# DENSE FORESTS WITH LOW VISIBILITY

BY

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# ABSTRACT

A dense forest in $\mathbb{R}^d$ is a set of points with finite density such that any line segment of length $V(\varepsilon)$, the visibility function of the forest, comes within distance $\varepsilon$ of an element of the set—the points with a ball of radius $\varepsilon$ centred at them can be viewed as trees obstructing visibility, hence the name. Such sets have been shown to exist. The problem now is to determine how small the visibility can be made and how many additional properties can be required from a dense forest. This thesis presents results in those directions.

A dense forest is constructed by modifying a base set obtained from a Poisson point process by adding some points to it such that the visibility is in $O\left(\varepsilon^{-(d-1)} \ln \varepsilon^{-1}\right)$. The constant implied by the big-$O$ notation depends on dimension only. Further, the Poisson point process can be adjusted so that the base set is uniformly discrete while maintaining the same final visibility. Due to the latter addition of points, the resulting dense forest is not uniformly discrete.

Through a slight change of perspective—from eliminating lines of sight to eliminating empty 'tubes'—dense forests are closely related to Danzer sets. The methods used to find dense forests in this thesis do not rely on the specific shape of the empty regions being examined. As a result, it is particularly suited to approaching Danzer's problem. A new approach to finding Danzer sets is examined where the volume of the convex sets is allowed to grow as their eccentricity increases. Bounds on the growth of

the volume are obtained by adapting the approach taken for dense forests earlier.

# ACKNOWLEDGEMENTS

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# PUBLICATIONS

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# FORESTS

Consider a forest in which all the trees have the same trunk radius, $\varepsilon$. Can the trees be planted in such a way so as to guarantee that no matter where you stand and where you look, you cannot see farther than some fixed distance that depends on $\varepsilon$? To make the problem non-trivial, the arrangement of trees must be the same for all $\varepsilon$, otherwise a fine enough grid for each $\varepsilon$ will do. There also needs be a bound on the density of the trees in the forest, otherwise a tree planted at each, say, rational point will work. A set of points $X \subset \mathbb{R}^d$ has finite density if

$$\limsup_{T\to\infty} \frac{\#\left(X \cap B(0,T)\right)}{T^d} < \infty.$$

where $\#(\cdot)$ is the cardinality of a set and $B(x,T)$ is the ball centred at $x$ of radius $T$. The variable $d$ will always refer to the dimension of the ambient space. A tree is a ball of radius $\varepsilon$. The precise definition of a dense forest is given below.

*Definition* 1.0.1. Given a set of points $F \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$, $v \in S^{d-1}$, and $\varepsilon > 0$, the **visibility** in $F$ at scale $\varepsilon$ from $x$ in direction $v$ is the smallest $t \in \mathbb{R}_{\geq 0}$ such that

$$||x + tv - y|| \leq \varepsilon$$

for some $y \in F$.

Here, $||\cdot||$ is the Euclidean norm. When they are unambiguous, the parameters in the above notation will be suppressed. With few exceptions, the visibility is presented in big-O notation, ignoring constants. Given two functions $f$ and $g$, $f$ is big-O of $g$, $f \in O(g)$, if $\limsup_{x\to 0} \frac{f(x)}{g(x)} < \infty$. The constants hidden behind the big-O notation depend on—among other things—dimension, and can be quite large.

*Definition* 1.0.2. A set of points $F \subset \mathbb{R}^d$ is a **dense forest** if it has finite density and there exists a visibility function $V : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ such that for any $x \in \mathbb{R}^d$, $v \in S^{d-1}$, and $\varepsilon > 0$, the visibility is at most $V(\varepsilon)$.

The points of $F$ can be thought of as the centres of the trees in the forest. The dense forest problem is the following:

*Question* 1.0.1 (Dense forest problem). Does there exist a dense forest? how small can the visibility function be made?

Note that the 'dense' component of the name of dense forests refers to them obstructing visibility, not to the finite density requirement. Some authors do not include the finite density in the definition. To exclude irrelevant sets, such as all of $\mathbb{R}^d$, being called dense forests, the convention of including the finite density requirement as in [32] is adopted.

The 'optimal' visibility, in the sense that it is the lowest possible for a set of finite density, for a dense forest in $\mathbb{R}^d$ is $O\left(\varepsilon^{-(d-1)}\right)$. To see this, assume that $F$ is a dense forest with visibility function $V$. $V$ is necessarily nondecreasing as $\varepsilon \to 0$ as otherwise any bounded set would need to contain infinitely many points of $F$. Let the density of $F$ be $M$ and let $T$ be large enough so that $\#(B(0,t) \cap F) \leq 2MT^d$ for all $t \geq T$. Let $\varepsilon$ be small enough so that $V(\varepsilon) \geq T$. Let $C$ be a cube centred at the origin with side

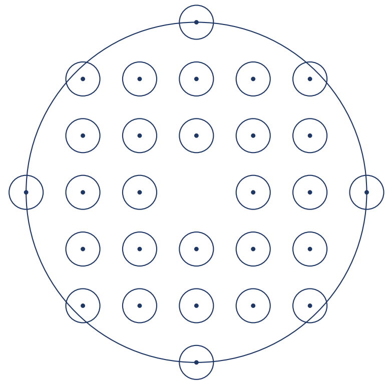

**Figure 1.1:** Pólya's orchard. A sketch of an orchard having radius 3. The trees in the orchard are placed at the integer points within the orchard and have some small, identical, radius. Is it possible to see outside of the orchard while standing at its centre?

length $V(\varepsilon)$. Then, there are at least $(\varepsilon^{-1}V(\varepsilon))^{d-1}$ lines of visibility that will require distinct trees to block their sight contained in the cube. Also,

$$\frac{(\varepsilon^{-1}V(\varepsilon))^{d-1}}{(V(\varepsilon))^d} \leq \frac{\#(F \cap B(0, V(\varepsilon))}{(V(\varepsilon))^d} \leq 2M.$$

Rearranging gives $\frac{(\varepsilon^{-1})^{d-1}}{2M} \leq V(\varepsilon)$ for $\varepsilon$ small enough. The visibility functions presented here will be of the form $O(\varepsilon^{-(d-1)}E(\varepsilon))$ where $E$ is an error function that measures how far sway from the optimal visibility the result is. So, an optimal dense forest would have $E(\varepsilon) \in O(1)$. It is not known if the optimal visibility is achievable.

The problem can be connected to a wide range of visibility problems that have been studied over many generations. The problem considered the foundational one is one posed by Pólya, in [26, problem 239, page 151], called Pólya's Orchard Problem.

*Question* 1.0.2 (Pólya's orchard problem). If trees are placed on the integer lattice inside a circular orchard, how thick must their trunks be to completely block the view from the centre?

He only gives a bound for the requisite thickness when the circle had integer radius. Later, in [5], Allen solved the orchard problem using careful trigonometric analysis. If the radius of the orchard is $s$, then the trees must have radius at least $\frac{1}{\sqrt{\lambda}}$, where $\lambda$ is the first integer greater than $s^2$ that can be written as the sum of squares of two coprime integers. Later, Hening and Kelly in [20] extended Allen's result to more general orchards. Those being orchards $O \subset \mathbb{R}^2$ which are compact, convex, and for which consecutive rays that pass through integer points in $O$ form acute angles. In [21], Kruskal considered some related problems. He dealt with the situation where the trees lie not necessarily on the integer lattice, but on any parallelogram lattice. Also, he examined what the radius of trees needs to be in order to block the line of sight between some two trees in the orchard. Independently, calling them view-obstruction problems in [16], Cusick formulates the problem in higher dimensions and where the trees are arbitrary convex bodies. He examines the cases of spheres and cubes, noting that even for simple shapes, the problem becomes quite difficult in higher dimensions.

A related problem can be derived from the so called circle problem of Gauss(see [18, page 104]). In two dimensions, it asks for an estimate of how many points of $\mathbb{Z}^2$ are contained in a ball centred at the origin of radius $r$. Some variations ask how many such points $(m, n)$ have coprime coordinates. If the points are considered as obstructing the line of sight, then this is the same thing as asking how many points of $\mathbb{Z}^2$ are visible from the origin. It turns out, in [24] for instance, that the accuracy of the estimate is related to the Riemann Hypothesis. Variants of the circle and orchard problems in hyperbolic space were studied by Chamizo in [14].

Yet another class of related problems are illumination problems. A recent one, called Mitchell's dark forest conjecture and shown to be true

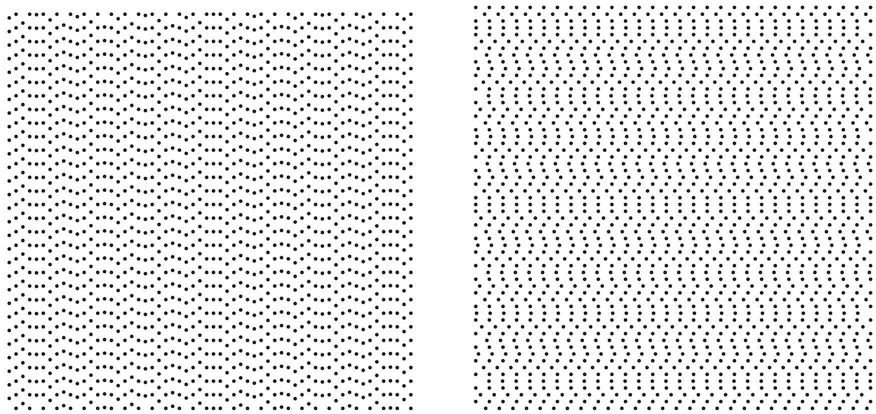

**Figure 1.2:** Peres forest 1. On the left, the centres of the trees in Peres' forest in $[-15, 15]^2$ that handle all mostly horizontal lines. It is the union of two sets, a lattice and a lattice that has had its points vertically shifted by various amounts. On the right, the same set that has been rotated by 90 degrees to handle all mostly vertical lines. Their union, shown in Figure 1.3, will then handle all lines.

by Dumitrescu and Jiang in [17], asked the following. Suppose the unit ball in the plane contains a disjoint set of smaller balls, called trees, of radius $\varepsilon$ which obscure line of sight. Is there always a point in the unit ball not visible from the outside? Equivalently, is there always a point from which one cannot see outside the ball? Amusingly, the set of trees in the unit ball was also referred to as a dense forest by the authors; it is in the same spirit, but not equivalent to the definition used here. Boshernitzan and Solomon in [13] examine a similar problem on $\mathbb{R}^d$. They show that if there is a set of trees of radius $\varepsilon$ in $\mathbb{R}^d$ without arbitrarily large clearings, then there are always points are not illuminated by a light at infinity.

The modern concept of a dense forest was first introduced by Bishop in [12]. There are some major differences between the original formulation of the problem and the one used here. The first is that the forest is planar and may be sought only on a subset of $\mathbb{R}^2$, not on the whole space. The other distinction is that he asked for the forest to be uniformly discrete. A set $X \subset \mathbb{R}^d$ is uniformly discrete if there exists $r > 0$ such that for any

$x, y \in X$, $|x - y| \geq r$. In other words, there is some guaranteed distance by which any two points in the set are separated. A uniformly discrete set necessarily has finite density, but the reverse is not true.

The same paper also contains the first proof—supplied by Peres— of the existence of a dense forest with an explicit visibility bound, the first one found.

*Theorem* 1.0.1 (Bishop, [12]). There exists a dense forest in $\mathbb{R}^2$ with visibility $V(\varepsilon) \in O\left(\varepsilon^{-4}\right)$.

The general idea in the approach taken in the proof has been the basis for several future constructions of dense forests. The forest is obtained by, for each basis vector $e_i$, constructing a set $F_i$ that interrupts lines that lie mostly in direction $e_i$. The final forest is obtained by taking the union of all the $F_i$. Each $F_i$ is constructed as the union of a translated lattice and a shear of a lattice by the golden ratio. For $e_1$, the set $F_1$ is

$$(\mathbb{Z}^2 + (1/2, 0)) \cup \{(m, n + m\varphi) | m, n \in \mathbb{Z}\},$$

or, equivalently,

$$\left(\mathbb{Z}^2 + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}\right) \cup \begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix} \mathbb{Z}^2$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

The set $F_2$ for $e_2$ is obtained by rotating $F_1$ by and angle of $\pi/2$(see Figure 1.2). Treating the directions separately allows you to restrict focus to a smaller set of lines. Basing the construction on a lattice gives some periodicity to the construction, which helps deal with lines in arbitrary positions. Forests that are based on the same principles, dealing with the directions separately and constructing the set as a union of modified

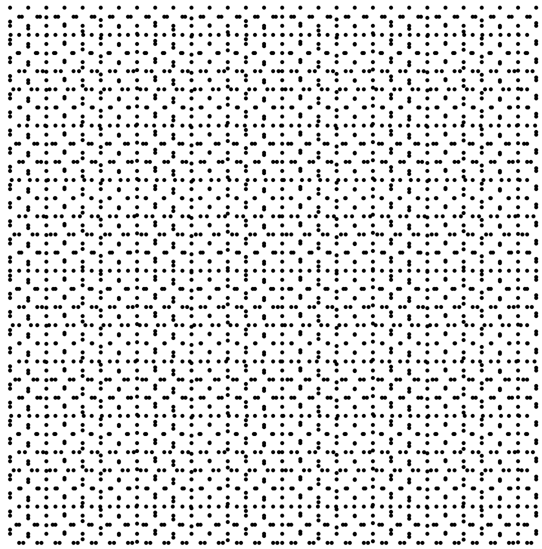

**Figure 1.3:** Peres forest 2. Peres' forest in $[-15, 15]^2$ that handles lines in all directions. It is the union of the two sets shown in Figure 1.2.

lattices are sometimes called Peres type forests in the literature. While the visibility of this first forest is far from the desired $O(\varepsilon^{-1})$ in $\mathbb{R}^2$, a more careful analysis of this forest in [3] showed that it has a visibility $V(\varepsilon) \in O\left(\varepsilon^{-3}\right)$.

Adiceam wrote the first paper published on the topic of dense forests after it was introduced by Bishop, [1]. In it, he introduced much of the notation and terminology of dense forests used today, expanded the concept to higher dimensions, and construed forests with improved visibility bounds. The main result of the paper is the following.

*Theorem* 1.0.2 (Adiceam, [1]). There exists a dense forest $\mathcal{F} \subset \mathbb{R}^d$ with visibility $V(\varepsilon) \in O\left(\varepsilon^{-2(d-1)-\eta}\right)$ for any $\eta > 0$.

Since, up to a dimension dependent constant, which is already implied by the big-O notation, norms are equivalent in $\mathbb{R}^d$, he works with the $\ell^\infty$ norm in place of the standard Euclidean one. The forest he constructs is a Peres type forest and, as such, only of the form $x_t v$ where $v \in S_\infty^{d-1}$, the unit ball in $\ell_\infty$, with $v_1 = 1$ need to be considered. That is, those lines that extend mostly in the $e_1$ direction. Mirroring the construction about the hyperplane $x_1 = 0$ and adding rated copies of it to account for different orientations of lines completes the construction. To construct the blueprint, a sequence of points satisfying certain growth properties is used to shift the points of the integer lattice in certain directions,

$$\mathcal{F}_1 = \{(k_1, a_{|k_1|} + k_2, \ldots, a_{|k_1|} + k_d) | (k_1, \ldots, k_d) \in \mathbb{Z}^d\}.$$

To show that the union of such sets is indeed a forest, Adiceam first discretises the lines that need to be examined by considering only those of the form $x + tv$ with $v_1 = a/b$ with a bound on the possible values of $b$ for fixed $\varepsilon$. Each tree is then replaced by an indicator function for its set. For a given point $x$, consider a new function, $L$, defined as the sum of the indicator functions between $V + 1$ and $2V$ in the $x_1$ direction from $x$. The function $L$ is nonzero if and only if a line of length $2V$ from $x$ intersects a tree. To show that $L$ is nonzero, the indicator functions are approximated from below by trigonometric polynomials using the Beurling-Selberg external functions. Using Fourier analytic techniques, the function $L$ becomes bounded below by the difference of two terms, the minuend depending on the visibility $V$, the subtrahend on the sequence $a_k$. The requirement of this difference being positive dictated the growth condition on $a_k$. The proof uses $a_k = \theta k!$ for some real number $\theta$. It is also proved that any sequence of the form $a_k = \theta\alpha_k$ with $\alpha_k \in o(\alpha_{k+1}$ will work. While the sequence is almost fully specified, in the end, the forest is not explicit. This is because the constant $\theta$ is not specified, as its validity is shown using the Borel-Cantelli lemma. The freedom to choose $a_k$ leads to a family of forests, although they will not have a different visibility,

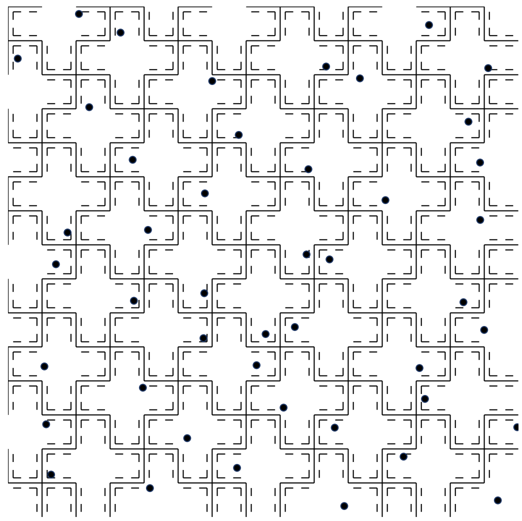

**Figure 1.4:** Alon's cross tessellation. An illustration of how the points are placed inside the cross tessellation. The solid crosses tessellate $\mathbb{R}^2$. The dashed crosses inside of them are where the points are placed—one in each. The distance separating the boundary of each dotted cross ensures that the points will be at least a certain distance apart.

which—while being the best at the time of being published—still has a significant error term. Adiceam also mentions in the paper that the bounds obtained are the best that one can expect from the Fourier analytic proof used. So, further refinement of the result is unlikely.

In the spirit of the original question posed by Bishop, some authors continue to construct uniformly discrete dense forests. However, this is rare. Currently, the construction with the best visibility bound for a uniformly discrete dense forest is due to Alon in [6]. The construction is probabilistic and is only proved in the planar case—although, it easily generalises to $\mathbb{R}^d$.

*Theorem* 1.0.3 (Alon, [6]). There is a positive constant $C$ such that there exists a uniformly discrete dense forest in $\mathbb{R}^2$ with visibility $V(\varepsilon) = 2^{C\sqrt{\ln \varepsilon^{-1}}}\varepsilon^{-1}$.

The forest is constructed by first tiling the plane with crosses, each consisting of five unit squares. Inside each cross, at least some small distance away from the boundary, exactly one point is placed randomly, uniformly, and independently. Placing the points away from the boundary of the crosses ensures that the forest is uniformly discrete(see Figure 1.4). To analyse the visibility, Alon examines the different scales of $\varepsilon$ separately, using $\varepsilon_i = 2^{-i^2}$ for he scales, for reasons that will become apparent later. For each scale, points in the crosses in horizontal and vertical strips near the horizontal and vertical lines $2^i b$ with $b$ odd are used to handle the lines of scale $\varepsilon_i$. Any line of length at least $V(\varepsilon_i)$ will pass through enough of the interiors of each cross in these strips so that the probability the the line has none of the randomly placed points within distance $\varepsilon_i$ of it is at most $e^{-ci^2}$ for a fixed constant $c$. The proof is concluded with a compactness argument and an application of the Lovász local lemma to show that there exists a placement of the points for which all lines of length $V(\varepsilon)$ have a point within distance $\varepsilon$ of them. The product space of all placements of one point in each cross is compact by Tikhonoff's theorem. For any line, the set of all placements of points in that space that contain a point within distance $\varepsilon$ of it is closed. This allows Alon to only consider a finite set of lines and show that the intersection of sets of placements of points in crosses that come within distance $\varepsilon_i$ of each line is non-empty This is accomplished using the symmetric version of the Lovász local lemma.

*Lemma* 1.0.1. Let $A_i$, $1 \leq i \leq n$ be events in the same probability space. Suppose that the probability of each event occurring is at most $p$ and that each $A_i$ is independent of all but at most $d$ other events. If $ep(d+1) \leq 1$, then none of the events occur with positive probability.

Any line is independent of all other lines except those that intersect the same crosses as it. After approximating the number of such lines at scale $\varepsilon_i$, using a discrete set, the assumptions of the Lovász local lemma are met and there exists, possibly with probability zero, a placement of points that doesn't miss any line. Using this approach, the visibility is a result of needing to sufficiently separate the scales of $\varepsilon$. In particular, using a different random process to place the points in the crosses of using a different tessellation will not have a meaningful effect on the visibility. There is a particular weakness of the Lovász local lemma, which is also acknowledged by Alon in the paper. It ignores the sign and size of he dependence. There is an asymmetric version of the Lovász local lemma which would, in principle, allow one to deal with different scales of $\varepsilon$ all at once.

*Lemma* 1.0.2. Let $\mathcal{A} = \{A_i\}_{i \in I}$ be a finite family of events in a probability space. Let $\Gamma(A)$ denote the neighbours of $A$ in the dependency graph. If there exists a function $x : \mathcal{A} \to [0, 1)$ such that

$$\forall A \in \mathcal{A}, P(A) \leq x(A) \prod_{B \in \Gamma(A)} (1 - x(B))$$

then with positive probability, none of the $A \in \mathcal{A}$ occur.

The dependency graph, is a graph whose vertices are the elements of $\mathcal{A}$. Two vertices are connected by an edge if they are dependent.

However, again, there are too many intersections between lines at different scales and so, too many dependent events. Nonetheless, for two lines of different scales—and even for lines of similar scales— the dependence is very small, which is ignored by the Lovász local lemma. The combination of the Lovász local lemma with a compactness argument has been used in other contexts(cf. [7], chapter 5). The

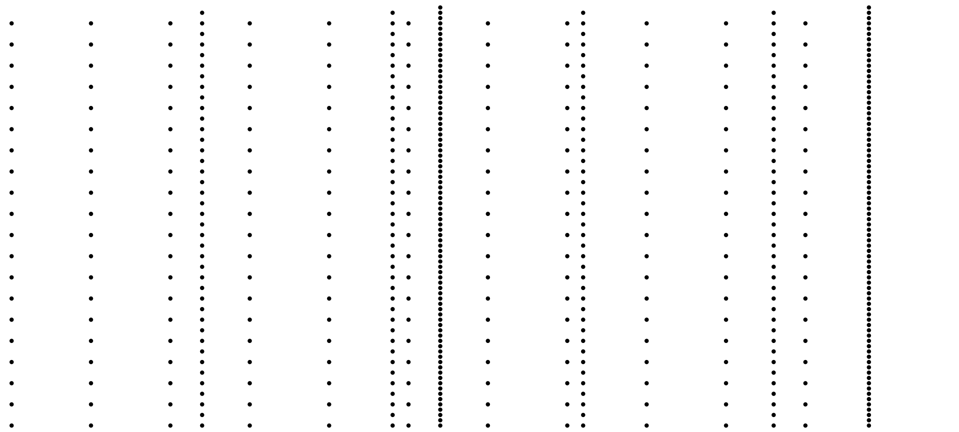

**Figure 1.5:** Tsokanos' curtains. A sketch of the curtains that catch the mostly horizontal lines for different scales of $\varepsilon$. The ones that deal with larger $\varepsilon$ can have their points farther apart, but need to themselves be closer together. The ones that handle the lines for smaller $\varepsilon$ can be farther apart, but need to have their points closer together to ensure that a line that passes through them comes sufficiently close to a point.

The best known results, that is, the forests with lowest visibilities, have probabilistic constructions. It is natural to ask what bounds can be obtained for a deterministic construction. The dense forest with a deterministic construction with the lowest visibility is given by Tsokanos in [32]. The construction shares some similarities with Peres type forests, but is quite different. Deterministic Peres type forests and what Tsokanos calls optical forests are also constructed in the paper. The different coordinate directions are separated and the final forest is a union of sets in each direction, like in Peres type forests. For each direction, regularly spaced 'curtains' are placed for each scale of $\varepsilon$. The curtains have the form

$$S_j = \left\{ \left( kV(e_j), \ell_2 \frac{e_j}{\sqrt{d}}, \dots, \ell_d \frac{e_j}{\sqrt{d}} \right) | k \in \mathbb{Z} \setminus \{0\}, \ell_2, \dots, \ell_d \in \mathbb{Z} \right\}.$$

The points on them are dense enough so that any line passing through them will have a point within distance $e_j$ of it. This differs markedly from the Peres type forests, which require the line to pass through sufficiently many curtains in order to guarantee a point near it. Since the points in Tsokanos' curtains are much closer together, the curtains need to be spaced much farther apart to ensure finite density. This is what gives rise to the visibility in this case. This is in contrast to Peres type forests, where the visibility is a result of needing to pass through sufficiently many curtains. The construction is done using arbitrary scales of $\varepsilon$. Choosing $e_i = 2^{-i}$ gives the following result.

*Theorem* 1.0.4 (Tsokanos, [32]). There exists a deterministic dense forest with visibility

$$V(\varepsilon) \in O\left(\varepsilon^{-(d-1)} \ln \varepsilon^{-1} (\ln \ln \varepsilon^{-1})^{1+\eta}\right)$$

for any $\eta > 0$.

It should be noted that the forest is constructed for the given visibility, not the other way around. This is the best known visibility for an explicit dense forest. The concept of an optical forest that appears is base on the approach to finding Danzer sets. An optical forest is a dense forest with optimal visibility, but where the density is allowed to grow away from the origin. The result combines theorem 1.4 of [31] with a placement of points very similar to the one given above to show the existence of a deterministic optical forest with growth $g(T) \in O\left(T^d \ln T\right)$, the same as the best know one for Danzer sets. The growth of a set $X \subset \mathbb{R}^d$ is defined to be

$$g_X(T) := \#\left(X \cap B(0,T)\right).$$

When unambiguous, the subscript will be suppressed.

Periodic sets, as well as their unions and translates, have been studied as a potential source of dense forests. Adiceam, Solomon, and Weiss in [3] give several results to that effect. They show that while a cut and project set cannot be a dense forest, there is a finite union of cut and project sets in the plane that is a dense forest. Moreover, the dense forest is uniformly discrete and explicit. Unfortunately, their proof does not give a bound on the visibility. In the paper, they also examine unions of lattices and get two results concerning them. They give an explicit construction of a uniformly discrete dense forest that is a union of three grids in $\mathbb{R}^2$ that is a dense forest with visibility $V(\varepsilon) \in O\left(\varepsilon^{-(5+\eta)}\right)$ for any $\eta > 0$. For higher dimensions, they show that for any $d \geq 2$ and each $s \geq d$ and each $\eta > 0$, for a.e. choice of $d \cdot s$ lattices in $\mathbb{R}^d$, their union is a dense forest with visibility $V(\varepsilon) \in O\left(\varepsilon^{-(d-1+\alpha_d(s)+\eta)}\right)$ where $\alpha_d(s) \xrightarrow[s\to\infty]{} 0$. The construction is a generalisation of the Peres forest from [12] and relies on a random choice of vector $\Theta \in \mathbb{R}^{d-1}$.

Unions of grids were also examined by Shirandami in [28]. He gives an equivalent condition to a finite union of grids being a dense forest in terms of the matrices that define them. He also shows that for any $\eta > 0$, there is some $k > (d-1)^2$ such that almost all unions of $k$ grids are dense forests with visibility $V(\varepsilon) \in O\left(\varepsilon^{-(d-1)-\eta}\right)$.

Visibility properties of spiral sets were studied by Adiceam and Tsokanos in [4]. A spiral set is one of the form $\{\sqrt[d]{n}u_n\}_{n\in\mathbb{N}}$ where $\{u_n\}_{n\in\mathbb{N}}$ is a sequence on the unit sphere $S^{d-1}$. They are able to find an equivalent characterisation for a spiral set to be a dense forest. However, it is still unknown whether a spiral set exists which meets those conditions. Since the origin is a 'special' point for spiral sets, the authors also investigated the visibility through the origin of such sets, similarly to Pólya's orchards.

Many of the dense forests mentioned here satisfy more properties than required by the definition. The extended goals for the problem of dense forests can be viewed as follows, find a dense forest such that:

- The error term in the visibility is small; the visibility is as close to $O(\varepsilon^{-(d-1)})$ as possible.
- The forest is uniformly discrete,
- The construction is explicit; it does not rely on randomness.

There is another concept of density which is stronger than finite density, but weaker than uniform discreteness. A set $X \subset \mathbb{R}^d$ has finite upper uniform density if

$$\limsup_{T\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\#\left(X \cap B(x,T)\right))}{T^d} < \infty.$$

The definition requires there to be a bound on density around every point in $\mathbb{R}^d$, not just centred at the origin. In addition to uniformly discrete forests, Peres type forests, and any forests constructed as the union of uniformly discrete sets, have finite upper uniform density. The idea of imposing this additional requirement has not drawn much interest and is only mentioned here for completeness.

Most of the dense forests presented here have used the definition given at the start of this section to verify their construction. That is, they showed that any line of length $V(\varepsilon)$ contains a point of the forest within distance $\varepsilon$ of it. An important equivalent characterisation of this property will now be given. For a line $L$, consider the convex body

$$C = \{x \in \mathbb{R}^d \,|\, |x - y| \leq \varepsilon \text{ for some } y \in L\}.$$

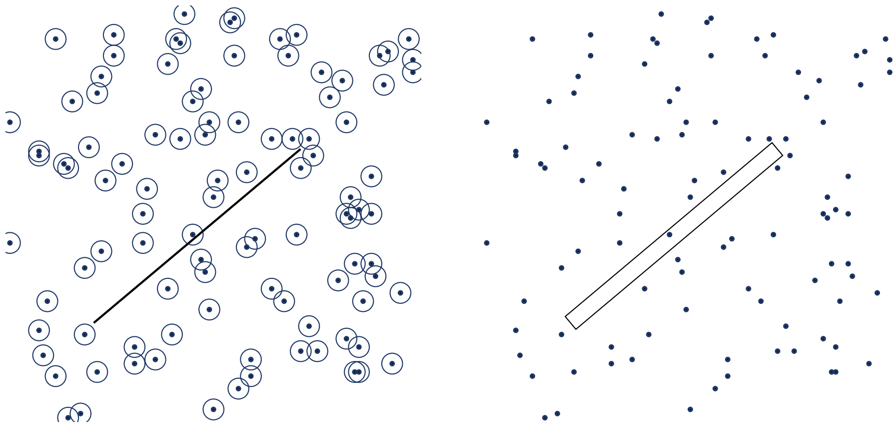

**Figure 1.6:** Equivalence of forest definitions. On the left is a sketch illustrating the standard definition of a dense forest. The trees are depicted as circles of radius $\varepsilon$ with a point at their centre. The illustrated line collides with one tree in the forest. On the right is a sketch illustrating the equivalent definition of a dense forest. Only the centres of the trees are left. The lines are replaced by boxes of the same length and width $2\varepsilon$. The box that represents the line that collided with one tree contains the point that was the centre of that tree.

The line $L$ contains a point of a dense forest within distance $\varepsilon$ of it if and only if $C$ contains that point of the forest in it. The set $C$ has the shape of a cylinder with round caps. It will be seen later that, WLOG, it is enough to look at all boxes with $d-1$ sides of length $\varepsilon$ and one side of length $V(\varepsilon)$. A box here, and throughout, refers to a product of intervals. An equivalent definition of a dense forest is given below(see also Figure 1.6).

*Definition* 1.0.3. A set of points $F \subset \mathbb{R}^d$ is a **dense forest** if it has finite density and there exists a visibility function $V : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ such that for any $\varepsilon > 0$, all boxes with $d-1$ sides of length $\varepsilon$ and one side of length $V(\varepsilon)$ contain a point of $F$.

This equivalence paves the way for connection the dense forest problem to many others in convex geometry.

2

# DANZER SETS

The dense forest problem can be thought of as a relaxation of an old problem in convex geometry. In the 1960's, Danzer asked if there exists a set of finite density which intersects ever convex set in $\mathbb{R}^2$ with fixed volume. The property that a set intersects every convex set of some fixed volume will be referred to as the Danzer property. A set that satisfies Danzer's property and has finite density will be called a Danzer set.

*Question* 2.0.1 (Danzer's problem). Does there exists a Danzer set in $\mathbb{R}^d$ with $d \geq 2$?

The problem is restricted to dimension at least 2 since in $\mathbb{R}^1$, $\mathbb{Z}$ will do. A set satisfying Danzer's property for volume $C$ may be rescaled to get a set that satisfies Danzer's property for larger or smaller volume. So, typically, a volume of 1 is used. John's theorem [9] says that any convex set in $\mathbb{R}^d$ contains an ellipse and is itself contained in a different ellipse such that the ration of the volumes of the two is bounded by a dimension dependent constant. Combining John's theorem with the rescaling property means that it is enough to find a set that intersects every box—equivalently, every ellipse, simplex, etc.—in $\mathbb{R}^d$ of volume 1. Rescaling that set will then give a Danzer set which intersects every convex set of any desired, fixed, volume.

In $\mathbb{R}^2$ Danzer's problem is equivalent to the problem of finding a dense forest with optimal visibility $O(\varepsilon^{-1})$. Since the volume of the boxes examined for dense forests is exactly the error term in the visibility, $E(\varepsilon)$, an optimal dense forest is one for which all boxes with $d-1$sides of length $\varepsilon$ and of volume 1 contain a point of the forest in them. In $\mathbb{R}^2$, every box of volume 1 has one side of length $\varepsilon$ and the other of length $\varepsilon^{-1}$. However, in dimension greater than 2, the two problems are not equivalent since the boxes considered for dense forests are a strict subset of all boxes in $\mathbb{R}^d$. The search for a Danzer set has been difficult.

There have been results on similar problems published even before Danzer's problem was stated, notably, Minkowski's theorem(see [18], chapter 3). It says that any convex body in $\mathbb{R}^d$ of volume at least $2^d$ which is symmetric about the origin will contain a nonzero point of the integer lattice. So, it gives something of a positive result, if translation is not included in Danzer's problem. Pólya's original results for his orchard problem relied on Minkowski's theorem.

As with dense forests, there is a version of the Danzer problem that asks for the set satisfying Danzer's property to be uniformly discrete. It was asked seemingly independently by Conway and is known as Conway's dead fly problem [15].

*Question* 2.0.2 ( Conway's dead fly problem). Does there exist a uniformly discrete Danzer set in $\mathbb{R}^d$ with $d \geq 2$?

The name originates from the wallpaper in Conway's childhood bedroom which was adorned with flowers resembling dead flies. He was looking for the largest convex region on the wallpaper that did not contain any fly. The problem used to carry a $1000 reward for solving it.

The first progress was made by Bambah and Woods in [10]. The paper contains two important results. The first shows that a finite union of translated lattices, hereafter called grids, cannot satisfy the Danzer property. The proof relies on two properties. The first is that the Danzer property is affinely invariant. That is, if a set satisfies Danzer's property, any volume preserving affine transform of it will too. The second is that the limit of sets satisfying Danzer's property will also satisfy Danzer's property. So, assuming that a union of grids could satisfy Danzer's property, they show that a sequence of volume preserving affine transforms can b found such that the transform of the union converges to a hyperplane, which is obviously not a Danzer set. The second main result is the construction of a set satisfying Danzer's property whose growth is $O\left(T^d(\log T)^{d-1}\right)$. One of the keys to the proof is the observation that a Danzer set is a covering set for every convex set of a fixed volume. A covering set $S$ for a convex set $K$ is one which has the property that $\mathbb{R}^d = \bigcup_{x\in S}(K+x)$. The set they construct is an infinite union of grids, each of which handles part of the convex sets. This paper went on to be the only significant result on this topic for over 40 years. The approach of finding a set that satisfies Danzer's property but having growth higher than the optimal $O(T^d)$ and trying to minimise the growth remains one of the main approaches for trying to find a Danzer set.

The other way to relax Danzer's problem is to change the set of boxes being examined. Call a set that has finite density and intersects every axis aligned box of fixed volume an aligned Danzer set. In [29], Simmons and Solomon show that such sets exist and give two constructions. The first is based on the binary van der-Corput sequence and can be written explicitly as

$$\left\{ \left( \pm \sum_{n\in\mathbb{Z}} a_n 2^n, \pm \sum_{n\in\mathbb{Z}} a_n 2^{-n} \right) \middle| (a_n) \in \{0,1\}^{\mathbb{Z}}_{\mathrm{Fin}} \right\}.$$

However, such a construction only works in $\mathbb{R}^2$. For arbitrary dimensions, they show that certain lattices are aligned Danzer sets. In $\mathbb{R}^d$, let

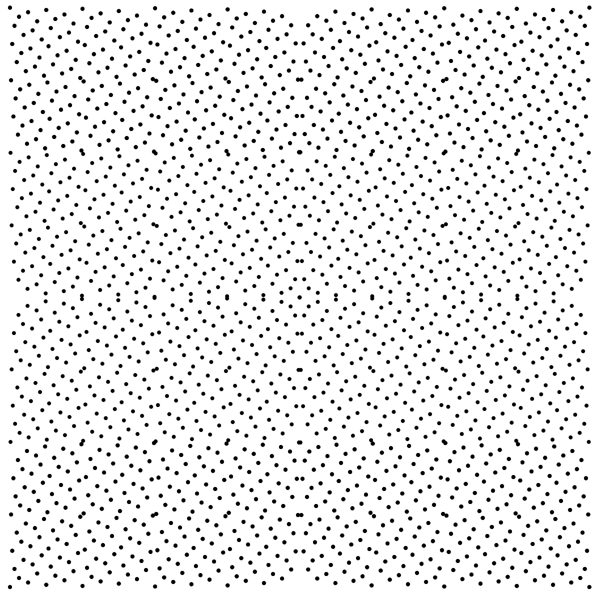

**Figure 2.1:** Aligned Danzer set. The aligned Danzer set constructed from the van der-Corput sequence in $[-16, 16]^2$ which intersects any axis aligned box of fixed volume. Empty diagonal strips can be easily seen, demonstrating that this is not a usual Danzer set.

$\phi_1, \ldots, \phi_d$ be the Galois embeddings of a totally real number field $K$ of degree $d$ into $\mathbb{R}$. Let $\phi$ be their direct sum. Let $O$ be the ring of integers of $K$. They show that $\phi(O)$ satisfies Mahler's compactness criteria and is therefore an admissible lattice. They then show that every admissible lattice is an aligned Danzer set. Despite this, they are not normal Danzer sets, as pointed out in their paper. Both aligned Danzer sets they gave were already known to exist in different contexts. This problem is closely related to the problem of minimal dispersion. Of note, the constructions of th aligned Danzer set are explicit, not random. While for most related problems the 'best' constructions are random, this is one of the few for which that is not the case.

When considered together with Minkowski's theorem, this suggests that Danzer's problem lies on the boundary of possibility. There is a set that intersects all boxes of volume 1 when allowing translation but not rotation, and a set that intersects all boxes of volume 1 when allowing rotation but not translation.

If, on the other hand, the set of boxes under consideration in Danzer's problem is expanded to include some sets which are not quite convex, then—at least in $\mathbb{R}^2$—there does not exist a set of finite density that intersects each of them. This was shown by Pach and Tardos in [25] in the context of the Danzer Rogers problem. In addition to all boxes of volume $\varepsilon$ in the unit square, consider also all quasi-rectangles of volume $\varepsilon$. A quasi-rectangle is defined as follows, fix a small number $\delta$ and a line segment $s$. Move $s$, without rotation, such that the angle between the trajectories of the centre of $s$ and its normal vector is at most $\delta$. The region traced by $s$ is a quasi-rectangle. They show that for any set of $\Omega\left(\varepsilon^{-1}\log\varepsilon^{-1}\right)$ points in the unit square, there must exist a 'staircase' composed of small squares which contains none of the points. In such a staircase, a quasi-rectangle with $\delta = \frac{\pi}{4}$ may be placed. Such a quasi-rectangle may be turned into one for arbitrarily small $\delta$ through an affine transform. This result again suggests that the Danzer problem lies on the boundary of possibility.

A theorem of [31] relates the Danzer problem to one about $\varepsilon$-nets on the unit cube. It makes the problem equivalent to the Danzer Rogers problem. To properly introduce and contextualise it, some concepts will be introduced. The measure $\lambda$ throughout will be the Lebesgue measure. Let $X \subset \mathbb{R}^d$ be a set and let $\mathcal{R}$ be a family of measurable subsets of $X$, called ranges. The pair $(X, \mathcal{R})$ is called a range space. Given a finite set of points, $P$, its dispersion is defined as

$$\mathrm{disp}_{(X,\mathcal{R})}(P) := \sup\left\{\lambda(R) | R \cap P = \emptyset,\, R \subset \mathcal{R}\right\}.$$

In other words, the dispersion of $P$ in the range space $(X, \mathcal{R})$ is the size of the largest range that does not contain a point of $P$. Given $\varepsilon > 0$, an $\varepsilon$-net for the range space $(X, \mathcal{R})$ is a set of points $P$ such that $\text{disp}_{(X,\mathcal{R})}(P) \leq \varepsilon$. The concept of $\varepsilon$-nets was introduced by Haussler and Welzl in [19] in the context of geometric range query problems. These problems can be summarised as follows, given an $n$ point set $P$, preprocess it and build a suitable data structure structure such that for a given range $R$, you can quickly determine the number of points—or just whether or not it contains a point—of $P$ laying in $R$. The ranges $R$ can be thought of as queries. In practice, the range space is some collection of geometric objects such as half spaces, boxes, or simplexes. The typical problem in the field is, given a range space $(X, \mathcal{R})$, what is the minimal size of an $\varepsilon$-net, in terms of $\varepsilon$, that can be found for it? It turns out that the size of a minimal $\varepsilon$-net for a given range space is related to its Vapnik-Chervonenkis(VC) dimension. Let $(X, \mathcal{R})$ be a range space. A finite set $S \subset X$ is shattered by $\mathcal{R}$ if $|\{S \cap R | R \in \mathcal{R}\}| = 2^{|S|}$. That is, $S \cap \mathcal{R}$ contains all subsets of $S$. The VC-dimension is defined as

$$\text{VCdim}(X, \mathcal{R}) := \sup\{|S| | S \text{ is shattered by } \mathcal{R}\}.$$

If a range space has VC dimension $d$, then with high probability a randomly and uniformly selected set of $O\left(\frac{d}{\varepsilon} \log \frac{d}{\varepsilon}\right)$ points from $X$ will be an $\varepsilon$-net. It was shown that this is a lower bound. The question is then whether the same is true for other range spaces. It is conjectured in the conclusion of [27] that this is the case, which, in particular, would give a negative answer to Danzer's problem. In such problems, the dimension is usually fixed, that is, the size of an $\varepsilon$-net is only examined in terms of $\varepsilon$. However, it also makes sense to examine the dependence on dimension.

If the ambient space $X$ is taken to be the unit cube in $\mathbb{R}^d$ and $\mathcal{R}$ is taken to be the set of axis aligned boxes in the cube, then one arrives at the notion of minimal dispersion. More precisely, the minimal dispersion is defined as

$$\operatorname{disp}^*(n,d) := \inf_{|P|=n} \operatorname{disp}_{(X,\mathcal{R})}(P).$$

Some of the techniques used to study it are similar to the ones used for finding dense forests and Danzer sets. Indeed, a typical approach for bounding the dispersion of a set is to first approximate the set of all axis aligned boxes of some volume in the unit cube by a finite set of slightly smaller boxes, such that every large box contains one of the smaller boxes. From there, it is enough to examine whether a given point set hits every one of the smaller boxes. Typically a random set is used and it is shown that one of sufficiently large size hits all the the boxes with high probability. So, improvements to the bounds on minimal dispersion often come from improvements in constructing the set of text boxes. However, finding the exact minimal dispersion, even for very few points and in low dimension, is quite difficult; the minimal dispersion of five points in $\mathbb{R}^2$ is not known. An overview of results in the field is given in [23]. If the dimension dependence is ignored then the problem is solved, there exists an optimal $\varepsilon$-net of size $O(\varepsilon^{-1})$[29].

If the ambient space $X$ is a again the unit cube and $\mathcal{R}$ is all convex sets in it—equivalently, boxes, ellipses, simplices, etc.—then one arrives at the Danzer-Rogers problem.

A question called—for reasons unknown to the author—the dual Danzer problem has also been proposed([11] page 285).

*Question* 2.0.3 (Dual Danzer problem). For a fixed integer $M \geq 2$, must a set of points $X \subset \mathbb{R}^d$ such that $\#(K \cap X) \leq M$ for any ellipse in the plane, or more generally, in $\mathbb{R}^d$, of volume 1 have density zero?

When $M \leq d - 1$, the question can be easily answered—although in a way that suggests that the question should be stated more precisely and require that $M \geq d$. If $M \leq d - 1$, then a set $X$ that meets the assumptions of the question can have at most $d$ points. To see this, note that the convex hull of $d$ points is a $d - 1$ dimensional object. So, if $X$ has more than $d - 1$ points, then the convex hull of $d$ of them can be 'fattened' to give a convex set of volume 1 containing $M + 1$ points of $X$.

Another question closely related to Danzer's problem, but asked independently of it, is called Gower's problem.

*Question* 2.0.4 (Gower's problem). Does there exist a set $S \subset \mathbb{R}^d$ with $d \geq 2$ such that for every convex set $K$ of volume 1, $1 \leq \#(K \cap S) \leq c$ for some constant $c$?

Put another way, does there exist a Danzer set such that the number of points in each convex set of volume 1 is bounded? The question was answered in the negative by Solan, Solomon, and Weiss in [30]. The result is curious enough on its own and takes great advantage of the affine invariance of the problem. In point of fact, they show more. They show that if there exists a Danzer set $D$, then for every $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists a convex set $E_n$ with $\lambda(E_n) \leq \varepsilon$ and $\#(E_n \cap D) \geq n$. So there are convex sets with arbitrarily small volume that contain arbitrarily many points of a Danzer set. Two proofs are supplied, a dynamical one involving group actions and a direct one. The second more clearly conveys the geometry involved and is the one that will be described here. The proof is by induction on $n$. The base case is a direct consequence of the

Danzer property. Assume that a convex set of volume $< \varepsilon$ with $n$ points in it exists. Due to the affine invariance of the Danzer set, this set can be assumed to be a ball centred at the origin. A ball beside, but disjoint from, it of sufficiently large radius is guaranteed to contain another point of the Danzer set. Then the ball centred at the origin can be deformed into a convex set which includes the new point without increasing its volume past $\varepsilon$, giving the desired result.

In [31], Solomon and Weiss provided a wealth of results on dense forests and Danzer sets. There are two negative ones concerning families of sets which cannot be Danzer sets, a construction of a dense forest, although without explicit visibility bounds, the equivalence of the Danzer Rogers problem with the Danzer problem and a construction of a set satisfying the Danzer property with the lowest known growth. In [10], Brambah and Woods showed that finite unions of grids cannot be Danzer sets. It is natural to ask whether taking the vertices of some other tiling can work. A tile is a set which is the closure of its interior. A tiling of $\mathbb{R}^d$ is a set of tiles that only intersect at their boundary and cover $\mathbb{R}^d$. Given a finite collection of tiles $\{T_1, \ldots, T_n\}$, a substitution is a map, or rule, that assigns to each $T_i$ a tiling by scaled down copies of $T_1, \ldots, T_n$. A substitution tiling is one that is obtained by repeatedly applying a substitution and scaling the tiles up. Such a tiling is called primitive if, essentially, it cannot be replicated using a smaller subset of tiles. They show that a primitive substitution tiling, due to the similarity at different scales caused by the substitutions, it is possible to find an arbitrarily long strip that runs near the edge of one of the tiles. That is, the vertices of such a tiling cannot form a Danzer set. By a very similar argument, they show that arbitrarily large empty strips can be found no matter where in the tile a point is picked from, as long as the choice is consistent. While the finite union of grids is not a Danzer set, it remains an open question whether a finite union of sets coming from arbitrary tilings will work.

Another category of discrete sets studied in the paper are cut and project sets. A cut and project set is defined as follows. Let $d > 1$ and $k \geq 1$ be integers and let $n = d + k$. Consider $\mathbb{R}^n$ as the direct sum of $\mathbb{R}^d$ and $\mathbb{R}^k$. Let $\pi_d$ and $\pi_k$ be the projections from $\mathbb{R}^n$ onto $\mathbb{R}^d$ and $\mathbb{R}^k$ respectively. Let $L$ be a grid in $\mathbb{R}^n$ and $W \subset \mathbb{R}^k$ a bounded subset. A cut and project set $\Lambda(L, W)$ is defined as

$$\Lambda(L, W) := \pi_d \left( L \cap \pi_k^{-1}(W) \right).$$

They show that finite unions of cut and project sets cannot be Danzer sets. The construction given of the set satisfying the Danzer property with the slowest known growth relies on the equivalence of the problem with the Danzer Rogers problem.

*Theorem* 2.0.1 (Solomon and Weiss, [31]). [1] For a fixed $d \geq 2$, and a doubling function $g(x)$ that satisfies $\frac{g(x)}{x^d}$ is non decreasing, then the following are equivalent.

i) There exists a set $D \subset \mathbb{R}^d$ satisfying Danzer's property with growth in $O(g(T))$.

ii) For every $\varepsilon > 0$ there exists an epsilon net for the range space of boxes in the unit cube $N_\varepsilon \subset [-\frac{1}{2}, \frac{1}{2}]^d$, such that $\#N_\varepsilon \in O(g(\varepsilon^{-1/d}))$.

One direction of the proof is quite simple. If $D$ is a set satisfying Danzer's property, then a large enough part of it can be scaled down to intersect all boxes of volume $\varepsilon$ in the unit cube. The other direction is harder, since it is not possible to scale up a single $\varepsilon$-net to cover all

1 The description of this result, and the rest of this paper should be redone somewhat.

of $\mathbb{R}^d$. Instead, a series of $\varepsilon$-nets for decreasing $\varepsilon$ are scaled up to form increasing layers around the origin. Any convex set of volume 1 in $\mathbb{R}^d$ will have enough of its volume contained in one of the layers that it will contain a point of the scaled up $\varepsilon$ net in it. There is an additional consequence to this equivalence. Namely, an $\varepsilon$-net for all boxes in the unit cube can be perturbed by an arbitrarily small amount to get an $\varepsilon$-net with rational coordinates. So, if there exists a Danzer set at all, then there exists a Danzer set contained in $\mathbb{Q}^d$. The construction of the set satisfying Danzer's property they give is a special case of a result in [19]. They use the equivalence with the Danzer-Rogers problem and construct an $\varepsilon$-net for boxes in the unit cube. To this end, they consider a fine grid on the cube and consider two boxes equivalent if they contain the same set of grid points. They calculate the VC dimension of the range space of these boxes which in turn gives a bound on the cardinality of the range space. Finally, they show that a random placement of points at the grid points, with some probability, gives the desired $\varepsilon$-net.

An important point ot note is that throughout, the measure that is used to talk about the volume is always the Lebesgue measure. It is natural to make the question even more complicated than it already is by examining Danzer's problem or the problem of dense forests in different measures. It was shown by Alon in [6] that for an arbitrary measure, Danzer's problem has a negative answer, at least in $\mathbb{R}^2$. He shows that a discrete probability distribution on the unit square can be contrived such that there is no $\varepsilon$-net for convex sets of size $O(\frac{1}{\varepsilon})$.

A further, extensive, overview of dense forests, Danzer sets, and related topics is given by Adiceam in [2].

All of the above mentioned problems measure, in some sense, how well 'spread out' a set of points is. The standard way this is measured is using discrepancy theory. The discrepancy of a finite sequence of points $x_1, \dot{,} x_N$ in the unit cube $C\mathbb{R}^d$ is

$$D_N = \sup_J \left| \frac{\#(\{x_1, \ldots, x_N\} \cap J)}{N} - \lambda(J) \right|$$

where $J$ is of the form

$$J = \{(x_1, \ldots, x_d) \in C | \alpha_i \leq x_i \leq \beta_i, 1 \leq i \leq d\}.$$

The discrepancy can also be considered for infinite sequences. Then $D_N$ measures the discrepancy of the initial segment of $N$ terms of the sequence. If the supremum above is taken over all convex subsets of $C$, then the resulting quantity is called the isotropic discrepancy.

Discrepancy measures, in a sense, how close a set of points is to being uniformly distributed. That is, how close the number of points in a box is to the volume of the box. In particular, having too many points in a box would result in a higher discrepancy. This is in contrast to the Danzer problem, which does not set any upper bound on the number of points in a convex set. In fact, as was shown in [30], there is no upper bound on the number of points of a Danzer set in a convex set. In particular, this means that the isotropic discrepancy of such a set is not bounded. See [22] for an overview of results in the field.

# 3

# RESULTS

## 3.1 DESCRIPTION OF RESULTS

The primary result of this thesis is the following theorem and the lemma that appears as part of its proof:

*Theorem* 3.1.1. Let $d \geq 2$. There exists a dense forest in $\mathbb{R}^d$ with visibility

$$V(\varepsilon) \in O\left(\varepsilon^{-(d-1)} \ln \varepsilon^{-1}\right).$$

This is the slowest growing visibility function known for a dense forest. The formulation of the dense forest problem in terms of boxes with $d-1$ sides equal is used for it. A Poisson point process lies at the base of the construction. The Poisson point process works well for this purpose since its density is finite and the probability that a box contains a point can be calculated based on only the volume of the box. Both are controlled by the parameter of the process. In fact, the probability of a box being empty depends on the error in the visibility function, since that is what determines the growth of the volume of the boxes. The exact probability is $e^{-\frac{\lambda}{(4\sqrt{d})^d}E(\varepsilon^{-1})}$. However, on its own, it is not a dense forest; it is possible to find arbitrarily large gaps in a Poisson point process. To correct this, additional points are manually added to the boxes that are left empty. Care needs to be taken when doing this as adding a point for every empty box would result in the density of the resulting point set no longer being finite.

The concept of modified random constructions is not new and similar ideas was used recently by Arman and Litvak in [8] for the problem of minimal dispersion. There are also further examples using this concept in [7]. Since the probability of a box being empty depends only on its volume, not on its shape, this approach to finding dense forests is more evocative of Danzer's problem then earlier ones which relied specifically on the boxes having one long direction. Indeed, the visibility in this forest is a result of the volume needing to grow sufficiently quickly as $\varepsilon \to 0$ in order to guarantee finite density. Once it is determined how many additional points are needed in relation to the parameter of the Poison process, the parameter can be adjusted to make sure that, in expectation, few additional points are placed. Indeed, the change in density can be made arbitrarily small, in expectation, by choosing a large enough parameter. In that sense, a Poisson point process is almost a dense forest.

While a Poisson point process has finite density, it is certainly not uniformly discrete, and it does not even have finite upper uniform density. One way to ameliorate this would be to use a different point process. However, that process would not lend itself to estimates as easily as the Poisson process. Another option is to modify a Poisson point process to ensure that none of its points are too close together. This can be accomplished by removing all points of the Poisson point process that have another point of the process within some small distance $\delta > 0$ of them. The issue with this approach is that it removes the complete independence enjoyed by the Poisson point process, which is crucial to the proof.

Despite this, the process still maintains some independence. Namely, the probability of there being a point of the process in some set $C$ is independent of anything that is distance at least, say $2\delta$ from $C$. With this observation, it is possible to get an upper bound of $e^{-\lambda c E(\varepsilon^{-1})}$, for a constant $c$ not dependent on $\varepsilon$, on the probability that a given visibility box is empty. In particular, by adjusting the parameter of the Poisson process, this probability can be made equal to the one for the regular unmodified Poisson point process, albeit, with a different parameter. From here, the same modification as for the regular Poisson process can be applied to obtain a dense forest with the same bound on visibility. However, the modification destroys the uniform discreteness of the forest, since there is no control over where, or how close together, the additional points are placed.

There are two typical ways that the requirements of a Danzer set are relaxed. The first is the approach taken in [10] and [31] which is to drop the finite density requirement and allow the density to grow. The other is to only consider boxes with $d-1$ sides equal and the length of the last side is allowed to grow, which is the problem of dense forests. A less restrictive variant of the second relaxation is now introduced[1].

*Definition* 3.1.1. A set of points $D \subset \mathbb{R}^d$ with finite density is called a **relaxed Danzer set** if for some non decreasing error function $E$ and any $0 < \varepsilon \leq 1$, $D$ intersects every box in $\mathbb{R}^d$ with shortest side of length $\varepsilon$ and volume $E(\varepsilon^{-1})$.[2]

The aim with finding relaxed Danzer sets is to minimise the error function. A relaxed Danzer set with $E(\varepsilon^{-1}) \in O(1)$ is a Danzer set. The earlier remark about the proof of theorem 3.1.1 relying mostly on the volume of the boxes, not on their shape, introduces the following theorem.

1 The author is not aware of it appearing previously in the literature.
2 The definition may need to be adjusted slightly to not be as general for the result to be valid.

*Theorem* 3.1.2. Let $d \geq 2$. There exists a relaxed Danzer set in $\mathbb{R}^d$ with error function $E(\varepsilon^{-1}) \in O(\log \varepsilon^{-1})$.

## 3.2 PROOF OF RESULTS

The forest will be constructed by first placing down points randomly and then adding points where the visibility exceeds the desired amount. In this case, the desired amount is $V(\varepsilon) = \varepsilon^{-(d-1)} E(\varepsilon^{-1})$, where $E(\varepsilon^{-1})$ will be assumed to be non decreasing and $\varepsilon^{-1} \geq E(\varepsilon^{-1}) \geq \varepsilon$ for $\varepsilon \leq \frac{1}{2}$ and will be chosen later. The number of additional points placed will be small enough such that the resulting set will have finite density.

The following lemma will be used to help count regions where the visibility is too large. A box will refer to a product of intervals.

*Lemma* 3.2.1. Let $E$ be as above. For any unit cube $C \subset \mathbb{R}^d$ and any $0 < \varepsilon \leq \frac{1}{2}$, there exists a finite set of boxes with $d-1$ sides of length $\frac{\varepsilon}{4\sqrt{d}}$ and one side of length $\frac{1}{4\sqrt{d}} \varepsilon^{-(d-1)} E(\varepsilon^{-1})$ of cardinality

$$\left(\frac{4\sqrt{d}}{\varepsilon}\right)^d \left(\frac{\pi}{2\varepsilon^{d+1}}\right)^{d-1}$$

such that any box with $d-1$ sides of length $2\varepsilon$ and one side of length $\varepsilon^{-(d-1)} E(\varepsilon^{-1})$ with centre in $C$ entirely contains one of the smaller boxes.

*Proof of Lemma 3.2.1.* The boxes with the smaller size described above will be called test boxes. Let $C$ be a unit cube in $\mathbb{R}^d$. The cube may be assumed to be axis aligned. Consider a $\frac{\varepsilon}{4\sqrt{d}}$ axis aligned grid of points on the cube. It will be used to position the test boxes in $C$. Multiple copies of the test boxes will be placed centred at each grid point to account for all possible rotations of the larger boxes.

For each grid point, an axis aligned test box with its long axis parallel to the $x_1$ axis is taken. Next, for each grid point, copies of the test box centred at it and rotated by all positive integer multiples of $\theta = 2\varepsilon^{d+1}$ such that $k\theta \leq \pi$ along all axis parallel axes, except the long axis, are added. There are

$$\left(\frac{4\sqrt{d}}{\varepsilon}\right)^d$$

grid points and

$$\left(\frac{\pi}{2\varepsilon^{d+1}}\right)^{d-1}$$

test boxes centred at each grid point.

It remains to show that each large box contains a test box. Any large box with its centre in $C$ contains a grid point, also in $C$, within distance at most $\frac{\varepsilon}{2}$ of its centre and of distance at least $\frac{\varepsilon}{2\sqrt{d}}$ from its boundary. Due to the $\frac{1}{\sqrt{d}}$ factor, it can be assumed that the large box is axis aligned. Call that grid point $G$. A copy of the large box scaled down to the size of a test box is placed centred at $G$. Let $P$ be the point on the border of the large box with minimal distance to $G$. There is a maximal angle, $\varphi$, by which the scaled down box may be rotated toward the face on which $P$ lies while remaining inside the large box. Let $I$ be the point of intersection of the long axis of the rotated scaled down box with the large box (see Figure 3.1).

Consider the right triangle formed by the points $G$, $P$, and $I$. The length of $GP$ is at least $\frac{\varepsilon}{2\sqrt{d}}$ and the length of $GI$ is at most $\frac{1}{4\sqrt{d}}\varepsilon^{-(d-1)}E(\varepsilon^{-1})$. The angle $\varphi$ is $GIP$. The angle $\theta$ is at most $\varphi$ since

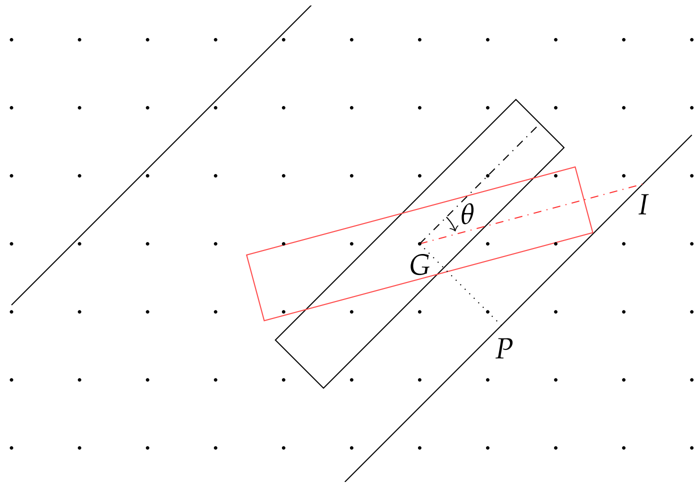


**Figure 3.1:** Derivation of $\theta$. A sketch of how the angle $\theta$ was derived. Two sides of the large box being approximated are shown with the long, solid, diagonal lines. A copy of it scaled down to the size of a test box and centred at a grid point is shown in black. The smaller box's maximum rotation is shown in red. The grid point $G$, intersection with the central axis $I$, and projection $P$ used to construct the triangle to estimate $\theta$ are also shown.

$$\begin{aligned}\varphi \geq \sin\varphi &\geq \frac{\frac{\varepsilon}{2\sqrt{d}}}{\frac{1}{4\sqrt{d}}\varepsilon^{-(d-1)}E(\varepsilon^{-1})}\\ &= \frac{2\varepsilon^d}{E(\varepsilon^{-1})}\\ &\geq 2\varepsilon^{d+1}\\ &= \theta.\end{aligned}$$

So, the scaled down box may be rotated by an angle of $\theta$ about the $d-1$ short axes and by any angle about its long axis while remaining inside the large box. □

*Proof of Theorem 3.1.1.* To begin, take a point set $F \subset \mathbb{R}^d$ obtained from a Poisson point process on $\mathbb{R}^d$ with density $\lambda$, where $\lambda$ depends on the dimension and will be chosen later. The visibility in $F$ will be unbounded, since the Poisson process on $\mathbb{R}^d$ will produce arbitrarily large gaps between points. The rest of the proof is concerned with adding points to $F$ to remove those gaps while maintaining finite density. This is done by discretising over scales of $\varepsilon$ and, for each scale, adding points to reduce the visibility at that scale to at most $V(\varepsilon)$. The sequence $\varepsilon_k = 2^{-k}, k \geq 2$ will be used for the scales.

For a given $\varepsilon$, the probability that the visibility in $F$ from some point $x \in \mathbb{R}^d$ in some direction $v \in S^{d-1}$ is greater than $V(\varepsilon)$ is the probability that there are no points in $F$ within distance $\varepsilon$ of the line segment $\{x + tv : 0 \leq t \leq V(\varepsilon)\}$. By John's theorem, up to a dimension dependent constant, this is equal to the probability of there being an empty box with $d-1$ sides of length $2\varepsilon$ and one side of length $V(\varepsilon)$ with its base centred at $x$ and extending in direction $v$. To limit the visibility in $F$, a point must be added to all such empty boxes. This is accomplished by constructing a set of test boxes for each scale of $\varepsilon$. The test boxes are chosen so that each of the larger boxes contains a test box. That way, a point is placed in the centre of each empty test box ensures that none of the original boxes are empty.

Since the density of the Poisson process is fixed, it is enough to restrict the calculations to a unit cube. At scale $k$, the boxes that will be examined are those with $d-1$ sides of length $2\varepsilon_k$ and one of length $V(\varepsilon_k)$ with their centres in an arbitrary fixed unit cube $C$. By lemma 3.2.1, there is a set of test boxes with $d-1$ sides of length $\frac{\varepsilon}{4\sqrt{d}}$ and one side of length $\frac{1}{4\sqrt{d}}\varepsilon_k^{-(d-1)}E(\varepsilon_k^{-1})$ of cardinality

$$\left(\frac{4\sqrt{d}}{\varepsilon_k}\right)^d \left(\frac{\pi}{2\varepsilon_k^{d+1}}\right)^{d-1}$$

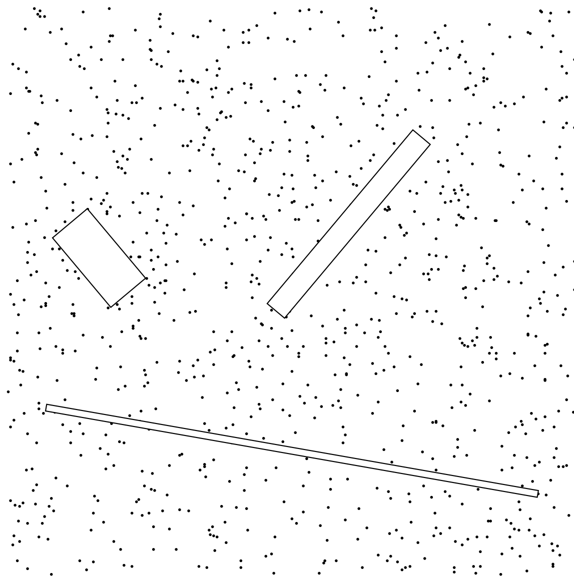

**Figure 3.2:** Visibility is too large. The sketch depicts a random placement of points. The highlighted boxes show some of the places where the visibility is too large for various values of $\varepsilon$. As $\varepsilon \to 0$, the boxes, in all dimensions, begin to resemble lines. Each such box needs a point added inside of it for the set of points to be a dense forest.

such that each of the large boxes with their centre in $C$ entirely contains one of the test boxes. The probability of such a test box being empty is

$$e^{-\frac{\lambda}{(4\sqrt{d})^d} E(\varepsilon_k^{-1})}.$$

So, the expected number of points added at scale $k$ is

$$N_k = \left(\frac{4\sqrt{d}}{\varepsilon_k}\right)^d \left(\frac{\pi}{2\varepsilon_k^{d+1}}\right)^{d-1} e^{-\frac{\lambda}{(4\sqrt{d})^d} E(\varepsilon_k^{-1})}.$$

It should be noted that the points added for $\varepsilon_k$ will also reduce the visibility for all $\varepsilon$ with $\varepsilon_k \leq \varepsilon < \varepsilon_{k-1}$.

Summing over all the scales, the total amount—in expectation—of points that were added is

$$\sum_{k=2}^{\infty} N_k = \sum_{k=2}^{\infty} \left(\frac{4\sqrt{d}}{2^{-k}}\right)^d \left(\frac{\pi}{2(2^{-k})^{d+1}}\right)^{d-1} e^{-\frac{\lambda}{(4\sqrt{d})^d} E(2^k)}$$
$$\lesssim \sum_{k=2}^{\infty} 2^{dk+(d+1)(d-1)k} e^{-\frac{\lambda}{(4\sqrt{d})^d} E(2^k)}$$
$$\leq \sum_{k=2}^{\infty} 2^{2(d+1)^2 k} e^{-\frac{\lambda}{(4\sqrt{d})^d} E(2^k)}.$$

Let $E(\varepsilon) = \ln \varepsilon$. The two main terms in the above sum are exponentials of the same order. For sufficiently large, depending on the dimension, $\lambda$, the negative exponent is much larger than the positive one and the sum converges. Therefore, the expected number of points added to each unit cube is finite and the density remains finite.

□

The set of test boxes in 3.2.1 was constructed rather wastefully and no attempt was made to optimise its size. However, that does not have any effect on the final result apart from the dimension dependent constants, which are mostly ignored anyway. First, note that any face of the unit cube in $\mathbb{R}^d$ may be tiled by $2^{k(d-1)}$ cubes in $\mathbb{R}^{d-1}$ of side length $2^{-k}$. These cubes can be viewed as the cross section of disjoint large boxes having their centres inside the unit cube. So, there are at least $2^{k(d-1)}$ large boxes with disjoint interiors being examined. Therefore, in order to have a test box in each large box, the number of test boxes needed at scale $\varepsilon_k$ is at least $2^{k(d-1)}$. This shows that the bounds on visibility cannot be improved by improving the approximation in Lemma 3.2.1.

The next lemma shows that the Poisson point process used above can be modified into a uniformly discrete set without affecting the probability fo a text box being empty too much. A construction very similar to the one of theorem 3.1.1 can then be applied to the modified Poisson point process to obtain a dense forest with the same visibility bound.

Let $F_0$ be the set of points obtained from a Poisson process with density $\lambda$ where $\lambda$ depends on dimension and will be chosen later. Let $\delta$ be a constant which is much smaller than $\lambda$ and will also be chosen later. Let $F_0^\delta$ be the set of points in $F_0$ which do not have any other point of $F_0$ within distance $\delta$ of them. That is,

$$F_0^\delta = \{x \in F_0 || x - y| \geq \delta \text{ for all } y \in F_0, x \neq y\}.$$

The quantities $\varepsilon_k = 2^{-k}$ with $k \geq 0$ will again be used for the scales of $\varepsilon$. Let $\varepsilon_\delta = 2\delta$ and $k_0$ be the smallest integer such that $\varepsilon_{k_0} < \varepsilon_\delta$.

At scale $k$, with $k$ such that $\varepsilon_k < \varepsilon_\delta$, consider boxes with $d-1$ sides of length $2\varepsilon_k$ and one side of length $\varepsilon_k^{-(d-1)} E(\varepsilon_k^{-1})$.

*Lemma* 3.2.2. For any $\lambda > 0$, there exists $\delta > 0$ such that for any $\varepsilon < 2\delta$, the probability that a box with $d-1$ sides of length $2\varepsilon$ and one side of length $\varepsilon^{-(d-1)} E(\varepsilon^{-1})$ does not contain a point of $F_0^\delta$, constructed as above, is at most

$$e^{-\lambda c E(\varepsilon^{-1})}$$

where $c > 0$ is an absolute constant.

*Proof of Lemma 3.2.2.* Let $0 < \varepsilon < 2\delta$. Consider a box with $d-1$ sides of length $2\varepsilon$ and one side of length $\varepsilon^{-(d-1)} E(\varepsilon^{-1})$, $B$. Divide it crosswise into alternating sub boxes, independent boxes having $d-1$ sides of length $2\varepsilon$ and one side of length $\delta$, and separating boxes having $d-1$ sides of length $2\varepsilon$ and one side of length $2\delta$. The probability of having, or not having, points in one independent box is—as the name suggests—independent of

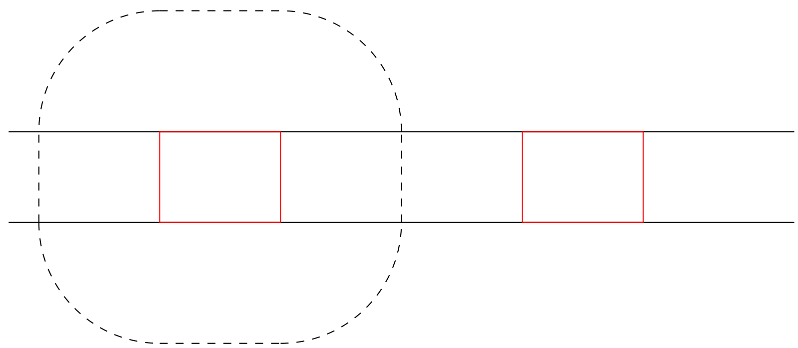

**Figure 3.3:** Dividing a box into independent regions. A sketch of a large box divided into independent and separating boxes. The events of there being—or not being—points in a red, independent, box is dependent only on points in the dashed shape.

the same events in any other independent box. The separating boxes serve to ensure that the independent boxes are far enough apart.

The probability that a given independent box $I$ in $B$ is not empty is at least the probability that it contains exactly one point of $F_0^\delta$. This is the probability that there is exactly one point of $F_0$ in $I$ and no other points of $F_0$ in the ball of radius $\delta$ around that point. Since $F_0$ was obtained from a Poisson point process, this probability is at least

$$p := \lambda\delta\varepsilon^{d-1}e^{-\lambda\delta\varepsilon^{d-1}} \cdot e^{-\lambda\gamma_d\delta^d}$$

where $\gamma_d$ is the volume of the unit ball in $d$ dimensions. The probability that an independent box is empty is therefore at most $1-p$. Note that

$$\begin{aligned}
p &= \lambda\delta\varepsilon^{d-1}e^{-\lambda\delta\varepsilon^{d-1}}e^{-\lambda\gamma_d\delta^d} \\
&\geq \lambda\delta\varepsilon^{d-1}e^{-\delta^{-(d-1)}\varepsilon^{d-1}}e^{-\gamma_d} && (\text{Take } \delta \leq \lambda^{-\frac{1}{d}}) \\
&> \lambda\delta\varepsilon^{d-1}e^{-2^{-(d-1)}}e^{-\gamma_d} && (\varepsilon < 2\delta) \\
&= \frac{\lambda c_0}{\varepsilon^{-(d-1)}\delta^{-1}} && (c_0 := e^{-2^{-(d-1)}}e^{-\gamma_d})
\end{aligned}$$

The number of independent boxes in the large box is

$$n = \frac{\varepsilon^{-(d-1)} E(\varepsilon^{-1})}{3\delta}.$$

Since in the end $\delta$ is taken to be sufficiently small, it may also be chosen such that $n$ is an integer. So, the probability that $B$ is empty is at most

$$P := \prod_{i=1}^{n}(1-p) = (1-p)^n < \left[\left(1 - \frac{\lambda c_0}{\varepsilon^{-(d-1)}\delta^{-1}}\right)^{\varepsilon^{-(d-1)}\delta^{-1}}\right]^{\frac{E(\varepsilon^{-1})}{3}}.$$

Now, the approximation of $e^x$ by $\left(1 + \frac{x}{m}\right)^m$ with $m = \varepsilon^{-(d-1)}\delta^{-1}$ gives the final bound. For small enough $\delta$, there is an absolute positive constant $c$, such that the above is bound by

$$P \leq e^{-\lambda c E(\varepsilon^{-1})}.$$

□

This bound is similar to the one obtained for an unmodified Poisson process. Indeed, the probability that a box at scale $\varepsilon$ does not contain any points of an unmodified Poisson process with parameter $\lambda$ is exactly $e^{-\lambda E(\varepsilon^{-1})}$. Since $\delta$ is taken to be very small—as small as one wants, in fact—very few points end up being removed from the Poisson point process.

Replace the regular Poisson point process in theorem 3.1.1 with the point process above with visibility in $O\left(\varepsilon^{-(d-1)} \ln \varepsilon^{-1}\right)$ for boxes at scale $k \geq k_0$. It remains to deal with the empty boxes in $F_0^\delta$ at scale $k < k_0$. Note that by choice of $k_0$, every empty box at scale $k$ will contain a cube of side length $\varepsilon_\delta$ which is at least distance $\delta$ from any point in $F_0^\delta$. Let $\Lambda$ be a lattice with a small enough fundamental region such that every cube of side length $\varepsilon_\delta$ in $\mathbb{R}^d$ contains a point from $\Lambda$. Finally, let

$$F = F_0^\delta \cup \{x \in \Lambda | |x-y| > \delta \text{ for all } y \in F_0^\delta\}.$$

However, the resulting dense forest is not itself uniformly discrete because, as mentioned earlier, the points added to account for the boxes missed by the point process are added without regard for the already existing points.

*Proof of Theorem 3.1.2.* [3] The proof mimics the proof of theorem 3.1.1 and lemma 3.2.1. After a small enough set of test boxes is constructed, the probability argument from the end of the proof of theorem 3.1.1 gives the final result. To begin, take a point set $F \subset \mathbb{R}^d$ obtained from a Poisson point process on $\mathbb{R}^d$ with density $\lambda$, where $\lambda$ depends on the dimension and will be chosen later. Again, fix some unit cube and examine all the boxes with their centres in it.

The values $\varepsilon_k = 2^{-k}$ are used for the scales of $\varepsilon$. A box $B$ with shortest side $\varepsilon_k$ and volume $E(\varepsilon_k^{-1})$ is considered 'close' to a rotation and translation of a box from the following set

$$S_k := \Big\{[0,a_1] \times \cdots \times [0,a_d] | a_1 \times \cdots \times a_d = E(\varepsilon_k^{-1}),$$
$$\varepsilon_k = a_1 \le \cdots \le a_d \le \varepsilon_k^{-(d-1)} E(\varepsilon_k^{-1}),\ a_2, \ldots, a_{d-1} \in \{2^i | i \in \mathbb{Z}\}\Big\}.$$

in the sense that a test box for some translation and rotation of an element of $S_k$ will also work as a test box for $B$. Since the boxes under consideration here do not have the same 'shape' as they do in the problem

3 Needs more details

of dense forests, a much larger set of test boxes is needed. The test boxes are translations and rotations of elements of

$$T_k := \Bigg\{ [0,a_1] \times \cdots \times [0,a_d] | a_1 \times \cdots \times a_d = \frac{E(\varepsilon_k^{-1})}{(4\sqrt{d})^d}, \\ \frac{\varepsilon_k}{4\sqrt{d}} = a_1 \leq \cdots \leq a_d \leq \frac{\varepsilon_k^{-(d-1)} E(\varepsilon_k^{-1})}{4\sqrt{d}}, a_2, \ldots, a_{d-1} \in \left\{ \frac{2^i}{4\sqrt{d}} | i \in \mathbb{Z} \right\} \Bigg\}.$$

For each element $B$ of $T_k$—following lemma 3.2.1—a set of test boxes consisting of rotations and translations of elements of $T_k$ can be constructed of cardinality $\varepsilon_k^{-P_B(d)}$ where $P_b(d)$ is a polynomial in $d$. At scale $k$, the number of elements in $S_k$ is at most the number of ways to choose $d-2$ numbers from the $kd+1$ element set $\{-k, \ldots, k(d-1)\}$. So,

$$|S_k| \leq \binom{kd + (d-2)}{kd+1} \leq (kd)^{R(d)}$$

for some polynomial $R$. In total, at scale $k$, there are at most

$$(kd)^{Q(d)} \varepsilon_k^{-P_B(d)} \leq \varepsilon_k^{-Q(d)}$$

test boxes for some polynomial $Q$. Applying the union bound the the probability of needing to place an additional point in a test box and summing over all the scales, the expected number of points that were added is at most

$$\sum_{k=2}^{\infty} 2^{kQ(d)} e^{-\frac{\lambda}{(4\sqrt{d})^d} E(2^k)}.$$

Let $E(\varepsilon) = \ln \varepsilon$. The two main terms in the above sum are again exponentials of the same order. For sufficiently large, depending on the dimension, $\lambda$, the sum again converges. Therefore, the expected number of points added to each unit cube is finite and the density remains finite.

□

# 4

# CONCLUSION

[1] The final chapter of this thesis contains discussions of work that was not fully successful and some open problems.

Instead of adding points to the modified Poisson point process, taking inspiration from Alon, it would have been nice to show that the Poisson point process on its own can be a dense forest. Since the set of points is constructed differently, an approach different from the one in [6] needed to be taken to achieve a different bound. Despite both base sets being the uniformly discrete, there is less control on the locations of the points when using a Poisson point process. So, instead of examining all of $\mathbb{R}^d$ at once, cubes of side length $2^k$ are examined and the limit $k \to \infty$ can be taken. Under the vague topology, the placements of points for each box is closed and compact. So, if it can be shown that for any finite number of scales, the modified Poisson point process intersects all boxes at those scales, then the infinite intersection of the sets that work will be nonempty. While for each box, the Lovász local lemma shows that for a finite number of scales of $\varepsilon$, there is some modified Poisson point process which is a dense forest for those scales, this is only achieved by increasing the density of the process.

As a result, the final set does not have finite density. This is the because as the number of scales under consideration increases, the number of

1 WIP

intersections between boxes of different scales becomes far to great. The same issue that was pointed out by Alon in [6]. The problem is, in fact, worse than that. In a regular Poisson point process, disjoint boxes are independent. However, in the modified Poisson point process, boxes that are within $\delta$ of each other are now dependent. The question remains open whether a uniformly discrete forest with visibility in $O\left(\varepsilon^{-(d-1)} \ln \varepsilon^{-1}\right)$ exists.

Keeping in line with the guiding questions about dense forests, another question is whether a non probabilistic construction of a dense forest with visibility in $O(\varepsilon^{-(d-1)} \ln \varepsilon^{-1})$ exists.

While the work here improves the known bounds on visibility, it provides no guidance on the existence of a non-trivial lower bound for the visibility in a dense forest; the question of what is the lowest achievable visibility for a dense forest remains open.

There is no unified approach, or preferred method for approaching the study of dense forests or Danzer sets. The tools that each mathematician uses are those that they are familiar with, leading to sets with different advantages over each other. Often, constructions have appeared in other contexts previously and their special status was revealed later. Despite this, a common feature—and consequently, a common issue—of many constructions of dense forests is the separation of scales of $\varepsilon$. While it makes the proof easier—that is to say, possible—it is also plays a role in prescribing the visibility. There should be something to gain if all scales of $\varepsilon$ can be considered all at once.

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## COLOPHON

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