Topology of real matroid Schubert varieties

by

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A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy

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2025

Abstract

In this thesis we study the topology of real matroid Schubert varieties, which are closures of real linear spaces in products of projective lines. We show that the topology of these varieties is controlled by the combinatorics of real hyperplane arrangements. More precisely, we exhibit homeomorphisms from real matroid Schubert varieties to quotients of zonotopes. Further, this combinatorial model for the topology of the variety can be generalised to define a topological space for any oriented matroid. As a consequence, we are able to compute the fundamental group and integral cohomology of these spaces, obtaining virtual Coxeter groups (in special cases) and signed analogues of the graded Möbius algebra respectively.

Acknowledgements

The content of this thesis is part of a work in progress joint with Yu Li. Parts were previously published in the extended abstract [20].

Contents

Chapter 1

Introduction

A recurring theme in algebraic combinatorics is the fruitful interaction between geometry and combinatorics. The geometric properties of certain combinatorially defined algebraic varieties (toric varieties, Schubert varieties,...) contain information about related combinatorial objects (polytopes, Weyl groups,...). However, the existence of these varieties (or at least their nice properties) is often restricted to a certain class of "realisable" combinatorial objects. The geometric methods can nevertheless often be combinatorialised to prove results even in the nonrealisable case.

This philosophy has recently been especially successful in the study of matroids. In this case the realisable object is a hyperplane arrangement $\mathcal{A}$, or equivalently a linear subspace of a (finite-dimensional) vector space k^n. The combinatorially relevant variety which is the subject of this thesis is the multiprojective variety obtained as the closure of this subspace in a product of projective lines. These so-called *matroid Schubert varieties* $Y_{\mathcal{A}}$ (introduced by Ardila and Boocher [1]) have been related to several major results in matroid theory:

- The *Dowling–Wilson top-heavy conjecture* states that the lattice of flats $\mathcal{L}(M)$ of a matroid M is "top-heavy": for $k \leqslant \operatorname{rk} M/2$, the number of flats of rank k is bounded from above by the number of flats of rank $\operatorname{rk} M - k$. When M is realisable by a hyperplane arrangement over a field, Huh and Wang proved the conjecture using the intersection cohomology of matroid Schubert varieties [17].

  For simplicity, assume that we work over $\mathbb{C}$. Then the cohomology of $Y_{\mathcal{A}}$ is the *graded Möbius algebra*: $H^*(Y_{\mathcal{A}};\mathbb{Q}) = \bigoplus_{F\in\mathcal{L}(\mathcal{A})} \mathbb{Q}y_F$ with multiplication

$$y_F y_G = \begin{cases} y_{F\vee G} & \text{if } \operatorname{rk} F + \operatorname{rk} G = \operatorname{rk} F \vee G \\ 0 & \text{otherwise.} \end{cases}$$

The Dowling–Wilson conjecture then follows from the construction of injective maps $H^{2k}(Y_{\mathcal{A}};\mathbb{Q}) = \bigoplus_{F\in\mathcal{L}^k(\mathcal{A})}\mathbb{Q}y_F \to H^{2(\operatorname{rk} M-k)}(Y_{\mathcal{A}};\mathbb{Q}) = \bigoplus_{F\in\mathcal{L}^{\operatorname{rk} M-k}(\mathcal{A})}\mathbb{Q}y_F$ via the hard Lefschetz theorem. A purely combinatorial approach inspired by this strategy was later developed to resolve the conjecture for all matroids [8].

- The intersection cohomology of $Y_{\mathcal{A}}$ also shares analogies with the intersection cohomology of Schubert varieties, in that it forms the geometric basis for a theory of *matroidal Kazhdan–Lusztig polynomials* $P_M(t)$ [12]. (More precisely, $Y_{\mathcal{A}}$ gives rise to the related Z-polynomial of a matroid [26]; $P_M(t)$ itself comes from the geometry of an open subset of $Y_{\mathcal{A}}$ called the reciprocal plane.) The nonnegativity of the coefficients of matroidal Kazhdan–Lusztig polynomials (as well as Z-polynomials) for any matroid M was also confirmed in [8] by realising them as Poincaré polynomials of vector spaces related to the combinatorially defined *intersection cohomology module of a matroid* $\mathrm{IH}(M)$.

- Recently Berget and Fink introduced the notion of the *Schubert variety of a pair of linear spaces* Y_{L_1,L_2} [5]. This generalises the construction of Ardila–Boocher: the matroid Schubert variety of $\mathcal{A}$ (thought of as a linear subspace L) is recovered when $L_1 = L$ and L_2 is the 1-dimensional subspace spanned by $(1,\dots,1)$. By studying the K-polynomial of such varieties, Berget and Fink showed that Speyer's g-invariant $g_M(t)\in\mathbb{Z}[t]$ of a matroid M has nonnegative coeffients [5, Theorem E]. As a corollary they obtained a proof of Speyer's f-vector conjecture on the number of faces in a subdivision of a hypersimplex into matroid base polytopes [28].

Beyond their applications in matroid combinatorics, matroid Schubert varieties can also be thought of as additive analogues of toric varieties, in that the vector space being compactified acts equivariantly on the variety [10]. A related viewpoint underlies their appearance (when $\mathcal{A}$ is the braid arrangement) in work of Ilin–Kamnitzer–Li–Przytycki–Rybnikov [18]. The "multiplicative" story involves the Deligne–Mumford moduli space $\overline{M_{0,n}}$ of stable n-pointed genus 0 curves and its construction as an iterated blowup of the permutohedral variety (the toric variety of the braid fan). When replacing the permutohedral variety with $Y_{\mathcal{A}}$, the analogue of $\overline{M_{0,n}}$ constructed in [18] is the moduli space $\overline{F}_n$ of *cactus flower curves.*

The motivation of [18] is representation theory; the connection to the spaces described above is through the topology of their real points. Henriques and Kamnitzer showed that the S_n-equivariant fundamental group of $\overline{M_{0,n+1}}(\mathbb{R})$ (called the *cactus group*) acts on tensor products of crystals [16, Theorem 7], and Halacheva–Kamnitzer–Rybnikov–Weekes showed that this action agrees with the monodromy action of $\pi_1^{S_n}(\overline{M_{0,n+1}}(\mathbb{R}))$ on certain covers of $\overline{M_{0,n+1}}(\mathbb{R})$ [14, Theorem 1.4] (this was also independently proven by White [29, Theorem 1.1]). A similar story holds for the *virtual cactus group* $\pi_1^{S_n}(\overline{F}_n)$ [19].

One is led to consider common generalisations of the combinatorial and representation theoretic results from the appearance of the matroid Schubert variety in both. A natural first goal (which is the object of this thesis) is to understand the topology of real matroid Schubert varieties. Besides the results in [18], related results were previously obtained by He–Simpson–Xie [15], who showed that the totally nonnegative part of $Y_{\mathcal{A}}$ is homeomorphic to a ball for all real arrangements $\mathcal{A}$.

While working over the non-algebraically closed field $\mathbb{R}$ has some disadvantages from a geometric point of view, the sign patterns coming from the total order on the field induce an *orientation* on the matroid of $\mathcal{A}$. This additional combinatorial structure of an *oriented matroid* turns out to be sufficiently rich for our purposes. Beyond clarifying the structure in the realisable case, it allows us to easily extend results to the nonrealisable setting. In fact, we are able to generalise not only the combinatorial statements or the algebraic invariants (as is typical in the geometry of matroids), but also the topological spaces themselves. These results parallel the study of the Salvetti complex Sal_M of an oriented matroid M [27, 13], a combinatorially defined regular CW complex which, when M comes from a real hyperplane arrangement $\mathcal{A}$, is homotopy equivalent to the complement of the complexification $\mathcal{A}_{\mathbb{C}}$.

The outline of this thesis is as follows. After summarising the necessary combinatorial and topological background in Chapter 2, we define in Chapter 3 the matroid Schubert variety $Y_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ (in particular over the real numbers). The first main result is a polyhedral model for real matroid Schubert varieties:

Theorem (Theorem 3)**.** *The real matroid Schubert variety $Y_{\mathcal{A}}(\mathbb{R})$ is homeomorphic to the zonotope $Z_{\mathcal{A}}$ with "parallel faces" identified by translation.*

Given this result, we are then motivated to generalise this model beyond real arrangements. Using results from the topology of oriented matroids, for an arbitrary oriented matroid we define a CW complex which specialises to the polyhedral model in the realisable case (Definition 8).

We proceed to compute topological invariants of this "real matroid Schubert variety" Y_M for an arbitrary oriented matroid M. In Chapter 4, we compute the fundamental group $\pi_1(Y_M)$. When M comes from a root system, we show that this group and the W-equivariant fundamental group are Coxeter analogues of the (pure) virtual Artin groups of Bellingeri–Paris–Thiel [4].

Theorem (Corollary 4 and Theorem 7). *Let Φ be a root system with Coxeter group W. The ordinary and W-equivariant fundamental groups $\pi_1(Y_{\Phi^+})$ and $\pi_1^W(Y_{\Phi^+})$ are isomorphic to the pure virtual Coxeter group* PVW *and the virtual Coxeter group* VW *respectively.*

Finally, in Chapter 5, we use a combination of cellular and simplicial methods to compute a presentation for the integral cohomology ring $H^*(Y_M;\mathbb{Z})$. The major difficulty of determining precisely the signs in the cup product is overcome in a surprisingly elegant manner by the combinatorics of the oriented matroid.

Theorem (Theorem 8). *Let M be an oriented matroid with no loops. Then $H^*(Y_M;\mathbb{Z})$ is isomorphic to*

$$\mathrm{OB}(M) := \bigwedge [y_e \colon e \in E] \,/ \big\langle \chi^F(B) y_B - \chi^F(B') y_{B'} \colon F \in \mathcal{L}(M), B, B' \in \mathcal{B}(M^F) \big\rangle.$$

Chapter 2

Background

In this chapter we recall some notions from topology and combinatorics and fix the notation and conventions that will be used in the sequel. The material covered is standard, with the exception of the explicit description of orientations in Section 2.4.

2.1 Posets

A *partially ordered set* or *poset* comprises a set P together with a *partial order* $\leqslant$ on P (a reflexive, antisymmetric, and transitive binary relation). For $x, y \in P$, we write $x < y$ if $x \leqslant y$ and $x \neq y$. In particular, we say that y *covers* x (denoted by $x \lessdot y$) if $x < y$ and no $z \in P$ satisfies $x < z < y$. In this thesis, we will only consider partial orders on finite sets. Further, when the partial order is clear from context, we will refer to a poset by its underlying set.

A *map of posets* from P_1 to P_2 is a map $f\colon P_1 \to P_2$ of the underlying sets, such that $p \leqslant p'$ in P_1 implies $f(p) \leqslant f(p')$ in P_2 (it is *order-preserving*). Such a map is an isomorphism of posets if there is a map of posets $g\colon P_2 \to P_1$ such that gf and fg are the identity.

If Q is a subset of P, then a partial order on P restricts to a partial order on Q, and unless otherwise noted we consider Q as a poset with this partial order. In particular, for $x, y \in P$ we write $P_{\geqslant x} = \{p \in P\colon x \leqslant p\}$ (the *principal order filter* generated by x), $P_{\leqslant y} = \{p \in P\colon p \leqslant y\}$ (the *principal order ideal* generated by y) and $[x, y] = P_{\geqslant x} \cap P_{\leqslant y} = \{p \in P\colon x \leqslant p \leqslant y\}$ (the *(closed) interval* from x to y).

A *chain* Δ in a poset P is a subset of P such that the induced partial order is a total order: for every $x, y \in \Delta$, we have $x \leqslant y$ or $y \leqslant x$. A chain of *length* k is a chain of cardinality k; if its elements are labelled $x_1, \ldots, x_k$ such that $x_i < x_j$ if and only if $i < j$, then we also denote the chain by $(x_1 < \ldots < x_k)$. It is clear that a subset of a chain is a chain; hence we can consider the *order complex* $\Delta(P)$ of a poset P, which is the (abstract) simplicial complex on vertex set P with k-simplices given by chains of length k. If every maximal chain in P has the same length, then it is called *pure*. Such a poset has a well-defined *rank function* rk which assigns to $x \in P$ the length of any maximal chain in $P_{\leqslant x}$.

If a poset has a unique minimal and maximal element, it is *bounded*. The *join* of $x, y \in P$, denoted by $x \vee y$, is the minimal element of $P_{\geqslant x} \cap P_{\geqslant y}$, if it exists and is unique. Dually, their *meet* $x \wedge y$ is the maximal element of $P_{\leqslant x} \cap P_{\leqslant y}$ if it exists and is unique. A poset is called a *lattice* if every pair of elements $x, y \in P$ has a join and a meet. As we consider only finite posets, lattices are bounded with minimal element $\bigwedge_{p \in P} p$ and maximal element $\bigvee_{p \in P} p$. The *atoms* of a lattice are the elements which cover the minimal element. We say that a lattice is *atomic* if every element is the join of atoms. If the rank function of an atomic lattice is *submodular* ($\mathrm{rk}(x) + \mathrm{rk}(y) \geqslant \mathrm{rk}(x \vee y) + \mathrm{rk}(x \wedge y)$ for all $x, y \in P$), then the lattice is called *geometric*.

2.2 Regular CW complexes

We assume familiarity with CW complexes (which we assume to be finite), following for example [23]. Certain CW complexes with sufficiently nice attaching maps can be treated in a more combinatorial way:

Definition 1 ([6, Definition 4.7.4])**.** *Let X be a Hausdorff space. A* (finite) regular CW complex *on X is the data of a finite set $\mathcal{F}(X)$ of subspaces $\sigma \subseteq X$ (*cells*), each homeomorphic to a closed ball, such that $X = \cup_{\sigma \in \mathcal{F}(X)} \sigma$, the interiors σ° of the cells partition X, and the boundary $\partial\sigma$ of each cell is a union of cells.*

The set $\mathcal{F}(X)$ is partially ordered by inclusion and called the *face poset* of X.

Remark 1. Often we will conflate the complex with its underlying topological space.

The combinatorial nature of regular CW complexes is captured in the following result. Recall the notation $\|\Delta\|$ for the geometric realisation of a simplicial complex Δ.

Proposition 1 ([6, Proposition 4.7.8])**.** *If X is a regular CW complex, then there exists a homeomorphism $X \to \|\Delta(\mathcal{F}(X))\|$ which restricts to homeomorphisms $\sigma \to \|\Delta(\mathcal{F}(X)_{\leqslant \sigma})\|$ for every $\sigma \in \mathcal{F}(X)$.*

Face posets of regular CW complexes have the following nice property.

Proposition 2 ([6, Corollary 4.7.12]). *Let $\mathcal{F}(X) \cup \{\hat{0}\}$ be the poset obtained from $\mathcal{F}(X)$ by adjoining a bottom element $\hat{0}$. Then this poset is* thin*: every closed interval of length 2 has exactly 4 elements.*

Orientations on a regular CW complex are encoded by the following data:

Definition 2 ([9, Definition I.1.8]). *An* incidence function *on a regular CW complex X is a function $(\sigma, \tau) \mapsto [\sigma : \tau] \in \{-1, 0, 1\}$ on ordered pairs of cells in $\mathcal{F}(X)$ such that:*

(i) $[\sigma, \tau]$ *is nonzero if and only if* $\tau \lessdot \sigma$*;*

(ii) if σ is a 1-cell incident to 0-cells p and q, then $[\sigma : p] + [\sigma : q] = 0$;

(iii) if $\rho \lessdot \sigma_1, \sigma_2 \lessdot \tau$ are the cells of an interval of length 2 in $\mathcal{F}(X)$, then

$$[\tau : \sigma_1][\sigma_1 : \rho] + [\tau : \sigma_2][\sigma_2 : \rho] = 0.$$

Proposition 3 ([23, Theorem V.4.2]). *Incidence functions on a regular CW complex X are in bijection with choices of orientations for the cells of X.*

We will be interested in certain constructions of quotients of regular CW complexes.

Definition 3 ([9, Chapter III.1, p. 66]). *An* identification *on a regular CW complex X is a homeomorphism $\sigma \to \tau$ for some $\sigma, \tau \in \mathcal{F}(X)$ which restricts to homeomorphisms $\sigma' \to \tau' \subseteq \tau$ for all $\sigma' \leqslant \sigma$.*

Definition 4 ([9, Definition III.1.1]). *A collection Ω of identifications on a regular CW complex X is a* family of identifications *on X if the following conditions hold:*

1. *for each $\sigma \in \mathcal{F}(X)$, the identity homeomorphism $\sigma \to \sigma$ is in Ω;*
2. *if $f \in \Omega$, then $f^{-1} \in \Omega$;*
3. *if $f\colon \rho \to \sigma$ and $g\colon \sigma \to \tau$ are in Ω, then $gf\colon \rho \to \tau$ is in Ω;*
4. *if $f\colon \sigma \to \sigma$ is in Ω, then f is the identity homeomorphism;*
5. *if $f\colon \sigma \to \tau$ is in Ω and $\sigma_0 < \sigma$, then $f|_{\sigma_0}$ is in Ω.*

A family of identifications Ω on X defines an equivalence relation on the topological space X by $x \sim y$ if there exists $f \in \Omega$ such that $f(x) = y$. Let X/Ω denote the quotient space of X by this equivalence relation.

Proposition 4 ([9, Chapter III.1, p. 67])**.** *If Ω is a family of identifications on a regular CW complex X, then X/Ω is a CW complex with open cells the images of open cells in X.*

Certain orientations of cells of X descend to orientations of cells of X/Ω:

Definition 5 ([9, Definition III.2.1])**.** *An incidence function on a regular CW complex X is* invariant *under a family of identifications Ω if whenever $f\colon \rho \to \sigma$ is in Ω and $\tau < \rho$, then $[\rho\colon \tau] = [f\rho\colon f\tau]$.*

Proposition 5 ([9, Chapter III.2, p. 72])**.** *Let X be a regular CW complex with an incidence function invariant under a family of identifications Ω. Then the formula $\partial([\sigma]) = \sum_{\tau \lessdot \sigma}[\sigma\colon \tau][\tau]$ is well-defined and computes the differential in the cellular chain complex of X/Ω. Here $[\sigma]$ denotes the cell in X/Ω which is the image of $\sigma \in \mathcal{F}(X)$.*

2.3 Matroids and oriented matroids

Matroids are mathematical structures which abstract combinatorial properties of linear (in)dependence. Notably, their definition can be formulated in many different but equivalent ("cryptomorphic") ways, reflecting the various instances in which matroids arise across mathematics. A standard reference on matroids is [25]; we summarise the necessary definitions and properties below.

Recall the notation 2^E for the set of subsets of a set E, which is partially ordered by inclusion. A *matroid* M on a finite set E (called the *ground set*) is defined by a *rank function* $\mathrm{rk}\colon 2^E \to \mathbb{N}$ satisfying the following axioms:

- if $S \subseteq E$, then $\mathrm{rk}(S) \leqslant |S|$;
- if $S \subseteq S' \subseteq E$, then $\mathrm{rk}(S) \leqslant \mathrm{rk}(S')$;
- rk is submodular.

The subsets $S \subseteq E$ which satisfy $\mathrm{rk}(S) = |S|$ are called *independent*. Maximal independent sets (with respect to the inclusion partial order) are called *bases*, and minimal non-independent sets are called *circuits*. A subset of E which is not contained in a larger subset of the same rank is called a *flat*. The set of flats of M forms a geometric lattice $\mathcal{L}(M)$ under inclusion, and the rank function of this lattice is the (restriction of the) rank function of the matroid.

A *loop* in a matroid M is a element $e \in E$ such that $\mathrm{rk}(\{e\}) = 0$. Non-loops $e, e' \in E$ are *parallel* if $\mathrm{rk}(\{e, e'\}) = 1$.

Example 1. Let V be a vector space over a field k, and let $\mathcal{A} = \{\alpha_e \in V : e \in E\}$ be a collection of vectors indexed by the finite set E. Then there is a matroid $M(\mathcal{A})$ (called the *vector matroid* associated to $\mathcal{A}$) whose rank function is $S \mapsto \dim\langle \alpha_e : e \in S\rangle$. The independent sets of $M(\mathcal{A})$ are exactly the subsets of E indexing linearly independent subsets of $\mathcal{A}$. A matroid which arises as the vector matroid of some $\mathcal{A}$ is called *realisable*.

Oriented matroids encode more specifically the combinatorics of linear (in)dependence over $\mathbb{R}$. The basic definitions of the theory are built on certain structures on the set $\{+, -, 0\}$ of signs. This set has notions of multiplication and negation coming from the identifications $+ \leftrightarrow +1$, $- \leftrightarrow -1$, $0 \leftrightarrow 0$. Further, we always consider $\{+, -, 0\}$ with the partial order where $+$ and $-$ are incomparable and $+, - < 0$.

Remark 2. This choice of partial order on $\{+, -, 0\}$ is the opposite of the usual convention in oriented matroid theory. For various reasons it will prove to be more convenient for our purposes.

Let E be a finite set. A *sign vector* is an element of $\{+, -, 0\}^E$, which we often think of as a function $E \to \{+, -, 0\}$. Its *support* is the subset of E which is mapped to $+$ or $-$. Given $X, Y \in \{+, -, 0\}^E$, their *composition* $X \circ Y$ is the sign vector defined by $(X \circ Y)(e) = X(e)$ if $X(e) \neq 0$ and $Y(e)$ otherwise. Their *separation set* is $S(X, Y) = \{e \in E : X(e) = -Y(e) \neq 0\}$.

A set $\mathcal{C}(M) \subseteq \{+, -, 0\}^E$ is the set of *covectors* of an oriented matroid M if

1. $0^E \in \mathcal{C}(M)$;
2. if $X \in \mathcal{C}(M)$, then $-X \in \mathcal{C}(M)$;
3. if $X, Y \in \mathcal{C}(M)$, then $X \circ Y \in \mathcal{C}(M)$;
4. if $X, Y \in \mathcal{C}(M)$ and $e \in S(X, Y)$, then there exists $Z \in \mathcal{C}(M)$ such that $Z(e) = 0$ and $Z(f) = (X \circ Y)(f) = (Y \circ X)(f)$ for all $f \notin S(X, Y)$.

Example 2. The fundamental example is that of real hyperplane arrangements. Let V be a vector space over $\mathbb{R}$, and let $\mathcal{A} = \{\alpha_e \in V^* : e \in E\}$ be a collection of vectors *in the dual vector space* indexed by the finite set E. The vector matroid $M(\mathcal{A})$ does arise from an oriented matroid, the covectors of which are exactly the sign vectors of the form $(\mathrm{sgn}(\alpha_e(v)))_{e \in E}$ for $v \in V$. The composition $X \circ Y$ can be interpreted in this setting as starting at a generic point with sign vector X and moving ϵ in the direction of Y. As for matroids, an oriented matroid which can obtained from some real arrangement $\mathcal{A}$ is called *realisable*.

We consider $\mathcal{C}(M)$ with the partial order induced from the product partial order on $\{+,-,0\}^E$. The relation between oriented matroids and matroids is given by the following:

Proposition 6 ([6, Proposition 4.1.13]). *The* zero map $z\colon X \mapsto X^{-1}(0) \subseteq E$ *is a cover-preserving (and hence order-preserving) surjection from* $\mathcal{C}(M)$ *to the lattice of flats of a matroid.*

By abuse of notation we also use M to denote this matroid, and freely apply matroid terminology to oriented matroids.

Just as covectors are related to flats, oriented matroids can also be formulated using structures related to bases. Let $\mathcal{B}(M)$ be the set of totally ordered bases of M. A *chirotope* of M is a map $\chi\colon \mathcal{B}(M) \to \{\pm 1\}$ such that χ is *alternating*, $\chi(b_1, \dots, b_n) = (\operatorname{sgn} \pi)\chi(b_{\pi(1)}, \dots, b_{\pi(n)})$ for all $(b_1, \dots, b_n) \in \mathcal{B}(M)$ and $\pi \in S_n$, and satisfies

$$C(s)\chi(b_1, \dots, b_{n-1}, s) = C(t)\chi(b_1, \dots, b_{n-1}, t), \tag{PV*}$$

for any two ordered bases of the form $(b_1, \dots, b_{n-1}, s) \neq (b_1, \dots, b_{n-1}, t)$. Here C is a corank 1 covector such that $\{b_1, \dots, b_{n-1}\} \subseteq z(C)$. In fact, chirotopes exist and are unique up to negation [6, Proposition 3.5.2].

Remark 3. Strictly speaking the above defines a *basis orientation*, which is the restriction of the chirotope from all $\operatorname{rk}(M)$-tuples of E to the set of ordered bases.

Example 3. A chirotope of $M(\mathcal{A})$ is given by fixing coordinates on the vector spsace and defining $\chi(b_1, \dots, b_n) = \operatorname{sgn}(\det(b_1, \dots, b_n))$.

The minimal covectors of an oriented matroid M are called its *topes*. If M has a tope in $\{+,0\}^E$, then it is said to be *acyclic*. Often one transforms M into an acyclic oriented matroid by *reorienting*: for covectors, reorienting $A \subseteq E$ involves replacing each covector C with a new sign vector ${}_{-A}C$ obtained by flipping the sign of each coordinate in A.

Circuits also have signed generalisations. We will need only a specific instance. Given a basis B (unordered) of the matroid M and $e \in E \setminus B$, there is a unique sign vector $C(e, B) \in \{+,-,0\}^E$ such that $C(e, B)(e) = +$ and its support is a circuit of M. It is called the *fundamental circuit* of e with respect to B, and it is *orthogonal* to all covectors [6, p. 115]. This means that if $C \in \mathcal{C}(M)$ is a covector, then either the supports of C and $C(e,B)$ are disjoint, or there exist f, f' in both of their supports such that $C(f)C(e,B)(f) = -C(f')C(e,B)(f')$.

Finally, we will need the following procedure to construct related (oriented) matroids. For every flat $F \in \mathcal{L}$, the *localisation* of M at F is the oriented matroid M^F on ground set F with covectors $\mathcal{C}(M^F) = \{X|_F \colon X \in \mathcal{C}(M)\}$. Observe that $X \mapsto X|_F$ defines a surjective poset map $\mathcal{C}(M) \to \mathcal{C}(M^F)$. A chirotope of M^F is computed (up to negation) by choosing elements $a_1, \ldots, a_{n-r} \in E \setminus F$ which extend bases of M^F to bases of M, and setting

$$\chi^F(b_1, \ldots, b_r) = \chi(b_1, \ldots, b_r, a_1, \ldots, a_{n-r}).$$

2.4 Topology of oriented matroid posets

It is often useful to study oriented matroids through the topology of various posets associated to them (see for example [6, Section 4.3]). The relevant space for our purposes is the one guaranteed by the following:

Theorem 1 ([6, Corollary 4.3.4])**.** *There exists a regular CW complex Z_M homeomorphic to a ball of dimension* $\operatorname{rk} M$ *such that $\mathcal{F}(Z_M) \cong \mathcal{C}(M)$.*

The cell of Z_M associated to $C \in \mathcal{C}(M)$ will be denoted by σ_C. When the meaning is clear, we may simplify notation by writing C for σ_C.

Remark 4. There is a similar statement for the opposite poset of $\mathcal{C}(M)$ [6, Theorem 4.3.3]. The poset isomorphism of Theorem 1 (as opposed to anti-isomorphism) is one reason for our nonstandard partial order on signs (see Remark 2). Note also that these results are called "Sphericity Theorems" in [6, Section 4.3]; it seems that their convention is to remove top elements from their posets. Finally, Z_M is in fact a PL ball, but we will not need this fact.

When $M = M(\mathcal{A})$ for a real arrangement $\mathcal{A}$, the regular CW complex Z_M can in fact be realised as a polytope (see Figure 2.1 for an example).

Definition 6. *The* zonotope *associated to $\mathcal{A}$ is the Minkowski sum*

$$Z_{\mathcal{A}} = \sum_{e \in E} [-1,1]\alpha_e = \left\{ \sum_{e \in E} c_e \alpha_e \colon -1 \leqslant c_e \leqslant 1 \text{ for all } e \in E \right\} \subset V^*.$$

Equivalently, it is the image of the cube $[-1,1]^E$ under the projection $(c_e)_{e \in E} \mapsto \sum_{e \in E} c_e \alpha_e$.

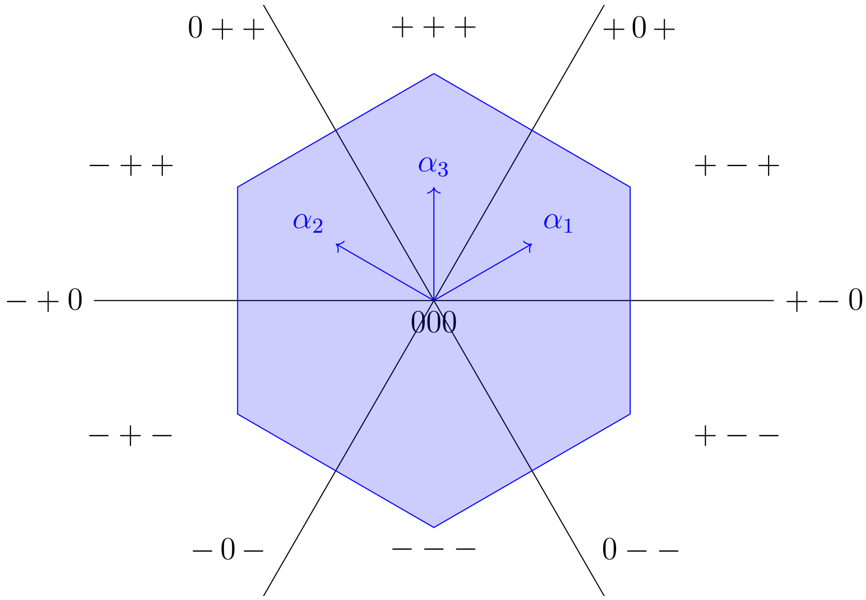

Figure 2.1: The zonotope $Z_{\mathcal{A}}$ for the rank 2 braid arrangement.

Proposition 7 ([6, Proposition 2.2.2]). *The map*

$$C \mapsto \sigma_C = \sum_{C(e)=+} \alpha_e - \sum_{C(e)=-} \alpha_e + \sum_{C(e)=0} [-1,1]\alpha_e$$

defines a poset isomorphism $\mathcal{C}(M(\mathcal{A})) \to \mathcal{F}(Z_{\mathcal{A}})$.

Corollary 1. $Z_{\mathcal{A}}$ *is cellularly homeomorphic to* $Z_{M(\mathcal{A})}$.

We now give explicit incidence functions on Z_M (adapting an idea in [11, §2.4.3]). For every $F \in \mathcal{L}(M)$, choose a chirotope χ^F of M^F. Then for $C \lessdot C'$, choose $(b_1, \ldots, b_n) \in \mathcal{B}(M^{z(C)})$ and $s \in z(C') \setminus z(C)$ and define

$$[C' : C] = \frac{C(s)\chi^{z(C')}(b_1, \ldots, b_n, s)}{\chi^{z(C)}(b_1, \ldots, b_n)}. \tag{2.1}$$

Proposition 8. *The value of the expression* (2.1) *is independent of the choices of* $(b_1, \ldots, b_n)$ *and* s*, and it defines an incidence function on* Z_M.

Proof. To show well-definedness, assume without loss of generality (by localising at the flat $z(C')$) that $C' = 0^E$. Then C is a covector of M of corank 1, so (PV*) implies that $C(s)\chi^{z(C')}(b_1, \ldots, b_n, s)$ (and therefore $[C' : C]$) is independent of s for any fixed choice of $(b_1, \ldots, b_n)$. Further, observe from the definition of localisation that for every s there exists $\epsilon(s) \in \{\pm\}$ such that $\chi^{z(C')}(b_1, \ldots, b_n, s) = \epsilon(s)\chi^{z(C)}(b_1, \ldots, b_n)$.

It follows that $[C' : C] = C(s)\epsilon(s)$ is independent of $(b_1, \ldots, b_n)$ for any fixed s, which verifies the first claim.

If $\operatorname{rk} C' = 1$ and the two covectors it covers are C_1 and C_2, then there must exist $s \in z(C')$ such that $C_1(s) = -C_2(s) \neq 0$. As $z(C_1) = z(C_2) = \emptyset$, it follows that

$$[C' : C_1] + [C' : C_2] = \frac{C_1(s)\chi^{z(C')}(s)}{\chi^{\emptyset}(\emptyset)} + \frac{C_2(s)\chi^{z(C')}(s)}{\chi^{\emptyset}(\emptyset)} = 0.$$

Finally, if $C'' \lessdot C_1, C_2 \lessdot C'$ is an interval of length 2 in $\mathcal{C}(M)$ and $(b_1, \ldots, b_n)$ a basis of $z(C'')$, choose $s_1 \in z(C_1) \setminus z(C'')$ and $s_2 \in z(C_2) \setminus z(C'')$. Then $(b_1, \ldots, b_n, s_1, s_2)$ is a basis of $z(C')$. Since $s_2 \in z(C') \setminus z(C_1)$, we have

$$\begin{aligned}
[C' : C_1][C_1 : C''] &= \frac{C'(s_2)\chi^{z(C')}(b_1, \ldots, b_n, s_1, s_2)}{\chi^{z(C_1)}(b_1, \ldots, b_n, s_1)} \frac{C_1(s_1)\chi^{z(C_1)}(b_1, \ldots, b_n, s_1)}{\chi^{z(C'')}(b_1, \ldots, b_n)} \\
&= \frac{C''(s_2)C''(s_1)\chi^{z(C')}(b_1, \ldots, b_n, s_1, s_2)}{\chi^{z(C'')}(b_1, \ldots, b_n)},
\end{aligned}$$

using that $C'(s_2) = C''(s_2)$ and $C_1(s_1) = C''(s_1)$ as $C'' < C_1, C'$. A similar computation gives

$$[C' : C_2][C_2 : C''] = \frac{C''(s_1)C''(s_2)\chi^{z(C')}(b_1, \ldots, b_n, s_2, s_1)}{\chi^{z(C'')}(b_1, \ldots, b_n)} = -[C' : C_1][C_1 : C''],$$

as $\chi^{z(C')}$ is alternating. Hence (2.1) defines an incidence function on Z_M. □

Remark 5. The choices of χ^F should be thought of as fixing orientations on the cells of Z_M such that if $z(C) = z(C')$ then σ_C and $\sigma_{C'}$ have the same orientation. More generally, one could construct an incidence function as above with independent choices of chirotopes for each non-tope covector (the topes must have the same orientation for the formula to satisfy (ii) in Definition 2). However, the incidence functions of Proposition 3 will be invariant under a family of identifications to be constructed in Proposition 12.

Chapter 3

Real matroid Schubert varieties

3.1 Matroid Schubert varieties

Let V be a finite-dimensional vector space over a field k, and let $\mathcal{A} = (\alpha_e)_{e\in E} \in (V^*)^E$ be a collection of linear forms on V indexed by a finite set E. Without loss of generality, assume that the α_e span V^*; the map $V \to k^E$ defined by $v \mapsto (\alpha_e(v))_{e\in E}$ is then an embedding of V as a linear subspace of k^E.

Remark 6. The reason for working with elements of the dual space is so that (the simplification of) $M(\mathcal{A})$ coincides with the matroid of the hyperplane arrangement $\{\ker \alpha_e \subseteq V : e \in E\}$. In particular, the condition that the α_e span V^* is equivalent to the corresponding hyperplane arrangement being *essential* (the intersection of all hyperplanes is $\{0\} \subseteq V$).

Definition 7 ([1, Definition 1.2])**.** *The* matroid Schubert variety $Y_\mathcal{A}$ *of* $\mathcal{A}$ *is the closure of* V *in* $(\mathbb{P}^1)^E$ *(in the Zariski topology) under the embedding* $v \mapsto ([\alpha_e(v) : 1])_{e\in E}$.

Remark 7. A more accurate name for $Y_\mathcal{A}$ would be the *Schubert variety of the arrangement* $\mathcal{A}$, since the definition really depends on $\mathcal{A}$ and not just the underlying matroid $M(\mathcal{A})$. It should also be noted that $Y_\mathcal{A}$ is *not* a Schubert variety in a Grassmannian or partial flag variety.

Equations for $Y_\mathcal{A}$ can be obtained from the combinatorics of $\mathcal{A}$. For every circuit $C \subset E$ of (the vector matroid of) $\mathcal{A}$, there exist $(a_e)_{e\in C} \in (k^\times)^C$ (unique up to scaling) such that $\sum_{e\in C} a_e\alpha_e = 0$.

Theorem 2 ([1, Theorem 1.3(a)])**.** *The multihomogeneous defining ideal of* $Y_\mathcal{A}$ *is*

$$\left\langle \sum_{e\in C} a_e x_e \prod_{e'\in C\setminus e} y_{e'} : C \textit{ a circuit of } \mathcal{A} \right\rangle \subseteq k[x_e, y_e]_{e\in E}. \tag{3.1}$$

Remark 8. In [1], the matroid associated to $\mathcal{A}$ is the dual of the vector matroid. Hence [1, Theorem 1.3(a)] is stated there in terms of cocircuits, not circuits.

A key combinatorial consequence of (3.1) is the construction of an affine stratification for $Y_{\mathcal{A}}$.

Proposition 9 ([26, Lemmas 7.5 and 7.6])**.** *The matroid Schubert variety $Y_{\mathcal{A}}$ has a stratification $Y_{\mathcal{A}} = \bigsqcup_{F\in\mathcal{L}(\mathcal{A})} Y_{\mathcal{A}}^F$, where*

$$Y_{\mathcal{A}}^F = \{(p_e)_{e\in E} \in Y_{\mathcal{A}} \colon p_e = \infty \text{ if and only if } e \notin F\} \cong V/\left(\cap_{e\in F} \ker \alpha_e\right) \cong k^{\operatorname{rk} F}.$$

Further, $\overline{Y_{\mathcal{A}}^G} = \bigsqcup_{F\leqslant G} Y_{\mathcal{A}}^F \cong Y_{\mathcal{A}^G}$ for every $G \in \mathcal{L}(\mathcal{A})$.

3.2 Combinatorial model for the real locus

Henceforth we restrict our attention to real arrangements $\mathcal{A}$ and the corresponding real locus $Y_{\mathcal{A}}(\mathbb{R}) \subseteq (\mathbb{P}^1(\mathbb{R}))^E$. Our first main result is a combinatorial model for $Y_{\mathcal{A}}(\mathbb{R})$ as a topological space.

Observe from Proposition 7 that if C and C' are covectors of $M(\mathcal{A})$ with $z(C) = z(C')$, then the corresponding faces of the zonotope $Z_{\mathcal{A}}$ are translates of each other. It is easy to verify the following (see Proposition 12 for the proof of a more general statement):

Proposition 10. *Given $C, C' \in \mathcal{C}(M(\mathcal{A}))$ with $z(C) = z(C')$, the map $\sigma_C \to \sigma_{C'}$ defined by*

$$x \mapsto x + \left(\sum_{C'(e)=+} \alpha_e - \sum_{C'(e)=-} \alpha_e\right) - \left(\sum_{C(e)=+} \alpha_e - \sum_{C(e)=-} \alpha_e\right)$$

is an identification. Further, the set of such maps for all pairs of covectors (C, C') with $z(C) = z(C')$ form a family of identifications on $Z_{\mathcal{A}}$.

Let $Z_{\mathcal{A}}/{\sim}$ denote the corresponding quotient CW complex. The cells of $Z_{\mathcal{A}}/{\sim}$ are indexed by flats $F \in \mathcal{L}(M)$. Let σ_F be the cell which is the image of $\sigma_C \in Z_M$ for any $C \in \mathcal{C}(M)$ with $z(C) = F$.

Remark 9. This construction was previously observed by Bartholdi–Enriquez–Etingof–Rains [3, §8.2] when $\mathcal{A}$ is the set of (positive) roots of a type A root system. More recently, Ilin–Kamnitzer–Li–Przytycki–Rybnikov [18, Appendix A] considered $Z_{\mathcal{A}}/{\sim}$ for crystallographic root systems.

Theorem 3. *The real matroid Schubert variety $Y_{\mathcal{A}}(\mathbb{R})$ (with the analytic topology) is homeomorphic to $Z_{\mathcal{A}}/\sim$.*

Proof. We construct explicit homeomorphisms $Y_{\mathcal{A}}(\mathbb{R}) \to Z_{\mathcal{A}}/\sim$. For every $e \in E$, fix an increasing homeomorphism $f_e \colon \mathbb{R} \to (-1,1)$ (so in particular $\lim_{x\to\infty} f_e(x) = 1$ and $\lim_{x\to-\infty} f_e(x) = -1$). We claim that the map $\phi\colon V \to \mathbb{R}^E \to (-1,1)^E \to Z^\circ_{\mathcal{A}}$ defined by $v \mapsto (\alpha_e(v))_{e\in E} \mapsto (f_e(\alpha_e(v)))_{e\in E} \mapsto \sum_{e\in E} f_e(\alpha_e(v))\alpha_e$ extends to a well-defined continuous map $Y_{\mathcal{A}}(\mathbb{R}) \to Z_{\mathcal{A}}/\sim$. In fact, if $y = (y_e)_{e\in E} \in Y^F_{\mathcal{A}} \subseteq (\mathbb{P}^1(\mathbb{R}))^E$, there are several possible values for $(f_e(y_e))_{e\in E} \in [-1,1]^E$ allowed by continuity, but they correspond to different covectors with the same zero set F and hence $\sum_{e\in E} f_e(y_e)\alpha_e$ is well-defined in the quotient $Z_{\mathcal{A}}/\sim$.

Since $Y_{\mathcal{A}}(\mathbb{R})$ is compact and $Z_{\mathcal{A}}/\sim$ is Hausdorff, the continuous map ϕ is a homeomorphism if it is a bijection. Observe that by construction ϕ sends the stratum $Y^F_{\mathcal{A}}$ to the cell σ°_F. Hence it is enough to verify bijectivity for each $F \in \mathcal{L}(M)$ separately. Further, it is enough to check the top-dimensional stratum $Y^E_{\mathcal{A}} \cong V$, since Proposition 9 also implies that bijectivity on $Y^F_{\mathcal{A}}$ is the same as bijectivity on the top-dimensional stratum of $Y_{\mathcal{A}^F}$.

For injectivity, let $v, w \in V$ and consider $(d_e) = (f_e(\alpha_e(v)) - f_e(\alpha_e(w))) \in (-2,2)^E$. As the f_e are increasing, the sign of d_e is the same as the sign of $\alpha_e(v-w)$. So $\sum_{e\in E} d_e\alpha_e(v-w)$ is non-negative, and it is zero if and only if $d_e = \alpha_e(v-w) = 0$ for every $e \in E$. But if v and w map to the same point in σ°_E, then $\sum_{e\in E} d_e\alpha_e = 0$. It follows that $\alpha_e(v-w) = 0$ for every $e \in E$, and thus $v = w$ as the α_e span V^*.

To show surjectivity, consider the quotients of $Y_{\mathcal{A}}(\mathbb{R})$ and $Z_{\mathcal{A}}/\sim$ identifying all strata/cells of positive codimension to a point ∞. Both quotients are homeomorphic to spheres, with induced cell decompositions $V \sqcup \{\infty\}$ and $\sigma^\circ_E \sqcup \{\infty\}$ respectively. Since ϕ sends strata to (open) cells, it descends to a continuous cellular map $\overline{\phi}$ between the quotients. If ϕ were not surjective, then the image of $\overline{\phi}$ would be contained in the sphere minus one point and hence be homeomorphic to (a subset of) V. By the Borsuk–Ulam theorem $\overline{\phi}$ would not be injective. In particular, cellularity of $\overline{\phi}$ implies that $\overline{\phi}|_V = \phi|_V$ would not be injective, contradicting what was shown above. □

Remark 10. In the case when $\mathcal{A}$ is a crystallographic root system, an independent proof of Theorem 3 was obtained in [18, Appendix A] using somewhat involved root system computations.

Example 4. If $\dim V = \dim V^* = 2$, then $Z_{\mathcal{A}}$ is a $2n$-gon (where $n \geqslant 2$ is the number of rank 1 flats). Identifying parallel edges of $Z_{\mathcal{A}}$ gives a connected compact orientable surface without boundary. The resulting cell structure on the surface has one 0-cell if n is even and two 0-cells if n is odd. By an Euler characteristic computation and the classification of surfaces, it follows that $Y_{\mathcal{A}}(\mathbb{R})$ is homeomorphic to Σ_g (if $n = 2g$ is even) or Σ_g with two (distinct) points identified (if $n = 2g + 1$ is odd). For example, the matroid Schubert variety corresponding to the rank 2 braid arrangement of Figure 2.1 is homeomorphic to the torus with two points identified.

3.3 Extension to oriented matroids

An immediate consequence of Theorem 3 and Corollary 1 is the following:

Corollary 2. *The homeomorphism type of $Y_{\mathcal{A}}(\mathbb{R})$ depends only on the oriented matroid $M(\mathcal{A})$.*

It is then natural to try and generalise $Y_{\mathcal{A}}(\mathbb{R})$ to an arbitrary oriented matroid M. While the variety does not exist in this setting, it is possible to extend the definition of the CW complex $Z_{\mathcal{A}}/\sim$. The key ideas are contained in the following elementary observations about $\mathcal{C}(M)$, which follow immediately from the definitions:

Lemma 1. *If $C, D \in \mathcal{C}(M)$, then $D \leqslant C$ if and only if $C \circ D = D$.*

Lemma 2. *If $C, C' \in \mathcal{C}(M)$ satisfy $z(C) = z(C')$, then $C \circ C' = C$ and $C' \circ C = C'$.*

Lemma 3. *If $C, C' \in \mathcal{C}(M)$ satisfy $z(C) = z(C')$, then $i_{C,C'} \colon D \mapsto C' \circ D$ is a poset isomorphism $\mathcal{C}(M)_{\leqslant C} \to \mathcal{C}(M)_{\leqslant C'}$ with inverse $i_{C',C}$.*

Proof. One can again show directly from the definitions that $i_{C,C'}$ is order-preserving. Lemma 1 and Lemma 2 combine with associativity of composition to show that $i_{C',C} i_{C,C'}$ is the identity. □

Lemma 4. *If $C, C', C'' \in \mathcal{C}(M)$ have the same zero set, then $i_{C',C''} i_{C,C'} = i_{C,C''}$.*

Proof. This is an immediate corollary of Lemma 2. □

Lemma 5. *If $C, C', D \in \mathcal{C}(M)$ satisfy $z(C) = z(C')$ and $D \leqslant C$, then the restriction of $i_{C,C'}$ to $\mathcal{C}(M)_{\leqslant D}$ equals $i_{D,C' \circ D}$.*

Proof. The fact that $z(D) = z(C' \circ D)$ follows from the fact that $z(D) \subseteq z(C) = z(C')$ and the definition of composition. If $E \leqslant D$, then $i_{C,C'}(E) = C' \circ E = C' \circ (D \circ E) = (C' \circ D) \circ E = i_{D,C' \circ D}(E)$ by Lemma 1. □

The poset isomorphism $i_{C,C'}$ defined in Lemma 3 induces a canonical simplicial homeomorphism $\|\Delta(\mathcal{C}(M)_{\leqslant C})\| \to \|\Delta(\mathcal{C}(M)_{\leqslant C'})\|$, where the map is defined on vertices by the poset isomorphism and is otherwise defined on simplices by (affine) linearity. By abuse of notation we also denote this homeomorphism by $i_{C,C'}$.

Proposition 11. *The collection*

$$\{i_{C,C'}|_\sigma \colon C, C' \in \mathcal{C}(M), z(C) = z(C'), C \in \sigma \in \Delta(\mathcal{C}(M)_{\leqslant C})\}$$

is a family of identifications on $\|\Delta(\mathcal{C}(M))\|$.

Proof. Definition 41–3 follow from Lemma 3 and Lemma 4, and uniqueness of the identity identification (Definition 44) follows from the restriction of the collection of identifications with domain σ to $i_{C,C'}|_\sigma$ satisfying $C \in \sigma \in \Delta(\mathcal{C}(M)_{\leqslant C})$. Finally, Definition 45 follows from Lemma 5. □

It is possible to upgrade this to a family of identifications on Z_M, generalising Proposition 10.

Proposition 12. *Let* $h\colon Z_M \to \|\Delta(\mathcal{C}(M))\|$ *be a homeomorphism satisfying the properties of Proposition 1. Then the collection*

$$\left\{h^{-1} i_{C,C'} h|_{\sigma_C} \colon C, C' \in \mathcal{C}(M), z(C) = z(C')\right\}$$

is a family of identifications on Z_M.

Proof. Definition 41–3 follow from Lemma 3 and Lemma 4 as above, and Definition 44 is now immediate. Definition 45 follows from Lemma 5 together with the properties of h under restriction. □

The spaces obtained from these families of identifications are homeomorphic:

Proposition 13. *Let* Y_M^Δ *be the CW (in fact semisimplicial) complex obtained as the quotient of* $\|\Delta(\mathcal{C}(M))\|$ *by the family of identifications in Proposition 11, and let* Y_M^{CW} *be a CW complex obtained as the quotient of* Z_M *by the family of identifications in Proposition 12. Then the topological spaces underlying* Y_M^Δ *and* Y_M^{CW} *are homeomorphic. In particular,* Y_M^{CW} *is (up to cellular homeomorphism) independent of* h.

Proof. By the universal property of quotient spaces, the homeomorphism $h\colon Z_M \to \|\Delta(\mathcal{C}(M))\|$ induces a homeomorphism between the relevant topological spaces. The uniqueness of Y_M^{CW} up to cellular homeomorphism follows from Proposition 1 on the properties of h under restriction to cells. $\square$

Hence we can make the following definition:

Definition 8. *For an oriented matroid M, let Y_M be the topological space constructed in Proposition 13.*

The conclusion of Theorem 3 can then be restated as follows:

Corollary 3. *If $\mathcal{A}$ is a real arrangement, then $Y_{\mathcal{A}}(\mathbb{R}) \cong Y_{M(\mathcal{A})}$.*

Chapter 4

Virtual Coxeter groups

4.1 Fundamental groups

We can immediately compute $\pi_1(Y_M)$ from the 2-skeleton of Y_M^{CW}. The k-cells of Y_M^{CW} are in bijection with the rank k flats of M. In particular, there is a unique 0-cell which we take to be the basepoint. Then there is a presentation of $\pi_1(Y_M)$ with generators indexed by rank 1 flats and relations indexed by rank 2 flats.

To compute the relations more explicitly, it is helpful to work instead with an acyclic reorientation of M (this does not affect the isomorphism type of $\mathcal{C}(M)$, so does not change the CW structure on Y_M^{CW}). Further, localisations of acyclically oriented matroids remain acyclically oriented. Every oriented matroid of rank 2 is realisable over $\mathbb{R}$ [6, Corollary 8.3.3(i)]. Hence if $F \in \mathcal{L}^2(M)$, then Z_{M^F} is a $2n$-gon (where $n = |\mathcal{L}^1(M^F)|$). One vertex of this $2n$-gon has a covector without any $-$ coordinates (this follows from the choice of acyclic orientation). A length n sequence of edges to this vertex from its opposite vertex defines a total order $F_1 < \ldots < F_n$ on the rank 1 flats contained in F. There are two such sequences, giving opposite orders. The relation corresponding to F then says that the two paths $x_{F_1} \cdots x_{F_n}$ and $x_{F_n} \cdots x_{F_1}$ determined by these sequences are equal. This proves the following:

Theorem 4. *The fundamental group $\pi_1(Y_M)$ has a presentation with generators $\{x_F \colon F \in \mathcal{L}^1(M)\}$ and relations $x_{F_1} \cdots x_{F_n} x_{F_1}^{-1} \cdots x_{F_n}^{-1}$ for every rank 2 flat F, where $F_1, \ldots, F_n$ are the rank 1 flats contained in F ordered as above.*

Example 5. For the running example (pictured in Figure 4.1), the fundamental group of the matroid Schubert variety has a presentation $\langle x_1, x_2, x_{12} \mid x_1 x_{12} x_2 = x_2 x_{12} x_1 \rangle$.

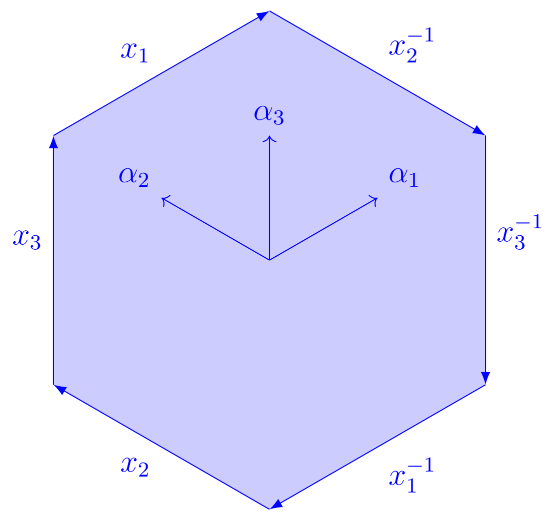

Figure 4.1: An example of the relations in $\pi_1(Y_M)$.

4.2 Coxeter arrangements

We now specialise to real matroid Schubert varieties coming from Coxeter arrangements, which have the additional symmetry of a Coxeter group action. Let Φ be a root system (not necessarily crystallographic) with simple roots Π and positive roots Φ^+. Further, let $(m_{\alpha,\beta})_{\alpha,\beta\in\Pi}$ be the Coxeter matrix associated to Φ, and let $\Sigma = \{\sigma_\alpha \colon \alpha \in \Pi\}$ and $S = \{s_\alpha \colon \alpha \in \Pi\}$ be abstract sets indexed by Π.

The Artin group $A = A(\Phi)$ has a presentation with generators Σ and relations $\operatorname{Prod}(\sigma_\alpha, \sigma_\beta, m_{\alpha,\beta}) = \operatorname{Prod}(\sigma_\beta, \sigma_\alpha, m_{\alpha,\beta})$ for all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$. Here $\operatorname{Prod}(b, a, m)$ is the word $\dots aba$ of length m. Similarly, the Coxeter group $W = W(\Phi)$ has a presentation with generators S and relations $s_\alpha^2 = 1$ for all $\alpha \in \Pi$ and $\operatorname{Prod}(s_\alpha, s_\beta, m_{\alpha,\beta}) = \operatorname{Prod}(s_\beta, s_\alpha, m_{\alpha,\beta})$ for all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$.

Fix a bilinear form on the vector space V^* with basis Π, defined by $\langle \alpha, \alpha' \rangle = -2\cos(\pi/m_{\alpha,\alpha'})$. As this is positive definite, we may identify V with V^*. We consider W as acting on $V \cong V^*$ by $s_\alpha(v) = v - \langle v, \alpha \rangle \alpha$.

Bellingeri–Paris–Thiel [4] have recently defined the *virtual Artin group* VA as the free product of W and A modulo some "mixed relations" coming from the action of W on Φ. Their definition unifies the Coxeter-theoretic and knot-theoretic generalisations of the classical braid group to Artin groups and virtual braid groups respectively.

Definition 9 ([4])**.** *The* virtual Artin group VA *is the free product of W and A modulo relations* $\operatorname{Prod}(s_\alpha, s_\beta, m_{\alpha,\beta} - 1)\sigma_\alpha = \sigma_\gamma \operatorname{Prod}(s_\alpha, s_\beta, m_{\alpha,\beta} - 1)$ *for all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$. In these relations, the simple root γ is defined as α if $m_{\alpha,\beta}$ is even and*

β if $m_{\alpha,\beta}$ is odd.

Remark 11. The definition in [4] applies more generally to Coxeter graphs with countably many vertices. Since we restrict our attention to finite type, we make some simplifications in our presentation (in particular, we do not need to exclude certain cases where $m_{\alpha,\beta} = \infty$).

Example 6. When Φ is of type A_{n-1}, the corresponding virtual Artin group is the *virtual braid group* $\mathrm{V}B_n$. The above presentation has generators σ_i, s_i for $1 \leqslant i \leqslant n-1$, with relations as follows:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ (the σ_i generate a copy of the braid group B_n);
- $s_is_j = s_js_i$ for $|i-j| > 1$ and $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ (the s_i generate a copy of the symmetric group S_n);
- $s_i\sigma_j = \sigma_js_i$ for $|i-j| > 1$ and $s_is_{i+1}\sigma_i = \sigma_{i+1}s_is_{i+1}$ (the *mixed relations*).

Note that the mixed relations are asymmetric in the σ_i and the s_i.

We are interested in a quotient of VA that can be considered as a virtual analogue of the corresponding Coxeter group.

Definition 10. *The* virtual Coxeter group VW *is the quotient of* VA *by the relations $\sigma_\alpha^2 = 1$ for all $\alpha \in \Pi$.*

Remark 12. Specialising to type A gives the *virtual symmetric group* $\mathrm{V}S_n$ of [18], which is better known in the literature as the *flat virtual braid group* [21].

There is a surjective group homomorphism VW $\to W$ defined on generators by $\sigma_\alpha, s_\alpha \mapsto s_\alpha$ for all $\alpha \in \Pi$, and we call its kernel the *pure virtual Coxeter group* PVW. The map $\pi_P\colon \mathrm{VA} \to W$ is the composition of this map with the quotient VA $\to$ VW, and its kernel is the *pure virtual Artin group* PVA. We describe a presentation of PVA obtained by Bellingeri–Paris–Thiel [4, Section 2].

Lemma 6 ([4, Lemma 2.2])**.** *For $w \in W$ and $\alpha \in \Pi$, the group element $ws_\alpha\sigma_\alpha w^{-1} \in \mathrm{PVA} \leqslant \mathrm{VA}$ depends only on $\beta = w(\alpha) \in \Phi$.*

An immediate corollary of the above lemma is that $\zeta_\beta = ws_\alpha\sigma_\alpha w^{-1} \in \mathrm{PVA}$ is well-defined. These will be shown to generate PVA. To construct the relations, for

$\beta \neq \gamma \in \Phi$ such that $\beta = w(\alpha)$ and $\gamma = w(\alpha')$ for some $w \in W$ and $\alpha, \alpha' \in \Pi$, let $\beta_1 = \beta$ and define $\beta_i \in \Phi$ for $2 \leqslant i \leqslant m_{\alpha,\alpha'}$ by

$$\beta_i = \begin{cases} \begin{aligned} &\mathrm{Prod}(ws_{\alpha'}w^{-1}, ws_\alpha w^{-1}, i-1)(\gamma) \\ &\qquad = w\,\underbrace{s_\alpha s_{\alpha'} \dots s_{\alpha'} s_\alpha}_{i-1}(\alpha') \end{aligned} & \text{if } i \text{ is even,} \\ \begin{aligned} &\mathrm{Prod}(ws_\alpha w^{-1}, ws_{\alpha'}w^{-1}, i-1)(\beta) \\ &\qquad = w\,\underbrace{s_\alpha s_{\alpha'} \dots s_\alpha s_{\alpha'}}_{i-1}(\alpha) \end{aligned} & \text{if } i \text{ is odd.} \end{cases}$$

Then define $Z(\gamma, \beta) = \zeta_{\beta_{m_{\alpha,\alpha'}}} \cdots \zeta_{\beta_1}$. By [4, Lemma 2.5], $Z(\beta, \gamma) = \zeta_{\beta_1} \dots \zeta_{\beta_{m_{\alpha,\alpha'}}}$.

Proposition 14. *Let $\Phi_{\beta,\gamma}$ be the rank 2 dihedral root subsystem with simple roots β, γ. Then the β_i are in bijection with $\Phi^+_{\beta,\gamma}$ and the total order on the indices coincides with the rotational order from β to γ on $\Phi^+_{\beta,\gamma}$.*

Proof. We compute the angle between β_i and β_{i+1}. Assume that i is even (the other case is similar). The elements of W act by isometries, so $\langle \beta_i, \beta_{i+1} \rangle = \langle \alpha', s_{\alpha'}(\alpha) \rangle = -\langle \alpha', \alpha \rangle$. Since $\langle \beta_i, \beta_{i+2} \rangle = \langle \alpha', s_{\alpha'} s_\alpha(\alpha') \rangle$ and $s_{\alpha'} s_\alpha$ is rotation by $\pi / m_{\alpha,\alpha'}$, the conclusion follows. $\square$

Theorem 5 ([4, Theorem 2.6])**.** *The pure virtual Artin group* PVA *is generated by $\{\zeta_\beta \in \mathrm{PVA} \colon \beta \in \Phi\}$, subject to relations $Z(\gamma, \beta) = Z(\beta, \gamma)$ for $\beta \neq \gamma \in \Phi$ such that $\beta = w(\alpha)$, $\gamma = w(\alpha')$ for some $w \in W$ and $\alpha, \alpha' \in \Pi$. Further, this defines a presentation of* PVA*.*

We will use this presentation of PVA to obtain a presentation of PVW.

Theorem 6. *The pure virtual Coxeter group* PVW *is the quotient of the pure virtual Artin group* PVA *by the relations $\zeta_\beta \zeta_{-\beta} = 1$.*

Proof. We briefly recall the Reidemeister–Schreier method for computing presentations of subgroups [24, Section 2.3]. Let G be a group presented as the quotient of a free group F (with basis X indexed by generators of the presentation) by N (the normal closure of the set of relations R). Given a subgroup $H \leqslant G$, the presentation of which is required, let $\tilde{H} \leqslant F$ be the preimage of H under the quotient map $F \to G = F/N$, and let $T \subset F$ be a *Schreier transversal* of $\tilde{H}$ (a complete set of (right) coset representatives of $\tilde{H}$ which is closed under taking initial words). For $w \in F$, let $\overline{w} \in T$ be such that $\tilde{H}w = \tilde{H}\overline{w}$. Then there is a presentation of H with generators indexed by the nontrivial words of the form $tx(\overline{tx})^{-1}$ with $t \in T$,

$x \in X$ [24, Theorem 2.7]. The relations in this presentation are obtained by "rewriting" the relations in R [24, Theorem 2.8].

We partially apply this method to PVA $\leqslant$ VA and PVW $\leqslant$ VW simultaneously. Both VA and VW have as set of generators $\{s_\alpha, \sigma_\alpha \colon \alpha \in \Pi\}$, and the relations of VW are exactly those of VA with the addition of $\sigma_\alpha^2 = 1$ for all $\alpha \in \Pi$. To construct Schreier transversals for these subgroups, first consider the subgroup $\{e\} \leqslant W$, where W is presented as above with generators s_α, $\alpha \in \Pi$. Schreier transversals always exist [24, Lemma 2.2]; it is clear that this Schreier transversal T (thought of as a set of words in the s_α) will also be a Schreier transversal for PVA and PVW. Then PVA and PVW are generated by elements of the form $w\sigma_\alpha(\overline{w\sigma_\alpha})^{-1} = w\sigma_\alpha s_\alpha w^{-1}$, where w is a coset representative (words of the form $ws_\alpha(\overline{ws_\alpha})^{-1}$ are trivial in PVA and PVW). Further, PVW is the quotient of PVA by the rewritings of the relations $\sigma_\alpha^2 = 1$. These rewritings are of the form $(w\sigma_\alpha s_\alpha w^{-1})((ws_\alpha)\sigma_\alpha s_\alpha (ws_\alpha)^{-1})$ for $w \in T$.

Observe that for PVA the above set of generators contains the generators appearing in Theorem 5, as $\zeta_\beta = ws_\alpha\sigma_\alpha w^{-1} = (ws_\alpha)\sigma_\alpha s_\alpha (ws_\alpha)^{-1}$. If $\beta = w(\alpha)$, then $-\beta = ws_\alpha(\alpha)$, so $\zeta_{-\beta} = (ws_\alpha)s_\alpha\sigma_\alpha(ws_\alpha)^{-1} = w\sigma_\alpha s_\alpha w^{-1}$, and hence PVW has a presentation which is obtained from that in Theorem 5 by adding the relations $\zeta_{-\beta}\zeta_\beta = 1$. □

Remark 13. Bardakov applied the Reidemeister–Schreier method to obtain the presentation of the pure virtual braid group in Theorem 5 [2, Theorem 1]. In particular, he constructed an explicit Schreier transversal.

Corollary 4. *The fundamental group $\pi_1(Y_{M(\Phi^+)})$ is isomorphic to* PVW.

Proof. By applying the relations $\zeta_\beta\zeta_{-\beta} = 1$ we can reduce the generating set of PVW to $\{\zeta_\beta \colon \beta \in \Phi^+\}$, which bijects naturally onto the set of generators of $\pi_1(Y_{M(\Phi^+)})$.

We show that the relations in Theorem 5 give the relations in Theorem 4. A root subsystem $\Phi' \subset \Phi$ is *parabolic* if $\Phi' \cap \Phi^+$ corresponds to a flat of the Coxeter arrangement. The relations in Theorem 5 are in bijection with choices of simple roots for rank 2 parabolic root subsystems of Φ. Applying the relation $\zeta_{-\beta}\zeta_\beta = 1$, it follows that the relations corresponding to the same parabolic subsystem are equivalent. By Proposition 14, they are exactly the relations in Theorem 4. Hence the presentations of PVW and $\pi_1(Y_{\Phi^+})$ define the same group. □

Remark 14. When Φ is the root system of type A_n, the fundamental group $\pi_1(Y_{M(\Phi^+)})$ was computed in [3, Theorem 8.1] and called the *triangular group* $\mathbf{Tr}_{n+1}$.

We can also consider an equivariant version of the fundamental group:

Definition 11 ([18, Definition 11.1])**.** *Let G be a finite group acting on a path-connected, locally simply-connected space X, and let $x \in X$ be a basepoint. Then the* G-equivariant fundamental group $\pi_1^G(X, x)$ *is*

$$\pi_1^G(X, x) = \{(g, p) \colon g \in G, p \text{ a homotopy class of paths } x \to gx\}.$$

Multiplication in $\pi_1^G(X, x)$ is defined by $(g, p)(g', p') = (gg', p \cdot g(p'))$.

We apply this definition to the action of W on $Y_{M(\Phi^+)}$ obtained by extending the action of W on V by reflections (and their compositions). (Recall that Φ^+ is realisable, and hence $Y_{M(\Phi^+)} = Y_{\Phi^+}(\mathbb{R})$ is the closure of some real vector space V.) As the unique 0-cell is fixed by the action of W, taking it as the basepoint gives a semidirect product decomposition $\pi_1^W(Y_{\Phi^+}) \cong W \ltimes \pi_1(Y_{\Phi^+})$. The homomorphism $W \to \mathrm{Aut}(\pi_1(Y_{\Phi^+}))$ defining the semidirect product is exactly the W-action indicated above. Explicitly, an element $w \in W$ acts on generators of $\pi_1(Y_{\Phi^+})$ by $\zeta_\beta \mapsto \zeta_{w(\beta)}$.

Theorem 7. *The W-equivariant fundamental group $\pi_1^W(Y_{\Phi^+})$ is isomorphic to the virtual Coxeter group* VW*.*

Proof. The virtual Coxeter group has a semidirect product decomposition $W \ltimes \mathrm{PVW}$ coming from the section $i_W \colon W \to \mathrm{VW}$ defined by $w \mapsto w$ (the analogous statement for VA is [4, Proposition 2.1]). The corresponding action of W on PVW is defined on the generators ζ_β by $w(\zeta_\beta) = w\zeta_\beta w^{-1}$. But this is equal to $\zeta_{w(\beta)}$, so the actions of W on $\pi_1(Y_{\Phi^+})$ and PVW coincide under the isomorphism $\pi_1(Y_{\Phi^+}) \cong \mathrm{PVW}$ of Theorem 6. Hence the semidirect products $\pi_1^W(Y_{\Phi^+})$ and VW must be isomorphic. □

Remark 15. Theorem 7 generalises [18, Lemma 11.6], which proved the result in type A.

Chapter 5

Cohomology

We proceed in several steps to compute the integral cohomology ring $H^*(Y_M; \mathbb{Z})$.

5.1 Cohomology groups

Lemma 7. *The incidence function* (2.1) *on Z_M from Proposition 8 is invariant under the family of identifications in Proposition 12.*

Proof. It is required to show that $[C' : C] = [D \circ C' : D \circ C]$ for $C, C', D \in \mathcal{C}(M)$ such that $C \lessdot C'$ and $z(C') = z(D)$. Since $z(D \circ C') = z(C')$ and $z(D \circ C) = z(C)$, one can compute (2.1) for both $[C' : C]$ and $[D \circ C' : D \circ C]$ using the same choices of $b_1 \ldots, b_n$ and s. The conclusion follows from the observation that for $s \in z(C') \setminus z(C) = z(D) \setminus z(C)$, we have $(D \circ C)(s) = C(s)$. $\square$

Corollary 5. *The differential in the cellular chain complex of Y_M^{CW} is zero.*

Proof. Given a cell $\sigma_C \in Z_M$ mapping to a cell $\sigma_{z(C)} \in Y_M^{\mathrm{CW}}$, the computation in cellular homology (Proposition 5) gives

$$\partial(\sigma_{z(C)}) = \sum_{D \lessdot C} [C : D] \sigma_{z(D)} = 0,$$

as the set of D covered by C can be partitioned into pairs $\{D = C \circ D, C \circ (-D)\}$ which cancel in the sum. $\square$

Corollary 6. *$H_i(Y_M; \mathbb{Z}) \cong \mathbb{Z}^{\mathcal{L}^i(M)}$ and $H^i(Y_M; \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(H_i(Y_M; \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}^{\mathcal{L}^i(M)}$.*

Proof. Corollary 5 implies that $H_i(Y_M; \mathbb{Z})$ is equal to the ith cellular chain group, and the cells of dimension i are indexed by the flats of rank i. Dualising the cellular chain complex gives the result for the cohomology groups. $\square$

Remark 16. When M comes from the type A root system, Corollary 5 and Corollary 6 were observed in [3, Proposition 8.3].

5.2 Cup product via simplicial cohomology

We now consider the computation of the cup product in cohomology. To this end, we turn from cellular to simplicial methods (cf. [7, §7]). As a first step we construct explicit simplicial cycles which represent a basis for the homology.

Fix a choice of chirotope x^F for every $F \in \mathcal{L}(M)$, and hence an incidence function on Z_M by (2.1). We need some explicit signs:

Proposition 15. *Given $C \in \mathcal{C}(M)$ and a maximal simplex $\Delta = (C_0 \lessdot \ldots \lessdot C_i = C) \in \Delta(\mathcal{C}(M)_{\leqslant C})$, choose $(b_1, \ldots, b_i) \in \mathcal{B}(M^{z(C)})$ such that $(b_1, \ldots, b_j) \in \mathcal{B}(M^{z(C_j)})$ for all $1 \leqslant j \leqslant i$. Then $\epsilon_\Delta = C_0(b_1) \ldots C_0(b_i) \chi^{z(C)}(b_1, \ldots, b_n) \in \{\pm 1\}$ does not depend on the choice of $(b_1, \ldots, b_i)$.*

Proof. This is exactly $[C_i \colon C_{i-1}] \ldots [C_1 \colon C_0]$. □

To minimise the notational burden, we will write simplices of Y_M^Δ as chains of covectors of $\mathcal{C}(M)$, with the understanding that some such simplices are identified.

Proposition 16. *For $F \in \mathcal{L}(M)$ and $C \in \mathcal{C}(M)$ with $z(C) = F$, the simplicial chain*

$$x_F = \sum_{\Delta = (C_0 \lessdot \ldots \lessdot C_i = C) \in \Delta(\mathcal{C}(M)_{\leqslant C})} \epsilon_\Delta \Delta$$

is independent of C and represents a cycle in $H_i(Y_M^\Delta; \mathbb{Z})$.

Proof. Independence of C is clear from the definition of the identifications.

Without loss of generality, assume (by localising at $z(C)$) that $C = 0^E$. Consider a simplex $(C_0 \lessdot \ldots \lessdot \hat{C}_j \lessdot \ldots \lessdot C_i)$, where $0 \leqslant j < i$. Thinness of the poset $\mathcal{C}(M)$ implies that it appears in the image of the simplicial differential applied to exactly two summands Δ_1, Δ_2 in the above expression. By the same argument as in Proposition 3, we have $\epsilon_{\Delta_1} = -\epsilon_{\Delta_2}$. Hence its coefficient in $\partial(x_C)$ is zero.

A simplex of the form $(C_0 \lessdot \ldots \lessdot C_{i-1})$ only appears once as a face of a maximal simplex in $\Delta(\mathcal{C}(M))$. However, such a simplex is identified with exactly one other such simplex $((-C_{i-1}) \circ C_0 \lessdot \ldots \lessdot (-C_{i-1}) \circ C_{i-1} = -C_{i-1})$, and hence appears in the image of the differential applied to $\Delta_1 = (C_0 \lessdot \ldots \lessdot C_{i-1} \lessdot 0^E)$ and $\Delta_2 = (-C_{i-1} \circ C_0 \lessdot \ldots \lessdot -C_{i-1} \lessdot 0^E)$. One can compute ϵ_{Δ_1} and ϵ_{Δ_2} using the same choice of $(b_1, \ldots, b_i)$; but $C_0(b_j) = ((-C_{i-1}) \circ C_0)(b_j)$ for $j < i$ as $b_j \in z(C_{i-1})$, and $C_0(b_i) = C_{i-1}(b_i) = -(-C_{i-1})(b_i) = -(-C_{i-1} \circ C_0)(b_i)$. Hence $\epsilon_{\Delta_1} = -\epsilon_{\Delta_2}$, as required. □

Proposition 17. *The x_F are the images of the cellular chains σ under the map induced by the barycentric subdivision $h\colon Z_M \to \|\Delta(\mathcal{C}(M))\|$.*

Proof. This is exactly an observation of Björner and Ziegler [7, Section 7.2] together with the definition of the incidence function (2.1). □

By Corollary 6, we can construct a basis for the simplicial cohomology by constructing simplicial cocycles representing the Kronecker duals of the cycles constructed in Proposition 16.

Definition 12. *Fix a flat $F \in \mathcal{L}^k(M)$ of rank k and a maximal chain of covectors $\Delta = (X_0 \lessdot \ldots \lessdot X_k) \in \Delta(\mathcal{C}(M^F))$. Define the simplicial cochain $y_\Delta \in C^k(Y_M^\Delta; \mathbb{Z})$ to be the sum of $1_{\Delta'}$, where the sum is taken over k-simplices $\Delta' \in Y_M^\Delta$ such that Δ is the image of Δ' (thought of as a chain in $\Delta(\mathcal{C}(M))$) under the map induced by the localisation $\mathcal{C}(M) \to \mathcal{C}(M^F)$ on each vertex.*

Proposition 18. *The cochains y_Δ are well-defined.*

Proof. This is again clear from the definition of the identifications. □

Proposition 19. *The y_Δ are cocycles.*

Proof. We check that the image of y_Δ under the cochain differential is zero on all $k+1$-simplices σ. Given such a simplex, for $y_\Delta(\sigma)$ to be nonzero there must be some codimension 1 face σ' of σ which projects to Δ after localisation at F. If instead we apply localisation at F to σ, the $k+2$ vertices of σ are mapped surjectively to a chain with $k+1$ vertices, so there must be two covectors, one a cover of the other, mapped to the same covector of M^F. Hence applying the chain differential to σ results in exactly two relevant Δ's appearing with opposite sign. Then the evaluations by the $1_{\Delta'}$ cancel to give zero. □

We then have the following by construction.

Proposition 20. *$y_\Delta(x_F) = \epsilon_\Delta = \pm 1$ and $y_\Delta(x_{F'}) = 0$ for rank k flats $F' \neq F$. In particular, choices of Δ corresponding to the same flat $F \in \mathcal{L}(M)$ give the same cohomology class up to sign.*

Proposition 21. *Let $F, G \in \mathcal{L}(M)$ be flats, and choose maximal chains of covectors $\Delta_1 \in \Delta(\mathcal{C}(M^F))$, $\Delta_2 \in \Delta(\mathcal{C}(M^G))$. Then the cup product of y_{Δ_1} and y_{Δ_2} is*

$$y_{\Delta_1} y_{\Delta_2} = \begin{cases} y_{\Delta_3} & \operatorname{rk} F + \operatorname{rk} G = \operatorname{rk} F \vee G \\ 0 & \textit{otherwise,} \end{cases} \tag{5.1}$$

where $\Delta_3 \in \Delta(\mathcal{C}(M^{F \vee G}))$ is a simplex defined in the course of the proof.

Proof. First note that either $\operatorname{rk} F + \operatorname{rk} G = \operatorname{rk} F \vee G$ or $\operatorname{rk} F + \operatorname{rk} G > \operatorname{rk} F \vee G$ by submodularity. We show that the second option is not possible for a nonzero product.

If $y_{\Delta_1} y_{\Delta_2}$ is nonzero on some simplex $\sigma = (X_0 < \ldots < X_n)$ of dimension $\operatorname{rk} F + \operatorname{rk} G$, then by definition of the simplicial cup product we require $(X_0 < \ldots < X_{\operatorname{rk} F})$ to be sent to Δ_1 after localisation at F, and similarly for $(X_{\operatorname{rk} F} < \ldots < X_n)$ and Δ_2 after localisation at G. But both of these factor through the localisation at $F \vee G$; if $\operatorname{rk} F \vee G < \operatorname{rk} F + \operatorname{rk} G$ then the localisation of σ at $F \vee G$ must be a degenerate simplex with repeated vertices, and then it would be impossible to localise further to both F and G without degeneracy.

Hence assume $\operatorname{rk} F + \operatorname{rk} G = \operatorname{rk} F \vee G$. We wish to show that the signs $X_i(e)$ for $e \in F \vee G$ are uniquely determined. Equivalently, for $e \in F \vee G$ this is the data of the sign $X_0(e)$ and the index i such that $X_{i-1}(e) \neq 0$ and $X_i(e) = 0$ by the structure of the partial order on covectors. Observe that the required localisations to F and G (and the definition of the partial order on covectors) determine these completely for $e \in F \cup G$.

Pick an ordered basis $B = (b_1, \ldots, b_{\operatorname{rk} F + \operatorname{rk} G})$ of $F \vee G$ which is contained in $F \cup G$ such that $(b_1, \ldots, b_k)$ is a basis of $z(X_k)$. (The elements $(b_1, \ldots, b_{\operatorname{rk} F})$ and $(b_{\operatorname{rk} F+1}, \ldots, b_n)$ can be determined from Δ_1 and Δ_2.) The index i at which $X_i(e)$ is first 0 for $e \in F \vee G \setminus F \cup G$ is determined by the underlying chain of flats $(z(X_0) < \ldots < z(X_n))$. It remains to determine the signs for $e \in F \vee G - F \cup G$. Let $X = X_{i-1}$ be the last covector in the maximal chain where $X(e)$ is nonzero. Setting $Y = C(e, B)$, the intersection of the supports is exactly e and one element $b \in B$ (in fact $b \in G \cap B$). Then by orthogonality we have $X(e) = X(e)Y(e) = -X(b)Y(b)$, so the sign is determined and the simplex Δ_3 constructed in this way is the unique one in $\Delta(\mathcal{C}(M^{F \vee G}))$ which localises to Δ_1 and Δ_2. $\square$

Example 7. A cup product computation for the rank 2 braid arrangement is shown in Figure 5.1. The 2-simplex $(+++ \lessdot +0+ \lessdot 000)$ is the only one for which the localisation of the first two vertices at $\{2\}$ and the localisation of the last two at $\{3\}$

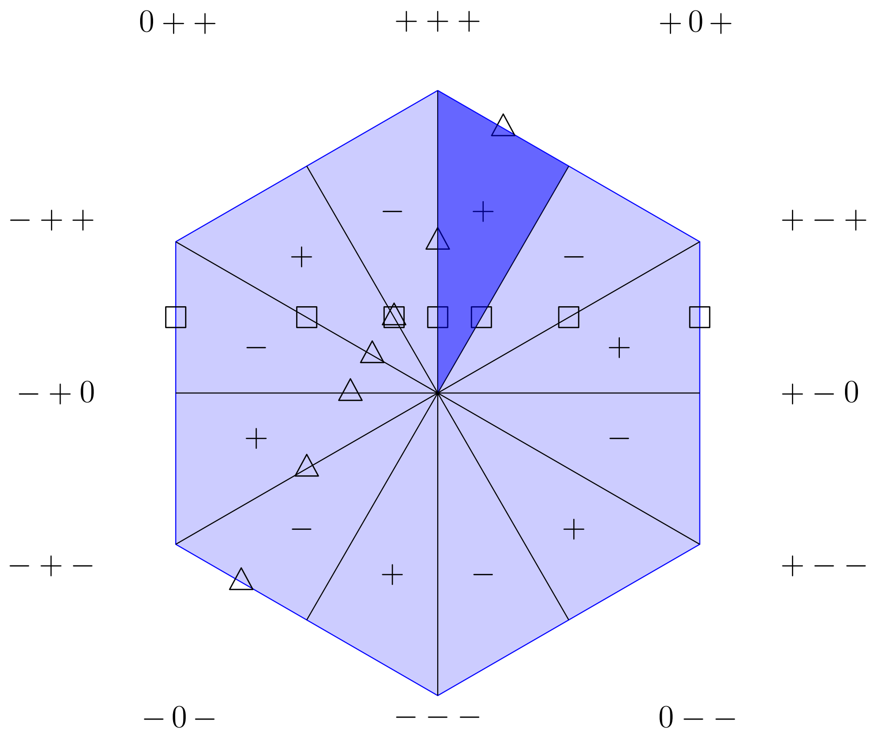

Figure 5.1: The cup product $y_\Delta y_{\Delta'}$ of cocycles y_Δ (support indicated by $\triangle$) and $y_{\Delta'}$ (support indicated by $\square$), where $\Delta = (+ \lessdot 0) \in \Delta(\mathcal{C}(M^{\{2\}}))$ and $\Delta' = (+ \lessdot 0) \in \Delta(\mathcal{C}(M^{\{3\}}))$. Signs of the simplicial cycle x_E are indicated.

both give $(+ \lessdot 0)$.

Corollary 7. *$H^*(Y_M;\mathbb{Z})$ is generated in degree* 1.

Proof. This follows from the above formula for cup product and the fact that $\mathcal{L}(M)$ is atomic. $\square$

5.3 A presentation for $H^*(Y_M;\mathbb{Z})$

Theorem 8. *Let M be an oriented matroid with no loops. Then $H^*(Y_M;\mathbb{Z})$ is isomorphic to*

$$\mathrm{OB}(M) := \bigwedge [y_e \colon e \in E] \,/ \big\langle \chi^F(B) y_B - \chi^F(B') y_{B'} \colon F \in \mathcal{L}(M), B, B' \in \mathcal{B}(M^F) \big\rangle.$$

Here $y_B = y_{b_1,\ldots,b_k}$ if $B = (b_1, \ldots, b_k)$.

Remark 17. The deletion (or addition) of loops to M does not change the topology of Y_M; the requirement that M is loopless is to ensure that the presentation of $\mathrm{OB}(M)$

omits variables corresponding to loops. Observe also that the ideal generated by the relations is insensitive to the choice of a chirotope or its negation.

Remark 18. The algebra $\mathrm{OB}(M)$ appears to be new. The notation reflects its similarities with the Orlik–Solomon algebra $\mathrm{OS}(M)$ and graded Möbius algebra $B(M)$ of a matroid.

Proof of Theorem 8. Observe that if $e, e' \in E$ are parallel then the relations in $\mathrm{OB}(M)$ imply that $y_e = \pm y_{e'}$. Hence without loss of generality we can assume that M has no parallel elements.

Choose simplicial representatives x_e^* for a basis of the degree 1 cohomology coming from Definition 12. In particular, choose such representatives associated to the maximal simplices $(+ \lessdot 0)$ in the rank 1 localisations. We claim that $y_e \mapsto x_e^*$ defines a ring homomorphism $\mathrm{OB}(M) \to H^*(Y_M; \mathbb{Z})$. To verify this, note that Proposition 21 implies that the product x_B^* corresponds to a single simplex (perhaps after localisation); by Proposition 16, comparing the signs of x_B^* and $x_{B'}^*$ amounts to computing the signs ϵ_Δ of Proposition 15 of these simplices. But our choice of x_e^* means that choosing B and B' to be the ordered bases in the computation of the signs gives us the benefit that the $C_0(b_i)$ are all $+$. Hence the signs associated to x_B^* and $x_{B'}^*$ are $\chi^F(B)$ and $\chi^F(B')$ respectively, whence the x_e^* satisfy the relations of $\mathrm{OB}(M)$. Corollary 7 implies that this homomorphism is surjective; since $\operatorname{rk} \mathrm{OB}^i(M) \leqslant |\mathcal{L}^i(M)| = \operatorname{rk} H^i(Y_M; \mathbb{Z})$, it follows that it is injective and hence an isomorphism. □

We obtain distinguished bases and their structure constants for the cup product as an immediate corollary:

Proposition 22. *Fix a total order on E, and let $B(F)$ be the lex-minimal ordered basis of $F \in \mathcal{L}(M)$. Then $\mathrm{OB}(M)$ has a basis $\{y_F := y_{B(F)} \colon F \in \mathcal{L}(M)\}$, and multiplication in this basis is defined by*

$$y_F y_G = \begin{cases} \epsilon_{F,G} y_{F \vee G} & \text{if } \operatorname{rk} F + \operatorname{rk} G = \operatorname{rk} F \vee G \\ 0 & \text{otherwise.} \end{cases}$$

Here $\epsilon_{F,G} = \chi^{F \vee G}(B(F \vee G))\chi^{F \vee G}(B(F) \cdot B(G))$, where $\cdot$ is concatenation.

Remark 19. When M comes from the type A root system, a presentation for $H^*(Y_M; \mathbb{Z})$ was claimed in [3] and proven by Lee [22]. Their presentation has fewer relations than ours (only rank 2 flats are required); it is not yet clear to what extent their presentation generalises to other oriented matroids.

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