope of this text.

We now want to present a result which proves useful in finding the normalization of an affine neighbourhood of a circulant point. This result is a corollary of Zariski's main theorem. Given the many versions of this result, let us state the version we need.

**Theorem 2.73** (Zariski's main theorem; see [AM99], p. 209, version I)**.** *Let $X$ be a normal variety over $K$ and let $f : X' \to X$ be a birational morphism with finite fibres. Then, there exists an open subvariety $U \subset X$ such that $f : X' \to U$ is an isomorphism.*

**Corollary 2.74.** *If $X, Y$ are irreducible affine varieties where $Y$ is smooth and if $q : Y \to X$ is a finite birational morphism, then $\tilde{X} \simeq Y$.*

*Proof.* Given that $q$ is a birational map, we have that $q$ is dominant. Thus, by the universal property of normalization, there exists a map $Y \xrightarrow{p} \tilde{X}$ such that $q = p \circ \pi$. Therefore, $p$ is a finite birational morphism, and thus with finite fibres. By Zariski's main theorem we have that $p$ is an isomorphism onto an open subvariety $U \subset Y$. On the other hand, we know that $p$ is finite, and thus surjective. □

## 2.6 MARKED IDEALS

The central objects in which we will develop the ideas for partial resolution are marked ideals. They present a convenient way of encoding the necessary information for the problem of finding an embedded resolution.

The problem of finding a resolution sequence of a variety $X$ can be restated purely in terms of algebraic terms: Consider an ideal sheaf $\mathcal{I} \subset \mathcal{O}$ generated by elements $f_1, \dots, f_m$ of order $\geqslant d$, we would like to find an explicit birational morphism $\sigma^* : \mathcal{O} \to \mathcal{O}'$ such that at least one of $\sigma^*(f_1), \dots, \sigma^*(f_m)$ has order strictly smaller than $d$. Assuming we are able to do so then, by applying said strategy finitely many times, we obtain a way of reducing the order of the ideal until $V(\mathcal{I})$ is smooth.

To help keeping track of all these pieces of information, we use *marked ideals*. The term marked ideal appears as early as the work of [Wlo04]. In the present text, the term *marked ideal* does not refer to the concept present in [Wlo04], but rather the one that appears in [BM08].

**Definition 2.75** (Marked ideal)**.** A *marked ideal* $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ is an algebraic structure given by 5 objects, where:

- $Z$ is a variety without singular points.
- $N$ is a smooth closed subvariety of $Z$.
- $E$ is an ordered collection of divisors such that $(N, E)$ is snc at every point $a \in N \cap E$, and $N \not\subset E$.
- $\mathcal{I}$ is a coherent ideal sheaf defined on $N$.
- $d$ is a positive integer.

If $Y$ is an open subvariety of $N$, we will use the notation $\underline{\mathcal{I}}|_Y$ to consider the marked ideal $(Z, N \cap Y, E|_Y, \mathcal{I}|_{N \cap Y}, d)$, where $E|_Y$ and $\mathcal{I}|_{N \cap Y}$ denote the respective restrictions to $Y$. In case $Z, N$ and $E$ are understood we simplify the notation by expressing $\underline{\mathcal{I}}$ as $(\mathcal{I}, d)$.

The guiding strategy for finding the resolution sequence is to blow-up at each step the collection of "most singular points". The Hilbert-Samuel function is the main tool that we will use for measuring how singular a point is. But for a family of varieties we can dispense with this function and use instead the vanishing order.

**Definition 2.76** (Cosupport of a marked ideal)**.** Given a marked ideal $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ we define the *cosupport* of $\underline{\mathcal{I}}$ as

$$\operatorname{cosupp}(\underline{\mathcal{I}}) := \{p \in N : \operatorname{ord}_{\mathcal{I}}(p) \geqslant d\}.$$

In these terms, the resolution strategy is to find a subset of the set of points with maximal vanishing order, in other words, when $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ is arbitrary, we reduce the problem of finding a desingularization sequence for $\underline{\mathcal{I}}$, to finding a desingularization sequence for the marked ideal with the highest possible order, that is, the marked ideal given by $(Z, N, E, \mathcal{I}, m)$, where $m$ is such that $\operatorname{cosupp}((Z, N, E, \mathcal{I}, m + 1)) = \varnothing$.

The number $d$ in the definition of marked ideal serves the purpose of bookkeeping the desired order of vanishing for the ideal $\mathcal{I}$. As such we require the definition of admissibility to be related to this number.

**Definition 2.77** (Admissible blow-up of a marked ideal)**.** Given a marked ideal $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$, we say that a blow-up $\sigma : Z' \to Z$ with smooth centre $C$ is $\underline{\mathcal{I}}$-*admissible* if $C \subset \operatorname{cosupp}(\underline{\mathcal{I}})$. We also say that $C$ is an $\underline{\mathcal{I}}$-*admissible centre*.

**Remark 2.78.** Later in this text, we present the construction of the desingularization invariant inv (see Subsection 2.7.4, c.f. [BM97]). For this procedure we only consider marked ideals where the integer $d$ is the maximum value of the vanishing order of $\mathcal{I}$ in $Z$ and so, this procedure only considers equimultiple centres. Nonetheless, if $d$ is an arbitrarily chosen integer, then a $\underline{\mathcal{I}}$-admissible blow-up does not need to be equimultiple.

**Definition 2.79** (Resolution of singularities of a marked ideal)**.** Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ be a marked ideal. A *resolution of singularities of* $\underline{\mathcal{I}}$ is a sequence of blowings-up

$$Z_t \xrightarrow{\sigma_t} \dots \xrightarrow{\sigma_1} Z_0 := Z,$$

such that, if $\underline{\mathcal{I}}_k$ denotes the controlled transform of $\underline{\mathcal{I}}$ in year $k$, then, for all $k < t$, $\sigma_{k+1}$ is $\underline{\mathcal{I}}_k$-admissible and if $\operatorname{cosupp}(\underline{\mathcal{I}}_t) = \varnothing$.

### 2.6.1 *Test morphisms and equivalence of marked ideals*

**Definition 2.80** (Exceptional blowing-up)**.** Given a blow-up morphism $\sigma : Z' \to Z$ with smooth centre $C$, we say that $\sigma$ is a $E$-*exceptional blowing-up* if there exist irreducible components $E_1, \dots, E_r$ of $E$ such that $C$ is the intersection $\cap_{i=1}^r E_i$.

Because we need three different kinds of morphisms we will give a name to refer to any one of them.

**Definition 2.81** (Test morphism)**.** Given a marked ideal $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$, we say that a morphism $\sigma : Z' \to Z$ is a *test morphism* for $\underline{\mathcal{I}}$ if $\sigma$ is either

- a $\underline{\mathcal{I}}$-admissible blowing-up.
- an $E$-exceptional blowing-up.
- a projection morphism of the form $\sigma : Z \times \mathbb{A}^r \to Z$.

Because we want to establish a result for resolution of singularities using blowings-up, we need to provide the notion of *transforms* of marked ideals under a blowing-up.

**Definition 2.82** (Controlled transform, total transform)**.** Let

$$\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$$

be a marked ideal, where $E$ is a reduced ordered divisor $E_1, E_2, \ldots, E_r$. Let $\sigma : Z' \to Z$ be an admissible blowing-up morphism for $\underline{\mathcal{I}}$. We define

1. the *controlled transform* $\underline{\mathcal{I}}'$ as the marked ideal determined by the following information:
   - $N'$ as the strict transform of $N$ under $\sigma$.
   - $E'$ as the divisor given by $E_1' + E_2' + \ldots + E_r' + E_{r+1}$, where each $E_j'$ is the strict transform of $E_j$ under $\sigma$ and $E_{r+1}$ is the exceptional divisor of $\sigma$.
   - $\mathcal{I}'$ as the ideal sheaf such that, if $g_1, \ldots, g_s$ are a family of local generators of $\mathcal{I}$ in $U \subset Z$ and if $w$ denotes a local generator of the exceptional divisor of $\sigma$, then the family of local generators of $\tilde{\mathcal{I}}$ in $U' \subset Z'$ is given by $w^{-d} g_1 \circ \sigma, \ldots, w^{-d} g_s \circ \sigma$.
   - $d' = d$.
2. the *total transform* $\underline{\mathcal{I}}^*$ as the marked ideal determined by the following information:
   - $N^*$ as the strict transform of $N$ under $\sigma$.
   - $E^*$ as the divisor given by $E_1' + E_2' + \ldots + E_r' + E_{r+1}$, where each $E_j'$ is the strict transform of $E_j$ under $\sigma$ and $E_{r+1}$ is the exceptional divisor of $\sigma$.
   - $\mathcal{I}^*$ as the pull-back $\sigma^*(\mathcal{I})$.
   - $d^* = d$.

We also need to establish a similar notion for the other two types of test morphisms.

**Definition 2.83** (Transform by exceptional blowings-up)**.** Let

$$\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$$

be a marked ideal, where $E$ is an ordered reduced divisor with components $E_1, E_2, \ldots, E_r$. Let $\sigma : Z' \to Z$ be an exceptional blowing-up for $\underline{\mathcal{I}}$. We define the *(controlled) transform* $\underline{\mathcal{I}}'$ as the marked ideal determined by the following information:

- $N'$ as the strict transform of $N$ under $\sigma$.
- $E'$ as the divisor given by $E_1' + E_2' + \ldots + E_r' + E_{r+1}$, where each $E_j'$ is the strict transform of $E_j$ under $\sigma$ and $E_{r+1}$ is the exceptional divisor of $\sigma$.
- $\mathcal{I}'$ as the ideal sheaf $\sigma^*(\mathcal{I})$.
- $d' = d$.

**Definition 2.84** (Transform by projections)**.** Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ be a marked ideal, where $E$ is an ordered reduced divisor with components $E_1, E_2, \dots, E_r$. Let $\sigma : Z' := Z \times \mathbb{A}^1_k \to Z$ be the projection onto $Z$. We define the *(controlled) transform* $\underline{\mathcal{I}}'$ as the marked ideal determined by the following information:

- $N' = \sigma^{-1}(N)$.
- $E'$ as the divisor given by $E'_1 + E'_2 + \dots + E'_r + E_{r+1}$, where each $E'_j$ is $\sigma^{-1}(E_j)$ under $\sigma$ and $E_{r+1}$ is the exceptional divisor of $\sigma$.
- $\mathcal{I}'$ as the ideal sheaf $\sigma^*(\mathcal{I})$.
- $d' = d$.

We finalize this subsection with the definition of *test sequence*.

**Definition 2.85** (Test sequence)**.** A *test sequence* of the marked ideal $\underline{\mathcal{I}}$, is a sequence of morphisms

$$Z_t \xrightarrow{\sigma_t} Z_{t-1} \to \dots \xrightarrow{\sigma_1} Z_0. \tag{2.4}$$

such that each $\sigma_k$ is a test morphism of $\underline{\mathcal{I}}_k$, where $\underline{\mathcal{I}}_k$ is the transform of $\underline{\mathcal{I}}_{k-1}$ by $\sigma_{k-1}$, and $\underline{\mathcal{I}}_0 := \underline{\mathcal{I}}$.

**Definition 2.86** (Equivalent marked ideals)**.** Let $\underline{\mathcal{I}} = (Z, N_1, E_1, \mathcal{I}, d_1)$ and $\underline{\mathcal{J}} = (Z, N_2, E_2, \mathcal{J}, d_2)$ be two marked ideals. We say that $\underline{\mathcal{I}}$ and $\underline{\mathcal{J}}$ are *equivalent marked ideals* if for any open subset $U \subset Z$ we have that any test sequence for $(U, N_1 \cap U, E_1 \cap U, \mathrm{res}_U(\mathcal{I}), d_1)$ is also a test sequence for $(U, N_2 \cap U, E_2 \cap U, \mathrm{res}_U(\mathcal{J}), d_2)$, and vice versa.

In this work, the notation $\underline{\mathcal{I}} = \underline{\mathcal{J}}$ strictly means that both quintuplets are equal.

### 2.6.2 *Constructions associated to marked ideals*

**Definition 2.87** (Product of marked ideals)**.** Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d_1), \underline{\mathcal{J}} = (Z, N, E, \mathcal{J}, d_2)$ be two marked ideals. We define the *product of* $\underline{\mathcal{I}}$ *and* $\underline{\mathcal{J}}$ as

$$\underline{\mathcal{I}} \cdot \underline{\mathcal{J}} := (Z, N, E, \mathcal{I} \cdot \mathcal{J}, d_1 + d_2),$$

where $\mathcal{I} \cdot \mathcal{J}$ denotes the product of the ideal sheaves $\mathcal{I}$ and $\mathcal{J}$.

For a given marked ideal we define $\underline{\mathcal{I}}^1 := \underline{\mathcal{I}}$, and for any $n \in \mathbb{N}$, we define recursively $\underline{\mathcal{I}}^n := \underline{\mathcal{I}} \cdot \underline{\mathcal{I}}^{n-1}$.

**Definition 2.88** (Sum of marked ideals)**.** Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d_1)$ and let $\underline{\mathcal{J}} = (Z, N, E, \mathcal{J}, d_2)$ be two marked ideals. We define the *sum of* $\underline{\mathcal{I}}$ *and* $\underline{\mathcal{J}}$ as

$$\underline{\mathcal{I}} + \underline{\mathcal{J}} := (Z, N, E, \mathcal{I}^{d_2/\gcd(d_1,d_2)} + \mathcal{J}^{d_1/\gcd(d_1,d_2)}, \mathrm{lcm}(d_1, d_2)),$$

where $\mathcal{I} + \mathcal{J}$ denotes the sum of the ideal sheaves $\mathcal{I}$ and $\mathcal{J}$.

**Remark 2.89.** Notice that, given how we defined the sum of marked ideals we obtain that $\underline{\mathcal{I}} + \underline{\mathcal{I}} \neq \underline{\mathcal{I}}$. Nonetheless, when we identify equivalent marked ideals, some fundamental properties of operations of ideal sheaves are recovered for marked ideals.

For example, we have that for any marked ideal $\underline{\mathcal{I}}$, we have that $\underline{\mathcal{I}}+\underline{\mathcal{I}}$ is equivalent to the marked ideal $\underline{\mathcal{I}}$.

Also, for any marked ideals $\underline{\mathcal{I}}_1, \underline{\mathcal{I}}_2, \underline{\mathcal{I}}_3$ we have

$$(\underline{\mathcal{I}}_1+\underline{\mathcal{I}}_2)+\underline{\mathcal{I}}_3 \neq \underline{\mathcal{I}}_1+(\underline{\mathcal{I}}_2+\underline{\mathcal{I}}_3).$$

But, $(\underline{\mathcal{I}}_1+\underline{\mathcal{I}}_2)+\underline{\mathcal{I}}_3, \underline{\mathcal{I}}_1+(\underline{\mathcal{I}}_2+\underline{\mathcal{I}}_3)$ are equivalent. Moreover, for any $n \in \mathbb{N}$ and for any marked ideal $\underline{\mathcal{I}}$, we obtain that the marked ideals $\underline{\mathcal{I}}^n$ and $\underline{\mathcal{I}}$ are equivalent.

**Proposition 2.90** (Lemma 3.8 in [BM08]). *Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$, $\underline{\mathcal{J}} = (Z, N, E, \mathcal{J}, e)$ be two marked ideals. A centre $C \subset Z$ is $(\underline{\mathcal{I}}+\underline{\mathcal{J}})$-admissible if and only if $C$ is $\underline{\mathcal{I}}$- and $\underline{\mathcal{J}}$-admissible.*

*Proof.* Let $a \in C$, and let $\mathfrak{m}_a$ denote the maximal ideal of the local ring $\mathcal{O}_{X,a}$.

Assume that $a \in \operatorname{cosupp}(\underline{\mathcal{I}}) \cap \operatorname{cosupp}(\underline{\mathcal{J}})$. Then for every $f \in \underline{\mathcal{I}}$ we have that $\operatorname{ord}_f(a) \in \mathfrak{m}_a^d$, and for every $g \in \underline{\mathcal{J}}$ we have that $\operatorname{ord}_g(a) \in \mathfrak{m}_a^e$. In particular, $\operatorname{ord}_{f^{e/\gcd(d,e)}}(a) \in \mathfrak{m}_a^{de/\gcd(d,e)}$, and $\operatorname{ord}_{g^{d/\gcd(d,e)}}(a) \in \mathfrak{m}_a^{ed/\gcd(d,e)}$. Given that $\operatorname{lcm}(d,e)\gcd(d,e) = de$, we obtain that $a \in \operatorname{cosupp}(\underline{\mathcal{I}}+\underline{\mathcal{J}})$.

The other implication follows a very similar argument. □

**Remark 2.91.** By successively applying Proposition 2.90 we obtain that $C$ is admissible for $\underline{\mathcal{I}}_1, \ldots, \underline{\mathcal{I}}_n$ if and only if it is $\underline{\mathcal{I}}_1+\ldots+\underline{\mathcal{I}}_n$-admissible.

We now present a couple of results that help us in simplifying the computations of the centres of blow-up of circulant singularities.

**Lemma 2.92.** *Let $\underline{\mathcal{I}} = (\mathcal{I}, d)$ and $\underline{\mathcal{J}} = (\mathcal{J}, e)$ be two principal marked ideals of maximal vanishing order, that is, $d$ is such that*

$$\operatorname{cosupp}(\underline{\mathcal{I}}) \neq \varnothing \textit{ and } \operatorname{cosupp}((\mathcal{I}, d+1)) = \varnothing,$$

*and similarly for $\underline{\mathcal{J}}$. Then,*

$$\operatorname{cosupp}(\underline{\mathcal{I}} \cdot \underline{\mathcal{J}}) = \operatorname{cosupp}(\underline{\mathcal{I}}) \cap \operatorname{cosupp}(\underline{\mathcal{J}}).$$

*In particular, $\underline{\mathcal{I}} \cdot \underline{\mathcal{J}}$ has maximal vanishing order if and only if $\operatorname{cosupp}(\underline{\mathcal{I}}) \cap \operatorname{cosupp}(\underline{\mathcal{J}}) \neq \varnothing$.*

*Proof.* Let $f$ and $g$ be local generators of $\mathcal{I}$ and $\mathcal{J}$, respectively. Notice that

$$\begin{aligned}\operatorname{cosupp}(\underline{\mathcal{I}} \cdot \underline{\mathcal{J}}) &= \{a \in N : f_a \cdot g_a \in \mathfrak{m}_a^{e+d}\} \\ &= \bigcup_{k=0}^{d+e} \left(\{a \in N : f_a \in \mathfrak{m}_a^k\} \cap \{a \in Z : g_a \in \mathfrak{m}_a^{d+e-k}\}\right)\end{aligned}$$

Notice that if $k > d$ then

$$\{a \in N : f_a \in \mathfrak{m}_a^k\} = \varnothing,$$

and if $k < d$ then

$$\{a \in N : g_a \in \mathfrak{m}_a^{d+e-k}\} = \varnothing.$$

Thus, $\bigcup_{k=0}^{d+e} \left(\{a \in Z : f_a \in \mathfrak{m}_a^k\} \cap \{a \in N : g_a \in \mathfrak{m}_a^{d+e-k}\}\right)$ is equal to the set

$$\{a \in N : f_a \in \mathfrak{m}_a^d, g_a \in \mathfrak{m}_a^e\},$$

which is what we wanted to prove. □

By successively applying Lemma 2.92 we obtain the following.

**Corollary 2.93.** *Let* $\{\underline{\mathcal{I}}_j = (\mathcal{I}_j, d_j)\}_{j=1}^r$ *be a finite set of principal marked ideals of maximal vanishing order. Then,*

$$\operatorname{cosupp}\left(\prod_{j=1}^{r}\underline{\mathcal{I}}\right) = \bigcap_{j=1}^{r}\operatorname{cosupp}(\underline{\mathcal{I}}_j).$$

**Definition 2.94** (Derivative of a marked ideal)**.** Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ be a marked ideal. We define the *ideal of derivatives* of $\mathcal{I}$ as the ideal generated by the elements $f \in \mathcal{I}$ together with their first order derivatives that preserve the ideal $\mathcal{I}_E$ associated to the divisor $E$.

The *marked ideal of derivatives* of $\underline{\mathcal{I}}$ is the marked ideal

$$\underline{\mathcal{D}}_E(\underline{\mathcal{I}}) := (Z, N, E, \mathcal{D}_E(\underline{\mathcal{I}}), d-1).$$

We also define for each $n > 1$,ab

$$\underline{\mathcal{D}}_E^n(\underline{\mathcal{I}}) := \underline{\mathcal{D}}_E(\underline{\mathcal{D}}_E^{n-1}(\underline{\mathcal{I}})),$$

and when $n = 0$ we define $\underline{\mathcal{D}}_E^0(\underline{\mathcal{I}}) := \underline{\mathcal{I}}$. When $E = \varnothing$ we denote $\mathcal{D}_E^n(\underline{\mathcal{I}})$ and $\underline{\mathcal{D}}_E^n(\underline{\mathcal{I}})$ by $\mathcal{D}^n(\underline{\mathcal{I}})$ and $\underline{\mathcal{D}}^n(\underline{\mathcal{I}})$, respectively.

**Lemma 2.95** (Lemma 3.2 in [BM08])**.** *Let* $\underline{\mathcal{I}}$ *be a marked ideal. Then*

$$\operatorname{cosupp}(\underline{\mathcal{I}}) \subset \operatorname{cosupp}(\underline{\mathcal{D}}_E^k(\underline{\mathcal{I}})),$$

*for any* $0 \leqslant k \leqslant d-1$*. Moreover, if* $E = \varnothing$ *then equality holds.*

*Proof.* By definition, if $a \in N$ is such that $\operatorname{ord}_{\mathcal{I}}(a) \geqslant d$, we have that $\operatorname{ord}_f(a) \geqslant d$ for any $f \in calI$. Thus, for any $f' \in \mathcal{D}(\underline{\mathcal{I}})$ we have that $\operatorname{ord}_{f'}(a) \geqslant d-1$, in particular for those derivatives that preserve $\mathcal{I}_E$. Applying this successively, we obtain the first part of the claim.

Assume now that $E = \varnothing$ and consider $f \in \mathcal{I}$ such that one of its derivatives $\operatorname{ord}_{f'}(a) < d-1$. Then, $\operatorname{ord}_f(a) < d$, and so $f \notin \operatorname{cosupp}(\underline{\mathcal{I}})$. □

Let us now present the derivation rules for blowings-up. Let $\sigma : Z' \to Z$ be an admissible blow-up for $X \hookrightarrow Z$ with centre $C$. We can find an appropriate local coordinate systems $(x_1, \ldots, x_n)$ in $Z$ centred at $\sigma(a)$ and $(y_1, \ldots, y_n)$ in $Z'$ centred at $a$ such that for some $r$ we have

$$\sigma^*(x_k) = \begin{cases} y_k & \text{if } 0 \leqslant k \leqslant r \\ y_r y_k & \text{if } r < k \end{cases}$$

Because $\sigma$ defines an isomorphism outside some strict subvariety of $Z'$ we have that the derivation rules for functions in $\mathcal{O}_Z$ and $\mathcal{O}_{Z'}$ are the same. In particular, we can apply the chain rule in order to obtain

$$\frac{\partial}{\partial x_k} = \begin{cases} \dfrac{\partial}{\partial y_k} & \text{, if } 0 \leqslant k < r \\ -\displaystyle\sum_{j=r+1}^{n} \frac{x_j}{x_r^2}\frac{\partial}{\partial y_k} + \frac{\partial}{\partial y_k} & \text{, if } k = r \\ \dfrac{1}{x_r}\dfrac{\partial}{\partial y_k} & \text{, if } k > r \end{cases}$$

Or equivalently,

$$\begin{aligned} \frac{\partial}{\partial x_k} &= \frac{\partial}{\partial y_k} && \text{, for } 0 \leqslant k < r \\ x_r\frac{\partial}{\partial x_k} &= -\sum_{k=r+1}^{n} y_k\frac{\partial}{\partial y_k} + y_r\frac{\partial}{\partial y_r} && \text{, for } k = r \\ x_k\frac{\partial}{\partial x_k} &= y_k\frac{\partial}{\partial y_k} && \text{, for } k > r \end{aligned}$$

and consequently for all $f \in \mathcal{O}_Z$ we have that

$$\frac{1}{y_r^{d-1}}\frac{\partial f}{\partial x_k} = y_r\frac{\partial}{\partial y_k}\left(\frac{f\circ\sigma}{y_r^d}\right), \tag{2.5}$$

$$\frac{1}{y_r^{d-1}}\frac{\partial f}{\partial x_k} = d\frac{f\circ\sigma}{y_r^d} + y_r\frac{\partial}{\partial y_r}\left(\frac{f\circ\sigma}{y_r^d}\right) - \sum_{k=r+1}^{n} y_k\frac{\partial}{\partial y_k}\left(\frac{f\circ\sigma}{y_r^d}\right), \tag{2.6}$$

$$\frac{1}{y_r^{d-1}}\frac{\partial f}{\partial x_k} = y_k\frac{\partial}{\partial y_k}\left(\frac{f\circ\sigma}{y_r^d}\right), \tag{2.7}$$

depending on the value of $k$.

**Proposition 2.96** (Lemma 3.3 in [BM08]). *Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ be a marked ideal and let*

$$\sigma : Z' \to Z,$$

*be a $\underline{\mathcal{I}}$-admissible blow-up. Let $\underline{\mathcal{I}}'$ denote the transform of $\underline{\mathcal{I}}$ by $\sigma$ and let $\underline{\mathcal{D}}_E(\underline{\mathcal{I}})'$ denote the strict transform of $\underline{\mathcal{D}}_E(\underline{\mathcal{I}})$ by $\sigma$. Then, $\sigma$ is $\underline{\mathcal{D}}_E(\underline{\mathcal{I}})$-admissible and $\underline{\mathcal{D}}_{E'}(\underline{\mathcal{I}}') \subset \underline{\mathcal{D}}_E(\underline{\mathcal{I}})'$.*

*Proof.* The fact that $\sigma$ is $\underline{\mathcal{D}}_E(\underline{\mathcal{I}})$-admissible is a consequence of Lemma 2.95. We know that any element of $\underline{\mathcal{D}}_{E'}(\underline{\mathcal{I}}')$ is a partial derivative preserving $E'$ of an element of $\underline{\mathcal{I}}'$. Let $f \in \underline{\mathcal{I}}'$ and assume that $y_k$ is such that $\frac{\partial f}{\partial y_k}$ preserves $E'$, then there exists an affine subvariety $U \subset Z$ where the local expression of the strict transform of $\frac{\partial f}{\partial y_k}$ is the left-hand side of one of the equations (2.5), (2.6) or (2.7). Given that $\frac{f\circ\sigma}{y_r^d}$ and its partial derivatives are elements of $\underline{\mathcal{D}}_E(\underline{\mathcal{I}})'$, we obtain the second claim. □

**Corollary 2.97** (Corollary 3.6 in [BM08]). *Any test morphism for $\underline{\mathcal{I}}$ is a test morphism for $\underline{\mathcal{D}}^k(\underline{\mathcal{I}})$ for any $0 \leqslant k \leqslant d-1$. Moreover, for any test sequence $Z_t \xrightarrow{\sigma_t} \ldots \xrightarrow{\sigma_1} Z_0 := Z$, if $E_t$ denotes* *the transform of* $E$ *by the composition* $\sigma := \sigma_t \circ \ldots \circ \sigma_1$, $\underline{\mathcal{I}}_t$ *denotes the transform of* $\underline{\mathcal{I}}$ *by* $\sigma$, *and* $\underline{\mathcal{D}}^k_E(\mathcal{I})_t$ *denotes the transform of* $\underline{\mathcal{D}}^k_E$ *by* $\sigma$, *then*

$$\underline{\mathcal{D}}^k_E(\underline{\mathcal{I}})_t \subset \underline{\mathcal{D}}^k_{E_t}(\underline{\mathcal{I}}_t).$$

*Proof.* Notice that exceptional blowings-up and projection morphisms are not stated in terms of the ideal sheaf $\mathcal{I}$, and so they are test morphisms for $\underline{\mathcal{D}}^k_E(\underline{\mathcal{I}})$ too. The rest of the claims follow by successively applying Proposition 2.96. □

**Definition 2.98** (Coefficient Ideal)**.** We define the *coefficient ideal* associated to $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ as

$$\underline{\mathcal{C}}_E(\underline{\mathcal{I}}) = \sum_{k=0}^{d-1} \mathcal{D}^k(\underline{\mathcal{I}}).$$

Similarly as before, when $E = \varnothing$ we simplify the notation to $\underline{\mathcal{C}}(\underline{\mathcal{I}})$.

**Remark 2.99.** By Corollary 2.97 we have that any $\underline{\mathcal{I}}$-admissible centre is also $\underline{\mathcal{D}}^k_E(\underline{\mathcal{I}})$-admissible, for any $k$. Thus, any $\underline{\mathcal{I}}$-admissible centre is $\underline{\mathcal{C}}_E(\underline{\mathcal{I}})$-admissible. On the other hand, given that $\underline{\mathcal{D}}^0_E(\underline{\mathcal{I}}) = \underline{\mathcal{I}}$ we have that any $\underline{\mathcal{C}}_E(\underline{\mathcal{I}})$-admissible centre is $\underline{\mathcal{I}}$-admissible.

**Theorem 2.100** (Theorem 3.10 in [BM08])**.**

1. *Any test morphism of* $\underline{\mathcal{I}}$ *is also a test morphism of* $\underline{\mathcal{C}}_E(\underline{\mathcal{I}})$.
2. *Let* $\sigma$ *be an* $\underline{\mathcal{I}}$*-admissible blow-up, let* $E'$ *be the strict transform of* $E$ *by* $\sigma$ *together with the exceptional divisor created by* $\sigma$ *and let* $\mathtt{ucalC}_E(\underline{\mathcal{I}})'$ *be the transform of* $\underline{\mathcal{C}}_E(\underline{\mathcal{I}})$ *by* $\sigma$. *Then*

$$\operatorname{cosupp}(\underline{\mathcal{C}}_{E'}(\underline{\mathcal{I}}')) = \operatorname{cosupp}(\underline{\mathcal{C}}_E(\underline{\mathcal{I}})').$$

3. *Any test sequence for* $\underline{\mathcal{I}}$ *is a test sequence for* $\underline{\mathcal{C}}_E(\underline{\mathcal{I}})$.

*Proof.*

1. This is an immediate consequence of Remark 2.99.
2. By Corollary 2.97 we have that $\operatorname{cosupp}(\underline{\mathcal{C}}_{E'}(\underline{\mathcal{I}}')) \subset \operatorname{cosupp}(\underline{\mathcal{C}}_E(\underline{\mathcal{I}})')$. On the other hand, we have that

$$\begin{aligned}\operatorname{cosupp}(\underline{\mathcal{C}}_E(\underline{\mathcal{I}})') &= \cap_{k=0}^{d-1}\operatorname{cosupp}(\underline{\mathcal{D}}^k_E(\underline{\mathcal{I}})') \\ &=\subset \operatorname{cosupp}(\underline{\mathcal{I}}') \\ &=\subset \cap_{k=0}^{d-1}\operatorname{cosupp}(\underline{\mathcal{D}}^k_{E'}(\underline{\mathcal{I}}')) \\ &= \operatorname{cosupp}(\underline{\mathcal{C}}_{E'}(\underline{\mathcal{I}}'))\end{aligned}$$

3. This follows by successively applying item 2 finitely many times.

□

**Proposition 2.101** (Corollary 3.11 in [BM08])**.** $\underline{\mathcal{I}}$ *and* $\underline{\mathcal{C}}_E(\underline{\mathcal{I}})$ *are equivalent ideals.*

*Proof.* Given that $\underline{\mathcal{I}}$ is one of the summands of $\underline{\mathcal{C}}_E(\underline{\mathcal{I}})$, we have that any test sequence of $\underline{\mathcal{C}}_E(\underline{\mathcal{I}})$ is a test sequence of $\underline{\mathcal{I}}$. We thus obtain what we wanted to show. □

**Definition 2.102** (Maximal contact hypersurface)**.** Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ be a marked ideal. Let $z \in \mathcal{O}_N$ be such that $H := V(z)$ is smooth and irreducible in an open subvariety $U \subset N$. We say that $H$ is a *maximal contact hypersurface* of $\underline{\mathcal{I}}$ at $a \in U$ if

- the pair $(H, E)$ is snc in $U$ and $H \not\subset E$,
- there is some open subvariety $a \in V \subset N$ such that

$$\underline{\mathcal{I}}|_V + (Z, N|_V, E|_V, \langle z \rangle|_V, 1) \equiv \underline{\mathcal{I}}|_V.$$

**Remark 2.103.** Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ be a marked ideal such that $E = \varnothing$. Let $a \in \operatorname{cosupp}(\underline{\mathcal{I}})$ be such that $a \notin \operatorname{cosupp}((Z, N, E, \mathcal{I}, d+1))$. We claim that there exists at least one maximal contact hypersurface $H$ of $\underline{\mathcal{I}}$ at $a$. Given that $a \notin \operatorname{cosupp}((Z, N, E, \mathcal{I}, d+1))$, we have that the generators of $\underline{\mathcal{D}}^{d-1}(\underline{\mathcal{I}})$ have vanishing order 1 in some open subvariety $V \subset N$. Let $z \in \underline{\mathcal{D}}^{d-1}$ be such an element. Given that $E = \varnothing$ the first condition for maximal contact hypersurface is trivially verified. On the other hand, we know that

$$\begin{aligned}\underline{\mathcal{I}}|_V &\equiv \underline{\mathcal{C}}(\underline{\mathcal{I}}) \\ &= \underline{\mathcal{C}}(\underline{\mathcal{I}}) + (Z, N, \varnothing, \langle z \rangle, 1) \\ &= \underline{\mathcal{I}} + (Z, N, \varnothing, \langle z \rangle, 1)\end{aligned}$$

and thus, $V(z)$ is a maximal contact hypersurface for $\underline{\mathcal{I}}$ at $a$.

Notice that if

$$Z_t \xrightarrow{\sigma_t} \dots \xrightarrow{\sigma_1} Z_0,$$

is a test sequence for $\underline{\mathcal{I}} = (Z_0, N, E, \mathcal{I}, d)$ consisting of admissible blowings-up, and if $\underline{\mathcal{I}}_t = (Z_t, N_t, E_t, \mathcal{I}_t, d_t)$ is a marked ideal equivalent to a principal ideal of vanishing order 1, then the test sequence defines a resolution of singularities of $V(\mathcal{I})$.

Informally speaking, finding a resolution of singularities of marked ideals helps us find a resolution for a variety, because if $X \hookrightarrow Z$ is a variety embedded in a smooth variety, and if $\mathcal{I}$ is the reduced ideal associated to $X$ then we can consider the sequence of blowings-up that blow-up the stratum of points with maximal value of the desingularization invariant for the marked ideal $\underline{\mathcal{I}} := (Z, Z, \varnothing, \mathcal{I}, d)$ with $d = \operatorname{maxord}(\mathcal{I})$. Notice that the marked ideal $\underline{\mathcal{J}} = (Z', Z', E, \mathcal{J}, d)$ given by the controlled transform of $\underline{\mathcal{I}}$ under this sequence of blowings-up satisfies that $\operatorname{cosupp}(\underline{\mathcal{J}}) = \varnothing$. Thus, we now consider the ideal $\underline{\mathcal{I}}_1 := (Z', Z', E, \mathcal{J}, d-1)$ and use the desingularization invariant to find a sequence of blowings-up such that the controlled transform of $\underline{\mathcal{I}}_1$ under this sequence has empty cosupport. We continue iterating this process on like this until we obtain a marked ideal with maximal order equal to 1. The strict transform $X'$ of $X$ under the composition of these sequences of blowings-up gives us a resolution of singularities of $X$.

## 2.7 DESINGULARIZATION INVARIANT

In this section, we present some key concepts involved in the proof of the main desingularization theorems in [BM97] and [BM08]. The key object in the main result of both works is the *desingularization invariant*, $\operatorname{inv} := \operatorname{inv}_{\underline{\mathcal{I}}}$. Given that the language of marked ideals is more convenient for the presentation of the tools and techniques involved in this process, the construction we present is in terms of marked ideals, as in [BM08]. Let us provide a short description of inv. Consider a marked ideal $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$, and let $X$ denote $\operatorname{cosupp}(\underline{\mathcal{I}})$. Given an $\underline{\mathcal{I}}$-admissible blow-up sequence

$$\begin{array}{lllll} Z_t & Z_{t-1} & & & Z_0 := Z \\ X_t \xrightarrow{\sigma_t} & X_{t-1} & \to \dots & \xrightarrow{\sigma_1} & X_0 := X, \\ E_t & E_{t-1} & & & E_0 := E \end{array} \tag{2.8}$$

there exist functions $\operatorname{inv} : X_k \to A$ for each $0 \leqslant k \leqslant t$ such that each function $\operatorname{inv} : X_k \to A$ depends on the functions $\operatorname{inv} : X_j \to A$ for all $j < k$; where $A$ is a fixed partially ordered set. This construction can be carried out even if the sequence of blowings-up is empty, that is, when $t = 0$. Two important properties of inv are

- the set of points $S_t$ with highest inv-value is a smooth closed subvariety of $X_t$,
- the blow-up $\sigma_{t+1}$ with centre $S_t$ is $\underline{\mathcal{I}}_t$-admissible.

The main desingularization theorem of [BM97] and [BM08] states that given any marked ideal $\underline{\mathcal{I}}$, we can successively blow-up the collection of points with maximal value of inv, and after finitely many blowings-up, we obtain that $\operatorname{cosupp}(\underline{\mathcal{I}}) = \varnothing$. In particular, if we apply this theorem to a marked ideal of the form $(Z, Z, E, \mathcal{I}_X, 1)$, we obtain that after finitely many blowings-up we can resolve the singularities of a pair $(X, E)$[1].

The goal of this section is to present a procedure that allows us to compute a simplified version of inv. We do not provide a proof that this construction satisfies the properties that are involved in the proof of the main desingularization theorems of [BM97] or [BM08]. We also refer the interested reader to the *Crash course on the desingularization invariant* in the Appendix of [BM12], where we can find a brief presentation of the construction of inv.

The construction of inv that allows to find a resolution procedure for all varieties involves the computation of the *Hilbert-Samuel function*. In order to provide a simple procedure for computing the invariant, it is much better to present a simpler version which only involves computing the *order* of an ideal.

### 2.7.1 *Posets and the space of values of the desingularization invariant*

The purpose of this subsection is to construct the space of values of both version of the desingularization invariant.

**Definition 2.104** (Partially ordered set, totally ordered set)**.** Given a set $A$, a relation $\leqslant \subset A \times A$ is said to be a *partial order* on $A$ if it is

- transitive, that is, if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$.
- reflexive, that is, $x \leqslant x$ for any $x$.
- asymmetric, that is, if $x \leqslant y$ and $y \leqslant x$ then $x = y$.

If $\leqslant$ is a partial order of $A$ then we say that $A$ is a *poset*.

1 More precisely, the resolution of singularities of the pair is given by the sequence of blowings-up of the resolution of the marked ideal $\underline{\mathcal{I}}$ *before the last blow-up*

If $\leqslant$ is a partial order on $A$ such that for all $x, y \in A$ with $x \neq y$ we have that either $x \leqslant y$ or $y \leqslant x$ is satisfied, we say that the order $\leqslant$ is a *total order*. In this case we use the notation $x < y$.

**Remark 2.105.** If $X$ is a totally ordered set with respect to $\leqslant$ then any subset $Y$ is also a totally ordered set with respect to $\leqslant$.

**Example 2.106.** The extended real numbers $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$ form a totally ordered set with the standard order.

**Example 2.107.** The natural numbers $\mathbb{N} \subset \mathbb{R}$ form a totally ordered set.

**Example 2.108.** $\mathbb{Q}_{\geqslant 1} := \{q \in \mathbb{Q} : q \geqslant 1\} \subset \overline{\mathbb{R}}$ is a totally ordered set.

**Example 2.109.** The set $\{0, \infty\} \subset \overline{\mathbb{R}}$ with the relation given by $0 < \infty$ is a totally ordered set. Notice that $\{0, \infty\}$ satisfies that for all $x \in \mathbb{Q}_{\geqslant 1}$ and for all $y \in \{0, \infty\}$, $x < y$ or $y < x$.

**Example 2.110.** The set

$$\mathbb{H} := \{f : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}\}$$

can be given the structure of a totally ordered set. For this, consider the order given by $f < g$ if and only if there exists $m \in \mathbb{Z}_{\geqslant 0}$ such that $f(m) < g(m)$ and, for all $k < m$, $f(k) = g(k)$.

Notice also that the set of increasing finite (possibly empty) sequences

$$\mathbb{J} := \{(y_1, \ldots, y_r) : y_k \in \mathbb{Z}_{\geqslant 0}, \text{ for all } 1 \leqslant k \leqslant r, \text{ and } y_1 \leqslant \ldots \leqslant y_r\}$$

can be given the structure of a totally ordered set. This is because there is an injective map $\mathbb{J} \to \mathbb{H}$ mapping $(y_1, \ldots, y_r)$ to the function $f(k) := y_{k+1}$ for all $0 \leqslant k \leqslant r-1$ and $f(k) = 0$, otherwise. This allows us to compare any two finite (increasing) sequences, and using this relation, we obtain $(y_1, \ldots, y_r) \leqslant (y_1, \ldots, y_r, a_1, \ldots, a_{r'})$.

**Example 2.111.** Given a totally ordered set $X$, the set $X \times \mathbb{Z}_{\geqslant 0}$ is a totally ordered set with respect to the relation $(x, n) \leqslant (y, m)$ if and only if $x < y$ or ($x = y$ and $n < m$).

**Definition 2.112** (Dictionary set, lexicographic order)**.** Let $n \in \mathbb{Z}_{\geqslant 0}$. Consider $n+3$ totally ordered sets

$$(A^0, \leqslant_0), (A^1, \leqslant_1), \ldots, (A^n, \leqslant_n), (A^{n+1}, \leqslant_{n+1}), (A^{n+2}, \leqslant_{n+2})$$

such that there exists a totally ordered set $X$ satisfying that for all $1 \leqslant k \leqslant n+1$ we have that $A^k \subset X$ and for all $x \in A^k$ and for all $y \in A^{n+1}$ we have that $x < y$ or $x > y$. We define the *dictionary set*

$$A = \bigcup_{k=1}^{n} A^0 \times A^1 \times \ldots \times A^k \times A^{n+1} \times A^{n+2},$$

and we equip it with the *lexicographic order*, which we define as follows. Let $a = (a^0, \ldots, a^{k_1})$, $b = (b^0, \ldots, b^{k_2}) \in A$, we say that $a \leqslant b$ if and only if one of the two following cases holds:

- there exists $k \leqslant \min\{k_1, k_2\}$ such that for each $j < k$, we have $a^j = b^j$ and $a^k <_k b^k$.
- $k_1 < k_2$ and for each $j \leqslant k_1$, we have $a^j = b^j$.

**Example 2.113.** Fix $n \in \mathbb{N}$ and consider the totally ordered sets

- $A^0 := \mathbb{H} \times \mathbb{Z}_{\geqslant 0}$,
- $A^k := \mathbb{Q}_{\geqslant 1} \times \mathbb{Z}_{\geqslant 0}$, for all $1 \leqslant k \leqslant n$,
- $A^{n+1} := \{0, \infty\}$,
- $A^{n+2} := \mathbb{J}$.

See Example 2.110 for the definition of $\mathbb{H}$ and $\mathbb{J}$. Notice that if $(x, d) \in A^k$ $(1 \leqslant k \leqslant n)$ and $y \in A^{n+1}$, then we may compare them by means of the relation $(x, d) < y$ if and only if $x < y$. Define $\Sigma$ as the dictionary set on $A^0, \ldots, A^{n+2}$.

**Example 2.114.** Fix $n \in \mathbb{N}$ and consider the totally ordered sets

- $A^0 := \mathbb{Z}_{\geqslant 1} \times \mathbb{Z}_{\geqslant 0}$,
- $A^k := \mathbb{Q}_{\geqslant 1} \times \mathbb{Z}_{\geqslant 0}$, for all $1 \leqslant k \leqslant n$,
- $A^{n+1} := \{0, \infty\}$,
- $A^{n+2} := \mathbb{J}$.

See Example 2.110 for the definition of $\mathbb{J}$. Define $\Sigma_h$ as the dictionary set on $A^0, \ldots, A^{n+2}$.

In general, the values that $\mathrm{inv}_{\underline{\mathcal{I}}}$ adopts will be of the form

$$((H, s^1), (\nu^2, s^2), \ldots, (\nu^n, s^n), 0, (y_1, \ldots, y_r)),$$

or

$$((H, s^1), (\nu^2, s^2), \ldots, (\nu^n, s^n), \infty),$$

with the understanding that the latter has an empty sequence in $\mathbb{J}$, where $H$ is a Hilbert-Samuel function (see Subsection 2.7.2) of a closed point.

**Definition 2.115** (Upper semi-continuity)**.** Let $\iota : X \to \Sigma$ be a function from a topological space $X$ to a poset $\Sigma$. We say that $\iota$ is *upper semi-continuous* if

- The image $\iota(X)$ is a finite set.
- For any $s \in \Sigma$, $\{x \in X : \iota(x) \geqslant s\}$ is a closed subset of $X$.

The space of values of $\mathrm{inv}_{\underline{\mathcal{I}}}$ is the dictionary set $\Sigma$ (see Example 2.113) as it uses the Hilbert-Samuel function to construct an *infinitesimal presentation* (see [BM97] for more details). For simplicity, the construction we present in Subsection 2.7.4 is a *simplified version* inv that can be used for the desingularization of *hypersurfaces*.

### 2.7.2 *Hilbert-Samuel function*

The purpose of this subsection is to present the definition and fundamental properties of the Hilbert-Samuel function, as this function is the first entry of $\mathrm{inv}_{\underline{\mathcal{I}}}$.

**Definition 2.116** (Hilbert-Samuel function)**.** Let $(X, \mathcal{O}_X)$ be a variety over an algebraically closed field $\mathbb{K}$, let $a \in X$ be a closed $\mathbb{K}$-point and let $\mathfrak{m}_a$ be the maximal ideal of the stalk $\mathcal{O}_{X,a}$ at $a$. We define the *Hilbert-Samuel function* of $\mathcal{O}_X$ at $a$ as the function $H : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ given by

$$H_{X,a}(n) := \dim_{\mathbb{K}} \mathcal{O}_{X,a}/\mathfrak{m}_a^{n+1}.$$

There are a couple of remarks worth pointing out.

**Remark 2.117.** Let $X$ be a variety over an algebraically closed field $\mathbb{K}$, and assume that $H(1) = d+1$. Let $g_1, \dots, g_d \in \mathcal{O}_{X,a}$ be such that their equivalence classes generate $\mathfrak{m}_a/\mathfrak{m}_a^2$. This allows us to construct an embedding $U \hookrightarrow \mathbb{A}^d_{\mathbb{K}}$, for some neighbourhood $U \ni a$. On the other hand, any set of generators of $\mathfrak{m}_a/\mathfrak{m}_a^2$ may be lifted to a set of representatives $g_1, \dots, g_d \in \mathcal{O}_{X,a}$ by Nakayama's lemma. Thus, $d$ is the smallest dimension of a smooth variety $Z$ admitting a closed embedding $U(\subset X) \hookrightarrow Z$. Because of the above, we say that $d$ is the *local embedding dimension of* $X$ *at* $a$, which we denote by $e_X(a)$.

**Remark 2.118.** In the special case where $X \hookrightarrow Z$ is a hypersurface, $Z$ is smooth and $a$ is singularity of $X$, we have that $e_X(a) > \dim X$ by definition of singular point. On the other hand, because $X$ admits a local embedding into a variety of dimension $\dim X + 1$, we have $\dim X + 1 \geqslant e_X(a) > \dim X$. Thus,

$$H(1) = \dim X + 2.$$

**Remark 2.119.** Assume $X$ is a hypersurface where $\operatorname{ord}_X(a) = d$. Notice that for any $k \in \mathbb{N}$,

$$H_{X,a}(k) - H_{X,a}(k-1) = \dim_{\mathbb{K}}(\mathfrak{m}_a^k/\mathfrak{m}_a^{k+1}).$$

Moreover, if $1 < k < d$, we have

$$\dim_{\mathbb{K}}(\mathfrak{m}_a^k/\mathfrak{m}_a^{k+1}) = \binom{k + \dim_a Y - 1}{k},$$

and if $k \geqslant d$, we have

$$\dim_{\mathbb{K}}(\mathfrak{m}_a^k/\mathfrak{m}_a^{k+1}) = \binom{k + \dim_a Y - 1}{k} - \binom{k - d + \dim_a Y - 1}{k - d}.$$

Thus, if $X$ is a hypersurface at $a$ and $b$ of the same dimension, then $H_{X,a} < H_{X,b}$ (see $\mathbb{H}$ in Example 2.113) if and only if $\operatorname{ord}_X(a) < \operatorname{ord}_X(b)$.

Let us now present the properties of the Hilbert-Samuel function that ensure that the desingularization algorithm is well-defined and terminates are the following.

**Proposition 2.120** (Bierstone, Milman; Theorem 1.14 and Theorem 9.2 in [BM97])**.** *Let* $X$ *be a variety, and let* $H : X \to \mathbb{H}$ *denote the Hilbert-Samuel function, that is*

$$H(a)(n) = \dim_k \left(\mathcal{O}_{X,a}/\mathfrak{m}_a^{n+1}\right).$$

*Then,*

1. $H$ *is upper-semicontinuous.*

2. $H$ *is infinitesimally upper-semicontinuous, that is, if* $C \subset \{a \in X : S(a) = h\}$ *is closed, and if* $\sigma$ *denotes the blow-up of* $X$ *with centre* $C$*, then for any* $a' \in \sigma^{-1}(a)$ *we have that* $H(a') \leqslant H(a)$.
3. *The image* $H(X)$ *is a finite set.*

The proofs of item 1 and item 3 can be found in Theorem 9.2 and Lemma 3.10 of [BM97], respectively. The proof of item 2 can be found in Proposition 6.13 of [BM97] (cf. Theorem 1.14 in [BM97]).

### 2.7.3 *Main desingularization theorem and relevance of hypersurface case*

Let us provide the statement of the main desingularization theorem. Given that the language of marked ideals is better suited for the techniques involved in the proof, we present the statement in terms of a resolution of marked ideals..

**Theorem 2.121** (Theorem 7.1 in [BM08]). *Let* $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ *be a marked ideal, with* $d > 0$. *Then,* $\underline{\mathcal{I}}$ *admits a resolution of singularities*

$$\begin{matrix} Z_t \\ \underline{\mathcal{I}}_t \end{matrix} \xrightarrow{\sigma_t} \cdots \xrightarrow{\sigma_1} \begin{matrix} Z_0 := Z \\ \underline{\mathcal{I}}_0 := \underline{\mathcal{I}} \end{matrix},$$

*satisfying the following properties:*

1. *There exist upper-semicontinuous functions* inv *defined on* $\operatorname{cosupp}(\underline{\mathcal{I}}_j)$ *for each* $j$*, taking values in the partially ordered set defined in Example 2.113,*
2. *Each centre of blow-up is given by the locus of points with maximal value of* inv*,*
3. *The functions* inv *are* infinitesimally upper-semicontinuous*, that is, if* $a \in \operatorname{cosupp}(\underline{\mathcal{I}}_{j+1})$, $b = \sigma_{j+1}(a)$ *then*
   - $\operatorname{inv}(b) = \operatorname{inv}(a)$ *if* $b \notin C_j$,
   - $\operatorname{inv}(b) < \operatorname{inv}(a)$ *if* $b \in C_j$.

In order to obtain a result similar to Theorem 2.121, for pairs $(X, E)$, we can construct the functions $\operatorname{inv}_{(X,E)}$ by defining $\operatorname{inv}_{(X,E)}$ as the invariant inv associated to the marked ideal $(Z, Z, E, \mathcal{I}_X, 1)$, where $\mathcal{I}_X$ denotes the ideal sheaf associated to $X$.

These functions $\operatorname{inv}_{\underline{\mathcal{I}}}$ are constructed in such a way that the first entry of $\operatorname{inv}_{\underline{\mathcal{I}}}(a_j)$ is the Hilbert-Samuel function of $\mathcal{O}_{X_j}$ at $a$, where $X_j := V(\mathcal{I}_j)$. Let us recall the fact that the Hilbert-Samuel function at a point $a \in X$ is of the form

$$(1, e_X(a) + 1, \ldots),$$

(see Remark 2.117 and Remark 2.118). Thus, after finitely many steps, we may assume that the sequence of blowings-up with centres of maximal values of $\operatorname{inv}_{\underline{\mathcal{I}}}$ will necessarily take us from a general marked ideal $\underline{\mathcal{I}}$ to a marked ideal $\underline{\mathcal{I}}'$ such that the entries of the Hilbert-Samuel function, at any point $a \in X$, are of the form

$$(1, \dim_a X + 2, \ldots).$$

In other words, we can always reduce the general case, to a hypersurface case.

The main relevance of the hypersurface case is that we can exchange inv by an invariant that can be computed in a simpler way. In particular, it does not require computing the Hilbert-Samuel function, yet it still determines the same centres of blow-up. More precisely, we may replace the first entry of inv by the vanishing order of a suitable ideal related to $\underline{\mathcal{I}}$ in the respective year (see Remark 2.119).

### 2.7.4 *Recursive definition of the invariant in the hypersurface case*

Consider a hypersurface $X$ of a smooth variety $Z$, together with an ordered exceptional divisor $E = D_1 + D_2 + \ldots + D_r$. Consider a blow-up history

$$X' := X_t \xrightarrow{\sigma_t} X_{t-1} \to \ldots \xrightarrow{\sigma_1} X_0 := X,$$

and let $E_{t'}$ be the respective transform of $E$ at year $t'$, that is, $E_{t'}$ is the collection of strict transforms of $D_1, \ldots, D_r$ in year $t'$ together with the respective transforms of the divisors created by the blowings-up in all years $< t'$.

We can define functions $\mathrm{inv} : X_{t'} \to \Sigma_h$ (see Example 2.114) such that $\mathrm{inv}_{(X_{t'},E_{t'})}(a') < \mathrm{inv}_{(X_{t'},E_{t'})}(b')$ if and only if $\mathrm{inv}(a') = \mathrm{inv}(b')$ for all $a', b' \in X_{t'}$. That is, we can replace the desingularization invariant which uses the Hilbert-Samuel function in its first entry, by a simpler desingularization invariant that can be constructed directly using only information of the marked ideal. Because of this, from this point on, we deal exclusively with the case where $X$ is a hypersurface embedded in a smooth space $Z$.

The goal of this subsection is to present a procedure that allows us to compute inv in year $t$ after having computed inv in years $t-1, \ldots, 0$. Said procedure is recursive in dimension. Let us be more specific. Assume that we have computed inv at every year $t' < t$, and that for each year $0 \leqslant t' < t$ the centre of the blow-up in year $t'$ is inside the locus of points $S_{t'} \subset X_{t'}$ with maximal value of inv. Given a marked ideal of the form $\underline{\mathcal{I}}^0 = (Z, Z, E, \mathcal{I}, d) = (Z_t, Z_t, E_t, \mathcal{I}_t, d_t)$ we can compute the first entry $(\nu_1, s_1)$ of $\mathrm{inv}(a)$, and the rest of the entries are determined using a marked ideal $\underline{\mathcal{I}}^1 = (Z, N^1, E^1, \mathcal{J}, e)$ where $\dim N^1 = \dim Z - 1$, and $\mathcal{I}^1$ and $e$ are defined in terms of $\underline{\mathcal{I}}^0$. In general, once we have determined the first $k$ entries $(\nu_1, s_1), \ldots, (\nu_k, s_k)$ of $\mathrm{inv}(a)$, we may use the marked ideal $\underline{\mathcal{I}}^k$ to construct $(\nu_{k+1}, s_{k+1})$. To proceed to the next step we replace $\underline{\mathcal{I}}^k$ with the appropriate marked ideal $\underline{\mathcal{I}}^{k+1}$. The construction of these marked ideals is such that, after finitely many steps, we have that $\mathcal{I}^n = 0$ or $\mathcal{I}^n$ is a principal ideal generated by monomial ideals in the variables that define the irreducible components of $E$. Using an inductive argument, we may assume that $\mathrm{ord}_{\mathcal{I}^k}(a) \geqslant d$. Thus, we have two cases:

1. the marked ideal $\underline{\mathcal{I}}^k$ has maximal order at $a$, in other words, $\mathrm{ord}_{\mathcal{I}^k}(a) = d^k$ or
2. $\mathrm{ord}_{\mathcal{I}^k}(a) > d^k$.

Assume that we have already computed $(\nu_1, s_1), \ldots, (\nu_k, s_k)$ entries of $\mathrm{inv}(a)$. Express the ideal $\underline{\mathcal{I}}^k = (Z, N^k, E, \mathcal{I}^k, d^k)$. We have two possibilities, either $\mathrm{ord}_{\mathcal{I}^k}(a) > d$ or $\mathrm{ord}_{\mathcal{I}^k}(a) = d$. In the case where $\mathrm{ord}_{\mathcal{I}^k}(a) = d^k$, we define $\nu^{k+1} := 1$, and we do not need to modify $\underline{\mathcal{I}}^k$.

Assume that $\operatorname{ord}_{\mathcal{J}^k}(a) > d^k$. Our goal is to construct a marked ideal $\underline{\mathcal{G}}(\underline{\mathcal{J}}^k) = (Z, N^k, E, \mathcal{G}(\underline{\mathcal{J}}^k), e^k)$, which we call *companion ideal*, whose cosupport is inside the cosupport of $\underline{\mathcal{J}}^k$ and such that

$$\operatorname{ord}_{\mathcal{G}(\underline{\mathcal{J}}^k)}(a) = e^k.$$

In this case, $\nu^k$ is given by $\operatorname{ord}_{\mathcal{J}^k}(a)/d^k$.

Let $y_1, \dots, y_r \in \mathcal{O}_{N^k}$ be the generators of all the irreducible components of $E$ passing through $a$ (given that the information for inv is local, we may assume that all components $D_1, \dots, D_r$ pass through $a$), and define $\mathcal{M}(\underline{\mathcal{J}}^k)$ as the principal ideal sheaf generated by the monomial in $y_1, \dots, y_r$ such that if $\mathcal{R}(\underline{\mathcal{J}}^k)$ is the ideal sheaf satisfying

$$\mathcal{J}^k = \mathcal{M}(\underline{\mathcal{J}}^k) \cdot \mathcal{R}(\underline{\mathcal{J}}^k),$$

then no element $\mathcal{R}(\underline{\mathcal{J}}^k)$ is divisible by any $y_1, \dots, y_r$. We say that $\mathcal{M}(\underline{\mathcal{J}}^k)$ is the *monomial part* of $\underline{\mathcal{J}}^k$ and $\mathcal{R}(\underline{\mathcal{J}}^k)$ is the *residual part* of $\underline{\mathcal{J}}^k$.

If $\underline{\mathcal{M}}(\underline{\mathcal{J}}^k) = \underline{\mathcal{J}}^k$, then we are in one of the special cases indicated in the paragraph above. We indicate how to deal with this particular case at the end.

Thus, let us assume that $\mathcal{R}(\underline{\mathcal{J}}^k)$ is not the zero ideal. We are now presented with two possibilities, either $\operatorname{ord}_{\mathcal{R}(\underline{\mathcal{J}})}(a) \geqslant d^k$ or $\operatorname{ord}_{\mathcal{R}(\underline{\mathcal{J}})}(a) < d^k$. If $d' := \operatorname{ord}_{\mathcal{R}(\underline{\mathcal{J}})}(a) < d^k$ we define the ideal

$$\underline{\mathcal{G}}(\underline{\mathcal{J}}^k) := (Z, N^k, E, \mathcal{R}(\underline{\mathcal{J}}^k), d') + (Z, N^k, E, \mathcal{M}(\underline{\mathcal{J}}^k), d^k - d'). \tag{2.9}$$

In the case where $\operatorname{ord}_{\mathcal{R}(\underline{\mathcal{J}})}(a) \geqslant d^k$ we define

$$\underline{\mathcal{G}}(\underline{\mathcal{J}}^k) := (Z, N^k, E, \mathcal{R}(\underline{\mathcal{J}}^k), \operatorname{ord}_{\mathcal{R}(\underline{\mathcal{J}}^k)}(a)). \tag{2.10}$$

In short, we may replace $\underline{\mathcal{J}}^k$ by a related marked ideal

$$\underline{\mathcal{G}}(\underline{\mathcal{J}}^k) = (Z, N^k, E, \mathcal{G}(\underline{\mathcal{J}}^k), e^k) \tag{2.11}$$

such that $\operatorname{ord}_{\mathcal{G}(\underline{\mathcal{J}}^k)} = e^k$. In order to compute $s^k$, we have to consider the first year $t'$ such that the image $a' \in X_{t'}$ of $a$ satisfies that the first $k$ couples of the invariant at $a'$ are the same as those of $a$ and $\nu^{k+1}$ at $a'$ is the same as that of $a$. We call this the *year of birth* of $\nu^{k+1}$. Let $D'_{i_1} + \dots + D'_{i_s}$ be all the divisors of $E_{t'}$ passing through $a'$. We define $s^k := s$.

Define $E^{k+1} := E - D_{i_1} - \dots - D_{i_s}$, and let $\underline{\mathcal{C}} = (Z, N, E^{k+1}, \mathcal{C}, D)$ be the coefficient ideal with respect to $E^{k+1}$ of the marked ideal

$$(Z, N^k, E^{k+1}, \mathcal{G}, 1) + \sum_{j=1}^{s} (Z, N^k, E^{k+1}, \langle y_{i_j} \rangle, 1),$$

where $\underline{\mathcal{G}} := \underline{\mathcal{G}}(\underline{\mathcal{J}}^k)$. We define the ideal

$$\underline{\mathcal{J}}^{k+1} := (Z, N^k \cap V(z), E^{k+1}, \mathcal{C}(\underline{\mathcal{J}}^k), D),$$

where $z$ is a maximal contact hypersurface of $\underline{\mathcal{G}}$ on a neighbourhood of $a$. Because $\underline{\mathcal{G}}$ has maximal order at $a$, we have that $\underline{\mathcal{J}}^{k+1}$ also has maximal order at $a$.

After finitely many steps, we reach the case $\dim N^n = 0$. In this case, the only possible proper ideal of $\mathcal{O}_{N^n}$ is $0$, and we define $\nu^{k+1} := \infty$.

Let us finally address the case where $\mathcal{M}(\underline{\mathcal{I}}^k) = \mathcal{I}^k$. In this case, we define $\nu^{k+1} := 0$. Express $\mathcal{M}(\underline{\mathcal{I}}^k) = \langle y_1^{\alpha_1} \cdot \ldots \cdot y_r^{\alpha_r} \rangle$ and express the components of $E$ passing through $a$

$$D_1 + \ldots + D_r = V(y_1) + \ldots + V(y_r).$$

Notice that a centre of the form $C = V(y_{i_1}, \ldots, y_{i_\ell}) \subset N^k$ is equimultiple if and only $\sum_{j=1}^{\ell} \alpha_{i_j} \geqslant d^k$. In order to uniquely identify an equimultiple centre from all the options, we select the one minimizing $\ell$. In case there are two distinct ordered sets $i_1 \leqslant \ldots \leqslant i_\ell$, $i'_1 \leqslant \ldots \leqslant i'_\ell$ of indices that are minimal, and such that the sum of the respective powers exceeds $d^k$, we choose the smaller one with respect to the lexicographic order comparing $(i_1, \ldots, i_\ell)$ and $(i'_1, \ldots, i'_\ell)$. If $(i_1, \ldots, i_\ell)$ is the smallest set, we define the J-invariant as $(i_1, \ldots, i_\ell)$, and in this case inv is of the form

$$((\nu^1, s^1), (\nu^2, s^2), \ldots, (\nu^k, s^k), 0, (i_1, \ldots, i_\ell)).$$

For homogeneity of notation, if $\nu^k = \infty$, then inv can be thought of as having appended a J-invariant with an empty sequence of indices.

From now on, we drop the inner round brackets of the invariant, and we drop the J-invariant from the notation, as this is only dealt with when $\underline{\mathcal{I}}^k$ is monomial, for some $k$. In other words, we will use the notation

$$(\nu^1, s^1, \nu^2, s^2, \ldots, \nu^k, s^k, \infty),$$

or

$$(\nu^1, s^1, \nu^2, s^2, \ldots, \nu^k, s^k, 0).$$

# 3

# CIRCULANT SINGULARITIES

A class of singularities that plays a central role in the partial desingularization procedure we present is that of *circulant singularities*, an example of which is the *pinch-point* of the Whitney umbrella. Circulant singularities are named after *circulant matrices*, as circulant matrices can be used to define a simple local expression of a circulant singularity. One of the defining properties of a circulant singularity is the existence of a local action by a cyclic group.

This chapter is divided in two parts. The first part concerns the notion of circulant singularities, and the proofs of the relevant properties to the main results of this thesis. In the second part of this chapter, we present *group precirculant singularities*, which are a class of singularities admitting a local action by an *abelian group*, thus generalizing the notion of *precirculant singularities*. The latter part of the chapter is not relevant to the main results of this thesis, but I believe that the notion of precirculant singularities could be useful for partial desingularization in higher dimensions.

## 3.1 CIRCULANT MATRICES

In this work, we use circulant matrices to define the notion of circulant singularities. Circulant matrices enjoy a lot of properties, and because of this, let us present the ones that are used in this work. Two good references on the properties of circulant matrices are [KS12] and [Dav79]. Let us begin by recalling the notion of *permutation matrices*.

**Definition 3.1** (Permutation matrices)**.** Let $R$ be a ring and fix $n \in \mathbb{N}$. Given a permutation $\sigma \in S_n$ we define the *permutation matrix* associated to $\sigma$ as the matrix $P_\sigma \in \mathrm{Mat}_{n\times n}(R)$ whose entries are defined as $(P_\sigma)_{ij} := \delta_{\sigma(i),j}$, where $\delta$ denotes the Kronecker delta function.

**Lemma 3.2.** *The map $\sigma \mapsto P_\sigma$ is a group homomorphism $S_n \to GL_n(R)$.*

*Proof.* Let $\sigma, \tau \in S_n$ be two permutations. Notice that, for any $i, j \in \{1, \dots, n\}$ we have

$$\begin{aligned}(P_\sigma \cdot P_\tau)_{ij} &= \sum_{k=1}^{n} (P_\sigma)_{ik}(P_\tau)_{kj} \\ &= (P_\sigma)_{i\sigma(i)}(P_\tau)_{\sigma(i)\tau(\sigma(i))} \\ &= (P_{\sigma\circ\tau})_{ij}.\end{aligned}$$

Consequently, the association $\sigma \mapsto P_\sigma$ is a group homomorphism. Moreover, this homomorphism is injective, as $P_\sigma = I$ if and only if $\sigma(i) = i$ for all $i \in \{1, \dots, n\}$. □

**Definition 3.3.** Given a positive integer $n \in \mathbb{N}$, we consider the $n$-cycle $\sigma = (1\ 2\ \dots\ n) \in S_n$, and we define the matrix $E := P_\sigma$.

Let us depict the matrices $E, E^2, E^3, E^4$, in the case $n = 4$.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Remark 3.4.** Because the group homomorphism that we defined is injective, and given that $E$ is the matrix associated to a cyclic permutation of order $n$, we have that the minimal polynomial of $E$ is $m_E(\lambda) = \lambda^n - 1$.

**Definition 3.5** (Algebra of circulant matrices, circulant matrix)**.** Let $R$ be a ring and let $n \in \mathbb{N}$. We define the algebra of *circulant matrices* as the $R$-algebra generated by $E$, that is,

$$\mathrm{Circ}_n(R) := R[E] \simeq R[x]/\langle x^n - 1\rangle.$$

Any element of $\mathrm{Circ}_n(R)$ is called a *circulant matrix*.

**Definition 3.6** (Induced circulant matrix, $\Delta_n$)**.** Given indeterminates $x_0, \dots, x_{n-1}$ we define the circulant matrix associated to $(x_0, \dots, x_{n-1})$ as

$$C(x_0, \dots, x_{n-1}) = \begin{pmatrix} x_0 & x_1 & x_2 & \dots & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \dots & x_{n-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ x_1 & x_2 & \dots & \dots & x_0 \end{pmatrix}$$

We also define $\Delta_n(x_0, \dots, x_{n-1}) := \det(C(x_0, \dots, x_{n-1}))$.

Given $(r_0, r_1, \dots, r_{n-1}) \in R^n$ we define the *circulant matrix associated to* $(r_0, \dots, r_{n-1})$ as

$$C(r_0, \dots, r_{n-1}) := r_0 I + r_1 E + r_2 E^2 + \dots + r_{n-1} E^{n-1}.$$

**Remark 3.7.** Assume that $R$ is a ring containing a primitive $n$-th root of unity, and call it $\varepsilon$. Then, notice that, for each $k \in \{0, \dots, n-1\}$ we have

$$E \begin{pmatrix} 1 \\ \varepsilon^k \\ \dots \\ \varepsilon^{(n-1)k} \end{pmatrix} = \varepsilon^k \begin{pmatrix} 1 \\ \varepsilon^k \\ \dots \\ \varepsilon^{(n-1)k} \end{pmatrix}.$$

Notice that $E$ admits $n$ linearly independent eigenvectors. Thus, $E$ is diagonalizable. In fact, $\mathrm{Circ}_n(R) \subset \mathrm{Mat}_{n \times n}(R)$ is a collection of *simultaneously diagonalizable matrices*. Let us formulate a more precise statement (see Proposition 3.10), but let us first provide a useful definition.

**Definition 3.8** (Discrete Fourier Transform)**.** Let $\varepsilon \in R$ be a primitive $n$-th root of unity. We define the *discrete Fourier transform* (DFT, for short) as the matrix $F$ whose $ij$-th entry is $F_{ij} = \varepsilon^{(i-1)(j-1)}$, in other words,

$$F = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \dots & \varepsilon^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \varepsilon^{n-1} & \dots & \varepsilon^{(n-1)(n-1)} \end{pmatrix}$$

**Remark 3.9.** Given $i, j$ such that $i \neq j$ notice that

$$\begin{aligned} \sum_{k=1}^{n} \varepsilon^{(n+1-i)(k-1)} \varepsilon^{(k-1)(j-1)} &= \sum_{k=1}^{n} \varepsilon^{(n-i+j)(k-1)} \\ &= \begin{cases} 0 & \text{if } n \nmid (j-i) \\ n & \text{else} \end{cases} . \end{aligned}$$

An immediate consequence of the previous observation is

$$F^{-1} = \frac{1}{n} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \varepsilon^{n-1} & \dots & \varepsilon \\ \vdots & \vdots & \dots & \vdots \\ 1 & \varepsilon^{(n-1)(n-1)} & \dots & \varepsilon^{n-1} \end{pmatrix} .$$

**Proposition 3.10.** *Let $R$ be a ring containing a primitive root of unity $\varepsilon$, and let $(r_0, \dots, r_{n-1}) \in R^n$. Then,*

$$F^{-1} C(r_0, \dots, r_{n-1}) F = \begin{pmatrix} \lambda_0 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1} \end{pmatrix} \tag{3.1}$$

*where*

$$\lambda_k := r_0 + r_1 \varepsilon^k + \dots + r_{n-1} \varepsilon^{k(n-1)}, \textit{ for each } 0 \leqslant k \leqslant n-1.$$

*Consequently, we have*

$$\det(C(r_0, \dots, r_{n-1})) = \prod_{k=0}^{n-1} \left( \sum_{j=0}^{n-1} r_j \varepsilon^{kj} \right) .$$

## 3.2 CIRCULANT SINGULARITIES

**Example 3.11.** Let $\varepsilon$ be a primitive $n$-th root of unity, let $G$ be the cyclic group of order $n$ generated by $g \in G$, let $R = \mathbb{C}[w, x_1, \dots, x_{n-1}, z]$, and consider the finite extension

$S = \mathbb{C}[v, x_1, \ldots, x_{n-1}, z]$, where $v$ is a formal $n$-th root of $w$. We can define a $G$-action on $S$ by extending the maps

$$(g, v) \mapsto \varepsilon v, (g, x_k) \mapsto x_k, (g, z) \mapsto z.$$

For each $0 \leqslant k \leqslant n-1$ define

$$f_k := z + \varepsilon^k v^k x_1 + \varepsilon^{2k} v^{2k} x_2 + \ldots + \varepsilon^{(n-1)k} v^{(n-1)k} x_{n-1}. \tag{3.2}$$

Notice that $g \cdot f_k = f_{k+1}$. Then,

$$\prod_{k=0}^{n-1} f_k = \det(C(z, v x_1, v^2 x_2, \ldots, v^{n-1} x_{n-1})).$$

**Remark 3.12.** Let us carry out a slightly more general construction as the one in the previous example, as this will prove useful in Chapter 4. Consider the rings $R = \mathbb{C}[\![w_1, \ldots, w_r, x, z]\!]$ and $S = \mathbb{C}[\![v_1, \ldots, v_r, x, z]\!]$, where each $v_j$ is a formal $n$-th root of $w_j$. Assume that $f \in R$ is irreducible but $f = f_0 \ldots f_{n-1} \in S$. Let $\varepsilon$ denote a primitive $n$-th root of unity. Let $G$ denote the cyclic group of order $n$ with generator $g$ and notice that, for each $1 \leqslant j \leqslant r$ there is a $G$ action on $S$ induced by $g \cdot v_j \mapsto \varepsilon v_j$, and acting trivially in the rest of the coordinates. Let us focus on the action of $G$ in one of the coordinates, say $v_1$, as the following remarks work the same for the rest. Because $G$ acts trivially in $v_1^n$ we have that $G$ leaves $f$ invariant, but permutes the roots $f_j$. We may rearrange the indices of the $f_j$ in such a way that $g^k f_j = f_{j+k}$ for all $j, k$, where $g$ is a generator of $G$. Then, the entries $M_{i,1}$ of the matrix of the diagonal lift are given by

$$y_i := \frac{1}{n} \sum_{j=0}^{n-1} \varepsilon^{-ij} f_j. \tag{3.3}$$

Notice that

$$\begin{aligned} g \cdot \left(v_1^{n-i} y_i\right) &= \frac{1}{n} \sum_{j=0}^{n-1} \varepsilon^{-i} \varepsilon^{-ij} v^{n-i} (g \cdot f_j) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \varepsilon^{-i(j+1)} v_1^{n-i} f_{j+1} \\ &= v_1^{n-i} y_i. \end{aligned}$$

Thus, there exist $m_0, \ldots, m_{n-1} \in \mathbb{Z}_{\geqslant 0}$ with $m_k \equiv k \mod n$ such that

$$y_i = v_1^{m_i} \overline{y}_i,$$

where $\overline{y}_i \in R$ and $w \nmid \overline{y}_i$. By Corollary 3.47 we have that

$$f = \det(C(\overline{y}_0, w_1^{m_1/n} \overline{y}_1, \ldots, w_1^{m_{n-1}/n} \overline{y}_{n-1})).$$

**Definition 3.13** (Circulant singularity, Product of circulants)**.** Let $X$ be a hypersurface of a smooth $\mathbb{K}$-variety $Z$. We say that $a \in X$ is a *circulant singularity* if there exists a regular coordinate system $w, x_1, \ldots, x_{n-1}, z, u_1, \ldots, u_q$ at $a$ such that the ideal $\mathcal{I}_X$ restricted to some affine neighborhood $U$ of $a$ is of the form

$$\mathcal{I}_X|_U = \langle \Delta_n(z, w^{1/n}x_1, \ldots, w^{(n-1)/n}x_{n-1})) \rangle.$$

If there exists a coordinate system $w, x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{d,1}, \ldots, x_{d,n_d}, z_1, \ldots, z_d, u_1, \ldots, u_q$ such that the local expression of $\mathcal{I}_X$ in some affine neighborhood $U$ around $a$ is of the form

$$\left\langle \prod_{k=1}^{d} \Delta_{n_k}(z_k, w^{1/n_k}x_{k,1}, \ldots, w^{(n_k-1)/n_k}x_{k,n_k-1}) \right\rangle,$$

we say that $X$ is a *product of circulants* at $a$.

As a word of warning, notice that the definition above is not an arbitrary product of circulants. In particular, we require that all the factors share a single common coordinate which adopts fractional powers.

Let us finalize this section by showing that if $X$ is the zero locus of

$$f(w, x, z) := \Delta_n(z, w^{1/n}x_1, \ldots, w^{(n-1)/n}x_{n-1}),$$

then the normalization $\tilde{X}$ of $X$ is smooth.

To achieve this, we want to construct a smooth variety $Y$ together with a finite birational morphism $\pi : Y \to X$. Let $R_0 := \mathbb{K}[\![w, x_1, \ldots, x_{n-1}]\!][z]$, and let $X := \mathrm{Spec}(R_0/\langle f \rangle)$. Consider the morphism

$$R_0 \xrightarrow{\phi^\sharp} R_1 := R_0[v]/\langle v^n - w \rangle.$$

By (3.3), $\phi(f)$ splits into $n$ linear factors $f_1, \ldots, f_n$ in $R_1$, and so the morphism

$$R_1/\langle f(v^n, x, z) \rangle \xrightarrow{\psi^\sharp} R_2 := R_1/\langle f_1 \rangle$$

is well-defined. Notice that $\phi^\sharp$ descends to the quotient, giving us a morphism

$$R_0/\langle f \rangle \xrightarrow{\overline{\phi}^\sharp} R_2.$$

Because $f_1$ is an element of vanishing order 1, we have that $\mathrm{Spec}(R_2) \simeq \mathrm{Spec}(\mathbb{K}[\![w, x]\!])$. The variety $Y$ we are looking for is $\mathrm{Spec}(\mathbb{K}[\![w, x]\!])$.

We now claim that the morphism $Y \xrightarrow{\psi \circ \phi} X$ is birational. To achieve this, we want to show that $\mathrm{Frac}(\mathcal{O}_Y) \simeq \mathrm{Frac}(\mathcal{O}_X)$, and given that $R_2 = R_0 + R_0 v + \ldots + R_0 v^{n-1}$, it suffices to show that $v \in \mathrm{Frac}(\mathcal{O}_X)$. More precisely, we want to show that there exists $s \in \mathrm{Frac}(\mathcal{O}_X)$ such that $s^n = w$.

**Remark 3.14.** In the same fashion we construct the field of fractions for an integral domain, we may construct the *total quotient ring* $Q(R)$ of a ring $R$, by localizing all the non-zero divisors of $R$. If $R$ is a noetherian reduced ring, and $p_1, \ldots, p_n$ are its minimal prime ideals, then $Q(R)$ is isomorphic to the product $\mathrm{Frac}(R/p_1) \times \ldots \times \mathrm{Frac}(R/p_n)$. Notice that the minimal prime ideals of $R$ uniquely determine the number of factors in the product.

**Proposition 3.15.** *Let* $f \in K[x]$, $g \in K[y]$ *be irreducible polynomials. Let* $L_f$ *be the splitting field of* $f$ *and let* $L_g$ *be the splitting field of* $g$*. Then, the number of irreducible factors of* $g$ *in* $L_f[y]$ *is the same as the number of irreducible factors of* $f$ *in* $L_g[x]$*.*

*Proof. Due to Pierre Lairez.* Notice that the number of irreducible factors of $g$ in $L_f[y]$ equals the number of minimal prime ideals of $L_f[y]/\langle g\rangle$, which is the number of components of $Q(L_f[y]/\langle g\rangle)$. On the other hand, we have

$$L_f[y]/\langle g\rangle \simeq K[x,y]/\langle f(x), g(y)\rangle \simeq L_g[x]/\langle f\rangle,$$

giving us what we wanted. □

Define $K := \mathrm{Frac}(\mathbb{K}[\![w,x]\!])$, and consider the polynomials $g := v^n - w \in K[v]$ and $f \in K[z]$. By Proposition 3.15, the number of irreducible factors of $g$ in $L_f[v]$ is the same as the number of irreducible factors of $f$ in $K[v]$. But by (3.3), we have that $f \in L_g[z]$ splits into $n$ irreducible factors. In other words, $v \in L_f[v]/\langle g\rangle$ is algebraic over $L_f$. Notice also that $L_g[z]/\langle f\rangle \simeq R_0$. Given that $g$ is a monic polynomial, we have that $v$ is integral. Let us summarize the previous arguments in the following.

**Proposition 3.16.** *Let* $X$ *be a variety, let* $\tilde{X} \xrightarrow{\tau}$ *be the normalization of* $X$*, let* $a \in X$ *be a* $\mathrm{cp}(n)$ *point of* $X$*. Then,* $\tilde{X}$ *is smooth at all points* $a' \in \tau^{-1}(a)$*.*

## 3.3 FURTHER PROGRESS

In this section we use the tools of representation theory to present the construction of *group circulant matrices* and *group* precirculant singularities.

The notion of group circulant matrices can be found in [DR90] and [KW13]. The notion of circulant singularities can be found in [BM12], under the name of *cyclic points*. The notion of *group precirculant singularities* is an attempt to generalize the notion of circulant singularities, in the case where the group acting on the splitting factors is an abelian group. In this case, we may use representation theory in order to define the Discrete Fourier Transform, allowing us to express a product as the determinant of a group circulant matrix.

Notice that we do not provide a definition of *group circulant singularity*. This is because we want to reserve the definition of group circulant singularities for the minimal family of local normal forms obtained after an application of a splitting theorem followed by a cleaning procedure, as in Theorem 1.4 and Theorem 5.1; a cleaning procedure remains to be developed for group circulant singularities.

The objects presented in this section are not relevant to the main result in this text. We present them nonetheless, as they could be useful for partial desingularization in higher dimension. At the end of this chapter, we an example of a group precirculant singularity, and the problems that appear when using our approach to the moving away procedure in this more general family of singularities.

### 3.3.1 *Elements of Representation Theory*

One of the tools that help us in the study circulant singularities is the Fourier transform, which can be established in the language of representation theory. While the definitions that we present are given in a general context, we mainly focus on the case of representations of a finite abelian group G in a finite dimensional vector space V over $\mathbb{C}$.

**Definition 3.17** (Representation of a group)**.** Let G be a group. We call a pair $(\rho, V)$ a *representation of* G if

$$\rho : G \to GL(V),$$

is a homomorphism of groups.

**Example 3.18.** Fix $N \in \mathbb{Z}$, let $Z_N$ denote the group of N-th roots of unity, and let j denote the generator of $Z_N$. Notice that we can induce an action of $Z_N$ on $\mathbb{C}[v, x_1, \dots, x_m]$ by considering the linear extension of the mappings

$$j \cdot v^{\alpha} x^{\beta} := (jv)^{\alpha} x^{\beta}.$$

**Definition 3.19** (Sub-representation)**.** If $(\rho, V)$ is a representation of a group G, and $W \subset V$ is a vector subspace such that $\rho(G)(W) \subset W$ we say that W is a sub-representation of V.

**Remark 3.20.** Notice that in Example 3.18 we have that the collection of scalar multiples of a monomial is a sub-representation of the action.

**Definition 3.21** (Irreducible representation)**.** Given a representation $(\rho, V)$ we say that it is *irreducible* if the only sub-representations are the trivial ones, that is, if $(\rho_0, \{0\})$ and $(\rho, V)$ are the only sub-representations of $(\rho, V)$.

**Lemma 3.22** (Schur's lemma)**.** *If* G *is a finite abelian group then any irreducible representation* $(\rho, V)$ *satisfies that* $\dim V = 1$.

*Sketch of proof.* Let us show that any irreducible representation of a finite abelian group is finite dimensional. Let $v \neq 0$ be an element of V. Notice that the orbit $G \cdot v \subset V$ is a finite subset which is invariant under the action of G. In particular, $\text{span}(G \cdot v)$ is a finite dimensional subspace invariant under G. Thus, any irreducible representation has finite dimension.

By Lemma 1.7 of [FH13], $\rho$ is a multiple of the identity, and by irreducibility $\dim V = 1$. ☐

**Definition 3.23** (Class function)**.** We say that a function $\chi : G \to \mathbb{C}$ is a *class function* if $\chi(ghg^{-1}) = \chi(h)$ for all $g, h \in G$. We denote the collection of all class functions as $C[G]$.

**Remark 3.24.** Notice that $C[G]$ is a finite $\mathbb{C}$-algebra, where the dimension as a vector space is

$$\dim C[G] = |G|.$$

Moreover, in the case where G is an abelian group, then any function $\chi : G \to \mathbb{C}$ is a class function.

**Remark 3.25.** If G is an abelian group, for each $g \in G$, we may define the class function $\mathbb{1}_g : G \to K$ given by

$$\mathbb{1}_g(h) := \begin{cases} 1 & \text{if } h = g \\ 0 & \text{otherwise} \end{cases}.$$

Notice then, that

$$\{\mathbb{1}_g : g \in G\}$$

is a basis of the collection of class functions.

But this is not the only useful basis that can be used to describe the space of class functions. Another good candidate is the collection of characters of G.

**Definition 3.26** (Character of a representation)**.** Given a representation $(\rho, V)$ of a finite group G, we define the *character induced by* $\rho$ as the function $\theta : G \to \mathbb{C}$ given by $\theta(g) := \operatorname{tr}(\rho(g))$.

From now on, we drop the reference to the representation $\rho$ that gives rise to $\theta$, and we simply say that $\theta$ is a character of G. In the case that $\rho$ is an irreducible representation, we say that $\theta$ is an *irreducible character*.

**Remark 3.27.** In the particular case where G is an abelian group, we have that any irreducible representation is one-dimensional, and so any character $\theta$ can be identified with a group homomorphism $G \to \mathbb{C}^\times$.

This leads us to consider the following.

**Definition 3.28** (Dual group)**.** Given a finite group G, we define the *dual group* as the collection of group homomorphisms

$$\hat{G} = \hom(G, \mathbb{C}^\times).$$

Notice that $\hat{G}$ is a group where the product is given by point-wise multiplication, that is, if $\chi, \theta \in \hat{G}$ then

$$(\chi \cdot \theta)(g) := \chi(g)\theta(g),$$

for all $g \in G$. In the case where G is an abelian group, by Remark 3.27, we have that $\hat{G}$ is the collection of characters of G.

**Remark 3.29.** Using the notation of Example 3.18, we have that for each character $\chi : Z_N \to \mathbb{C}$, there exists $k \in \mathbb{Z}$ such that $\chi(z) = z^k$. Moreover, each $k$ is unique up to equivalence modulo N. In other words, the map

$$\begin{aligned} \psi : \mathbb{Z} &\to \widehat{Z_N}, \\ \alpha &\mapsto (z \mapsto z^\alpha), \end{aligned}$$

is a surjective group homomorphism satisfying $\ker \psi = N\mathbb{Z}$. Using this notation, notice that for any $h \in Z_N$

$$h \cdot v^\alpha = \psi(\alpha)(h) v^\alpha. \tag{3.4}$$

Given a finite group G we can consider the $\sigma$-algebra of all the subsets of G. Let us call this $\sigma$-algebra $2^G$. Then, we can consider any finite measure $\mu$ defined on $2^G$. Notice then that $(G, 2^G, \mu)$ is a measure space.

**Remark 3.30.** Notice that, when we consider the $\sigma$-algebra $2^G$, the space of measurable functions

$$L^+(2^G) := \{f : G \to \mathbb{C} : f \text{ is measurable}\},$$

is the collection of all functions $f : G \to \mathbb{C}$. Moreover, if we consider any finite measure $\mu$ defined on $2^G$ then the collection of p-integrable functions

$$L^p(\mu) := \left\{ f : G \to \mathbb{C} : \int_G |f|^p \, d\mu < \infty \right\},$$

satisfies that

$$L^p(\mu) = L^+(2^G),$$

for all $p \geqslant 1$. Because these $L^p$ spaces are the same independently of the finite measure $\mu$ considered, we use the notation

$$L^p(G) := L^p(\mu).$$

**Definition 3.31** (Haar measure)**.** We define the *Haar measure* as the $\sigma$-additive extension $\mu$ of the constant function $\mu_\lambda(\{g\}) = \frac{1}{|G|}$, for each $g \in G$.

**Definition 3.32** (Inner product)**.** Let $G$ be a finite group and let $f_1, f_2 \in L^2(G)$. We define the *inner product* of $f_1$ and $f_2$ as

$$\langle f_1, f_2 \rangle := \int_G f_1 \overline{f_2} d\mu.$$

**Theorem 3.33** (Schur's orthogonality relations, see Theorem 6 in Section 2.5 of [SS96])**.** *If $G$ is a finite group, then the collection of all irreducible characters of $G$ is an orthonormal basis of $C[G]$.*

**Corollary 3.34.** *If $G$ is an abelian group, then $\hat{G}$ is an orthonormal basis of $L^2(G)$.*

### 3.3.2 *The Discrete Fourier Transform as a Change of Basis*

**Definition 3.35** (Group algebra)**.** Let $G$ be a group. We define the *group algebra* $\mathbb{C}[G]$ as the collection

$$\mathbb{C}[G] := \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{C} \right\}$$

**Remark 3.36.** The group algebra $\mathbb{C}[G]$ is a finite $\mathbb{C}$-algebra, whose dimension as a vector space is

$$\dim \mathbb{C}[G] = |G|.$$

**Remark 3.37.** Consider the injective map $\iota : G \hookrightarrow L^2(\hat{G})$ given by $g \mapsto (\chi \mapsto \chi(g))$. Assume that $G$ is an abelian group. Notice then that $\dim L^2(\hat{G}) = |\hat{G}| = |G|$, and by Theorem 3.33 we have that the images $\{\iota(g_1), \ldots, \iota(g_n)$ is a basis of $L^2(\hat{G})$, where $G = \{g_1, \ldots, g_n\}$. Taking the linear extension of $\iota$ we obtain a linear transformation $\mathbb{C}[G] \hookrightarrow L^2(\hat{G})$ which is an isomorphism between $\mathbb{C}[G]$ and $L^2(\hat{G})$.

**Remark 3.38.** In a similar way, we can embed $L^2(G)$ into $\mathbb{C}[G]^*$ by noticing that a map $f : G \to \mathbb{C}$ can be linearly extended to a map $L_f : \mathbb{C}[G] \to \mathbb{C}$, and this extension is unique. By noticing that $\dim(\mathbb{C}[G]^*) = |G| = \dim(L^2(G))$ we obtain that this embedding is an isomorphism.

**Definition 3.39** (Endomorphism space)**.** Let $G$ be a finite abelian group and let $\hat{G}$ denote its dual. We define the *endomorphism space* of $\mathbb{C}[G]$ as

$$\mathrm{End}(\mathbb{C}[G]) := \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}[G]^*.$$

**Remark 3.40.** Notice that we can endow $\mathrm{End}(\mathbb{C}[G])$ with a module structure over $\mathbb{C}[G]$, where the action is given by

$$s \cdot \left( \sum_{i,j} \lambda_{i,j} g_i \otimes f_j \right) \mapsto \sum_{i,j} \lambda_{i,j} (s g_i) \otimes f_j.$$

Notice then that if $\beta = \{f_1, \dots, f_{|G|}\}$ is a basis of the vector space $\mathbb{C}[G]^*$ then $\tilde{\beta} := \{1 \otimes f_1, \dots, 1 \otimes f_{|G|}\}$ is a free basis of $\mathrm{End}(\mathbb{C}[G])$ as a $\mathbb{C}[G]$-module.

**Definition 3.41** (Diagonal lift)**.** Let

$$L : \mathrm{End}(\mathbb{C}[G]) \to \mathrm{End}(\mathbb{C}[G])$$

be the $\mathbb{C}[G]$-module homomorphism given by the $\mathbb{C}[G]$-linear extension of the maps

$$L(1 \otimes \mathbb{1}_g) = g \otimes \mathbb{1}_g.$$

We call this $\mathbb{C}[G]$-module homomorphism the *diagonal lift of* $G$.

**Remark 3.42.** Consider an enumeration of an abelian group $G = \{e, g, g^2, \dots, g^{n-1}\}$, and consider the basis $\beta := \{1 \otimes g_1, \dots, 1 \otimes g_n\}$ of $\mathrm{End}(\mathbb{C}[G])$. Notice that the matrix $D := [L]_\beta$ of $L$ expressed in the basis $\beta$ is

$$\begin{pmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_n \end{pmatrix} \tag{3.5}$$

**Remark 3.43.** Let $G = \{g_1, \dots, g_n\}$ be an enumeration of a finite abelian group, and let $\hat{G} = \{\chi_1, \dots, \chi_n\}$ be an enumeration of its dual group. Two notable bases that we can use to express the diagonal lift as a matrix in $\mathrm{Mat}_{n \times n}(\mathbb{C}[G])$ are

$$\gamma := \{1 \otimes \mathbb{1}_{g_1}, \dots, 1 \otimes \mathbb{1}_{g_n}\},$$

and

$$\beta := \{1 \otimes \chi_1, \dots, 1 \otimes \chi_n\}.$$

Notice then

$$\begin{aligned} L(1 \otimes \chi_j) &= L\left(1 \otimes \left(\frac{1}{n}\sum_{g\in G}\langle \chi_j, \delta_g\rangle \delta_g\right)\right) \\ &= \frac{1}{n}\sum_{g\in G}\langle \chi_j, \delta\rangle L(1\otimes \delta_g) \\ &= \frac{1}{n}\sum_{g\in G}\langle \chi_j, \delta_g\rangle g \otimes \delta_g \\ &= \frac{1}{n}\sum_{g\in G}\sum_{i=1}^{s}\langle \chi_j, \delta_g\rangle\langle \delta_g, \chi_i\rangle g\otimes \chi_i \\ &= \frac{1}{n}\sum_{i=1}^{s}\sum_{g\in G}\chi_j(g)\overline{\chi}_i(g) g\otimes \chi_i. \end{aligned}$$

Thus, the matrix $M := [L]_\beta$ expressing $L$ in the basis $\beta$ is such that

$$M_{i,j} = \frac{1}{n}\sum_{g\in G}\chi_j(g)\overline{\chi}_i(g)g$$

**Remark 3.44.** Let $\{\chi_1, \dots, \chi_n\}$ be an enumeration of the elements of $\hat{G}$ and let $i_1, i_2, j_1, j_2, k \in \{1, \dots, n\}$ be such that $\chi_{i_1}\chi_k = \chi_{i_2}$ and $\chi_{j_1}\chi_k = \chi_{j_2}$. If $M = [L]_\beta$ where $\beta = \{1 \otimes \chi_1, \dots, 1 \otimes \chi_n\} \subset \mathrm{End}(\mathbb{C}[G])$, then

$$M_{i_2,j_2} = M_{i_1,j_1}.$$

In particular, notice that given $0 \leqslant i, j \leqslant n-1$, we may define $k \in \{0, \dots, n-1\}$ as the index satisfying $\chi_k = \chi_1\chi_j\overline{\chi}_i$. Then,

$$M_{i,j} = M_{1,k},$$

in other words, all entries of $M$ are uniquely determined by its first row. A similar remark shows that $M$ is uniquely determined by its first column.

**Remark 3.45.** Given that det is multiplicative, we may define $\det L := \det[L]_\alpha$, for any $\mathbb{C}[G]$-basis $\alpha$ of $\mathrm{End}(\mathbb{C}[G])$. In particular,

$$\det[L]_\beta = \det[L]_\gamma,$$

where $\beta, \gamma$ are the bases defined in Remark 3.43. Let us single out the following case. Let $\tilde{\beta}$ be the basis of $\mathrm{End}(\mathbb{C}[G])$ induced by a different enumeration of $\hat{G}$, then there exists a permutation matrix $P$ such that $P^{-1}[L]_\beta P = [L]_{\tilde{\beta}}$, and so

$$\det([L]_{\tilde{\beta}}) = \det([L]_\beta),$$

and a similar remark holds for a different enumeration of $G$.

**Proposition 3.46.** *Let* $G$ *be an abelian group of order* $n$. *Then, there exists a* $\mathbb{C}[G]$*-basis* $\beta$ *of* $\mathrm{End}(\mathbb{C}[G])$ *such that the matrix* $M$ *expressing the diagonal lift* $L : \mathrm{End}(\mathbb{C}[G]) \to \mathrm{End}(\mathbb{C}[G])$ *in the basis* $\beta$ *is circulant.*

*Proof.* Let $g$ be a generator of $G$, let $\varepsilon := e^{2\pi i/n} \in \mathbb{C}$ and define $\beta = \{1 \otimes \chi_0, \dots, 1 \otimes \chi_{n-1}\}$ as the $\mathbb{C}[G]$-basis of $\mathrm{End}(\mathbb{C}[G])$ such that $\chi_k(g) = \varepsilon^k$ for all $k \in \{0, \dots, n-1\}$. Consider $M = [L]_\beta$, and notice that

$$\begin{aligned} M_{i,j} &= \frac{1}{n}\sum_{k=0}^{n-1} \chi_{j-1}(g^k)\overline{\chi}_{i-1}(g^k)g^k \\ &= \frac{1}{n}\sum_{k=0}^{n-1} \varepsilon^{k(j-1)-k(i-1)}g^k \\ &= \frac{1}{n}\sum_{k=0}^{n-1} \varepsilon^{k(j-i)}g^k \end{aligned}$$

In particular, notice that $M_{i,j} = M_{i+\ell,j+\ell}$ for all $i, j, \ell$, if we consider the indices to be the respective representative modulo $n$ in $\{1, \dots, n\}$. In other words, $M \in \mathrm{Mat}_{n\times n}(\mathbb{C}[G])$ is a circulant matrix. □

**Corollary 3.47.** *Let* $S$ *be a finitely generated* $\mathbb{C}$*-algebra admitting a* $G$*-action, for some cyclic group* $G$ *of order* $n$. *Then, the product of all the elements of the orbit of* $f_0 \in S$ *can be expressed as the determinant of a circulant matrix.*

*Proof.* Assume that $f = \prod_{g\in G} g \cdot f_0$. Let $g$ be a generator of $G$. Then,

$$\begin{aligned} f &= \det \begin{pmatrix} 1\cdot f_0 & 0 & \dots & 0 \\ 0 & g\cdot f_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g^{n-1}\cdot f_0 \end{pmatrix} \\ &= n^{-n} \det \sum_{k=0}^{n-1} \begin{pmatrix} g^k\cdot f_0 & \varepsilon^k g^k \cdot f_0 & \ddots & \varepsilon^{k(n-1)}g^k\cdot f_0 \\ \varepsilon^{-k}g^k\cdot f_0 & g^k\cdot f_0 & \ddots & \varepsilon^{k(n-2)}g^k\cdot f_0 \\ \vdots & \ddots & \ddots & \ddots \\ \varepsilon^{-k(n-1)}g^k\cdot f_0 & \varepsilon^{-k(n-2)}g^k\cdot f_0 & \ddots & g^k\cdot f_0 \end{pmatrix} \end{aligned}$$

which is what we wanted to show. □

### 3.3.3 *Group circulant matrices and group precirculant singularities*

Notice that the constructions done in Remark 3.42 and Remark 3.43 can be carried out for any abelian group, and so we can use an enumeration of the dual group $\hat{G}$ to generalize the identity in Corollary 3.47. Moreover, Remark 3.44 leads us to a definition of $G$-circulant matrices for abelian groups. Let us provide the explicit construction in this context.

**Definition 3.48** (Group circulant matrix, cf. *Group matrix* in the introduction of [KW13] and *Permutation matrix representation* in p. 298 of [DR90])**.** Let $G$ be a finite abelian group and fix enumerations $G = \{g_1, \dots, g_n\}$, $\hat{G} = \{\chi_1, \dots, \chi_n\}$. Define the permutations $\sigma_i : \{1, \dots, n\} \to \{1, \dots, n\}$ for $1 \leqslant i \leqslant n$ satisfying that $\chi_{\sigma_i(j)} = \overline{\chi}_i \chi_j$. Given elements $r_1, \dots, r_n$ we define $X(r_1, \dots, r_n)$ as the matrix $A$ whose entries are given by $A_{i,j} := r_{\sigma_i(j)}$. We say that $X(r_1, \dots, r_n)$ is the $G$-*circulant matrix* associated to the vector $r_1, \dots, r_n$.

**Example 3.49.** Given an integer $n$, let $Z_n$ denote the group of $n$-th roots of unity. We define the group $P_{n,d}$ as the direct product

$$P_{n,d} := \prod_{i=1}^{d} Z_n,$$

which is a finite abelian group. Notice that we can induce an action of $P_{N,d}$ on $\mathbb{C}[v_1, \dots, v_d, x_1, \dots, x_n]$ by considering the linear extension of the maps

$$(j_1, \dots, j_d) \cdot v^{\alpha} x^{\beta} := \left( \prod_{k=1}^{d} (j_k v_k)^{\alpha_k} \right) x^{\beta}.$$

This action induces a representation of $P_{n,d}$, and any 1-dimensional subspace spanned by a monomial is a subrepresentation.

For each character $\chi : P_{n,d} \to \mathbb{C}$, there exists a vector of integers $k = (k_1, \dots, k_d)$ such that

$$\chi(z_1, \dots, z_d) = z^k := z_1^{k_1} \cdot \dots \cdot z_d^{k_d}.$$

Each $k_i$ is unique up to equivalence modulo $n$. In other words, the map

$$\begin{aligned} \psi : \mathbb{Z}^d &\to \widehat{P_{n,d}}, \\ \alpha &\mapsto (z \mapsto z^{\alpha_1} \cdot \dots \cdot z^{\alpha_d}), \end{aligned}$$

is a surjective group homomorphism satisfying $\ker \psi = (n\mathbb{Z})^d$. Notice that for any $h \in P_{n,d}$

$$h \cdot v^{\alpha} = \psi(\alpha)(h) v^{\alpha}. \tag{3.6}$$

The following is relevant example later in this chapter, that allows us to showcase the problems that arise when following our approach to the moving away algorithm for group precirculant singularities.

**Example 3.50.** Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We want to construct the matrix

$$X(z, w_1^{1/2} x_1, w_1^{1/2} w_2^{1/2} x_2, w_1 w_2^{1/2} x_3)$$

associated to the enumeration of $\hat{G} = \{\chi_1, \chi_2, \chi_3, \chi_4\}$, given by $\chi_1 \equiv 1$, $\chi_2(g, \cdot) = -1$, $\chi_3(\cdot, g) = -1$ and $\chi_4 = \chi_2 \chi_3$, where $g$ is the generator of $\mathbb{Z}/2\mathbb{Z}$. Consider the permutations $\sigma_i \in S_4$ given by

$$\sigma_1 = \mathrm{Id}, \sigma_2 = (12)(34), \sigma_3 = (13)(24), \sigma_4 = (14)(23),$$

define

$$\begin{aligned} f_0 &:= z + w_1^{1/2}x_1 + w_1^{1/2}w_2^{1/2}x_2 + w_1 w_2^{1/2}x_3, \\ f_1 &:= z - w_1^{1/2}x_1 - w_1^{1/2}w_2^{1/2}x_2 + w_1 w_2^{1/2}x_3, \\ f_2 &= z + w_1^{1/2}x_1 - w_1^{1/2}w_2^{1/2}x_2 - w_1 w_2^{1/2}x_3, \\ f_3 &:= z - w_1^{1/2}x_1 + w_1^{1/2}w_2^{1/2}x_2 - w_1 w_2^{1/2}x_3, \end{aligned} \tag{3.7}$$

and notice that

$$f_0 f_1 f_2 f_3 = \det(X(z, w_1^{1/2}x_1, w_1^{1/2}w_2^{1/2}x_2, w_1 w_2^{1/2}x_3)) \tag{3.8}$$

$$= \det \begin{pmatrix} z & w_1^{1/2}x_1 & w_1^{1/2}w_2^{1/2}x_2 & w_1 w_2^{1/2}x_3 \\ w_1^{1/2}x_1 & z & w_1 w_2^{1/2}x_3 & w_1^{1/2}w_2^{1/2}x_2 \\ w_1^{1/2}w_2^{1/2}x_2 & w_1 w_2^{1/2}x_3 & z & w_1^{1/2}x_1 \\ w_1 w_2^{1/2}x_3 & w_1^{1/2}w_2^{1/2}x_2 & w_1^{1/2}x_1 & z \end{pmatrix}. \tag{3.9}$$

**Remark 3.51.** Let $G$ be a finite abelian group, let $N := |G|$ and fix $f \in \mathbb{C}[v, x]$. Let

$$\psi : \mathbb{Z}^d \to \hat{G},$$

be a surjective homomorphism. Let $\varphi$ be the inverse of the isomorphism induced by taking $\psi$ modulo its kernel. Given an enumeration

$$\{\chi_1, \dots, \chi_N\}$$

of $\hat{G}$, which allows us to define the ordered basis $\{1 \otimes \chi_1, \dots, 1 \otimes \chi_N\}$ of $\mathrm{End}(\mathbb{C}[G])$, we can define the map

$$\begin{aligned} \Phi : \hat{G} &\to (\mathbb{Z}/n\mathbb{Z})^d \\ \rho &\mapsto \varphi(\chi_1) - \varphi(\rho). \end{aligned}$$

Notice that $\Phi$ is a bijective map but not a group morphism unless $\chi_1 = 1$, as $\Phi(1) = \psi(\chi_1)$. Instead, notice that $\Phi$ is the analog of an affine transformation, as the map $\Phi(\cdot) - \Phi(1) = -\varphi(\cdot)$ is indeed an isomorphism of abelian groups. Let us also consider the map

$$\iota : (\mathbb{Z}/n\mathbb{Z})^d \to \mathbb{Z}^d$$

such that $\iota$ maps each entry in $\mathbb{Z}/n\mathbb{Z}$ to its reduced representative mod $n$ in the integers. Notice that $\Phi$ and $\iota$ allow us to associate to any $\rho \in \hat{G}$ some non-negative exponent vector $\alpha_\rho := (\iota \circ \Phi)(\rho)$.

Given $\rho \in \hat{G}$, and $f \in \mathbb{C}[v, x]$, we may define

$$f_\rho := \frac{1}{N} \sum_{h \in G} \chi_1(h)\overline{\rho}(h) h \cdot f. \tag{3.10}$$

Fix $\rho \in \hat{G}$ and let $f \in \mathbb{C}[\nu, x]$. Notice that for any $h_0 \in G$ we have

$$\begin{aligned} h_0 \cdot (\nu^{\alpha_\rho} f_\rho) &= \frac{1}{N}\psi(\iota(\Phi(\rho)))(h_0)\nu^{\alpha_\rho} \sum_{h \in G} \chi_1(h)\overline{\rho}(h)h_0 \cdot h \cdot f \\ &= \frac{1}{N}(\chi_1\overline{\rho})(h_0)\nu^{\alpha_\rho} \sum_{h \in G} \chi_1(h)\overline{\rho}(h)h_0 \cdot h \cdot f \\ &= \frac{1}{N}\nu^{\alpha_\rho} \sum_{h \in G} \chi_1(h_0 h)\overline{\rho}(h_0 h)h_0 \cdot h \cdot f \\ &= \frac{1}{N}\nu^{\alpha_\rho} \sum_{h \in G} \chi_1(h)\overline{\rho}(h)h \cdot f \\ &= \nu^{\alpha_\rho} f_\rho \end{aligned} \tag{3.11}$$

Because $\nu^{\alpha_\rho} f_\rho$ is invariant under the action of G, we have that

$$\nu^{\alpha_\rho} f_\rho \in \mathbb{C}[\nu^n, x] = \mathbb{C}[\nu_1^n, \ldots, \nu_d^n, x_1, \ldots, x_m].$$

In particular, we have that for each $\rho \in \hat{G}$ there exists a vector of integers $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, where $n$ divides each of the entries of $k$, and an element $\zeta_\rho \in \mathbb{C}[\nu^n, x]$ such that

$$f_\rho = \nu^{k + (\varphi(\rho) - \varphi(\chi_1))_0} \zeta_\rho, \tag{3.12}$$

where $(\varphi(\rho) - \varphi(\chi_1))_0 \in \mathbb{Z}^d$ is some representative of $\varphi(\rho) - \varphi(\chi_1) \bmod (n\mathbb{Z})^d$ with positive entries, and such that for any $i$ and any $\rho$ we have that $\nu_i$ does not divide $\zeta_\rho$.

(3.12) is a generalization of Example 3.11 and Remark 3.12. In the cyclic case, these identities are useful in the procedure of reduction to normal forms. But in order to adequately generalize this result to arbitrary finite abelian groups more work is needed.

We conclude this section with the definition of group precirculant singularity.

**Definition 3.52** (Group Precirculant Singularity)**.** Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let X be a hypersurface over $\mathbb{K}$ embedded in a smooth space Z, and let $a \in X$ be a singular point. Let G be an abelian group, and fix an enumeration $\hat{G} = \{\chi_0, \ldots, \chi_{n-1}\}$. We say that X is a G-*precirculant singularity* at $a$ if there exists an étale coordinate system $u_1, \ldots, u_q, w_1, \ldots, w_r, \zeta_0, \ldots, \zeta_{n-1}$ locally defined at $a$ such that X is the zero locus of the determinant of a G-circulant matrix of the form $X(\zeta_0, w^{\gamma_1/n}\zeta_1, \ldots, w^{\gamma_{n-1}/n}\zeta_{n-1})$, where $\gamma_j$ is an integer (associated to the irreducible character $\chi_j$) of the form (3.12), and $\zeta_0, \ldots, \zeta_{n-1}$, and where $N = |G|$.

Unfortunately, this definition is not restrictive enough to be a good family of normal forms for the limits of the normal crossings locus. More specifically, notice that the polynomials

$$\Delta_3 = \det(C(z, w^{1/3}x_1, w^{2/3}x_2)),$$

and

$$\Delta_3' = \det(C(z, w^{4/3}x_1, w^{8/3}x_2)),$$

both satisfy to express the origin as a $\mathbb{Z}/3\mathbb{Z}$-circulant singularity, but only the former satisfies minimality in the powers of $w$, which is a restriction we ask to our definition of circulant singularities. Thus, in order to provide a definition of group circulant singularity an extra reducedness condition on the powers $\alpha_1, \dots, \alpha_{n-1}$ is necessary.

### 3.3.4 *Notable Examples and Future Endeavors*

In this work we present the techniques required for a partial desingularization preserving normal crossings for varieties of dimension at most 4, but new techniques are needed for any attempt at partial desingularization in dimension 5. Let us follow through the first steps in our moving away procedure (cf. Chapter 7) for a particular case of a group precirculant singularity, and let us point out the precise problem that appears in this case.

**Example 3.53.** Consider a singularity $a \in X$ such that, in some affine chart $U$ with local coordinates $w_1, w_2, x_1, x_2, x_2, z$ centred at $a = 0 \in U$ we have that the local expression of $X$ is the vanishing locus of

$$f(w,x,z) := \det(X(z, w_1^{1/2}x_1, w_1^{1/2}w_2^{1/2}x_2, w_1 w_2^{1/2}x_3)),$$

where the matrix $X(z, \dots, w_1 w_2^{1/2}x_3)$ denotes the $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$-circulant matrix in (3.9), and where $D_{-2} = \{w_1 = 0\}$ and $D_{-1} = \{w_2 = 0\}$ locally describe exceptional divisors passing through the origin.

We claim that there exists a suitable sequence of admissible and equimultiple blowings-up after which, we can cover the fibre of $a = 0 \in U$ with affine charts $\{U_k\}_{k \in K}$, where the singularity at the origin in each affine chart can be expressed in local coordinates as the vanishing locus of

$$\Delta_2\Bigg(\Delta_2(z, y^{1/2}x_1^{\delta_1}) + ywu^{\alpha}\Delta_2(x_2^{\delta_2}, y^{1/2}x_3^{\delta_3}), \\ 2y^{1/2}w^{1/2}u^{\beta}\det\begin{pmatrix} z & y^{1/2}x_3^{\delta_3} \\ y^{1/2}x_1^{\delta_1} & x_2^{\delta_2}\end{pmatrix}\Bigg), \tag{3.13}$$

where $\{y = 0\}$ is one of the *distinguished divisors* for which the original normal crossings locus is a collection of neighboring singularities, $\{w = 0\}$ is an exceptional divisor, each $\{u_i = 0\}$ is an exceptional divisor intersecting the chart, $\alpha_i \in \mathbb{Z}$, and $\delta_j \in \{0, 1\}$ where $\delta_j = 0$ if and only if $\{x_j = 0\} \cap U_k = \varnothing$.

Let us show our claim. Let us denote by $Y_0$ the locus of points locally given by $\{z = 0\}$. Similarly, we define $Y_k$ as the locus of points $\{x_k = 0\}$ for $k \in \{1, 2, 3\}$. Let us denote by $D_{-2}$ and $D_{-1}$ the exceptional divisors associated to $w_1$ and $w_2$, respectively.

Let us declare the year in which we begin this process as year zero. We begin by moving away $D_{-2}$ from the limit points of the stratum with invariant inv(nc(4, 0)) by blowing-up $D_{-2} \cap S_{nc(4)}$. This introduces *a distinguished divisor*, which we call $D_0$.

The following is a table containing the local information of the ideal in the standard charts that intersect the strict transform of X, together with the list of exceptional divisors that intersect the respective chart. The exceptional divisors intersecting the respective chart are listed to the right, and below the name of each divisor, we write the local coordinate defining said divisor.

| Name of chart | Ideal in local coordinates | Exc. divs. |
|---|---|---|
| $U(w_1)$ | $\langle z^4, w_1^2x_1^4, w_1^2w_2^2x_2^4, w_1^4w_2^2x_3^4\rangle$ | $D_{-1}: w_2 \quad D_0: w_1$ |
| $U(x_1)$ | $\langle z^4, w_1^2x_1^2, w_1^2w_2^2x_1^2x_2^4, w_1^4w_2^2x_1^4x_3^4\rangle$ | $D_{-2}: w_1 \quad D_{-1}: w_2 \quad D_0: x_1$ |
| $U(x_2)$ | $\langle z^4, w_1^2x_1^4x_2^2, w_1^2w_2^2x_2^2, w_1^4w_2^2x_2^4x_3^4\rangle$ | $D_{-2}: w_1 \quad D_{-1}: w_2 \quad D_0: x_2$ |
| $U(x_3)$ | $\langle z^4, w_1^2x_1^4x_3^2, w_1^2w_2^2x_2^4x_3^2, w_1^4w_2^2x_3^4\rangle$ | $D_{-2}: w_1 \quad D_{-1}: w_2 \quad D_0: x_3$ |

We now move away $D_{-1}$ from the limit points of inv(nc(4, 0)) by blowing-up $D_{-1} \cap S_{nc(4)}$. This introduces another distinguished divisor, which we call $D_1$.

| | | |
|---|---|---|
| $U(w_1, w_2)$ | $\langle z^4, w_1^2x_1^4, w_1^2w_2^2x_2^4, w_1^4w_2^2x_3^4\rangle$ | $D_0: w_1 \quad D_1: w_2$ |
| $U(w_1, x_1)$ | $\langle z^4, w_1^2, w_1^2w_2^2x_1^2x_2^4, w_1^4w_2^2x_1^2x_3^4\rangle$ | $D_{-1}: w_2 \quad D_0: w_1 \quad D_1: x_1$ |
| $U(w_1, x_2)$ | $\langle z^4, w_1^2x_1^4, w_1^2w_2^2x_2^2, w_1^4w_2^2x_2^2x_3^4\rangle$ | $D_{-1}: w_2 \quad D_0: w_1 \quad D_1: x_2$ |
| $U(w_1, x_3)$ | $\langle z^4, w_1^2x_1^4, w_1^2w_2^2x_2^4x_3^2, w_1^4w_2^2x_3^2\rangle$ | $D_{-1}: w_2 \quad D_0: w_1 \quad D_1: x_3$ |

From this point and on, blowing-up the centre $C_k$ in year $k$ creates an exceptional divisor $D_k$. Some of these divisors will work as distinguished divisors, but they will be declared as such later in the process.

Centre of blow-up: $C_2$=$D_{-2} \cap D_0 \cap Y_0$.

Charts intersecting $C_2$: $U(x_1)$, $U(x_2)$, $U(x_3)$.

| | | |
|---|---|---|
| $U(x_1, w_1)$ | $\langle z^4, x_1^2, w_2^2x_1^2x_2^4, w_1^4w_2^2x_1^4x_3^4\rangle$ | $D_{-1}: w_2 \quad D_0: x_1 \quad D_2: w_1$ |
| $U(x_2, w_1)$ | $\langle z^4, x_1^4x_2^2, w_2^2x_2^2, w_1^4w_2^2x_2^4x_3^4\rangle$ | $D_{-1}: w_2 \quad D_0: x_2 \quad D_2: w_1$ |
| $U(x_3, w_1)$ | $\langle z^4, x_1^4x_3^2, w_2^2x_2^4x_3^2, w_1^4w_2^2x_3^4\rangle$ | $D_{-1}: w_2 \quad D_0: x_3 \quad D_2: w_1$ |

Centre of blow-up: $C_3$=$D_{-1} \cap D_1 \cap Y_0 \cap Y_1$.

Charts intersecting $C_3$: $U(w_1, x_2)$, $U(w_1, x_3)$.

| | | |
|---|---|---|
| $U(w_1, x_2, w_2)$ | $\langle z^4, w_1^2x_1^4, w_1^2x_2^2, w_1^4x_2^2x_3^4\rangle$ | $D_0: w_1 \quad D_1: x_2 \quad D_3: w_2$ |
| $U(w_1, x_2, x_2)$ | $\langle z^4, w_1^2x_1^4, w_1^2w_2^2, w_1^4w_2^2x_3^4\rangle$ | $D_{-1}: w_2 \quad D_0: w_1 \quad D_3: x_2$ |
| $U(w_1, x_2, x_1)$ | $\langle z^4, w_1^2, w_1^2w_2^2x_2^2, w_1^4w_2^2x_2^2x_3^4\rangle$ | $D_{-1}: w_2 \quad D_0: w_1 \quad D_1: x_2$<br>$D_3: x_1$ |
| $U(w_1, x_3, w_2)$ | $\langle z^4, w_1^2x_1^4, w_1^2x_2^4x_3^2, w_1^4x_3^2\rangle$ | $D_0: w_1 \quad D_1: x_3 \quad D_3: w_2$ |
| $U(w_1, x_3, x_3)$ | $\langle z^4, w_1^2x_1^4, w_1^2w_2^2x_2^4, w_1^4w_2^2\rangle$ | $D_{-1}: w_2 \quad D_0: w_1 \quad D_3: x_3$ |
| $U(w_1, x_3, x_1)$ | $\langle z^4, w_1^2, w_1^2w_2^2x_2^4x_3^2, w_1^4w_2^2x_3^2\rangle$ | $D_{-1}: w_2 \quad D_0: w_1 \quad D_1: x_3$<br>$D_3: x_1$ |

Notice that at this stage all of the affine charts have a local expression of the form

$$\det(X(z, y^{1/2}x_1^{\delta_1}, y^{1/2}u^{\alpha}x_2^{\delta_2}, yu^{\beta}x_3^{\delta_3})),$$

where $y$ is the local expression of $D_0$, $u$ is a monomial in the exceptional divisors that intersect $V$, $\delta_i \in \{0, 1\}$ and $\delta_i = 1$ if and only if $Y_i \cap V \neq \varnothing$, and $|\alpha|, |\beta| \geqslant \frac{1}{2}$. We want to continue blowing-up in those charts that contain more than two exceptional divisors which do not have powers divisible by 4 in every monomial. For this, we need a remark.

**Remark 3.54.** For any $a, b, c, d$ we have that

$$\Delta_{cp(2)\times cp(2)}(a,b,c,d) = \Delta_2\left(\Delta_2(a,b)+\Delta_2(c,d), 2\det\begin{pmatrix} a & d \\ b & c\end{pmatrix}\right)$$

In particular, we get that $\Delta := \det(X(z, y^{1/2}, y^{1/2}u^{\alpha}x_2^{\delta_2}, yu^{\beta}x_3^{\delta_3}))$ is equal to

$$\begin{aligned}\Delta_2\Bigg(&\Delta_2(z,y^{1/2}) + yu^{2\gamma}\Delta_2(u^{\alpha-\gamma}x_2^{\delta_2}, y^{1/2}u^{\beta-\gamma}x_3^{\delta_3}),\\ &2y^{1/2}u^{\gamma}\det\begin{pmatrix} z & y^{1/2}u^{\beta-\gamma}x_3^{\delta_3} \\ y^{1/2} & u^{\alpha-\gamma}x_2^{\delta_2}\end{pmatrix}\Bigg),\end{aligned} \tag{3.14}$$

where $\gamma$ is the exponent where the entry $\gamma_i$ is $\min\{\alpha_i, \beta_i\}$.

We want to continue blowing-up these expressions until the powers in all monomials are divisible by 4, except for at most 2 exceptional divisors at each chart. Notice that we cannot continue blowing-up with combinatorial centres using the coordinates $z, x_1, x_2, x_3, u, y$, because we will not be able to reduce $\alpha$ nor $\beta$ with equimultiple blowings-up.

Instead, notice that $\dfrac{\partial\Delta}{\partial y} = 0$ is a maximal contact hypersurface of the form

$$z^2 - y + u^{\gamma}s = 0,$$

for some function $s$. Thus, if we want to continue moving away all non-minimal singularities we need to consider centres which are no longer coordinate subspaces in these variables. Thus, a new approach is required for further progress.

This problem of our approach to the moving away algorithm appears for a general group precirculant singularity.

# 4

# SPLITTING RESULTS

This chapter is dedicated to the proof of the splitting theorem. As mentioned in Chapter 1, there are two versions of splitting that are needed in this text. One of the statements is Theorem 1.3. Let us provide a statement for the other splitting result.

**Theorem 4.1** (Splitting theorem of order three)**.** *Let $(X, E)$ be a pair consisting of a hypersurface $X \hookrightarrow Z$ where $Z$ is a smooth variety over $\mathbb{K}$ with $\dim Z = n+1$, that is, $\dim X = n$, and $E \subset Z$ is a* snc *divisor. Assume that for some open $U \subset Z$, and after a finite sequence of* inv*-admissible blowings-up, the maximal value of* inv *of $(X, E)$ in $U$ is $\mathrm{inv}(\mathrm{nc}(3,0)) = (3,0,1,0,1,0,\infty)$, so that the stratum $S_{3,0} \subset U$ of points with $\mathrm{inv} \geqslant \mathrm{inv}(\mathrm{nc}(3,0))$ is a (closed in $X$) smooth subvariety of dimension $n-2$. Assume moreover, that $X$ is* nc *generically in $S_{3,0}$. Then, there is a finite sequence of* inv*-admissible blowings-up such that, if the transform $(X', E')$ of $(X, E)$ is not* nc *at $a'$ in the strict transform $S'_{3,0}$ of $S_{3,0}$, then there exist étale coordinates $u_1, \dots, u_q, w_1, \dots, w_r, x_1, x_2, z$ defined at $a'$ satisfying that $S_{3,0}$ is locally defined by $\{z = x_1 = x_2 = 0\}$ and $f(u, w_1^{1/6}, \dots, w_r^{1/6}, x, z)$ splits into $3$ irreducible factors of vanishing order $1$, where $\{f = 0\}$ is a local equation defining $X'$.*

Theorem 1.3 is used as stepping stone in the proof of Theorem 1.4. Similarly, Theorem 4.1 is involved in the proof of reduction to normal form for the limits of triple normal crossings. Let us motivate the structure of the hypotheses of the theorems above.

Let $X \hookrightarrow Z$ be a hypersurface of a smooth variety $Z$ over an algebraically closed field $\mathbb{K}$ of characteristic zero, where $n = \dim X$. Given that a normal crossings singularity $a \in X$ of order $k$ satisfies that $\mathrm{inv}_X(a) = \mathrm{inv}(\mathrm{nc}(k,0))$, where $k \leqslant n$, if we perform the blow-up of any inv-admissible stratum of singularities with invariant strictly greater than $\mathrm{inv}(\mathrm{nc}(n,0))$, then all normal crossings singularities are preserved. Thus, the first step of the partial desingularization procedure is to follow the classical desingularization algorithm until $\mathrm{inv}_X$ at all points of the strict transform $X'$ of $X$ is $\leqslant \mathrm{inv}(\mathrm{nc}(n,0))$.

Notice that if $X'$ is normal crossings of order $k$ at $a'$, then there is an open neighbourhood $U' \subset Z'$ around $a'$, where $X'$ is normal crossings at any point $b' \in U'$. In short, the condition " $X'$ is normal crossings at $a'$ " is an open condition. Thus, any irreducible component $S \subset X'$ of the stratum with highest value of the invariant is either generically $\mathrm{nc}(n)$ or does not contain any normal crossings singularities. We then apply the classical desingularization algorithm on the irreducible components that do not contain any normal crossings point, allowing us to assume that, after performing an adequate inv-admissible sequence of blowings-up if necessary, every irreducible component of the stratum with highest value of the invariant is generically $\mathrm{inv}(\mathrm{nc}(n))$.

Fix an irreducible component $S$ of the stratum whose value of inv is $\mathrm{inv}(\mathrm{nc}(n,0))$ and assume there is a point $a' \in S$ which is not $\mathrm{nc}(n)$. As inv is constant on $S$, $S$ is the intersection of $n$ hypersurfaces in snc, thus $\dim S = 1$. We can now use Weierstrass' preparation theorem to find an adequate coordinate system $w, x_1, \dots, x_{n-1}, z$ at $a'$ defined on an affine neighborhood $U \subset Z'$ of $a'$ such that the local generator $f$ of the ideal $\mathcal{I}_{X'}$

is an element of $\mathbb{K}[\![w,x]\!][z]$, where $\{z = 0\}$ is a maximal contact hypersurface of $X$ and $S \cap U = \{z = x = 0\}$.

Given that we want to find a finite family of local normal forms for all singularities that remain after partial desingularization preserving the normal crossings locus, in particular we need a finite family of normal forms for the limits of the normal crossings locus. And so, as a preliminary step, one would like to find a ring extension of $\mathbb{K}[\![w,x]\!][z]$ together with an adequate coordinate system, that expresses the ideal $\mathcal{I}_{X'}$ associated to $X'$ in a simple fashion. More precisely, that we can split a local equation defining $X$ into terms of order 1, even if we need to consider to a finite formal extension of the original ring of functions.

## 4.1 SPLITTING AT THE LIMIT POINTS OF THE NORMAL CROSSINGS LOCUS OF TOP VANISHING ORDER

One of the technical tools used in our approach for finding the splitting results is the main result found in [SV11]. Let us briefly discuss the notions that appear in said work.

### 4.1.1 *Newton-Puiseux Theorem and Polyhedral Transformations*

The main theorem in [SV11] provides an *effective* way of splitting a polynomial $f(x,z) = z^n + c_{n-1}(x)z^{n-1} + \ldots + c_0(x) \in \mathbb{K}[x,z]$ into irreducible factors in the polynomial ring over the Puiseux series field $\mathbb{K}\{x\}[z]$, after an adequate transformation of the monomials defining $f$.

**Definition 4.2.** We define $\mathrm{SL}^{(+)}_{\mathrm{lex}}(n,\mathbb{Z})$, as the multiplicative semigroup of $\mathrm{SL}(n,\mathbb{Z})$ of upper triangular matrices of with positive entries above the diagonal, and 1 at every entry of the diagonal.

**Remark 4.3.** Notice that $\mathbb{Z}^n_{\geqslant 0}$ admits a right semigroup action by $\mathrm{SL}^{(+)}_{\mathrm{lex}}(n,\mathbb{Z})$, given by $(\alpha, A) \mapsto \alpha A$. Similarly, given a ring $R$, the ring $R[x_1,\ldots,x_n]$ admits a right monoid action by $\mathrm{SL}^{(+)}_{\mathrm{lex}}(n,\mathbb{Z})$ by letting $\mathrm{SL}^{(+)}_{\mathrm{lex}}(n,\mathbb{Z})$ act on the exponents, that is, by the linear extension of $(x^\alpha, A) \mapsto x^{\alpha A}$. Given $A \in \mathrm{SL}^{(+)}_{\mathrm{lex}}(n,\mathbb{Z})$ let us denote by $\Psi_A$ the $R$-algebra homomorphism $\Psi_A : R[x_1,\ldots,x_n] \to R[x_1,\ldots,x_n]$ induced by $A$.

The main result of the paper [SV11] can then be stated as follows.

**Theorem 4.4** (Soto, Vicente; [SV11])**.** *Let $\mathbb{K}$ denote an algebraically closed field of characteristic zero. For each monic polynomial*

$$f(x,z) = z^d + c_1(x)z^{d-1} + \ldots + c_d(x)$$

*where each $c_k(x) \in \mathbb{K}[\![x_1,\ldots,x_n]\!]$, there exist*

- *a positive integer $p$,*
- $A \in \mathrm{SL}^{(+)}_{\mathrm{lex}}(n,\mathbb{Z})$*,*

*such that $\Psi_A(f)$ splits in the ring $\mathbb{K}[\![x_1,\ldots,x_n]\!][z]$.*

**Remark 4.5.** An important caveat about this theorem is that the polyhedral transformation depends on the choice of an ordering in the variables.

### 4.1.2 *Splitting at the limit points of a normal crossings locus of dimension one*

Let $\mathbb{K}(w)$ denote the fraction field of the polynomial ring $\mathbb{K}[w] := \mathbb{K}[w_1, \ldots, w_r]$. Let $\mathbb{K}(\!(w)\!)$ denote the fraction field of the formal power series ring $\mathbb{K}[\![w]\!] := \mathbb{K}[\![w_1, \ldots, w_r]\!]$, and let $\overline{\mathbb{K}(\!(w)\!)}$ denote the algebraic closure of $\mathbb{K}(\!(w)\!)$. Given that there exists a subfield of $\mathbb{K}(\!(w)\!)$ which is isomorphic to $\mathbb{K}(w)$, we can identify $\overline{\mathbb{K}(w)}$ as a subfield of $\overline{\mathbb{K}(\!(w)\!)}$. Moreover, under this identification, $\overline{\mathbb{K}(w)}$ is the collection of elements $\overline{\mathbb{K}(\!(w)\!)}$ which are algebraic over $\mathbb{K}(w)$.

Notice that in the case $r = 1$, $\mathbb{K}(\!(w)\!)$ is the field of Laurent series. By the Newton-Puiseux theorem (see *Newton's theorem* of Lecture 12 in [Abh90]) we have that $\overline{\mathbb{K}(\!(w)\!)} \simeq \bigcup_{k=1}^{\infty} \mathbb{K}(\!(w^{1/k})\!)$, and so any field $L \subset \overline{\mathbb{K}(\!(w)\!)}$ which is a finite field extension $L|\mathbb{K}(\!(w)\!)$ is contained in $\mathbb{K}(\!(w^{1/k})\!)$, for some $k$.

**Definition 4.6** (Formal splitting at a point)**.** Let

$$\begin{aligned} f(w, x, z) &= f(w_1, \ldots, w_r, x_1, \ldots, x_m, z) \\ &= z^d + c_1(w, x) + \ldots + c_d(w, x) \end{aligned} \tag{4.1}$$

be a monic polynomial in $z$, where the coefficients $c_k(w, x)$ are regular functions in an open neighbourhood $U$ of the origin $0 \in \mathbb{A}^{r+m}_{\mathbb{K}}$. We say that $f$ *splits formally* at $a = (w_0, x_0, 0) \in U$ (or $f$ splits in $\mathbb{K}[\![w - w_0, x - x_0]\!][z]$, or $f$ splits over $\mathbb{K}[\![w - w_0, x - x_0]\!]$) if the expansion in formal power series of $f$ at $a$, that is, the expansion as an element of $\mathbb{K}[\![w - w_0, x - x_0]\!][z]$, factors as

$$f(w, x, z) = \prod_{j=1}^{d} (z - b_j(w - w_0, x - x_0)), \tag{4.2}$$

where for each $j$, $b_j \in \mathbb{K}[\![w - w_0, x - x_0]\!]$ and $b_j$ vanishes at $\tilde{a} := (w_0, x_0)$.

We also consider splittings in $\overline{\mathbb{K}(\!(w)\!)}[\![x - x_0]\!][z]$, but we do not assign a name for this type of splitting.

**Lemma 4.7.** *Let $f$ be as in (4.1). If $f$ splits formally at a point $(w, x, z) = (w_0, x_0, 0)$ then $f$ splits in $\overline{\mathbb{K}(w)}[\![x - x_0]\!][z]$.*

*Proof.* By translating if necessary, we may assume that $x_0 = 0$. Given that $\overline{\mathbb{K}(w)}$ is isomorphic to $\overline{\mathbb{K}(w - w_0)}$ it suffices to show that $f$ splits in $\overline{\mathbb{K}(w - w_0)}[\![x]\!][z]$.

Because $f$ is monic, we have that each $b_j$ (as in (4.2)) is algebraic over $\mathbb{K}[w - w_0, x]$. By Corollary 2.68 we have that the partial derivatives $\frac{\partial^{|\alpha|} b_j}{\partial x^{\alpha}}$ together with their evaluations at $y = 0$ are also algebraic for all $\alpha$ and all $j$. Thus, the coefficients in $w$ of the power series expansion of each $b_j$ can be expressed as elements of $\overline{\mathbb{K}(w)}$ □

**Lemma 4.8.** *Let $f$ be as in (4.1). Suppose that $f$ splits in $\overline{\mathbb{K}(w)}[\![x - x_0]\!][z]$. Then, there is a finite normal extension $L|\mathbb{K}(w)$ such that $f$ splits in $L[\![x - x_0]\!][z]$.*

*Proof.* By translating if necessary we can assume $x_0 = 0$. Given that power series rings with coefficients over a field are unique factorization domains, we have that $\overline{\mathbb{K}(w)}[\![x]\!][z]$ is a unique factorization domain. Let us show the claim for a polynomial $f$ which is irreducible in $\mathbb{K}[w, x, z]$, as the general case follows from this case.

Let $f_1, \dots, f_d \in \overline{\mathbb{K}(w)}[\![x]\!][z]$ be irreducible elements such that $f = f_1 \dots f_d$, and $f_j = z - b_j$, where the elements $b_j$ are given as in (4.2). Express each $b_j$ in formal power series expansion

$$b_j(w, x) = \sum_{\gamma \in \mathbb{N}^m} b_{j,\gamma}(w) x^\gamma,$$

where each $b_{j,\gamma} \in \overline{\mathbb{K}(w)}$. Let L be the subfield generated by

$$\{\{b_{j,\gamma} : j \in \{1, \dots, d\}, \gamma \in \mathbb{N}^m\}.$$

We claim that L is a finite normal extension of $\mathbb{K}(w)$.

Consider the group $\Gamma := \mathrm{Aut}_{\mathbb{K}(w)}(\overline{\mathbb{K}(w)})$ of automorphisms of $\overline{\mathbb{K}(w)}$ that fix the subfield $\mathbb{K}(w)$. Given $1 \leqslant k \leqslant d$ and $\alpha \in \mathbb{Z}^m_{\geqslant 0}$ we have that $\sigma \in \Gamma$ maps $b_{k,\gamma}$ to $b_{k',\gamma}$ for some $1 \leqslant k' \leqslant d$, by the uniqueness of the power series expansion. In particular, $\sigma(L) = L$.

We now want to show that L is a normal extension. Given an irreducible polynomial $p \in \mathbb{K}(w)[t]$, with a root $\alpha \in L$. Notice that the different elements of the orbit $O_\Gamma(\alpha)$ of $\alpha$ under the action of $\Gamma$ are also roots of p. Because L is invariant under the action of $\Gamma$ we have that $O_\Gamma(\alpha)$ is the set of distinct roots of p in L. Notice that $q(t) := \prod_{\beta \in O_\Gamma(\alpha)} (t - \beta)$ is invariant under the action of $\Gamma$, and so $q \in \mathbb{K}(w)[t]$, and $q|p$. Thus, $q = p$, and therefore all the roots of p are elements of L.

Consider now the group $\mathrm{Aut}_{\mathbb{K}(w)}(L)$ of automorphisms of L fixing $\mathbb{K}(w)$. Notice that given $\sigma \in \mathrm{Aut}_{\mathbb{K}(w)}(L)$ and a root $b_i$, we have that $\sigma(b_i) = b_j \in \{b_1, \dots, b_d\}$. Notice also that if $\sigma(b_i) = b_i$ for all $1 \leqslant i \leqslant d$, then we have that $\sigma$ fixes $b_{i,\gamma}$ for all i and all $\gamma$, and so, $\sigma = \mathrm{Id}_{\mathbb{K}(w)}$. Thus, there is an injective morphism $\mathrm{Aut}_{\mathbb{K}(w)}(L) \to S_d$ onto a subgroup of the symmetric group $S_d$. In particular, $\mathrm{Aut}_{\mathbb{K}(w)}(L)$ is a finite group, and thus L is a finite extension of $\mathbb{K}(w)$. □

**Lemma 4.9.** *Let f be a polynomial satisfying the hypotheses of Lemma 4.8, then there exists $p \in \mathbb{N}$ such that the roots (as in (4.2)) satisfy $b_1, \dots, b_d \in \mathbb{K}(w^{1/p})[\![x - x_0]\!]$. Moreover, if $f = \prod_{i=1}^s f_i$ where each $f_i \in \mathbb{K}[\![w, x - x_0]\!][z]$ is irreducible, then $f_i$ splits into $d_i$ factors*

$$f_i = \prod_{j=1}^{d_i} (z - r_{i,j}),$$

*where $d_i = \deg_z(f_i)$ and $r_{i,j} \in \mathbb{K}(w^{1/d_i})[\![x - x_0]\!]$.*

*Proof.* By translating if necessary we can assume that $x_0 = 0$.

Let L be the finite normal extension of $\mathbb{K}(w)$ such that f splits in $L[\![x]\!][z]$, as in Lemma 4.9. Because L is a finite extension of $\mathbb{K}(w)$, we can identify L as a subfield of $\overline{\mathbb{K}(\!(w)\!)}$. Let $\beta$ be a basis of L as a $\mathbb{K}(w)$-vector space. For each $\nu \in \beta$ there exists q such that $\nu \in \mathbb{K}(\!(w^{1/q})\!)$. By taking the least common multiple of all such q, we obtain p such that f splits over $\mathbb{K}(\!(w^{1/p})\!)[\![x]\!]$. Given that L is a finite extension of $\mathbb{K}(w)$, each of the roots $b_j$ is an element of $\mathbb{K}(w^{1/p})[\![x]\!]$.

Let us now assume that $f = \prod_{i=1}^s f_i$ where each $f_i \in \mathbb{K}[\![w, x]\!][z]$ is irreducible. Let $p \in \mathbb{N}$ be such that f splits in $\mathbb{K}(\!(w^{1/p})\!)[\![x]\!][z]$. Let $Z_p$ denote the finite cyclic group of p-th roots of unity, with generator $\zeta := e^{\frac{2\pi i}{p}}$. Notice that $\mathbb{K}(\!(w^{1/p})\!)[\![x]\!][z]$ admits a $Z_p$-action given by $\zeta \cdot g(w, x) = g(\zeta w, x)$. By relabeling if necessary, we may assume that $b_1$ is a root of

$f_1$. Notice then that the distinct elements of the orbit of $b_1$ by $Z_p$ is the collection of all the roots of $f_1$ (as $f_1$ is invariant under the action of $Z_p$). By relabeling if necessary, we may assume that $b_1, \ldots, b_{d_1}$ are the distinct roots of $f_1$ and that $\zeta^k \cdot b_1 = b_{k+1}$ for all $0 \leqslant k \leqslant d_1 - 1$. By the orbit-stabilizer theorem, there exists a morphism $Z_p \xrightarrow{\varphi_1} C_1$, where $C_1$ is bijective to the orbit of $b_1$. Given that $f_1$ is irreducible, we have that $C_1$ is isomorphic to some subgroup of $S_{d_1}$. Repeating this argument for each $f_i$ we obtain surjective group homomorphisms $Z_p \xrightarrow{\varphi_i} C_i$ where each $C_i$ is a cyclic subgroup of the symmetric group $S_{d_i}$. Given that $C_i$ must act transitively in the collection of roots of $f_i$ we have that $|C_i| = d_i$. Thus, $f_i$ splits in $\mathbb{K}(\!(w^{1/d_i})\!)[\![x]\!][z]$. $\square$

Similarly as in Lemma 4.7, the statements of Lemma 4.8 and Lemma 4.9 hold for the rings $\overline{\mathbb{K}(w)}[\![x - x_0]\!][z]$, $L[\![x - x_0]\!][z]$ and $\mathbb{K}(\!(w^{1/p})\!)$. Using this lemma and the main theorem from [SV11] we can obtain the following.

*Proof of Theorem 1.3.*
Fix $a \in C$ such that $X$ is not $\mathrm{nc}(n)$ at $a$. We know that $C$ is smooth at $a$ and so there exists a regular coordinate system $w, x_1, \ldots, x_{n-1}, z$ at $a$ such that $C$ is locally described by $\{z = x = 0\}$. By Weierstrass' preparation theorem, and by performing a coordinate change if necessary, we may express $X$ in local coordinates as the vanishing locus of a polynomial

$$f(w, x, z) = z^n + c_1(w, x) + \ldots + c_n(w, x),$$

where each $c_j \in \mathbb{K}[\![w, x]\!]$. Because $C$ is generically $\mathrm{nc}(n, 0)$, there exists $b = (w_0, 0, 0)$ such that $f$ formally splits at $b$, where $w_0 \in \mathbb{K}^\times$. By Lemma 4.7, there exist $b_1, \ldots, b_n \in \overline{\mathbb{K}(w)}[\![x]\!][z]$ as in (4.2). By Lemma 4.9, there exists $p \in \mathbb{N}$ such that $f$ splits in $\mathbb{K}(\!(w^{1/p})\!)[\![x]\!][z]$.

Now, using the ordering

$$x_1 < x_2 < \ldots < x_{n-1} < w$$

and Theorem 4.4 we can find an upper triangular matrix $A \in \mathrm{SL}^{(+)}_{\mathrm{lex}}(n, \mathbb{Z})$ with non-negative entries such that $\Psi_A(f)$ splits in $\mathbb{K}[\![w^{1/q}, x^{1/q}]\!]$ for some $q \in \mathbb{N}$. By replacing $p$ with $\mathrm{lcm}(p, q)$, we may assume that $f$ splits in $\mathbb{K}(\!(w^{1/p})\!)[\![x]\!]$ and $\Psi_A(f)$ splits in $\mathbb{K}[\![w^{1/p}, x^{1/p}]\!]$. Notice then

$$\Psi_A(f)(w, x, z) = (z - \Psi_A(b_1)(w, x)) \cdot \ldots \cdot (z - \Psi_A(b_n)(w, x)).$$

Express $A$ as

$$A = \left( \begin{array}{c|c} \tilde{A} & c \\ \hline 0 & 1 \end{array} \right),$$

where $\tilde{A} \in \mathrm{SL}^{(+)}_{\mathrm{lex}}(n-1, \mathbb{Z})$ is an upper triangular matrix with non-negative entries, and $c \in \mathbb{Z}^n_{\geqslant 0}$. Notice then

$$\Psi_A(b_k(w, x)) = \sum_{\alpha} b_{k,\gamma}(w) w^{\gamma \cdot c} x^{\gamma \tilde{A}},$$

and so we can assume that the Puiseux series expansion of $b_{k,\gamma}$ satisfy that

$$b_{k,\gamma}(w) w^{\gamma \cdot c} \in \mathbb{K}[\![w^{1/p}]\!],$$

for each $k$ and each $\gamma$.

Consider a blow-up $\sigma_1$ with centre

$$\{z = x_1 = x_2 = \ldots = x_{n-1} = w = 0\}.$$

Notice that the strict transform of the nc locus of $X$ is inside the standard chart $U_w$. Moreover, in $U_w$ we have

$$\sigma_1^*(b_k) = \sum_{\gamma} \tilde{q}_{k,\gamma}(w) w^{|\gamma|} x^{\gamma}.$$

Notice also that the origin in $U_w$ is a limit point of the nc locus, and so the strict transform of $f$ is of the form

$$\prod_{k=1}^{n} (z - \sum_{\gamma} \tilde{q}_{k,\gamma}(w) w^{|\gamma|-1} x^{\gamma}),$$

and $|\gamma| - 1 \geqslant 0$. Notice that $\tilde{q}_{k,\gamma} \neq q_{k,\gamma}$ but because $\tilde{q}_{k,\gamma}$ is still a regular function, for economy of notation we still denote the function obtained after considering the total transform by $q$. And so, after taking $t$ consecutive blow-ups each with centre locally described by

$$\{z = x_1 = \ldots = x_{n-1} = w = 0\},$$

we obtain the following local expression of the total transform

$$\sigma^*(b_k) = \sum_{k,\alpha} q_{k,\gamma}(w) w^{t|\alpha|} x^{\gamma},$$

in the unique chart $U''$ that intersects the strict transform of the normal crossings locus. Because $t|\gamma| \geqslant c \cdot \gamma$ for any $\gamma \in \mathbb{N}^n$ we have that $\sigma^*(b_k) \in \mathbb{K}[\![w^{1/p}]\!]$, and because after each blow-up the vanishing order of the respective origin is $d$, we have that the factors of the strict transform of $f$ are also in $\mathbb{K}[\![w^{1/p}]\!]$.

The argument above shows the desired splitting at the isolated point $a'$ over $a$ in the strict transform $C'$ of $C$. By repeating this argument for each isolated point of $C$, we obtain the desired result. □

## 4.2 SPLITTING AT THE LIMITS OF TRIPLE NORMAL CROSSINGS IN ANY DIMENSION

In this section, we provide the details of the proof of Theorem 5.1. One crucial object that we use to prove Lemma 4.14 is the discriminant. More precisely, we require the discriminant with respect to $z$ of the defining equation of $X$ to be a square. As such, we provide a blowing-up procedure, using only inv-admissible centres after which we can assume the discriminant to be a square. This is described in Lemma 4.11.

### 4.2.1 *Lemmas on the Discriminant*

Given that we want to find a blow-up sequence after which the discriminant of the local equation of $X$ is a square, it is reasonable to first establish a factorization result for the discriminant for equations satisfying to split generically (see Lemma 4.10). We then use that factorization to determine the sequence of blowings-up (see Lemma 4.11).

**Lemma 4.10** (Lemma 3.2 in [BLM12]). *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and let*

$$f = z^m + a_1(w,x)z^{m-1} + \ldots + a_m(w,x), \tag{4.3}$$

*where $a_j \in \mathbb{K}[\![w_1,\ldots,w_r,x_1,\ldots,x_{m-1},u_1,\ldots,u_q]\!]$ satisfies $\operatorname{ord}(a_j) \geqslant j$. Assume that $f$ is in the ideal generated by $z, x_1, \ldots, x_{m-1}$ and that $f$ completely splits into $m$ formal factors at every point of*

$$\{z = x_1 = \ldots = x_{m-1} = 0\} \setminus \{w_1 \ldots w_r = 0\}.$$

*Let $D$ denote the discriminant of $f$ with respect to $z$. Then, $D$ factors as*

$$D = \Psi\Phi^2,$$

*where $\Psi, \Phi \in \mathbb{K}[\![w,x,u]\!]$, $\Phi \in \langle x_1,\ldots,x_{m-1}\rangle$ and $\Psi$ does not vanish outside $\{w_1 \ldots w_r = 0\}$.*

*Proof.* Let $b \in \{z = x = 0\} \setminus \{w_1 \ldots w_r = 0\}$. Given that $f$ is a polynomial in $z$, and given that $f$ splits into linear factors, we have that $D$ is a unit times a square in an étale neighbourhood $U$ of $b$. Notice also that if $D = g_1g_2$, where $g_1$ vanishes at some point $p \in U$ outside $\{w_1 \ldots w_r = 0\}$, then $g_1$ cannot be a unit in a neighbourhood of $p$. Thus $g_1^2 | D$, giving us the desired result. □

**Lemma 4.11** (Lemma 3.4 of [BLM12]). *Assume that $f$ satisfies the hypotheses of Lemma 4.10 for some étale coordinate system at $a = 0$. Then, there is a finite sequence of blowings-up with centres of the form $\{z = x_1 = \ldots = x_{m-1} = 0\} \cap \{w_j = 0\}$ such that if $D'$ denotes the strict transform of $D$ then $D'(v_1^2,\ldots,v_r^2,x,u)$ is a square in some étale neighbourhood of the point $a'$ over $a$.*

*Proof.* Fix $j = 1$ and consider the blow-up $\sigma_1$ induced by the centre $C = \{z = x = 0\} \cap \{w_1 = 0\}$. Let $f'$ denote the strict transform of $f$ by $\sigma_1$. Notice that $f'$ still satisfies the generic splitting hypothesis of Lemma 4.10, and so the strict transform of $\Phi^2$ is still a square. On the other hand, if we express

$$\Psi = w_1^{\alpha_1} \cdot \ldots \cdot w_r^{\alpha_r} g_0(w,u) + x_1g_1(w,x,u) + \ldots + x_{m-1}g_{m-1}(w,x,u),$$

where $g_0$ is a unit outside $\{u_1 \ldots u_q = 0\}$, we have that the total transform $\overline{\Psi}$ of $\Psi$ satisfies

$$\overline{\Psi} = w_1(w_1^{\alpha_1-1} \cdot \ldots \cdot w_r^{\alpha_r}\tilde{g}_0 + x_1\tilde{g}_1 + \ldots + x_{m-1}\tilde{g}_{m-1}),$$

for some regular functions $\tilde{g}_0, \tilde{g}_1, \ldots, \tilde{g}_{m-1}$. Thus, after performing $\alpha_1$ blowings-up with centres of the form $\{z = x = 0\} \cap \{w_1 = 0\}$ we may assume $w_1^{\alpha_1} | \overline{\Psi}$. Following a similar sequence of blow-ups for each $\alpha_j$, where each centre of blow-up is locally given by

$$\{z = x = 0\} \cap \{w_j = 0\}, \tag{4.4}$$

repeated $\alpha_j$ times, we may assume that $w_1^{\alpha_1} \cdot \ldots \cdot w_r^{\alpha_r} | \overline{\Psi}$. After $|\alpha|$ blowings-up we may assume that $\overline{\Psi}$ is a unit times a monomial in $w$, outside $\{u_1 \ldots u_q = 0\}$. The desired result follows by considering an étale neighbourhood in which the unit $\tilde{g}_0 + \sum_{i=1}^{m-1} x_i\overline{g}_i$ is a square. □

**Remark 4.12.** Notice that the closure of the local centres in (4.4) can be globally described as

$$S_{3,0} \cap F,$$

for some irreducible component $F \subset E$. Notice also that the order in which we perform the blow-ups does not affect the final expression of the total (or strict) transform of $\Psi$. Thus, by blowing-up the component $F \subset E$ with the largest year of birth such that

$$\operatorname{ord}_{S_{3,0} \cap F} \Psi > \operatorname{ord}_{S_{3,0}} \Psi,$$

after finitely many blowings-up we obtain the desired result.

### 4.2.2 *Splitting theorem for triple normal crossings*

The main goal of this subsection is to provide the proof of a local version of Theorem 4.1, from which Theorem 4.1 follows.

**Remark 4.13.** Let $(X, E)$ be a pair where $X \hookrightarrow Z$ is a hypersurface of a smooth variety $Z$, and $E \subset Z$ is a snc divisor. Assume that in a neighbourhood $U \subset Z$, the pair $(X, E)$ satisfies the hypothesis of Theorem 4.1. Assume that after finitely many inv-admissible blowings-up we have that $\operatorname{inv}(b) \leqslant (3, 0, 1, 0, 1, 0, \infty)$ for all points $b$ in (the strict transform of) $X$. By blowing-up the irreducible components of the stratum $S \subset U$ of points with inv-value equal to $(3, 0, 1, 0, 1, 0, \infty)$ which are not generically $\mathrm{nc}(3, 0)$, we may assume that $S$ is generically $\mathrm{nc}(3, 0)$. Let $f$ be a local equation defining $X$ at a non-nc point $a \in S$. Given that $\operatorname{inv}(a) = (3, 0, 1, 0, 1, 0, \infty)$ there is an étale coordinate system $(u_1, \dots, u_q, w_1, \dots, w_r, x_1, x_2, z)$ at $a$, in which we can express a local equation $\{f = 0\}$ defining $X$ as $f \in \mathbb{K}[\![u, w, x, z]\!]$ with $f \in \langle w, x, z \rangle$ and where $\{w_1 \dots w_r = 0\}$ is a local equation of $E$ at $a$. By Weierstrass' preparation theorem, we may assume $f \in \mathbb{K}[\![u, w, x]\!][z]$, and by applying a *Tschirnhaus transformation* we can write

$$f(u, w, x, z) = z^3 - 3B(w, x)z + C(w, x), \tag{4.5}$$

and where $f$ splits into 3 factors of order 1 in $\{z = x = 0\} \setminus \{w_1 \dots w_r = 0\}$.

Let us now provide the statement of the local version of Theorem 4.1.

**Lemma 4.14.** *Let $Z, X, U$ be as in Remark 4.13, and express $X$ at a point $a = 0$ with $\operatorname{inv}(a) = \operatorname{inv}(\mathrm{nc}(3, 0))$ as in (4.5). Then, there exists a finite sequence of blowings-up with centres of the form $\{z = x = w_j = 0\}$ for some $1 \leqslant j \leqslant r$, such that the strict transform $g$ of $f$ satisfies that $g(u, v_1^6, \dots, v_r^6, x_1, x_2, z)$ splits into 3 factors with vanishing order 1.*

The lemma above, in turn, relies on the following splitting lemma, which is conditional on the discriminant of $f$ being a square.

**Lemma 4.15.** *Under the hypotheses of Lemma 4.14, let $f$ be as in (4.5). Let $D$ be the discriminant of $f$ as a polynomial in $z$. If $D$ is a square, then $f(u, v_1^3, \dots, v_r^3, x_1, x_2, z)$ splits.*

*Proof.* Express $f$ as in (4.5) of Remark 4.13. We know that

$$D := \operatorname{Disc}_z(f) = -27(C^2(w, x) - 4B^3(w, x)).$$

Let $G := \mathrm{Aut}_{\mathbb{K}(w)}(\overline{\mathbb{K}(w)})$ be the group of automorphisms of the algebraic closure $\overline{\mathbb{K}(w)}$ fixing $\mathbb{K}(w)$. The action of $G$ on $\overline{\mathbb{K}(w)}$ induces an action on $\mathbb{K}[\![w,x]\!]$ and this induced action permutes the roots of $f$. Given that the discriminant is a square, we may identify the action of $G$ on the roots of $f$ with the action of the cyclic group $A_3$. Thus, we can enumerate the roots $b_0, b_1, b_2$ of $f$ (as a polynomial in $z$) in such a way that $j \cdot b_i = b_{i+1}$, where $j$ denotes a generator of $A_3$, and where the indices are taken to be reduced modulo 3.

Then, we can define $y_0, y_1, y_2$ as in (3.3) with $n = 3$. Given the condition that $z$ is a maximal contact hypersurface, we have that $b_0 + b_1 + b_2 = 0$ and so $y_0 = z$. Notice also that if $\varepsilon$ denotes a primitive cubic root of unity, then

$$\begin{aligned} f &= (y_0 + y_1 + y_2)(y_0 + \varepsilon y_1 + \varepsilon^2 y_2)(y_0 + \varepsilon^2 y_1 + \varepsilon y_2) \\ &= z^3 - 3y_1 y_2 z + y_1^3 + y_2^3. \end{aligned} \tag{4.6}$$

We claim that $y_1, y_2$ each are a monomial in $w_1^{1/3}, \ldots, w_r^{1/3}$ times an element in $\mathbb{K}[\![w,x]\!]$. To achieve this, we show that $y_1^3, y_2^3 \in R$ and that they do not share any common irreducible factors.

We can express the discriminant in terms of $y_1, y_2$ as $D = (y_1^3 - y_2^3)^2$. Let $S = \frac{1}{i\sqrt{27}}(y_1^3 - y_2^3)$. Notice that

$$C + i\sqrt{27}S = 2y_1^3, \qquad C - i\sqrt{27}S = 2y_2^3, \qquad 4B^3 = (C-S)(C+S).$$

Given that both $C, S \in \mathbb{K}[\![w,x]\!]$, it suffices to verify that the only common irreducible factors (in $R$) of $C + i\sqrt{27}S$ and $C - i\sqrt{27}S$ are monomials in $w$. For this, let us show that there are no common irreducible factors of $C$ and $S$ in $\overline{\mathbb{K}(w)}[\![x]\!]$, and then apply Lemma 4.16 that we write below.

Notice that the vanishing order of each $y_i$ at $a$ is one, as $(y_0, y_1, y_2)$ can be obtained by multiplying $(z - b_0, z - b_1, z - b_2)$ by an invertible matrix, and each $z - b_i$ has vanishing order one at $a$. On the other hand, if $\{x = 0\}$ denotes the subspace $\{x_1 = x_2 = 0\}$, we have that $y_i|_{x=0} = 0$ for $i = 1, 2$. By the implicit function theorem, we deduce the following identity of ideals

$$\langle y_1, y_2 \rangle = \langle x_1, x_2 \rangle \subset \overline{\mathbb{K}(w)}[\![u,x]\!]. \tag{4.7}$$

Thus, $C$ and $S$ have no common irreducible factors in $\overline{\mathbb{K}(w)}[\![u,x]\!]$.

Consider the marked ideal $\mathcal{I} := (Z, Z, E, \langle f \rangle, 3)$, and notice that the coefficient ideal $\underline{\mathcal{J}} := \underline{\mathcal{C}}(\mathcal{I})$ restricted to the maximal contact subspace $\{z = 0\}$ admits a local expression of the form $\underline{\mathcal{J}} = (Z, \{z = 0\}, E, \langle B^3, C^2 \rangle, 6)$, or equivalently, $\underline{\mathcal{J}} = (Z, \{z = 0\}, E, \langle S^2, C^2 \rangle, 6)$. Let $\gamma$ be the largest exponent such that $\underline{\mathcal{J}} = w^\gamma \underline{\tilde{\mathcal{J}}}$, for some ideal $\underline{\tilde{\mathcal{J}}}$. Notice that both generators of $\underline{\mathcal{J}}$ are squares, and so each entry of $\gamma$ is even. Let $\alpha$ be such that $2\alpha = \gamma$. In particular we have that $w^\alpha$ is the largest common monomial in $w$ of $C$ and $S$.

Given that $\mathrm{inv}(a) = (3, 0, 1, \ldots)$, we have that

$$\mathrm{ord}(w^{-\gamma} S^2) = \mathrm{ord}_x(w^{-\gamma} S^2) \quad \text{or} \quad \mathrm{ord}(w^{-\gamma} C^2) = \mathrm{ord}_x(w^{-\gamma} C^2).$$

By Lemma 4.16, we obtain that $w^\alpha$ is the only common factor in $R$ of $S$ and $C$. From this, the result follows. $\square$

Let us provide the proof of the following, to finalize the proof of Lemma 4.15.

**Lemma 4.16.** *Let* $F \in \mathbb{K}[\![u, w, x]\!]$*, and suppose that* $\mathrm{ord}(F) = \mathrm{ord}_x(F)$*. If* $\theta$ *is an irreducible factor of* $F$ *in* $\mathbb{K}[\![u, w, x]\!]$*, then* $\theta$ *is an irreducible factor of* $F$ *in* $\overline{\mathbb{K}(w)}[\![u, x]\!]$*.*

*Proof.* Let $\prod_{j=1}^{n} \theta_j$ be a factorization in irreducible elements of $F$. By hypothesis, we have that

$$\mathrm{ord}\left(\prod_{j=1}^{n} \theta_j\right) = \mathrm{ord}_x\left(\prod_{j=1}^{n} \theta_j\right) \tag{4.8}$$

By the properties of ord, we have that $\mathrm{ord}(\theta_j) \leqslant \mathrm{ord}_x(\theta_j)$, for each $j$. If at least one $\theta_j$ satisfies that $\theta_j$ is a unit in $\overline{\mathbb{K}(w)}[\![u, x]\!]$ then $\mathrm{ord}_x(\theta_j) > \mathrm{ord}(\theta_j)$, contradicting (4.8). □

*Proof of Lemma 4.14.*
By Lemma 4.11, there is a finite sequence of blowings-up with centres inside the stratum $S_{3,0}$ (and thus inv-admissible in $U$) after which we may assume that the discriminant $D$ of a local equation $\{f = 0\}$ defining $X$ in $U$, is a square in $\mathbb{K}[\![u, w_1^{1/2}, \ldots, w_r^{1/2}, x]\!]$. Then, by Lemma 4.15 we have that $f$ splits in $\mathbb{K}[\![u, w_1^{1/6}, \ldots, w_r^{1/6}, x]\!]$. □

Notice that Theorem 4.1 follows by performing the blowings-up with centres at each limit point of the curves $C$ which are generically $\mathrm{nc}(n, 0)$. Thus, after finitely many blowings-up with this description of the centres, we obtain the desired splitting.

**Remark 4.17.** In Section 7.7 we make use of a slightly more general version of Lemma 4.14, where instead of assuming that inv in $U$ is at most inv(nc(3, 0)), we assume that inv in $U$ is at most inv(nc(3, $r$)) but $X$ is generically $\mathrm{nc}(3, r)$ in the stratum of points with inv = inv(nc(3, $r$)). A similar statement is needed for inv(nc(2, $r$)) in Section 7.8.

**Remark 4.18.** Let $f \in \mathbb{K}[u, w, x, z]$ as in Theorem 4.1 and assume $f$ is irreducible. By Lemma 4.14, we may consider the actions of the cyclic group of order 6 generated by $g$ on $\mathbb{K}[\![w_1^{1/6}, \ldots, w_r^{1/6}, x]\!][z]$, where the j-th action satisfies

$$g \cdot w_j \mapsto e^{\frac{1}{3}\pi i} w_j,$$

and leaves the rest of the generators of $\mathbb{K}[\![u, w_1^{1/6}, \ldots, w_r^{1/6}, x]\!][z]$ fixed. Each of these actions permutes the roots of $f$ and leaves $f$ invariant. Given that $f$ is irreducible, if one of these actions does not act trivially, then it acts transitively on the roots $b_0, b_1, b_2$ of $f$. By the orbit stabilizer theorem, an action which is not trivial descends to an action of the cyclic group of order 3. In particular, we obtain that $f$ splits in $\mathbb{K}[\![u, w_1^{\delta_1}, \ldots, w_r^{\delta_r}, x]\!][z]$, where each $\delta_j \in \{1, 1/3\}$.

# 5

# REDUCTION TO CIRCULANT NORMAL FORM

The main goal of this chapter is to provide the tools needed for our approach at establishing normal forms for the limits of the normal crossings locus. Just as in Chapter 4, the results we obtained are divided in two cases: for the (isolated) limit singularities of the locus $S_{\mathrm{nc}(n)}$ of $\mathrm{nc}(n)$ singularities of a variety of dimension $n$ (see Theorem 1.4), and for the limit singularities of the stratum $\mathrm{nc}(3)$ in arbitrary dimension. The statement for the latter is the following.

**Theorem 5.1.** *Let $(X, E)$ be a pair consisting of a hypersurface $X \hookrightarrow Z$, where $Z$ is a smooth variety over $\mathbb{K}$ and $E \subset Z$ is a* snc *divisor. Assume that there exists an open $U \subset Z$ such that, after a finite sequence of* inv*-admissible blowings-up, the stratum with highest* inv *value in $U$ is $S_{3,0} := \{\mathrm{inv} = \mathrm{inv}(\mathrm{nc}(3, 0))\}$, and that $S_{3,0}$ is generically $\mathrm{nc}(3, 0)$. Then, there is a finite sequence of admissible and equimultiple blowings-up for $(X, E)$ that restrict to an isomorphism on the $(X, E)$-normal crossings locus, after which (the strict transform of) $X$ is one of the following normal forms at every point in (the strict transform of) $S_{3,0}$:*

1. $\mathrm{cp}(3)$.
2. $\mathrm{nc}(1) \times \mathrm{cp}(2)$.
3. $\mathrm{nc}(3)$.

Because the proofs are clearer in the irreducible case, we provide the proof of the irreducible case first and later we treat the general case.

The proofs of these theorems follow a similar structure. We first use a splitting theorem to establish a factorization of $f$ into functions of order 1, we then find an equivalent marked ideal whose generators are expressed in terms of the factors of $f$ and use the properties of inv together with Corollary 2.93 to successively find more terms of the power series expansion of these generators. Finally, we use a *cleaning sequence* to reduce the monomials in the variables defining the local equations of the exceptional divisors. After this, the desired theorem follows.

A *cleaning sequence* of a pair $(X, E)$ is a sequence of admissible blowings-up, after which, the marked ideal associated $(X, E)$ is *clean*. A *clean marked ideal* is a marked ideal satisfying that the *monomial part* of the companion ideals in the sequential restrictions to maximal contact have empty cosupport (see Subsection 2.7.4 for the definition of monomial part). Let us provide a more precise definition.

**Definition 5.2** (Cleaning ideal, Clean marked ideal)**.** Using the notation in Subsection 2.7.4, we define the *cleaning ideal* of $\underline{\mathcal{I}}$ in codimension $k$ as

$$\underline{\mathcal{L}}(\underline{\mathcal{I}}^k) := (Z, N, E, \mathcal{M}(\underline{\mathcal{I}}^k), d).$$

See Subsection 2.7.4 for the definition of the monomial part $\mathcal{M}(\underline{\mathcal{I}})$ and for the definition of $\underline{\mathcal{I}}^k$. We refer to the (non-marked) ideal sheaf associated to $\underline{\mathcal{L}}(\mathcal{I})$ by $\mathcal{L}(\underline{\mathcal{I}}) := \mathcal{M}(\underline{\mathcal{I}})$.

We say that $\underline{\mathcal{I}}$ is *clean* if the cleaning ideals at every codimension level have empty cosupport.

**Remark 5.3.** If the cosupport of a cleaning ideal is non-trivial, then it is a hypersurface. Also, if we express $\underline{\mathcal{I}}^k = (Z, N, E, \mathcal{I}, d)$ then

$$\operatorname{cosupp}(\underline{\mathcal{L}}(\underline{\mathcal{I}}^k)) \subset \operatorname{cosupp}(\underline{\mathcal{I}}^k).$$

## 5.1 NORMAL FORMS OF LIMITS OF THE NC LOCUS, TOP ORDER CASE

We are now ready for the proof of Theorem 1.4. We first provide the proof in the case where f is irreducible, and later on we provide the argument for the general case.

*Proof of Theorem 1.4 in the irreducible case.*

Let $a \in C$ be such that $X'$ is not $\mathrm{nc}(n)$ at $a$. By Theorem 1.3, there exists a sequence of inv-admissible blowings-up after which, there are étale coordinates $w, x_1, \ldots, x_{n-1}, z$, the local equation $f = 0$ defining (the transform of) $X$ in $U$ splits into formal factors $f = f_1 \cdot \ldots \cdot f_n$, each with $\operatorname{ord}(f_i) = 1$ in $\mathbb{K}[\![w^{1/p}, x]\!][z]$, for some $p \in \mathbb{N}$. Let $R := \mathbb{K}[\![w, x]\!]$, define $v := w^{1/p}$ and let $S := \mathbb{K}[\![v, x]\!]$.

We can now consider the action of the cyclic group of order $p$ generated by $g$ on $\mathbb{K}[\![w^{1/p}, x]\!][z]$, given by $g \cdot w^{1/p} \mapsto e^{\frac{2\pi i}{p}} w^{1/p}$. This action permutes the roots of f and leaves f invariant. Given that f is irreducible, by the orbit stabilizer, following a reasoning similar to the one in Remark 4.18 we may assume that $p = n$ (and so $v = w^{1/n}$ and $S = \mathbb{K}[\![w^{1/n}, x]\!]$).

Let $\varepsilon$ be a primitive N-th root of unity and let $\phi_0, \ldots, \phi_{n-1}$ denote the collection of roots of f in $K[\![v, x]\!]$. We may assume the chosen indices satisfy

$$m \cdot \phi_j = \phi_{j+m},$$

for all $m \in \mathbb{Z}$, and where $j + m$ is taken reduced modulo $n$. Define $y_0, \ldots, y_{n-1}$ as in (3.3). By Remark 3.12, there exist $\zeta_j \in R$ and $m_j \in \mathbb{Z}_{\geqslant 0}$ such that

$$y_j(v, x) = v^{m_j n + j} \zeta_j(v^n, x). \tag{5.1}$$

Notice that $y_0 = z$, and that $y_0$ satisfies an identity similar to (5.1), but in this particular case we have $m_j = j = 0$. Combining the above with Proposition 3.10 we obtain

$$f = \det(C(\zeta_0, v^{1+m_1 n}\zeta_1, \ldots, v^{n-1+m_{n-1} n}\zeta_{n-1})). \tag{5.2}$$

Notice that the marked ideal $(\langle f \rangle, n)$ is equivalent to its coefficient ideal which in turn is equivalent to the ideal

$$\underline{\mathcal{J}} := \sum_{j=0}^{n-1} (\langle w^{m_j n + j} \zeta_j^n \rangle, n), \tag{5.3}$$

as the vector of functions $(\zeta_0, \ldots, v^{n-1+m_{n-1}n}\zeta_{n-1})$ is the product of an invertible matrix (with coefficients in $\mathbb{K}$) times the vector $(z+\Phi_0, \ldots, z+\Phi_{n-1})$.

We now want to prove that there exists a finite sequence of blowings-up that avoid the normal crossings locus such that the strict transform of $f$ under the composition of these blowings-up has the origin as a circulant singularity. For this, we first find an ordered sequence of smooth sections $\zeta^1_{\pi(1)}, \ldots, \zeta^{n-1}_{\pi(n-1)}$ that help determine the centres of the sequence of *cleaning blowings-up*. Let us emphasize that the superscripts $j$ in $\zeta^j_{\pi(j)}$ are not exponents.

We follow a sequential reasoning. In the rest of the proof, the superscript in $\zeta^k_j$ and $\alpha^k_j$ denotes the codimension (or nesting level) at which we are considering these objects. The hypothesis we work with is the following.

For $k \geqslant 0$ there exist

- a bijective function $\pi : \{0, \ldots, k-1\} \to S_k \subset \{0, \ldots, n-1\}$ with $|S_k| = k$,
- a finite set of functions $\{\zeta^k_j : j \notin S_k\}$,
- a finite set of exponents $\{\alpha^k_j : j \notin S_k\}$.

such that

- $N^{k-1} := \{\zeta^0_{\pi(0)} = \zeta^1_{\pi(1)} = \ldots = \zeta^{k-1}_{\pi(k-1)} = 0\}$ is a smooth maximal contact subspace of $\underline{\mathcal{I}}$ of codimension $k$,
- the exceptional divisor $w$ does not divide $\zeta^d_j$ for all $j$,
- the restriction of the coefficient ideal of $(\langle(\zeta_0)^n, \ldots, w^{n-1+m_{n-1}n}(\zeta_{n-1})^n\rangle, n)$ to $N^{k-1}$ is equivalent to $(Z, N^{k-1}, E, \langle\{w^{\alpha^k_j}(\zeta^k_j)^n : j \notin S_k\}\rangle, n)$,
- each function $\zeta^k_j$ has order $\geqslant 1$ and the collection of exponents $\{\alpha^k_j : j \notin S_k\}$ contains at most one representative for each class of integers modulo $n$.

This information allows us to construct then the functions $\zeta^{k+1}_j$, as well as the exponents $\alpha^{k+1}_j$ and consequently, to extend the function $\pi : \{0, \ldots, k\} \to S_{k+1} \subset \{0, \ldots, n-1\}$.

For homogeneity of notation, define $\zeta^0_j := \zeta_j$ for all $0 \leqslant j \leqslant n-1$, $\alpha^0_0 := 0$, and $\alpha^0_j := j + m_j n$ for all $1 \leqslant j \leqslant n-1$. Notice that the conditions are then trivially satisfied for the case $k = 0$.

Let us also work out the case $k = 1$. Given that $\alpha^0_0 = 0$ and $\mathrm{inv}(a) = (n, \ldots)$, $N^1 := \{\zeta^0_0 = 0\}$ is a maximal contact hypersurface of the marked ideal $(\langle\{w^{\alpha^0_j}(\zeta^0_j)^n : 0 \leqslant j \leqslant n-1\}\rangle, n)$. This leads us to define $\pi(0) := 0$. We want to restrict to maximal contact, and so, let us express

$$\zeta^0_j := \xi^0_j + w^{\beta^1_j}\zeta^1_j, \tag{5.4}$$

for each $j$, where $\xi^0_j \in \langle\zeta^0_0\rangle$ and $w$ does not divide any of the functions $\zeta^1_j$. Notice that the restriction of $\underline{\mathcal{I}}$ to $\{\zeta^0_0 = 0\}$ is the ideal

$$(\langle w^{\alpha^0_1+\beta^1_1}(\zeta^1_1)^n, \ldots, w^{\alpha^0_{n-1}+\beta^1_{n-1}}(\zeta^1_{n-1})^n\rangle, n).$$

Define $\alpha^1_j := \alpha^0_j + \beta^1_j n$. Notice in this case that $\{\alpha^1_1, \ldots, \alpha^1_{n-1}\}$ contains exactly one representative for each class of integers modulo $n$, except 0. In short, the case $k = 1$ is also satisfied.

Let us consider an integer $1 \leqslant k \leqslant n-1$ satisfying the inductive hypothesis. Given that $\operatorname{inv}(a) = \operatorname{inv}(\mathrm{nc}(n,0))$, we have that

$$\pi(k) := \arg\min\{\alpha_j^k : j \notin S_k\},$$

satisfies that $N^k := N^{k-1} \cap \{\zeta^k_{\pi(k)} = 0\}$ is a maximal contact hypersurface of the marked ideal $(\langle\{w^{\alpha_j^k}(\zeta_j^k)^n : 0 \leqslant j \leqslant n-1\}\rangle, n)$. In other words, we define $\pi(k)$ such that $w^{\alpha^k_{\pi(k)}}$ is the generator of the monomial part $\mathcal{M}(\underline{\mathcal{I}}^k)$. We want to restrict to maximal contact, and so, let us express

$$\zeta_j^k := \xi_j^k + w^{\beta_j^{k+1}}\zeta_j^{k+1}, \tag{5.5}$$

for each $j \notin S_k$, where $\xi_j^k \in \langle \zeta^0_{\pi(0)}, \dots, \zeta^k_{\pi(k)}\rangle$ and $w$ does not divide any of the functions $\zeta_j^{k+1}$. Thus, after removing the monomial part from the marked ideal, and then going to maximal contact we obtain the following exponents (in $w$)

$$\alpha_j^{k+1} := \alpha_j^k + n\beta_j^{k+1} - \alpha^k_{\pi(k)}.$$

Given that there is at most one representative of each class of the integers modulo $n$ in $\{\alpha_j^k : j \notin S_k\}$, the same holds for $\{\alpha_j^{k+1} : j \notin S_{k+1}\}$.

Thus, we verify the inductive hypotheses for $k+1$.

Let us now consider the effect of performing a single blowing-up $\sigma$ with centre

$$C = \{\zeta^0_{\pi(0)} = \zeta^1_{\pi(1)} = \ldots = \zeta^{n-2}_{\pi(n-2)} = w = 0\}. \tag{5.6}$$

First, let us notice that the strict transform of $\underline{\mathcal{I}}$ over a point contained in the $z$-chart gives rise to a regular ideal. Now, let us consider the pullback of $\underline{\mathcal{I}}$ over a point $\tilde{a}$ in the $\zeta^r_{\pi(r)}$-chart. Consider the identities,

$$\sigma^*(\zeta_j^k) := \sigma^*(\xi_j^k) + w^{\beta_j^{k+1}}(\zeta^r_{\pi(r)})^{\beta_j^{k+1}}\sigma^*(\zeta_j^{k+1}),$$

for $k < r-1$, and

$$\sigma^*(\zeta^{r-1}_{\pi(r)}) := \sigma^*(\xi^{r-1}_{\pi(r)}) + w^{\beta^r_{\pi(r)}}(\zeta^r_{\pi(r)})^{\beta_j^k+1},$$

A particular consequence of this sequence of identities is that the restriction to the maximal contact subspace $N^r$ gives us a monomial ideal, and so inv has been decreased in the chart corresponding to $\zeta^r_{\pi(r)}$.

There is only one chart left to consider, and so let us consider a point $\tilde{a}$ contained in the $w$-chart of $\sigma$. Notice then

$$\sigma^*(\zeta_j^k) = \sigma^*(\xi_j^k) + w^{\beta_j^{k+1}}\sigma^*(\zeta_j^{k+1})$$

for all $k < n-2$, and for $k = n-2$ we obtain

$$\begin{aligned}\sigma^*(\zeta^{n-2}_{\pi(n-1)}) &= \sigma^*(\xi^{n-2}_{\pi(n-1)}) + w^{\beta^{n-1}_{\pi(n-1)}}\sigma^*(\zeta^{n-1}_{\pi(n-1)})\\ &= w(\tilde{\xi}^{n-2}_{\pi(n-1)} + w^{\beta^{n-1}_{\pi(n-1)}-1}\tilde{\zeta}^{n-1}_{\pi(n-1)}),\end{aligned}$$

for some regular functions $\tilde{\xi}^{n-2}_{\pi(n-1)}, \tilde{\zeta}^{n-1}_{\pi(n-1)}$.

And so, if we perform $\lfloor \dfrac{\alpha^{n-2}_{\pi(n-1)}}{n} \rfloor$ successive blowings-up with centre

$$\{\zeta^0_0 = \zeta^1_{\pi(1)} = \ldots = \zeta^{n-2}_{\pi(n-2)} = w = 0\} \tag{5.7}$$

we reduce to the case where $\alpha^{n-2}_{\pi(n-1)} \in \{1, \ldots, n-1\}$. Similarly, we can now consider a sequence of $\lfloor \dfrac{\alpha^{n-2}_{\pi(n-3)}}{n} \rfloor$ successive blowings-up with centre

$$\{\zeta^0_0 = \zeta^1_{\pi(1)} = \ldots = \zeta^{n-3}_{\pi(n-3)} = w = 0\}$$

to assume $\alpha^{n-2}_{\pi(n-2)} \in \{1, \ldots, n-1\}$. Continuing this cleaning sequence, we may assume that all the exponents $\beta^k_j$ are zero, for $k \geqslant 2$. We can now perform $m_{\pi(1)}$ blowings-up with centre $\{z = w = 0\}$ to reduce to the case $m_{\pi(1)} = 0$.

By (5.5), we may perform a change of variables to express

$$f = \det(C(\zeta_0, v^{r(1)}\zeta_1, v^{r(2)}\zeta_2, \ldots, v^{r(n-1)}\zeta_{n-1})),$$

where $r$ is a permutation of the set $\{1, \ldots, n-1\}$, and each $\zeta_0, \ldots, \zeta_{n-1}$ can be completed to an étale coordinate system. Expressing $f$ as a product we obtain

$$\begin{aligned} f &= \prod_{k=0}^{n-1} \left( \sum_{j=0}^{n-1} \xi^{jk} v^{r(j)} \zeta_j \right) \\ &= \prod_{k=0}^{n-1} \left( \sum_{j=0}^{n-1} \xi^{r^{-1}(j)k} v^j \zeta_{r^{-1}(j)} \right). \end{aligned}$$

Let $\varphi_0 := \sum_{j=0}^{n-1} v^j \zeta_{r^{-1}(j)}$ and notice that the rest of the factors of $f$ can be obtained by letting the $n$-th roots of unity act on $\varphi_0$. Then, following a reasoning similar to the one in Proposition 3.10, and by performing a relabelling of the variables if necessary, we obtain that the transform of $X$ at the point $a'$ over $a$ is circulant of order $n$. □

**Remark 5.4.** Notice that our selection of centres of blowings-up in the cleaning procedure is equivalent to a desingularization procedure similar to the sequential maximal contact subspace construction for the invariant, but instead of considering the marked ideal $\underline{\mathcal{R}}(\underline{\mathcal{I}}^k)$ we consider the ideal $\underline{\mathcal{L}}(\underline{\mathcal{I}}^k)$. In other words, each of the centres considered in our cleaning procedure are $\iota$-admissible, where $\iota$ is a truncation of inv at some codimension level $k \geqslant 1$.

*Sketch of proof of Theorem 1.4 in the general case.*
The hypothesis of $f$ being irreducible is only used to obtain (5.2), as we claim that the rest of the argument follows similarly for the case where $f$ is not irreducible. Assume also that $f$ is given by

$$f = f_1 \cdot \ldots \cdot f_s,$$

where each $f_\ell$ is an irreducible element of $\mathcal{O}_Z$. For each $\ell \in \{1, \ldots, s\}$, let $n_\ell$ denote $\mathrm{ord}_{f_\ell}(a)$. Notice that $f$ satisfies the hypotheses of Theorem 1.3, and so there is a suitable sequence of blowings-up each of which has a centre given by a single point, after which $f$ completely splits in a punctured étale neighbourhood centred at $a$. Moreover, given that $n = \sum_{\ell=1}^{s} n_\ell = \dim X$, by Weierstrass' preparation theorem we can express $f \in \mathbb{K}[\![w, x]\!][z]$. This allows us to express each $f_\ell$ as

$$f_\ell = \prod_{j=0}^{n_\ell - 1} (z - \phi_{\ell,j}),$$

where each $\phi_{\ell,j} \in \mathbb{K}[\![v, x]\!][z]$, $x$ and $z$ are understood to be multi-index variables, and where $v$ is a formal $N$-th root of $w$, for some positive integer $N$. Because each of the factors $f_\ell$ is irreducible, we may assume WLOG that $N = \mathrm{lcm}(n_1, n_2, \ldots, n_s)$. Notice that we have an action of $\mathbb{Z}/N\mathbb{Z}$ on $\mathbb{K}[\![v, x]\!][z]$ that leaves each $f_\ell$ invariant. We can use this action to re-label the sections $\phi_{\ell,j}$ in such a way that if $g$ is a generator of $\mathbb{Z}/N\mathbb{Z}$ then $g^m \cdot \phi_{\ell,j} = \phi_{\ell,j+m}$, where the subindices are considered to be reduced modulo $n_\ell$. We then use the Discrete Fourier Transform with respect to the relevant group action to define,

$$y_{\ell,j} := \frac{1}{n_\ell} \sum_{k=0}^{n_\ell - 1} \varepsilon^{-\frac{N}{n_\ell} kj} (z - \phi_{\ell,k}),$$

where $\varepsilon$ is a primitive $N$-th root of unity. Similarly to the irreducible case, we may notice that

$$v^{N - \frac{jN}{n_\ell}} y_{\ell,j} = \frac{1}{n_\ell} \sum_{k=0}^{n_\ell - 1} \left( \varepsilon^{\frac{N}{n_\ell} k} v^{\frac{N}{n_\ell}} \right)^{n_\ell - j} (z - \phi_{\ell,k}),$$

and consequently, for a generator $g$ of $\mathbb{Z}/N\mathbb{Z}$ we have

$$\begin{aligned} g \cdot v^{N - \frac{N}{n_\ell} j} y_{\ell,j} &= \frac{1}{n_\ell} \sum_{k=0}^{n_\ell - 1} \left( \varepsilon^{\frac{N}{n_\ell}(k+1)} v^{\frac{N}{n_\ell}} \right)^{n_\ell - j} (z - \phi_{\ell,k+1}) \\ &= v^{N - \frac{N}{n_\ell} j} y_{\ell,j}. \end{aligned}$$

Thus, given $\ell \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, n_\ell - 1\}$ there exist $\zeta_{\ell,j} \in \mathbb{K}[\![w, x]\!][z]$ and $m_{\ell,j} \in \mathbb{Z}_{\geqslant 0}$ such that

$$y_{\ell,j} = v^{m_{\ell,j} N + \frac{N}{n_\ell} j} \zeta_{\ell,j}(v^N, x). \tag{5.8}$$

We also define $m_{\ell,0} := 0$ for all $\ell$. We then have that

$$f_\ell = \Delta_{n_\ell} \left( \zeta_{\ell,0}, \left( v^{\frac{N}{n_\ell}} \right)^{m_{\ell,1} n_\ell + 1} \zeta_{\ell,1}, \ldots, \left( v^{\frac{N}{n_\ell}} \right)^{m_{\ell,n_\ell - 1} n_\ell + n_\ell - 1} \zeta_{\ell,n_\ell - 1} \right),$$

for each $\ell \in \{1, \ldots, s\}$. By Corollary 2.93, we have that the marked ideal $(\langle f \rangle, n)$ is equivalent to the marked ideal

$$\sum_{\ell=1}^{s} \sum_{j=0}^{n_\ell - 1} \left( \left\langle w^{m_{\ell,j} N + \frac{N}{n_\ell} j} \zeta_{\ell,j}^{n_\ell} \right\rangle, n_\ell \right).$$

We now need to construct a permutation $\pi$ for this case. This construction, is also carried out sequentially, but the index family is different. Consider the index family given by

$$I := \{(\ell, j) : \ell \in \{1, \dots, s\}, j \in \{0, \dots, n_\ell - 1\}\}.$$

We want to sequentially construct a function $\pi$ that allows us to index the sequential maximal contact subspaces, as this helps us determine the centres of blow-up. The hypothesis we use is that there exists a subset $S_k \subset I$ of size $k$ together with

- a family of smooth sections $\{\zeta^k_{\ell,j} : (\ell, j) \notin S_k\}$,
- a family of exponents $\{\alpha^k_{\ell,j} : (\ell, j) \notin S_k\}$,
- an injective function $\pi : \{0, \dots, k-1\} \to S_k \subset I$

such that

- the subspace $N^k := \{\zeta^1_{\pi(0)} = \dots = \zeta^k_{\pi(k-1)} = 0\}$ is a smooth maximal contact subspace $\underline{\mathcal{I}}$ of codimension $k$,
- the exceptional divisor $w$ does not divide $\zeta^k_{(\ell,j)}$ for all $(\ell, j)$,
- the marked ideals $\underline{\mathcal{I}}$ and $\sum_{(\ell,j)\notin S_k} \left(\left\langle w^{\alpha^k_{(\ell,j)}}(\zeta^k_{\ell,j})^{n_\ell}\right\rangle, n_\ell\right)$ are equivalent,
- each section $\zeta^k_{\ell,j}$ has order $\geqslant 1$ and for each $\ell$ we have that the set of exponents $\{\alpha^k_{\ell,j} : j \in \{0, \dots, n_\ell - 1\}\}$ contains at most one representative for each class of integers modulo $n_\ell$.

The base case is verified by taking $\zeta^0_{\ell,j} := \zeta_{\ell,j}$, with $\alpha^0_{\ell,j} = j + m_{\ell,j} n_\ell$ and $S_1$ defined as the singleton determined by a subindex of the functions $\zeta^0_{\ell,0}$, as any maximal contact hypersurface of $\{f_j = 0\}$ is a maximal contact hypersurface of $\{f = 0\}$. In order to verify that given $k \leqslant |I| - 1$, we can construct $S_{k+1}$ together with the rest of the data, it suffices to follow a similar reasoning as in the irreducible case, *mutatis mutandis*. And so, let us provide the identities satisfied by the analogous constructions for the non-irreducible case, without repeating the arguments for their existence, as they are exactly the same. If the inductive hypothesis is satisfied for some $k < |I|$, we may then define

$$\pi(k) := \arg\min \left\{ \frac{\alpha^k_{\ell,j}}{n_\ell} : (\ell, j) \in I \setminus S_k \right\}.$$

Just as in the irreducible case, we want to restrict to a maximal contact hypersurface. Notice that $\{\zeta^k_{\pi(k)} = 0\}$ is maximal contact, and so we consider the identity

$$\zeta^k_{\ell,j} = \xi^k_{\ell,j} + w^{\beta^{k+1}_{\ell,j}} \zeta^{k+1}_{\ell,j},$$

where $\xi^k_{\ell,j} \in \langle \zeta^0_{\pi(0)}, \dots, \zeta^k_{\pi(k)} \rangle$. This in turn allows us to define

$$\alpha^{k+1}_{\ell,j} := \beta^{k+1}_{\ell,j} + \alpha^k_{\ell,j} - \alpha^k_{\pi(k)}.$$

Let us emphasize that two consecutive indices

$$\pi(k) = (\ell_k, j_k), \qquad \pi(k+1) = (\ell_{k+1}, j_{k+1})$$

do not need to satisfy that $\ell_k = \ell_{k+1}$, as what determines these indices are the exponents $\alpha^{k+1}_{\ell,j}$.

We can then use the function $\pi$ to verify that blowing-up with a centre of the form

$$\{\zeta^0_{\pi(0)} = \zeta^1_{\pi(1)} = \ldots = \zeta^{n-2}_{\pi(n-2)} = 0\},$$

reduces the value $\alpha^{n-2}_{\pi(n-2)}$ by $n_\ell$. Thus, we can successively perform blowings-up of this form until $\alpha^{n-2}_{\pi(n-2)}$ is reduced modulo $n_\ell$, where $n_\ell$ is the first entry of $\pi(n-2)$. In general, we perform blowings-up with centres of the form

$$\{\zeta^0_{\pi(0)} = \zeta^1_{\pi(1)} = \ldots = \zeta^{d}_{\pi(d)} = 0\}, \tag{5.9}$$

until we reduce to the case where $\alpha^d_{\pi(d)}$ is a residue modulo the first entry of $\pi(d)$. We repeat this for $d = n-2, n-3, \ldots, 1$. After cleaning and changing coordinates we obtain that $f$ can be expressed as the product of circulant expressions

$$f = f_1 \cdot \ldots \cdot f_s,$$

where

$$f_j = \det(C(z_j, w^{1/n_j} x_{j,1}, \ldots, w^{(n_j-1)/n_j} x_{j,n_j-1})),$$

which is what we wanted to prove. □

## 5.2 NORMAL FORMS OF LIMITS OF TRIPLE NORMAL CROSSINGS

**Lemma 5.5** (Reduction to circulant normal form of order three, local version). *Let $X$ be a variety defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. Assume that, after a finite sequence of* inv*-admissible blowings-up, the maximum value of* inv *is $(3,0,1,0,1,0,\infty)$ and the stratum $S$ of points with highest* inv *value satisfies that $X$ is generically* nc(3) *on $S$. Let $a \in S$ be such that $X$ is not* nc *at $a$. Let $w_1,\ldots,w_r,u_1,\ldots u_q,x_1,x_2,z$ be an étale coordinate system defined at $a$ where $\{w_1 \ldots w_r = 0$ is the local expression of the exceptional divisor $E$. Then, there is a finite sequence of admissible and equimultiple blowings-up that restrict to an isomorphism over the normal crossings locus after which the limit points of the* nc(3) *locus are products of circulants. Moreover, if $f$ is irreducible then the limit points of the* nc(3) *locus are circulant singularities of order 3.*

*Proof in the irreducible case.*
This proof follows a similar structure as that of the proof of Theorem 1.4.

Following the reasoning in (4.7), the ideal $\langle z-b_0, z-b_1, z-b_2\rangle$ can be generated by the sections $y_0 = z, y_1, y_2$ (see (4.6)). Assume that $\mathrm{inv}_X(a)$ is the highest possible value as in the hypotheses. Then, we can factor a monomial

$$w^\alpha = \prod_{i=1}^r w_i^{\alpha_i}$$

from both $y_1$ and $y_2$ and at least one of the pair $w^{-\alpha}y_1, w^{-\alpha}y_2$ has vanishing order 1. We follow the case where $w^{-\alpha}y_1$ has vanishing order one, and we do not present the other case, as it follows from a similar reasoning. By (3.4) we have that $\alpha_i = 3m_i + k_i$, for some non-negative integer $m_i$ and for some $k_i \in \{0, 1, 2\}$. Similarly to the previous proof, let us express

$$y_2 = w^{\alpha} h_1 + w^{\beta} \zeta_2,$$

for some $\beta$ with $\beta_i \geqslant \alpha_i$ for all $i$, for some regular section $h_1 \in \langle y_1 \rangle$, and for some smooth section $\zeta_2$.

Let $\gamma := \beta - \alpha$ and express $\gamma_i = 3\ell_i + s_i$, where $\ell_i \in \mathbb{Z}_{\geqslant 0}$ and $s_i \in \{0, 1, 2\}$. Let us now provide a local description of smooth centres that, after blowing-up, allow us to reduce to the case where $\sum_{i=1}^{r} \gamma_i < 3$. For this, let $J$ denote the smallest (with respect to cardinality) subset of $\{1, \ldots, r\}$ such that $\sum_{i \in J} \gamma_i \geqslant 3$. In case there are two subsets with equal cardinality satisfying this condition we pick the one with the highest value for $\sum_{i \in J} \gamma_i$; and in case there are two subsets $J_1, J_2$ with same cardinality and with $\sum_{i \in J_1} \gamma_i = \sum_{i \in J_2} \gamma_i$ we select the one with the highest lexicographic order. We then consider the centre

$$C := \{z = y_1 = 0\} \cap \bigcap_{i \in J} \{w_i = 0\}. \tag{5.10}$$

**Remark 5.6.** Notice that this selection for the components of $E$ is similar to the selection done in the J-part of the invariant, but truncated to some codimension level prior to the final one. In other words, the closure of these centres of blow-up can be globally described as the component of

$$\{\mathrm{inv} \geqslant (3, 0, 1)\} \cap \operatorname{cosupp}((\mathcal{M}(\underline{\mathcal{I}}^2), 3)),$$

with highest J-invariant (see Subsection 2.7.4).

Notice that $C$ is an equimultiple centre. Notice also that the only charts of the strict transform that contain the normal crossings locus of order 3 are those associated to the $w$ variables, and by the minimality condition for $J$, we have that the value of $\gamma_i$ in the strict transform is strictly reduced for any $i \in J$.

By sequentially performing blowings-up with centres selected as we mentioned, we can reduce to the case where each chart centred at limit points of the $\mathrm{nc}(3)$ locus intersects at most 2 exceptional divisors, and the power of $\sum_i (\beta_i - \alpha_i) \in \{0, 1, 2\}$. After this, we can now select the smallest subset $I$ of $\{1, \ldots, r\}$ (we use the same disambiguation criteria as before) such that $\sum_{i \in I} \alpha_i \geqslant 3$. We now blow-up with centre

$$C := \{z = 0\} \cap \bigcap_{i \in I} \{w_i = 0\}, \tag{5.11}$$

and sequentially performing these blowings-up allow us to reduce to the case where $\sum_{i=1}^{r} \alpha_i < 3$.

**Remark 5.7.** Similarly to Remark 5.6, the topological closure of the centres of blow-up defined as in (5.11) can be globally described as the component of

$$\{\mathrm{ord} \geqslant (3)\} \cap \operatorname{cosupp}((\mathcal{M}(\underline{\mathcal{I}}^1), 3)),$$

with highest J-invariant.

Thus, after these sequences of blowings-up, we may assume that the local expression of $f$ is one of the following

1. $\det(C(z, y_1, y_2))$,
2. $\det(C(z, w^{1/3}y_1, w^{2/3}y_2))$,
3. $\det(C(z, w^{2/3}y_1, w^{4/3}y_2))$,
4. $\det(C(z, w_1^{1/3}w_2^{1/3}y_1, w_1^{2/3}w_2^{2/3}y_2))$,

Notice that all the singularities of item 1 are $nc(3)$, and that item 2 is a circulant singularity of order 3. Notice that for item 3 we can blow-up one more time with centre $C = \{z = y_1 = w = 0\}$ to reduce to item 2. Finally, notice that we can perform 3 more blow-ups for points with a local expression of the form item 4. The first centre is described by $C_0 = \{z = y_1 = w_1 = w_2 = 0\}$ and if we denote by $E_0$ the exceptional divisor created by blowing-up then the next centres are given by $C_i = \{z = w_i = 0\} \cap E_0$. This allows us to reduce item 4 to the item 2 in all the relevant affine charts. □

**Remark 5.8.** Notice that in the case where $f$ is not irreducible, factorizing $f$ in its irreducible components, we obtain that either $f$ splits into 2 factors, one of order 2, or into 3 factors, all of them of order 1. The latter yields a factorization of the form $nc(3)$. Thus, assume that $f$ factorizes into 2 factors. We can still apply the same reasoning on the orbits of the roots of $f$, allowing us to express $f$ as the product of a smooth section with a generalized circulant of order 2. A similar sequence of blowings-up, that is, blowings-up with centres of the form

$$\{\mathrm{ord} \geqslant 2\} \cap \operatorname{cosupp}((\mathcal{M}(\underline{\mathcal{I}}^1), 2))$$

determined by the highest value of the J-invariant, allows us to reduce to a product of a smooth section with a circulant of order 2.

**Remark 5.9.** Because the centres of blow-up in the previous algorithm are all given by a maximal contact hypersurface and a globally well-defined selection of components of $E$, we may use truncation of inv together with the J-part of the invariant and blow-up admissible centres for this upper-semicontinuous function to deduce Theorem 5.1 at every point of (the strict transform of) $X$ (see Remark 5.6 and Remark 5.7).

**Remark 5.10.** There is a technical generalization of Lemma 4.14 and Lemma 5.5. More precisely, if $X$ satisfies that after an inv-admissible sequence of blowings-up the stratum with highest value of the invariant is $S_{3,r} := \{\mathrm{inv} = \mathrm{inv}(nc(3, r))\}$, then there is a suitable sequence of admissible and equimultiple blowings-up restricting to an isomorphism over the nc locus after which the limit points of the $nc(3)$ locus are the product of the form $\exp^r \times cp(3)$ or $\exp^r \times nc(1) \times cp(2)$. To deduce this, it suffices to notice that the local equation $\{g = 0\}$ defining $X \cup E$ at a limit point of $S_{3,r}$ can be expressed as

$$g = u_1 \ldots u_r f,$$

where $f$ is as in Lemma 5.5. The sequence of blowings-up that lead to the normal forms is almost the same, with the only modification being that we need to blow-up inside $\{u_1 = \ldots = u_r = 0\}$ to ensure that the centres are inside the $r$ components of $E$ indicated by inv.

# 6 PARTIAL DESINGULARIZATION RESULTS

The goal of this chapter is to provide the proofs of Theorem 1.1 and Theorem 1.2, assuming the existence of a moving away algorithm as in Algorithm 1.5. We postpone the proof of Algorithm 1.5 until Chapter 7, as we need multiple versions of it for different specific cases of products of circulants, and the statements indicating the necessary hypotheses are clearer once the procedure for partial desingularization has been explained.

The proof we present for Theorem 1.1 and Theorem 1.2 rely on proving the following claim.

**Claim 6.1.** *Let $X$ be a hypersurface of a smooth variety $Z$ over an algebraically closed field $\mathbb{K}$ of characteristic zero, and let $E \subset Z$ be an* snc *divisor. Given $(p, r) \in \mathbb{N}^2$ there exists a sequence*

$$X' := X_t \xrightarrow{\sigma_t} \dots \xrightarrow{\sigma_1} X_0 := X \tag{6.1}$$

*of admissible and equimultiple blowings-up such that*

1. *the sequence (6.1) restricts to an isomorphism over the locus of* nc *points of order at most $(p, r)$,*
2. *if $(X_0, E_0)$ is* nc *in $U \subset Z$, then the centres $C_k \subset U$ of the sequence (6.1) are the centres given by the classical desingularization algorithm (see [BM97]) until we have $\mathrm{inv} \leqslant \mathrm{inv}(\mathrm{nc}(p, r))$, after this point, the centres are empty,*
3. *all the singularities of the pair $(X_t, E_t)$ are minimal singularities.*

**Remark 6.2.** If $X$ is an $n$-dimensional variety, then the statement of Theorem 1.1 for dimension $n$ follows from Claim 6.1 in the case $(p, r) = (n + 1, 0)$, as the highest possible order for a nc singularity of $X$ is the embedding dimension of $X$, which is $n + 1$.

The proof of Claim 6.1 we present follows an inductive structure over the pairs $(p, r)$ ordered lexicographically.

Notice that Claim 6.1 in the case $(p, r) = (1, 0)$ is a consequence of the classical desingularization theorem (see [BM97]) if we stop before blowing-up the stratum $S_{1,0}$ of points with $\mathrm{inv} = \mathrm{inv}(\mathrm{nc}(1, 0)) := (1, 0, \infty)$. And so, we verify Claim 6.1 in the base case $(p, r) = (1, 0)$. For uniformity of notation, let us define $D_{1,0} = \varnothing$ and $\Sigma_{1,0} := S_{1,0}$.

For the inductive step, we proceed as follows. Assume that we want to verify Claim 6.1 for $(p, r)$. We proceed in *stages*.

*First stage.* We start by following the classical desingularization algorithm until $\mathrm{inv} \leqslant \mathrm{inv}(\mathrm{nc}(p, r))$ at all points. We now split the stratum $S_{p,r} := \{\mathrm{inv} = \mathrm{inv}(\mathrm{nc}(p, r))\}$ into irreducible components, and we blow-up those components that do not contain any $\mathrm{nc}(p, r)$ points. Given that $X$ is nc at $a$ implies that $X$ is nc in a neighbourhood $U$ around $a$, after finitely many steps, we may assume that $S_{p,r}$ only contains irreducible components in which $X$ is generically $\mathrm{nc}(p,r)$. Furthermore, we may assume that the non-nc points in $S_{p,r}$ are inside $E' \subset E$, where $E'$ is transverse to $S_{p,r}$.

*Second stage.* We use a splitting result, to deduce the existence of an inv-admissible blow-up sequence after which, we can split the local equation defining $X$, in a finite extension of $\mathcal{O}_Z$.

*Third stage.* We follow a cleaning procedure, that is, we perform a sequence of admissible and equimultiple blowings-up as in (5.9) (p. 81) after which all limit points of the $\mathrm{nc}(p,r)$ locus reduce to a (circulant) normal form. This sequence of blowings-up is admissible and equimultiple but it is not inv-admissible. Notice that Theorem 1.4 and Theorem 5.1 both account for the application of the second stage followed by the third stage[1].

*Fourth stage.* We follow a moving away procedure to deduce the existence of a sequence of admissible and equimultiple blowings-up after which, there are a strict subdivisor $D_0 \subset E$, which we call *distinguished divisor*, together with an open set $U$ containing $D_0$ such that

- $D_0$ contains all non-$\mathrm{nc}(p,r)$ singularities in $U$,
- all the singularities of $(X,E)$ in $D_0$ are minimal singularities,
- all the singularities of $(X,E)$ in $U \setminus D_0$ are $\mathrm{nc}(p,r)$.

All the centres of blow-up involved in the stages above avoid all nc points of $(X,E)$, and as such, they satisfy item 2 of Claim 6.1. Define $D_{p,r}$ as the union of all the distinguished divisors created in the fourth stage. Define $\Sigma_{p,r}$ as the union of the strict transform of $S_{p,r}$ and $D_{p,r}$. We now apply Claim 6.1 to the pair $(X \setminus \Sigma_{p,r}, E - D_{p,r})$ at the predecessor $(p^-, r^-)$[2] on $X \setminus \Sigma_{p,r}$ to deduce Claim 6.1 for $(p,r)$.

**Remark 6.3.** In stages 3 and 4 we performed blowings-up with centres that might fail to be inv-admissible. As such, in order to resume performing the classical desingularization algorithm so that we may drop the maximal value of inv in $X \setminus \Sigma_{p,r}$ to be at most $(p^-, r^-)$ we need to reset to year zero.

**Remark 6.4.** By construction, the centres of blow-up of the classical desingularization procedure are closed in $X \setminus \Sigma_{p,r}$ but they will, in general, fail to be closed in $X$. To correct this, we blow-up with centre given by the topological closure in $X$. This introduces a new problem. Namely, there are singularities of $(X,E)$ in $\Sigma_{p,r}$ that will be blown-up, and so, we need to verify that, for all the normal forms that admit a curve of singularities which are not nc, all future inv-admissible blowings-up do not introduce new normal forms. We address this problem in Section 7.7 and Section 7.8.

## 6.1 PARTIAL DESINGULARIZATION IN DIMENSION 4

When $\dim X \leqslant 4$, the sequence of pairs that need to be treated in the strategy of partial desingularization we presented in the introduction to this chapter is $(5,0), (4,1), (4,0), (3,2), (3,1), (3,0), (2,3), \dots, (2,0), (1,4), \dots, (1,0)$. Let us present the details of the first iterative steps as in out strategy, and provide the general argument for the cases $\leqslant (3,1)$.

1 Recall that in the proof of these theorems, we begin by applying a splitting theorem.

2 We define the predecessor of $(p,r)$ as $(p, r-1)$ if $r > 0$ or $(p-1, r')$ for some $r' \geqslant 0$ in the case $r = 0$.

Let $X$ be a variety over an algebraically closed field $\mathbb{K}$ of characteristic zero of dimension 4. Because the normal crossings locus of $X$ is contained in the collection of points where the embedding dimension of $X$ is 5, we can follow the classical desingularization procedure where we blow-up the stratum of points with maximal value of inv until the first entry of inv given by the Hilbert-Samuel function $H$ satisfies $H(1) = 5$, at all points of the strict transform of $X$. Thus, we may assume that the ideal sheaf $\mathcal{I}$ associated to $X$ is principal.

Notice that the maximal possible value of a nc point $a$ is

$$\mathrm{inv}(\mathrm{nc}(5)) = (5, 0, 1, 0, \dots, 1, 0, \infty),$$

where the invariant has 11 entries. Thus, we may continue blowing-up the stratum with highest inv-value until the stratum $S_{5,0}$ of points with inv value equal to $\mathrm{inv}(\mathrm{nc}(5, 0))$ is the stratum with highest inv value. Notice also that, by the construction of inv and $S_{5,0}$, we have that $S_{5,0}$ has codimension 5 in $Z$. Thus, $\dim S_{5,0} = 0$. This implies that $\{a \in S_{5,0} : a \notin \mathrm{nc}(5)\}$ is a discrete collection of points of $S$. Notice also that if $a$ is $\mathrm{nc}(5)$, then there is an open neighbourhood $U$ of $a$ such that the only singularities of $X$ in $U \setminus \{a\}$ are nc of order $< 5$. Consequently, we can continue blowing-up the stratum of points with maximal value of inv outside the $\mathrm{nc}(5)$ locus. For homogeneity of notation, let us define $D_{5,0} := \varnothing$, $T_{5,0} := S_{5,0}$, and $\Sigma_{5,0} := T_{5,0} \cup D_{5,0}$.

We continue blowing-up the stratum with maximal value of inv in $X$ outside $\Sigma_{5,0}$ until the maximal value of the invariant is $\mathrm{inv}(\mathrm{nc}(4, 1))$. Notice that the locus of points with value of the invariant equal to $\mathrm{inv}(\mathrm{nc}(4, 1))$ is zero-dimensional, and so we can blow-up only those isolated points that are not in the $\mathrm{nc}(4, 1)$ locus. Similarly as before, there is an open neighbourhood $U$ around each $\mathrm{nc}(4, 1)$ point $a$ such that (the strict transform of) $X$ is nc in $U \setminus \{a\}$. We define $S_{4,1}$ as the stratum of points outside $\Sigma_{5,0}$ with inv value equal to $\mathrm{inv}(\mathrm{nc}(4, 1))$, $T_{4,1} := S_{4,1} \cup T_{5,0}$, $D_{4,1} := D_{5,0}$, and $\Sigma_{4,1} := T_{4,1} \cup D_{4,1}$.

We have now reached a case that exemplifies the general structure of our argument. We begin by applying the classical desingularization algorithm outside $\Sigma_{4,1}$ until the stratum with maximal value of inv is $S_{4,0}$. Furthermore, we can continue blowing-up the locus of points with maximal value outside the locus of $\mathrm{nc}(4, 0)$ singularities, which is open in $S_{4,0}$. This allows us to reduce to the case where $S_{4,0}$ is generically $\mathrm{nc}(4, 0)$. Notice that the locus of non-$\mathrm{nc}(4)$ points inside the stratum $S_{4,0}$ of points with inv value equal to $\mathrm{inv}(\mathrm{nc}(4, 0))$ is discrete. By Theorem 1.4, for each non-nc point $a \in S_{4,0}$, and after a suitable sequence of admissible and equimultiple blowings-up, we may express (the strict transform of) $X$ (locally at the point $a'$ over) $a$ as the product of circulant expressions with orders $n_1, \dots, n_s$ (see Remark 5.1), where $\sum_{k=1}^{s} n_k = n$. In this particular case, we obtain that the normal forms for the limits of the $\mathrm{nc}(4, 0)$ locus are

- $\mathrm{cp}(4)$,
- $\mathrm{cp}(2) \times \mathrm{cp}(2) : \Delta_2(z_1, w^{1/2}x_1)\Delta_2(z_2, w^{1/2}x_2)$.
- $\mathrm{nc}(1) \times \mathrm{cp}(3)$,

We address a moving away algorithm for each of the above:

- $\mathrm{cp}(4)$: Section 7.4,
- $\mathrm{cp}(2) \times \mathrm{cp}(2)$:Section 7.6,
- $\mathrm{nc}(1) \times \mathrm{cp}(3)$: Section 7.7, case $r = 1$.

In the case of $\mathrm{cp}(4)$, the moving away algorithm finds a sequence of admissible and equimultiple blowings-up after which, the limit singularities of the neighbouring locus of $\mathrm{cp}(4)$ can be expressed, in some local coordinate system of the form $x_1, x_2, x_3, y, z$, as the vanishing locus of one of the following:

- $\mathrm{cp}(4) : \Delta_4(z, y^{1/4}x_1, y^{2/4}x_2, y^{3/4}x_3)$
- $\Delta_4(z, y^{1/4}x_1, y^{2/4}x_2, y^{3/4})$,
- $\Delta_4(z, y^{1/4}x_1, y^{2/4}, y^{3/4}x_2x_3)$,
- $\Delta_4(z, y^{1/4}x_1, y^{2/4}, y^{3/4}x_3)$,
- $\Delta_4(z, y^{1/4}x_1, y^{2/4}, y^{3/4})$,

where the latter 4 are analogues of the degenerate pinch-point in dimension 4. Notice that, for each of the normal forms above, $X$ is nc at every point outside of $\{y = 0\}$. Moreover, the moving away procedure is such that $\{y = 0\}$ is the local equation of a component $D_0$ of the exceptional divisor $E$.

In the case of $\mathrm{nc}(1) \times \mathrm{cp}(3)$ the moving away procedure (see Section 7.7 in the case $r = 1$) helps us express $X$ in some local coordinate system $u, x_1, x_2, y, z$ as:

- $\mathrm{nc}(1) \times \mathrm{cp}(3) : u\Delta_3(z, y^{1/3}x_1, y^{2/3}x_2)$
- $\mathrm{cp}(3) : \Delta_3(z, y^{1/3}x_1, y^{2/3}x_2)$,
- $\mathrm{dpp} : \Delta_3(z, y^{1/3}x_1, y^{2/3})$,
- $\mathrm{nc}(1) \times \mathrm{dpp} : u\Delta_3(z, y^{1/3}x_1, y^{2/3})$.

Again, the moving away algorithm is such that $\{y = 0\}$ is the local expression of a component $D_0$ of the exceptional divisor $E$, and all singularities outside $\{y = 0\}$ are nc.

In the case of $\mathrm{cp}(2) \times \mathrm{cp}(2)$, the moving away procedure (see Section 7.6) leads us to two more normal forms:

- $\mathrm{nc}(1) \times \mathrm{cp}(2)$,
- $\mathrm{cp}(2)$.

Similarly, as in the other cases, the normal forms obtained by the moving away algorithm are contained in an exceptional divisor $D_0 \subset E$.

We define $\mathscr{D}_{4,0}$ as the union of the components of $E$ that contain the normal forms (that is, those divisors whose local expressions we denoted by $\{y = 0\}$), we also define $T_{4,0} := S_{4,0} \cup T_{4,1}$, $D_{4,0} := \mathscr{D}_{4,0} \cup D_{4,1}$ and $\Sigma_{4,0} := T_{4,0} \cup D_{4,0}$.

We then move on to the next pair $(3, 2)$. We resume the classical desingularization algorithm outside of $\Sigma_{4,0}$(by resetting to year zero) until inv drops to be at most inv(nc(3, 2)). The stratum $S_{3,2}$ is discrete, and so we may resolve all non-nc points. Again, we define $T_{3,2} := S_{3,2} \cup T_{4,0}$, $D_{3,2} := D_{4,0}$ and $\Sigma_{3,2} := T_{3,2} \cup D_{3,2}$.

We then move on to the pair $(3, 1)$, and we follow the 4 stages of the process. There are no new normal forms found by Theorem 5.1 nor the moving away algorithm. Notice that when we move on to the pair $(3, 0)$ the issue of having limits of the stratum $S_{3,0}$ inside $\Sigma_{3,0}$ appears. Nonetheless, we take care of this issue at the end of sections Section 7.7 and Section 7.8.

We continue this process for the pairs $(3, 0), (2, 3), \ldots, (1, 4), \ldots, (1, 0)$. The only new normal form found by the moving away algorithm (see Section 7.7 and Section 7.8) is in the case $(2, 2)$, where we add the normal form $\mathrm{nc}(2) \times \mathrm{cp}(2)$.

Putting all these normal forms together we obtain Table 1.1, thus completing the proof of Theorem 1.1.

## 6.2 PARTIAL DESINGULARIZATION PRESERVING TRIPLE NORMAL CROSSINGS

*Proof of Theorem 1.2.* The proof follows a similar structure to that of Theorem 1.1. More precisely, we blow-up with inv-admissible centres until the stratum $S$ with highest inv-value is generically nc of order $(p, r)$ with $p \leqslant 3$. We then apply Theorem 5.1 to express all the non-nc points in $S$ as $\exp^r \times \mathrm{cp}(3)$ or $\exp^r \times \mathrm{nc}(1) \times \mathrm{cp}(2)$. We then follow the moving away procedure stated in Section 7.7 (for the former) or Section 7.8 (for the latter)[3] Let $\mathscr{D}_{3,r}$ be the union of all the exceptional divisors that admit a deleted neighbourhood in which all singularities of $(X, E)$ are nc, as indicated by the moving away procedure, and define $\Sigma_{3,r} := S_{3,r} \cup \mathscr{D}_{3,r}$. We then apply Claim 6.1 to $(X \setminus \Sigma_{3,r}, E - D_{3,r})$. By the remarks at the end of each of those two sections, future blow-up whose centre is the closure of an inv-admissible centre outside $\Sigma_{3,r}$ will necessarily preserve the local expression of the minimal singularities we have established. Thus, we obtain the desired result.

□

3 Notice that the moving away sequence of $\exp^r \times \mathrm{nc}(1) \times \mathrm{cp}(2)$ is the same as the moving away sequence of a singularity of the form $\exp^{r+1} \times \mathrm{cp}(2)$.

# 7

# MOVING AWAY ALGORITHM AND NORMAL FORMS

The main goal of this chapter is to provide the details of Algorithm 1.5. Given that this algorithm needs to desingularize a local expression of the form $\{\Delta_n = 0\}$, we first find a simpler ideal that allows us to determine centres of blow-up in a more straightforward way. More precisely, we replace the marked ideal sheaf $(\langle\Delta_n\rangle, n)$ associated to a circulant singularity by a (marked) monomial ideal sheaf $(\mathcal{M}, n)$. We then show the existence of a desingularization procedure for the ideal $(\mathcal{M}, n)$

Given that we need to deduce explicit local normal forms, we need to apply Algorithm 1.5 to the cases that are relevant in Theorem 1.1 (and Theorem 1.2). Thus, we provide the explicit sequence of blowings-up leading to the normal forms of limits of the neighbours of $\mathrm{cp}(4)$ in Section 7.4. For varieties of dimension $\leqslant 4$ we also need to find normal forms for the limits of the neighbours of singularities which are locally of the form $\mathrm{cp}(2) \times \mathrm{cp}(2)$ (see Section 7.6).

Finally, because part of our procedure involves performing inv-admissible blowings-up after having performed equimultiple (but not necessarily inv-admissible) blowings-up, we need to reset the desingularization history to year zero, but we need to verify that blowing-up these strata does not modify the normal forms that we have previously obtained. And so, for those normal forms that admit a curve of singularities that are not nc that could be blown-up in subsequent years we verify that the local expression does not get modified. These details are provided in Section 7.7 for singularities that are locally of the form $\exp^r \times \mathrm{cp}(3)$ and in Section 7.8 for singularities that are locally of the form $\exp^r \times \mathrm{cp}(2)$ and $\exp^r \times \mathrm{nc}(1)$.

## 7.1 LOCAL EQUATION AT SINGULARITIES RELEVANT TO MOVING AWAY

A key intermediate step in our moving away algorithm is to be able to reduce the local expression of X to one of the following.

**Definition 7.1.** [Raw family of local equations, $\overline{N}$] Consider a hypersurface X of a smooth variety Z over an algebraically closed field $\mathbb{K}$ of characteristic zero, with $\dim Z = n+1$. We say that $a \in X$ is a singularity in the *raw family of local equations* $\overline{N}$ if there exist smooth functions[1] $w_1, \dots, w_r, x_1, \dots, x_{n-1}, y, z$ satisfying that the local equation of X at $a$ is

$$\Delta_n(z, y^{1/n}x_1, y^{2/n}x_2, \dots, y^{(d-1)/n}x_{d-1}, y^{d/n}, \\ y^{(d+1)/n}w^{p_{d+1}}x_{d+1}^{\delta_{d+1}}, \dots, y^{(n-1)/n}w^{p_{n-1}}x_{n-1}^{\delta_{n-1}})) = 0,$$

1 some of the functions $x_k$ may fail to vanish at $a$

where $d+r<n$, $p_{d+1},\dots,p_{n-1}\in(\mathbb{Z}_{\geqslant 0})^r$, and $\delta_j\in\{0,1\}$ with $\delta_j=1$ if and only if $x_j$ vanishes at $a$, and

$$\{w_1,\dots,w_r,y,x_0,x_1,\dots,x_{d-1}\}\cup\{x_j : d<j<n \text{ with } \delta_j=1\},$$

can be completed to a étale coordinate system at $a$.

**Remark 7.2.** Notice that the vanishing order of a singularity in $\overline{N}$ equals the power of the monomial with the lowest degree that is only divisible by $y$.

**Remark 7.3.** Notice also that if $U$ is an affine chart that realizes $a=0\in U$ as a singularity in $\overline{N}$ of order $d$, and if $C\subset U$ is such that $C\not\subset Y_j$ for some $j<d$, then the maximal value of the vanishing order of $\mathcal{I}$ along $C$ is at most $j$.

**Remark 7.4.** Let $a\in X$ be such that $a$ is a singularity in $\overline{N}$ and $a\notin\{y=0\}$. Because $a\notin\{y=0\}$ we can find an étale neighbourhood $V$ of $a$ such that the local expression of $\mathcal{I}$ in $V$ is

$$\langle\det(C(z,vx_1,v^2x_2,\dots,v^{d-1}x_{d-1},v^d,\dots))\rangle,$$

where $v^n=y$. We claim that $\mathrm{Sing}\,(X)\cap V\setminus\{v=0\}$ is a subset of the normal crossings locus of order at most $d$. To verify this, define

$$F_j(y,x_1,\dots,x_{d-1},z,\dots)=z+(\varepsilon v)^j x_1+\dots+(\varepsilon v)^{(d-1)j}x_{d-1}+\dots,$$

where $j\in\{0,\dots,n-1\}$. Because the vanishing order of

$$\det(C(z,vx_1,v^2x_2,\dots,v^{d-1}x_{d-1},v^d,\dots))$$

is at most $d$, there exist at most $d$ functions $F_{j_1},\dots,F_{j_d}$ that vanish at $a$. Notice also that the matrix of partial derivatives of

$$F(y,x_1,\dots,x_{d-1},z,\dots):=(F_0,\dots,F_{n-1})$$

with respect to $z,x_1,\dots,x_{d-1}$ is

$$\begin{pmatrix} 1 & v & v^2 & \dots & v^{d-1}\\ 1 & \varepsilon v & \varepsilon^2 v^2 & \dots & \varepsilon v^{d-1}\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 1 & \varepsilon^{n-1}v & \varepsilon^{2(n-1)}v^2 & \dots & \varepsilon^{(n-1)^2}v^{(d-1)} \end{pmatrix},$$

which is a matrix of maximal rank as $v\neq 0$. Using the Implicit Function Theorem, we obtain that $\{\{F_{j_1}=0\},\dots,\{F_{j_d}=0\}\}$ are normal crossings.

**Remark 7.5.** Fix a combination $(p_{d+1},\dots,p_{n-1},\delta_{d+1},\dots,\delta_{n-1})$, and fix an affine chart $U$ in which a coordinate system $w,x,y,z$ as in Definition 7.1 is defined. Notice that the local expression at all singularities of $X$ in $U$ can be parametrized by the coordinates in $\{x_2,\dots,x_{n-1}\}$ which vanish in $U$.

**Definition 7.6** (Neighbouring singularity)**.** Let $X$ be a variety, fix $a \in \mathrm{Sing}\,(X)$. We say that $b \in \mathrm{Sing}\,(X)$ is a *neighbouring singularity* of $a$ if for every open neighbourhood $U$ of $a$ there exist $c \in \mathrm{Sing}\,(X) \cap U$ and an étale isomorphism

$$\mathcal{O}_{X,b} \to \mathcal{O}_{X,c}.$$

Equivalently, if on every neighbourhood $U$ of $a$ we can find a representative of the equivalence class of étale isomorphisms of the stalk $\mathcal{O}_{X,b}$.

**Example 7.7** (Local expression of neighbouring singularities of $\mathrm{cp}(4)$)**.** Let $X$ be the complex variety given by the zero locus of $\Delta_4(y, x_1, x_2, x_3, z)$. Let $a$ be a non-zero singularity of $X$ inside $\{y = 0\}$. We claim that $a$ is a neighbouring singularity of the origin. Let us first notice that in order for $a$ to be a singular point we have that $a = (0, 0, \lambda_0, \mu_0, 0)$ for some $\lambda_0, \mu_0 \in \mathbb{C}$. Assume that $\lambda_0, \mu_0$ are both non-zero. Notice that we can also describe $X$ as the vanishing locus of

$$\frac{x_2^{12}}{x_3^8}\Delta_4(y', x_1', 1, 1, z'),$$

where $y' = \dfrac{x_3^4}{x_2^4}y$, $x_1' = \dfrac{x_3}{x_2^2}x_1$, $z' = \dfrac{x_3^2}{x_2^3}z$. This shows that the stalks of $X$ of singularities outside all coordinate subspaces but inside $\{y = 0\}$ are equivalent up to étale isomorphism. Similar computations allow us to find the normal forms described by

$$\Delta_4(y, x_1, x_2, 1, z) \text{ and } \Delta_4(y, x_1, 1, x_3, z).$$

While the family $\overline{N}$ is infinite (there is one element per combination $(p_{d+1}, \dots p_{n-1}, \delta_{d+1}, \dots, \delta_{n-1})$), the moving away algorithm we present satisfies that after performing it, all limit singularities of the neighbouring locus of $\mathrm{cp}(n)$ belong to a collection of finitely many elements of $\overline{N}$.

We may notice that performing the necessary computations for maximal contact subspaces of $\Delta_n$ and its subsequent companion ideals is a considerably difficult task, as we need to consider arbitrary years in a blow-up sequence and arbitrary values of $n$. Because of this problem, we reduce the problem of finding such an algorithm for $\Delta_n$ to finding a similar algorithm but for a specific monomial ideal that we can construct using $\Delta_n$.

## 7.2 REDUCTION TO A MONOMIAL IDEAL

In this subsection we consider $X$ to be a variety of dimension $n + 1 \geqslant 2$ such that for any $b \in X$ we have

$$\mathrm{inv}(b) \leqslant \mathrm{inv}(\mathrm{nc}(n)),$$

where the latter expression has $2n + 1$ entries. Notice that $\mathrm{nc}(n)$ has dimension at most 1, as $\dim X = n + 1$, and by Theorem 1.3 we have that the $\mathrm{cp}(n)$ locus has dimension 0.

Let $a$ be a $\mathrm{cp}(n)$ singularity of $X$ and let $U$ be an affine chart centred at $a$ expressing $X$ as the vanishing locus of

$$\Delta := \Delta_n(C(z, w^{1/n}x_1, \dots, w^{(n-1)/n}x_{n-1})) = 0. \tag{7.1}$$

In particular, notice that the origin is given by intersection of $n+1$ divisors, 1 of which corresponds to an exceptional divisor. Let $Y_\ell$ denote the locus $\{x_\ell = 0\}$, let $Y_0$ denote the locus $\{z = 0\}$ and let $W$ denote the divisor associated to $w$. For simplicity of notation, if $\sigma$ is a blow-up we use the same notation for the strict transform of each of these divisors by $\sigma$.

Notice that we can define a ramified $n$-covering of $U \simeq \mathbb{A}^1$ given by

$$\begin{aligned} \pi_0 : C' := \mathbb{A}^1 &\to C := \mathbb{A}^1 \\ v &\mapsto v^n. \end{aligned}$$

Using $\pi_0$ we can construct a ramified $n$-covering

$$\begin{aligned} \pi : \mathbb{A}^{n+1} &\to \mathbb{A}^{n+1} \\ (v, x, z) &\mapsto (v^n, x, z) \end{aligned}$$

**Remark 7.8.** We formally define the vanishing order of $v$ at the origin as $\frac{1}{n}$, and extend this definition to the algebra $\mathbb{K}[\![v, x, z]\!]$. This allows us to express the vanishing order of any function in the variables $w, x, z$ in terms of the vanishing order of $v, x, z$.

**Remark 7.9.** Notice that the pullback $X_n$ of $X \hookrightarrow Z$ by $\pi$ factors into $n$ irreducible components

$$\Delta = f_0 \dots f_{n-1},$$

where $f_j := z + \varepsilon^j v x_1 + \dots + \varepsilon^{j(n-1)} v^{n-1} x_{n-1}$.

Define $\underline{\mathcal{I}} := (\langle \Delta \rangle, n)$ and notice,

$$\begin{aligned} \operatorname{cosupp}(\underline{\mathcal{I}}) &= \operatorname{cosupp}\left(\left(\left\langle \prod_{j=0}^{n-1} f_j \right\rangle, n\right)\right) \\ &= \operatorname{cosupp}\left(\prod_{j=0}^{n-1} (\langle f_j \rangle, 1)\right) \\ &= \bigcap_{j=0}^{n-1} \operatorname{cosupp}(\langle f_j \rangle, 1) && \text{by Corollary 2.93} \\ &= \operatorname{cosupp}\left(\sum_{j=0}^{n-1} (\langle f_j \rangle, 1)\right) && \text{by Remark 2.91} \\ &= \operatorname{cosupp}((\langle f_0, \dots, f_{n-1} \rangle, 1)) \\ &= \operatorname{cosupp}(\langle f_0, \dots, f_{n-1} \rangle) && (7.2) \end{aligned}$$

On the other hand, notice that

$$\frac{1}{n} \sum_{j=0}^{n-1} \varepsilon^{-j\ell} f_j = v^\ell x_\ell.$$

In other words, the vector $(z, vx_1 v^2 x_2, \dots, v^{n-1}x_{n-1})^\top \in \mathbb{K}[v,x,z]^n$ can be obtained by multiplying the vector $(f_0, \dots, f_{n-1})^\top$ by an invertible $n \times n$ matrix with coefficients in $\mathbb{K}$. Therefore,

$$\begin{aligned}\operatorname{cosupp}(\underline{\mathcal{I}}) &= \operatorname{cosupp}((\langle z, vx_1, \dots, v^{n-1}x_{n-1}\rangle, 1)) \\ &= \operatorname{cosupp}((\langle z^n, wx_1^n, \dots, w^{n-1}x_{n-1}^n\rangle, n)).\end{aligned} \tag{7.3}$$

Notice that if we blow-up the ideal $\underline{\mathcal{I}}$ and the monomial ideal

$$\underline{\mathcal{M}} := (\langle z^n, wx_1^n, \dots, w^{n-1}x_{n-1}^n\rangle, n)$$

by a coordinate subspace inside $\operatorname{cosupp}(\underline{\mathcal{M}})$ then the cosupports of $\underline{\mathcal{I}}'$ and $\underline{\mathcal{M}}'$ are also the same. This identity is preserved after any finite sequence of blowings-up with equimultiple centres given by coordinate subspaces.

**Proposition 7.10.** *Let $X$ be a variety embedded in a smooth space $Z$, and let $U$ be an affine chart with a regular coordinate system $w, x_1, \dots, x_{n-1}, z$ on $U$. Assume that the local expression of the ideal sheaf $\mathcal{I}$ associated to $X$ is*

$$\mathcal{I}(U) = \langle \Delta' := \det(C(m_0, \dots, m_{n-1}))\rangle,$$

*where $m_0, \dots, m_{n-1}$ are monomials in $w^{1/n}, x_1^{1/n}, \dots, x_{n-1}^{1/n}, z^{1/n}$. Define $\mathcal{M}$ as the ideal sheaf defined on $U$ given by*

$$\langle m_0^n, m_1^n, \dots, m_{n-1}^n\rangle. \tag{7.4}$$

*If $d$ denotes the vanishing order of $\mathcal{I}$ at $a$ then,*

$$\operatorname{cosupp}((\mathcal{I}, d)) \cap U = \operatorname{cosupp}((\mathcal{M}, d)) \cap U.$$

*Proof.*
For each $j$, define

$$f_j := \sum_{k=0}^{n-1} \varepsilon^{jk} m_k,$$

where $\varepsilon$ is a primitive $n$-th root of unity.

Our first claim is that these functions have a common maximal vanishing order. Assume that $d_0$ is the maximal vanishing order of $f_0$. Notice that if $m$ is a monomial of $f_0$ such that its associated coefficient is not zero, then $m$ is also a monomial of $f_j$ with non-zero associated coefficient.

We know that

$$\Delta' = \prod_{j=0}^{n-1} f_j,$$

thus,

$$\begin{aligned}\operatorname{cosupp}((\langle\Delta'\rangle, d)) &= \operatorname{cosupp}\left(\left(\left\langle\prod_{j=0}^{n-1} f_j\right\rangle, d\right)\right)\\ &= \operatorname{cosupp}\left(\prod_{j=0}^{n-1}(\langle f_j\rangle, d/n)\right)\\ &= \bigcap_{j=0}^{n-1}\operatorname{cosupp}\left(\langle f_j\rangle, d/n\right) && \text{by Corollary 2.93}\\ &= \operatorname{cosupp}\left(\sum_{j=0}^{n-1}\left(\langle f_j\rangle, d/n\right)\right) && \text{by Remark 2.91}\\ &= \operatorname{cosupp}((\langle f_0,\dots,f_{n-1}\rangle, d/n)),\end{aligned}$$

But we can use a similar identity as before:

$$\frac{1}{n}\sum_{j=0}^{n-1}\varepsilon^{-j\ell}f_j = m_\ell.$$

Therefore,

$$\begin{aligned}\operatorname{cosupp}((\langle\Delta'\rangle, d)) &= \operatorname{cosupp}((\langle m_0, m_1,\dots,m_{n-1}\rangle, d/n))\\ &= \operatorname{cosupp}((\langle m_0^n, m_1^n,\dots,m_{n-1}^n\rangle, d)).\end{aligned}\tag{7.5}$$

□

**Remark 7.11.** Notice in particular that at points satisfying the hypotheses of Proposition 7.10, the local candidates for a centre of an equisingular blow-up are given by coordinate subspaces. Moreover, if $\sigma$ is a blow-up whose centre $C$ is equimultiple, then we can cover the strict transform $X'$ by affine charts $\{U_\alpha\}$ satisfying that the strict transform of the coordinates $w, x_1, \dots, x_{n-1}, z$ together with the newly created exceptional divisor $E$ form an étale coordinate system that still satisfies the hypotheses of Proposition 7.10.

In the paper [BM06], the authors define an algorithm that finds a resolution of toric singularities. The main idea behind this resolution is to reduce this problem to the problem of finding a resolution of singularities of monomial ideals. In particular, in Theorem 8.5 (p. 32-36), the authors establish that we can use a simplified version of the desingularization invariant (see Subsection 2.7.4) that does not depend on the previous history of blow-ups to find a resolution of singularities of monomial ideals. More precisely, this *monomial desingularization invariant*, which we denote by minv is the same as inv after removing the entries $s_k$. Because this algorithm is applied to monomial ideals, we can assume that in each of the affine charts in which we perform our computations, the local centre of maximal value of minv is a coordinate subspace. In particular, the stratum with maximal value of the invariant passes through the origin of each of the standard affine charts that it intersects.

Notice that Proposition 7.10 makes the subsets given by coordinate subspaces defined by the monomial ideal $\underline{\mathcal{M}}$ good candidates for centres of blow-up. Nonetheless, an arbitrary centre determined by the ideal $\underline{\mathcal{M}}$ does not satisfy the condition of preserving the closure of the normal crossings locus, so we need to impose certain restrictions which we discuss in the next subsection.

**Remark 7.12.** Because we want to perform blowings-up that are contained inside $D_0$, instead of performing (partial) monomial desingularization to $\underline{\mathcal{M}}$ directly, we instead perform (partial) monomial desingularization to the monomial ideal *plus boundary*

$$\underline{\mathcal{N}} := \underline{\mathcal{M}} + (Z, Z, E, \langle y \rangle, 1).$$

The reason behind this is that, in the charts where this is possible, we want to separate $\text{cosupp}(\underline{\mathcal{M}})$ from $D_0$, and in any other chart we want to express the origin in normal form.

We finish this subsection by providing a couple of useful remarks about the powers of generators of exceptional divisors defining $\mathcal{M}$.

**Remark 7.13.** Notice that the couple $(U, a)$, where $U$ is the affine chart we use to express $a$ in circulant form, satisfies that for any local coordinate $s$ of $U$, there exist two non-negative integers $D, R$ such that if $p_j$ denotes the power of $s$ in the monomial at the $j$-th position of the ideal $\mathcal{M}$ then for all $j$ we have

$$p_j \equiv jD + R \mod n.$$

This is a consequence of the fact that $\text{cp}(n)$ singularities satisfy this, and this fact is preserved by any equimultiple blow-up.

## 7.3 NORMAL FORMS ASSOCIATED TO THE CIRCULANT LOCUS

In this section, $X$ denotes an $n$-dimensional variety that can be locally embedded into an $(n+1)$-dimensional smooth ambient space $Z$, and $\mathcal{I}$ denotes the ideal sheaf associated to $X$. Moreover, we assume that for any point $b \in X$ we have that either

- $b \in \text{nc}(n+1)$, or
- $\text{inv}(b) \leqslant \text{inv}(\text{nc}(n))$.

*Proof of Algorithm 1.5.*
Notice that $\text{cp}(n) \subset X$ is a collection of isolated singularities. Thus, it suffices to show that the theorem holds for a single element $a \in \text{cp}(n)$. Recall that $a$ can be described as the intersection of the closure of the $\text{nc}(n)$ locus with some exceptional divisor, which we denote by $F_{-1}$. As we mentioned in the introduction to this chapter, we begin by performing an initial blow-up centred at $a$,

$$\sigma_0 : X' \to X.$$

This creates an exceptional divisor, which we call $D_0$. Because this centre of blow-up is given by the intersection $\cap_{k=0}^{n-1} Y_k \cap W$, (the strict transform of) $Y_0, \ldots, Y_{n-1}, W$ are well-defined on all of the standard affine charts associated to this blow-up, allowing us to treat these loci as divisors.

**Remark 7.14.** Notice in particular that any further sequence of blowings-up whose centre is inside (the strict transform of) $D_0$ preserves the strict transform of any nc singularity of $X$.

We now proceed to describe inductively a sequence of blowings-up, as well as a condition that indicates when there are no more valid centres of blow-up, such that,

1. the sequence is finite,
2. at each step, the centre of blow-up is smooth and equimultiple,
3. if $\sigma$ denotes the composition of the blowing-up maps in the sequence, all the singularities in a punctured neighbourhood of the strict transform (under $\sigma$) of $D_0$ are nc.
4. all the singularities inside the strict transform of $D_0$ can be covered by affine charts, each of which is centred at a singularity in normal form.

Notice that for each standard affine chart $U$ from the open cover of the strict transform $X'$ of $X$ under $\sigma_0$ we are able to express the ideal sheaf $\mathcal{M}$ in $U$ as an ideal generated by $n$ monomials of the form

$$\mathcal{M}|_U = \langle m_0, m_1 y, m_2 y^2, \dots, m_{n-1} y^{n-1} \rangle,$$

where $y$ denotes the local expression of $D_0$ in $U$. For example, the standard affine charts associated to the initial blow-up $\sigma_0$ can be described as $U(w)$, $U(x_j)$ for each $j \in \{1, \dots, n-1\}$ and $U(z)$. In $U(w)$, the local expression of $\mathcal{M}$ is

$$\langle z^n, x_1^n y, x_2^n y^2, \dots, x_{n-1}^n y^{n-1} \rangle.$$

In $U(z)$, the local expression of $\mathcal{M}$ is

$$\langle 1, x_1^n w y, x_2^n w^2 y^2, \dots, x_{n-1}^n w^{n-1} y^{n-1} \rangle.$$

For a given $j \in \{1, \dots, n-1\}$, the local expression of $\mathcal{M}$ in $U(x_j)$ is

$$\langle z^n, x_1^n w y, \dots, x_{j-1}^n w^{j-1} y^{j-1}, w^j y^j, x_{j+1}^n w^{j+1} y^{j+1}, \dots, x_{n-1}^n w^{n-1} y^{n-1} \rangle.$$

Notice that $\operatorname{cosupp}(\mathcal{M}|_{U(z)}) = \varnothing$, and so it is a chart in which we do not need to continue to perform blowings-up. We call such charts *irrelevant*. Let us provide a precise definition of all the charts we consider irrelevant.

**Definition 7.15** (Irrelevant Chart)**.** Given a standard chart $U$ of a (the strict transform $X'$ under a blow-up sequence of a) variety $X$, we say that $U$ is an *irrelevant chart for* $X$ if one of the following conditions is satisfied

- $a = 0 \in U$ is a singularity in $\overline{N}$,
- $a = 0 \in U$ is a $\mathrm{cp}(n)$ point of $X$,
- (the strict transform of) $D_0$ does not intersect $U$,

- (the strict transform of) X does not intersect U.

We perform partial desingularization to the marked monomial ideal plus boundary given by

$$\underline{\mathcal{N}}_n = (U, U, E, \mathcal{M}, n) + (U, U, E, y, 1),$$

where $(U, U, E, \mathcal{M}, n)$ is a marked ideal with maximal vanishing order. For each non-irrelevant standard chart U, compute the maximal value of minv of $\underline{\mathcal{N}}$ at points inside U, together with the respective local centre of blow-up $C_U$ inside U.

In this case, the points with maximal value for the desingularization invariant can be found in $U(w)$, but we may notice that $U(w)$ is an irrelevant chart for X and so we need to compute the values of minv in the rest of the standard affine cover. In particular, we claim that the locus of points with maximal value of minv of $\underline{\mathcal{N}}$ inside the union

$$U(x_1) \cup U(x_2) \cup \ldots \cup U(x_{n-1})$$

is the intersection

$$C_1 := F_{-1} \cap D_0 \cap Y_0 \cap \ldots \cap Y_{n-2}.$$

Consider the origin $a_j \in U(x_j)$, and notice that

$$\operatorname{minv}_{\underline{\mathcal{N}}}(a_j) = (1, 1, \ldots, 1, 0),$$

where the total length of the invariant is $j+1$. In particular, we have that the locus of points with maximal value of the desingularization invariant is in $U(x_{n-1})$ and the centre is locally described by

$$C_1 = \{y = z = x_1 = \ldots = x_{n-1} = w = 0\}.$$

Notice that $C_1$ is closed in X and that $C_1$ does not intersect the closure of the $\mathrm{nc}(n)$ locus (or more specifically, $C_1$ does not intersect any irrelevant chart), and so we perform the blow-up $\sigma_1$ with centre $C_1$. This finalizes the description of the first blow-up step. Let us describe the rest.

In the following, if D is a divisor intersecting $\operatorname{cosupp}(\underline{\mathcal{N}})$, we also use D to denote its strict transform after performing any sequence of blowings-up.

Consider the finite covering $\mathscr{U}$ of *relevant* standard charts of the blowing-up sequence. Let $\mathscr{G}$ denote the collection of irrelevant charts in $\mathscr{U}$ and let $\mathscr{B}$ denote the relevant charts. If

$$\operatorname{cosupp}((Z, Z, E, \mathcal{M}, d) + (Z, Z, E, D_0, 1)) \subset \bigcup_{U \in \mathscr{G}} U$$

then we cannot further improve $\underline{\mathcal{N}}_d := (Z, Z, E, \mathcal{M}, d) + (Z, Z, E, D_0, 1)$ while preserving the singularities in normal form, and so the next step in the algorithm is to consider the partial desingularization of

$$\underline{\mathcal{N}}_{d-1} := (Z, Z, E, \mathcal{M}, d-1) + (Z, Z, E, D_0, 1).$$

In this sense, our proof is inductive in d.

Thus, let us assume that $\operatorname{cosupp}(\underline{\mathcal{N}} := \underline{\mathcal{N}}_d)$ is not completely covered by irrelevant charts. Then, there exists at least one standard chart U for which the origin is not in $\overline{\mathrm{N}}$. We then compute the locus C of points with the highest value of minv inside $\bigcup_{U \in \mathscr{B}} U$, and consider its closure $\overline{C}$ in Z. If $\overline{C}$ does not intersect an irrelevant chart then we proceed to blow-up $\overline{C}$.

Otherwise, there is an irrelevant chart V such that $\overline{C} \cap V$ is a coordinate subspace of V. Let us first determine some properties of the general expression of $\mathcal{M}$ restricted to U.

In general, the local expression of $\mathcal{M}$ in U is of the form

$$\mathcal{M}|_U = \langle z^{n\delta_0}u^{\alpha_0}, yx_1^{n\delta_1}u^{\alpha_1}, \ldots, y^d x_d^{n\delta_d}u^{\alpha_d}, \ldots \rangle,$$

where each $u^{\alpha_j}$ is a monomial in terms of the non-distinguished exceptional divisors that intersect U, and where $\delta_j \in \{0, 1\}$ and $\delta_j = 1$ if and only if $Y_j \cap U \neq \varnothing$.

By the construction of minv, we have that all the points in C have vanishing order d, and by upper-semicontinuity of the vanishing order we have that $d \leqslant \ell$, where $\ell$ is the vanishing order of $\mathcal{M}$ at the origin of V.

It is convenient to also consider the local expression of $\mathcal{M}$ in V. By the definition of irrelevant chart we have that

$$\mathcal{M}|_V = \langle z^n, yx_1^n, \ldots, y^{\ell-1}x_{\ell-1}^n, y^\ell, \ldots \rangle.$$

Let $D_r$ be the exceptional divisor associated to the newest exceptional divisor in U given by $u_r$. If $C \subset \{u_r = 0\}$, notice that the local expression of $u_r$ in the first $\ell - 1$ monomials in V is zero, and so $\alpha_{0,r} = \ldots = \alpha_{d,r}$. In particular, the only positive exponents in $\alpha_0, \ldots, \alpha_r$ are those of exceptional divisors that do not intersect V. In particular, we have that

$$C \subset D_0 \cap Y_0 \cap Y_1 \cap \ldots \cap Y_{d-1},$$

as otherwise we would have that the vanishing order of $\mathcal{M}$ along C is strictly smaller than d.

We now claim that $Y_d \cap U = \varnothing$. Assume that $Y_d \cap U$ is non-empty, and notice that the maximal value of the vanishing order for points in U is then $\geqslant d + 1$. A very similar argument shows that $\alpha_d$ is the zero vector.

Briefly, we have that

$$\mathcal{M}|_U = \langle z^n u^{\alpha_0}, yx_1^n u^{\alpha_1}, y^{d-1}x_{d-1}^n u^{\alpha_{d-1}}, \ldots, y^d, \ldots \rangle,$$

where $\{y, x_1, \ldots, x_{d-1}, u_1, \ldots, u_r, z\}$ denotes a collection of sections that can be completed to a local coordinate system of U and $u^{\alpha_0}, \ldots, u^{\alpha_{d-1}}$ are monomials purely in terms of exceptional divisors that intersect U but not V.

For each coordinate $u_i$, let $\alpha_{j,i}$ denote the power of $u_i$ in the j-th generator of $\mathcal{M}$. Fix now the coordinate $u = u_i$ such that

$$\max_{0 \leqslant j \leqslant d-1} \alpha_{j,i} = \max_k \max_{0 \leqslant j \leqslant d-1} \alpha_{j,k}.$$

In case there are two elements achieving the maximum, we select the newest exceptional divisor created in the blowing-up process. Let

$$A := \max\{\alpha_{j,i} : j \in \{0, \dots, d-1\}\}.$$

Let $S$ be the closure of the vanishing locus of the ideal generated by

$$\{y = u = 0\} \cap \bigcap_j \{x_j = 0 : \alpha_{j,i} + j < d\}. \tag{7.6}$$

By construction of $S$, for all $b \in S$ we have that $\mathrm{ord}_X(b)$ is constant. On the other hand, the vanishing order of the points in $S \cap U$ is maximal inside the relevant charts, using the fact that the vanishing order is upper semi-continuous we deduce that all the points in $S$ have constant vanishing order.

We claim that $S$ does not intersect any irrelevant chart and that, if $\sigma_S$ denotes the blow-up of $X$ with centre $S$, then for any standard chart $U'$ such that $\sigma^{-1}(S) \cap U' \neq \varnothing$ we have that one and only one of the following is satisfied:

1. $U' \cap D_0 = \varnothing$.
2. $U'$ intersects the strict transform of $\{u = 0\}$. In this case, there is some $k \in \{0, \dots, d-1\}$ such that $U'$ does not intersect at least one of sets $Y_k$. Let $C$ denote the locus of points with maximal value of minv in $U'$. If the closure $\overline{C}$ of $C$ in $Z$ intersects an irrelevant chart then the vanishing order of the points in $C$ is less than $d$.
3. $U'$ is the standard chart associated to the local coordinate $u$. In this case, we further separate in two cases.
   - 3.a) If $A \geqslant n$, we let $\alpha'_{j,i}$ denote the power of the local expression $u'$ of the exceptional divisor created by $\sigma_S$ in $U'$, and define
     $$A' := \max\{\alpha'_{j,i} : j \in \{0, \dots, d-1\}\}.$$
     Then, $A' < A$.
   - 3.b) If $0 < A < n$, then $A' < n$ and $\alpha'_{d-1,i} = \alpha_{d-1,i} - 1$. Moreover, after at most $n-1$ blowings-up with a centre defined as in (7.6), we obtain that the residue of $\alpha'_{j,i}$ mod $n$ is constant in $j$ for $j \in \{0, \dots, n-1\}$.

First we want to show that $S$ does not intersect any irrelevant chart. Assume that $W$ is an irrelevant chart for which $S \cap W \neq \varnothing$, and consider the local expression of $\mathcal{M}$ in $W$ given by

$$\langle z^n, yx_1^n, \dots, y^\ell, \dots \rangle. \tag{7.7}$$

But notice that the local expression of the exceptional divisor associated to $u$ in $W$ satisfies that $\alpha_{j,i} = 0$ for all $j \in \{0, \dots, d-1\}$, leading to a contradiction.

Let us now focus on the case item 2. If $U'$ is a standard chart associated to the strict transform of $X$ with $\sigma^{-1}(S) \cap U' \neq \varnothing$ and if $U'$ intersects the distinguished divisor $D_0$ and the strict transform of the divisor associated to $\{u = 0\}$ then there is some $k \in \{0, \dots, d-1\}$ such that the strict transform $Y'_k$ of $Y_k$ under $\sigma$ satisfies that $U' \cap Y'_k = \varnothing$. Thus, we claim that we can continue blowing-up the strata with maximal value of minv until we have reduced the maximal value of the vanishing order of $\mathcal{M}$ in $U$ below $d$. Indeed, this is the case as per Remark 7.3 we have that the centre $C$ of maximal value inside $U'$ of minv satisfies that $\overline{C}$ intersects an irrelevant chart only if the vanishing order of $\mathcal{M}$ at the points of $C$ is $\leqslant k < d$.

For item 3.a), it suffices to notice that

$$\alpha'_{j,i} = \begin{cases} \alpha_{j,i} + j - d & \text{if } \alpha_{j,i} + j \geqslant d \\ \alpha_{j,i} + j + n - d & \text{if } \alpha_{j,i} + j < d \end{cases},$$

and so if for some $j \in \{0, \ldots, d-1\}$ we have that $\alpha_{j,i} \geqslant n$ then $\alpha'_{j,i} < \alpha_{j,i}$, and if for some $j$ we have that $\alpha_{j,i} < n$ then $\alpha'_{j,i} < n$. Notice that this also shows that when $A < n$, and if $0 < \alpha_{d-1,i}$ we then have that

$$\alpha'_{d-1,i} = \alpha_{d-1,i} - 1.$$

And so, after at most $d-1$ blowings-up with centres defined as in (7.6) we can assume that $\alpha_{d-1,i} = \alpha_{d,i} = 0$. From Remark 7.13, we deduce that there is some integer $R$ such that for all $j$ we get

$$\alpha_{j,i} \equiv j \cdot 0 + R \mod n.$$

Because $\alpha_{d-1,i} = \alpha_{d,i} = 0$ we obtain that $R \equiv 0 \mod n$, and given that we can assume that $0 \leqslant \alpha_{j,i} < n$ for all $j \in \{0, \ldots, d-1\}$ we obtain that

$$\alpha_{0,i} = \alpha_{1,i} = \ldots = \alpha_{d,i} = 0.$$

We have now cleaned the local expression of $\underline{\mathcal{N}}$ from the exceptional divisor with local expression given by $\{u = 0\}$. We repeat this process for the next exceptional divisor with the highest exponent in the chart of type item 3, until we obtain a chart $U'$ satisfying

$$\max_k \max_{0 \leqslant j \leqslant d-1} \alpha_{j,k} = 0.$$

When this is the case, $U'$ is an irrelevant chart.

**Remark 7.16.** Notice also that it may occur that minv after some blow-up in this sequence is strictly increased. Nonetheless, if $U$ is an affine chart that does not intersect the divisor $Y_j$ and if the global locus of points $C$ with maximal value of minv is such that $C \cap U \neq \varnothing$, then the vanishing order of any point $a \in C$ is at most $j$, by a similar reasoning as in (7.7).

After this, we continue blowing-up the centre $C$ with maximal value of minv inside relevant charts until $\overline{C}$ intersects an irrelevant chart, in which case, we apply again the *cleaning blowings-up* defined as in (7.6). Eventually, all standard charts covering (the strict transform of) $D_0$ become irrelevant.

This proves the first part of Algorithm 1.5. In order to verify that all singularities in the irrelevant charts that intersect (the strict transform of) $D_0$, for a variety $X$ with $\dim X \leqslant 4$, can be also expressed as a singularity in $\overline{\mathcal{N}}$ we first find the explicit finite list of singularities of $\overline{\mathcal{N}}$ which will be part of the minimal family of local normal forms. As such, we postpone the proof of the second part of the statement until Section 7.5 (see Remark 7.17). □

## 7.4 EXPLICIT BLOW-UP SEQUENCE FOR CP(4) AND NORMAL FORMS

Given that we need to find the explicit family of local expression for the minimal singularities, we need to carry out explicitly the algorithm for $\mathrm{cp}(4)$.

Assume that $X$ is such that the first value of the desingularization invariant is at most inv(nc$(4,0)$) at all points $a \in X$. Let $S_{4,0}$ be an irreducible component of the closure of the locus of points $\mathrm{nc}(4,0)$ containing a point $a$ whose local expression is circulant of order 4, that is, the local expression of $X$ at $a$ is of the form

$$\det(C(z, w^{1/4}x_1, w^{2/4}x_2, w^{3/4}x_3)),$$

where $w$ is the local expression of an exceptional divisor.

Let $U$ denote the chart in which $a$ admits this local expression. Notice that the origin can be identified as the intersection of 5 locally smooth divisors:

- $Y_0$ whose local expression in $U$ is $\{z = 0\}$,
- $Y_j$ whose local expression in $U$ is $\{x_j = 0\}$ for $j \in \{1, 2, 3\}$,
- the exceptional divisor $F_{-1}$ whose local expression is $\{w = 0\}$.

In order to simplify our notation, we replace the circulant expression for the ideal associated to $X$ by instead considering the monomial ideal expression as in (7.4). In other words, the local expression of the ideals we consider from now on are ideals generated by 4 monomials, for example the ideal associated to a $\mathrm{cp}(4)$ point is

$$\langle z^4, wx_1^4, w^2x_2^4, w^3x_3^4\rangle.$$

**Blow-up 0**. We start by blowing-up the origin in $U$. This blow-up creates an exceptional divisor which we will call $D_0$, which is what we refer to as a *distinguished divisor*. From this point and on, all the subsequent blowings-up that we will perform will be inside $D_0$. We can cover the inverse image of $U$ under this blow-up with 5 standard affine charts, which we now describe.

$U(z)$**-chart**

This chart intersects the strict transform of the divisors $Y_1, Y_2, Y_3, D_0$ but not $Y_0$. One family of uniformizing parameters for $U(z)$ is

$$\{W, X_1, X_2, X_3, y\},$$

where $x_i = yX_i$, $w = yW$ and $z = y$. In order to compute the local expression of the strict transform of $X$ in $U(z)$ we perform the previous substitutions, and we divide by the highest power of $y$ that divides the ideal. Giving us the local expression

$$\langle 1, WX_1^4y, W^2X_2^4y^2, W^3X_3^4y^3\rangle.$$

The local expression of the strict transform of the divisor $Y_i$ is $\{X_i = 0\}$, the local expression of $F_{-1}$ is $\{w = 0\}$, and the local expression of $D_0$ is $\{y = 0\}$. Notice in particular that $U(z)$ does not contain any point of the strict transform of the variety $X$.

For economy of notation, from now on we use the same notation

$$w, x_1, x_2, x_3, z$$

for the collection of local coordinates of the affine charts covering the strict transform, and we reserve the variable $y$ to denote the local expression of the strict transform of $D_1$ in the given chart. To prevent adding too many variables to keep track of the exceptional divisors created by the sequence of blowings-up, and because the divisor $D_0$ is important in our procedure, $y$ is the only new variable that we will add, that is, the rest of the exceptional divisors will be named the same as the local expression of the variable that was blown-up.
$U(w)$**-chart**

This chart intersects the strict transform of the divisors $Y_0, Y_1, Y_2, Y_3, Z$ but not $F_{-1}$. The local expression of the monomial ideal in $U(w)$ is

$$\langle z^4, yx_1^4, y^2x_2^4, y^3x_3^4\rangle.$$

*This chart contains a circulant point of order* 4 *and so it contains a singularity in* $\overline{N}$*, and outside* $D_0$ *all singularities are* $\mathrm{nc}(k)$ *for some* $k \leqslant 4$*.*
$U(x_1)$**-chart**

The local expression of the strict transform of $X$ in $U(x_1)$ is

$$\langle z^4, wy, w^2y^2x_2^4, w^3y^3x_3^4\rangle,$$

and the local expression of the exceptional divisor $F_{-1}$ is $\{w = 0\}$.
$U(x_2)$**-chart**

The local expression of the strict transform of $X$ in $U(x_1)$ is

$$\langle z^4, wyx_1^4, w^2y^2, w^3y^3x_3^4\rangle.$$

From now on, we use the notation $F_{-1} : w$ to signify that the local expression of $F_{-1}$ in the given chart is $\{w = 0\}$.
$U(x_3)$**-chart**

The local expression of the strict transform of $X$ in $U(x_1)$ is

$$\langle z^4, wyx_1^4, w^2y^2x_2^4, w^3y^3\rangle,$$

$F_{-1} : w = 0$.

From this point on, we blow-up with centres inside $D_0$, allowing us to treat $Y_j$ and $F_{-1}$ as globally defined divisors. Moreover, all centres will be given by intersections of subsets of the strict transforms of these divisors and newly created exceptional divisors, allowing us to treat all of each of these divisors as a coordinate hyperplane.

**Blow-up 1**. We blow-up with centre equal to

$$F_{-1} \cap D_0 \cap Y_0 \cap Y_1 \cap Y_2.$$

Notice that this centre of blow-up is the origin in the affine chart $U(x_3)$. Because we want our sequence of blowings-up to be inside the respective strict transform of $D_0$ when we blow-up, we do not need to pay close attention at the local expression of the strict transform of $X$ in $U(x_3, x_3)$. Also, similarly to what happened in the affine chart $U(z)$, the strict transform of $X$ does not intersect the chart $U(x_3, z)$. From this point on, we only mention the relevant charts without explicitly stating the reason as to why we do not consider those charts we do not present. This leaves us to consider only 3 affine charts, and the relevant local information at each chart is:

| Name of chart | Local expression of ideal | Exc. Divs. |
|---|---|---|
| $U(x_3, w)$ | $\langle z^4, w^2x_1^4x_3, w^4x_2^4x_3^2, w^2x_3^3\rangle$ | $D_1 : w$ |
| $U(x_3, x_1)$ | $\langle z^4, wx_1^2x_3, w^2x_1^4x_2^4x_3^2, w^3x_1^2x_3^3\rangle$ | $F_{-1} : w \quad D_1 : x_1$ |
| $U(x_3, x_2)$ | $\langle z^4, wx_1^4x_2^2x_3, w^2x_2^4x_3^2, w^3x_2^2x_3^3\rangle$ | $F_{-1} : w \quad D_1 : x_2$ |

**Blow-up 2**. We blow-up with centre equal to

$$D_0 \cap D_1 \cap Y_0 \cap Y_1.$$

This centre intersects the charts $U(x_3, w)$ and $U(x_3, x_2)$. The new relevant affine charts that we obtain from this blow-up are thus,

| | | |
|---|---|---|
| $U(x_3, w, w)$ | $\langle z^4, w^3x_1^4x_3, w^2x_2^4x_3^2, wx_3^3\rangle$ | $D_2 : w$ |
| $U(x_3, w, x_1)$ | $\langle z^4, w^2x_1^3x_3, w^4x_1^2x_2^4x_3^2, w^2x_1x_3^3\rangle$ | $D_1 : w \quad D_2 : x_1$ |
| $U(x_3, x_2, x_1)$ | $\langle z^4, wx_1^3x_2^2x_3, w^2x_1^2x_2^4x_3^2, w^3x_1x_2^2x_3^3\rangle$ | $F_{-1} : w \quad D_1 : x_2 \quad D_2 : x_1$ |
| $U(x_3, x_2, x_2)$ | $\langle z^4, wx_1^4x_2^3x_3, w^2x_2^2x_3^2, w^3x_2x_3^3\rangle$ | $F_{-1} : w \quad D_2 : x_2$ |

**Blow-up 3**. We blow-up with centre equal to

$$D_0 \cap D_2 \cap Y_0.$$

This centre intersects all the charts created after performing Blow-up 2. The new relevant affine charts that we obtain from this blow-up are thus,

| | | |
|---|---|---|
| $U(x_3, w, w, w)$ | $\langle z^4, x_1^4x_3, x_2^4x_3^2, x_3^3\rangle$ | $D_4 : w$ |
| $U(x_3, w, x_1, x_1)$ | $\langle z^4, w^2x_3, w^4x_2^4x_3^2, w^2x_3^3\rangle$ | $D_1 : w \quad D_3 : x_1$ |
| $U(x_3, x_2, x_1, x_1)$ | $\langle z^4, wx_2^2x_3, w^2x_2^4x_3^2, w^3x_2^2x_3^3\rangle$ | $F_{-1} : w \quad D_1 : x_2 \quad D_3 : x_1$ |
| $U(x_3, x_2, x_2, x_2)$ | $\langle z^4, wx_1^4x_3, w^2x_3^2, w^3x_3^3\rangle$ | $F_{-1} : w \quad D_3 : x_2$ |

*Notice that the chart $U(x_3, w, w, w)$ contains a singularity in $\overline{N}$, and that all singularities outside $D_0$ are* nc *(see Remark 7.4).*

**Blow-up 4**. We blow-up with centre equal to

$$F_{-1} \cap D_0 \cap Y_0 \cap Y_1.$$

This centre intersects the charts $U(x_2)$ and $U(x_3, x_2, x_2, x_2)$. The new relevant affine charts that we obtain from this blow-up are thus,

| | | |
|---|---|---|
| $U(x_2, w)$ | $\langle z^4, w^2x_1^4x_2, x_2^2, w^2x_2^3x_3^4\rangle$ | $D_4 : w$ |
| $U(x_2, x_1)$ | $\langle z^4, wx_1^2x_2, w^2x_2^2, w^3x_1^2x_2^3x_3^4\rangle$ | $F_{-1} : w \quad D_4 : x_1$ |
| $U(x_3, x_2, x_2, x_2, w)$ | $\langle z^4, w^2x_1^4x_3, x_3^2, w^2x_3^3\rangle$ | $D_3 : x_2 \quad D_4 : w$ |
| $U(x_3, x_2, x_2, x_2, x_1)$ | $\langle z^4, wx_1^2x_3, w^2x_3^2, w^3x_1^2x_3^3\rangle$ | $F_{-1} : w \quad D_3 : x_2 \quad D_4 : x_1$ |

**Blow-up 5**. We blow-up with centre equal to

$$F_{-1} \cap D_0 \cap D_4 \cap Y_0.$$

This centre intersects the charts $U(x_2, x_1)$ and $U(x_3, x_2, x_2, x_2, x_1)$. The new relevant affine charts that we obtain from this blow-up are thus,

| | | |
|---|---|---|
| $U(x_2, x_1, w)$ | $\langle z^4, x_1^2x_2, x_2^2, w^4x_1^2x_2^3x_3^4\rangle$ | $D_4 : x_1 \quad D_5 : w$ |
| $U(x_2, x_1, x_1)$ | $\langle z^4, wx_2, w^2x_2^2, w^3x_1^4x_2^3x_3^4\rangle$ | $F_{-1} : w \quad D_5 : x_1$ |
| $U(x_3, x_2, x_2, x_2, x_1, w)$ | $\langle z^4, x_1^2x_3, x_3^2, w^4x_1^2x_3^3\rangle$ | $D_3 : x_2 \quad D_4 : x_1 \quad D_5 : w$ |
| $U(x_3, x_2, x_2, x_2, x_1, x_1)$ | $\langle z^4, wx_3, w^2x_3^2, w^3x_1^4x_3^3\rangle$ | $F_{-1} : w \quad D_3 : x_2 \quad D_5 : x_1$ |

**Blow-up 6**. We blow-up with centre equal to

$$F_{-1} \cap D_0 \cap D_1 \cap Y_0.$$

This centre intersects the charts $U(x_3, x_1)$ and $U(x_3, x_2, x_1, x_1)$. The new relevant affine charts that we obtain from this blow-up are thus,

| | | |
|---|---|---|
| $U(x_3, x_1, w)$ | $\langle z^4, x_1^2x_3, w^4x_1^4x_2^4x_3^2, w^4x_1^2x_3^3\rangle$ | $D_1 : x_1 \quad D_6 : w$ |
| $U(x_3, x_1, x_1)$ | $\langle z^4, wx_3, w^2x_1^4x_2^4x_3^2, w^3x_1^4x_3^3\rangle$ | $F_{-1} : w \quad D_6 : x_1$ |
| $U(x_3, x_2, x_1, x_1, w)$ | $\langle z^4, x_2^2x_3, w^4x_2^4x_3^2, w^4x_2^2x_3^3\rangle$ | $D_1 : x_2 \quad D_3 : x_1 \quad D_6 : w$ |
| $U(x_3, x_2, x_1, x_1, x_2)$ | $\langle z^4, wx_3, w^2x_2^4x_3^2, w^3x_2^4x_3^3\rangle$ | $F_{-1} : w \quad D_3 : x_1 \quad D_6 : x_2$ |

**Blow-up 7**. We blow-up with centre equal to

$$D_0 \cap D_1 \cap Y_0.$$

This centre intersects the charts $U(x_3, w, x_1, x_1)$, $U(x_3, x_1, w)$, and $U(x_3, x_2, x_1, x_1, w)$. No new relevant affine charts are obtained after blowing-up. More precisely, $X$ is either smooth or has been separated from $D_0$ in all of the standard charts created by Blow-up 7.

In this point of the moving away procedure, we may notice that the closure of the locus of points with maximal value of minv outside the irrelevant charts is

$$D_0 \cap Y_0 \cap Y_1,$$

which intersects the closure of the nc locus of points. Consequently, we blow-up instead the closure of the local centre defined as in (7.6).

**Blow-up 8**. We blow-up with centre equal to

$$D_0 \cap D_4 \cap Y_0.$$

This centre intersects the charts $U(x_2, w)$, $U(x_3, x_2, x_2, x_2, w)$, $U(x_2, x_1, w)$, and $U(x_3, x_2, x_2, x_2, x_1, w)$. The new relevant affine charts that we obtain from this blow-up are thus,

| | | |
|---|---|---|
| $U(x_2, w, w)$ | $\langle w^2z^4, wx_1^4x_2, x_2^2, w^3x_2^3x_3^4\rangle$ | $D_8 : w$ |
| $U(x_2, w, z)$ | $\langle z^2, w^2x_1^4x_2z, x_2^2, w^2x_2^3x_3^4z^3\rangle$ | $D_4 : w$ $D_8 : z$ |
| $U(x_3, x_2, x_2, x_2, w, w)$ | $\langle w^2z^4, wx_1^4x_3, x_3^2, w^3x_3^3\rangle$ | $D_3 : x_2$ $D_8 : w$ |
| $U(x_3, x_2, x_2, x_2, w, z)$ | $\langle z^2, w^2x_1^4x_3z, x_3^2, w^2x_3^3z^3\rangle$ | $D_3 : x_2$ $D_4 : w$ $D_8 : z$ |
| $U(x_2, x_1, w, x_1)$ | $\langle x_1^2z^4, x_1x_2, x_2^2, w^4x_1^3x_2^3x_3^4\rangle$ | $D_5 : w$ $D_8 : x_1$ |
| $U(x_2, x_1, w, z)$ | $\langle z^2, x_1^2x_2z, x_2^2, w^4x_1^2x_2^3x_3^4z^3\rangle$ | $D_4 : x_1$ $D_5 : w$ $D_8 : z$ |
| $U(x_3, x_2, x_2, x_2, x_1, w, x_1)$ | $\langle x_1^2z^4, x_1x_3, x_3^2, w^4x_1^3x_3^3\rangle$ | $D_3 : x_2$ $D_5 : w$ $D_8 : x_1$ |
| $U(x_3, x_2, x_2, x_2, x_1, w, z)$ | $\langle z^2, x_1^2x_3z, x_3^2, w^4x_1^2x_3^3z^3\rangle$ | $D_3 : x_2$ $D_4 : x_1$ $D_5 : w$ $D_8 : z$ |

We perform another blow-up defined as in (7.6).

**Blow-up 9**. We blow-up with centre equal to

$$D_0 \cap D_8.$$

This centre intersects all the charts created after performing Blow-up 8. The new relevant affine charts that we obtain from this blow-up are thus,

| | | |
|---|---|---|
| $U(x_2, w, w, w)$ | $\langle z^4, x_1^4x_2, x_2^2, w^4x_2^3x_3^4\rangle$ | $D_9 : w$ |
| $U(x_3, x_2, x_2, x_2, w, w, w)$ | $\langle z^4, x_1^4x_3, x_3^2, w^4x_3^3\rangle$ | $D_3 : x_2$ $D_9 : w$ |

*Notice that both charts contain a singularity in* $\overline{N}$*, and outside* $D_0$ *all singularities are* nc *(see Remark 7.4).*

**Blow-up 10**. We blow-up with centre equal to

$$F_{-1} \cap D_0 \cap Y_0.$$

This centre intersects the charts $U(x_1)$, $U(x_2, x_1, x_1)$, $U(x_3, x_2, x_2, x_2, x_1, x_1)$, $U(x_3, x_1, x_1)$, and $U(x_3, x_2, x_1, x_1, x_2)$. The new relevant affine charts that we obtain from this blow-up are thus,

| | | |
|---|---|---|
| $U(x_1, z)$ | $\langle z^2, wx_1, w^2x_1^2x_2^4z^2, w^3x_1^3x_3^4z^4\rangle$ | $F_{-1}: w$ $D_{10}: z$ |
| $U(x_2, x_1, x_1, z)$ | $\langle z^2, wx_2, w^2x_2^2z^2, w^3x_1^4x_2^3x_3^4z^4\rangle$ | $F_{-1}: w$ $D_5: x_1$ $D_{10}: z$ |
| $U(x_3, x_2, x_2, x_2, x_1, x_1, z)$ | $\langle z^2, wx_3, w^2x_3^2z^2, w^3x_1^4x_3^3z^4\rangle$ | $F_{-1}: w$ $D_3: x_2$ $D_5: x_1$ $D_{10}: z$ |
| $U(x_3, x_1, x_1, z)$ | $\langle z^2, wx_3, w^2x_1^4x_2^4x_3^2z^2, w^3x_1^4x_3^3z^4\rangle$ | $F_{-1}: w$ $D_6: x_1$ $D_{10}: z$ |
| $U(x_3, x_2, x_1, x_1, x_2, z)$ | $\langle z^2, wx_3, w^2x_2^4x_3^2z^2, w^3x_2^4x_3^3z^4\rangle$ | $F_{-1}: w$ $D_3: x_1$ $D_6: x_2$ $D_{10}: z$ |

**Blow-up 11**. We blow-up with centre equal to

$$F_{-1} \cap D_0 \cap D_{10}.$$

This centre intersects all the charts created after performing blow-up 10.
No new relevant affine charts are created.

To summarize, at the end of this procedure, the local expression of the singularity at the origin of those standard charts intersecting the distinguished divisor $D_0$ are:

a) $\Delta_4(z, y^{1/4}x_1, y^{2/4}x_2, y^{3/4})$.
b) $\Delta_4(z, y^{1/4}x_1, y^{2/4}, y^{3/4}wx_3)$, where $\{w = 0\}$ corresponds to an exceptional divisor and $Y_3 = \{x_3 = 0\}$ is the local expression of a locally smooth non-exceptional divisor.
c) $\Delta_4(z, y^{1/4}x_1, y^{2/4}, y^{3/4}w)$, where $\{w = 0\}$ corresponds to an exceptional divisor.

## 7.5 normal forms for limit singularities

While it is not true that the neighbouring singularities of a singularity in normal form is in normal form, we can append to our list of normal forms new normal forms that encompass all the neighbours of the given normal forms, at least when $n \leqslant 4$.

Let us focus on the case item a). Notice that all neighbouring singular points of item a) satisfy that $x_2 \neq 0$. In this case, if we consider the collection of coordinates

$$\tilde{y} = \frac{y}{x_2^4}$$
$$\tilde{z} = \frac{z}{x_2^3}$$
$$\tilde{x_1} = \frac{x_1}{x_2^2}$$

we obtain that

$$\begin{aligned}\Delta_4(z, y^{1/4}x_1, y^{2/4}x_2, y^{3/4}) &= \Delta_4(x_2^3\tilde{z}, x_2^3\tilde{y}^{1/4}\tilde{x_1}, x_2^3\tilde{y}^{2/4}, x_2^3\tilde{y}^{3/4}) \\ &= x_2^{\frac{3}{4}}\Delta_4(\tilde{z}, \tilde{y}^{1/4}\tilde{x_1}, \tilde{y}^{2/4}, \tilde{y}^{3/4})\end{aligned}$$

and so we append to our list of normal forms the new normal form given by

d) $\Delta_4(z, y^{1/4}x_1, y^{2/4}, y^{3/4})$.

**Remark 7.17.** We can apply a similar reasoning to the other singularities found in Section 7.4. This completes the proof of Algorithm 1.5.

We now analyze the normal form item b). Notice that, for the neighbouring singularities of item b) given by $y = z = x_1 = w = 0$, $x_3 \neq 0$, by means of the change of coordinates

$$\tilde{y} = yx_3^4$$
$$\tilde{z} = zx_3^2$$
$$\tilde{x_1} = x_1x_3,$$

we obtain that

$$\Delta_4(z, y^{1/4}x_1, y^{2/4}, y^{3/4}wx_3)) = x_3^{-\frac{1}{2}}\Delta_4(\tilde{z}, \tilde{y}^{1/4}\tilde{x_1}, \tilde{y}^{2/4}, \tilde{y}^{3/4}w),$$

which is in the normal form item c). A similar reasoning gives us that the neighbouring singularities of item b) given by $\{y = z = x_1 = x_3 = 0\}$ require us to append a new normal form to the aforementioned list, given by

e) $\Delta_4(z, y^{1/4}x_1, y^{2/4}, y^{3/4}x_3)$, where $Y_3 = \{x_3 = 0\}$ is a locally smooth non-exceptional divisor.

The neighbouring singularities of item c) can all be expressed in one of the previous normal forms.

In short, at the end of the moving away procedure, applied to $\mathrm{cp}(4)$, we obtain that all the singularities inside $D_0$ can be locally expressed in one of the following normal forms:

a) $\Delta_4(z, y^{1/4}x_1, y^{2/4}x_2, y^{3/4})$.
b) $\Delta_4(z, y^{1/4}x_1, y^{2/4}, y^{3/4}wx_3)$.
c) $\Delta_4(z, y^{1/4}x_1, y^{2/4}, y^{3/4}w)$.
d) $\Delta_4(z, y^{1/4}x_1, y^{2/4}, y^{3/4})$.
e) $\Delta_4(z, y^{1/4}x_1, y^{2/4}, y^{3/4}x_3)$.

## 7.6 PRODUCT OF TWO CIRCULANTS OF ORDER TWO

One essential component of our argument for the existence of a partial desingularization algorithm requires finding a procedure for *moving away* all the limit singularities of the nc locus that are not in normal form. In particular, to complete the proofs of Theorem 1.1 and Theorem 1.2 we need to establish a moving away procedure for the product of circulants $\mathrm{cp}(2) \times \mathrm{cp}(2)$. Let us carry this out in this section. Let $S$ be an irreducible component of the closure of the locus of points $\mathfrak{nc}(4, 0)$ containing a point whose local expression is a product circulants points of the form

$$\Delta_2(z_1, w^{1/2}x_1)\Delta_2(z_2, w^{1/2}x_2).$$

Notice that the origin in this chart can be identified as the intersection of 5 locally smooth divisors:

- $Z_j$ whose local expression in $U$ is $\{z_j = 0\}$ for $j \in \{1, 2\}$,
- $Y_j$ whose local expression in $U$ is $\{x_j = 0\}$ for $j \in \{1, 2\}$,
- the exceptional divisor $F_{-1}$ whose local expression is $\{w = 0\}$.

Similarly as before, we first blow-up $S$ to introduce a divisor $D_0$, and from that point onwards, we blow-up with centres in (the strict transform of) $D_0$. Our goal is to find a suitable sequence of admissible and equimultiple blowings-up after which, all the singularities of $(X, E)$ in $D_0$ are in $\overline{N}$, and in a deleted neighbourhood $U$ around $D_0$ all singularities of $(X, E)$ are nc. We follow the same monomial ideal notation as in the previous subsection.

**Blow-up 0**. We start by blowing-up the origin in $U$. This blow-up creates an exceptional divisor which we will call $D_0$ and we follow the same notation for $D_0$ as in the previous subsection. From this point and on, all the subsequent blowings-up that we will perform will be inside $D_0$. The new relevant charts are:

| | | |
|---|---|---|
| $U(w)$ | $\langle z_1^2, yx_1^2\rangle\langle z_2^2, yx_2^2\rangle$ | |
| $U(x_1)$ | $\langle z_1^2, wy\rangle\langle z_2^2, wyx_2^2\rangle$ | $F_{-1} : w$ |
| $U(x_2)$ | $\langle z_1^2, wyx_1^2\rangle\langle z_2^2, wy\rangle$ | $F_{-1} : w$ |

Notice that the origin in the $U(w)$-chart has a local expression of the form $\mathrm{cp}(2) \times \mathrm{cp}(2)$, and moreover, notice that any equisingular locus inside $U(w)$ is $\mathrm{nc}(4)$ for the pair $(X', E_{\mathrm{old}})$. Notice that the local expression for the origin in the chart $U(x_2)$ is symmetric to that of the origin in the chart $U(x_1)$, as such the local resolution procedure for the chart $U(x_2)$ is the same as that of the chart $U(x_1)$ by exchanging the role of $Y_2$ with $Y_1$.

**Blow-up 1**. We blow-up with centre equal to

$$F_{-1} \cap D_0 \cap Z_1 \cap Z_2.$$

This centre intersects the charts: $U(x_1)$ and $U(x_2)$. The new relevant affine charts that we obtain from this blow-up are thus,

| | | |
|---|---|---|
| $U(x_1, w)$ | $\langle z_1^2, y\rangle\langle z_2^2, yx_2^2\rangle$ | $D_1 : w$ |
| $U(x_1, z_1)$ | $\langle z_2^2, wyx_2^2\rangle$ | $F_{-1} : w \quad D_1 : z_1$ |
| $U(x_1, z_2)$ | $\langle z_1^2, wyx_1^2\rangle$ | $F_{-1} : w \quad D_1 : z_2$ |

Notice that the singularities in $U(x_1, w)$ admit a local expression of the form $\exp^\delta \times \mathrm{nc}(1) \times \mathrm{cp}(2)$, where $\delta \in \{0, 1\}$. Notice also that the local expression of the origin in the chart $U(x_1, z_2)$ is the same as the local expression of the origin in $U(x_1, z_1)$ after substituting $z_1$ by $z_2$, as such we only need to consider the resolution sequence of the origin in $U(x_1, z_1)$.

**Blow-up 2**. We blow-up with centre equal to

$$F_{-1} \cap D_0 \cap Z_2.$$

This centre intersects the chart $U(x_1, z_1)$. The new relevant affine chart that we obtain from this blow-up is,

| | | |
|---|---|---|
| $U(x_1, z_1, w)$ | $\langle z_2^2, yx_2^2 \rangle$ | $D_1 : w \quad D_2 : x_2$ |

## 7.7 PRODUCT EXP(R)XCP(3)

In this section we focus on the proof of one of the inductive steps needed for the proofs of Theorem 1.1 and Theorem 1.2. More precisely, we want to establish a moving away procedure for the points of the stratum $S_{3,r} = \{\mathrm{inv} = \mathrm{inv}(\mathrm{nc}(3, r))\}$ where the local expression of $X \cup E$ is of the form

$$u_1 \ldots u_r \Delta_3(z, w^{1/3}x_1, w^{2/3}x_2).$$

We can partition the set of exceptional divisors $E$ into two subsets: $E_{\mathrm{old}}$ given by the collection of $r$ divisors determined by the second entry of inv, and $E_{\mathrm{new}}$ which is the collection of exceptional divisors introduced in the sequence of blowings-up performed after reaching $\mathrm{inv}(\mathrm{nc}(3, r))$.

Let $E_{\mathrm{old}}$ denote the collection of exceptional divisors associated to the second entry of $\mathrm{inv}(\mathrm{nc}(3, r))$, and express $E_{\mathrm{old}} = E_1 + \ldots + E_r$. In the following, we denote by $u_i$ the local coordinate associated to the exceptional divisor $E_i$. By hypothesis, the local expression of $(X, E)_{\mathrm{old}}$ at $a$ has the form

$$u_1 \ldots u_r \Delta_3(z, w^{1/3}x_1, w^{2/3}x_2),$$

where $\{w = 0\}$ is the local expression of one of the components $F_{-1}$ of $E_{\mathrm{new}}$.

**Year 0.** We perform an initial blow-up $\sigma_0$ whose centre is given by $C = S_{3,r} \cap F_{-1}$, which in local coordinates can be expressed as

$$C = \{u_1 = \ldots = u_r = z = x_1 = x_2 = w = 0\}$$

This creates $r + 4$ affine charts. Because all the charts associated to exceptional coordinates $u_i$ are symmetrical, we present only the procedure for $u_r$ with the understanding that an analogous sequence of blowings-up leads us to the desired result.

We know that $\sigma_0$ creates an exceptional divisor. This works as one of our distinguished divisors. Let us denote it by $D_0$, and let $y$ denote the local expression of $D_0$ in all the subsequent affine charts. From this point on, all the blowings-up have centres inside $D_0$, and we want the sequence to satisfy that all the non-nc singularities of the final transform $X'$ of $X$ whose first entry of the desingularization invariant is 3 to be inside the strict transform of $D_0$. The local information of the collection of affine charts created by $\sigma_0$ can be summarized with the following table.

| | | |
|---|---|---|
| $U(w)$ | $\Delta_3(z, x_1y^{1/3}, x_2y^{2/3})$ | $E_{old} : u_1 \dots u_r$<br>$E_{new} : yv_1 \dots v_s$ |
| $U(u_r)$ | $\Delta_3(z, w^{1/3}x_1y^{1/3}, w^{2/3}x_2y^{2/3})$ | $F_{-1} : w$<br>$E_{old} : u_1 \dots u_{r-1}$<br>$E_{new} : wyv_1 \dots v_s$ |
| $U(x_1)$ | $\Delta_3(z, w^{1/3}y^{1/3}, w^{2/3}x_2y^{2/3})$ | $F_{-1} : w$<br>$E_{old} : u_1 \dots u_r$<br>$E_{new} : wyv_1 \dots v_s$ |
| $U(x_2)$ | $u_1 \dots u_r\Delta_3(z, w^{1/3}x_1y^{1/3}, w^{2/3}y^{2/3})$ | $F_{-1} : w$<br>$E_{old} : u_1 \dots u_r$<br>$E_{new} : wyv_1 \dots v_s$ |

Notice that the chart $U(w)$, contains the strict transform of the stratum $S_{3,r}$, and all the non-nc singularities with vanishing order 3 are contained inside $D_0$.

**Year 1.** We now perform the blow-up $\sigma_1$ with centre

$$C = \{b \in X : \mathrm{ord}_X(b) \geqslant 3\} \cap \bigcap_{i=1}^{r} E_i \cap F_{-1} \cap D_0.$$

Notice that this centre of blow-up intersects the chart $U(x_2)$.

| | | |
|---|---|---|
| $U(x_2, w)$ | $\Delta_3(z, x_1y^{1/3}w^{2/3}, y^{2/3}w^{1/3})$ | $D_1 : w$<br>$E_{old} : u_1 \dots u_r$<br>$E_{new} : wyv_1 \dots v_s$ |
| $U(x_2, u_r)$ | $\Delta_3(z, w^{1/3}x_1y^{1/3}, w^{2/3}y^{2/3})$ | $F_{-1} : w \quad D_1 : u_r$<br>$E_{old} : u_1 \dots u_{r-1}$<br>$E_{new} : u_ryv_1 \dots v_s$ |
| $U(x_2, x_1)$ | $\Delta_3(z, w^{1/3}y^{1/3}x_1^{2/3}, w^{2/3}y^{2/3}x_1^{1/3})$ | $F_{-1} : w \quad D_1 : x_1$<br>$E_{old} : u_1 \dots u_r$<br>$E_{new} : wx_1yv_1 \dots v_s$ |

**Year 2.** Let us now consider the blow-up $\sigma_2$ with centre

$$C = \{b \in X : \mathrm{ord}_X(b) \geqslant 3\} \cap \bigcap_{i=1}^{r} E_i \cap D_0 \cap D_1$$

This creates the collection of charts

| | | |
|---|---|---|
| $U(x_2, w, w)$ | $\Delta_3(z, x_1 y^{1/3}, y^{2/3})$ | $D_2 : w$<br>$E_{\text{old}} : u_1 \ldots u_r$<br>$E_{\text{new}} : wyv_1 \ldots v_s$ |
| $U(x_2, w, u_r)$ | $\Delta_3(z, x_1 y^{1/3} w^{2/3}, y^{2/3} w^{1/3})$ | $D_1 : w \quad D_2 : u_r$<br>$E_{\text{old}} : u_1 \ldots u_{r-1}$<br>$E_{\text{new}} : u_r wyv_1 \ldots v_s$ |
| $U(x_2, x_1, u_r)$ | $\Delta_3(z, w^{1/3} y^{1/3} x_1^{2/3}, w^{2/3} y^{2/3} x_1^{1/3})$ | $F_{-1} : w \quad D_1 : x_1 \quad D_2 : u_r$<br>$E_{\text{old}} : u_1 \ldots u_{r-1}$<br>$E_{\text{new}} : wx_1 u_r yv_1 \ldots v_s$ |
| $U(x_2, x_1, x_1)$ | $\Delta_3(z, w^{1/3} y^{1/3}, w^{2/3} y^{2/3})$ | $F_{-1} : w \quad D_2 : x_1$<br>$E_{\text{old}} : u_1 \ldots u_r$<br>$E_{\text{new}} : wx_1 yv_1 \ldots v_s$ |

Notice that the singularities in $U(x_2, w, w)$ are in normal form.
**Year 3.** We now perform the blow-up $\sigma_3$ whose centre is given by

$$C = \{b \in X : \text{ord}_X(b) \geqslant 2\} \cap \bigcap_{i=1}^{r} E_i \cap F_{-1} \cap D_0.$$

This creates the charts

| | | |
|---|---|---|
| $U(x_1, w)$ | $\Delta_3(zw^{1/3}, y^{1/3}, x_2 y^{2/3} w^{2/3})$ | $D_3 : w$<br>$E_{\text{old}} : u_1 \ldots u_r$<br>$E_{\text{new}} : wyv_1 \ldots v_s$ |
| $U(x_1, u_r)$ | $\Delta_3(z, w^{1/3} y^{1/3}, w^{2/3} x_2 y^{2/3})$ | $F_{-1} : w \quad D_3 : u_r$<br>$E_{\text{old}} : u_1 \ldots u_{r-1}$<br>$E_{\text{new}} : wu_r yv_1 \ldots v_s$ |
| $U(x_1, z)$ | $\Delta_3(z^{1/3}, w^{1/3} y^{1/3}, w^{2/3} x_2 y^{2/3} z^{2/3})$ | $F_{-1} : w \quad D_3 : z$<br>$E_{\text{old}} : u_1 \ldots u_r$<br>$E_{\text{new}} : wyzv_1 \ldots v_s$ |
| $U(x_2, x_1, x_1, w)$ | $\Delta_3(zw^{1/3}, y^{1/3}, y^{2/3} w^{2/3})$ | $D_2 : x_1 \quad D_3 : w$<br>$E_{\text{old}} : u_1 \ldots u_r$<br>$E_{\text{new}} : wx_1 yv_1 \ldots v_s$ |
| $U(x_2, x_1, x_1, u_r)$ | $\Delta_3(z, w^{1/3} y^{1/3}, w^{2/3} y^{2/3})$ | $F_{-1} : w \quad D_2 : x_1 \quad D_3 : u_r$<br>$E_{\text{old}} : u_1 \ldots u_{r-1}$<br>$E_{\text{new}} : wx_1 u_r yv_1 \ldots v_s$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

| | | |
|---|---|---|
| $U(x_2, x_1, x_1, z)$ | $\Delta_3(z^{1/3}, w^{1/3}y^{1/3}, w^{1/3}y^{2/3}z^{2/3})$ | $F_{-1} : w \quad D_2 : x_1 \quad D_3 : z$ <br> $E_{\text{old}} : u_1 \dots u_r$ <br> $E_{\text{new}} : wx_1yzv_1 \dots v_s$ |

Notice that X is smooth on any of the charts intersecting $E_r$.

**Year 4.** We now perform the blow-up $\sigma_4$ with centre given by

$$C = \{b \in X : \text{ord}_X(b) \geqslant 3\} \cap \bigcap_{i=1}^{r-1} E_i \cap F_{-1} \cap D_0.$$

This creates $r+3$ charts. Again, the charts associated to the components $E_i$ are symmetrical, and so we just write the one associated to $E_{r-1}$. The relevant charts are the following.

| | | |
|---|---|---|
| $U(u_r, w)$ | $\Delta_3(z, w^{2/3}x_1y^{1/3}, w^{1/3}x_2y^{2/3})$ | $D_4 : w$ <br> $E_{\text{old}} : u_1 \dots u_{r-1}$ <br> $E_{\text{new}} : u_rwyv_1 \dots v_s$ |
| $U(u_r, u_{r-1})$ | $\Delta_3(z, w^{1/3}x_1y^{1/3}, w^{2/3}x_2y^{2/3})$ | $F_{-1} : w \quad D_4 : u_{r-1}$ <br> $E_{\text{old}} : u_1 \dots u_{r-2}$ <br> $E_{\text{new}} : u_{r-1}wyv_1 \dots v_s$ |
| $U(u_r, x_1)$ | $\Delta_3(z, w^{1/3}x_1^{2/3}y^{1/3}, w^{2/3}x_1^{1/3}x_2y^{2/3})$ | $F_{-1} : w \quad D_4 : x_1$ <br> $E_{\text{old}} : u_1 \dots u_{r-1}$ <br> $E_{\text{new}} : wx_1yv_1 \dots v_s$ |

**Year 5.** We now perform the blow-up $\sigma_5$ with centre given by

$$C = \{b \in X : \text{ord}_X(b) \geqslant 3\} \cap \bigcap_{i=1}^{r-1} E_i \cap D_4 \cap D_0.$$

The relevant charts are the following.

| | | |
|---|---|---|
| $U(u_r, w, w)$ | $\Delta_3(z, x_1y^{1/3}, x_2y^{2/3})$ | $D_5 : w$ <br> $E_{\text{old}} : u_1 \dots u_{r-1}$ <br> $E_{\text{new}} : wyv_1 \dots v_s$ |
| $U(u_r, w, u_{r-1})$ | $\Delta_3(z, w^{2/3}x_1y^{1/3}, w^{1/3}x_2y^{2/3})$ | $D_4 : w \quad D_5 : u_{r-1}$ <br> $E_{\text{old}} : u_1 \dots u_{r-2}$ <br> $E_{\text{new}} : u_{r-1}wyv_1 \dots v_s$ |
| $U(u_r, x_1, u_{r-1})$ | $\Delta_3(z, w^{1/3}x_1^{2/3}y^{1/3},$ <br> $w^{2/3}x_1^{1/3}x_2y^{2/3})$ | $F_{-1} : w \quad D_4 : x_1 \quad D_5 : u_{r-1}$ <br> $E_{\text{old}} : u_1 \dots u_{r-2}$ <br> $E_{\text{new}} : u_{r-1}wx_1yv_1 \dots v_s$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

| | | |
|---|---|---|
| $U(u_r, x_1, x_1)$ | $\Delta_3(z, w^{1/3}y^{1/3}, w^{2/3}x_2y^{2/3})$ | $F_{-1} : w \quad D_5 : x_1$<br>$E_{old} : u_1 \dots u_{r-1}$<br>$E_{new} : wx_1yv_1 \dots v_s$ |

Notice that the singularities in the chart $U(u_r, w, w)$ are all in normal form.

**Year 6.** We now perform the blow-up $\sigma_6$ with centre given by

$$C = \{b \in X : \mathrm{ord}_X(b) \geqslant 2\} \cap \bigcap_{i=1}^{r-1} E_i \cap D_5 \cap D_0.$$

The relevant charts are the following.

| | | |
|---|---|---|
| $U(u_r, x_1, x_1, w)$ | $\Delta_3(w^{1/3}z, y^{1/3}, w^{2/3}x_2y^{2/3})$ | $D_5 : x_1 \quad D_6 : w$<br>$E_{old} : u_1 \dots u_{r-1}$<br>$E_{new} : wx_1yv_1 \dots v_s$ |
| $U(u_r, x_1, x_1, u_{r-1})$ | $\Delta_3(z, w^{1/3}y^{1/3}, w^{2/3}x_2y^{2/3})$ | $F_{-1} : w \quad D_5 : x_1 \quad D_6 : u_{r-1}$<br>$E_{old} : u_1 \dots u_{r-2}$<br>$E_{new} : u_{r-1}wx_1yv_1 \dots v_s$ |
| $U(u_r, x_1, x_1, z)$ | $\Delta_3(z^{1/3}, w^{1/3}y^{1/3}, w^{2/3}x_2y^{2/3}z^{2/3})$ | $F_{-1} : w \quad D_5 : x_1 \quad D_6 : z$<br>$E_{old} : u_1 \dots u_{r-1}$<br>$E_{new} : wx_1yzv_1 \dots v_s$ |

$X$ is smooth in the charts $U(u_r, x_1, x_1, w)$ and $U(u_r, x_1, x_1, z)$.

Notice that the singularities of the chart $U(u_r, u_{r-1})$ in $D_0$ can be taken into normal form by performing a similar sequence of blowings-up to the one taking the singularities in $U(u_r)$ to normal form, but where the centre is defined by one divisor less. A similar remark applies to the charts $U(x_2, u_r)$, $U(x_2, w, u_r)$, $U(x_2, x_1, u_r)$, $U(x_1, u_r)$, $U(x_2, x_1, x_1, u_r)$, $U(u_r, x_1, u_{r-1})$, $U(u_r, x_1, x_1, u_{r-1})$. At the end of this finite sequence of blowings-up all singularities in $D_0$ have been put in a local expression of the form

1. $u_{i_1} \dots u_{i_k}\Delta_3(z, y^{1/3}x_1, y^{2/3}x_2)$,
2. $u_{i_1} \dots u_{i_k}\Delta_3(z, y^{1/3}x_1, y^{2/3})$,
3. $u_{i_1} \dots u_{i_k}s$, where $\mathrm{ord}(s) = 1$.

In short, for each irreducible component of the stratum $S_{3,r}$ consisting entirely of limits of $\exp^r \times \mathrm{nc}(3)$ we create an irreducible distinguished divisor $D_0$. The divisor $D_{3,r}$ (see Chapter 6) is the union of all the divisors $D_0$ that are created for each irreducible component of $S_{3,r}$ whose limit singularities are of the form $\exp^r \times \mathrm{cp}(3)$, together with the divisors created for each irreducible component of $S_{3,r}$ whose limit singularities are of the form $\exp^r \times \mathrm{nc}(1) \times \mathrm{cp}(2)$. Notice that the moving away procedure can be carried out in a similar way to the case $\exp^{r+1} \times \times \mathrm{cp}(2)$, which we cover in Section 7.8.

Now we need to verify that the subsequent blowings-up with centres given by the strata of inv after resetting to year zero do not modify the local expression of the singularities we have found.

### 7.7.1 *Subsequent blowings-up with limit points in distinguished divisor*

It is important to our partial desingularization strategy to continue performing the classical desingularization algorithm outside $D_{3,r} \cup S_{3,r}$. More precisely, we want to continue performing the sequence of blowings-up given by centres with the highest value of inv outside $\bigcup_{(p,r')\geqslant(3,r)} D_{p,r'} \cup \bigcup_{(p,r')\geqslant(3,r)} S_{p,r'}$. Thus, we need to verify that future inv-admissible blowings-up do not modify the singularities we found in Section 7.7, even if they have limits inside an irreducible component $D_0 \subset D_{3,r}$ .

Consider an affine chart in which $(X, E)$ has a local expression of the form item 1 (e.g. the local expression found in the chart $U(u_r, w, w)$). Notice that any inv-admissible centre with inv-value greater than inv(nc$(3, r-1)$) is necessarily inside at least one component of $E_{new}$, and by hypothesis, outside $\{y = 0\}$. Thus, any blowings-up with inv-admissible centres with inv-value larger than inv(nc$(3, r-1)$) preserve the collection of nc points of the (strict transform of the) original pair $(X_0, E_0)$. Once we reach inv = inv(nc$(3, r-1)$) the only inv-admissible centre is the collection of nc points. For example, in the chart $U(u_r, w, w)$ we have that

$$\{w = v = u = x = z = 0\}$$

is an inv-admissible collection of points (after resetting to year zero), and the collection $S := \{z = x_1 = x_2 = y = 0\}$ will be preserved by any inv-admissible blow-up until inv drops to inv(nc$(3, r-1)$). In this case, there is a deleted neighbourhood around (the strict transform of) $S$ containing only nc singularities of (the strict transform of) $(X_0, E_0)$.

In the chart $U(x_2, w, w)$, we may continue blowing-up with inv-admissible centres, until inv drops to inv(nc$(2, r+1)$). At this point, there is a deleted neighbourhood around $\{z = x_1 = y = 0\}$ in which there are only nc singularities of $(X_0, E_0)$. This deals with singularities of the form item 2.

We postpone the case item 3 to Remark 7.19.

## 7.8 PRODUCT EXP(R)XCP(2)

In this section we present the argument for moving away the singularities whose local expression is of the form $\exp^r \times \mathrm{cp}(2)$. The overall argument follows a similar structure as $\exp^r \times \mathrm{cp}(3)$, but the sequence is shorter.

Let $S_{2,r}$ be an irreducible component of the closure of the locus of points $\exp^r \times \mathrm{nc}(2)$ such that $S_{2,r}$ contains a singularity $a$ which is circulant of order 2. Let $E_{old}$ denote the components of $E$ that are associated to the second entry of inv(nc$(2, r)$), and express $E_{old} = E_1 + \ldots + E_r$. In the following, we denote by $u_i$ the local coordinate associated to the exceptional divisor $E_i$. By hypothesis, the local expression of $X$ at $a$ has the form

$$u_1 \ldots u_r \Delta_2(z, w^{1/2}x),$$

where $\{w = 0\}$ is the local expression of one of the components $F_{-1}$ of the exceptional divisor $E_{new} := E \setminus E_{old}$.

We begin by performing an initial blow-up $\sigma_0$ whose centre is given by $C = S_{2,r} \cap F_{-1}$, which in local coordinates can This creates $r + 3$ affine charts, one of them consisting entirely of monomial singularities, that is, singularities of the form $\exp^r$. Similarly as before, we only follow the computations for the chart $U(u_r)$ which models the case for all the charts $U(u_i)$.

| | | |
|---|---|---|
| $U(w)$ | $u_1 \dots u_r \Delta_2(z, xy^{1/2})$ | |
| $U(u_r)$ | $u_1 \dots u_{r-1} \Delta_2(z, w^{1/2}xy^{1/2})$ | $F_{-1} : w$ |
| $U(x)$ | $u_1 \dots u_r \Delta_2(z, w^{1/2}y^{1/2})$ | $F_{-1} : w$ |

We now perform the blow-up $\sigma_1$ whose centre is given by

$$C = S_{2,r} \cap F_{-1} \cap D_0,$$

giving us the following charts.

| | | |
|---|---|---|
| $U(x, w)$ | $u_1 \dots u_r \Delta_2(z, y^{1/2})$ | $D_1 : w$ |
| $U(u_r, w)$ | $u_1 \dots u_{r-1} \Delta_2(z, xy^{1/2})$ | $F_{-1} : w$ |

Notice that the singularities in the chart $U(x, w)$ are $nc(1)$ and, restricted to $U(u_r, w)$, the singularities inside $D_0$ are $cp(2)$ and the singularities outside $D_0$ are of the form $nc(k)$ for some $0 \leqslant k \leqslant 2$.

**Remark 7.18.** Similarly as in the case of $\exp^r \times cp(3)$, we need to verify that the centres of blow-up after resetting to year zero do not modify the local expression of the singularities in normal form inside $\mathscr{D}_{2,r}$. Similar remarks to the ones done in Subsection 7.7.1 apply to the chart $U(u_r, w)$.

**Remark 7.19.** Notice that an inv-admissible centre in a chart where the local equation of $X$ is $s = 0$, for smooth $s$, is of the form $\{u = v = s = 0\}$. A centre of this form cannot contain any nc point of $(X_0, E_0)$ or it consists entirely of points of this form. Thus, we preserve the local expression of these charts.

# BIBLIOGRAPHY


[Abh90] S.S. Abhyankar, *Algebraic geometry for scientists and engineers*, Mathematical surveys and monographs, American Mathematical Society, 1990.

[AGM96] V. Arnautov, S. Glavatsky, and A.V. Mikhalev, *Introduction to the theory of topological rings and modules*, Monographs and textbooks in pure and applied mathematics 197, New York : M. Dekker, 1996.

[AM69] Michael Francis Atiyah and I. G. MacDonald, *Introduction to commutative algebra.*, Addison-Wesley-Longman, 1969.

[AM99] E. Arbarello and D. Mumford, *The red book of varieties and schemes: Includes the michigan lectures (1974) on curves and their jacobians*, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 1999.

[BdSBRL25] André Belotto da Silva, Edward Bierstone, and Ramon Ronzon Lavie, *Partial desingularization*, Trans. Amer. Math. Soc. **378** (2025), no. 4, 2829–2879. MR 4880463

[BdSMV14] Edward Bierstone, Sergio da Silva, Pierre D. Milman, and Franklin Vera Pacheco, *Desingularization by blowings-up avoiding simple normal crossings*, Proc. Amer. Math. Soc. **142** (2014), no. 12, 4099–4111. MR 3266981

[BLM12] Edward Bierstone, Pierre Lairez, and Pierre D. Milman, *Resolution except for minimal singularities II. The case of four variables*, Adv. Math. **231** (2012), no. 5, 3003–3021. MR 2970471

[BM97] Edward Bierstone and Pierre D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302. MR 1440306

[BM06] ———, *Desingularization of toric and binomial varieties*, J. Algebraic Geom. **15** (2006), no. 3, 443–486. MR 2219845

[BM08] ———, *Functoriality in resolution of singularities*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 609–639. MR 2426359

[BM12] ———, *Resolution except for minimal singularities I*, Adv. Math. **231** (2012), no. 5, 3022–3053. MR 2970472

[Bou98] N. Bourbaki, *Commutative algebra: Chapters 1-7*, Elements de mathematique. English, no. v. 1, Springer, 1998.

[Dav79] P.J. Davis, *Circulant matrices*, Monographs and textbooks in pure and applied mathematics, Wiley New York, 1979.

[DR90] Persi Diaconis and Daniel Rockmore, *Efficient computation of the Fourier transform on finite groups*, J. Amer. Math. Soc. **3** (1990), no. 2, 297–332. MR 1030655

[Eis95] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, Springer, 1995.

[FH13] W. Fulton and J. Harris, *Representation theory: A first course*, Graduate Texts in Mathematics, Springer New York, 2013.

[GH14] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, Wiley, 2014.

[GW20] U. Görtz and T. Wedhorn, *Algebraic geometry i: Schemes: With examples and exercises*, Springer Studium Mathematik - Master, Springer Fachmedien Wiesbaden, 2020.

[Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Springer, 1977.

[Kol07] János Kollár, *Lectures on resolution of singularities (am-166)*, Princeton University Press, 2007.

[Kol08] János Kollár, *Semi log resolution*, arXiv preprint arXiv:0812.3592 [math.AG] (2008).

[KS12] Irwin Kra and Santiago R. Simanca, *On circulant matrices*, Notices Amer. Math. Soc. **59** (2012), no. 3, 368–377. MR 2931628

[KW13] Shigeru Kanemitsu and Michel Waldschmidt, *Matrices of finite abelian groups, finite Fourier transform and codes*, Number theory—arithmetic in Shangri-La, Ser. Number Theory Appl., vol. 8, World Sci. Publ., Hackensack, NJ, 2013, pp. 90–106. MR 3089011

[Mum99] David Mumford, *The red book of varieties and schemes*, Lecture Notes in Mathematics, vol. 1358, Springer-Verlag, Berlin, Heidelberg, 1999.

[Sok09] Alan D. Sokal, *A ridiculously simple and explicit implicit function theorem*, arXiv preprint arXiv:0902.0069 [math.CV] (2009).

[SS96] L.L. Scott and J.P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, Springer New York, 1996.

[SV11] M. J. Soto and J. L. Vicente, *The Newton procedure for several variables*, Linear Algebra Appl. **435** (2011), no. 2, 255–269. MR 2782778

[Vak24] Ravi Vakil, *The rising sea*, Published online `https://math.stanford.edu/~vakil/216blog/FOAGsep0824public.pdf`, September 2024.

[Vas70] Wolmer V. Vasconcelos, *Flat modules over commutative noetherian rings*, Trans. Amer. Math. Soc. **152** (1970), 137–143. MR 265341

[Wlo04] Jaroslaw Wlodarczyk, *Simple Hironaka resolution in characteristic zero*, arXiv preprint arXiv:math/0401401 [math.AG] (2004).

# INDEX

## Colophon

This thesis was typeset using the typographical look-and-feel `classicthesis` developed by André Miede and Ivo Pletikosić.

The style was inspired by Robert Bringhurst's seminal book on typography "*The Elements of Typographic Style*".

Here you can insert things like "Figures were created with..."

[ Insert version number/description, if you want ]