FLAT SYMPLECTIC BUNDLES OVER A RIEMANN SURFACE

by

MATTHEW KOSTER

A thesis submitted in conformity with
the requirements for the degree of
Doctor of Philosophy

Graduate Department of Mathematics
University of Toronto

ABSTRACT

Flat symplectic bundles over a Riemann surface

Matthew Koster

Doctor of Philosophy

Graduate Department of Mathematics

University of Toronto

2025

This thesis studies flat projective symplectic bundles over closed Riemann surfaces. A Milnor-Wood type inequality is established using the Maslov index to classify the isomorphism classes of principal $\mathrm{PSp}(2n;\mathbb{R})$-bundles admitting a flat connection. Special attention is given to the case $n = 1$ corresponding to $\mathrm{PSL}_2(\mathbb{R})$ where explicit topological descriptions of commutator fibers are obtained, revealing their structure as 3-manifolds. This thesis also explores the interpretation of flat $\mathrm{PSL}_2(\mathbb{R})$-bundles in terms of singular hyperbolic geometry via holonomy representations, providing an explicit correspondence between flat connections and hyperbolic structures (with singularities). These geometric techniques are applied to analyze the maximal components of the moduli space where Teichmüller space is a distinguished connected component corresponding to maximal representations.

To my younger self,
for having courage and curiosity,
And to Jess,
for joining me.

ACKNOWLEDGEMENTS

I am forever indebted to my supervisor Eckhard Meinrenken. He allowed me a great deal of latitude in where and how to direct my work, providing guidance through insight and wisdom. I would like to thank him for his remarkable patience and support with an emphasis on the attention he puts into the success of his students, starting with the excellent notes so graciously provided for each course he teaches. I feel privileged to have had the chance to work with him, and I remain deeply appreciative of all I have learned.

I would also like to thank Anton Alekseev for taking an interest in me and my work, and Marco Gualtieri, for his support and encouragement, as well as the inspirational enthusiasm and energy he brings to teaching.

I am grateful to my friends for eagerly working on problems with me and for many interesting mathematical discussions and suggestions.

I thank my friends and family for their unconditional love and support. I particularly thank my parents for maintaining stability and regularity in my life.

CONTENTS

1 BACKGROUND AND INTRODUCTION TO THE THESIS 1
1.1 Flat Connections . . . 1
1.2 The Moduli Space . . . 3
1.3 Teichmüller Theory . . . 4
1.4 Summary of the thesis . . . 7

2 THE PROJECTIVE SYMPLECTIC GROUP 9
2.1 The Maslov Index . . . 9
2.2 The projective symplectic Milnor-Wood inequality . . . 14
2.3 Existence of flat projective symplectic bundles . . . 15

3 TOPOLOGY IN LOW GENUS 21
3.1 Commutator fibers . . . 22

4 SINGULAR HYPERBOLIC GEOMETRY 30
4.1 Fenchel-Nielsen coordinates . . . 39
4.2 Hyperbolic pants . . . 40
4.3 Hyperbolic punctured torii . . . 48
4.4 The isometry groups . . . 51

5 APPENDIX 55
5.1 The classification of principal bundles . . . 55
5.2 Technical lemmas of Chapter 2. . . . 56

1

BACKGROUND AND INTRODUCTION TO THE THESIS

1.1 FLAT CONNECTIONS

Let Σ_g be a closed, connected, orientable, genus g surface. The study of flat connections over Σ_g can trace its roots back to the Gauss-Bonnet theorem, starting with the following elementary result:

Theorem 1.1. *If there exists a flat Riemannian metric on Σ_g then $g = 1$, or equivalently $\chi(\Sigma_g) = 0$.*

After re-characterizing theorem 1.1 in terms of flat connections on the tangent bundle of the surface, or equivalently flat principal connections on the associated frame bundle, there are many directions it could be generalized. By the Chern-Gauss-Bonnet theorem:

Theorem 1.2. *Let M be a compact, connected, orientable, Riemannian manifold of even dimension. If there exists a flat (principal) connection on the orthonormal frame bundle of M then $\chi(M) = 0$.*

We call a flat, torsion free connection on the tangent bundle of a manifold an *affine structure*. In 1955, Chern proposed a conjecture vastly generalizing theorem 1.2:

Conjecture 1.3 (Chern)**.** *Let M be a compact manifold. If there exists an affine structure on M then $\chi(M) = 0$.*

This was solved for closed surfaces in Benzecri's 1955 PhD thesis [Ben55], generalizing theorem 1.1 in a different direction:

Theorem 1.4 (Benzecri)**.** *The surface Σ_g admits an affine structure if and only if $g = 1$.*

Principal $SL_2(\mathbb{R})$-bundles P over Σ_g are in bijection with the integers and we denote this association by $e(P) \in \mathbb{Z}$. In a seminal paper two years after Benzecri's thesis, Milnor [Mil58] (who was on Benzecri's thesis committee) generalized Benzecri's result to:

Theorem 1.5 (Milnor). *There exists a flat connection on the principal* $\mathrm{SL}_2(\mathbb{R})$*-bundle P if and only if the Euler number of P is in* $|e(P)| \leq \max(g-1, 0)$.

Since $|e(F(T\Sigma_g))| = 2(g-1) > g-1$ for $g \geq 2$, Benzecri's result is recovered. A sketch of one (the "only if") direction of Milnor's argument is as follows: Let $\pi\colon \widetilde{G} \to G$ denote the universal covering map and let $\Phi_g^{\widetilde{G}}\colon \widetilde{G}^{2g} \to \widetilde{G}$ be the map:

$$\Phi_g^{\widetilde{G}}(A_1, B_1, ..., A_g, B_g) = \prod_{i=1}^{g}[A_i, B_i] = \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1}. \qquad (1.1)$$

Since $\ker(\pi) \cong \pi_1(G)$, the isomorphism class of P can be viewed either as an element of $\mathrm{Cent}(\widetilde{G})$ or as an element of $\pi_1(G)$. To distinguish these two situations we will use the notation $\deg(P) \in \mathrm{Cent}(\widetilde{G}) \cong \frac{1}{2}\mathbb{Z}$ and $e(P) \in \pi_1(G) \cong Z$ (cf. below). Examining the details of the bijection between principal G-bundles over Σ_g and $\pi_1(G) \cong \ker(\pi)$, when the dust settles one finds that there exists a flat connection on P if and only if:

$$\deg(P) \in \mathrm{Im}(\Phi_g^{\widetilde{G}}) \cap \ker\pi \qquad (1.2)$$

so what remains to be done is a calculation of $\mathrm{Im}(\Phi_g^{\widetilde{G}}) \cap \ker\pi$ when $G = \mathrm{SL}_2(\mathbb{R})$. Inspecting $\Phi_g^{\widetilde{G}}$, we see that it is factored into a product of elements having a particular form - commutators. A strategy then presents itself; concoct a function $\widetilde{G} \to \mathbb{R}$ that restricts to the identity on $\ker(\pi)$ and is sufficiently straightforward to work out on 1) commutators and 2) products. One way of producing such a function uses the Cartan decomposition $G = K \times \exp(\mathfrak{p})$ where $K = SO(2)$ and $\mathfrak{p}$ are the symmetric matrices of trace 0. Let $\mathrm{proj}_{\widetilde{K}}\colon \widetilde{G} \to \tilde{K} \cong \mathbb{R}$ be the lift of proj_K. Milnor shows that $\mathrm{proj}_{\widetilde{K}}$ satisfies:

$$|\mathrm{proj}_{\widetilde{K}}(AB) - \mathrm{proj}_{\widetilde{K}}(A) - \mathrm{proj}_{\widetilde{K}}(B)| < 1/4 \qquad (1.3)$$

so $|\deg(P)| \leq g-1$. What is shown more generally by Milnor's argument is that:

$$\mathrm{Im}(\Phi_g^{\widetilde{G}}) \cap \mathrm{Cent}(\widetilde{G}) \subseteq \{k \in \frac{1}{2}\mathbb{Z} : |k| \leq g - \frac{1}{2}\} \qquad (1.4)$$

where the half-integers correspond to paths in $\mathrm{SL}_2(\mathbb{R})$ ending at $-I$ (i.e. elements of $\mathrm{Cent}(\widetilde{G}) \setminus \ker(\pi)$). This was sharpened by Wood [Woo71] to:

$$\mathrm{Im}(\Phi_g^{\widetilde{G}}) \cap \mathrm{Cent}(\widetilde{G}) = \{k \in \frac{1}{2}\mathbb{Z} : |k| \leq g-1\} \qquad (1.5)$$

providing a complete classification of principal bundles supporting a flat connection for any quotient of $\widetilde{SL}_2(\mathbb{R})$:

Theorem 1.6 (Milnor-Wood)**.** *If $G = \widetilde{SL}_2(\mathbb{R})/\langle k\rangle$ for a subgroup $\langle k\rangle \leq \mathrm{Cent}(\widetilde{G})$ then there exists a flat connection on the principal G-bundle $P \to \Sigma_g$ if and only if $|\deg(P)| \leq \max(g-1, 0)$.*

For $G = \widetilde{SL}_2(\mathbb{R})/\langle k\rangle$, deg and e are related by $\deg(P) = ke(P)$ in which case the above inequality becomes $|e(P)| \leq (g-1)/k$. In particular taking $k = \frac{1}{2}$ we have the *classical Milnor-Wood inequality*:

Corollary 1.7 (Milnor-Wood inequality)**.** *There exists a flat connection on the principal $\mathrm{PSL}_2(\mathbb{R})$-bundle $P \to \Sigma_g$ if and only if $|e(P)| \leq \max(2g-2, 0)$.*

Because it will be relevant later, we present a trick from [BIW14] (see the discussion after theorem 3.4) to achieve Wood's contribution (1.5) directly from Milnor's result (theorem 1.5). For any integer $m \geq 1$ there exists an m-fold covering map $\pi_m \colon \Sigma_{m(g-1)+1} \to \Sigma_g$. But this means:

$$m|\deg(P)| = |\pi_m^* \deg(P)| = |\deg(\pi_m^*(P))| \leq m(g-1) + 1 - \frac{1}{2} \tag{1.6}$$

by theorem 1.5. Dividing by m and taking $m \to \infty$ the result follows.

1.2 THE MODULI SPACE

For G a connected Lie group we define the *moduli space of flat G-bundles* by:

$$\mathcal{M}(G, \Sigma_g) = \frac{\{(P,\theta) : P \to \Sigma_g \text{ a principal } G\text{-bundle, } \theta \text{ a flat connection on } P\}}{\sim} \tag{1.7}$$

where $(P_1, \theta_1) \sim (P_2, \theta_2)$ if there exists a bundle isomorphism $\varphi \colon P_1 \to P_2$ such that $\varphi^*\theta_2 = \theta_1$. By fixing a basepoint $x_0 \in \Sigma_g$ and taking holonomies:

$$\mathcal{M}(G, \Sigma_g) = \mathrm{Hom}(\pi_1(\Sigma_g, x_0), G)/G \tag{1.8}$$

where G acts on $\mathrm{Hom}(\pi_1(\Sigma_g, x_0), G)$ by $g \cdot \phi = \mathrm{Ad}_g(\phi)$. With the standard presentation:

$$\pi_1(\Sigma_g, x_0) = \langle a_1, b_1, ..., a_g, b_g : \prod_{i=1}^{g} [a_i, b_i] = e\rangle$$

we have that:

$$\mathrm{Hom}(\pi_1(\Sigma_g, x_0), G) \cong (\Phi_g^G)^{-1}(e) \tag{1.9}$$

where $\Phi_g^G\colon G^{2g} \to G$ is the product of commutators map:

$$\Phi_g^G(A_1, B_1, ..., A_g, B_g) = \prod_{i=1}^{g}[A_i, B_i] = \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1}. \tag{1.10}$$

Thus:

$$\mathcal{M}(G, \Sigma_g) \cong \mathrm{Hom}(\pi_1(\Sigma_g, x_0), G)/G \cong (\Phi_g^G)^{-1}(e)/G \tag{1.11}$$

where G acts on G^{2g} by diagonal conjugation.

The map Φ_g^G has a lift to the universal cover $\widetilde{\Phi}_g^G\colon G^{2g} \to \widetilde{G}$. Its restriction to $(\widetilde{\Phi}_g^G)^{-1}(e)$ takes values in $\mathrm{Cent}(\widetilde{G})$ and descends to the map :

$$\mathcal{M}(G, \Sigma_g) \to \mathrm{Cent}(\widetilde{G}), \qquad [(P, \theta)] \mapsto \deg(P).$$

By continuity, this map is constant on connected components of the moduli space but it does not in general provide a bijection between the connected components and $\mathrm{Cent}(\widetilde{G})$. An investigation of the various connected components of $\mathcal{M}(G, \Sigma_g)$ has become an active area of research and is a major theme in this thesis. We will almost exclusively take the holonomy perspective on $\mathcal{M}(G, \Sigma_g)$ and concern ourselves with:

$$(\Phi_g^G)^{-1}(e) = (\widetilde{\Phi}_g^G)^{-1}(\mathrm{Cent}(\widetilde{G})), \tag{1.12}$$

often calling $\phi \in (\Phi_g^G)^{-1}(e)$ a *representation*. It was first observed by Atiyah-Bott [AB83] that for any connected Lie group G whose Lie algebra $\mathfrak{g}$ has an Ad-invariant metric, there is a natural symplectic structure on $\mathcal{M}(G, \Sigma_g)$. Further results (e.g. topological) generally involve imposing additional restrictions on G and exploiting some underlying structure, and with the number of connected components being perhaps the simplest invariant of a topological space, a Milnor-Wood type inequality is typically the first step in the analysis.

For example, it was shown by Alekseev-Malkin-Meinrenken [AMM98] that Φ_g^G is an example of a *group-valued moment map* and moreover when G is compact and simply connected such maps have connected fibers. Hence for G compact with finite center (so that $\widetilde{G}$ is compact simply connected) $(\widetilde{\Phi}_g^G)^{-1}(k)$ is connected for all $k \in \mathrm{Cent}(\widetilde{G})$.

1.3 TEICHMÜLLER THEORY

Study of the moduli space $\mathcal{M}_g$ of complex structures on Σ_g up to diffeomorphism dates back to Riemann, who had shown that it was described by

$6g-6$ parameters [Rie57] (where he called them moduli), but a rigorous description of $\mathcal{M}_g$ was unclear - topology as an area of mathematics did not exist in Riemann's time and $\mathcal{M}_g$ is quite complicated. A key breakthrough was the introduction of *Teichmüller space* $\mathcal{T}_g$, a coarser yet still finite dimensional quotient on the set of complex structures on Σ_g such that $\mathcal{T}_g \to \mathcal{T}_g/\mathrm{MCG}(\Sigma_g) = \mathcal{M}_g$ is the universal cover. Fixing a complex structure on Σ_g, a point $([S,f]) \in \mathcal{T}_g$ is an equivalence class of pairs (S,f) where S is a Riemann surface, $f\colon \Sigma_g \to S$ is an orientation preserving diffeomorphism (called a *marking*), and (S_1,f_1), (S_2,f_2) are declared equivalent if $f_1 \circ f_2^{-1}$ is homotopic to a biholomorphism. A fundamental result from which $\mathcal{T}_g$ begins to derive its utility is the existence of *Fenchel-Nielsen coordinates* [FN03]; a homeomorphism $\mathcal{T}_g \cong \mathbb{R}^{6g-6}$ giving $\mathcal{M}_g$ an *orbifold* structure with $\mathcal{T}_g$ as its *universal covering orbifold*.

Let $\langle \alpha_1, \beta_1, ..., \alpha_g, \beta_g : \prod_{i=1}^{g}[\alpha_i, \beta_i] = e\rangle$ be the standard representation for $\pi_1(\Sigma_g)$. Consider a pair (S,f) representing a point in $\mathcal{T}_g$. Choosing a Fuchsian representation $S = \mathbb{H}/\Gamma$ for a Fuchsian group $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$, the marking induces an isomorphism:

$$f_*\colon \pi_1(\Sigma_g) \to \pi_1(S) = \Gamma \leq \mathrm{PSL}_2(\mathbb{R}) \tag{1.13}$$

so:

$$(f(\alpha_1), f(\beta_1), ..., f(\alpha_g), f(\beta_g)) \in (\Phi_g^{\mathrm{PSL}_2(\mathbb{R})})^{-1}(e) \subset \mathrm{PSL}_2(\mathbb{R})^{2g} \tag{1.14}$$

and equivalent pairs (S_1,f_1), (S_2,f_2) differ by the diagonal conjugation action of G. After taking quotients we have therefore defined an embedding:

$$\mathcal{T}_g \to (\Phi_g^{\mathrm{PSL}_2(\mathbb{R})})^{-1}(e)$$

identifying Teichmüller space in $\mathcal{M}(G,\Sigma_g)$ where, under this identification, $\mathcal{T}_g$ is identified with $(\widetilde{\Phi}_g^G)^{-1}(2g-2)$, distinguishing Teichmüller space with a set of representations maximizing the Milnor-Wood inequality.

For other Lie groups G, a common idea after producing a Milnor-Wood type inequality has been the identification of a distinguished $\mathcal{T}_g$-esque subset in $(\Phi_g^G)^{-1}(e)$ maximizing it. When G is a split real group there is a distinguished component *Hitchin component* discovered by Hitchin [Hit92] and studied by Labourie [Lab06] and Fock-Goncharov [FG06] (among others) with Fock-Goncharov providing a generalization of Fenchel-Nielsen coordinates. When G is of Hermitian type the so-called *Toledo invariant* is defined and the distinguished subset maximizing it has been studied by Bradlow - García-Prada - Gothen [BGPG03] [BGPG06] and Burger-Iozzi-Wienhard [BIW03] At the intersection of these two families are central extensions of $\mathrm{PSp}(2n;\mathbb{R})$ (i.e. having universal cover $\widetilde{\mathrm{Sp}}(2n;\mathbb{R})$); characterized as being the only Lie groups that are both split real and Hermitian.

As was previously mentioned, the map Φ_g^G is already factored so we are given a natural strategy of starting with commutators and building up, as Milnor did. When $G \leq \mathrm{SL}_n(\mathbb{R})$ is a matrix group with $n > 2$, the entries of a commutator are not particularly easy to work with so one is tempted to return to the flat connection perspective. Viewed as flat principal bundles, factoring Φ_g^G into a product of commutators is equivalent to decomposing Σ_g into a union of g punctured torii T_i fused together at a g-hored sphere, and then adding things together. But every principal bundle over a surface with boundary is trivial so care is needed (one cannot simply add up contributions to $\deg(P)$ from $\deg(P|_{T_i})$ as $\deg(P|_{T_i}) = 0$).

For $G = \mathrm{PSL}(2;\mathbb{R})$, Goldman in [Gol88] defined a *relative* version of $e(P)$ for surfaces with boundary such that

$$e(P|_{T_1}) + e(P|_{T_2}) = e(P|_{T_1 \cup T_2}) \tag{1.15}$$

and moreover is equal to the $e(P)$ when the boundary is empty. The relative Euler class was used to show that $(\widetilde{\Phi}_g^G)^{-1}(k)$ is connected for all $k \in \mathrm{Cent}(\widetilde{G})$ hence:

Theorem 1.8 (Goldman)**.** *Equivalence classes of flat G-bundles* $([P_1, \theta_1])$, $([P_2, \theta_2])$ *lie in the same connected component of* $\mathcal{M}(G, \Sigma_g)$ *if and only if* $P_1 \cong P_2$.

Around the same time as Goldman's result, Hitchin in [Hit87] introduced what is now called a *Higgs bundle*: a principal G-bundle P with connection θ together with a section Φ of $\mathrm{Ad}\, P \otimes \mathbb{C}$ called a *Higgs field* satisfying the *self-duality* equations $\mathrm{curv}_\theta = -[\Phi, \Phi^*]$ and $d''_\theta \Phi = 0$. For a given principal bundle P, the set of pairs (θ, Φ) satisfying the self-duality equations modulo gauge transformations of P is given the structure of a hyperkähler manifold $\mathcal{M}$. Following Frankel [Fra59], the Higgs field is used to define a Morse function $2i \int_{\Sigma_g} \mathrm{tr}(\Phi\Phi^*)$ on $\mathcal{M}$ from which the Betti numbers of $\mathcal{M}$ are found. Of particular interest to us, the hyperkähler structure on $\mathcal{M}$ is used to identify $\mathcal{M}(\mathrm{PSL}_2(\mathbb{R}), \Sigma_g)$ as a fixed point set of an involution and providing additional geometric structure:

Theorem 1.9 (Hitchin)**.** *For* $G = \mathrm{PSL}_2(\mathbb{R})$ *and each nonidentity* $k \in \mathrm{Cent}(\widetilde{G})$, $(\widetilde{\Phi}_g^G)^{-1}(k)/G$ *is the total space of a complex vector bundle of rank* $g - 1 + |k|$ *over the* $|k|$*-fold symmetric product* $(\Sigma_g)^{2g-2-|k|}/S_{2g-2-|k|}$. *Moreover, this bundle is trivial if and only if (1)* $|k| = 2g - 2$ *or (2)* $|k| = 2g - 3$ *and* g *is even.*

For $n > 1$, the Toledo invariant was generalized by Burger-Iozzi-Wienhard in [BIW10] to Σ_g having nonempty boundary using the theory of bounded cohomology to function in a similar way to Goldman's relative Euler class, while Kim-Pansu-Wan in [KPW22] generalized it to the *signature*, developing the theory using the Atiyah-Singer index theorem. In any case (see e.g. theorem 1 in [BIW10] or theorem 4.5 in [KPW22]) the Milnor-Wood inequality for $\mathrm{Sp}(2n;\mathbb{R})$ takes the form:

Theorem 1.10 (Symplectic Milnor-Wood)**.** *A principal* $\mathrm{Sp}(2n;\mathbb{R})$*-bundle* P *over* Σ_g *supports a flat connection if and only if* $|e(P)| \leq \max(0, n(g-1))$.

1.4 SUMMARY OF THE THESIS

Let $G = \mathrm{Sp}(2n;\mathbb{R})$ be the symplectic group with universal cover $p\colon \widetilde{G} \to G$. In this thesis we start with quotients of $\widetilde{G}$ where we take advantage of the *Maslov index*:

$$\mu\colon \widetilde{G} \to \frac{1}{2}\mathbb{Z}$$

to produce a Milnor-Wood inequality. In particular we prove:

Theorem 1.11. *For G a quotient of* $\widetilde{\mathrm{Sp}}(2n;\mathbb{R})$ *there exists a flat connection on the principal G-bundle* $P \to \Sigma_g$ *if and only if* $|\mu(\deg(P))| \leq \max(0, n(2g-2))$.

The center of $\widetilde{G}$ is $\mathbb{Z}$ for n odd and $2\mathbb{Z} \times Z/2\mathbb{Z}$ for n even. In the first case $|\mu(z)| = |z|$ while in the second $|\mu(z,w)| = |z|$ so for the projective symplectic group $\mathrm{PSp}(2n;\mathbb{R}) = \widetilde{G}/\operatorname{Cent}(\widetilde{G})$:

Corollary 1.12. *There exists a flat connection on the projective* $\mathrm{PSp}(2n;\mathbb{R})$*-bundle* $P \to \Sigma_g$ *if and only if* $|e(P)| \leq \max(0, n(2g-2))$ *for* n *odd or* $|proj_{2\mathbb{Z}}(e(P))| \leq \max(0, n(2g-2))$ *for* n *even.*

The first direction of the proof is thematically similar to Milnor's original argument for $n = 1$ in [Mil58]. The Maslov index is shown to satisfy:

$$|\mu[A,B]| \leq n, \qquad |\mu(AB) - \mu(A) - \mu(B)| \leq n \tag{1.16}$$

which combine to give the rough estimate $|e(P)| \leq n(2g-1)$. Using the trick from [BIW14] this is immediately improved to $|e(P)| \leq n(2g-2)$. For the other direction we show that it is almost enough to consider $n = 1$. All that is missing is the existence of $X \in \mathrm{Sp}(4;\mathbb{R})$ and $x_1, x_2, y_1, y_2 \in \widetilde{\mathrm{Sp}}(4;\mathbb{R})$ with:

$$\mu[x_1, y_1] = \mu[x_2, y_2] = 2, \quad p[x_1, y_1] = X, \quad p[x_2, y_2] = -X^{-1} \tag{1.17}$$

The construction of x_1, y_1 again reduces to $n = 1$ but x_2, y_2 does not.

We then specialize to $n = 1$ and make explicit progress on the topology of $\mathcal{M}(G, \Sigma_g)$ in low genus, starting with the topology of commutator fibers by proving:

Theorem 1.13. *For $G = \mathrm{PSL}_2(\mathbb{R})$ and $g \in \widetilde{G}$, if $(\widetilde{\Phi}_g^G)^{-1}(g)$ is not empty then:*

$$(\widetilde{\Phi}_g^G)^{-1}(g) = \begin{cases} \mathbb{R}^3 & -\infty < \mathrm{tr}(g) \leq -2 \\ S^1 \times \mathbb{R}^2 & -2 < \mathrm{tr}(g) \leq 2 \\ (T^2 \setminus (1,1)) \times \mathbb{R} & 2 < \mathrm{tr}(g) < \infty \end{cases}$$

where $\mathrm{tr}(g)$ is the trace of $p(g) \in \mathrm{SL}_2(\mathbb{R})$.

We then use these ideas to study $\mathcal{T}_g = (\widetilde{\Phi}_g^G)^{-1}(2 - 2g)$ explicitly from the holonomy perspective, using a convenient Iwasawa-like decomposition of $\mathrm{SL}_2(\mathbb{R})$ to elucidate the relationship between the holonomies of a flat connection and hyperbolic geometry. In particular, given:

$$\phi \in (\widetilde{\Phi}_g^G)^{-1}(2 - 2g) \tag{1.18}$$

we show directly how to read off the Fenchel-Nielsen coordinates of the associated Riemann surface in $\mathcal{T}_g$.

2 THE PROJECTIVE SYMPLECTIC GROUP

A Milnor-Wood inequality for principal $\mathrm{Sp}(2n;\mathbb{R})$-bundles is known but the proofs involve sophisticated geometric constructions (the Higgs bundle approach of [GPGMiR13], [Got01], bounded cohomlogy in e.g. [BIW10]) or following from arguments in [Tur84] as in [Dup79]. We use the Maslov index to provide an elementary argument for arbitrary quotients of $\widetilde{\mathrm{Sp}}(2n;\mathbb{R})$.

Let G be a quotient of $\widetilde{\mathrm{Sp}}(2n;\mathbb{R})$ by a central subgroup. Our main result is the following:

Theorem 2.1. *There exists a flat connection on the principal G-bundle $P \to \Sigma_g$ if and only if $|\mu(\deg(P))| \leq \max(0, n(2g-2))$ where $\mu\colon \widetilde{\mathrm{Sp}}(2n;\mathbb{R}) \to \frac{1}{2}\mathbb{Z}$ is the Maslov index.*

In Section 2.1 we review the definition of the Maslov index and present the basic properties used in the subsequent sections. The proof of theorem 2.1 is separated in half; in section 2.2 we prove:

Theorem 2.2. *If there exists a flat connection on the principal G-bundle $P \to \Sigma_g$ then $|\mu(\deg(P))| \leq \max(0, n(2g-2))$*

and in section 2.3 we prove:

Theorem 2.3. *If $|\mu(\deg(P))| \leq \max(0, n(2g-2))$ then there exists a flat connection on P.*

2.1 THE MASLOV INDEX

We begin with a review of the Maslov index. See e.g. [Arn67], [LV80], or [Mei94] for additional details.

Let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and define the symplectic group:

$$\mathrm{Sp}(2n;\mathbb{R}) = \{M \in \mathrm{Mat}(2n;\mathbb{R}) : M^t J M = J\} \tag{2.1}$$

Then:

$$((u_1, u_2), (v_1, v_2)) \mapsto u_1^t J v_1 - u_2^t J v_2$$

defines a symplectic form on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ such that $\Gamma_T \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ is Lagrangian for every $T \in \mathrm{Sp}(2n;\mathbb{R})$.

Given three Lagrangian subspaces L_1, L_2, L_3 of a symplectic vector space (E, ω_0) we denote by $s(L_1, L_2, L_3)$ the signature of the quadratic form:

$$\begin{aligned} &Q(L_1, L_2, L_3)\colon L_1 \times L_2 \times L_3 \to \mathbb{R} \\ &(x_1, x_2, x_3) \mapsto \omega_0(x_1, x_2) + \omega_0(x_2, x_3) + \omega_0(x_3, x_1). \end{aligned}$$

Definition 2.4. The map $r\colon \mathrm{Sp}(2n;\mathbb{R}) \times \mathrm{Sp}(2n;\mathbb{R}) \to \mathbb{Z}$:

$$r(A, B) = s(\Gamma_I, \Gamma_A, \Gamma_{AB})$$

is called the **Maslov cocycle**.

The *cocycle identity* for r reads $\delta r = 0$ where:

$$\delta r(A, B, C) = r(B, C) - r(AB, C) + r(A, BC) - r(A, B) \tag{2.2}$$

Thus r is a 2-cocycle on $\mathrm{Sp}(2n;\mathbb{R})$ with values in $\mathbb{Z}$. The cocycle identity may be seen from the fact that its pullback to $\widetilde{\mathrm{Sp}}(2n;\mathbb{R})$ is a coboundary; see item 4 of proposition 2.9.

Lemma 2.5. *$r(A, B)$ is equal to the signature of the symmetric matrix:*

$$\begin{pmatrix} 0 & J - JA & JAB - J \\ (J - JA)^t & 0 & J - JB \\ (JAB - J)^t & (J - JB)^t & 0 \end{pmatrix}.$$

Proof. Starting with the canonical symplectic basis $\{e_1, f_1, ..., e_n, f_n\}$ for $\mathbb{R}^{2n}$ and any $M \in \mathrm{Sp}(2n;\mathbb{R})$ we define the ordered basis:

$$\beta_M = \{(e_1, M(e_1), (f_1, M(f_1))), ..., (e_n, M(e_n)), (f_n, M(f_n))\}$$

for Γ_M. With respect to $\beta = \beta_I \cup \beta_A \cup \beta_{AB}$ it is readily verified that the symmetric bilinear form associated to $Q(\Gamma_I, \Gamma_A, \Gamma_{AB})$ has matrix representation:

$$\begin{pmatrix} 0 & J - JA & JAB - J \\ (J - JA)^t & 0 & J - JB \\ (JAB - J)^t & (J - JB)^t & 0 \end{pmatrix}.$$

□

Corollary 2.6. *$r(A,B)$ is equal to the signature of the symmetric matrix:*

$$J(B^{-1}-I)(I-A)^{+}(AB-I) \tag{2.3}$$

where M^{+} denotes the Moore-Penrose pseudoinverse of M. In particular:

$$|r| \leq 2n \tag{2.4}$$

Recall that the Moore-Penrose pseudoinverse of M is the unique matrix M^{+} satisfying:

1. $MM^{+}M = M$.
2. $M^{+}MM^{+} = M^{+}$.
3. $(MM^{+})^{t} = MM^{+}$.
4. $(M^{+}M)^{t} = M^{+}M$.

Proof. The signature of any symmetric matrix $\begin{pmatrix} X & Y \\ Y^{t} & W \end{pmatrix}$ is equal to $\operatorname{sgn}(X) + \operatorname{sgn}(W - Y^{t}X^{+}Y)$ (see e.g. lemma 4.1.2 and the surrounding discussion in [Dui11]). Taking:

$$X = \begin{pmatrix} 0 & J-JA \\ (J-JA)^{t} & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} JAB-J \\ J-JB \end{pmatrix}, \quad W = 0 \tag{2.5}$$

we have $\operatorname{sgn}(X) = 0$ so:

$$\begin{aligned}
r(A,B) &= \operatorname{sgn}\begin{pmatrix} 0 & J-JA & JAB-J \\ (J-JA)^{t} & 0 & J-JB \\ (JAB-J)^{t} & (J-JB)^{t} & 0 \end{pmatrix} \\
&= -\operatorname{sgn}\left(\begin{pmatrix} (JAB-J)^{t} & (J-JB)^{t} \end{pmatrix} \begin{pmatrix} 0 & J-JA \\ (J-JA)^{t} & 0 \end{pmatrix}^{+} \begin{pmatrix} JAB-J \\ J-JB \end{pmatrix} \right) \\
&= -\operatorname{sgn}\left(\begin{pmatrix} (JAB-J)^{t} & (J-JB)^{t} \end{pmatrix} \begin{pmatrix} 0 & ((J-JA)^{+})^{t} \\ (J-JA)^{+} & 0 \end{pmatrix} \begin{pmatrix} JAB-J \\ J-JB \end{pmatrix} \right) \\
&= \operatorname{sgn}\left(J\left((I-(AB)^{-1})(I-A^{-1})^{+}(B-I) + (B^{-1}-I)(I-A)^{+}(AB-I) \right) \right).
\end{aligned}$$

Finally, it follows from a short calculation that:

$$(I-(AB)^{-1})(I-A^{-1})^{+}(B-I) = (B^{-1}-I)(I-A)^{+}(AB-I) \tag{2.6}$$

□

Remark 2.7. The identity 2.6 is a purely algebraic fact and holds in any ring where both sides are well defined (e.g. if the pseudoinverses are replaced by inverses).

Definition 2.8. Let $A_t\colon [0,1] \to \mathrm{Sp}(2n;\mathbb{R})$ be a path with $A_0 = I$. Choose a partition $0 = t_0 \leq t_1 \leq \cdots \leq t_m = 1$ such that for each $1 \leq j \leq m$ there exists a Lagrangian subspace $U_i \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ transverse to Γ_I and Γ_{A_t} for $t \in [t_{i-1}, t_i]$. We define the **Maslov index** μ of A_t:

$$\mu(A_t) = \frac{1}{2}\sum_{i=1}^{m}\left(s(\Gamma_I, \Gamma_{A_{t_{i-1}}}, U_i) - s(\Gamma_I, \Gamma_{A_{t_i}}, U_i)\right). \tag{2.7}$$

This quantity does not depend on the choice of partition nor on the choice of U_i for a given partition and is invariant under homotopies of A_t so it descends to a well defined function:

$$\widetilde{\mathrm{Sp}}(2n;\mathbb{R}) \to \frac{1}{2}\mathbb{Z} \tag{2.8}$$

that we also call the Maslov index and denote by μ.

Let $\pi\colon \widetilde{\mathrm{Sp}}(2n;\mathbb{R}) \to \mathrm{Sp}(2n;\mathbb{R})$ denote the covering map. The Maslov index satisfies a number of properties that we collect here for convenience.

Proposition 2.9.

1. $\mu(gAg^{-1}) = \mu(A)$.
2. $\mu(A^{-1}) = -\mu(A)$.
3. *For* $A \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})$, $B \in \widetilde{\mathrm{Sp}}(2m;\mathbb{R})$, *their direct sum* $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \widetilde{\mathrm{Sp}}(2(m+n);\mathbb{R})$ *satisfies:*

$$\mu\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \mu(A) + \mu(B) \tag{2.9}$$

   where the symplectic form on $\mathbb{R}^{2(m+n)}$ *is the direct sum (rather than the symplectic direct sum) of those on* $\mathbb{R}^{2m}$ *and* $\mathbb{R}^{2n}$.
4. $\mu(AB) = \mu(A) + \mu(B) + \frac{1}{2}r(\pi(A), \pi(B))$.
5. *For* $X \in \mathfrak{sp}(2n;\mathbb{R})$ *with no purely imaginary eigenvalues,* $\mu(\exp(tX)) = 0$.

We will refer to property (4) as the *coboundary identity*. As a simple but helpful first application we evaluate the Maslov index on commutators:

Corollary 2.10. *For* $A, B \in \widetilde{\mathrm{Sp}}(2n; \mathbb{R})$*:*

$$\mu([A, B]) = \frac{1}{2} r(\pi(A), \pi(BA^{-1}B^{-1}))$$

Proof. By properties (1), (2), and (4) of proposition 2.9:

$$\begin{aligned}\mu([A, B]) = \mu(ABA^{-1}B^{-1}) &= \mu(A) + \mu(BA^{-1}B^{-1}) + \frac{1}{2} r(A, BA^{-1}B^{-1}) \\ &= \frac{1}{2} r(\pi(A), \pi(BA^{-1}B^{-1}))\end{aligned}$$

□

Lemma 2.11. *If* z *is a generator of* $\mathrm{Cent}(\widetilde{\mathrm{Sp}}(2; \mathbb{R}))$ *then* $|\mu(z)| = 1$ *(compare [Mei94] example 7).*

Proof. Let $R_t = \begin{pmatrix} \cos(t\frac{\pi}{2}) & \sin(t\frac{\pi}{2}) \\ -\sin(t\frac{\pi}{2}) & \cos(t\frac{\pi}{2}) \end{pmatrix}$ for $0 \leq t \leq 1$. Then Γ_{-I} is transverse to Γ_I and Γ_{R_t} for all t. When $U = \Gamma_T$ we have:

$$s(\Gamma_I, \Gamma_S, U) = s(\Gamma_I, \Gamma_S, \Gamma_T) = r(S, S^{-1}T) \tag{2.10}$$

so by corollary 2.6:

$$\begin{aligned}\mu(R_t) &= \frac{1}{2} \left(s(\Gamma_I, \Gamma_I, \Gamma_{-I}) - s(\Gamma_I, \Gamma_J, \Gamma_{-I}) \right) \\ &= -\frac{1}{2} r(J, J) \\ &= -\frac{1}{2} \operatorname{sgn} \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \\ &= 1.\end{aligned}$$

Using the coboundary identity:

$$\mu(z) = \mu(R_t^2) = 2\mu(R_t) + \frac{1}{2} r(J, J) = 1$$

and by proposition 2.9 (2) $\mu(z^{-1}) = -1$. □

Corollary 2.12. *If* z *is a generator of* $\pi_1(\mathrm{Sp}(2n; \mathbb{R}))$ *then* $|\mu(z)| = 2$

Proof. Let w be a generator of $\mathrm{Cent}(\widetilde{\mathrm{Sp}}(2; \mathbb{R}))$. By the previous lemma, the coboundary identity, and corollary 2.6:

$$\mu(z) = \mu(w^2) = 2\mu(w) + \frac{1}{2} r(-I, -I) = 2.$$

By proposition 2.9 (3) the result follows for $n > 1$. □

Corollary 2.13. *If n is even then $\mu(z)$ is even for $z \in \mathrm{Cent}(\widetilde{\mathrm{Sp}}(2n;\mathbb{R}))$.*

Proof. When n is even, $\mathrm{Cent}(\widetilde{\mathrm{Sp}}(2;\mathbb{R}))$ is generated by w, z where z is a generator of $\pi_1(\mathrm{Sp}(2;\mathbb{R}))$ and w has order 2. Then for $wz^k \in \mathrm{Cent}(\widetilde{\mathrm{Sp}}(2;\mathbb{R}))$ we compute using the coboundary identity:

$$\mu(wz^k) = \mu(w) + \mu(z^k) + \frac{1}{2}r(-I, I) = \mu(w) + k\mu(z).$$

Since w has order 2, $\mu(w) = 0$ as:

$$0 = \mu(e) = \mu(w^2) = 2\mu(w) + \frac{1}{2}r(-I, -I) = 2\mu(w)$$

so $\mu(wz^k) = k\mu(z)$ is even by corollary 2.12. □

2.2 THE PROJECTIVE SYMPLECTIC MILNOR-WOOD INEQUALITY

Proposition 2.14. *For any $(A_1, B_1, ..., A_g, B_g) \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})^{2g}$:*

$$|\mu(\prod_{i=1}^{g}[A_i, B_i])| \leq n(2g-1). \tag{2.11}$$

Proof. Let $Z_i = [A_i, B_i]$. By corollary 2.10:

$$\mu(Z_i) = \frac{1}{2}r(\pi(A_i), \pi(B_i A_i^{-1} B_i^{-1}))$$

so by corollary 2.6 $|\mu(Z_i)| \leq n$. Then by the coboundary identity:

$$\begin{aligned} \left|\mu\left(\prod_{i=1}^{g} Z_i\right)\right| &= \left|\mu\left(\prod_{i=1}^{g-1} Z_i\right) + \mu\left(Z_g\right) + \frac{1}{2}r\left(\pi(\prod_{i=1}^{g-1} Z_i), \pi(Z_g)\right)\right| \\ &\leq \left|\mu\left(\prod_{i=1}^{g-1} Z_i\right)\right| + 2n \end{aligned}$$

thus the result follows from inducting on g. □

Let G be any quotient of $\widetilde{\mathrm{Sp}}(2n;\mathbb{R})$ by a central subgroup.

Corollary 2.15. *If (P, θ) is a flat principal G-bundle over Σ_g then $|\mu(e(P))| \leq n(2g-1)$.*

Taking $n = 1$ and $g = 1$ and proposition 2.9 (5) we isolate the following observation:

Corollary 2.16. *For* $A, B \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$, $\mu([A,B]) = 0$ *if* $\mathrm{tr}[A,B] > 2$ *and* $|\mu[A,B]| = 1$ *if* $\mathrm{tr}[A,B] < -2$.

Using the trick from [BIW14] (who attribute it to Gromov) mentioned in the introduction, we improve the rough estimate of corollary 2.15:

Corollary 2.17. *If* (P, θ) *is a flat principal G-bundle over* Σ_g *then* $|\mu(e(P))| \leq n(2g-2)$.

Proof. Let $k > 1$ be an integer and let $\pi_k \colon \Sigma_{k(g-1)+1} \to \Sigma_g$ be a k-fold covering map. Then:

$$k|\mu(e(P))| = |\mu(e(\pi_k^* P))| \leq n(2(k(g-1)+1)-1) = nk(2g-2)+n$$

so $|\mu(e(P))| \leq n(2g-2) + n/k$. Taking $k \to \infty$ the result follows.

□

Corollary 2.18. *For any* $(A_1, B_1, ..., A_g, B_g) \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})^{2g}$, *if* $\prod_{i=1}^{g}[A_i, B_i] \in \mathrm{Cent}(\widetilde{\mathrm{Sp}}(2n;\mathbb{R}))$ *then:*

$$|\mu(\prod_{i=1}^{g}[A_i, B_i])| \leq n(2g-2) \tag{2.12}$$

2.3 EXISTENCE OF FLAT PROJECTIVE SYMPLECTIC BUNDLES

We now prove the converse to corollary 2.18:

Theorem 2.19. *For every* $|k| \leq n(2g-2)$ *and* $z \in \mathit{Cent}(\widetilde{G})$ *with* $\mu(z) = k$ *there exists* $(A_1, B_1, ..., A_g, B_g) \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})^{2g}$ *such that:*

$$\prod_{i=1}^{g}[A_i, B_i] = z$$

We start with a few reductions. First, observe that for block matrices:

$$\prod_{i=1}^{g}\left[\begin{pmatrix} A_{1,i} & & & \\ & A_{2,i} & & \\ & & \ddots & \\ & & & A_{m,i} \end{pmatrix}, \begin{pmatrix} B_{1,i} & & & \\ & B_{2,i} & & \\ & & \ddots & \\ & & & B_{m,1} \end{pmatrix}\right]$$
$$= \begin{pmatrix} \prod_{i=1}^{g}[A_{1,i}, B_{1,i}] & & & \\ & \prod_{i=1}^{g}[A_{2,i}, B_{2,i}] & & \\ & & \ddots & \\ & & & \prod_{i=1}^{g}[A_{m,i}, B_{m,i}] \end{pmatrix}.$$

It is therefore enough to prove the result for $n = 1$ and $n = 2$ as when $n > 2$ we may simply form a block matrix and use $\mu(A \oplus B) = \mu(A) + \mu(B)$. Next, we may assume $k \geq 0$ by a block-wise application of the following lemma:

Lemma 2.20. *If $X \in \mathrm{SL}_2(\mathbb{R})$ is hyperbolic there exists $g \in GL_2(\mathbb{R})$ such that $gXg^{-1} = X$ and $\mu(gX_tg^{-1}) = -\mu(X_t)$ for any path X_t with $X_0 = e$ and $X_1 = X$.*

Proof. There is no loss in generality in assuming X is diagonal in which case we simply let $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. □

Finally, for each g, n, it suffices assume $k \geq n(2g-4)+1$ by using the following lemma:

Lemma 2.21. *Fix $g > 2$ and suppose that for every $0 \leq k \leq n(2g-2)$ and $z \in \mathrm{Cent}(\widetilde{G})$ with $\mu(z) = k$ there exists $(A_1, B_1, ..., A_g, B_g) \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})^{2g}$ such that:*

$$\prod_{i=1}^{g}[A_i, B_i] = z.$$

Let $g' = g+1$. Then for every $0 \leq k \leq n(2g'-4)$ and $z \in \mathrm{Cent}(\widetilde{G})$ with $\mu(z) = k$, there exists $(C_1, D_1, ..., C_{g'}, D_{g'}) \in G^{2g'}$ such that:

$$\prod_{i=1}^{g'}[C_i, D_i] = z.$$

Proof. Suppose $0 \leq k \leq n(2g'-4) = n(2g-2)$. By the assumption there exists $(A_1, B_1, ..., A_g, B_g) \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})^{2g}$ such that:

$$\prod_{i=1}^{g}[A_i, B_i] = z.$$

For $1 \leq i \leq g$ let $C_i = A_i$ and define $C_{g'} = D_{g'} = e$. Then:

$$\prod_{i=1}^{g'}[C_i, D_i] = \prod_{i=1}^{g}[A_i, B_i] \cdot [C_{g'}, D_{g'}] = z \cdot [e, e] = z \tag{2.13}$$

□

Thus if we have proven theorem 2.19 for a fixed g, we have also proven it for all $0 \leq k \leq n(2g-2) = n(2g'-4)$ for $g' = g+1$.

Remark 2.22. It is not quite enough to assume $n = 1$ as there exists $z \in \mathrm{Cent}(\widetilde{\mathrm{Sp}}(4;\mathbb{R}))$ that cannot be reached as products of 2×2 block matrices with entries in $\widetilde{\mathrm{Sp}}(2;\mathbb{R})$.

We start with $n = 1$. From the earlier reductions we assume $k = 2g - 3$ or $k = 2g - 2$. For $g \in \widetilde{\mathrm{SL}_2}(\mathbb{R})$ we write $\mathrm{tr}(g)$ to mean the trace of its projection to $\mathrm{SL}_2(\mathbb{R})$. The next three technical lemmas follow from simple computations involving 2×2 matrices so we relegate their proofs to section 5.2 of the appendix.

Lemma 2.23. *The image of the commutator map in* $\mathrm{SL}_2(\mathbb{R})$ *is* $\mathrm{SL}_2(\mathbb{R}) \setminus \{-I\}$.

Lemma 2.24. *For every* $X \in \widetilde{\mathrm{SL}_2}(\mathbb{R})$ *with* $\mathrm{tr}(X) < -2$ *there exists* $A, B \in \widetilde{\mathrm{SL}_2}(\mathbb{R})$ *such that* $\mathrm{tr}(X[A,B]) < -2$ *and* $\mu(X[A,B]) = \mu(X) + 2$.

Lemma 2.25. *For every* $X \in \widetilde{\mathrm{SL}_2}(\mathbb{R})$ *with* $\mathrm{tr}(X) < -2$ *and* $\varepsilon \in \{-1, 0, 1\}$ *there exists* A, B *such that* $X[A,B] \in \mathrm{Cent}(\widetilde{G})$ *and* $\mu(X[A,B]) = \mu(X) + \varepsilon$.

To prove theorem 2.19 for $n = 1$, by lemma 2.23 there exists $A_1, B_1 \in \widetilde{\mathrm{SL}_2}(\mathbb{R})$ such that $\mathrm{tr}[A_1, B_1] < -2$ and $\mu([A_1, B_1]) = 1$. Applying lemma 2.24 $g - 2$ times, we obtain $(A_1, B_1, ..., A_{g-1}, B_{g-1})$ such that:

$$\mu\left(\prod_{i=1}^{g-1} [A_i, B_i]\right) = 2g - 3.$$

Finally, for $k = 2g - 3$ take $\epsilon = 0$ in lemma 2.25 and for $k = 2g - 2$ take $\epsilon = 1$ in lemma 2.25.

Now we consider $n = 2$. First, by corollary 2.13 we may assume k is even. Next, if $k \leq 4g - 6$ then we may use (following the discussion at the start of this section) 2×2 block matrices since $2g - 2 + 2g - 2 = 4g - 4$ and $2g - 3 + 2g - 3 = 4g - 6$. If $z \in \widetilde{\mathrm{Sp}}(4;\mathbb{R})$ with $\mu(z) = 4g - 4$ covers the identity in $\mathrm{Sp}(4;\mathbb{R})$ we may again use 2×2 block matrices as $2g - 2 + 2g - 2 = 4g - 4$. It therefore remains to prove theorem 2.19 for the unique $z \in \mathrm{Cent}(\widetilde{\mathrm{Sp}}(4;\mathbb{R}))$ with $\mu(z) = 4g - 4$ covering $-I$ in $\mathrm{Sp}(4;\mathbb{R})$.

For any $x, y > 1$, using the same method as in the proof for $n = 1$ we are able to construct $(A_1, B_1, ..., A_{g-1}, B_{g-1})$, $(C_1, D_1, ..., C_{g-1}, D_{g-1})$ with:

$$\pi\left(\prod_{i=1}^{g-1} [A_i, B_i]\right) = \begin{pmatrix} -\frac{1}{x} & 0 \\ 0 & -x \end{pmatrix}, \quad \pi\left(\prod_{i=1}^{g-1} [C_i, D_i]\right) = \begin{pmatrix} -\frac{1}{y} & 0 \\ 0 & -y \end{pmatrix}$$

and:

$$\mu(\prod_{i=1}^{g-1} [A_i, B_i]) = \mu(\prod_{i=1}^{g-1} [C_i, D_i]) = 2g - 3$$

so that:

$$\pi\left(\left[\begin{pmatrix} A_i & 0 \\ 0 & C_i \end{pmatrix}, \begin{pmatrix} B_i & 0 \\ 0 & D_i \end{pmatrix}\right]\right) = \begin{pmatrix} -\frac{1}{x} & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \\ 0 & 0 & -\frac{1}{y} & 0 \\ 0 & 0 & 0 & -y \end{pmatrix}$$

and:

$$\mu\left(\prod_{i=1}^{g-1}\left[\begin{pmatrix} A_i & 0 \\ 0 & C_i \end{pmatrix}, \begin{pmatrix} B_i & 0 \\ 0 & D_i \end{pmatrix}\right]\right) = \mu(\prod_{i=1}^{g-1}[A_i, B_i]) + \mu(\prod_{i=1}^{g-1}[C_i, D_i]) = 4g-6$$

By corollary 2.6:

$$r\left(\begin{pmatrix} -\frac{1}{x} & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \\ 0 & 0 & -\frac{1}{y} & 0 \\ 0 & 0 & 0 & -y \end{pmatrix}, \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \frac{1}{x} & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & \frac{1}{y} \end{pmatrix}\right) = 0$$

so for $X_g, Y_g \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})$ such that $\pi([X_g, Y_g]) = \mathrm{Diag}(x, 1/x, y, 1/y)$, using the coboundary identity:

$$\begin{aligned} &\mu\left(\left(\prod_{i=1}^{g-1}\left[\begin{pmatrix} A_i & 0 \\ 0 & C_i \end{pmatrix}, \begin{pmatrix} B_i & 0 \\ 0 & D_i \end{pmatrix}\right]\right)[X_g, Y_g]\right) \\ &= \mu\left(\prod_{i=1}^{g-1}\left[\begin{pmatrix} A_i & 0 \\ 0 & C_i \end{pmatrix}, \begin{pmatrix} B_i & 0 \\ 0 & D_i \end{pmatrix}\right]\right) + \mu\left([X_g, Y_g]\right) \\ &= 4g-6+\mu\left([X_g, Y_g]\right) \end{aligned}$$

and this is equal to $4g-4$ provided $\mu\left([X_g, Y_g]\right) = 2$. Since we are able to construct the A_i, B_i, C_i, D_i given any $x, y > 1$ we work in the other direction - we start by proving the existence of $X_g, Y_g \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})$ such that $\pi([X_g, Y_g])$ is diagonalizable with positive eigenvalues and having $\mu([X_g, Y_g]) = 2$ so the result follows. To help with the proof we employ the following lemma:

Lemma 2.26. *The matrix $A \in \mathrm{Sp}(4;\mathbb{R})$ is diagonalizable with positive real eigenvalues if and only if:*

$$\mathrm{tr}(A) > 4, \qquad \frac{\mathrm{tr}(A)^2 - 8}{2} \leq \mathrm{tr}(A^2) \leq (\mathrm{tr}(A) - 2)^2.$$

Proof. Eigenvalues of symplectic matrices come in quadruples $\{\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}\}$ so $A \in \mathrm{Sp}(4;\mathbb{R})$ has characteristic polynomial:

$$p_A(t) = t^4 - \mathrm{tr}(A)t^3 - \left(\frac{\mathrm{tr}(A^2) - \mathrm{tr}(A)^2}{2}\right) t^2 - \mathrm{tr}(A)t + 1 \tag{2.14}$$

as:

$$p_A(t) = (t^2 - (\lambda_1 + \frac{1}{\lambda_1}) + 1)(t^2 - (\lambda_2 + \frac{1}{\lambda_2}) + 1). \tag{2.15}$$

The roots of a polynomial of the form $t^4 - at^3 - (b - a^2)t^2/2 - at + 1$ can be solved for explicitly:

$$-\frac{1}{4}\sqrt{2b + 8 - a^2} \pm \frac{1}{2}\sqrt{\frac{b - a\sqrt{2b + 8 - a^2}}{2}} + \frac{a}{4}$$
$$\frac{1}{4}\sqrt{2b + 8 - a^2} \pm \frac{1}{2}\sqrt{\frac{b + a\sqrt{2b + 8 - a^2}}{2}} + \frac{a}{4}$$

and these are real and positive if and only if:

$$a > 4, \qquad \frac{a^2 - 8}{2} \leq b \leq (a - 2)^2.$$

□

Proposition 2.27. *There exists $\pi(X_g), \pi(Y_g) \in \mathrm{Sp}(4;\mathbb{R})$ such that $[\pi(X_g), \pi(Y_g)]$ is positive definite diagonalizable and $\mu([X_g, Y_g]) = 2$ for any $X_g, Y_g \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})$ covering $\pi(X_g)$, $\pi(Y_g)$.*

Proof. We claim that:

$$\pi(X_g) = \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad \pi(Y_g) = \begin{pmatrix} -\frac{18}{5} & \frac{27}{5} & \frac{38}{15} & \frac{18}{5} \\ 0 & -\frac{25}{13} & -\frac{157}{65} & -\frac{188}{65} \\ -\frac{24}{5} & \frac{36}{5} & \frac{31}{10} & \frac{24}{5} \\ 0 & -\frac{60}{13} & -\frac{171}{26} & -\frac{97}{13} \end{pmatrix}$$

(which were found with the use of a computer, aided by intuition from $n = 1$) satisfy these conditions. Indeed, computing that $\mathrm{tr}[\pi(X_g), \pi(Y_g)] = 17485172/68445$ and $\mathrm{tr}(\pi([X_g, Y_g])^2) = 230357194981156/4684718025$, by lemma 2.26 the first result follows. For the second we apply proposition 2.9 and corollary 2.6 and compute that for any $X_g, Y_g \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})$ covering $\pi(X_g)$, $\pi(Y_g)$:

$$\begin{aligned}\mu([X_g, Y_g]) &= \frac{1}{2} r(\pi(X_g), \pi(Y_g)) \\ &= \frac{1}{2}\operatorname{sgn}\begin{pmatrix} \frac{288694464}{211225} & \frac{605990648}{54925} & -\frac{-1956732032}{190125} & -\frac{3749091056}{823875} \\ \frac{605990648}{54925} & \frac{3902768036}{428415} & -\frac{456875886}{54925} & -\frac{8063436184}{2142075} \\ -\frac{1956732032}{190125} & -\frac{456875886}{54925} & \frac{4421169472}{570375} & \frac{8480848276}{2471625} \\ -\frac{3749091056}{823875} & -\frac{8063436184}{2142075} & \frac{8480848276}{2471625} & \frac{5554575632}{3570125} \end{pmatrix}.\end{aligned}$$

The smallest eigenvalue of the above symmetric matrix is bounded below by $10^{-3} > 0$ hence the signature is equal to 4 and the second result follows.

□

3 TOPOLOGY IN LOW GENUS

In this section we focus our attention on $G = \mathrm{PSL}_2(\mathbb{R})$. Denote by:

$$\widetilde{\Phi}_g^G \colon G^{2g} \to \widetilde{G} \tag{3.1}$$

the lift of Φ_g^G (1.1) to $\widetilde{G}$. Connectedness of the fibers of $\widetilde{\Phi}_g^G$ was first shown by Goldman in [Gol88] and around the same time, for $z \in \mathrm{Cent}(\widetilde{G})$, $(\widetilde{\Phi}_g^G)^{-1}(z)/G$ was shown to be the total space of a complex vector bundle by Hitchin [Hit87]. In this section we explicitly work out the topology of the fibers of:

$$\widetilde{\Phi}_1^G \colon G \times G \to \widetilde{G}, \qquad (A, B) \mapsto [\widetilde{A}, \widetilde{B}]. \tag{3.2}$$

summarized by:

Theorem 3.1. *The fibers of the lifted commutator map are:*

$$(\widetilde{\Phi}_1^G)^{-1}(y) = \begin{cases} \mathbb{R}^3 & -\infty < \mathrm{tr}(y) \leq -2 \\ S^1 \times \mathbb{R}^2 & -2 < \mathrm{tr}(y) \leq 2 \\ (T^2 \setminus q) \times \mathbb{R} & 2 < \mathrm{tr}(y) < \infty \end{cases}$$

where $\mathrm{tr}(y)$ *is defined to be the trace of the projection of* y *to* $\mathrm{SL}_2(\mathbb{R})$ *and* $T^2 \setminus q$ *is a punctured torus.*

Observe that $\widetilde{\Phi}_g^G = m \circ (\widetilde{\Phi}_1^G)^{\times g}$ is the composition of the two maps:

$$(\widetilde{\Phi}_1^G)^{\times g} \colon G^{2g} \to \widetilde{G}^g, \quad (A_1, B_1, ..., A_g, B_g) \mapsto ([\tilde{A}_1, \tilde{B}_1], ..., [\tilde{A}_g, \tilde{B}_g])$$

and:

$$m \colon \widetilde{G}^g \to \widetilde{G}, \quad (\tilde{X}_1, ..., \tilde{X}_g) \mapsto \prod_{i=1}^{g} \tilde{X}_i.$$

Together with Milnor-Wood inequality considerations, theorem 3.1 therefore becomes a useful tool in the study of the topology of the fibers of $\widetilde{\Phi}_g^G$.

We summarize the strategy of the proof of theorem 3.1 here. By the results of the previous section, if $x \in \mathrm{SL}_2(\mathbb{R})$ has $\mathrm{tr}(x) \geq -2$ then there is a unique element in the image of $\widetilde{\Phi}_1^G$ covering it so in this case the fibers of the commutator map lifted to $\widetilde{G}$ are equal to the fibers of the commutator map lifted to $\mathrm{SL}_2(\mathbb{R})$. When $\mathrm{tr}(x) < -2$, there are exactly two lifts of x in the image of $\widetilde{\Phi}_1^G$ and in this case we will see that the commutator map lifted to $\mathrm{SL}_2(\mathbb{R})$ has exactly two homeomorphic connected components.

For all $y \neq e$ the action of $\mathrm{Cent}_G(y)$ on $(\widetilde{\Phi}_1^G)^{-1}(y)$ by diagonal conjugation is smooth, free, and proper, making the quotient map $(\widetilde{\Phi}_1^G)^{-1}(y) \to (\widetilde{\Phi}_1^G)^{-1}(y)/\,\mathrm{Cent}_G(y)$ into a principal bundle that we trivialize. We then define a homeomorphism of the 2-manifold $(\widetilde{\Phi}_1^G)^{-1}(y)/\,\mathrm{Cent}_G(y)$ to either $\mathbb{R}^2$, $S^1 \times \mathbb{R}^2$, or a punctured torus. The first two types are done directly inside of $G \times G$ but the third is easier to see by first computing:

$$\{(A, B) \in \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) : [A, B] = y\}$$

and projecting into $G \times G$.

3.1 COMMUTATOR FIBERS

By equivariance it is enough to assume that y is one of the following three types:

$$\pm\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}. \tag{3.3}$$

The following lemma is used to simplify the calculation when $\mathrm{tr}(y) < 2$.

Lemma 3.2. *For $A, B, M \in \mathrm{SL}_2(\mathbb{R})$ we have $[MA, BM^{-1}] = -M^2 \neq I$ if and only if $\mathrm{tr}(A) = \mathrm{tr}(B) = \mathrm{tr}(ABM^{-1}) = 0$.*

Proof. For $y \in \mathrm{SL}_2(\mathbb{R})$, $y^2 = -I$ if and only if $\mathrm{tr}(y) = 0$ so if $\mathrm{tr}(A) = \mathrm{tr}(B) = \mathrm{tr}(ABM^{-1}) = 0$ then:

$$-I = A^2 = B^2 = ABM^{-1}ABM^{-1} = ABM^{-1}A^{-1}B^{-1}M^{-1}$$

so:

$$[MA, BM^{-1}] = MABM^{-1}A^{-1}B^{-1} = M(ABM^{-1}A^{-1}B^{-1}M^{-1})M = -M^2.$$

Conversely, if $[MA, BM^{-1}] = -M^2$ then $B(AM)^{-1}B^{-1} = -A^{-1}M$ so $\mathrm{tr}(AM) = \mathrm{tr}(A^{-1}M)$ i.e. $0 = \mathrm{tr}((A + A^{-1})M) = \mathrm{tr}(A)\,\mathrm{tr}(M)$. If $\mathrm{tr}(M) = 0$ then $M^2 = -I$ so $\mathrm{tr}(A) = 0$ and an identical calculation gives $\mathrm{tr}(B) = 0$. □

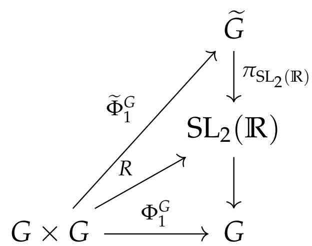
Let:

$$R\colon G \times G \to \mathrm{SL}_2(\mathbb{R}) \tag{3.4}$$

denote the lift of the commutator map Φ_1^G to $\mathrm{SL}_2(\mathbb{R})$ and let:

$$\pi_{\mathrm{SL}_2(\mathbb{R})}\colon \widetilde{G} \to \mathrm{SL}_2(\mathbb{R}) \tag{3.5}$$

be the canonical projection $\mathrm{SL}_2(\mathbb{R})$. We summarize these maps with the following diagram:



When $\mathrm{tr}(y) \geq -2$, by proposition 2.14 we have an equality:

$$R^{-1}(\pi_{\mathrm{SL}_2(\mathbb{R})}(y)) = \widetilde{\Phi}_1^{-1}(y). \tag{3.6}$$

When $\mathrm{tr}(y) < -2$:

$$R^{-1}(\pi_{\mathrm{SL}_2(\mathbb{R})}(y)) = \widetilde{\Phi}_1^{-1}(y) \sqcup \widetilde{\Phi}_1^{-1}(y^{-1}) \tag{3.7}$$

and lemma 2.20 together with equivariance gives a homeomorphism:

$$\widetilde{\Phi}_1^{-1}(y) \cong \widetilde{\Phi}_1^{-1}(y^{-1}). \tag{3.8}$$

It therefore suffices to compute the fibers of R. For brevity we henceforth use y to refer to an element of $\mathrm{SL}_2(\mathbb{R})$. We separate the proof of theorem 3.1 into four cases according to the trace of y, done in ascending order, and by equivariance we assume y is one of the matrices of 3.3

Proposition 3.3. *If:*

$$-y^2 = \begin{pmatrix} -t^2 & 0 \\ 0 & -\frac{1}{t^2} \end{pmatrix} \tag{3.9}$$

with $t \neq 1$ *then:*

$$R^{-1}(-y^2) \cong \mathbb{R}^3 \sqcup \mathbb{R}^3 \tag{3.10}$$

Proof. We apply lemma 3.2. If:

$$[yA, By^{-1}] = -y^2 \tag{3.11}$$

there exists a unique:

$$A' = \begin{pmatrix} s & \pm 1 \\ -s^2 - 1 & -s \end{pmatrix} \tag{3.12}$$

with $s > 0$ conjugate to A by an element of $C_G(-y^2) \cong \mathbb{R}$. Writing:

$$B = \begin{pmatrix} a & -\frac{a^2+1}{b} \\ b & -a \end{pmatrix}$$

then $\operatorname{tr}(A'By^{-1}) = 0$ if and only if:

$$b(sa + sat^2 + bt^2) = -(a^2+1)(s^2+1) \tag{3.13}$$

so:

$$R^{-1}(-y^2) \cong \mathbb{R}^3 \sqcup \mathbb{R}^3 \tag{3.14}$$

and the result follows. □

Proposition 3.4. *If:*

$$y = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \tag{3.15}$$

then:

$$R^{-1}(y) \cong \mathbb{R}^3 \tag{3.16}$$

Proof. For $(A, B) \in R^{-1}(y)$, there is a unique:

$$A' = \begin{pmatrix} 0 & \frac{1}{2s} \\ -2s & s \end{pmatrix}$$

conjugate to A by an element of $\operatorname{Cent}_G(g) \cong \mathbb{R}$. Writing:

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have $R(A', B) = g$ if and only if:

$$B = \begin{pmatrix} 2s^2(b-d) - d, b \\ 4s^2(b-d), d \end{pmatrix}, \qquad \det(B) = 1$$

and with the assumption that $s > 0$, $b > 0$, we can solve for s uniquely as a function of b and d:

$$s = \sqrt{-\frac{d^2+1}{2(d^2+2b^2-3bd)}}$$

hence:

$$R^{-1}(y) \cong \mathbb{R}^3$$

. □

Proposition 3.5. *If:*

$$-y^2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

then:

$$R^{-1}(-y^2) \cong S^1 \times \mathbb{R}^2$$

Proof. We apply lemma 3.2. If $[yA, By^{-1}] = -y^2$ there exists a unique

$$A' = \begin{pmatrix} 0 & s \\ -1/s & 0 \end{pmatrix}$$

conjugate to A by an element of $\mathrm{Cent}_G(-y^2) \cong S^1$. Writing:

$$B = \begin{pmatrix} \sqrt{-1-ab} & a \\ b & -\sqrt{-1-ab} \end{pmatrix}$$

it follows from a short computation that $\mathrm{tr}(A'By^{-1}) = 0$ if and only if:

$$s = \sqrt{\frac{b - \tan\theta\sqrt{-1-ab}}{a - \tan\theta\sqrt{-1-ab}}}$$

so:

$$R^{-1}(-y^2) \cong \mathbb{R}^2 \times S^1$$

□

Proposition 3.6. *If:*

$$y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then:

$$R^{-1}(y) \cong S^1 \times \mathbb{R}^2$$

Proof. If $R(A,B) = y$ then by trace invariance under conjugation A and B are necessarily upper triangular. Writing:

$$A = \begin{pmatrix} s & t \\ 0 & \frac{1}{s} \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}$$

with $s, a > 0$ we compute that $[A,B] = y$ if and only if:

$$st(1-a^2) + ab(s^2-1) = 1.$$

We claim that:

$$\{(s,a,t,b) \in \mathbb{R}^4 : st(1-a^2) + ab(s^2-1) = 1, s > 0, a > 0\} \cong S^1 \times \mathbb{R}^2. \tag{3.17}$$

Indeed:

$$(s,a) \mapsto (s(1-a^2), a(s^2-1))$$

defines a homeomorphism:

$$\mathbb{R}_{>0} \times \mathbb{R}_{>0} \setminus (1,1) \to \mathbb{R}^2 \setminus 0$$

and moreover

$$st(1-a^2) + ab(s^2-1) = \Big\langle \left(s(1-a^2), a(s^2-1)\right), (t,b) \Big\rangle$$

where the right hand side of this equality is the Euclidean dot product on $\mathbb{R}^2$. Combining these two observations together we conclude that there is a homeomorphism between the left hand side of 3.17 and:

$$\{(u,v) \in (\mathbb{R}^2 \setminus 0) \times \mathbb{R}^2 : \langle u,v \rangle = 1\} \cong (\mathbb{R}^2 \setminus 0) \times \mathbb{R} \cong S^1 \times \mathbb{R}^2$$

as needed. □

Proposition 3.7. *If:*

$$y = \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}$$

with $t > 0$ then:

$$R^{-1}(y) \cong (T^2 \setminus q) \times \mathbb{R}$$

where $T^2 \setminus q$ is a punctured torus.

Proof. Define:

$$C_y = \{(A,B) \in \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) : [A,B] = y,\ \mathrm{tr}(A), \mathrm{tr}(B) \geq 0\}$$

For $(A,B) \in C_y$ there exists a unique (A',B') in the $\mathrm{Cent}_G(y)$-orbit of (A,B) of the form:

$$(A',B') = \left(\begin{pmatrix} \sqrt{t(1+a\cos\theta)} & \frac{\cos\theta}{\sqrt{\frac{1+a\cos\theta}{t}}} \\ a & \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{1+b\sin\theta}{t}} & \frac{\sin\theta}{\sqrt{t(1+b\sin\theta)}} \\ b & \end{pmatrix} \right)$$

from which it follows that $[A',B'] = y$ if and only if:

$$1 + a\cos\theta \geq 0, \quad 1 + b\sin\theta \geq 0 \tag{3.18}$$

and:

$$(t-1)\sqrt{(1+a\cos\theta)(1+b\sin\theta)} = b\cos\theta - at\sin\theta \geq 0. \tag{3.19}$$

Consider the map $f\colon \mathbb{R}^2 \times S^1 \to \mathbb{R}$ defined by:

$$f(a,b,(\cos\theta,\sin\theta)) = (t-1)^2(1+a\cos\theta)(1+b\sin\theta) - b\cos\theta + at\sin\theta.$$

A straightforward computation reveals that f has exactly 4 critical points:, and these occur when:

$$\theta \in \left\{ \pm\arctan\left(\frac{1}{\sqrt{t}}\right), \pm\left(\arctan\left(\frac{1}{\sqrt{t}}\right) - \pi\right) \right\}$$

and:

$$1 + a\cos\theta = 1 + b\sin\theta = 0.$$

We therefore obtain 0 as a regular value for the restriction of f to the complement of these 4 points. But from 3.18 and 3.19:

$$C_y/\mathbb{R} \cong f^{-1}(0) \cap \{1 + a\cos\theta \geq 0\} \cap \{1 + b\sin\theta \geq 0\} \cap \{b\cos\theta \geq at\sin\theta\}.$$

Adding the four critical points back in we compute that:

$$C_y/\mathbb{R} \cong S^1 \times [0,\infty).$$

Projection:

$$C_y \to G \times G$$

is injective except along the boundary where it restricts to the usual quotient map identifying the boundary of the fundamental square to produce the 2-torus T^2. Indeed, the boundary of $C_y/\mathbb{R}$ is defined by the condition:

$$(1 + a\cos\theta)(1 + b\sin\theta) = 0.$$

When $1 + a\cos\theta = 0$ we can solve for a and b in terms of θ but 3.18 puts restrictions on θ:

$$-\arctan\left(\frac{1}{\sqrt{t}}\right) \leq \theta \leq \arctan\left(\frac{1}{\sqrt{t}}\right), \quad \pi - \arctan\left(\frac{1}{\sqrt{t}}\right) \leq \theta \leq \pi + \arctan\left(\frac{1}{\sqrt{t}}\right)$$

with θ taking values in $(-\pi, \pi)$. The second condition is equivalent to:

$$-\arctan\left(\frac{1}{\sqrt{t}}\right) \leq \pi - \theta \leq \arctan\left(\frac{1}{\sqrt{t}}\right).$$

but projection from $\mathrm{SL}_2(\mathbb{R})$ to $\mathrm{PSL}_2(\mathbb{R})$ identifies $M \sim -M$ so:

$$\begin{pmatrix} 0 & \cos\theta \\ -\sec\theta & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & \cos(\pi - \theta) \\ -\sec(\pi - \theta) & 0 \end{pmatrix}$$

Similarly, when $1 + b\sin\theta = 0$ we can solve for a and b in terms of θ but now the restriction of 3.18 on θ is:

$$\operatorname{arccot}(\sqrt{t}) \leq \theta \leq \operatorname{arccot}(-\sqrt{t}), \quad \operatorname{arccot}(\sqrt{t}) - \pi \leq \theta \leq \operatorname{arccot}(-\sqrt{t}) - \pi$$

and the second condition is equivalent to:

$$\operatorname{arccot}(\sqrt{t}) \leq \theta + \pi \leq \operatorname{arccot}(-\sqrt{t}).$$

but as above projection identifies:

$$\begin{pmatrix} 0 & \sin\theta \\ -\csc\theta & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & \sin(\pi + \theta) \\ -\csc(\pi + \theta) & 0 \end{pmatrix}.$$

In total we therefore see that, as a quotient map on S^1, the restriction of the projection map:

$$\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$$

to ∂C_y makes the two identifications:

$$-\arctan\left(\frac{1}{\sqrt{t}}\right) \leq \arg(z) \leq \arctan\left(\frac{1}{\sqrt{t}}\right) \sim -\arctan\left(\frac{1}{\sqrt{t}}\right) \leq \pi - \arg(z) \leq \arctan\left(\frac{1}{\sqrt{t}}\right) \tag{3.20}$$

and:

$$\operatorname{arccot}(\sqrt{t}) \leq \arg(z) \leq \operatorname{arccot}(-\sqrt{t}) \sim \operatorname{arccot}(\sqrt{t}) \leq \arg(z) + \pi \leq \operatorname{arccot}(-\sqrt{t}). \tag{3.21}$$

This is the usual construction of the 2-torus by identifying the boundary of the fundamental square hence:

$$R^{-1}(y) \cong \left(S^1 \times [0, \infty) / \sim\right) \times \mathbb{R} \cong (T^2 \setminus q) \cong \mathbb{R}.$$

□

4

SINGULAR HYPERBOLIC GEOMETRY

Let Σ_0 be a compact surface with (potentially empty) boundary and let Σ be the complement in Σ_0 of finitely many interior points. By a *hyperbolic metric* we mean a complete Riemannian metric with finite area, totally geodesic boundary, and constant Gaussian curvature $K \equiv -1$. We define the *Teichmüller space* $\mathcal{T}_\Sigma$ of Σ:

$$\mathcal{T}_\Sigma = \frac{\{\text{hyperbolic metrics on } \Sigma\}}{\text{Diff}_0(\Sigma)} \tag{4.1}$$

where $\text{Diff}_0(\Sigma)$ (diffeomorphisms of Σ homotopic to the identity) acts by pullback. When $\Sigma = \Sigma_g$ is a closed genus g surface we write $\mathcal{T}_g$ to mean $\mathcal{T}_{\Sigma_g}$. Fenchel-Nielsen [FN03] found what are now called *Fenchel-Nielsen* coordinates on Teichmüller space. First, some loops $\{C_i\}$ on Σ are chosen such that their complement is a disjoint union of 3-holed spheres (a *pants decomposition*). Given a class $[h] \in \mathcal{T}_\Sigma$, if one cuts along some C_i, rotates one of the two cylinder shaped pieces near C_i, and then glues the surface back together, a necessarily distinct class $[h'] \in \mathcal{T}_\Sigma$ is obtained, for if the rotation is not an integral multiple of 2π then the local geometry is changed, and if the rotation is a (nonzero) integral multiple of 2π then we have performed a Dehn twist which is not homotopic to the identity. Fenchel-Nielsen coordinates of $[h]$ are given by the geodesic *length* of the loops C_i together with the *twist* parameters, distinguished from each other by the aforementioned cutting and gluing.

Let $G = \text{PSL}_2(\mathbb{R})$ and $g \geq 2$. Recall that $\mathcal{T}_g$ was identified in $\mathcal{M}(G, \Sigma_g)$ as gauge classes of flat connections θ on the principal bundle $P \to \Sigma_g$ with $e(P) = 2 - 2g$. The gauge class of θ therefore determines and is determined by its associated length and twist parameters. Our main goal of this chapter is to make the identification of $\mathcal{T}_g$ in $\mathcal{M}(G, \Sigma_g)$ explicit. In doing so, we provide an interpretation of the non-Teichmüller subset of $\mathcal{M}(G, \Sigma_g)$ as degenerate or *singular* hyperbolic metrics on subsets of Σ_g.

We start by outlining how $\mathcal{T}_g$ is identified in $\mathcal{M}(G, \Sigma_g)$. from the holonomy perspective. Once a basepoint $x_0 \in \Sigma_g$ and loops:

$$(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$$

based at x_0 are chosen such that their homotopy classes (which for brevity we also denote by α_j and β_j) give the following presentation of the fundamental group of Σ_g:

$$\pi_1(\Sigma_g, x_0) = \langle \alpha_1, \beta_1, ..., \alpha_g, \beta_g : \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1 \rangle,$$

the holonomies of θ around these generators define:

$$(A_1, B_1, ..., A_g, B_g) \in (\widetilde{\Phi}_g^G)^{-1}(2g-2)$$

such that:

$$\prod_{i=1}^{g} [A_i, B_i] = e$$

and gauge transformations of θ correspond to simultaneous conjugation of the A_i and B_i by some $h \in G$. Choosing representative holonomies A_i, B_i in the gauge class of θ, they generate a torsion free Fuchsian group:

$$\Gamma := \langle A_i, B_i \rangle < \mathrm{PSL}_2(\mathbb{R})$$

and moreover there is a canonical identification between the fundamental group of $\mathbb{H}/\Gamma$ and Γ. Under this identification, there exists a diffeomorphism:

$$f\colon \Sigma_g \to \mathbb{H}/\Gamma$$

such that:

$$f_*([\alpha_i]) = A_i, \qquad f_*([\beta_i]) = B_i$$

and moreover f is unique up to homotopy. The hyperbolic metric on $\mathbb{H}$ descends to a hyperbolic metric on $\mathbb{H}/\Gamma$ and gauge transformations of θ correspond to conjugation of Γ by some $h \in \mathrm{PSL}_2(\mathbb{R})$ and this is equivalent to an isometry of $\mathbb{H}$ covering the isometry:

$$\mathbb{H}/\Gamma \to \mathbb{H}/h\Gamma h^{-1}.$$

The pullback to Σ_g by f of the hyperbolic metric on $\mathbb{H}/\Gamma$ is therefore a well defined class in Teichmüller space. In terms of representations, for:

$$\phi \in \mathrm{Hom}(\pi_1(\Sigma_g, x_0), G),$$

there is a unique homotopy class of maps:

$$f\colon \Sigma_g \to \mathbb{H}/\mathrm{Im}(\phi)$$

such that $f_* = \phi$, and pulling back the metric on $\mathbb{H}/\mathrm{Im}(\phi)$ defines a class in $\mathcal{T}_g$. The main goal of this chapter is to make explicit the composition of the map

$$(\widetilde{\Phi}_g^G)^{-1}(2g-2)/G \to \mathcal{T}_g$$

just described with Fenchel-Nielsen coordinates

$$\mathcal{T}_g \to \mathbb{R}^{6g-6}$$

in such a way that insight into the other components of $\mathcal{M}(G, \Sigma_g)$ can be gleaned.

As was mentioned at the start of this chapter, in order to define Fenchel-Nielsen coordinates we choose a pants decomposition of Σ_g. Our first observation is that the choice of loops generating the fundamental group of Σ_g prescribe a pants decomposition, as follows. Each commutator $[\alpha_i, \beta_i]$ has a simple closed curve γ_i in its free homotopy class bounding a punctured torus T_i on one side and a punctured genus $g-1$ surface on the other, and these simple closed curves can be taken disjoint. The complement of these simple closed curves is a union of g punctured torii and a g-holed sphere S_g. Cutting along γ_i and α_i gives a pants decomposition for T_i, and using an inductive procedure with the γ_i a pants decomposition of S_g is also obtained where the k'th pair of pants has boundary components corresponding to:

$$\prod_{i=1}^{k} \gamma_i, \qquad \gamma_{k+1}, \qquad \prod_{i=1}^{k+1} \gamma_i. \tag{4.2}$$

Having then a pants decomposition on Σ_g, the Teichmüller class of the hyperbolic metric defined by $(A_1, B_1, ..., A_g, B_g)$ can be found by working out the associated length and twist parameters. A direct calculation of the Fenchel-Nielsen coordinates from $(A_1, B_1, ..., A_g, B_g)$ was originally done by Maskit [Mas01]. We take a similar albeit different approach with an eye toward the non-Teichmüller components of $\mathcal{M}(G, \Sigma_g)$.

Each pair (A_i, B_i) generates a torsion free Fuchsian group:

$$\Gamma_i := \langle A_i, B_i \rangle < \mathrm{PSL}_2(\mathbb{R})$$

such that the quotient:

$$\mathbb{H}/\Gamma_i$$

is a punctured torus. A_i (resp. B_i) leaves invariant a unique geodesic L_{A_i} (resp. L_{B_i}) on $\mathbb{H}$ called its *axis* and the image C_{A_i} (resp C_{B_i}) of L_{A_i} (resp. L_{B_i}) in $\mathbb{H}/\Gamma_i$ is a geodesic loop. There exists a diffeomorphism:

$$f\colon T_i \to \mathbb{H}/\Gamma_i$$

such that:

$$f_*([\alpha_i]) = [C_{A_i}], \qquad f_*([\beta_i]) = [C_{B_i}],$$

and f is unique up to homotopy. Each pair (A_i, B_i) therefore determines one twist parameter (the twist around α_i) and two length parameters (the length of α_i and the length of γ_i). What remains are the $2g-3$ twist parameters in the pants decomposition of S_g (the loops of 4.2) and the $g-3$ remaining length parameters of 4.2 (the γ_i are already accounted for), and these are calculated similarly.

We summarize the above discussion as the following proposition:

Proposition 4.1. *Let:*

$$(A_1, B_1, ..., A_g, B_g) \in G^{2g}.$$

Then:

$$\Gamma := \langle A_1, B_1, ..., A_g, B_g \rangle < \mathrm{PSL}_2(\mathbb{R})$$

is a torsion free Fuchsian group and there exists a diffeomorphism:

$$f\colon \Sigma_g \to \mathbb{H}/\Gamma$$

such that:

$$f_*(\alpha_i) = A_i, \quad \textit{and} \quad f_*(\beta_i) = B_i$$

if and only if the following two hold:

1. *For each $1 \leq j \leq g$ the pair A_j, B_j generate a torsion free Fuchsian group and moreover:*

$$\mathbb{H}/\langle A_j, B_j \rangle$$

   is a punctured torus.

2. *For $1 \leq k \leq g-2$ the pair:*

$$\prod_{i=1}^{k} [A_i, B_i], \quad [A_{k+1}, B_{k+1}]$$

   generate a torsion free Fuchsian group K and moreover:

$$\mathbb{H}/K$$

is a 3*-holed sphere.*

Having reduced to one punctured torii Σ_1^1 and 3-holed spheres Σ_0^3, we therefore focus our attention on their respective Teichmüller spaces. That is, for $A, B \in G$ we aim to determine when $\langle A, B\rangle < \mathrm{PSL}_2(\mathbb{R})$ is a torsion free Fuchsian group such that:

$$\mathbb{H}/\langle A, B\rangle$$

is a punctured torus or a three-holed sphere.

We may actually narrow our scope a bit further. The generators in point (2) of proposition 4.1 are not totally arbitrary - they are products commutators of matrices that generate Fuchsian groups defining punctured torii. That is to say, we aim to determine when:

$$\mathbb{H}/\langle A, B\rangle$$

is a 3-holed sphere that moreover satisfies the property that there exists loops γ_A and γ_B freely homotopic to C_A and C_B that can be capped off with torii. This is exactly equivalent to C_A and C_B being disjoint simple closed curves.

To this end we now state the two main results of this chapter. Fix $A, B \in \mathrm{PSL}_2(\mathbb{R})$ and let $A', B' \in \mathrm{SL}_2(\mathbb{R})$ be lifts of A, B respectively with non-negative trace.

Theorem 4.2. *The subgroup* $\langle A, B\rangle < \mathrm{PSL}_2(\mathbb{R})$ *is torsion free Fuchsian and:*

$$\mathbb{H}/\langle A, B\rangle$$

is a punctured torus if and only if $\mathrm{tr}[A', B'] \leq -2$.

Theorem 4.3. *The subgroup* $\langle A, B\rangle < \mathrm{PSL}_2(\mathbb{R})$ *is torsion free Fuchsian and:*

$$\mathbb{H}/\langle A, B\rangle$$

is a 3*-holed sphere such that* C_A *and* C_B *are disjoint simple closed curves if and only if:*

$$\mathrm{tr}(A') \geq 2, \quad \mathrm{tr}(B') \geq 2, \quad \mathrm{tr}(A'B') \leq -2$$

.

Let (c_1, c_2, c_3) be the punctures of Σ_0^3 with simple closed curves L_1, L_2 around (c_1, c_2) respectively that generate the fundamental group of Σ_0^3, and let (α, β) be loops generating the fundamental group of Σ_1^1. Given a hyperbolic metric h on either surface and a loop γ let $\gamma^h \in G$ denote the associated deck transformation of $\mathbb{H}$. We have well defined maps:

$$\mathcal{T}_{\Sigma_0^3}, \mathcal{T}_{\Sigma_1^1} \to (G \times G)/G \tag{4.3}$$

$$[h] \mapsto [(L_1^h, L_2^h)], \qquad [h] \mapsto [(\alpha^h, \beta^h)]. \tag{4.4}$$

As in chapter 3 define:

$$R \colon G \times G \to \mathrm{SL}_2(\mathbb{R})$$

to be the lift of the commutator map Φ_1^G. For $[g_1], [g_2] \in \mathrm{PSL}_2(\mathbb{R})$ not elliptic let g_i be the representative with positive trace and define:

$$m([g_1], [g_2]) = g_1 g_2 \in \mathrm{SL}_2(\mathbb{R}). \tag{4.5}$$

Our first result of this chapter is:

Theorem 4.4. *The image of $\mathcal{T}_{\Sigma_0^3} \to (G \times G)/G$ is:*

$$(\mathrm{tr} \circ m)^{-1}(-\infty, -2]/G \tag{4.6}$$

and the image of $\mathcal{T}_{\Sigma_1^1} \to (G \times G)/G$ is:

$$(\mathrm{tr} \circ R)^{-1}(-\infty, -2]/G. \tag{4.7}$$

It is well known when $A, B \in \mathrm{PSL}_2(\mathbb{R})$ generate a Fuchsian group. A comprehensive treatment is given in [Gil95] where the author refers to the *Jørgensen-Poincaré* dichotomy. On the one hand there is Jørgensen's inequality [Jr76]:

Theorem 4.5 (Jørgensen's inequality)**.** *If A, B generate a discrete subgroup of* $\mathrm{SL}_2(\mathbb{C})$ *then:*

$$|\,\mathrm{tr}(A)^2 - 4| + |\,\mathrm{tr}[A, B] - 2| \geq 1. \tag{4.8}$$

On the other hand, there is the Poincaré polygon theorem: let $P \subset \mathbb{H}$ be a closed, convex polygon such that for each edge a_i of P there exists an edge b_i and $g_i \in \mathrm{PSL}_2(\mathbb{R})$ with $b_i = g_i(a_i)$, and the side of a_i in P gets mapped to the side of b_i not in P. Let $M = P/\sim$ with the induced metric from the hyperbolic metric on $\mathbb{H}$, where $\sim$ is the identification of edge a_i with edge b_i using g_i, and for v a vertex in P let $\theta(v)$ denote its angle.

Theorem 4.6 (Poincaré polygon)**.** *If M is a complete metric space and for each vertex $v \in P$:*

$$\frac{2\pi}{\sum_i \theta(g_i(v))}$$

is an integer then $\langle g_i \rangle$ is discrete with fundamental domain P.

The test for discreteness of [Gil95] is an algorithm. One first checks if A, B satisfy Jørgensen's inequality. If they don't, we are done. If they do, there is a subset of $\mathbb{H}$ associated to A and B one tries to apply the Poincaré polygon theorem to. If the polygon theorem does not apply, A' and B' are defined that generate a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ isomorphic to that generated by A and B and the process is started again, where A' and B' are chosen in such a way that this process is guaranteed to terminate after a finite number of steps.

Analysis typically emphasizes the axes of the generators (e.g. in [Gil95], [Mas01]), starting with whether or not they intersect. We instead prefer to work directly and only with lengths and traces.

Our main goal in proving theorem 4.4 is to make explicit the correspondence between the holonomies of the flat connection and the Fenchel-Nielsen coordinates of the associated hyperbolic metric. It will be a consequence of the following two pairs of results, each pair involving first a determination of the map from Teichmüller space to $(G \times G)/G$ written in Fenchel-Nielsen coordinates and second a characterization of their images in $(G \times G)/G$.

Starting with the 3-holed sphere: as above let (c_1, c_2, c_3) be the punctures and let L_i be a simple closed curve around c_i such that $L_1 L_2 L_3$ is trivial and L_1, L_2 generate the fundamental group of Σ_0^3.

Theorem 4.7. *The map:*

$$\begin{aligned} \mathcal{T}_{\Sigma_0^3} &\to (G \times G)/G \\ [h] &\mapsto [(L_1^h, L_2^h)] \end{aligned}$$

written in Fenchel-Nielsen coordinates is given by:

$$(\ell_1, \ell_2, s_3) \mapsto \left[\begin{pmatrix} \cosh(\ell_1) & \sinh(\ell_1) \\ \sinh(\ell_1) & \cosh(\ell_1) \end{pmatrix}, \begin{pmatrix} \cosh(\ell_2) & -s_3 \sinh(\ell_2) \\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2) \end{pmatrix} \right] \quad (4.9)$$

where:

$$\ell(L_i) = 2\ell_i, \quad \ell(C_3) = \log(s_3)$$

and C_3 is the seam connecting L_1 to L_2. Moreover we have $\ell(L_3) = \ell_3$ where ℓ_3 is the unique solution to:

$$\cosh(\ell_3) + \cosh(\ell_1)\cosh(\ell_2) = \frac{(1 + s_3^2)\sinh(\ell_1)\sinh(\ell_2)}{2s_3},$$

and the other geodesic seams C_1, C_2 have lengths $\ell(C_i)$ satisfying:

$$\sinh(\ell(C_2)) = \frac{\sinh(\ell(C_3))}{\sinh(\ell_3)}\sinh(\ell_2)$$
$$\sinh(\ell(C_1)) = \frac{\sinh(\ell(C_3))}{\sinh(\ell_3)}\sinh(\ell_1).$$

Theorem 4.8. *For $g_1, g_2 \in \mathrm{PSL}_2(\mathbb{R}) \setminus e$ not elliptic the following are equivalent:*

1. *There exists a homeomorphism:*

$$f\colon \Sigma_0^3 \to \mathbb{H}/\langle g_1, g_2\rangle$$

such that $f_*([L_i]) = C_{g_i}$.

2. $\mathrm{tr}(m(g_1, g_2)) \leq -2$.

3. *The positive trace lifts of g_1 and g_2 in $\mathrm{SL}_2(\mathbb{R})$ are simultaneously conjugate to*

$$\begin{pmatrix}\cosh(\ell_1) & \sinh(\ell_1)\\ \sinh(\ell_1) & \cosh(\ell_1)\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\cosh(\ell_2) & -s_3\sinh(\ell_2)\\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2)\end{pmatrix}$$

for some:

$$(\ell_1, \ell_2) \in \mathbb{R}^2_{>0}, \quad s_3 > \coth(\ell_1/2)\coth(\ell_2/2).$$

For the punctured torus something a bit stronger can be said: as above let (α, β) be loops representing generators of its fundamental group.

Theorem 4.9. *The map:*

$$\mathcal{T}_{\Sigma_1^1} \to (G\times G)/G$$
$$[h] \mapsto [(\alpha^h, \beta^h)]$$

written in Fenchel-Nielsen coordinates is given by:

$$(\ell_1, \ell_2, s_3) \mapsto \left[\begin{pmatrix}\cosh(\ell_1) & \sinh(\ell_1)\\ \sinh(\ell_1) & \cosh(\ell_1)\end{pmatrix}, \begin{pmatrix}\sqrt{s_3} & 0\\ 0 & \frac{1}{\sqrt{s_3}}\end{pmatrix}\begin{pmatrix}\cosh(\ell_2) & \sinh(\ell_2)\\ \sinh(\ell_2) & \cosh(\ell_2)\end{pmatrix}\right] \tag{4.10}$$

where:

$$\ell(\alpha) = 2\ell_1, \quad \ell(\beta) = \log(s_3) \quad \tau(\alpha) = 2\ell_2. \tag{4.11}$$

The geodesic loop L_3 *around the puncture has length* ℓ_3 *satisfying:*

$$\cosh(\ell_3) = \frac{1+s_3^2}{2s_3}\sinh(\ell_1)^2 - \cosh(\ell_1)^2$$

and the seams C_1, C_2 *connecting* α *to* L_3 *have the same length* $\ell(C)$ *satisfying:*

$$\sinh(\ell(C)) = \frac{\sinh(\ell(C_3))}{\sinh(\ell_3)}\sinh(\ell_1).$$

Theorem 4.10. *For* $g_1, g_2 \in \mathrm{PSL}_2(\mathbb{R})$ *the following are equivalent:*

1. *The quotient* $\mathbb{H}/\langle g_1, g_2\rangle$ *is a punctured torus and the quotient map is a covering map.*
2. $\mathrm{tr}(R(g_1, g_2)) \leq -2$.
3. *The positive trace lifts of* g_1 *and* g_2 *in* $\mathrm{SL}_2(\mathbb{R})$ *are simultaneously conjugate to*

$$\begin{pmatrix}\cosh(\ell_1) & \sinh(\ell_1)\\ \sinh(\ell_1) & \cosh(\ell_1)\end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} s_3 & 0\\ 0 & \frac{1}{s_3}\end{pmatrix}\begin{pmatrix}\cosh(\ell_2) & \sinh(\ell_2)\\ \sinh(\ell_2) & \cosh(\ell_2)\end{pmatrix}$$

for some:

$$(\ell_1, \ell_2) \in \mathbb{R}^2_{>0}, \quad s_3 > \coth(\ell_1/2)\coth(\ell_2/2).$$

Remark 4.11. The key difference between the punctured torus and the 3-holed sphere is that the loop around the puncture in the torus is distinguished. Given $A, B \in \mathrm{PSL}_2(\mathbb{R})$, if:

$$\mathbb{H}/\langle A, B\rangle$$

is a punctured torus then there is a canonical polygon $P \subset \mathbb{H}$ associated to $\langle A, B\rangle$ that we may immediately apply the Poincaré polygon theorem to, and P depends not on the specific choices of generators A and B but only on the Fuchsian group they generate. On the other hand, if:

$$\mathbb{H}/\langle A, B\rangle$$

is a 3-holed sphere, whether or not the Poincaré polygon theorem applies to P *does* depend on the choice of generators. This amounts to the observation that Nielsen equivalent generators always have the same commutator but not the same product.

We start by recalling a standard construction of Fenchel-Nielsen coordinates in section 4.1 to motivate what will come after. In section 2 we prove theorem 4.7 and in section 3 we prove theorem 4.9. Finally in section 4 we prove theorems 4.8 and 4.10.

The proofs of theorems 4.7 and 4.9 are fairly straightforward. Theorems 4.8 and 4.10 are essentially consequences of theorems 4.7 and 4.9. In both cases the proofs nearly follow the same structure. Equivalence of (2) and (3) is a routine calculation. We show that (3) implies (1) by applying the Poincaré polygon theorem and that (1) implies (3) in theorem 4.8 follows from uniqueness of hyperbolic hexagons. That (1) implies (3) in theorem 4.10 follows from 4.8 and a little covering space theory: Assuming (1) in theorem 4.10, under the covering space Galois correspondence between subgroups of the fundamental group and covering spaces, the subgroup $\langle g_1, g_2 g_1^{-1} g_2^{-1} \rangle$ corresponds to a 3-holed sphere. But by theorem 4.8 (2) this means $\mathrm{tr}([g_1, g_2]) \leq -2$ hence (1) implies (2) in 4.10, completing the proof.

4.1 FENCHEL-NIELSEN COORDINATES

There are several standard equivalent constructions of Fenchel-Nielsen coordinates and we recall one here, starting with Σ compact and having b boundary components $C_1, ..., C_b$.

There exist a collection of disjoint simple closed curves $\alpha_1, ..., \alpha_{3g-3+b}$ in $\mathrm{Int}(\Sigma)$ such that their complement in $\mathrm{Int}(\Sigma)$ is homeomorphic to the (finite) disjoint union of three holed spheres S_i. We call the closure P_i of each S_i in Σ a *pair of pants* and call the ordered tuple $(\alpha_1, ..., \alpha_{3g-3+b})$ a *pants decomposition* of Σ. We then choose a collection of disjoint simple closed curves $\beta_1, ..., \beta_{3g-3+b}$ intersecting the α_i transversely called *seams* such that for each pair of pants P and each α_i, α_j that is a boundary component of P (note that $\partial\Sigma$ is not included) there is a unique seam connecting α_i to α_j.

Fix a hyperbolic metric h on Σ. For each α_j there exists a unique geodesic loop γ_j freely homotopic to it and for each β_j there exists a unique geodesic loop δ_j freely homotopic to it. For each γ_j pick one of the two geodesic seams δ_j intersecting it. In each of the two pairs of pants P_{j_i} that γ_j is a boundary component of, δ_j connects γ_j to some other boundary component γ_{j_i}. There exists a unique length minimizing geodesic segment $c_{j,i}$ in P_{j_i} connecting γ_j to $\gamma_{j,i}$ that may or may not coincide with δ_j. A metric $d_{j,i}$ describing the failure for δ_j to coincide with $c_{j,i}$ in P_{j_i} is defined that moreover does not depend on the choice of seam δ_j, from which we define the *twist parameter* around γ_j:

$$\tau_{\gamma_j} := d_{j,1} - d_{j,2}.$$

Together with the *length parameters* ℓ of the geodesic loops (including the boundary components of Σ) we have a map:

$$\{\text{hyperbolic metrics on } \Sigma\} \to \mathbb{R}^b_{\geq 0} \times \mathbb{R}^{3g-3+b}_{>0} \times \mathbb{R}^{3g-3+b}$$

given by:

$$h \mapsto (\ell(C_1), ..., \ell(C_b), \ell(\gamma_1), ..., \ell(\gamma_{3g-3+b}), \tau(\gamma_1), ..., \tau(\gamma_{3g-3+b})).$$

This map is continuous and surjective and two metrics h_1, h_2 lie in the same fiber if and only if there exists $\varphi \in \mathrm{Diff}_0(\Sigma)$ such that $h_2 = \varphi^* h_1$ so it descends to a homeomorphism:

$$FN\colon \mathcal{T}_\Sigma \to \mathbb{R}^b_{\geq 0} \times \mathbb{R}^{3g-3+b}_{>0} \times \mathbb{R}^{3g-3+b}.$$

To extend this to non compact surfaces of finite type, a puncture is equivalent to a boundary component with length 0 so when Σ is the complement in Σ_0 of k points:

$$\mathcal{T}_\Sigma \cong \mathbb{R}^b_{\geq 0} \times \mathbb{R}^{3g-3+b+k}_{>0} \times \mathbb{R}^{3g-3+b+k}.$$

Remark 4.12. Note that a pants decomposition of Σ_g was chosen first so the Fenchel-Nielsen coordinates depend on them and different choices of pants decompositions will lead to different coordinate functions. When $b = 0$, pulling back the canonical symplectic form:

$$\sum_{i=1}^{3g-3} dx_i \wedge dx_{i+3g-3}$$

from $\mathbb{R}^{6g-6}$ defines a symplectic form ω_{FN} on $\mathcal{T}_g$. It is a remarkable fact due to Wolpert [Wol83] that although the coordinate functions do depend on the choice of pants decomposition, ω_{FN} does not.

4.2 HYPERBOLIC PANTS

We begin with a standard construction of Fenchel-Nielsen coordinates on the 3-holed sphere and the punctured torus but provide explicit equations and identities for everything. Our constructions essentially follow the ideas of chapter 3 in [IT92] and chapter 2 in [Hub06] but with a minor difference: when S is sphere with three punctures c_1, c_2, c_3 we prefer to take the Fenchel-Nielsen coordinates to be the lengths of the geodesics around c_1 and c_2 together with the length of the seam from c_1 to c_2 (the length of the loop around C_3 of course being determined by this data).

For an alternative approach in the disk rather than the half plane see e.g. chapter 4 in [Kat92]. For a comprehensive introduction to Teichmüller theory we recommend [Hub06] and [IT92].

Let S be the complement of three points c_1, c_2, c_3 in S^2 and let L_i be a loop around c_i such that L_1, L_2 generate the fundamental group of S. Let $\mathbb{H}$ denote the upper half plane with its hyperbolic metric g_h and distance function d_h. Although Fenchel-Nielsen coordinates:

$$FN : \mathcal{T}_{\Sigma_0^3} \to \mathbb{R}^3$$

are given by:

$$(\ell(L_1), \ell(L_2), \ell(L_3)),$$

this data is equivalent to the lengths of L_1, L_2, and of the seam C_3 connecting them, and this form is more convenient for our purposes. We therefore take that to be the Fenchel-Nielsen coordinate map:

$$FN([h]) = (\ell(L_1), \ell(L_2), \ell(C_3)).$$

The main result of this section is a determination of the inverse of FN:

Theorem 4.13. *Given:*

$$(\ell_1, \ell_2, s_3) \in \mathbb{R}^3_{>0}$$

with:

$$s_3 > \coth(\ell_1/2)\coth(\ell_2/2) > 1,$$

the unique hyperbolic structure on Σ_0^3 such that:

$$(\ell(L_1), \ell(L_2), \ell(C_3)) = (2\ell_1, 2\ell_2, \log(s_3))$$

is given by the quotient of $\mathbb{H}$ by the Fuchsian group:

$$\left\langle \begin{pmatrix} \cosh(\ell_1) & \sinh(\ell_1) \\ \sinh(\ell_1) & \cosh(\ell_1) \end{pmatrix}, \begin{pmatrix} \cosh(\ell_2) & -s_3\sinh(\ell_2) \\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2) \end{pmatrix} \right\rangle.$$

From this we have the stated result:

Corollary 4.14. *The map:*

$$\begin{aligned} \mathcal{T}_{\Sigma_0^3} &\to (G \times G)/G \\ [h] &\mapsto [(L_1^h, L_2^h)] \end{aligned}$$

written in Fenchel-Nielsen coordinates is given by:

$$(\ell_1, \ell_2, s_3) \mapsto \left[\begin{pmatrix} \cosh(\ell_1) & \sinh(\ell_1) \\ \sinh(\ell_1) & \cosh(\ell_1) \end{pmatrix}, \begin{pmatrix} \cosh(\ell_2) & -s_3 \sinh(\ell_2) \\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2) \end{pmatrix} \right]$$

To prove theorem 4.13 we apply the Poincaré polygon theorem 4.6. We construct a polygon with 4 geodesic sides C_1, C_1', C_2, C_2' such that the isometries E_3E_1 bringing C_1' to C_1 and E_2E_3 bringing C_2' to C_2 satisfy the hypotheses of the Poincaré polygon theorem (see figure 4.1). In the process of constructing this polygon, the boundary geodesic lengths as well as the seam lengths will be made clear. We then work out E_3E_1 and E_2E_3 explicitly and show that they are as claimed in theorem 4.13.

To this end, fix:

$$(\ell_1, \ell_2, s_3) \in \mathbb{R}^3_{>0}$$

with:

$$s_3 > \coth(\ell_1/2)\coth(\ell_2/2) > 1,$$

and let:

$$p_1 = (\tanh(\ell_1), \operatorname{sech}(\ell_1)) \in S^1$$

and note that $d_h(i, p_1) = \ell_1$. Similarly let:

$$p_2 = (s_3 \tanh(\ell_2), s_3 \operatorname{sech}(\ell_2))$$

and note that $d_h(s_3 i, p_2) = \ell_2$. Next, define:

$$p = \frac{(s_3^2+1)\coth(\ell_1) - 2s_3\coth(\ell_2)}{s_3^2 + 1 - 2s_3\coth(\ell_1)\coth(\ell_2) + 2\operatorname{csch}^2(\ell_1)}, \quad q = \sqrt{2p\coth(\ell_1) - p^2 - 1}$$

and:

$$m = \frac{p\coth\ell_1 - 1}{p - \coth\ell_1} = \frac{s_3^2 - 1}{2s_3\coth\ell_2 - 2\coth\ell_1}.$$

Define the circle L_3' by:

$$(x-m)^2 + y^2 = (m-p)^2 + q^2.$$

One can compute that:

$$(m-p)^2 + q^2 = \frac{(s_3^2+1)^2 + 4s_3(s_3\coth^2\ell_1 - (s_3^2+1)\coth\ell_1\coth\ell_2 + s_3\operatorname{csch}^2\ell_2)}{4(\coth\ell_1 - s_3\coth\ell_2)^2}$$

hence L_3' is given by:

$$x + (x^2+y^2-1)s_3\coth\ell_2 = xs_3^2 + (x^2+y^2-s_3^2)\coth\ell_1.$$

Let $L_1 \subset S^1$ be the arc from:

$$(-\tanh(\ell_1), \operatorname{sech}(\ell_1)) \text{ to } (\tanh(\ell_1), \operatorname{sech}(\ell_1))$$

and let $L_2 \subset s_3 S^1$ be the arc from:

$$(-s_3 \tanh(\ell_2), s_3 \operatorname{sech}(\ell_2)) \text{ to } (s_3 \tanh(\ell_2), s_3 \operatorname{sech}(\ell_2)).$$

Note that the length of L_1 is ℓ_1 and the length of L_2 is ℓ_2. Let:

$$C_3 = \{it : 1 \leq t \leq s_3\},$$

let C_2 be the geodesic segment from S^1 to L_3' and let C_1 be the geodesic segment from L_3' to $s_3 S^1$. More specifically:

$$C_2 = \{(x - \coth(\ell_1))^2 + y^2 = \operatorname{csch}^2(\ell_1) : y > 0,\ \tanh(s) \leq x \leq p\}$$

and:

$$C_1 = \{(x - s_3 \coth(\ell_2))^2 + y^2 = s_3^2 \operatorname{csch}^2(\ell_2) : y > 0,\ t \leq x \leq s_3 \tanh(\ell_2)\}$$

where:

$$t = \frac{s_3\left((s_3^2+1)\coth(\ell_2) - 2s_3\coth(\ell_1)\right)}{s_3^2 + 1 - 2s_3\coth(\ell_1)\coth(\ell_2) + 2s_3^2\operatorname{csch}^2(\ell_2)}.$$

Let $E_i \colon \mathbb{H} \to \mathbb{H}$ be reflection across C_i and define:

$$\Gamma_{\ell_1,\ell_2,s_3} \leq \mathrm{PSL}_2(\mathbb{R}) \tag{4.12}$$

to be the subgroup generated by:

$$\{E_2E_3, E_3E_1\}$$

so $\mathbb{H}/\Gamma_{\ell_1,\ell_2,s_3}$ is a three holed sphere. Let $C_i' = E(C_i)$. Notice that the region D bounded by:

$$C_1,\ C_2,\ C_1',\ C_2'$$

gives a fundamental domain for $\mathbb{H}/\Gamma_{\ell_1,\ell_2,s_3}$. Let $L_3 = L_3' \sqcup E_3(L_3')$. The image in $\mathbb{H}/\Gamma_{\ell_1,\ell_2,s_3}$ of the hyperbolic octagon O in D bounded by the geodesics:

$$C_1,\ C_2,\ C_1',\ C_2', L_1,\ L_2,\ L_3'$$

is a hyperbolic pair of pants whose boundary components are the images of L_1, L_2, and L_3 and have lengths $2\ell_1, 2\ell_2$, and $2\ell_3$ where:

$$\ell_3(s_3) \colon [\coth(\ell_1/2)\coth(\ell_2/2), \infty) \to [0, \infty)$$

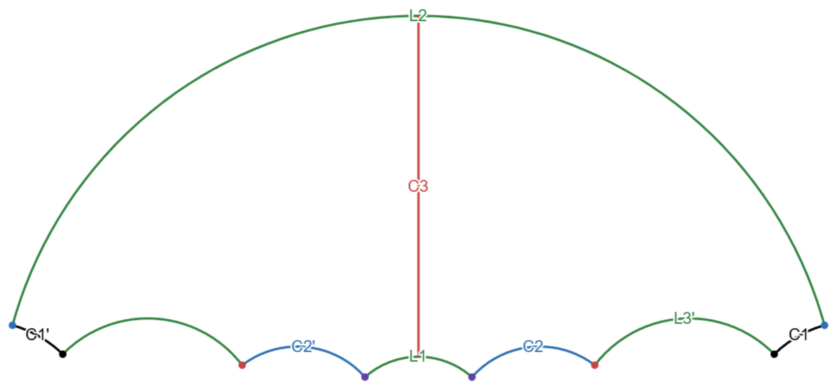


Figure 4.1: The hyperbolic octagon for a 3-holed sphere

is a homeomorphism defined implicitly by:

$$\begin{aligned}\cosh(\ell_3) + \cosh(\ell_1)\cosh(\ell_2) &= \cosh(\ell(C_3))\sinh(\ell_1)\sinh(\ell_2) \\ &= \frac{1+s_3^2}{2s_3}\sinh(\ell_1)\sinh(\ell_2).\end{aligned}$$

The curves C_i cover the seams on the hyperbolic pants and their lengths $\ell(C_i)$ can be computed using the sinh law for hyperbolic right-hexagons:

$$\frac{\sinh(a_1)}{\sinh(b_1)} = \frac{\sinh(a_2)}{\sinh(b_2)} = \frac{\sinh(a_3)}{\sinh(b_3)}$$

so we see:

$$\begin{aligned}\ell(C_3) &= \log(s_3) \\ \sinh(\ell(C_2)) &= \frac{\sinh(\ell(C_3))}{\sinh(\ell_3)}\sinh(\ell_2) \\ \sinh(\ell(C_1)) &= \frac{\sinh(\ell(C_3))}{\sinh(\ell_3)}\sinh(\ell_1)\end{aligned}$$

and:

$$\begin{aligned}\frac{\sinh(\ell(C_3))}{\sinh(\ell_3)} &= \frac{\sinh(\log(s_3))}{\sinh\left(\cosh^{-1}\left(\frac{1+s_3^2}{2s_3}\sinh(\ell_1)\sinh(\ell_2) - \cosh(\ell_1)\cosh(\ell_2)\right)\right)} \\ &= \frac{s_3 - \frac{1}{s_3}}{2\sqrt{\left(\frac{1+s_3^2}{2s_3}\sinh(\ell_1)\sinh(\ell_2) - \cosh(\ell_1)\cosh(\ell_2)\right)^2 - 1}}.\end{aligned}$$

When $\ell := \ell_1 = \ell_2$ these equations simplify; starting with:

$$p = \frac{(1-s_3)\sinh(2\ell)}{3+s_3+(1-s_3)\cosh(2\ell)}, \quad m = \frac{(s_3+1)\tanh(\ell)}{2}, \quad t = \frac{(s_3-1)s_3\sinh(2\ell)}{1+3s_3+(s_3-1)\cosh(2\ell)}$$

so:

$$(m-p)^2 + q^2 = \frac{(s_3-1)^2 - (s_3+1)^2\operatorname{sech}(\ell)^2}{4}$$

and therefore the equation defining L_3' reduces to:

$$x^2 + y^2 + s_3 = (s_3+1)x\tanh(\ell), \quad p \leq x \leq t.$$

We also have the simplification:

$$\frac{\sinh(\ell(C_3))}{\sinh(\ell_3)} = \frac{\sqrt{2}(s_3+1)\operatorname{csch}(\ell)}{\sqrt{-1-s_3(6+s_3)+(s_3-1)^2\cosh(2\ell)}}$$

so:

$$\ell(C_1) = \ell(C_2) = \sinh^{-1}\left(\frac{\sqrt{2}(s_3+1)}{\sqrt{-1-s_3(6+s_3)+(s_3-1)^2\cosh(2\ell)}}\right).$$

Next we work out the generators:

$$\gamma_1 = E_2E_3 \quad \text{and} \quad \gamma_2 = E_3E_1$$

of $\Gamma_{\ell_1,\ell_2,s_3}$ explicitly. Reflection across C_i can be decomposed by first translating C_i to the imaginary axis, reflecting across the imaginary axis and then translating the imaginary axis back to C_i. Since C_2 intersects the x-axis at $\tanh(\ell_1/2)$ and $\coth(\ell_1/2)$, and since $\tanh(\ell_1/2) < \coth(\ell_1/2)$, we have that for any $d \neq 0$:

$$\begin{pmatrix} \frac{\cosh^2\left(\frac{\ell_1}{2}\right)}{d} & d\tanh\left(\frac{\ell_1}{2}\right) \\ \frac{\sinh(\ell_1)}{2d} & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$$

is the transformation bringing the imaginary axis to C_2. Next, note that C_1 intersects the x-axis at $s_3\tanh(\ell_2/2)$ and $s_3\coth(\ell_2/2)$, so for $a \neq 0$:

$$\begin{pmatrix} \frac{\cosh^2\left(\frac{\ell_2}{2}\right)}{a} & as_3\tanh\left(\frac{\ell_2}{2}\right) \\ \frac{\sinh(\ell_2)}{2as_3} & a \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$$

brings the imaginary axis to C_1. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ let:

$$f_A \colon \mathbb{H} \to \mathbb{H}$$

be the associated isometry:

$$f_A(z) = \frac{az+b}{cz+d}.$$

Notice that:

$$\begin{aligned} f_A^{-1} \circ E_3 \circ f_A \circ E_3(z) &= f_A^{-1} \circ E_3 \circ f_A(-\bar{z}) \\ &= f_A^{-1} \circ E_3 \left(\frac{-a\bar{z}+b}{-c\bar{z}+d} \right) \\ &= f_A^{-1} \left(\frac{az-b}{-cz+d} \right) \\ &= \frac{(ad+bc)z - 2bd}{-2acz + (ad+bc)} \end{aligned}$$

corresponds to the matrix:

$$\begin{pmatrix} ad+bc & -2bd \\ -2ac & ad+bc \end{pmatrix} = \left[A^{-1}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

and:

$$\begin{aligned} E_3 \circ f_A^{-1} \circ E_3 \circ f_A(z) = E_3 \circ f_A^{-1} \circ E_3 \left(\frac{az+b}{cz+d} \right) &= E_3 \circ f_A^{-1} \left(-\frac{a\bar{z}+b}{c\bar{z}+d} \right) \\ &= E_3 \left(-\frac{(ad+bc)\bar{z} + 2bd}{2ac\bar{z} + (ad+bc)} \right) \\ &= \frac{(ad+bc)z + 2bd}{2acz + (ad+bc)} \end{aligned}$$

corresponds to the matrix:

$$\begin{pmatrix} ad+bc & 2bd \\ 2ac & ad+bc \end{pmatrix} = \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, A^{-1} \right].$$

Therefore we see:

$$\gamma_1 = \left[\begin{pmatrix} \frac{\cosh^2\left(\frac{\ell_1}{2}\right)}{d} & d\tanh\left(\frac{\ell_1}{2}\right) \\ \frac{\sinh(\ell_1)}{2d} & d \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} \cosh(\ell_1) & \sinh(\ell_1) \\ \sinh(\ell_1) & \cosh(\ell_1) \end{pmatrix}$$

$$\gamma_2 = \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{\cosh^2\left(\frac{\ell_2}{2}\right)}{a} & as_3\tanh\left(\frac{\ell_2}{2}\right) \\ \frac{\sinh(\ell_2)}{2as_3} & a \end{pmatrix} \right] = \begin{pmatrix} \cosh(\ell_2) & -s_3\sinh(\ell_2) \\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2) \end{pmatrix}.$$

Thus:

$$\Gamma_{\ell_1,\ell_2,s_3} = \left\langle \begin{pmatrix} \cosh(\ell_1) & \sinh(\ell_1) \\ \sinh(\ell_1) & \cosh(\ell_1) \end{pmatrix}, \begin{pmatrix} \cosh(\ell_2) & -s_3\sinh(\ell_2) \\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2) \end{pmatrix} \right\rangle$$

completing the proof of theorem 4.13.

Theorem 4.15. *The map:*

$$(\ell_1,\ell_2,s_3) \mapsto \left[\left(\begin{pmatrix} \cosh(\ell_1) & \sinh(\ell_1) \\ \sinh(\ell_1) & \cosh(\ell_1) \end{pmatrix}, \begin{pmatrix} \cosh(\ell_2) & -s_3\sinh(\ell_2) \\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2) \end{pmatrix} \right) \right]$$

is the composition of Fenchel-Nielsen coordinates with the Fuchsian representation $\mathcal{T}_{\Sigma_0^3} \to (G \times G)/G$.

Remark 4.16. Observe that $\cosh(\ell_3) = |\operatorname{tr}(\gamma_1\gamma_2)|/2$. Note also:

$$\begin{pmatrix} \cosh(\ell_2) & -s_3\sinh(\ell_2) \\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2) \end{pmatrix} = \begin{pmatrix} \sqrt{s_3} & 0 \\ 0 & \frac{1}{\sqrt{s_3}} \end{pmatrix} \begin{pmatrix} \cosh(\ell_2) & -\sinh(\ell_2) \\ -\sinh(\ell_2) & \cosh(\ell_2) \end{pmatrix} \begin{pmatrix} \sqrt{s_3} & 0 \\ 0 & \frac{1}{\sqrt{s_3}} \end{pmatrix}^{-1}$$

and moreover neither of γ_i depend on the choices of $a, d \in \mathbb{R} \setminus 0$ from before.

Conversely, suppose we have a hyperbolic metric on a pair of pants such that the boundary components L_i are geodesics with lengths $\ell_i = \ell(L_i)$ (equivalently, a complete hyperbolic metric on a 3-holed sphere with positive end lengths). Let C_3 be the geodesic segment connecting L_1 to L_2, let C_2 be the geodesic segment connecting L_1 to L_3 and let C_1 be the geodesic segment connecting L_2 to L_3. Cutting along C_i produces two hyperbolic hexagons U_j with sides $C_1, L_1^j, C_2, L_2^j, C_3, L_3^j$ for $j \in \{1,2\}$ such that all interior angles are $\pi/2$ (a "right" hexagon). Notice that $\ell(L_i^1) = \ell(L_i^2) = \ell_i/2$. We therefore may embed U_1 as the hexagon in O from before with $x \geq 0$ and embed U_2 as the hexagon in O with $x \leq 0$, meeting along the imaginary axis. Moreover, this procedure is the inverse to the previous construction. Thus we have the following:

Theorem 4.17. *The map:*

$$\begin{aligned} \mathcal{T}_{\Sigma_0^3} &\to (G \times G)/G \\ [h] &\mapsto [(L_1^h, L_2^h)] \end{aligned}$$

written in Fenchel-Nielsen coordinates is given by:

$$(\ell_1, \ell_2, s_3) \mapsto \left[\begin{pmatrix} \cosh(\ell_1) & \sinh(\ell_1) \\ \sinh(\ell_1) & \cosh(\ell_1) \end{pmatrix}, \begin{pmatrix} \cosh(\ell_2) & -s_3 \sinh(\ell_2) \\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2) \end{pmatrix} \right] \tag{4.13}$$

where:

$$\ell(L_i) = 2\ell_i, \quad \ell(C_3) = \log(s_3)$$

and C_3 is the seam connecting L_1 to L_2. Moreover we have $\ell(L_3) = \ell_3$ where ℓ_3 is the unique solution to:

$$\cosh(\ell_3) + \cosh(\ell_1)\cosh(\ell_2) = \frac{(1 + s_3^2)\sinh(\ell_1)\sinh(\ell_2)}{2s_3},$$

and the other geodesic seams C_1, C_2 have lengths $\ell(C_i)$ satisfying:

$$\begin{aligned} \sinh(\ell(C_2)) &= \frac{\sinh(\ell(C_3))}{\sinh(\ell_3)} \sinh(\ell_2) \\ \sinh(\ell(C_1)) &= \frac{\sinh(\ell(C_3))}{\sinh(\ell_3)} \sinh(\ell_1). \end{aligned}$$

4.3 HYPERBOLIC PUNCTURED TORII

Now suppose $\ell := \ell_1 = \ell_2$ and repeat the construction of a hyperbolic octagon in $\mathbb{H}$ but consider:

$$\Gamma = \left\langle \begin{pmatrix} \cosh(\ell) & \sinh(\ell) \\ \sinh(\ell) & \cosh(\ell) \end{pmatrix}, \begin{pmatrix} \cosh(\ell) & -s_3 \sinh(\ell) \\ -\frac{\sinh(\ell)}{s_3} & \cosh(\ell) \end{pmatrix}, \begin{pmatrix} \sqrt{s_3} & 0 \\ 0 & \frac{1}{\sqrt{s_3}} \end{pmatrix} \right\rangle$$

As was noted above:

$$\begin{pmatrix} \cosh(\ell) & -s_3 \sinh(\ell) \\ -\frac{\sinh(\ell)}{s_3} & \cosh(\ell) \end{pmatrix} = \begin{pmatrix} \sqrt{s_3} & 0 \\ 0 & \frac{1}{\sqrt{s_3}} \end{pmatrix} \begin{pmatrix} \cosh(\ell) & \sinh(\ell) \\ \sinh(\ell) & \cosh(\ell) \end{pmatrix}^{-1} \begin{pmatrix} \sqrt{s_3} & 0 \\ 0 & \frac{1}{\sqrt{s_3}} \end{pmatrix}^{-1}$$

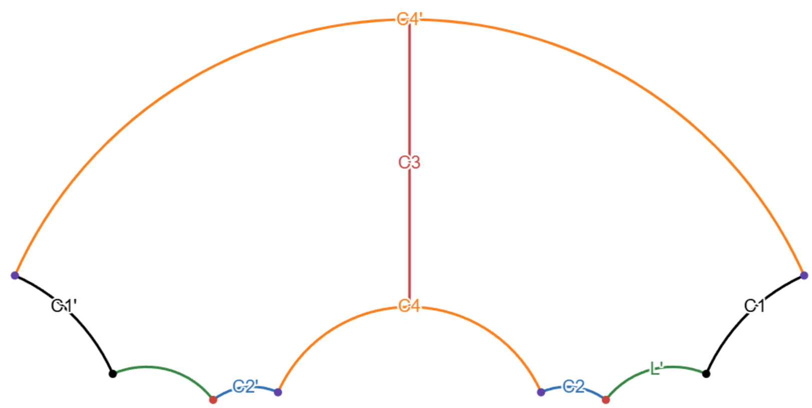


Figure 4.2: The hyperbolic octagon for a punctured torus

so:

$$\Gamma = \left\langle \begin{pmatrix} \cosh(\ell) & \sinh(\ell) \\ \sinh(\ell) & \cosh(\ell) \end{pmatrix}, \begin{pmatrix} \sqrt{s_3} & 0 \\ 0 & \frac{1}{\sqrt{s_3}} \end{pmatrix} \right\rangle .$$

A fundamental domain for Γ can be obtained from one for $\Gamma_{\ell_1,\ell_1,s_3}$ by adding to it the geodesic segments:

$$C_4 := L_1, \quad C_4' := L_2$$

defined in the previous section but now $\mathbb{H}/\Gamma$ is a punctured torus. The octagon bounded by:

$$C_1, \; C_2, \; C_4, \; L_3, \; C_1', \; C_2', \; C_4'$$

projects into $\mathbb{H}/\Gamma$ as a torus T with a single geodesic boundary component ∂T. The simple closed loops in $\mathbb{H}/\Gamma$ covered by C_4 and C_3 generate its fundamental group and have lengths:

$$\ell(C_4) = 2\ell, \quad \ell(C_3) = \log(s_3),$$

the boundary loop is covered by L_3 and has length:

$$\ell_3 = 2\left(\frac{1+s_3^2}{2s_3}\sinh(\ell)^2 - \cosh(\ell)^2\right) = -\operatorname{tr}\left[\begin{pmatrix} \cosh(\ell) & \sinh(\ell) \\ \sinh(\ell) & \cosh(\ell) \end{pmatrix}, \begin{pmatrix} \sqrt{s_3} & 0 \\ 0 & \frac{1}{\sqrt{s_3}} \end{pmatrix}\right].$$

A pants decomposition for T can be obtained by cutting along C_4 and will be the unique hyperbolic pair of pants with boundary lengths:

$$2\ell, \;\; 2\ell, \;\; \text{and} \;\; \frac{1+s_3^2}{s_3}\sinh(\ell)^2 - 2\cosh(\ell)^2.$$

All that remains is consideration of a twist parameter τ around C_4. To this end, note that applying a twist to C_4 will not have any effect on ℓ nor on ℓ_3. Thus any pair (a, b) generating a Fuchsian group $\Gamma = \langle a, b\rangle$ such that the length of the geodesics corresponding to a and $[a, b]$ are the same as T must be conjugate to

Suppose we have some other

hence the trace of:

$$\left[\begin{pmatrix}\cosh(\ell) & \sinh(\ell)\\ \sinh(\ell) & \cosh(\ell)\end{pmatrix}, M\right]$$

must be invariant.

But it can be shown that:

$$\operatorname{tr}\left[\begin{pmatrix}\cosh(\ell) & \sinh(\ell)\\ \sinh(\ell) & \cosh(\ell)\end{pmatrix}, \begin{pmatrix}\sqrt{s_3} & 0\\ 0 & \frac{1}{\sqrt{s_3}}\end{pmatrix}\right] = \operatorname{tr}\left[\begin{pmatrix}\cosh(\ell) & \sinh(\ell)\\ \sinh(\ell) & \cosh(\ell)\end{pmatrix}, M\right]$$

if and only if

$$M = \begin{pmatrix}\sqrt{s_3} & 0\\ 0 & \frac{1}{\sqrt{s_3}}\end{pmatrix}\begin{pmatrix}\cosh(x) & \sinh(x)\\ \sinh(x) & \cosh(x)\end{pmatrix}$$

for some $x \geq 0$. Thus we have the following:

Theorem 4.18. *The map:*

$$\mathcal{T}_{\Sigma_1^1} \to (G \times G)/G$$
$$[h] \mapsto [(\alpha^h, \beta^h)]$$

written in Fenchel-Nielsen coordinates is given by:

$$(\ell_1, \ell_2, s_3) \mapsto \left[\begin{pmatrix}\cosh(\ell_1) & \sinh(\ell_1)\\ \sinh(\ell_1) & \cosh(\ell_1)\end{pmatrix}, \begin{pmatrix}\sqrt{s_3} & 0\\ 0 & \frac{1}{\sqrt{s_3}}\end{pmatrix}\begin{pmatrix}\cosh(\ell_2) & \sinh(\ell_2)\\ \sinh(\ell_2) & \cosh(\ell_2)\end{pmatrix}\right] \tag{4.14}$$

where:

$$\ell(\alpha) = 2\ell_1, \quad \ell(\beta) = \log(s_3) \quad \tau(\alpha) = 2\ell_2. \tag{4.15}$$

The geodesic loop L_3 around the puncture has length ℓ_3 satisfying:

$$\cosh(\ell_3) = \frac{1+s_3^2}{2s_3}\sinh(\ell_1)^2 - \cosh(\ell_1)^2$$

and the seams C_1, C_2 connecting α to L_3 have the same length $\ell(C)$ satisfying:

$$\sinh(\ell(C)) = \frac{\sinh(\ell(C_3))}{\sinh(\ell_3)}\sinh(\ell_1).$$

Remark 4.19. The quotient of $\mathbb{H}$ by the subgroup:

$$\Gamma' = \left\langle \begin{pmatrix} \cosh(\ell) & \sinh(\ell) \\ \sinh(\ell) & \cosh(\ell) \end{pmatrix}, \begin{pmatrix} \cosh(\ell) & -s_3\sinh(\ell) \\ -\frac{\sinh(\ell)}{s_3} & \cosh(\ell) \end{pmatrix} \right\rangle < \Gamma$$

is a 3-holed sphere and:

$$\mathbb{H}/\Gamma' \to \mathbb{H}/\Gamma$$

is the (necessarily not normal) covering of a punctured torus by a 3-holed sphere.

4.4 THE ISOMETRY GROUPS

We start a further investigation into the group of isometries:

$$\Gamma = \left\langle \begin{pmatrix} \cosh(\ell_1) & \sinh(\ell_1) \\ \sinh(\ell_1) & \cosh(\ell_1) \end{pmatrix}, \begin{pmatrix} \cosh(\ell_2) & -s_3\sinh(\ell_2) \\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2) \end{pmatrix} \right\rangle$$

of the 3-holed sphere.

Proposition 4.20. *Two matrices $h_1, h_2 \in \mathrm{SL}_2(\mathbb{R})$ satisfy:*

$$\mathrm{tr}(h_1) > 2, \quad \mathrm{tr}(h_2) > 2, \quad \mathrm{tr}(h_1h_2) < -2$$

if and only if they are simultaneously conjugate to:

$$\begin{pmatrix} \cosh(\ell_1) & \sinh(\ell_1) \\ \sinh(\ell_1) & \cosh(\ell_1) \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \cosh(\ell_2) & -s_3\sinh(\ell_2) \\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2) \end{pmatrix}$$

for some:

$$(\ell_1, \ell_2) \in \mathbb{R}^2_{>0}, \quad s_3 > \coth(\ell_1/2)\coth(\ell_2/2).$$

Proof. Computing:

$$\operatorname{tr}\left(\begin{pmatrix}\cosh(\ell_1) & \sinh(\ell_1)\\ \sinh(\ell_1) & \cosh(\ell_1)\end{pmatrix}\begin{pmatrix}\cosh(\ell_2) & -s_3\sinh(\ell_2)\\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2)\end{pmatrix}\right)$$
$$= \operatorname{tr}\begin{pmatrix}\cosh(\ell_1)\cosh(\ell_2) - \frac{\sinh(\ell_1)\sinh(\ell_2)}{s_3} & \sinh(\ell_1)\cosh(\ell_2) - s_3\cosh(\ell_1)\sinh(\ell_2)\\ \sinh(\ell_1)\cosh(\ell_2) - \frac{\cosh(\ell_1)\sinh(\ell_2)}{s_3} & \cosh(\ell_1)\cosh(\ell_2) - s_3\sinh(\ell_1)\sinh(\ell_2)\end{pmatrix}$$
$$= 2\cosh(\ell_1)\cosh(\ell_2) - \frac{(s_3^2+1)\sinh(\ell_1)\sinh(\ell_2)}{s_3}$$

$$< -2.$$

Conversely suppose that:

$$\operatorname{tr}(h_1) > 2, \quad \operatorname{tr}(h_2) > 2, \quad \operatorname{tr}(h_1h_2) < -2.$$

Since h_1 is hyperbolic it is similar to $\begin{pmatrix}\cosh(\ell_1) & \sinh(\ell_1)\\ \sinh(\ell_1) & \cosh(\ell_1)\end{pmatrix}$ for some ℓ_1. Write:

$$h_2 = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

with $a + d > 2$. There exists:

$$h = \begin{pmatrix}\cosh(y) & \sinh(y)\\ \sinh(y) & \cosh(y)\end{pmatrix}$$

such that the diagonal entries of hh_2h^{-1} are equal if and only if:

$$-1 < \frac{d-a}{c-b} < 1$$

but this is implied by the conditions:

$$a + d > 2 \quad \text{and} \quad \cosh(a)(a+d) + \sinh(a)(b+c) < -2.$$

□

Now we consider the case of a single cusp. When $s_3 = \coth(\ell_1/2)\coth(\ell_2/2)$:

$$\Gamma = \left\langle \begin{pmatrix}\cosh(\ell_1) & \sinh(\ell_1)\\ \sinh(\ell_1) & \cosh(\ell_1)\end{pmatrix}, \begin{pmatrix}\cosh(\ell_2) & -\coth(\frac{\ell_1}{2})(1+\cosh(\ell_2))\\ \frac{\tanh(\frac{\ell_1}{2})}{1-\cosh(\ell_2)} & \cosh(\ell_2)\end{pmatrix}\right\rangle$$

and the product of these two generators is:

$$\begin{pmatrix} \cosh(\ell_1)+\cosh(\ell_2)-1 & -\coth(\frac{\ell_1}{2})\left(\cosh(\ell_1)+\cosh(\ell_2)\right) \\ \tanh(\frac{\ell_1}{2})(\cosh(\ell_1)+\cosh(\ell_2)) & -(\cosh(\ell_1)+\cosh(\ell_2))-1 \end{pmatrix}$$

which has trace equal to -2. Note that taking:

$$s_3 \to \coth(\ell_1/2)\coth(\ell_2/2)$$

shrinks L_3 on the hyperbolic pair of pants to a cusp (an end with length 0) thus obtaining a hyperbolic metric on a the 3-holed sphere having boundary lengths $\ell_1, \ell_2 > 0$, and $\ell_3 = 0$.

Finally, we consider the case of 2 or 3 cusps. Fix $c \leq -4$. The region in $\mathbb{H}$ bounded by:

$$y = \pm\frac{1}{2}, \quad \left(x \pm \frac{1}{c}\right)^2 + y^2 = \frac{1}{c^2}$$

forms a Dirichlet domain for:

$$\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \right\rangle$$

and $\mathbb{H}/\Gamma$ is a 3-holed sphere with two cusps and one end length equal to:

$$2\text{arccosh}\left(-\frac{c+2}{2}\right)$$

hence having 3 cusps when $c = -4$.

Collecting everything so far we obtain that for $\Gamma = \langle a, b\rangle < \mathrm{PSL}_2(\mathbb{R})$:

Theorem 4.21. *The quotient $\mathbb{H}/\Gamma$ defines a hyperbolic structure on a 3-holed sphere with a single cusp if and only if Γ is as described in 4.8 but with $s_3 = \coth(\ell_1/2)\coth(\ell_2/2)$.*

Theorem 4.22. *The quotient $\mathbb{H}/\Gamma$ defines a hyperbolic structure on a 3-holed sphere with two or three cusps if and only if a and b are simultaneously conjugate to:*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$$

for some $c \leq -4$ with the 3 cusp case corresponding to $c = -4$.

If one does not care about the explicit nature of the lengths, from a brief calculation one may combine 4.17 together with the degenerate (cusp) cases:

Theorem 4.23. *For $\Gamma = \langle a, b \rangle < \mathrm{PSL}_2(\mathbb{R})$ and $a', b' \in \mathrm{SL}_2(\mathbb{R})$ covering a, b respectively with $\mathrm{tr}(a'), \mathrm{tr}(b') \geq 0$, $\mathbb{H}/\Gamma$ is a 3-holed sphere and $\mathbb{H} \to \mathbb{H}/\Gamma$ is a covering map if and only if:*

$$\mathrm{tr}(a') \geq 2, \quad \mathrm{tr}(b') \geq 2, \quad \text{and} \quad \mathrm{tr}(a'b') \leq -2.$$

Finally, using an identical type of analysis we have:

Theorem 4.24. *For $\Gamma\langle a, b \rangle < \mathrm{PSL}_2(\mathbb{R})$ the quotient $\mathbb{H}/\Gamma$ is a punctured torus if and only if:*

$$\mathrm{tr}[a, b] < -2.$$

Theorem 4.25. *For $g_1, g_2 \in \mathrm{PSL}_2(\mathbb{R}) \setminus e$ not elliptic the following are equivalent:*

1. *There exists a homeomorphism:*

$$f \colon \Sigma_0^3 \to \mathbb{H}/\langle g_1, g_2 \rangle$$

such that $f_*([L_i]) = g_i$.

2. $\mathrm{tr}(m(g_1, g_2)) \leq -2$.

3. *The positive trace lifts of g_1 and g_2 in $\mathrm{SL}_2(\mathbb{R})$ are simultaneously conjugate to*

$$\begin{pmatrix} \cosh(\ell_1) & \sinh(\ell_1) \\ \sinh(\ell_1) & \cosh(\ell_1) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cosh(\ell_2) & -s_3 \sinh(\ell_2) \\ -\frac{\sinh(\ell_2)}{s_3} & \cosh(\ell_2) \end{pmatrix}$$

for some:

$$(\ell_1, \ell_2) \in \mathbb{R}^2_{>0}, \quad s_3 > \coth(\ell_1/2)\coth(\ell_2/2).$$

Theorem 4.26. *For $g_1, g_2 \in \mathrm{PSL}_2(\mathbb{R})$ the following are equivalent:*

1. *There exists a homeomorphism $f \colon \Sigma_1^1 \to \mathbb{H}/\langle g_1, g_2 \rangle$ such that $f_*(\alpha) = g_1$ and $f_*(\beta) = g_2$.*

2. $\mathrm{tr}(R(g_1, g_2)) \leq -2$.

3. *The positive trace lifts of g_1 and g_2 in $\mathrm{SL}_2(\mathbb{R})$ are simultaneously conjugate to*

$$\begin{pmatrix} \cosh(\ell_1) & \sinh(\ell_1) \\ \sinh(\ell_1) & \cosh(\ell_1) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} s_3 & 0 \\ 0 & \frac{1}{s_3} \end{pmatrix} \begin{pmatrix} \cosh(\ell_2) & \sinh(\ell_2) \\ \sinh(\ell_2) & \cosh(\ell_2) \end{pmatrix}$$

for some:

$$(\ell_1, \ell_2) \in \mathbb{R}^2_{>0}, \quad s_3 > \coth(\ell_1/2)\coth(\ell_2/2).$$

5 APPENDIX

5.1 THE CLASSIFICATION OF PRINCIPAL BUNDLES

Let G be a connected Lie group, Σ a closed, connected, orientable surface, $P \to \Sigma$ a principal G-bundle and θ a flat connection on P. In this section we show how to relate the holonomies of θ to the isomorphism class of P.

Fix $p \in \Sigma$, let U_1 be an open neighbourhood of p homeomorphic to a disk, and let $U_2 = \Sigma \setminus p$. Since U_1 is contractible $P|_{U_1}$ is trivial and since U_2 is homotopy equivalent to a wedge of circles $P|_{U_2}$ is also trivial. Pick trivializing sections $s_i \colon U_i \to P$ and let $g_{12} \colon U_{12} = U_1 \cap U_2 \to G$ denote their transition function. Homotopic transition functions define isomorphic bundles so we conclude (since U_{12} is an annulus):

Proposition 5.1. *There exists a bijection between isomorphism classes of principal G-bundles over Σ and $[U_{12}, G] = [S^1, G] = \pi_1(G)$.*

Remark 5.2. In general, $[S^1, X]$ is only the fundamental group of X up to conjugation but Lie groups have abelian fundamental groups.

Let θ be a flat connection on P and pick the neighbourhood U_1 of p so that $\theta|_{U_1} = \theta^L$. Let $\gamma_t \colon [0,1] \to U_{12}$ be a loop generating $\pi_1(U_{12})$ and let γ_h be its horizontal lift to P. Applying a gauge transformation if necessary there is no loss in generality in assuming $g_{12}(\gamma_0) = e$. In the trivialization over U_2 write $\gamma_h(t) = (\gamma(t), g_t)$ for $g_t \colon [0,1] \to G$ with $g_0 = e$. Since γ is a contractible loop in Σ we conclude that $g_1 = e$.

Lemma 5.3. *For $f, g \in C^\infty(X, G)$ we have $f^*\theta^L + g^*\theta^R = 0$ if and only if $f \cdot g$ is constant.*

Proof. Computing:

$$f^{-1} \cdot df = -dg \cdot g^{-1}$$
$$df \cdot g = -f \cdot dg$$

we conclude $0 = df \cdot g + f \cdot dg = d(f \cdot g)$. □

Since our connection 1-form is trivial in U_1 it is given by $A = g_{12}^* \theta^L$ in U_2 so γ_h being horizontal means:

$$0 = g_t^{-1} A(\dot\gamma_t) g_t + g_t^{-1} \dot g_t = A(\dot\gamma_t) + \dot g_t g_t^{-1} = g_{12}^* \theta^L(\dot\gamma_t) + \dot g_t g_t^{-1}$$
$$= (g_{12} \circ \gamma_t)^* \theta^L(\partial_t) + g_t^* \theta^R(\partial_t).$$

Since $g_0 = g_{12}(\gamma_0) = e$, by lemma 5.3 $(g_{12} \circ \gamma_t) \cdot g_t = e$ is constant. In particular their homotopy classes are inverses of each other so the isomorphism class of P can be computed via the horizontal lift of γ_t. We therefore conclude the following:

Corollary 5.4. *Let:*

$$\langle a_1, b_1, ..., a_g, b_g \rangle$$

be the standard presentation of the fundamental group of $\pi_1(\Sigma)$ and let $\{A_i, B_i\}_{i=1}^g$ be the holonomies of a_i, b_i. Then:

$$\left(\prod_{i=1}^g [A_i, B_i] \right)^{-1}$$

is a well defined element of $\pi_1(G)$ and its homotopy class is the isomorphism class of P.

5.2 TECHNICAL LEMMAS OF CHAPTER 2.

We present the proofs of the technical lemmas from chapter 2.

Lemma 2.23. The image of the commutator map in $SL_2(\mathbb{R})$ is $SL_2(\mathbb{R}) \setminus \{-I\}$.

Proof of Lemma 2.23. For $a, b \in SL_2(\mathbb{R})$, if $\mathrm{tr}(a) = \mathrm{tr}(b) \neq \pm 2$ then a is conjugate to b or b^{-1}, and since $[x,y]^{-1} = [y,x]$ it is enough (by equivariance) to show that the commutator map can produce any trace. But the trace of $\left[\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right]$ is equal to $2 - bc(t^2 + \frac{1}{t^2} - 2)$. To see that $-I$ is not a commutator note that $ABA^{-1}B^{-1} = -I$ implies that A, B both have trace zero hence we may assume $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and compute that the trace of $\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right]$ is $2a^2 + b^2 + c^2 \neq -2$. □

Lemma 2.24. For every $X \in \widetilde{SL}_2(\mathbb{R})$ with $\mathrm{tr}(X) < -2$ there exists $A, B \in \widetilde{SL}_2(\mathbb{R})$ such that $\mathrm{tr}(X[A,B]) < -2$ and $\mu(X[A,B]) = \mu(X) + 2$.

Proof of Lemma 2.24. There is no loss in generality in assuming the image of X in $\mathrm{SL}_2(\mathbb{R})$ is the diagonal matrix $\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}$ with $t < -1$. Then for:

$$Y = \begin{pmatrix} \frac{t-1}{t+1} & -1 \\ \frac{t^4+6t^2+1}{2(t+1)^2} & -\frac{t^3+t^2+5t+1}{2(t+1)} \end{pmatrix} \tag{5.1}$$

we compute:

$$\operatorname{tr}\left(\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} Y\right) = \frac{t^2 - 4t - 1}{2t} < -2$$

and by corollary 2.6:

$$\begin{aligned} & r\left(\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}, Y\right) \\ &= \operatorname{sgn}\begin{pmatrix} \frac{t^5-3t^4+6t^3-18t^2+t-3}{2t^3+2t^2-2t-2} & \frac{-t^4+t^3-5t^2+11t+2}{2t^2-2} \\ \frac{-t^4+t^3-5t^2+11t+2}{2t^2-2} & \frac{t^3+t^2+9t+1}{2t-2} \end{pmatrix} \\ &= 2. \end{aligned}$$

By lemma 2.23 there exists $A, B \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ such that $[A, B]$ covers Y and since:

$$\operatorname{tr}\begin{pmatrix} \frac{t-1}{t+1} & -1 \\ \frac{t^4+6t^2+1}{2(t+1)^2} & -\frac{t^3+t^2+5t+1}{2(t+1)} \end{pmatrix} = -\frac{t^2-3}{2} < -2$$

we must have $\mu[A, B] = \pm 1$. If $\mu[A, B] = -1$ apply lemma 2.20 to $\begin{pmatrix} \frac{t-1}{t+1} & -1 \\ \frac{t^4+6t^2+1}{2(t+1)^2} & -\frac{t^3+t^2+5t+1}{2(t+1)} \end{pmatrix}$ and replace A, B with gAg^{-1}, gBg^{-1}. Thus by the coboundary identity:

$$\begin{aligned} \mu(X[A,B]) &= \mu(X) + \mu[A,B] + \frac{1}{2} r\left(\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}, \begin{pmatrix} \frac{t-1}{t+1} & -1 \\ \frac{t^4+6t^2+1}{2(t+1)^2} & -\frac{t^3+t^2+5t+1}{2(t+1)} \end{pmatrix}\right) \\ &= \mu(X) + 2 \end{aligned}$$

□

Lemma 2.25. For every $X \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ with $\operatorname{tr}(X) < -2$ and $\varepsilon \in \{-1, 0, 1\}$ there exists A, B such that $X[A, B] \in \mathrm{Cent}(\widetilde{G})$ and $\mu(X[A, B]) = \mu(X) + \varepsilon$.

Proof of Lemma 2.25. As before there is no loss in generality in assuming X covers $\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}$ with $t < -1$. By lemma 2.23 there exists $A, B \in \widetilde{SL}_2(\mathbb{R})$ such that $[A, B]$ covers $\begin{pmatrix} -\frac{1}{t} & 0 \\ 0 & -t \end{pmatrix}$ and by corollary 2.6:

$$r\left(\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}, \begin{pmatrix} -\frac{1}{t} & 0 \\ 0 & -t \end{pmatrix}\right) = \operatorname{sgn}\begin{pmatrix} 0 & \frac{2t+2}{t-1} \\ \frac{2t+2}{t-1} & 0 \end{pmatrix} = 0.$$

Since $\operatorname{tr}\begin{pmatrix} -\frac{1}{t} & 0 \\ 0 & -t \end{pmatrix} > 2$ we must have $\mu[A, B] = 0$ so by the coboundary identity:

$$\mu(X[A,B]) = \mu(X) + \mu[A,B] + \frac{1}{2}r\left(\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}, \begin{pmatrix} -\frac{1}{t} & 0 \\ 0 & -t \end{pmatrix}\right) = \mu(X).$$

Using lemma 2.23 again there exists A, B such that $[A, B]$ covers $\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & t \end{pmatrix}$ and it is immediate by corollary 2.6 that $r(g, g^{-1}) = 0$ for any g. Then since $\operatorname{tr}\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & t \end{pmatrix} < -2$ we must have $\mu[A, B] = \pm 1$ so by the coboundary identity:

$$\mu(X[A,B]) = \mu(X) + \mu[A,B] = \mu(X) \pm 1$$

and if $\mu[A, B] = 1$ then we apply lemma 2.20 to $\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & t \end{pmatrix}$ and replace A, B with gAg^{-1}, gBg^{-1} (and conversely). □

BIBLIOGRAPHY

[AB83] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615. MR 702806

[AMM98] Anton Alekseev, Anton Malkin, and Eckhard Meinrenken, *Lie group valued moment maps*, J. Differential Geom. **48** (1998), no. 3, 445–495. MR 1638045

[Arn67] V. I. Arnold, *On a characteristic class entering into conditions of quantization*, Funkcional. Anal. i Priložen. **1** (1967), 1–14. MR 211415

[Ben55] Jean Paul Benzecri, *Varietes localement plates*, ProQuest LLC, Ann Arbor, MI, 1955, Thesis (Ph.D.)–Princeton University. MR 2612352

[BGPG03] Steven B. Bradlow, Oscar García-Prada, and Peter B. Gothen, *Surface group representations and $U(p,q)$-Higgs bundles*, J. Differential Geom. **64** (2003), no. 1, 111–170. MR 2015045

[BGPG06] ———, *Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces*, Geom. Dedicata **122** (2006), 185–213. MR 2295550

[BIW03] Marc Burger, Alessandra Iozzi, and Anna Wienhard, *Surface group representations with maximal Toledo invariant*, C. R. Math. Acad. Sci. Paris **336** (2003), no. 5, 387–390. MR 1979350

[BIW10] ———, *Surface group representations with maximal Toledo invariant*, Ann. of Math. (2) **172** (2010), no. 1, 517–566. MR 2680425

[BIW14] ———, *Higher Teichmüller spaces: from $SL(2,\mathbb{R})$ to other Lie groups*, Handbook of Teichmüller theory. Vol. IV, IRMA Lect. Math. Theor. Phys., vol. 19, Eur. Math. Soc., Zürich, 2014, pp. 539–618. MR 3289711

[Dui11] J. J. Duistermaat, *Fourier integral operators*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2011, Reprint of the 1996 edition [MR1362544], based on the original lecture notes published in 1973 [MR0451313]. MR 2741911

[Dup79] Johan L. Dupont, *Bounds for characteristic numbers of flat bundles*, Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), Lecture Notes in Math., vol. 763, Springer, Berlin, 1979, pp. 109–119. MR 561216

[FG06] Vladimir Fock and Alexander Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. (2006), no. 103, 1–211. MR 2233852

[FN03] Werner Fenchel and Jakob Nielsen, *Discontinuous groups of isometries in the hyperbolic plane*, De Gruyter Studies in Mathematics, vol. 29, Walter de Gruyter & Co., Berlin, 2003, Edited and with a preface by Asmus L. Schmidt, Biography of the authors by Bent Fuglede. MR 1958350

[Fra59] Theodore Frankel, *Fixed points and torsion on Kähler manifolds*, Ann. of Math. (2) **70** (1959), 1–8. MR 131883

[Gil95] Jane Gilman, *Two-generator discrete subgroups of psl(2;r)*, Mem. Amer. Math. Soc. **117** (1995), no. 561, x+204. MR 1290281

[Gol88] William M. Goldman, *Topological components of spaces of representations*, Invent. Math. **93** (1988), no. 3, 557–607. MR 952283

[Got01] Peter B. Gothen, *Components of spaces of representations and stable triples*, Topology **40** (2001), no. 4, 823–850. MR 1851565

[GPGMiR13] O. García-Prada, P. B. Gothen, and I. Mundet i Riera, *Higgs bundles and surface group representations in the real symplectic group*, J. Topol. **6** (2013), no. 1, 64–118. MR 3029422

[Hit87] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126. MR 887284

[Hit92] ———, *Lie groups and Teichmüller space*, Topology **31** (1992), no. 3, 449–473. MR 1174252

[Hub06] John Hamal Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*, Matrix Editions, Ithaca, NY, 2006, Teichmüller theory, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle. MR 2245223

[IT92] Y. Imayoshi and M. Taniguchi, *An introduction to Teichmüller spaces*, Springer-Verlag, Tokyo, 1992, Translated and revised from the Japanese by the authors. MR 1215481

[Jr76] Troels Jø rgensen, *On discrete groups of Möbius transformations*, Amer. J. Math. **98** (1976), no. 3, 739–749. MR 427627

[Kat92] Svetlana Katok, *Fuchsian groups*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1992. MR 1177168

[KPW22] Inkang Kim, Pierre Pansu, and Xueyuan Wan, *Signature, toledo invariant and surface group representations in the real symplectic group*, 2022.

[Lab06] Francois Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), no. 1, 51–114. MR 2221137

[LV80] Gérard Lion and Michèle Vergne, *The Weil representation, Maslov index and theta series*, Progress in Mathematics, vol. 6, Birkhäuser, Boston, MA, 1980. MR 573448

[Mas01] Bernard Maskit, *Matrices for Fenchel-Nielsen coordinates*, Ann. Acad. Sci. Fenn. Math. **26** (2001), no. 2, 267–304. MR 1833242

[Mei94] Eckhard Meinrenken, *Trace formulas and the Conley-Zehnder index*, J. Geom. Phys. **13** (1994), no. 1, 1–15. MR 1259446

[Mil58] John Milnor, *On the existence of a connection with curvature zero*, Comment. Math. Helv. **32** (1958), 215–223. MR 95518

[Rie57] B. Riemann, *Theorie der Abel'schen Functionen*, J. Reine Angew. Math. **54** (1857), 115–155. MR 1579035

[Tur84] A.T. Turaev, *A cocycle for the symplectic first chern class and the maslov index.*, Funct Anal Its Appl **18** (1984), 35–39.

[Wol83] Scott Wolpert, *On the symplectic geometry of deformations of a hyperbolic surface*, Ann. of Math. (2) **117** (1983), no. 2, 207–234. MR 690844

[Woo71] John W. Wood, *Bundles with totally disconnected structure group*, Comment. Math. Helv. **46** (1971), 257–273. MR 293655