APPROXIMATE SOLUTIONS TO THE SUPERCONDUCTING INTERFACE MODEL

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ABSTRACT

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We examine various aspects of the *Superconducting Interface Model*

$$\begin{aligned} \Box\varphi + \frac{1}{\varepsilon^2}\left[\lambda_\varphi(\varphi^2-1)+\beta|\sigma|^2\right]\varphi &= 0, \qquad \text{in } (0,T)\times\mathbb{R}^n,\\ \Box\sigma + \frac{1}{\varepsilon^2}\left[\lambda_\sigma(|\sigma|^2-m_\sigma^2)+\beta\varphi^2\right]\sigma &= 0, \qquad \text{in } (0,T)\times\mathbb{R}^n, \end{aligned}$$

for $0<\varepsilon\ll 1$, for certain choices of positive parameters $(\lambda_\varphi,\lambda_\sigma,m_\sigma,\beta)$, and with (φ,σ) taking values in $\mathbb{R}\times\mathbb{C}$. This model was introduced by Kyle Thompson in [34], who considers equivariant solutions with $n=2$ and proves, modulo a spectral assumption, the existence of solutions such that $\varphi\approx+1$ or $\varphi\approx-1$ except in a transition layer of thickness $O(\varepsilon)$ around a hypersurface Γ, while σ is exponentially small except near Γ. In this context, σ is interpreted as carrying a superconducting current confined to the transition layer around Γ. Moreover, as a result of the equivariant assumption, Γ is required to be a surface traced by a circle whose radius varies in time. A main contribution of [34] is to derive the laws of motion, in the form of an ODE, according to which the geometry of Γ changes due to the flow of current within Γ, in the limit $\varepsilon\to 0$.

In this thesis, we extend these results in several ways. First, we carry out a careful analysis of the static 1-dimensional superconducting interface model, including a proof of the spectral condition assumed in [34] for a range of the model parameters. This completes the rigorous derivation of the laws of motion in the 2-dimensional equivariant case. Additionally, in arbitrary dimensions and without any symmetry assumptions, we provide a rigorous construction of approximate solutions of arbitrary order for the superconducting interface model, yielding a formal derivation of the laws of motion for a superconducting interface in full generality. Finally, we prove the well-posedness for smooth data of these laws of motion for a suitable choice of gauge.

CONTENTS

1
INTRODUCTION

The present work is devoted to the study of the Superconducting Interface Model introduced by Kyle Thompson in [34] under less restrictive assumptions than the ones adopted therein. This model may be understood as a reduction of the Superconducting String Model introduced by Edward Witten in 1985 [35], which describes the behaviour of cosmic strings carrying a superconducting current. The end goal of the thesis is twofold and can be understood as the analysis of two systems of second order PDEs of hyperbolic character. The first system of equations (see Equation 1.7) arises naturally from the Lagrangian formulation of superconducting interfaces (higher dimensional analogues of superconducting strings) inspired by Witten's counterpart for superconducting strings, while the second system of equations (see Equation 1.26) describes the dynamics of these interfaces and of the electric current concentrated on them.

1.1 BACKGROUND AND PROBLEM STATEMENT

In what follows, an arbitrary element of $\mathbb{R}^{1+n}$ ($1+n$ dimensional spacetime) is represented by $(t,x) = (t, x_1, x_2, ..., x_n)$. For conciseness, we write ∂_0 (or ∂_t) and $\partial_i = \partial_{x_i}$ for $i = 1, 2, ..., n$, to represent the partial derivatives with respect to the time variable and with respect to the i^{th} spatial variable, respectively. Also, unless otherwise specified, greek indices $\alpha, \beta, ...$ run from 0 to n and latin indices $i, j, ...$ run from 1 to n. Furthermore, we use $\eta = (\eta_{\alpha\beta})^n_{\alpha,\beta=0} = (\eta^{\alpha\beta})^n_{\alpha,\beta=0} := \mathrm{diag}(-1, 1, 1, 1..., 1)$ to denote the canonical matrix representation of the Minkowski metric and its inverse on $\mathbb{R}^{1+n}$, and denote its associated bilinear form defined for $(v, w) \in \mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$ by $\langle v, w\rangle_m := v^T \eta w$.

Witten's model describes the behaviour of cosmic strings carrying a superconducting current. It can be described via the following Lagrangian defined in terms of the two complex-valued fields $\varphi : \mathbb{R}^{1+3} \to \mathbb{C}$ (the

Higgs field) and $\sigma : \mathbb{R}^{1+3} \to \mathbb{C}$ (the current carrying field), the gauge fields $A_\varphi, A_\sigma : \mathbb{R}^{1+3} \to \mathbb{R}^4$ associated to φ and σ, respectively, and the parameters $\Pi = (\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in (0, \infty)^4$:

$$L^s_\Pi[\varphi, \sigma, A_\varphi, A_\sigma] := K(\varphi, \sigma, A_\varphi, A_\sigma) + W(\varphi, \sigma; \Pi) \tag{1.1}$$

with

$$\begin{aligned} K(\varphi, \sigma, A_\varphi, A_\sigma) &= \frac{1}{2}(D_{\varphi,\alpha}\varphi, D^\alpha_\varphi \varphi) + \frac{1}{2}(D_{\sigma,\alpha}\sigma, D^\alpha_\sigma \sigma) + \frac{1}{4}F_{\varphi,\alpha\beta}F^{\alpha\beta}_\varphi + \frac{1}{4}F_{\sigma,\alpha\beta}F^{\alpha\beta}_\sigma, \\ W(\varphi, \sigma; \Pi) &= \frac{\lambda_\varphi}{4}\left(|\varphi|^2 - 1\right)^2 + \frac{\lambda_\sigma}{4}\left(|\sigma|^2 - 2m_\sigma^2\right)|\sigma|^2 + \frac{\beta}{2}|\varphi|^2|\sigma|^2, \end{aligned} \tag{1.2}$$

where, for $\alpha, \beta = 0, 1, 2, 3$:

- $F_{\varphi,\alpha\beta} := \partial_\alpha A_{\varphi,\beta} - \partial_\beta A_{\varphi,\alpha}$ and $F_{\sigma,\alpha\beta} := \partial_\alpha A_{\sigma,\beta} - \partial_\beta A_{\sigma,\alpha}$.
- Indices are raised and lowered using the Minkowski metric η, so that e.g., $F^{\alpha\beta}_\varphi = \eta^{\alpha\gamma}\eta^{\beta\delta}F_{\varphi,\gamma\delta}$.
- $D_{\varphi,\alpha} := \partial_\alpha - iq_\varphi A_{\varphi,\alpha}$ and $D_{\sigma,\alpha} := \partial_\alpha - iq_\sigma A_{\sigma,\alpha}$, where the parameters $q_\varphi, q_\sigma \in \mathbb{R}$ are called the gauge couplings of φ and of σ, respectively.
- $(v, w) = \mathrm{Re}(v\bar{w}) = \mathrm{Re}(\bar{v}w) = v_R w_R + v_I w_I$, for all $v = (v_R, v_I), w = (w_R, w_I) \in \mathbb{C}$.

In this model, we interpret the gauge potential A_σ as generating an electromagnetic field and σ as carrying an electric current. We define

$$j(\sigma, A_\sigma) := (i\sigma, D_{\sigma,\alpha}\sigma)dx^\alpha = \text{ current 1-form associated to } (\sigma, A_\sigma). \tag{1.3}$$

Roughly speaking, the role of φ is to form an interface, within which (we will see) the current is mostly confined. In a moment we will discard A_σ, and then we will write simply $j(\sigma)$.

The Superconducting Interface Model is obtained from the Superconducting String Model, (1.1), by making two assumptions:

1. The fields φ and σ are decoupled from their respective gauge fields (i.e., $q_\varphi = q_\sigma = 0$). In particular, this implies that there is no interaction between the electromagnetic field induced by σ and any other electromagnetic fields in the ambient space. This condition is referred to as the *neutral interface case*, making reference to the analogous concept of neutral superconducting strings [26].

2. The Higgs field φ is assumed to be real-valued. This change results in there being an interface (co-dimension 1 submanifold) instead a string (co-dimension 2 submanifold). This can be interpreted as a result in the change of symmetry of φ from $U(1)$ to the discrete symmetry corresponding to changes in sign.

Applying the above assumptions to (1.1), we obtain the Lagrangian for the Superconducting Interface Model:

$$\begin{aligned} L_\Pi[\varphi,\sigma] &:= \frac{1}{2}\left[\partial^\alpha\varphi\,\partial_\alpha\varphi + (\partial^\alpha\sigma,\partial_\alpha\sigma)\right] + W(\varphi,\sigma;\Pi) \\ &= \frac{1}{2}\Big[-(\partial_t\varphi)^2 + |\nabla_x\varphi|^2 - (\partial_t\sigma,\partial_t\sigma) + |\nabla_x\sigma_R|^2 + |\nabla_x\sigma_I|^2\Big] + W(\varphi,\sigma;\Pi). \end{aligned} \tag{1.4}$$

We consider the 4-parameter family of action functionals associated to (1.4) with an arbitrary number of spatial dimensions $n \geq 2$:

$$\mathcal{A}_\Pi[\varphi,\sigma] := \int_{\mathbb{R}^{1+n}} \left\{\frac{1}{2}\left[\partial^\alpha\varphi\,\partial_\alpha\varphi + (\partial^\alpha\sigma,\partial_\alpha\sigma)\right] + W(\varphi,\sigma;\Pi)\right\} dx\,dt, \tag{1.5}$$

where α now runs from 0 to n and $dt\,dx$ represents the standard Lebesgue measure on $\mathbb{R}^{1+n}$. We also introduce a (small) scaling parameter $\varepsilon > 0$ and consider the ε-weighted Lagrangian version of (1.5):

$$\mathcal{A}_{\Pi,\varepsilon}[\varphi,\sigma] := \int_{\mathbb{R}^{1+n}} \left\{\frac{\varepsilon}{2}\left[\partial^\alpha\varphi\,\partial_\alpha\varphi + (\partial^\alpha\sigma,\partial_\alpha\sigma)\right] + \frac{1}{\varepsilon}W(\varphi,\sigma;\Pi)\right\} dx\,dt. \tag{1.6}$$

As we will see, the scaling parameter ε represents the thickness of the region around the interface of φ in which the current carrying field σ is confined.

From now on, we will focus solely on $\mathcal{A}_{\Pi,\varepsilon}$, and more precisely on its Euler-Lagrange equations, which can be compactly written as

$$\begin{aligned} \Box\varphi + \frac{1}{\varepsilon^2}\partial_\varphi W(\varphi,\sigma;\Pi) &= 0, \qquad \text{in } \mathbb{R}^{1+n}, \\ \Box\sigma + \frac{1}{\varepsilon^2}\partial_\sigma W(\varphi,\sigma;\Pi) &= 0, \qquad \text{in } \mathbb{R}^{1+n}, \end{aligned} \tag{1.7}$$

where $\Box := \partial_0^2 - \Delta_x$ denotes the wave operator (with $\Delta_x = \partial_1^2 + \partial_2^2 + ... + \partial_n^2$) and $\partial_\sigma W$ is the vector whose entries are the derivatives of W with respect to the real and imaginary parts of σ. More specifically,

$$\begin{pmatrix} \partial_\varphi W(\varphi,\sigma;\Pi) \\ \partial_\sigma W(\varphi,\sigma;\Pi) \end{pmatrix} = \begin{pmatrix} \left[\lambda_\varphi(\varphi^2-1) + \beta|\sigma|^2\right]\varphi \\ \left[\lambda_\sigma(|\sigma|^2 - m_\sigma^2) + \beta\varphi^2\right]\sigma \end{pmatrix}.$$

Identifying σ with (σ_R, σ_I), where σ_R and σ_I are the real and imaginary parts of σ, respectively, and writing $\Phi := (\varphi, \sigma_R, \sigma_I)$ and $D_\Phi W := \begin{pmatrix} \partial_\varphi W \\ \partial_\sigma W \end{pmatrix}$, (1.7) can be written concisely as

$$\Box\Phi + \frac{1}{\varepsilon^2} D_\Phi W(\Phi; \Pi) = 0, \qquad \text{in } \mathbb{R}^{1+n}. \tag{1.8}$$

It is clear that the constant functions $\Phi_\pm \equiv (\pm 1, 0)$ are solutions to (1.8). Motivated by the underlying physical model, we shall be interested in solutions $\Phi = (\varphi, \sigma)$ to (1.8) that exhibit a sharp transition from one of these stationary solutions to the other. Particularly, we will be interested in the scenario where (φ, σ) are smooth solutions defined over $(0, T) \times \mathbb{R}^n$ for some $T > 0$, and the domain can be partitioned as

$$(0, T) \times \mathbb{R}^n = \Omega^+ \cup \Gamma \cup \Omega^-,$$

where Γ is a timelike smooth hypersurface and Ω^- and Ω^+ are open and connected disjoint regions with respect to which φ and σ present the following behaviour:

$$\varphi \approx \mathbb{I} \qquad \text{and} \qquad \begin{array}{l} |\sigma| \text{ is strictly positive and concentrated} \\ \quad \text{in an } \varepsilon\text{-neighbourhood of } \Gamma, \end{array} \tag{1.9}$$

with

$$\mathbb{I}(t, x) := \begin{cases} +1, & (t, x) \in \Omega^- \\ -1, & (t, x) \in \Omega^+. \end{cases} \tag{1.10}$$

Additionally, we will restrict most of our discussion to a particular scaling regime for which there is an ε-independent interaction between Γ and the leading order phase of σ. To present this choice of scaling and illustrate what motivates it, it is instrumental to introduce coordinates adapted to Γ. In particular, let $\psi : (0, T) \times \mathbb{R}^{n-1} \to \Gamma \subset (0, T) \times \mathbb{R}^n$ be local coordinates for Γ, and ν be a smooth Minkowski unit normal vector field to Γ (i.e., $\langle \nu, \nu \rangle_m \equiv 1$ and $\langle \nu(p), \tau \rangle_m$ for all $\tau \in T_p\Gamma$ and all $p \in \Gamma$). Furthermore, assume the existence of a neighbourhood $\mathcal{N}$ of Γ containing $\mathrm{Im}(\psi)$ in which we can project uniquely each point $(t, x) \in \mathcal{N}$ onto Γ in the Minkowski sense, i.e.,

$$(t, x) = \Psi(y, z) := \psi(y) + z\nu(y) \qquad \text{for all } (t, x) \in \mathcal{N}. \tag{1.11}$$

We call (y, z) Fermi coordinates adapted to Γ (see Section 3.1 for more details). Now, consider the case where Γ is a flat timelike hypersurface, say $\Gamma = \{x_n = 0\}$. In this case, writing $x = (\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we can simply set $\nu \equiv (0, ..., 0, 1) \in \mathbb{R}^{1+n}$, $y = (t, \bar{x})$ and $z = x_n$. We start by considering solutions of the form

$$\begin{pmatrix} \varphi(t,x) \\ \sigma(t,x) \end{pmatrix} = \begin{pmatrix} \varphi_0(z/\epsilon) \\ e^{i\alpha(y)/\varepsilon}\sigma_0(z/\epsilon) \end{pmatrix}, \tag{1.12}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}$ is a linear function and φ_0, σ_0 are real valued. Plugging in (1.12) into Equation 1.7 results in the following system of ODEs

$$\begin{aligned} &-\varphi_0'' + \partial_\varphi W(\varphi_0, \sigma_0; \Pi) = 0, && \text{in } \mathbb{R} \\ &-\sigma_0'' + \partial_\sigma W(\varphi_0, \sigma_0; \Pi) + p\sigma_0 = 0, && \text{in } \mathbb{R}, \end{aligned} \tag{1.13}$$

where $p := -(\partial_t \alpha)^2 + |D_{\bar{x}}\alpha|^2 = \langle \nabla_y \alpha, \nabla_y \alpha \rangle_m$ may be interpreted as the squared "Minkowski length" of the tangential gradient of α along Γ. It then follows that the presence of the phase α/ε in σ induces a virtual change in the parameters Π, in the sense that Equation 1.13 can be expressed as[1]

$$-\Phi_0'' + D_\Phi W(\Phi_0; \Pi_p) = 0 \quad \text{in } \mathbb{R}, \tag{1.15}$$

where $\Phi_0 = (\varphi_0, \sigma_0)$ and

$$\Pi_p := \left(\lambda_\varphi, \lambda_\sigma, \sqrt{m_\sigma^2 - \frac{p}{\lambda_\sigma}}, \beta\right), \text{ for some } p \in (-\infty, \lambda_\sigma m_\sigma^2). \tag{1.16}$$

Note that Equation 1.15 are the Euler-Lagrange equations of the functional $\mathcal{A}_\Pi^{1d}[\varphi, \sigma]$ for $\Pi = \Pi_p$ and σ real-valued, where

$$\mathcal{A}_\Pi^{1d}(\varphi, \sigma) := \int_{\mathbb{R}} \left\{ \frac{1}{2}\left[\varphi'^2(z) + |\sigma'|^2(z)\right] + W(\varphi(z), \sigma(z); \Pi) \right\} dz, \tag{1.17}$$

1 Indeed, a direct computation shows that

$$W(\Phi; \Pi_p) = W(\Phi; \Pi) + \frac{p}{2}|\sigma|^2, \qquad D_\Phi W(\Phi; \Pi_p) \quad = D_\Phi W(\Phi; \Pi) + p\mathcal{P}_\sigma(\Phi). \tag{1.14}$$

whose natural domain of definition is

$$\mathcal{H} := (\tanh, 0 + i0) + H^1(\mathbb{R}; \mathbb{R}) \times H^1(\mathbb{R}; \mathbb{C}). \tag{1.18}$$

In particular, any minimizer $(\varphi_{\Pi_p}, \sigma_{\Pi_p})$ of $\mathcal{A}^{1d}_{\Pi_p}$ over $\mathcal{H}$ satisfying (1.9) is a solution to (1.8) with the desired characteristics for the case $\Gamma = \{x_n = 0\}$ and $\langle \nabla_y \alpha, \nabla_y \alpha \rangle_m \equiv p$. For such a solution, the current 1-form defined in (1.3) becomes

$$j(\sigma) = \frac{1}{\varepsilon} \sigma^2_{\Pi_p} \left(\frac{z}{\varepsilon} \right) d\alpha.$$

which is tangent to the interface $\{z = 0\}$ and is concentrated in an ε-neighbourhood of the interface.

Next, consider the case where Γ is unknown and not necessarily flat. Motivated by the results corresponding to the case $\Gamma = \{z = 0\}$, we consider the ansatz

$$\begin{pmatrix} \varphi(t,x) \\ \sigma(t,x) \end{pmatrix} = \begin{pmatrix} \varphi_0(y, z/\varepsilon) \\ e^{i\theta(y)/\varepsilon}\, \sigma_0(y, z/\varepsilon) \end{pmatrix}, \qquad (t,x) = \Psi(y,z) \in \mathcal{N}, \tag{1.19}$$

where $(\varphi_0, \sigma_0) \in \mathcal{H}$ and θ is a smooth real-valued function. Plugging in the ansatz in (1.19) into the functional $\mathcal{A}_{\Pi,\varepsilon}$ and making a change of coordinates, we find that (see Section 3.2 for more details)

$$\mathcal{A}_{\Pi,\varepsilon}[\varphi, \sigma] = \underbrace{\int_\Gamma \mathcal{A}^{1d}_{\Pi_{\rho(y)}}[\varphi_0(y,\cdot), \sigma_0(y,\cdot)] \sqrt{-\det(g(y))}\, dy}_{\mathcal{A}_0[\varphi_0, \sigma_0, \Gamma, \theta_0]} + O(\varepsilon), \tag{1.20}$$

where g is the metric tensor induced by the Minkowski inner product $\langle \cdot, \cdot \rangle_m$ on Γ, and ρ is the squared "Minkowski length" of the tangential gradient of θ along Γ. More specifically, $g_{ab}(y) := \langle \partial_a \psi(y), \partial_b \psi(y) \rangle_m$ for $a, b = 0, ..., n-1$ and $\rho(y) := \langle \nabla_\Gamma \theta(y), \nabla_\Gamma \theta(y) \rangle_m$, where $\nabla_\Gamma \theta(y)$ is the tangential gradient of θ along Γ (see Definition 3.9). From this perspective, we are interested in configurations for which the ε-independent portion of $\mathcal{A}_{\Pi,\varepsilon}[\varphi, \sigma]$ (the first term in the right hand side of (1.20)) are critical. In other words,

$$(\varphi_0, \sigma_0, \theta_0, \Gamma) \text{ should be a critical point of } \mathcal{A}_0[\varphi_0, \sigma_0, \Gamma, \theta]. \tag{1.21}$$

One way to realize the heuristics in (1.21) is to chose φ_0 and σ_0 so that[2]

$$(\varphi_0(y,\cdot),\sigma_0(y,\cdot)) \quad \text{minimizes} \quad \mathcal{A}^{1d}_{\Pi_{\rho(y)}}, \qquad \text{for each } y\in\Gamma. \tag{1.22}$$

Provided that we can solve this minimization problem, the pair (φ_0,σ_0) depends on y only through the value of $\rho(y)$, which in turn depends solely on Γ and θ. Furthermore, if the pair $(\varphi_0(p,\cdot),\sigma_0(p,\cdot))$ is unique for each possible value of $p\in\mathrm{Im}(\rho)$, we may define the function

$$\mu_\Pi : \mathrm{Im}(\rho)\to(0,\infty), \qquad \mu_\Pi(p) := \mathcal{A}^{1d}_{\Pi_p}(\varphi_0(p,\cdot),\sigma_0(p,\cdot)), \tag{1.23}$$

so that $\mu_\Pi(p)$ is minimum value of the energy $\mathcal{A}^{1d}_{\Pi_p}$ over $\mathcal{H}$. Upon making this choice for (φ_0,σ_0), the functional $\mathcal{A}_0$ depends only on the pair (Γ,θ), and we are led to the following natural choice for (Γ,θ):

$$(\Gamma,\theta) \quad \text{is a critical point of} \quad \Sigma[\Gamma,\theta] := \int_\Gamma \mu_\Pi(\rho)\,d\lambda, \tag{1.24}$$

where $d\lambda$ represents integration against the area measure associated with $\langle\cdot,\cdot\rangle_m$. We stress the fact that the choices (1.22) and (1.24) are a result of the particular scaling presented in (1.19). In particular, $1/\varepsilon$ is the scaling on θ, phase of σ, which results in a non-trivial interaction between Γ and θ.

Motivated by the formal discussion above, we formulate the problem statement in terms of two objects of study:

1. The existence of admissible (Γ,θ) pairs according to (1.24) and the timelike condition on Γ.

2. The construction of solutions (φ,σ) to (1.7) which, given an admissible pair (Γ,θ), satisfy (1.9) are of the form

$$\Phi(t,x) = \begin{bmatrix} \varphi_0\left(y,z/\varepsilon\right) \\ e^{i\frac{\theta(y)}{\varepsilon}}\sigma_0\left(y,z/\varepsilon\right)\end{bmatrix} + O(\varepsilon), \quad (t,x)=\Psi(y,z)\in\mathcal{N}_0, \tag{1.25}$$

   inside a neighbourhood $\mathcal{N}_0\subset\mathcal{N}$ of Γ, where (φ_0,σ_0) are the functions defined via (1.22).

2 Hereafter, we use the abuse of notation $y\in\Gamma$ to mean that $\psi(y)\in\Gamma$ for convenience.

Description of Main Results

The main contributions of this thesis can be summarized as follows: 1) we provide a thorough analysis of (1.15), which represents the stationary one-dimensional problem corresponding to (1.8); 2) we provide affirmative results about the existence of admissible (Γ, θ) pairs according to the criterion in (1.24); and 3) we make substantial progress in proving the existence of the target solutions described in (1.25) by providing a construction of approximate solutions to (1.8) of arbitrary precision with the same characteristics.

In regards to the (1.15), our first main contribution is proving the existence of an open set of admissible parameters, $\mathcal{O} \subset (0, \infty)^4$, over which we can define a smooth map

$$\Phi : \mathcal{O} \times \mathbb{R} \to \mathbb{R} \times \mathbb{C},$$

such that, for each $\Pi \in \mathcal{O}$, $(\varphi, \sigma, 0) = \Phi(\Pi, \cdot)$ is the unique minimizer of $\mathcal{A}_{\Pi}^{1d}$ over $\mathcal{H}$ showcasing numerous desirable properties, including those mentioned in (1.9) with the magnitude of $(\varphi(z) - \operatorname{sign}(z), \sigma(z), 0)$ and of its derivatives decaying to 0 exponentially as $|z| \to \infty$ (see Theorem 2.3). Another important property of $\Phi(\Pi, \cdot)$ is its non-degeneracy (see Definition 2.2), a technical condition essential for the construction of dynamical approximate solutions of Equation 1.8. We note that these results provide the missing details for a full verification of the equivariant case where $n = 2$ studied in [34]. Additionally, we prove some useful properties of the map μ_{Π} from (1.23) (see Lemma 2.29), and address the invertibility of the operator $\mathcal{L}(\Phi; \Pi) : H^2(\mathbb{R}; \mathbb{R} \times \mathbb{C}) \to L^2(\mathbb{R}; \mathbb{R} \times \mathbb{C})$ given by

$$\mathcal{L}(\Phi; \Pi)[V] := -V'' + D_{\Phi}^2 W(\Phi(\Pi, \cdot); \Pi)V,$$

obtained by linearizing the operator in (1.15) about $\Phi(\Pi, \cdot)$ (see Theorem 2.4).

Next, we address the problem posed by the requirement in (1.24). As suggested by the heuristics from the previous section, the Euler-Lagrange equations corresponding to $\Sigma[\Gamma, \theta]$ represent the natural/correct laws of motion governing the dynamics of admissible interface/current pairs. These laws of motion can be succinctly expressed as the following second order quasilinear[3] system of PDEs:

$$\begin{cases} \vec{H}_\Gamma = -f_1(\rho)\vec{\mathrm{II}}(\nabla_\Gamma\theta, \nabla_\Gamma\theta) \\ \Box_\Gamma\theta = \frac{1}{2}f_2(\rho)\left\langle\nabla_\Gamma\rho, \nabla_\Gamma\theta\right\rangle_m , \end{cases} \tag{1.26}$$

where

- $\vec{H}_\Gamma$ is the mean curvature vector of Γ (see (3.55))
- $\vec{\mathrm{II}}$ is the second fundamental form of Γ (see (3.56))
- $\Box_\Gamma$ is the Laplace-Beltrami operator defined on Γ (see (3.26))
- $\nabla_\Gamma u$ is the tangential gradient of $u \in C^1(\Gamma)$ along Γ (see (3.28))
- $\rho = \left\langle\nabla_\Gamma\theta, \nabla_\Gamma\theta\right\rangle_m$, where $\left\langle u, v\right\rangle_m = u^i v^j \eta_{ij}$ is the Minkowski product of $u, v \in \mathbb{R}^{1+n}$.
- $f_1(\rho) = -2\frac{d}{d\rho}\log\left[\mu_\Pi(\rho)\right]$ and $f_2(\rho) = 2\frac{d}{d\rho}\log\left(\mu_\Pi'(\rho)\right)$, where μ_Π is given by (1.23).

The second main goal of this thesis is to establish the local well-posedness of the system in Equation 1.26 for arbitrary $n \in \mathbb{N}_{\geq 2}$ subject to suitable initial conditions on (Γ, θ) linked to the timelike character of Γ and the smallness of ρ. Our first result in this direction, Theorem 3.16, asserts that a (uniform) timelike condition on the submanifold $\Gamma_0 := \Gamma \cap \{(0, x) : x \in \mathbb{R}^n\}$ and a smallness condition on the initial value of ρ (see (3.73)) guarantee the existence of $T > 0$ (whose value depends on these initial conditions) for which Equation 1.26 admits solutions where Γ is the graph of a map $\gamma : [0, T] \times \mathbb{R} \to \mathbb{R}^{n-1}$.

Our second result, Theorem 3.22, provides a generalized version of Theorem 3.16 that applies to interfaces Γ of arbitrary codimension $k \in \mathbb{N}_{<n}$ under stricter conditions on the initial data (compare (3.73) to (3.92)). In particular, in addition to ρ being small, in this case the initial data are required to satisfy the following Lorentzian condition for some $\lambda \in (0, 1)$:

$$C_j^{ab} := g^{ab} + f_j(\rho)\partial^a\theta\partial^b\theta \in \mathcal{C}_{p,\lambda}, \quad j = 1, 2 \quad \text{and} \quad a, b = 0, ..., 1 + p, \tag{1.27}$$

3 The fact that this system is quasilinear is more easily seen by writing (1.26) in coordinates, see Lemma 3.12.

where $\left(g^{ab}\right)_{a,b=0}^{1+p}$ is the inverse of the metric on Γ induced by the Minkowski metric on $\mathbb{R}^{1+n}$ (see Section 3.1), $\partial^a\theta := g^{ab}\partial_b\theta$, and $\mathcal{C}_{p,\lambda}$ is the set $p \times p$ matrices with real entries which are canonical Lorentzian with bound λ (see Definition 3.21). Under the initial conditions on (Γ, θ) above, it is concluded that there exists $T > 0$ for which Equation 1.26 admits solutions where Γ is the graph of a map $\gamma : [0, T] \times \mathbb{R}^p \to \mathbb{R}^k$, where $p = n - k$.

In both Theorem 3.16 and Theorem 3.22, the solutions, expressed in terms of the coordinates ψ of the form (3.66) for Γ and the function θ, exhibit the following regularity[4] provided the initial data is of type H^s:

$$\theta \in C_b^r\left([0,T] \times \mathbb{R}^p; \mathbb{R}\right), \gamma \in C_b^r\left([0,T] \times \mathbb{R}^p; \mathbb{R}^{n-p}\right), \tag{1.28}$$

whenever $s \geq r + 1 + [p/2]$, with $[x] :=$ the integer part of $x \in \mathbb{R}$. The idea behind the proofs of these theorems consists in exploiting the structure of Equation 1.26, under the assumption that Γ is the graph of a function, to express them as a symmetric hyperbolic system of PDEs whose well-posedness is revealed thanks to the results from [19].

Our last result (see Theorem 4.2) is the construction of a sequence $(\Phi_k)_{k=0}^{\infty}$ of approximate solutions to (1.8) of the form (1.25), whose error of approximation is of order ε^k in the sense explained below. This construction is possible as long as the pair (Γ, θ) is admissible according to (1.24) and the condition (1.27) holds, the second condition being sufficient to ensure that the system (1.26) and its linearization about admissible (Γ, θ) pairs symmetric hyperbolic over Γ (see Definition 3.19 and Lemma D.1). From this perspective, the construction of approximate solutions gives an additional formal justification for the corresponding law of motion in (1.26) in the scaling regime (1.25) that we consider. In proving our results, we assume that (1.28) holds with $r = \infty$ for convenience, but analogous lower regularity results can be obtained with minor modifications to the proofs.

4 Let $r \in \mathbb{Z}_+$, $m, n \in \mathbb{N}$ and $U \subset \mathbb{R}^m$ be open. Here and in what follows, $C_b^r(U; \mathbb{R}^n)$ denotes the set of functions from U to $\mathbb{R}^n$ with bounded derivatives of order k for all $k \in \{0, 1, ..., r\}$. In a statement like $\theta \in C_b^r\left([0,T] \times \mathbb{R}^p; \mathbb{R}\right)$ we mean that θ can be extended to a function $\tilde{\theta} \in C_b^r(U; \mathbb{R})$, where $U \subset \mathbb{R}^{1+p}$ is open and $[0, T] \times \mathbb{R}^p \subset U$.

The functions Φ_k are approximate solutions to (1.8) in the following sense. First, let $\mathcal{N}$ be as in (1.11) and $\delta > 0$ be small enough so that $\mathcal{N}$ contains a tubular neighbourhood of radius δ around Γ, i.e.,

$$\mathcal{N}_\delta := \big\{(t,x) \in \mathbb{R}^{1+n} : (t,x) = p + z \cdot p \text{ for some } p \in \Gamma, z \in (-\delta, \delta)\big\} \subset \mathcal{N}.$$

Then, for any $k \geq 0$, we have that

$$\Box\Phi_k + \frac{1}{\varepsilon^2} D_\Phi W(\Phi_k; \Pi) = 0, \qquad \text{in } \left([0,T] \times \mathbb{R}^n\right) \setminus \mathcal{N}_{\delta/2}.$$

Near Γ, we have that for every indices $k, l \in \mathbb{Z}_+$, and every multi-index α, there exists constants $c > 0, C_{kl}, C_{kl\alpha} > 0$ depending on (Γ, θ, Π) but independent of ε, such that for all small enough $0 < \varepsilon \ll 1$ and all $(t,x) = \Psi(y,z) \in \mathcal{N}_{\delta/2}$:

$$\left|\partial_z^l \left[\Box\Phi_k + \frac{1}{\varepsilon^2} D_\Phi W(\Phi_k; \Pi)\right](t,x)\right| \leq C_{kl} \varepsilon^k (1 + |z/\varepsilon|^{k+2}) e^{-c(\Pi)\frac{|z|}{\varepsilon}} \tag{1.29}$$

and

$$\left|\partial_z^l D_y^\alpha \left[e^{-\frac{\theta}{\varepsilon}} \ltimes \left(\Box\Phi_k + \frac{1}{\varepsilon^2} D_\Phi W(\Phi_k; \Pi)\right)\right](t,x)\right| \\ \leq C_{kl\alpha} \varepsilon^k (1 + |z/\varepsilon|^{k+2}) e^{-c(\Pi)\frac{|z|}{\varepsilon}}, \tag{1.30}$$

where we have employed the notation

$$a \ltimes (b,c) := (b, ac) \qquad \text{for all } b \in \mathbb{R} \text{ and all } a, c \in \mathbb{C}.$$

The term $e^{-\frac{\theta}{\varepsilon}}$ premultiplying the expression for the error in the estimate (1.30) is required to compensate for the rapid oscillations associated with the θ/ε term in (1.25).

1.2 LITERATURE REVIEW

Most of the relevant literature to the present work pertains to the study of differential equations with solutions which have features (e.g., zero sets which are interfaces, point vortices, or vortex filaments) that can be described by means of an associated geometric problem. Below are some of the main themes in the literature that lie at the heart of this thesis.

In what follows, $0 < \varepsilon \ll 1$, and for $k = 1$ or $k = 2$, $W : \mathbb{R}^k \to \mathbb{R}_{\geq 0}$ denotes the function $W(u) := (|u|^2 - 1)^2/4$, which is the standard double-well potential for $k = 1$ (commonly referred to as the Allen-Cahn potential). Furthermore, let $\Omega \subset \mathbb{R}^n$ be an open bounded subset, where $n > k$.

1. Elliptic case: in the limit as $\varepsilon \to 0^+$, solutions to the equation

$$\Delta u(x) + \frac{1}{\varepsilon^2} W'(u(x)) = 0, \qquad x \in \Omega, \tag{1.31}$$

   which exhibit a phase transition do so around a surface of co-dimension k with zero Euclidean mean curvature (see e.g., [22, 33, 20, 25, 29, 28]).

2. Hyperbolic case: in the limit as $\varepsilon \to 0^+$ and for suitable initial data, solutions to the equation ((1.8) with $\sigma \equiv 0$)

$$\Box u(t,x) + \frac{1}{\varepsilon^2} W'(u(t,x)) = 0, \qquad (t,x) \in S_T, \tag{1.32}$$

   where $S_T = (0,T) \times \Omega$ or $(0,\infty) \times \Omega$, which exhibit a phase transition do so around a surface with co-dimension k with zero Minkowskian mean curvature (see e.g., [27, 16, 17]).

The scalar case (i.e., $k = 1$) is closest to our case of interest, in the sense that the phase transition of the unknown φ takes place close to a co-dimension 1 surface (i.e., a hypersurface). The case $k = 2$, on the other hand, is closer to the original model for superconducting cosmic strings proposed by Witten, in which φ exhibits vortex filaments corresponding to the cosmic strings themselves. A survey of relevant results in the hyperbolic setting for both cases is presented in [17], some of which we list below for the case $k = 1$.

In the case of $k = 1$, the general heuristic principle behind most, if not all, the cases above is that the solutions of interest are of the form

$$u \approx \tanh\left(\frac{z}{\sqrt{2}\varepsilon}\right), \tag{1.33}$$

where z is the signed-distance to a minimal hypersurface $\Gamma \subset \Omega$, just as in (1.11), where ν is a unit normal vector field to Γ in the Minkowski sense (resp. Euclidean sense) for the hyperbolic case (resp. elliptic case)[5].

5 In the elliptic case, $z : \Omega \to \mathbb{R}$ is characterized by the property that

$$z = 0 \text{ on } \Gamma, \qquad |\nabla z|^2 = 1 \qquad \text{near } \Gamma.$$

The fact that Γ is minimal can be understood, at least formally in the elliptic case, by noticing that (1.31) are the Euler-Lagrange equations of the functional

$$I_\varepsilon(u) := \int_\Omega \left[|\nabla u|^2 + \frac{1}{4\varepsilon^2} \left(|u|^2 - 1 \right)^2 \right] dx. \tag{1.34}$$

Heuristically (see e.g., [20], [32] for more details and precise statements), for u to be a local minimizer of I that exhibits a transition layer, u is expected to be approximately equal to ± 1 in most of Ω due to the cost associated with the potential term in I_ε. In this case, the energy corresponding to the term $|\nabla u|^2$ is expected to be concentrated around a hypersurface Γ (aka the interface of u), and therefore minimized if such hypersurface locally minimizes surface area. In the limiting case $\varepsilon \to 0$, it is expected that the thickness of the transition layer around Γ goes to 0 since the cost associated with the potential term in I_ε goes to ∞ in this case.

On the other hand, the specific form of u in (1.33) stems from the fact that the function $q(z) = \tanh(z/\sqrt{2})$ is the solution to the one dimensional analogue of (1.31) and (the static case of) (1.32), namely:

$$-q'' + W'(q) = 0, \qquad q(0) = 0, \text{ and } q(z) \to \pm 1 \text{ as } z \to \pm\infty. \tag{1.35}$$

In Fermi coordinates (y, z) adapted to Γ, one expects (formally) that $u(y, \cdot)$ showcases a similar behaviour to q in (1.35) for each $y \in \Gamma$, since u is required to have a transition around $\Gamma = \{z = 0\}$, in the same way that q has a transition around 0.

The above principle has been investigated in multiple settings. For instance, in the case of (1.31), the works of [32] and [20] rely on the idea of Γ-convergence to describe the limiting behaviour of the sequence of functionals I_ε appearing in (1.34), as well as the associated limiting geometrical problem describing the energy concentration in the elements of a sequence $(u_\varepsilon)_1^\infty \subset H^1(\Omega)$ of minimizers of these functionals.

In the hyperbolic case, z is a function defined on a neighborhood of $\Gamma \subset [0, T) \times \mathbb{R}^n$, characterized by

$$z = 0 \text{ on } \Gamma, \qquad -(\partial_t z)^2 + |\nabla_x z|^2 = 1 \qquad \text{near } \Gamma.$$

In regards to (1.32), in [5], the authors find travelling wave solutions for (1.32) for $k = 1$, each of which has its energy concentrated around a minimal surface in $\mathbb{R}^n$. They also show that, in the limit as $\varepsilon \to 0^+$ and under some technical assumptions, the energy density of solutions u_ε to (1.32) thought of as a measure on $\mathbb{R}^{1+n}$, concentrates around a co-dimension 1 set Γ as $\varepsilon \to 0$, where Γ is a timelike Minkowski minimal manifold whenever it is smooth. The conditions under which this last assertion holds, however, are not easily verified.

On the other hand, given a timelike minimal surface Γ which is smooth in a time interval $(-T, T)$, and well-prepared initial data close to the right hand side of (1.33), where z is the (Minkowski) signed distance to Γ, [16] establishes the existence of solutions to (1.32) whose energy is concentrated around Γ, at least up to time T. Further L^2 estimates are provided for the size of the deviation of these solutions to the right hand side of (1.35). More recently, in a collaboration by the same author in [10], these estimates are refined for the case $n = 2$.

In contrast, the existence of solutions of the form (1.33) is established in the elliptic setting in [25, 29, 28], and on the hyperbolic setting in [27] by linearizing (1.31) around an approximate solution, and using the spectral properties of a given operator to show that there is an exact solution which is ε-close to the approximate solution. Due to the nature of this approach, a very detailed description of the solutions is obtained. The general methodology followed in this works is often associated to the method of Lyapunov-Schmidt reduction (see e.g., [30]), and is identical in spirit to the general approach followed in [34] and in this thesis. We present a more detailed review of [27] and [34] below.

Finally, other two component systems, including the $k = 2$ case for (1.32) have been studied in [1, 2, 9, 8, 14], among others. We highlight the work of [14] in connection to the Abelian-Higgs model, due to the proximity between the general methodology followed in that work and the one followed in this thesis.

Most Relevant Literature

The most relevant literature on the subject consists of Kyle Thompson's doctoral thesis [34] and the work of del Pino, Jerrard and Musso in [27]. In [34], the system (1.8) is introduced and the problem of finding solutions satisfying (1.9) around a smooth timelike hypersurface Γ is considered. The main assumption made in this work is that

$$n = 2, \text{ so that } \Gamma \text{ is 2 dimensional. Also, } \Gamma \text{ is a surface of revolution about the } t \text{ axis.} \tag{1.36}$$

This assumption allows one to obtain solutions to (1.8) which are equivariant, in the sense that

$$\varphi(t,x) = u_\varepsilon(t,|x|), \qquad \text{and} \qquad \sigma(t,x) = e^{i\frac{1}{\epsilon}\arg(x)} v_\varepsilon(t,|x|),$$

for some functions $u_\varepsilon, v_\varepsilon : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\varepsilon \in \{1/k : k \in \mathbb{Z}\setminus\{0\}\}$. Furthermore, more detailed information is provided about the functions u_ε and v_ε in terms of Fermi coordinates adapted to Γ. In particular, one has that

$$\begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix}(y,z) = \text{Right hand side of } \textit{Equation } 1.25, \qquad (t,x) = \Psi(y,z) \in \mathcal{N}_\delta. \tag{1.37}$$

In contrast to our case, however, the leading order phase of σ is set to be equal to $\theta(y) = \arg(x)/\varepsilon$ upfront, and Γ is found by imposing the condition (1.26) given this choice of θ. In particular, this means that

1. θ is independent of time and linear on $\arg(x)$, which can be thought of as a variable parametrizing each time slice $\Gamma_t := \Gamma \cap \{(t,x) : x \in \mathbb{R}^2\}$.

2. Due to the radial symmetry of each time slice Γ_t, Γ can be parametrized as $(t, R(t)\cos(\alpha), R(t)\sin(\alpha))$ for $t \in [0,T]$ and $\alpha = \arg(x) \in [0, 2\pi)$. Consequently, solving (1.26) translates to solving an ordinary differential equation in terms of the time variable for the function R.

The existence of the solutions (φ, σ) described above is achieved in a four step process. Namely, the existence of choices for Γ that satisfy (1.36) and that solve the equation (1.26) is established appealing to results from classical ODE theory. Next, (1.8) is written as an expansion in terms of powers of epsilon, and it is shown that there exist “profiles” (φ_0, σ_0) with the characteristics outlined above which reduce the size of the left hand side of (1.8) (thought of as the error of approximation) by a factor of ε. Subsequently, a correction term of order ε is added to this profile to reduce the error by an extra factor of ε. Finally, calling the resulting approximation $\Phi(y,z)$, it is shown that for a suitable function h defined on Γ, the linearization of (1.8) about $\Phi_h(y,z) := \Phi(y, z - \varepsilon h(y))$ satisfies energy estimates which allows one to ultimately deduce that $\Phi(y,z)$ can be altered by a perturbation of order at most ε to attain an exact solution of (1.8), thereby proving the existence of the solutions with the desired properties.

On the other hand, [27] treats the particular instance of (1.8) where $\sigma \equiv 0$, namely:

$$\partial_t^2 \varphi - \Delta_x \varphi + \frac{1}{\varepsilon^2}(\varphi^2 - 1)\varphi = 0, \qquad \text{in } [0,T] \times \mathbb{R}^n. \tag{1.38}$$

The authors find solutions φ to Equation 1.38 which exhibit a transition layer close to a Minkowski minimal surface Γ which is timelike, smooth, and divides $[0,T] \times \mathbb{R}^n$ into two open and disjoint connected components Ω_- and Ω_+, with Ω_- being bounded. More precisely, using the same type of coordinates $(t,x) = \Psi(y,z)$ adapted to Γ as the ones described above (see also Definition 3.2), it is shown that (See Theorem 1 in [27]) for each $j \in \mathbb{N}$, there exists solutions φ_ε of (1.38) which satisfy[6]

$$\varphi_\varepsilon(t,x) = \tanh\left(\frac{z}{\sqrt{2}\varepsilon}\right) + O(\varepsilon), \qquad (t,x) = \Psi(y,z)$$

in a neighbourhood $\mathcal{N}_\delta$ of Γ, and such that $\varphi_\varepsilon(t,x) \to \mathbb{I}(t,x)$ in the C^j sense as $\varepsilon \to 0$ in compact subsets of $([0,T] \times \mathbb{R}^n) \backslash \mathcal{N}_\delta$, where $\mathbb{I}$ is as in (1.10).

In a nutshell, the procedure followed in [27] consists of three main steps. Firstly, an expansion in powers of epsilon of the left hand side of (1.38) (thought of as the error of approximation) in (y,z) coordinates is made, and it is found that setting $\varphi(y,z) = w(z) := \tanh\left(\frac{z}{\sqrt{2}\varepsilon}\right)$ reduces the size of the error by a factor of ε^2. To note is that the reduction of the error being of order ε^2 as opposed to a smaller power of ε is a consequence of the fact that Γ is a Minkowski minimal surface (cf. Step 1 of Section 4.2.1). Following this, it is shown via an inductive argument that for each $k \in \mathbb{N}$, there is an approximation ϕ_k with error of order at most ε^k. These approximations can be easily extended to the whole of $[0,T] \times \mathbb{R}^n$ to obtain a corresponding global approximation ϕ_k^* with error of order at most ε^k. Finally, it is shown via energy estimates associated to the linearization of (1.38) about ϕ_k^* that,

6 In fact, inside $\mathcal{N}_\delta$, $\varphi_\varepsilon(t,x) = \tanh(\frac{z}{\sqrt{2}\varepsilon}) + \phi_\varepsilon(t,x)$, with $\sum_{|\alpha|=1}^{j+1} |D_x^\alpha \phi_\varepsilon| + \sum_{|\alpha|=0}^{j} |D_x^\alpha \partial_t \phi_\varepsilon| \le C\varepsilon$.

as long as $k > 3n + 2$, ϕ_k^* can be perturbed into an exact solution of (1.38) by adding a perturbation of order at most $\varepsilon^{k/2}$.

As suggested in Section 1.1, our results share an intimate relation to the above two works; we further investigate and extend some of the results presented in [34], while shifting the focus to a strategy most closely related to that of [27]. In particular, we address the well-posedness of (1.26) in the absence of the symmetry assumption on Γ (alternatively, the equivariant assumption made on the form of (φ, σ)) and the assumption on the form of θ from [34]. Under these milder assumptions, we also broaden the analysis of the one dimensional static problem (1.15) presented in [34] and, in the same spirit as in [27], use these results to obtain approximate solutions of arbitrary degree of accuracy in the sense explained in Section 1.1. As noted throughout the thesis, some of the results presented here can also be found in [6] and in [34], albeit with a different presentation or argumentation in some cases.

1.3 MODEL ASSUMPTIONS AND NOTATION

Before proceeding, we state some assumptions on Γ and some important facts about the potential W which we will use throughout the thesis. First, for some $T > 0$, the interfaces Γ that we consider satisfy:

1. Γ is a timelike smooth hypersurface embedded in $[0, T] \times \mathbb{R}^n$.
2. Γ divides the space $[0, T] \times \mathbb{R}^n$ into two disjoint sets Ω_- and Ω_+.

Also, we always consider the potential $W : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ from (1.2), namely

$$W(\varphi, \sigma; \Pi) := \frac{\lambda_\varphi}{4}(\varphi^2 - 1)^2 + \frac{\lambda_\sigma}{4}(|\sigma|^4 - 2m_\sigma^2|\sigma|^2) + \frac{1}{2}\beta\varphi^2|\sigma|^2, \tag{1.39}$$

where $\Pi = (\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in (0, \infty)^4$.

Note that the potential (1.39) is a generalization of the Allen-Cahn potential $W_A(u) = \frac{1}{4}(1 - u^2)^2$. The key properties of W that we will abstract from W_A are provided by the following two lemmas whose proofs can be found in Appendix A. These lemmas apply to a special set of parameters called the set of ***regular parameters***, $\mathcal{O}^0$, given by

$$\mathcal{O}^0 := \left\{(\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in (0, \infty)^4 : \beta > \lambda_\sigma m_\sigma^2 \text{ and } \lambda_\varphi > \lambda_\sigma m_\sigma^4\right\}. \tag{1.40}$$

Lemma 1.1. *Let $W : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ be of the form (1.39). Then, $\Pi \in \mathcal{O}^0$ if and only if the following two conditions hold:*

$$D_\Phi^2 W(\pm 1, 0; \Pi) \text{ is positive definite.} \tag{1.41a}$$

$$W(\varphi, \sigma; \Pi) \geq 0, \text{ with equality only if } (|\varphi|, \sigma) = (1, 0). \tag{1.41b}$$

Lemma 1.2. *Let $W : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ be of the form (1.39). There exists $c \in C\left(\mathcal{O}^0; (0, \infty)\right)$ such that*

$$W(\varphi, \sigma; \Pi) \geq c(\Pi) \left[(\varphi^2 - 1)^2 + |\sigma|^2\right] \qquad \text{for all } (\varphi, \sigma) \in \mathbb{R} \times \mathbb{C}. \tag{1.42}$$

Some of our results apply to any smooth $W : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ of the form $W(\varphi, \sigma) = \tilde{W}(|\varphi|, |\sigma|)$ such that (1.41a), (1.41b), and (1.42) hold, where $\tilde{W}$ is always assumed to be even with respect to both arguments.

1.3.1 *Notation*

Throughout this thesis, we will write

$$\begin{aligned} \lambda_{\min}(\Pi) : &= \text{minimum eigenvalue of } D_\Phi^2 W((\pm 1, 0); \Pi) \\ &= \min\left(2\lambda_\varphi, \beta - \lambda_\sigma m_\sigma^2\right). \end{aligned} \tag{1.43}$$

We denote by $\eta = (\eta_{\alpha\beta})_{\alpha,\beta=0}^n = (\eta^{\alpha\beta})_{\alpha,\beta=0}^n := \operatorname{diag}(-1, 1, ..., 1)$ the canonical matrix representation of the Minkowski metric on $\mathbb{R}^{1+n}$, and write $\langle v, w \rangle_m := v^T \eta w$ for $(v, w) \in \mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$. Also, we denote by $\mathbb{R}_+$ the set of non-negative real numbers, and by $\mathbb{R}_{>x}$ (resp. $\mathbb{R}_{\geq x}$) all real numbers greater than (resp. or equal to) $x \in \mathbb{R}$. A similar notation is used with $< x$ and $\leq x$, and with $\mathbb{Z}$ and $\mathbb{N}$, the latter of which we identify with $\mathbb{Z}_{\geq 1}$.

We denote elements in the domain of a function $F : \mathbb{R} \times \mathbb{C} \times (0, \infty)^4 \to \mathbb{R}$ by (φ, σ, Π), where $\varphi \in \mathbb{R}$, $\sigma = \sigma_R + i\sigma_I \in \mathbb{C}$ and $\Pi = (\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in (0, \infty)^4$. Partial derivatives with respect to $\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta$ will be denoted by $\partial_{p_1}, \partial_{p_2}, \partial_{p_3}$, and ∂_{p_4}, respectively. Also, the gradient of F with respect to Π (resp. $\Phi = (\varphi, \sigma_R, \sigma_I) \in \mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}$) will be denoted as $D_\Pi F$ (resp. $D_\Phi F$), the Hessian by $D_\Pi^2 F$ (resp. $D_\Phi^2 F$), etc., and $D_\Pi^\alpha F := \partial_{p_1}^{\alpha_1} \partial_{p_2}^{\alpha_2} \partial_{p_3}^{\alpha_3} \partial_{p_4}^{\alpha_4} \Phi$ (resp. $D_\Phi^\beta F := \partial_\varphi^{\beta_1} \partial_{\sigma_R}^{\beta_2} \partial_{\sigma_I}^{\beta_3} F$) for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}_+^4$ (resp. $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{Z}_+^3$). Lastly, we write $\partial_\sigma F := \binom{\partial_{\sigma_R} F}{\partial_{\sigma_I} F}$.

On the other hand, we define

$$X_0 := L^2(\mathbb{R}) \qquad \text{and} \qquad X_k := H^k(\mathbb{R}), \qquad (k \in \mathbb{N}), \tag{1.44}$$

equipped with the norms

$$\|f\|_{X_k} := \sqrt{\sum_{j=0}^{k} \left\|f^{(j)}\right\|^2_{L^2(\mathbb{R})}}.$$

Also, we equip X_k^n with the norms $\|(f^1, f^2, ..., f^n)\|_{X_k^n} := \sum_1^n \left\|f^j\right\|_{X_k}$. We will often make the standard identification of X_0^3 (resp. H_k^3) with $L^2(\mathbb{R};\mathbb{R}^3)$ or $L^2(\mathbb{R};\mathbb{R}\times\mathbb{C})$ (resp. $H^k(\mathbb{R};\mathbb{R}^3)$ or $H^k(\mathbb{R};\mathbb{R}\times\mathbb{C})$).

The set of all bounded linear operators from the normed space $(X, \|\cdot\|_X)$ to the normed space $(Y, \|\cdot\|_Y)$ is denoted by $L(X,Y)$. As usual, $L(X,Y)$ is equipped with the standard operator norm defined on $\mathcal{L} \in L(X,Y)$ by

$$\|\mathcal{L}\|_{L(X,Y)} := \sup_{x\in X, \|x\|_X = 1} \|\mathcal{L}(x)\|_Y .$$

Also, we introduce the following sets corresponding to each $L \in L(X_2^3, X_0^3)$:

$$\begin{aligned} (\ker L)^\perp &:= \left\{ V \in X_0^3 : \int_{\mathbb{R}} V \cdot K = 0 \text{ for all } K \in \ker L \right\}, \\ \mathfrak{C}_k(L) &:= (\ker L)^\perp \cap X_k^3. \end{aligned} \tag{1.45}$$

In particular, $(\ker L)^\perp = \mathfrak{C}_0(L)$.

Additionally, for any $r \in \mathbb{Z}_+$, $m, n \in \mathbb{N}$ and $U \subset \mathbb{R}^m$ open, $C_b^r(U;\mathbb{R}^n)$ denotes the set of functions from U to $\mathbb{R}^n$ with bounded derivatives or order k for all $k \in \{0,1,...,r\}$. In a statement like $\theta \in C_b^r([0,T]\times\mathbb{R}^p;\mathbb{R})$ we mean that θ can be extended to a function $\tilde{\theta} \in C_b^r(U;\mathbb{R})$, where $U \subset \mathbb{R}^{1+p}$ is open and $[0,T]\times\mathbb{R}^p \subset U$.

On the other hand, the symbol $\mathcal{O}^0 \subset (0,\infty)^4$ always refers to the set defined in (1.40), and $I_\pm : \mathbb{R} \to \mathbb{R}\times\mathbb{C}$ denotes the function

$$I_\pm(z) := (\operatorname{sign}(z), 0) \in \mathbb{R}\times\mathbb{C} \text{ for each } z \in \mathbb{R},$$

where $\operatorname{sign} : \mathbb{R} \to \{-1,0,1\}$ is the standard sign function.

Finally, for any $\Phi = (\varphi, \sigma) \in \mathbb{R} \times \mathbb{C}$ and $\theta \in \mathbb{R}$, we define

$$\mathcal{P}_\sigma(\Phi) := (0, \sigma), \quad i\Phi := (\varphi, i\sigma) \quad \text{and} \quad e^{i\theta} \ltimes \Phi := (\varphi, e^{i\theta}\sigma), \tag{1.46}$$

and we often identify $\mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}$, so e.g., (1.46) is treated as

$$\mathcal{P}_\sigma(\Phi) := \begin{pmatrix} 0 \\ \sigma_R \\ \sigma_I \end{pmatrix}, i\Phi := \begin{pmatrix} \varphi \\ -\sigma_I \\ \sigma_R \end{pmatrix} \text{ and } e^{i\theta} \ltimes \Phi := \begin{pmatrix} \varphi \\ \sigma_R \cos(\theta) - \sigma_I \sin(\theta) \\ \sigma_R \sin(\theta) + \sigma_I \cos(\theta) \end{pmatrix}. \tag{1.47}$$

1.4 CHAPTER OVERVIEW AND ORGANIZATION OF THE THESIS

The rest of the thesis is organized as follows. Chapter 2 is devoted to the study of a particular family of solutions to (1.8) that serve as the fundamental building block for the construction of the leading order terms (i.e., the initial approximation) of the more general approximate solutions to (1.8) introduced in Chapter 4. These solutions, for which φ and σ are real-valued and depend on a single variable, as well as the properties of the linearized version of (1.8) about them, are described in detail. In Chapter 3, we investigate the coupling between the leading order phase of the current-carrying field, σ, and the interface, Γ, that needs to take place in order to improve the accuracy of the initial approximations. In particular, a heuristic argument justifying the role of (1.26) in this context is presented, relying on a "reduced action principle" closely resembling that utilized in [24] in relation to (1.32). Apart from the derivation of (1.26), the well-posedness of this system is established under the assumption that Γ is the graph of a function of timelike character, and some additional smallness conditions associated to the tangential gradient along Γ of the phase of σ (i.e., θ in Equation 1.25). In Chapter 4, we present a construction of approximate solutions to (1.8) which can be used to reduce the error to order $O(\varepsilon^k)$ for arbitrary $k \in \mathbb{N}$, and that fit the description of Φ in (1.25) for (Γ, θ) pairs of the type described in Chapter 3. Finally, Chapter 5 is devoted to conclusions and recommendations for future work.

2

1 DIMENSIONAL ANALYSIS

This chapter consists mostly of results from joint work with Mauro Bonafini, Giandomenico Orlandi, and Robert Jerrard and can also be found in [6]. Some of the results here can also be found in Kyle Thompson's thesis [34].

2.1 THE 1-DIMENSIONAL LAGRANGIAN

In this chapter, we consider solutions (φ, σ) of (1.7) of the form

$$\begin{pmatrix} \varphi(t,x) \\ \sigma(t,x) \end{pmatrix} = \begin{pmatrix} \varphi_0(z/\epsilon) \\ \sigma_0(z/\epsilon) \end{pmatrix}, \qquad z = x_n, \tag{2.1}$$

where $\Phi_0 = (\varphi_0, \sigma_0)$ belongs to the set

$$\begin{aligned} \mathcal{H} := \{(\varphi, \sigma) \in \dot{H}^1(\mathbb{R}) \times H^1(\mathbb{R};\mathbb{C}) :& \\ (\varphi^2 - 1) \in L^2(\mathbb{R}),\ \varphi(x) \to \pm 1 \text{ as } x \to \pm\infty\},& \end{aligned} \tag{2.2}$$

where $\dot{H}^1(\mathbb{R})$ is the set of functions with $L^2(\mathbb{R})$ weak derivative[1]. By the definition of the set $\mathcal{H}$, these special solutions, which we shall call ***one dimensional profiles***, are in the domain of (1.6) for $n = 1$ and are the desired solutions to (1.8) for the case where the interface Γ is the hyperplane $\{x_n = c\}$ for some $c \in \mathbb{R}$. We intend to use these special solutions as guiding models for building approximate solutions to (1.7) which exhibit the desired behaviour (1.9) with respect to interfaces Γ with more intricate geometries.

Plugging the ansatz (2.1) into (1.7), we obtain the following system of equations

$$\begin{aligned} &-\varphi_0'' + \partial_\varphi W(\varphi_0, \sigma_0; \Pi) = 0, \qquad \text{in } \mathbb{R} \\ &-\sigma_0'' + \partial_\sigma W(\varphi_0, \sigma_0; \Pi) = 0, \qquad \text{in } \mathbb{R} \\ &\lim_{z\to\pm\infty} (\varphi_0(z), \sigma_0(z)) = (\pm 1, 0). \end{aligned} \tag{2.3}$$

1 More simply, $\mathcal{H} = (\tanh, 0, 0) + H^1(\mathbb{R}; \mathbb{R} \times \mathbb{C})$ (see Lemma B.9).

The first two equations in (2.3) are the Euler Lagrange equations of the following Lagrangian defined over $\mathcal{H}$, which we shall refer to as ***the 1-dimensional Lagrangian*** corresponding to Π:

$$\mathcal{A}_{\Pi}^{1d}(\varphi,\sigma) := \int_{\mathbb{R}} \left\{ \frac{1}{2}\left[\varphi'^2(z) + |\sigma'|^2(z)\right] + W(\varphi(z),\sigma(z);\Pi) \right\} dz. \tag{2.4}$$

The main results of this chapter address the existence and basic properties of minimizers of $\mathcal{A}_{\Pi}^{1d}$ in $\mathcal{H}$ for all Π in either $\mathcal{O}^0$ or in an open subset $\mathcal{O} \subset \mathcal{O}^0$, where $\mathcal{O}^0$ is defined in (2.20). The properties that we consider are better expressed in terms of some operators related to $\mathcal{A}_{\Pi}^{1d}$ that we now define. Let $(\Phi,\Pi) \in \mathcal{H} \times \mathcal{O}^0$ and introduce the (quadratic) functional

$$\mathcal{Q}(\Phi;\Pi)[V] := \frac{d^2}{dh^2}\Big|_{h=0} \mathcal{A}_{\Pi}^{1d}(\Phi + hV) = \int_{\mathbb{R}} \left[\left|V'\right|^2 + V^T D_{\Phi}^2 W(\Phi;\Pi) V\right],$$

acting on functions $V \in H^1(\mathbb{R};\mathbb{R}\times\mathbb{C})$. Furthermore, let

$$\tilde{\mathcal{H}} := (\tanh,0,0) + X_2^3 \cong (\tanh,0,0) + H^2(\mathbb{R};\mathbb{R}\times\mathbb{C}),$$

and note that Equation 2.3 implies that[2] $\Phi \in \tilde{\mathcal{H}}$. Moreover, for any $\Psi \in X_2^3$:

$$\begin{aligned} \frac{d}{dh}\Big|_{h=0} \mathcal{A}_{\Pi}^{1d}(\Phi + h\Psi) &= \int_{\mathbb{R}} \mathcal{F}(\Phi;\Pi)\cdot\Psi, \\ \frac{d^2}{dh^2}\Big|_{h=0} \mathcal{A}_{\Pi}^{1d}(\Phi + h\Psi) &= \int_{\mathbb{R}} \mathcal{L}(\Phi;\Pi)[\Psi]\cdot\Psi, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} &\mathcal{F} : \tilde{\mathcal{H}} \times \mathcal{O}^0 \to X_0^3, && \mathcal{F}(V;\Pi) = -V'' + D_{\Phi}W(V;\Pi), \\ &\mathcal{L} : \tilde{\mathcal{H}} \times \mathcal{O}^0 \to L(X_2^3, X_0^3), && \mathcal{L}(V;\Pi)[\Psi] = -\Psi'' + D_{\Phi}^2 W(V;\Pi)\Psi. \end{aligned} \tag{2.6}$$

Thus, $\mathcal{F}(\Phi;\Pi) = 0$ are the Euler-Lagrange equations of $\mathcal{A}_{\Pi}^{1d}$ shown in (2.3), $\mathcal{L}(\Phi;\Pi)$ is the linear operator obtained by linearizing (2.3) about Φ, and since $\mathcal{Q}(\Phi;\Pi)[V] = \langle V, \mathcal{L}(\Phi;\Pi)V\rangle_{X_0^3}$, we refer to $\mathcal{Q}(\Phi;\Pi)$ as the quadratic form associated with $\mathcal{L}(\Phi;\Pi)$.

Once we address the question of existence of a minimizer of $\mathcal{A}_{\Pi}^{1d}$ in $\mathcal{H}$ for each $\Pi \in \mathcal{O}^0$, we prove that the operator $\mathcal{L}(\Phi;\Pi)$, where $\Phi \in \mathcal{H}$ is any such minimizer, exhibits a spectral gap in the following sense:

2 In fact, $\Phi \in (\tanh,0,0) + H^m(\mathbb{R};\mathbb{R}\times\mathbb{C})$ for any $m \in \mathbb{N}$ (see step 3 of Lemma 2.12).

Theorem 2.1 (H^1 Spectral Estimate)**.** *Let $\Pi \in \mathcal{O}^0$ and Φ be a minimizer of $\mathcal{A}^{1d}_{\Pi}$ in $\mathcal{H}$. Then, there exists $c = c(\Pi) > 0$ such that*

$$\mathcal{Q}(\Phi;\Pi)[V] > c(\Pi)\|V\|^2_{H^1(\mathbb{R};\mathbb{R}\times\mathbb{C})}, \qquad \textit{for all } V \in \mathfrak{C}_1\left(\mathcal{L}(\Phi;\Pi)\right). \tag{2.7}$$

A useful observation for proving Theorem 2.1 and some other properties of minimizers of $\mathcal{A}^{1d}_{\Pi}$ (e.g., their exponential decay established in Lemma 2.15) is that $\mathcal{Q}(\Phi;\Pi)$ can be viewed as a perturbation of the quadratic form

$$\begin{aligned}\mathcal{Q}^0\left(\Phi;\Pi\right)[V] &:= \int_{\mathbb{R}} \left[\left|V'\right|^2 + V^T D^2_{\Phi} W\left((1,0);\Pi\right) V\right] \\ &\geq \min\left(1, \lambda_{\min}(\Pi)\right) \|V\|^2_{H^1(\mathbb{R};\mathbb{R}\times\mathbb{C})}, \qquad \lambda_{\min}(\Pi) \text{ as in (1.43)}.\end{aligned}$$

Another crucial property of the minimizers that we will consider is a non-degeneracy condition that stems from the fact that the kernel of the map $\mathcal{L}(\Phi;\Pi)$ is at least two dimensional for any $\Phi \in \mathcal{H}$ which minimizes $\mathcal{A}^{1d}_{\Pi}$. Indeed, for any minimizer $\Phi = (\varphi, \sigma) \in \mathcal{H}$ of $\mathcal{A}^{1d}_{\Pi}$ we have that

$$\mathcal{F}(e^{i\theta} \ltimes \Phi(\cdot - h);\Pi) = 0, \qquad \text{for all } h, \theta \in \mathbb{R}, \tag{2.8}$$

which may be interpreted as a direct consequence of the translational and rotational symmetries exhibited by $\mathcal{A}^{1d}_{\Pi}$:

$$\mathcal{A}^{1d}_{\Pi}(\Phi) = \mathcal{A}^{1d}_{\Pi}(\Phi(\cdot - h)) = \mathcal{A}^{1d}_{\Pi}\left(e^{i\theta} \ltimes \Phi\right), \qquad \text{for all } h, \theta \in \mathbb{R}.$$

Using (2.8) we readily obtain

$$\begin{aligned}\frac{d}{dh}\Big|_{h=0} \mathcal{F}(\Phi(z-h);\Pi) &= \mathcal{L}\left(\Phi;\Pi\right)\left((\varphi', \sigma')\right) = 0 \\ \frac{d}{d\theta}\Big|_{\theta=0} \mathcal{F}(e^{i\theta} \ltimes \Phi(z);\Pi) &= \mathcal{L}\left(\Phi;\Pi\right)\left[(0, i\sigma)\right] = 0,\end{aligned}$$

and thus, in terms of the notation introduced in (1.46):

$$\Phi', \mathcal{P}_{\sigma}(i\Phi) \in \ker\left(\mathcal{L}\left(\Phi;\Pi\right)\right), \quad \text{for any miminizer } \Phi \text{ of } \mathcal{A}^{1d}_{\Pi} \text{ in } \mathcal{H}, \tag{2.9}$$

In view of the lower bound in the dimension of the kernel of $\mathcal{L}(\Phi;\Pi)$ implied by (2.9), we will be interested in minimizers of $\mathcal{A}^{1d}_{\Pi}$ which are non-degenerate in the following sense:

Definition 2.2. Let $\Pi \in \mathcal{O}^0$ and Φ be a minimizer (aka ground state) of $\mathcal{A}^{1d}_{\Pi}$ in $\mathcal{H}$. We say that Φ is ***nondegenerate with respect to*** Π if $\dim(\ker \mathcal{L}(\Phi;\Pi)) = 2$.

Our second main result, Theorem 2.3, establishes the existence of an open set of parameters for which $\mathcal{A}^{1d}_{\Pi}$ admits unique (modulo symmetries) non-degenerate minimizers in the set $\mathcal{H}^s \subset \mathcal{H}$ given by

$$\mathcal{H}^s = \left\{ (\varphi,\sigma) \in \mathcal{H} : \begin{array}{l} x\varphi(x) \geq 0 \text{ and } |\varphi(x)| \leq 1 \text{ for all } x \in \mathbb{R}, \varphi \text{ is odd,} \\ \sigma \text{ is real-valued}, \sigma(x) \in [0, m_\sigma] \text{ for all } x \in \mathbb{R}, \sigma \text{ is even} \end{array} \right\}.$$

A typical example of these 1D profiles is depicted in Figure 2.1.

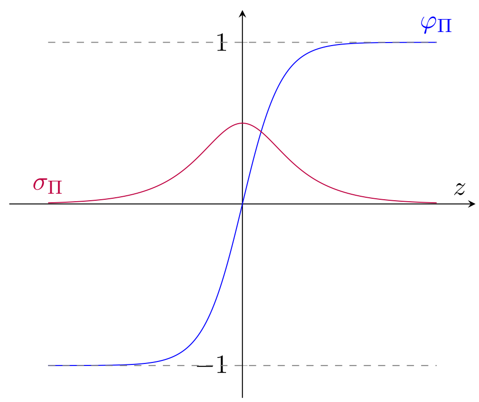


Figure 2.1: Sample 1D profiles.

Theorem 2.3. *There exists an open $\mathcal{O} \subset \mathcal{O}^0$ and a map $\Phi : \mathcal{O} \times \mathbb{R} \to \mathbb{R} \times \mathbb{C}$ such that for all $\Pi \in \mathcal{O}$, one has that $\Phi_\Pi = (\varphi_\Pi, \sigma_\Pi, 0) := \Phi(\Pi, \cdot) \in \mathcal{H}^s$ and:*

1. *Φ_Π minimizes $\mathcal{A}^{1d}_{\Pi}$ over $\mathcal{H}$, and thus $\mathcal{F}(\Phi_\Pi;\Pi) = 0$.*

2. *If (φ,σ) minimizes $\mathcal{A}^{1d}_{\Pi}$ in $\mathcal{H}$, then there exist $z_0, \alpha_0 \in \mathbb{R}$ such that*

$$\begin{pmatrix} \varphi \\ \sigma \end{pmatrix}(z) = \begin{pmatrix} \varphi_\Pi \\ e^{i\alpha_0}\sigma_\Pi \end{pmatrix}(z - z_0), \qquad \textit{for all } z \in \mathbb{R}.$$

3. *Φ_Π is non-degenerate for Π. In particular,*

$$\ker\left(\mathcal{L}\left(\Phi_\Pi;\Pi\right)\right) = \mathit{span}_{\mathbb{R}}\left\{\left(\varphi'_\Pi, \sigma'_\Pi, 0\right), (0,0,\sigma_\Pi)\right\}.$$

4. *σ_Π is a strictly positive function.*

5. *Φ is smooth.*

6. *For each compact set $K \subset \mathcal{O}$, $k \in \mathbb{Z}_+$ and multiindex α, there exists a constant $C_{\alpha k}$ depending only on K such that*

$$\left| D_\Pi^\alpha \partial_z^k \left[\Phi(\Pi, z) - I_\pm(z) \right] \right| \leq C_{\alpha k}(K) e^{-\frac{\sqrt{\lambda_{\min}(\Pi)}}{2}|z|}, \tag{2.10}$$

for all $(\Pi, z) \in K \times \mathbb{R}$, where $\lambda_{\min}(\Pi) > 0$ is the minimium eigenvalue of $D_\Phi^2 W(\pm 1, 0; \Pi)$.

Our strategy for proving Theorem 2.3 relies on identifying a 3-dimensional submanifold in the space of parameters Π (i.e., $\Sigma^{N,I}$ defined in (2.15)), for which we can explicitly identify minimizers of $\mathcal{A}_\Pi^{1d}$ and directly verify their uniqueness and nondegeneracy (see Theorem 2.5). We will refer to this special structure as the "integrable case." We then argue that this submanifold is contained in an open neighbourhood in which the same conclusions hold (i.e., $\mathcal{O}$ from Theorem 2.3 and Definition 2.22). The smooth dependence of ground states on parameters follows as a bi-product of the implicit function theorem, whose application will be partly justified by Theorem 2.1.

The next result deals with the solvability of a slightly more general version of the equation

$$\mathcal{L}(\Phi; \Pi)[V] = G, \tag{2.11}$$

where Φ is the map from Theorem 2.3 and $G \in C^\infty(\mathcal{O} \times \mathbb{R}; \mathbb{R} \times \mathbb{C})$ is given. In particular, it is shown that V inherits the smoothness and decay properties of G which will be necessary for the construction of approximate solutions in Chapter 4. Our main result in this regard is:

Theorem 2.4 (Inversion of Linearized Operator (part II))**.** *Let Φ_Π be as in Theorem 2.3, and $U \subset \mathbb{R}^n$ be open. Also, let $f \in C^\infty(U; \mathcal{O})$ and $G \in C^\infty(U \times \mathbb{R}; \mathbb{R} \times \mathbb{C})$ be such that:*

1. *The image of f is contained in a compact subset $K \subset \mathcal{O}$.*

2. *$G(y, \cdot) \in \ker \left(\mathcal{L} \left(\Phi_{f(y)}; f(y) \right) \right)^\perp$ for all $y \in U$.*

3. *There exists $c > 0$ such that for each $k \in \mathbb{Z}_+$ and each multi-index α,*

$$|\partial_z^k D_y^\alpha G(y, z)| \leq C_{\alpha k}(y) e^{-c|z|}, \qquad \text{for all } (y, z) \in U \times \mathbb{R}, \tag{2.12}$$

for some smooth function $C_{\alpha k}$.

Then, there exists a function $V \in C_b^\infty(U \times \mathbb{R}; \mathbb{R} \times \mathbb{C})$ such that

1. $\mathcal{L}\left(\Phi_{f(y)}; f(y)\right)[V(y,\cdot)] = G(y,\cdot)$*, for all $y \in U$.*
2. $V(y,\cdot) \in \ker\left(\mathcal{L}\left(\Phi_{f(y)}; f(y)\right)\right)^\perp$ *for all $y \in U$.*
3. *For each $k \in \mathbb{Z}_+$ and each multi-index α,*

$$|\partial_z^k D_y^\alpha V(y,z)| \leq D_{\alpha k}(y) e^{-\beta|z|}, \qquad \textit{for all } (y,z) \in U \times \mathbb{R}, \tag{2.13}$$

for some smooth function $D_{\alpha k}$ and $\beta = \min\left(\frac{1}{2}\sqrt{\lambda_{\min}(\Pi)}, c\right)$.

Finally, Section 2.3 contains some remarks about the so called *shifted potential* which will be used and built upon in Chapter 3 and Chapter 4.

2.2 EXISTENCE AND PROPERTIES OF MINIMIZERS OF A_{Π}^{1d}

2.2.1 *The Integrable Case*

We start by finding explicit solutions to (2.3) which will serve as "one-dimensional profiles" for solutions to the more general equation, (1.8). In particular, we find conditions on the parameters Π under which $\Phi = (\varphi, \sigma)$ solves (2.3), where

$$\varphi(z) = \tanh(az) \quad \text{ and } \quad \sigma(z) = b\,\text{sech}(az), \quad \text{for some } a, b > 0. \tag{2.14}$$

Furthermore, we will restrict our attention to solutions of this type that minimize $\mathcal{A}_{\Pi}^{1d}$ and that are non-degenerate in the sense of Definition 2.2. These additional properties play a central role in establishing the coercive character of $\mathcal{Q}(\Phi, \Pi)$ from (2.7). The results of this section can be summarized as:

Theorem 2.5 (Integrable Parameters)**.** *Consider the following sets of parameters:*

$$\Sigma^I := \Big\{(\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in (0,\infty)^4 : \beta \in \left(\lambda_\sigma m_\sigma^2, 2\lambda_\sigma m_\sigma^2\right), \quad \lambda_\varphi = \beta + \left(\frac{\beta}{\lambda_\sigma} - 1\right)\left(2\lambda_\sigma m_\sigma^2 - \beta\right)\Big\}, \tag{2.15}$$

$$\Sigma^{M,I} := \left\{(\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in \Sigma^I : \beta \le \lambda_\sigma(m_\sigma^2 + 2)\right\},$$
$$\Sigma^{N,I} := \left\{(\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in \Sigma^I : \beta < \lambda_\sigma(m_\sigma^2 + 2)\right\},$$

and the map $\mathfrak{m} : \Sigma^I \to \mathcal{H}^s$ *whose action on* $\Pi = (\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in \Sigma^I$ *is given by*

$$\mathfrak{m}(\Pi)(z) := \begin{pmatrix} \tanh(a(\Pi)z) \\ b(\Pi)\,\mathrm{sech}(a(\Pi)z) \end{pmatrix}, \qquad \text{for all } z \in \mathbb{R}, \tag{2.16}$$

where $a(\Pi) = \sqrt{\beta - \lambda_\sigma m_\sigma^2}$ *and* $b(\Pi) = \sqrt{2m_\sigma^2 - \frac{\beta}{\lambda_\sigma}}$.
Then,

1. *A pair* (φ, σ) *of the form* (2.14) *solves* (2.3) *if and only if* $(\varphi, \sigma) = \mathfrak{m}(\Pi)$ *for some* $\Pi \in \Sigma^I$.
2. $\mathfrak{m}(\Pi)$ *minimizes* $\mathcal{A}^{1d}_{\Pi}$ *for each* $\Pi \in \Sigma^{M,I}$. *Moreover, if* $\Pi \in \Sigma^{N,I}$, *any other minimizer of* $\mathcal{A}^{1d}_{\Pi}$ *is of the form* $e^{i\theta_0} \ltimes \mathfrak{m}(\Pi)(\cdot - z_0)$ *for some fixed* $\theta_0, z_0 \in \mathbb{R}$.
3. *For each* $\Pi \in \Sigma^{N,I}$, $\mathfrak{m}(\Pi)$ *is non-degenerate for* Π *and*

$$\ker \mathcal{L}\left(\mathfrak{m}(\Pi); \Pi\right) = \mathit{span}_{\mathbb{R}}\left(\left\{[\mathfrak{m}(\Pi)]', i\mathcal{P}_\sigma(\mathfrak{m}(\Pi))\right\}\right).$$

Remark 2.6. Item 2 from Theorem 2.5 implies that for each $\Pi \in \Sigma^{N,I}$, $\mathcal{A}^{1d}_{\Pi}$ admits a unique minimizer in $\mathcal{H}^s$ given by (2.16).

Definition 2.7. In view of Theorem 2.5, we will call the sets (three dimensional submanifolds of $(0,\infty)^4$) of parameters Σ^I and $\Sigma^{N,I}$ from (2.15) the set of ***integrable parameters*** and the set of ***non-degenerate integrable parameters***, respectively.

Remark 2.8. let $(\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in (0,\infty)^4$ be such that

$$\lambda_\varphi = \beta + (\beta/\lambda_\sigma - 1)\left(2\lambda_\sigma m_\sigma^2 - \beta\right). \tag{2.17}$$

Then,

$$\begin{aligned}\lambda_\varphi\lambda_\sigma - \left(\lambda_\sigma m_\sigma^2\right)^2 &= -\beta^2 + 2\lambda_\sigma\left(m_\sigma^2+1\right)\beta - \lambda_\sigma^2 m_\sigma^2\left(m_\sigma^2+2\right)\\ &= -\left(\beta - \lambda_\sigma m_\sigma^2\right)\left(\beta - \lambda_\sigma(m_\sigma^2+2)\right)\end{aligned}$$

which shows that, under the condition that (2.17) holds, $\lambda_\varphi > \lambda_\sigma m_\sigma^4$ if and only if $\beta \in (\lambda_\sigma m_\sigma^2, \lambda_\sigma(m_\sigma^2+2))$. As a result, $\Sigma^{N,I} = \Sigma^I \cap \mathcal{O}^0$. Also, if $m_\sigma > 0$, then $m_\sigma \leq \sqrt{2} \iff 2\lambda_\sigma m_\sigma^2 \leq \lambda_\sigma(m_\sigma^2+2)$. Therefore, defining $\mathcal{O}_\sigma := \left\{\Pi \in \mathcal{O}^0 : m_\sigma \in \left(0, \sqrt{2}\right]\right\}$, we have that (see Figure 2.2):

$$\Sigma^{N,I} \cap \mathcal{O}_\sigma = \Sigma^I \cap \mathcal{O}_\sigma \qquad \text{and} \qquad \Sigma^{N,I} \cap \mathcal{O}_\sigma^c \subsetneq \Sigma^I \cap \mathcal{O}_\sigma^c.$$

In other words, all integrable parameters with $m_\sigma \leq \sqrt{2}$ are non-degenerate. On the other hand, there are integrable parameters with $m_\sigma > \sqrt{2}$ which fail to be non-degenerate.

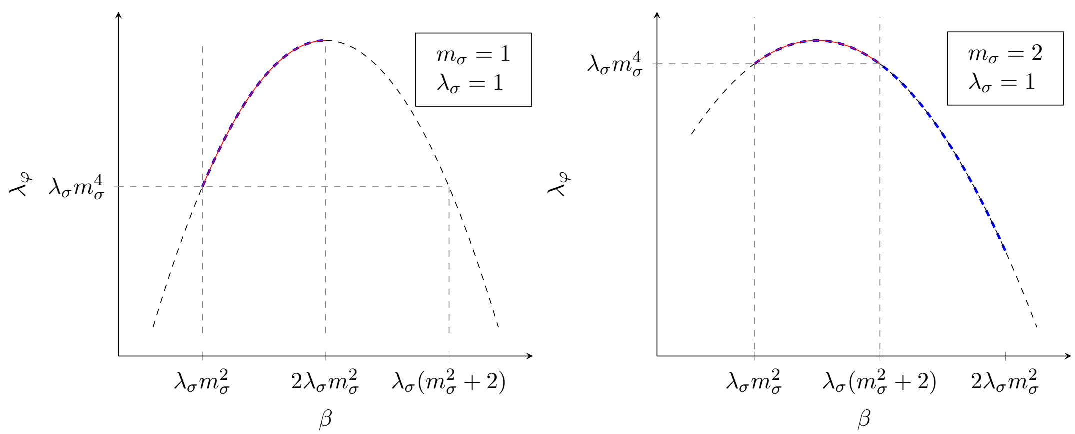


Figure 2.2: Representation of $\Sigma^{N,I} \cap U$ (red solid line) and $\Sigma^I \cap U$ (blue dashed line) for: $U = \left\{(\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in (0,\infty)^4 : \lambda_\sigma = m_\sigma = 1\right\}$ (left), $U = \left\{(\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in (0,\infty)^4 : \lambda_\sigma = 1, m_\sigma = 2\right\}$ (right).

Proof of Item 1 from Theorem 2.5. Plugging in (2.14) into (2.3), results in

$$\begin{aligned}\left(2a^2 + b^2\beta - \lambda_\varphi\right)\varphi(z) = 0\\ \left[a^2 - \lambda_\sigma(m_\sigma^2 - b^2)\right] + \left[\beta - \left(2a^2 + \lambda_\sigma b^2\right)\right]\varphi^2(z) = 0.\end{aligned}$$

Therefore, solutions to (2.3) of the form (2.14) exist if and only if the following relations are satisfied:

$$\begin{cases} b \in (0, m_\sigma) \\ a = \sqrt{\lambda_\sigma(m_\sigma^2 - b^2)} \\ \beta = 2a^2 + \lambda_\sigma b^2 \\ \lambda_\varphi = 2a^2(1+b^2) + \lambda_\sigma b^4. \end{cases} \tag{2.18}$$

A direct computation shows that these conditions can be rewritten as

$$\begin{cases} \beta \in \left(\lambda_\sigma m_\sigma^2, 2\lambda_\sigma m_\sigma^2\right) \\ a = \sqrt{\beta - \lambda_\sigma m_\sigma^2} \\ b = \sqrt{2m_\sigma^2 - \dfrac{\beta}{\lambda_\sigma}} \\ \lambda_\varphi = \beta + \left(\dfrac{\beta}{\lambda_\sigma} - 1\right)\left(2\lambda_\sigma m_\sigma^2 - \beta\right). \end{cases} \tag{2.19}$$

The first and last conditions in (2.19) are the defining properties of the parameters in Σ^I in (2.15), while the two middle ones are the conditions appearing in the definition of the map $\mathfrak{m}$ in (2.16). □

Remark 2.9. Based on (2.18), we may alternatively write

$$\Sigma^I = \bigcup_{\lambda_\sigma, m_\sigma \in (0,\infty)} \bigcup_{b \in (0, m_\sigma)} C(\lambda_\sigma, m_\sigma, b),$$

where

$$\begin{aligned} C(\lambda_\sigma, m_\sigma, b) &:= \{(\lambda_\varphi, u, v, \beta) \in (0,\infty)^4 : u = \lambda_\sigma, v = m_\sigma, \\ &\beta = 2a^2 + \lambda_\sigma b^2, \lambda_\varphi = 2a^2(1+b^2) + \lambda_\sigma b^4, \text{ with } a := \sqrt{\lambda_\sigma(m_\sigma^2 - b^2)}\}. \end{aligned} \tag{2.20}$$

Proof of Item 2 from Theorem 2.5. Let $\Pi \in \Sigma^I$ and let a and b be as in (2.18) (or equivalently as in (2.19)). Direct computations show that the functions $(\varphi_0, \sigma_0) = \mathfrak{m}(\Pi)$ satisfy the following relations

$$\begin{gathered} \varphi' = a(1-\varphi^2), \qquad \sigma' = -a\varphi\sigma, \qquad \sigma^2 = b^2(1-\varphi^2), \\ \lim_{z \to \pm\infty} \varphi(z) = \pm 1, \qquad \lim_{z \to \pm\infty} \sigma(z) = 0, \end{gathered} \tag{2.21}$$

and that

$$(\varphi,\sigma)\ \in\ \mathcal{H}\text{ solve (2.21)} \quad\Longleftrightarrow\quad \begin{pmatrix} (\varphi,\sigma) = (\varphi_0(\cdot - z_0), e^{i\alpha_0}\sigma_0(\cdot - z_0)) \\ \text{for some constant } z_0, \alpha_0 \in \mathbb{R} \end{pmatrix}. \tag{2.22}$$

Therefore, in order to show that (φ_0, σ_0) is the unique minimizer up to symmetries (i.e., up to constant translation and constant rotation) of $\mathcal{A}_{\Pi}^{1d}$, suffices to relate the functional $\mathcal{A}_{\Pi}^{1d}$ to a non-negative functional which vanishes at $(\varphi, \sigma) \in \mathcal{H}$ if and only if the conditions (2.21) are satisfied. To this end, note that

$$\Pi \in \Sigma^I \implies \mathcal{A}_{\Pi}^{1d}(\varphi,\sigma) = J_{\Pi}(\varphi,\sigma) - C(\Pi), \tag{2.23}$$

where $C(\Pi) = \frac{4}{3}a\left(1+\frac{b^2}{2}\right)$ with a and b as in (2.18), and

$$\begin{aligned} J_{\Pi}(\varphi,\sigma) := \frac{1}{2}\int_{\mathbb{R}} &\left[\left|\varphi' - a(1-\varphi^2) + \frac{a}{2}\left[|\sigma|^2 - b^2(1-\varphi^2)\right]\right|^2 + \left|\sigma' + a\varphi\sigma\right|^2\right] \\ &+ \frac{(2\lambda_\sigma - a^2)}{8}\int_{\mathbb{R}} \left[|\sigma|^2 - b^2(1-\varphi^2)\right]^2 . \end{aligned} \tag{2.24}$$

To see this, let $c = a\left(1+\frac{b^2}{2}\right)$ and $\theta = \frac{2\lambda_\sigma - a^2}{4}$ and rearrange the terms in J_{Π} as follows:

$$\begin{aligned} J_{\Pi}(\varphi,\sigma) &= \frac{1}{2}\int_{\mathbb{R}} \left\{\left[\varphi' + c(\varphi^2-1) + \frac{a}{2}|\sigma|^2\right]^2 + \left|\sigma' + a\varphi\sigma\right|^2 + \theta\left[|\sigma|^2 + b^2(\varphi^2-1)\right]^2\right\} \\ &= \frac{1}{2}\int_{\mathbb{R}} \left\{\left[(\varphi')^2 + (\sigma')^2\right] + \left(c^2 + b^4\theta\right)(\varphi^2-1)^2 + \left(\frac{a^2}{4}+\theta\right)|\sigma|^4\right\} \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} \left\{-\left(ac + 2b^2\theta\right)|\sigma|^2 + \left(ac + a^2 + 2b^2\theta\right)\varphi^2|\sigma|^2\right\} \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} \left[2c\left(\frac{\varphi^3}{3} - \varphi\right) + a\varphi|\sigma|^2\right]' . \end{aligned}$$

Then, (2.23) is obtained by plugging back the definitions of a, b, c, θ and by using the boundary conditions of φ, σ to evaluate the very last integral. In view of (2.23), (φ_0, σ_0) minimizes J_{Π} if and only if (φ_0, σ_0) minimizes A_{Π}^{1d}. Also, by looking at (2.24), it is clear that (φ, σ) is a global minimizer of J_{Π} over $\mathcal{H}$ if the following condition, written in three equivalent ways under the assumption that $\Pi \in \Sigma^I$, holds

$$a^2 \leq 2\lambda_\sigma, \qquad \text{i.e.,} \qquad \beta \leq \lambda_\sigma\left(m_\sigma^2 + 2\right), \qquad \text{i.e.,} \qquad m_\sigma^2 - 2 \leq b^2. \tag{2.25}$$

We thus have the following three possible cases for all $\Pi \in \Sigma^I$:

1. If (2.25) is not satisfied: a direct computation shows that for $\varphi(z) = \tanh(cz)$ and $\sigma(z) = 0$, one has that $J_\Pi(\varphi,\sigma) < J_\Pi(\varphi_0,\sigma_0) = 0$, and therefore (φ_0,σ_0) is not a minimizer over $\mathcal{H}$ in this case.

2. If $\beta = \lambda_\sigma\left(m_\sigma^2 + 2\right)$, then for $\varphi(z) = \tanh(cz)$ and $\sigma(z) = 0$, one has that $J_\Pi(\varphi,\sigma) = J_\Pi(\varphi_0,\sigma_0) = 0$, so (φ_0,σ_0) is not the unique minimizer (modulo symmetries) over $\mathcal{H}$.

3. If $\beta < \lambda_\sigma\left(m_\sigma^2 + 2\right)$, then $J_\Pi(\varphi,\sigma) = 0$ for $(\varphi,\sigma) \in \mathcal{H}$ if and only if (φ,σ) solves (2.21), which implies that (φ_0,σ_0) minimizes J_Π in this case, and it is the unique minimizer (modulo symmetries) over $\mathcal{H}$ due to (2.22).

The last two cases imply the result from item 2 of Theorem 2.5 by the definitions of $\Sigma^{M,I}$ and $\Sigma^{N,I}$. □

Remark 2.10. Figure 2.3 depicts the values of b and a as functions of m_σ for which $\mathfrak{m}(\Pi)$ from (2.16) is the unique minimizer of $\mathcal{A}_\Pi^{1d}$ modulo symmetries over $\mathcal{H}$ (i.e., whenever $\Pi \in \Sigma^{N,I}$) according to .

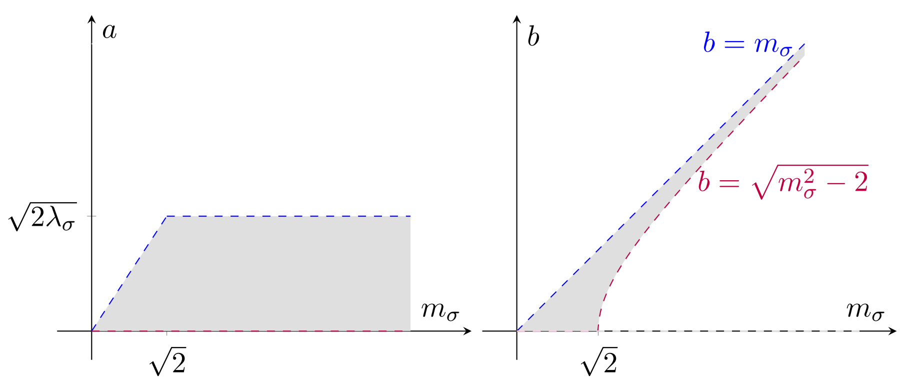


Figure 2.3: Ranges of values of a and b as functions of m_σ (gray areas) for which $(\tanh(az), b\,\mathrm{sech}(az))$ is the unique minimizer of $\mathcal{A}_\Pi^{1d}$ up to symmetries.

Proof of Item 3 from Theorem 2.5. Let $\Pi \in \Sigma^{N,I}$ and $(\varphi_0,\sigma_0) = \mathfrak{m}(\Pi)$. Using (2.23), we compute for any $\Phi = (\varphi,\sigma) = (\varphi,\sigma_1,\sigma_2) \in X_2^3$:

$$\int_{\mathbb{R}} \Phi \cdot \mathcal{L}\left(\mathfrak{m}(\Pi);\Pi\right)[\Phi] = \left.\frac{d^2}{dh^2}\right|_{h=0} \mathcal{A}_\Pi^{1d}\left((\varphi_0,\sigma_0) + h\Phi\right)$$

$$
\begin{aligned}
&= \frac{d^2}{dh^2}\bigg|_{h=0} J_{\Pi}\left((\varphi_0,\sigma_0)+h\Phi\right) \\
&= \int_{\mathbb{R}} \left[\left(\varphi'+2c\varphi_0\varphi+a\sigma_0\sigma_1\right)^2+\left(\sigma_1'+a\varphi_0\sigma_1+a\varphi\sigma_0\right)^2\right] \\
&\int_{\mathbb{R}} \left[\left(\sigma_2'+a\varphi_0\sigma_2\right)^2+\theta\left(\sigma_0\sigma_1+b^2\varphi_0\varphi\right)^2\right],
\end{aligned}
$$

where the last equality follows from a direct computation using the labels $c = a\left(1+b^2/2\right)$ and $\theta = \lambda_\sigma[2-(m_\sigma^2-b^2)]/4$. Therefore, if $\Phi = (\varphi,\sigma_1,\sigma_2) \in \ker\mathcal{L}(\mathfrak{m}(\Pi);\Pi)$, the above expression vanishes and one must have that

$$\varphi'+2c\varphi_0\varphi+a\sigma_0\sigma_1 = 0 \tag{2.26}$$

$$\sigma_1'+a\varphi_0\sigma_1+a\varphi\sigma_0 = 0 \tag{2.27}$$

$$\sigma_2'+a\varphi_0\sigma_2 = 0 \tag{2.28}$$

$$\sigma_0\sigma_1+b^2\varphi_0\varphi = 0. \tag{2.29}$$

Recall that $\sigma_0' = -a\varphi_0\sigma_0$, which implies that $(\sigma_0^2)' = -2a\varphi_0\sigma_0^2$. Now, combining (2.26) and (2.29) and noting that $ab^2-2c = 2a$ results in

$$\varphi' = -2a\varphi_0\varphi,$$

which means that φ solves the same linear ODE as $\sigma_0^2 = \frac{b^2}{a}\varphi_0'$, and thus

$$\varphi = c_1\varphi_0' \text{ for some constant } c_1. \tag{2.30}$$

On the other hand, (2.28) implies that σ_2 solves the same linear ODE as σ_0, and therefore

$$\sigma_2 = c_2\sigma_0 \text{ for some constant } c_2. \tag{2.31}$$

Finally, using (2.27) along with (2.30) and the identity $\varphi_0\sigma_0' = -a\varphi_0^2\sigma_0$, we obtain

$$
\begin{aligned}
\sigma_1'+a\varphi_0\sigma_1+a\varphi\sigma_0 &= \sigma_1'+a\varphi_0\sigma_1+ac_1\varphi_0'\sigma_0 \\
&= \sigma_1'+a\varphi_0\sigma_1+(ac_1\varphi_0\sigma_0)'-ac_1\varphi_0\sigma_0' \\
&= (\sigma_1+ac_1\varphi_0\sigma_0)'+a\varphi_0\left(\sigma_1+ac_1\varphi_0\sigma_0\right) \\
&= \left(\sigma_1-c_1\sigma_0'\right)'+a\varphi_0\left(\sigma_1-c_1\sigma_0'\right) = 0.
\end{aligned}
$$

By the same logic as the one behind (2.31), we conclude that $\sigma_1 = c_1\sigma_0' + c_3\sigma_0$ for some constant c_3. Now, plugging in this expression for σ_1 and (2.30) into (2.29), we find that

$$c_3\sigma_0^2 = -c_1\left(\sigma_0\sigma_0' + b^2\varphi_0\varphi_0'\right). \tag{2.32}$$

The expressions for φ_0 and σ_0 in (2.16) imply that the right hand side of (2.32) vanishes, and thus $c_3 = 0$. As a consequence,

$$\sigma_1 = c_1\sigma_0' \text{ for the same constant } c_1 \text{ from (2.30).} \tag{2.33}$$

Putting together (2.30), (2.31) and (2.33), we conclude that

$$\Phi \in \ker \mathcal{L}(\mathfrak{m}(\Pi);\Pi) \implies \Phi \in \operatorname{span}\left\{\begin{pmatrix}\varphi_0' \\ \sigma_0' \\ 0\end{pmatrix}, \begin{pmatrix}0 \\ 0 \\ \sigma_0\end{pmatrix}\right\}. \tag{2.34}$$

The reverse implication to (2.34) also holds in view of (2.9), and thus $\dim\ker\left(\mathcal{L}_\Pi\right) = 2$. □

2.2.2 *Other Ground States*

Some of the properties of the ground states from the integrable case are shared by ground states corresponding to parameters $\Pi \in \mathcal{O}^0\backslash\Sigma^{N,I}$ (see Lemmas 2.12 and 2.15). We begin with a useful compactness result.

Lemma 2.11. *Let* $((\varphi_n,\sigma_n))_{n=1}^{\infty} \subset \mathcal{H}^s$ *be such that* $\left(1-\varphi_n^2\right)_n, (\varphi_n')_n, (\sigma_n)_n,$ *and* $(\sigma_n')_n$ *are bounded in* $L^2(\mathbb{R})$*. Then, there exist* $(\varphi,\sigma) \in \mathcal{H}^s$ *such that, after possibly passing to a subsequence,*

$$\begin{cases} \varphi_n \to \varphi, \sigma_n \to \sigma & \textit{locally uniformly} \\ \varphi_n' \rightharpoonup \varphi' & \textit{weakly in } L^2(\mathbb{R}) \\ \sigma_n \rightharpoonup \sigma & \textit{weakly in } H^1(\mathbb{R}). \end{cases} \tag{2.35}$$

Proof. Let $M > 0$ be such that $\|\varphi_n'\|_{L^2}, \|\sigma_n\|_{H^1} < M$ for all $n \in \mathbb{N}$. By the Banach Alaoglu theorem, $(\varphi_n')_n$ (resp. $(\sigma_n)_n$) is weakly precompact in $L^2(\mathbb{R})$

(resp. $H^1(\mathbb{R})$). Thus, after passing to subsequences (still labelled (φ_n, σ_n)) we find that there exist $(\eta, \rho) \in L^2(\mathbb{R}) \times H^1(\mathbb{R})$ such that

$$\varphi_n' \rightharpoonup \eta \text{ weakly in } L^2(\mathbb{R}) \qquad \text{and} \qquad \sigma_n \rightharpoonup \rho \text{ weakly in } H^1(\mathbb{R}). \tag{2.36}$$

On the other hand, a standard Sobolev inequality implies that

$$|\varphi_n(y) - \varphi_n(x)| \leq |y-x|^{1/2} \left\|\varphi_n'\right\|_{L^2} \leq M\,|y-x|^{1/2} \qquad \text{a.e. } x, y \in \mathbb{R}, \tag{2.37}$$

As a result, the sequence $(\varphi_n)_n$ is equicontinous, and it is also pointwise bounded since $\varphi_n \in \mathcal{H}^s$ for each n. By the Arzela-Ascoli theorem (see e.g., [12, Theorem 4.44]), the last observations imply the existence of $\varphi \in C(\mathbb{R})$ such that, after possibly passing to a subsequence, $\varphi_n \to \varphi$ uniformly on compact sets of $\mathbb{R}$. Standard arguments then show that $\varphi' = \eta$. We conclude that $\varphi \in C(\mathbb{R}) \cap \dot{H}^1(\mathbb{R})$ is the function for which the statements in (2.35) involving $(\varphi_n)_n$ hold. A similar argument yields the existence of $\sigma \in H^1(\mathbb{R})$ such that the remaining statements in (2.35) hold.

It remains to show that (φ, σ) satisfies all other properties in the definition of $\mathcal{H}^s$. To this end, note that the boundedness of $\left(\left\|1-\varphi_n^2\right\|_{L^2}\right)_n$, along with the local uniform convergence of φ_n to φ (which implies pointwise convergence) and Fatou's lemma, imply that

$$\int_{\mathbb{R}} \left(1-\varphi^2\right)^2 \leq \liminf \int_{\mathbb{R}} \left(1-\varphi_n^2\right)^2,$$

so that $\left(\varphi^2 - 1\right) \in L^2(\mathbb{R})$. Also, by the local uniform convergence of (φ_n, σ_n) to (φ, σ) and the properties of each (φ_n, σ_n), we have that

$$(|\varphi(x)|, x\varphi(x), \sigma(x)) \in [0,1] \times [0,\infty) \times [0, m_\sigma] \text{ for all } x, \qquad \begin{pmatrix} \varphi \\ \sigma \end{pmatrix} \text{ is } \begin{pmatrix} \text{odd} \\ \text{even} \end{pmatrix}.$$

Finally, $(1-\varphi^2), \varphi' \in L^2(\mathbb{R})$ imply that $(1-\varphi^2) \in H^1 \subset C^{0,1/2}$, and it follows that $\varphi^2(x) \to 1$ as $x \to \pm\infty$ and hence, since $x\varphi(x) \geq 0$, that $\varphi(x) \to \pm 1$ as $x \to \pm\infty$. Thus $(\varphi, \sigma) \in \mathcal{H}^s$. $\square$

Lemma 2.12 (Existence, regularity and symmetry of minimizers)**.** *Let* $\Pi \in \mathcal{O}^0$. *Then,* $\inf_{\mathcal{H}} \mathcal{A}_{\Pi}^{1d}$ *is attained by a minimizer* (φ, σ) *in* $\mathcal{H}^s \cap C_b^\infty(\mathbb{R}; \mathbb{R} \times \mathbb{C})$. *Also, if* (φ_1, σ_1) *minimizes* $\mathcal{A}_{\Pi}^{1d}$ *in* $\mathcal{H}$, *then there exist* x_0 *and* $\alpha \in \mathbb{R}$ *and* $(\varphi, \sigma) \in \mathcal{H}^s$ *such that* $(\varphi_1, \sigma_1)(x) = (\varphi, e^{i\alpha}\sigma)(x - x_0)$.

Existence of a minimizer for more general multi-well potentials W is well-known, see for example [33], without the symmetry assertions, which require specific properties of the potentials W that we consider.

Proof of Lemma 2.12. Step 1. Symmetrization. We first claim that given any $(\varphi, \sigma) \in \mathcal{H}$, we can find $(\tilde{\varphi}, \tilde{\sigma}) \in \mathcal{H}^s$ such that

$$\mathcal{A}_{\Pi}^{1d}(\tilde{\varphi}, \tilde{\sigma}) \leq \mathcal{A}_{\Pi}^{1d}(\varphi, \sigma). \tag{2.38}$$

Since $\varphi(x) \to \pm 1$ as $x \to \pm\infty$, we can define $x_0 := \arg\min\{x \mid \varphi(x) = 0\}$ and $x_1 := \arg\max\{x \mid \varphi(x) = 0\}$. The definition of $\mathcal{H}$ implies that both x_0 and x_1 are finite. We have either

$$I_0 := \int_{-\infty}^{x_0} \left[\frac{1}{2}(\varphi'^2 + \sigma'^2) + W(\varphi, \sigma; \Pi)\right] dx \leq \frac{1}{2}\mathcal{A}_{\Pi}^{1d}(\varphi, \sigma)$$

or

$$I_1 := \int_{x_1}^{\infty} \left[\frac{1}{2}(\varphi'^2 + \sigma'^2) + W(\varphi, \sigma; \Pi)\right] dx \leq \frac{1}{2}\mathcal{A}_{\Pi}^{1d}(\varphi, \sigma).$$

Assume that the first inequality holds and define

$$\tilde{\varphi}(x) := \begin{cases} \max\{\varphi(x + x_0), -1\} & \text{if } x \leq 0 \\ \min\{-\varphi(-x + x_0), 1\} & \text{if } x \geq 0 \end{cases},$$

$$\tilde{\sigma}(x) := \begin{cases} \min\{|\sigma(x + x_0)|, m_\sigma\} & \text{if } x \leq 0 \\ \min\{|\sigma(-x + x_0)|, m_\sigma\} & \text{if } x \geq 0 \end{cases}.$$

The construction implies that $(\tilde{\varphi}, \tilde{\sigma}) \in \mathcal{H}^s$ and that (2.38) holds[3]. A symmetric argument applies when the second inequality holds, proving the claim.

Step 2. Existence of an odd/even minimizer. Consider a minimizing sequence $((\varphi_n, \sigma_n))_n \subset \mathcal{H}$. In view of this and Step 1, we can assume without loss of generality that $((\varphi_n, \sigma_n))_n \subset \mathcal{H}^s$. By property (1.42) of the potential W, we have

$$c(\Pi) \int_{\mathbb{R}} \left[\left(\varphi_n^2 - 1\right)^2 + \sigma_n^2 + \varphi_n'^2 + \sigma_n'^2\right] \leq \mathcal{A}_{\Pi}^{1d}(\varphi_n, \sigma_n) \leq \mathcal{A}_{\Pi}^{1d}(\varphi_1, \sigma_1), \tag{2.39}$$

3 Consider the functions $f(a) = \frac{\lambda_\sigma}{4}(a^2 - 2m_\sigma^2)a^2 + \frac{1}{2}\beta\varphi^2 a^2$ and $g(b) = \frac{\lambda_\varphi}{4}(b^2 - 1)^2 + \frac{1}{2}\beta|\sigma|^2 b^2$. Then, f and g are non-decreasing for $a \geq m_\sigma^2$ and $b \geq 1$ for all possible values of φ and of $|\sigma|$, respectively.

for all n and a continuous strictly positive function c. As a result, the sequence $\left((\varphi_n^2-1,\varphi_n')\right)_n$ (resp. $(\sigma_n)_n$) is bounded in the standard $L^2(\mathbb{R})\times L^2(\mathbb{R})$ norm (resp. $H^1(\mathbb{R})$ norm). Thus, after passing to subsequences (still labelled (φ_n,σ_n)), Lemma 2.11 implies the existence of $(\varphi,\sigma)\in\mathcal{H}^s$ such that

$$\begin{cases}\sigma_n \rightharpoonup \sigma \text{ weakly in } H^1(\mathbb{R}), & (\varphi_n,\sigma_n)\to(\varphi,\sigma) \text{ locally uniformly}\\ \varphi_n' \rightharpoonup \varphi' \text{ weakly in } L^2(\mathbb{R}) & \end{cases} \tag{2.40}$$

Then Fatou's Lemma and the weak-lower semicontinuity of the $L^2(\mathbb{R})$ norm imply that

$$\mathcal{A}_{\Pi}^{1d}(\varphi,\sigma)\le \liminf_n \mathcal{A}_{\Pi}^{1d}(\varphi_n,\sigma_n)=\inf_{\mathcal{H}}\mathcal{A}_{\Pi}^{1d}.$$

Step 3. Regularity of minimizers. Standard elliptic theory implies that

$$\Phi''=D_\Phi W(\Phi;\Pi)$$

in the weak sense, and then that $\Phi\in C_b^\infty(\mathbb{R})$. Let $\Phi=(\varphi,\sigma)\in\mathcal{H}$ be a critical point of $\mathcal{A}_{\Pi}^{1d}$. For each $\Psi\in X_2^3$, we have that

$$\frac{d}{dh}\Big|_{h=0}\mathcal{A}_{\Pi}^{1d}(\Phi+h\Psi)=0 \iff \int_{\mathbb{R}}\Phi'\cdot\Psi'=\int_{\mathbb{R}}\left[-D_\Phi W\left(\Phi;\Pi\right)\right]\cdot\Psi, \tag{2.41}$$

which implies that $\Phi''=D_\Phi W(\Phi;\Pi)$ in the weak sense. As a result, the fact that $\partial_\varphi W(\varphi,\sigma;\Pi)$ and $\partial_{\sigma_R}W(\varphi,\sigma;\Pi)$ are polynomial functions of $(1-\varphi^2),\varphi,\sigma\in H^1(\mathbb{R})$ and that $H^1(\mathbb{R})$ is a Banach algebra, readily provides $\varphi''\in H^1(\mathbb{R})$ and $\sigma''\in H^1(\mathbb{R})$, and hence $\varphi'\in H^2(\mathbb{R})$ and $\sigma'\in H^2(\mathbb{R})$. Differentiating the equation $\Phi''=D_\Phi W(\Phi;\Pi)$ repeatedly and applying a similar argument shows that $\varphi',\sigma'\in H^m(\mathbb{R})$ for all $m\in\mathbb{N}$, which implies that $\varphi,\sigma\in C_b^\infty(\mathbb{R})$ by the embedding $H^m(\mathbb{R})\hookrightarrow C_b^{m-1}(\mathbb{R})$.

Step 4. Parity of minimizers up to translation and rotation. Let (u,v) be a minimizer of $\mathcal{A}_{\Pi}^{1d}$ in $\mathcal{H}$ and note that for any $(\varphi,\sigma)\in\mathcal{H}$:

- $\mathcal{A}_{\Pi}^{1d}(\varphi,|\sigma|)\le\mathcal{A}_{\Pi}^{1d}(\varphi,\sigma)$, with equality if and only if $\sigma=e^{i\alpha}|\sigma|$ for some constant $\alpha\in\mathbb{R}$.
- If we define $\tilde\varphi(x):=\max\{-1,\min\{1,\varphi(x)\}\}$ and $\tilde\sigma(x):=\min\{m_\sigma,|\sigma(x)|\}$, thanks to (1.39) and (1.41), we have that $\mathcal{A}_{\Pi}^{1d}(\tilde\varphi,\tilde\sigma)\le\mathcal{A}_{\Pi}^{1d}(\varphi,\sigma)$ with equality if and only if $|\varphi(x)|\le 1$ and $|\sigma(x)|\le m_\sigma$ for all $x\in\mathbb{R}$.

In particular, we have that $(u, v) \in \mathcal{H}^b$, where

$$\mathcal{H}^b := \{(\varphi, \sigma) \in \mathcal{H} : \\ \exists \alpha \in \mathbb{R} \text{ s.t. } e^{-i\alpha}\sigma(x) \in \mathbb{R}, 0 \leq |\sigma(x)| \leq m_\sigma \text{ and } |\varphi(x)| \leq 1 \text{ for all } x \in \mathbb{R}\}.$$

Let $\alpha \in \mathbb{R}$ be the (constant) phase of v. We now argue that there is $x^* \in \mathbb{R}$ such that $u(\cdot + x^*)$ is odd and $e^{-i\alpha}v(\cdot + x^*)$ is even. To this end, define x_0, x_1, I_0, and I_1 as in Step 1. Also, let

$$\begin{aligned} u_L(x) &= \chi_{\{x<0\}}(x) \cdot u(x + x_0), \qquad & v_L(x) &= \chi_{\{x\leq 0\}}(x) \cdot e^{-i\alpha}v(x + x_0) \\ u_R(x) &= \chi_{\{x\geq 0\}}(x) \cdot u(x + x_1), \qquad & v_R(x) &= \chi_{\{x\geq 0\}}(x) \cdot e^{-i\alpha}v(x + x_1), \end{aligned}$$

and

$$\begin{aligned} \varphi_1(x) &= u_L(x) - u_L(-x), & \varphi_2(x) &= u_R(x) - u_R(-x), \\ \sigma_1(x) &= \begin{cases} v_L(x), & x < 0 \\ v_L(-x), & x \geq 0 \end{cases}, & \sigma_2(x) &= \begin{cases} v_R(-x), & x < 0 \\ v_R(x), & x \geq 0. \end{cases} \end{aligned}$$

We claim that

$$(u(\cdot + x_0), e^{-i\alpha}v(\cdot + x_0)) = (\varphi_1(\cdot), \sigma_1(\cdot)) = (\varphi_2(\cdot), \sigma_2(\cdot)) \in \mathcal{H}^s. \tag{2.42}$$

Indeed, note that the minimality of (u, v) requires that $I_0 = I_1$ and that $x_0 = x_1$, for otherwise either (φ_1, σ_1) or (φ_2, σ_2) would have strictly less energy than[4] (u, v). Therefore, both (φ_1, σ_1) and (φ_2, σ_2) are elements of $\mathcal{H}^s$ which minimize $\mathcal{A}^{1d}_{\Pi}$. As a result of this and Step 3, these functions are smooth and solve the same initial value problem given by

$$\begin{cases} -\begin{pmatrix} \varphi'' \\ \sigma'' \end{pmatrix} + \begin{pmatrix} \partial_\varphi W(\varphi, \sigma; \Pi) \\ \partial_{\sigma_R} W(\varphi, \sigma; \Pi) \end{pmatrix} = 0, & \text{in } \mathbb{R}, \\ (\varphi, \varphi')(0) = (0, u'(x_0)), (\sigma, \sigma')(0) = (e^{-i\alpha}v(x_0), 0). \end{cases} \tag{2.43}$$

By the boundedness of (φ_1, σ_1), (φ_2, σ_2), and of their first derivatives, and the fact that $D_\Phi W(\cdot, \cdot; \Pi)$ is locally Lipschitz continuous, standard ODE theory results imply that $(\varphi_1, \sigma_1) = (\varphi_2, \sigma_2)$. Using this and unravelling the definitions of (φ_1, σ_1) and (φ_2, σ_2), we conclude that (2.42) must hold. □

4 Note that $x_0 \neq x_1 \implies \int_{x_0}^{x_1} \left[\frac{1}{2}(u'^2 + |v'|^2) + W(u, v; \Pi)\right] dx > 0$, as u is continuous and $u(x_0) = u(x_1) = 0$.

Remark 2.13. Note that if $\Phi \in \mathcal{H}$ is a minimizer of $\mathcal{A}_{\Pi}^{1d}$, the fact that Φ satisfies the Euler-Lagrange equations $\Phi'' = D_\Phi W(\Phi;\Pi)$ implies the following ***equipartition of energy identity***:

$$\forall z \in \mathbb{R} \left(\frac{1}{2} \left| \Phi'(z) \right|^2 = W(\Phi(z);\Pi) \right) \quad \text{for all minimizers } \Phi \text{ of } \mathcal{A}_{\Pi}^{1d} \text{ in } \mathcal{H}. \tag{2.44}$$

This can be seen by multiplying both sides of the equation $\Phi'' = D_\Phi W(\Phi;\Pi)$ by Φ' and integrating from $-\infty$ to z for each $z \in \mathbb{R}$, while keeping in mind that both $\Phi'(z)$ and $W(\Phi(z);\Pi)$ tend to zero as $|z| \to \infty$ since $\Phi \in \mathcal{H}$.

Lemma 2.14 ("Stability" of minimizers). *Consider $\Pi \in \mathcal{O}^0$ and a sequence $(\Pi_n)_n \subset (0,\infty)^4$ converging to Π. For each n, let $\Phi_n = (\varphi_n, \sigma_n)$ be a minimizer of $\mathcal{A}_{\Pi_n}^{1d}$ in $\mathcal{H}^s$. Then, the sequence $((\varphi_n, \sigma_n))_n$ is pre-compact with respect to the $\left(H^2(\mathbb{R})\right)^2$ norm, and each corresponding cluster point is a minimizer of $\mathcal{A}_{\Pi}^{1d}$ in $\mathcal{H}^s$.*

Proof. For each n, write $\Pi_n = (\lambda_{\varphi_n}, \lambda_{\sigma_n}, m_{\sigma_n}, \beta_n)$. Also, let $c : (0,\infty)^4 \to (0,\infty)$ be the function appearing in Lemma 1.2. By the continuity of c and since $\Pi_n \to \Pi = (\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta)$, there exist $m, M, N > 0$ depending only on Π such that

$$\begin{cases} \sup_n \max\{\lambda_{\varphi_n}, \lambda_{\sigma_n}, m_{\sigma_n}, \beta_n\} \in (m, M), \\ \inf_n c(\Pi_n) > m, \qquad \text{for all } n > N. \end{cases} \tag{2.45}$$

We start with some consequences of (2.45). First, the definition of $\mathcal{H}^s$ and (2.45) imply that

$$|\varphi_n(z)| \leq 1, |\sigma_n(z)| \leq M, \qquad \text{for all } z \in \mathbb{R}, n \in \mathbb{N}. \tag{2.46}$$

Secondly, let $(\varphi, \sigma) \in \mathcal{H}^s$. The pointwise boundedness of φ and σ, together with the form of W in (1.39) and (2.45), imply that for all n:

$$\begin{aligned} W(\varphi, \sigma; \Pi_n) &= \frac{\lambda_{\varphi_n}}{4} \left(\varphi^2 - 1\right)^2 + \frac{\lambda_{\sigma_n}}{4} \left(|\sigma|^2 - 2m_{\sigma_n}^2\right) |\sigma|^2 + \frac{\beta_n}{2} \varphi^2 |\sigma|^2 \\ &\leq \frac{M}{4} \left[\left(\varphi^2 - 1\right)^2 + \left(m_\sigma^2 + 2\right) |\sigma|^2 \right] \in L^1(\mathbb{R}). \end{aligned}$$

Therefore, by the dominated convergence theorem,

$$\mathcal{A}_{\Pi_n}^{1d}(\varphi,\sigma) \to \mathcal{A}_{\Pi}^{1d}(\varphi,\sigma), \qquad \text{for all } (\varphi,\sigma) \in \mathcal{H}^s. \tag{2.47}$$

Also, a direct computation shows that $\mathcal{A}_{\Pi_n}^{1d}((\tanh,0)) = (2+\lambda_{\varphi_n})/3 \leq C = (2+M)/3$ independently of n. Combining this with the minimality of Φ_n we deduce that $\left(\mathcal{A}_{\Pi_n}^{1d}(\Phi_n)\right)_n$ is bounded (by C) and thus has a convergent subsequence. We will pass to this subsequence in the remainder of the argument.

Step 1. Stability in weaker sense. As a result of (2.45) and Lemma 1.2, we have, for all $n > N$,

$$m \int_{\mathbb{R}} \left[\left(\varphi_n^2 - 1\right)^2 + \sigma_n^2\right] + \frac{1}{2}\int_{\mathbb{R}} \left(\varphi_n'^2 + \sigma_n'^2\right) \leq \mathcal{A}_{\Pi_n}^{1d}\left(\varphi_n, \sigma_n\right) \leq C. \tag{2.48}$$

(2.48) implies that $\left(\varphi_n^2 - 1\right)_n, (\sigma_n)_n, (\varphi_n')_n$, and $(\sigma_n')_n$ are bounded in the $L^2(\mathbb{R})$ norm, and therefore, by Lemma 2.11, there exists $\Phi = (\varphi,\sigma) \in \mathcal{H}^s$ such that (2.35) holds. We claim that Φ minimizes $\mathcal{A}_{\Pi}$. To see this, note that by the positivity of $\mathcal{A}_{\Pi}^{1d}$ for all Π and since $\Phi_n \to \Phi$ locally uniformly (and thus pointwise), Fatou's lemma yields:

$$\mathcal{A}_{\Pi}^{1d}(\Phi) \leq \lim_n \mathcal{A}_{\Pi_n}^{1d}(\Phi_n). \tag{2.49}$$

On the other hand, let $\bar{\Phi} \in \mathcal{H}^s$ be a minimizer of $\mathcal{A}_{\Pi}^{1d}$, whose existence is guaranteed by Lemma 2.12. By the minimality of Φ_n for each Π_n and by (2.47), we have that

$$(\forall n > N)\left(\mathcal{A}_{\Pi_n}^{1d}(\Phi_n) \leq \mathcal{A}_{\Pi_n}^{1d}(\tilde{\Phi})\right) \overset{(2.47)}{\Longrightarrow} \lim_n \mathcal{A}_{\Pi_n}^{1d}(\Phi_n) \leq \mathcal{A}_{\Pi}^{1d}(\tilde{\Phi}). \tag{2.50}$$

Combining (2.50) and (2.49) and using the minimality of $\tilde{\Phi}$ we arrive at

$$\mathcal{A}_{\Pi}^{1d}(\tilde{\Phi}) \leq \mathcal{A}_{\Pi}^{1d}(\Phi) \overset{(2.49)}{\leq} \lim_n \mathcal{A}_{\Pi_n}(\Phi_n) \overset{(2.50)}{\leq} \mathcal{A}_{\Pi}^{1d}(\tilde{\Phi}),$$

which shows that $\lim_n \mathcal{A}_{\Pi_n}^{1d}(\Phi_n) = \mathcal{A}_{\Pi}^{1d}(\Phi) = \mathcal{A}_{\Pi}(\tilde{\Phi})$. In particular, Φ minimizes $\mathcal{A}_{\Pi}^{1d}$.

Step 2. Stability in $L^2(\mathbb{R})$ sense. In view of (2.44), the minimality of each Φ_n and the fact that $\mathcal{A}_{\Pi_n}^{1d}(\Phi_n) \to \mathcal{A}_{\Pi}^{1d}(\Phi)$ imply that

$$\int_{\mathbb{R}} W(\Phi_n;\Pi_n) \to \int_{\mathbb{R}} W(\Phi;\Pi) \qquad \text{and} \qquad \int_{\mathbb{R}} |\Phi_n'|^2 \to \int_{\mathbb{R}} |\Phi'|^2. \tag{2.51}$$

Also, (2.45) and the local uniform convergence of Φ_n to Φ yields the following conditions everywhere in $\mathbb{R}$:

$$\begin{cases} \left(\varphi_n^2 - 1\right)^2, \sigma_n^2 \le m^{-1} W(\Phi_n;\Pi_n), & \text{for all } n > N \\ \left(\varphi_n^2 - 1\right)^2 \to \left(\varphi^2 - 1\right)^2, \sigma_n^2 \to \sigma^2. \end{cases} \tag{2.52}$$

The first limit in (2.51), (2.52), and a variant of the Dominated Convergence Theorem (DCT) (see e.g., [11, Sec 1.3, Theorem 4]) imply that

$$\int_{\mathbb{R}} \left(\varphi_n^2 - 1\right)^2 \to \int_{\mathbb{R}} \left(\varphi^2 - 1\right)^2 \text{ and } \int_{\mathbb{R}} \sigma_n^2 \to \int_{\mathbb{R}} \sigma^2. \tag{2.53}$$

The weak $L^2(\mathbb{R})$ convergence $\sigma_n \rightharpoonup \sigma$, together with the second limit in (2.53), implies that $\sigma_n \to \sigma$ strongly in $L^2(\mathbb{R})$. Moreover, since $|\varphi_n(x)| \le 1$ for all $x \in \mathbb{R}$ and n, we have that $(\varphi_n^2 - 1)^2(\varphi^2 - 1)^2 \le (\varphi^2 - 1)^2 \in L^1(\mathbb{R})$. Thus, by (2.53) and the DCT we conclude that

$$\|\varphi_n^2 - \varphi^2\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \left[(\varphi_n^2 - 1) - (\varphi^2 - 1)\right]^2 \to 0. \tag{2.54}$$

To show that $\varphi_n \to \varphi$ strongly in $L^2(\mathbb{R})$, let $R > 0$ be such that $|\varphi(x)| \ge 1/2$ whenever $|x| > R$. Then, $[\varphi_n(x) + \varphi(x)]^2 \ge \frac{1}{4}$ whenever $|x| > R$ and thus

$$\begin{aligned} \|\varphi_n - \varphi\|_{L^2(\mathbb{R})}^2 &= \|\varphi_n - \varphi\|_{L^2(|x|\le R)}^2 + \|\varphi_n - \varphi\|_{L^2(|x|>R)}^2 \\ &\le \|\varphi_n - \varphi\|_{L^2(|x|\le R)}^2 + 4\|\varphi_n^2 - \varphi^2\|_{L^2(|x|>R)}^2, \end{aligned}$$

from which we conclude that $\varphi_n \to \varphi$ in $L^2(\mathbb{R})$ by the local uniform convergence of φ_n to φ and (2.54).

Step 3. Stability in $H^1(\mathbb{R})$ sense. The second limit in (2.51) together with the weak-$\left(L^2(\mathbb{R})\right)^2$ convergence of (φ_n', σ_n') to (φ', σ') yield

$$\|\varphi_n' - \varphi'\|_{L^2(\mathbb{R})}^2 + \|\sigma_n' - \sigma'\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \left|\Phi_n' - \Phi'\right|^2 \to 0.$$

Step 4. Stability in $H^2(\mathbb{R})$ sense. The minimality of Φ_n implies that

$$\Phi_n'' = D_\Phi W(\Phi_n;\Pi_n) = 0, \qquad \text{for all } n. \tag{2.55}$$

We obtain the following pointwise bounds holding in all of $\mathbb{R}$:

$$\begin{aligned}\partial_\varphi W^2(\Phi_n;\Pi_n) &\overset{(1.39)}{=} \left[\lambda_{\varphi_n}\left(\varphi_n^2-1\right)+\beta_n\sigma_n^2\right]^2\varphi_n^2 \\ &\overset{(2.45),(2.46)}{\leq} 2M^2(\Pi)\left[\left(\varphi_n^2-1\right)^2+\sigma_n^2\right] \\ &\overset{(1.42),(2.45),(2.46)}{\leq} 2\frac{M^2(\Pi)}{m(\Pi)}W(\Phi_n;\Pi_n),\end{aligned}$$

where m and M are the constants appearing in (2.45). Since the same inequality holds when (Φ_n,Π_n) is replaced by (Φ,Π), we have that

$$\begin{aligned}\left|\varphi_n''-\varphi''\right|^2 &= \left|\partial_\varphi W(\Phi_n;\Pi_n)-\partial_\varphi W(\Phi;\Pi)\right|^2 \\ &\leq C(\Pi)\left[W(\Phi_n;\Pi_n)+W(\Phi;\Pi)\right],\end{aligned} \tag{2.56}$$

everywhere in $\mathbb{R}$ for some constant $C(\Pi)>0$. Also, note that (2.55), the continuity of $D_\Phi W$ and the pointwise convergence of Φ_n to Φ imply that $\Phi_n''\to\Phi''$ pointwise. Using these facts together with (2.51), (2.56), and the variant of the DCT from [11], we conclude that $\varphi_n''\to\varphi''$ in $L^2(\mathbb{R})$. A similar argument shows that $\sigma_n''\to\sigma''$ in $L^2(\mathbb{R})$. $\square$

Lemma 2.15 (Exponential decay)**.** *Let $\Pi\in\mathcal{O}^0$, $I_\pm(z):=(sign(z),0)\in\mathbb{R}\times\mathbb{C}$ and Φ be a minimizer of $\mathcal{A}_\Pi^{1d}$ in $\mathcal{H}$. Then, for each $k\in\mathbb{N}$ and all $z\in\mathbb{R}\setminus\{0\}$:*

$$\left|\frac{d^k}{dz^k}\left(\Phi-I_\pm\right)(z)\right|\leq C_k e^{-\alpha|z|}, \qquad \textit{with } \alpha=\frac{3}{4}\sqrt{\lambda_{min}(\Pi)}, \tag{2.57}$$

where $\lambda_{min}(\Pi)>0$ is the minimum eigenvalue of $D_\Phi^2 W(\pm1,0;\Pi)$ and $C_k>0$ is a constant depending only on Π and on Φ.

Proof. Step 1. $k=0$. Let

$$\nu(z):=\frac{1}{2}\left|\Phi(z)-\mathrm{I}_\pm(z)\right|^2,$$

so that

$$\lim_{|z|\to\infty} \nu(z) = 0 \qquad \text{and} \qquad \nu''(z) = |\Phi'(z)|^2 + [\Phi(z) - \mathrm{I}_{\pm}(z)] \cdot \Phi''(z).$$

The minimality of Φ implies that (2.44) holds and that $\Phi'' = D_{\Phi}W(\Phi;\Pi)$. Using this and the fact that both $W(\mathrm{I}_{\pm};\Pi)$ and $D_{\Phi}W(\mathrm{I}_{\pm};\Pi)$ vanish everywhere, the computation of the Taylor expansion around $I_{\pm}(z)$ of $W(\Phi(z);\Pi)$ shows that

$$\begin{aligned} \nu''(z) &= \left|\Phi'(z)\right|^2 + [\Phi(z) - \mathrm{I}_{\pm}(z)] \cdot \Phi''(z) \\ &\overset{(2.44)}{=} 2W(\Phi(z);\Pi) + [\Phi(z) - \mathrm{I}_{\pm}(z)] \cdot D_{\Phi}W(\Phi(z);\Pi) \\ &= [\Phi(x) - \mathrm{I}_{\pm}] \cdot \left[2D_{\Phi}^2 W(\mathrm{I}_{\pm};\Pi) \cdot (\Phi(z) - \mathrm{I}_{\pm}(z)) + O(\nu)\right] \\ &\geq 4\lambda_{\min}(\Pi)\nu(z) + O(\nu^{3/2}(z)). \end{aligned} \tag{2.58}$$

Let $M > 0$ such that

$$\nu''(z) \geq (3/2)^2\, \lambda_{\min}(\Pi)\nu(z), \qquad \text{whenever } |z| > M. \tag{2.59}$$

It follows from (2.59), Lemma B.7, and the parity of φ and σ, that

$$\nu(z) \leq \nu(M)e^{\frac{3}{2}\alpha(M-|z|)} \quad \text{whenever } |z| \geq M,$$

where $\alpha = \sqrt{\lambda_{\min}(\Pi)}$ and M depends on Φ and Π. Additionally, since $\nu(x) \leq 2+m_\sigma^2$ for all $x \in \mathbb{R}$, we conclude that for $C_0(\Pi,\Phi) := \left(2+m_\sigma^2\right) e^{\frac{3}{2}\alpha M}$, one has that

$$|\Phi(\Pi,z) - I_{\pm}(z)| \leq C_0(\Pi,\Phi)e^{-\frac{3}{4}\alpha|z|}, \quad \text{for all } z \in \mathbb{R}. \tag{2.60}$$

The inequality (2.60) gives the estimate in (2.57) corresponding to $k = 0$.

Step 2. $k = 1$. Note that the fact that $|\varphi(z)| \leq 1$ for all $z \in \mathbb{R}$ implies that

$$\left(\varphi^2 - 1\right) \leq 2\left(\varphi - \mathrm{sign}\right).$$

Combining this with the fact that $\sigma(z) \in (0, m_\sigma)$ for all z, and the form (1.39) of W, we obtain

$$W(\Phi(z);\Pi) \leq \tilde{C}(\Pi)\left|\Phi(z) - \mathrm{I}_{\pm}(z)\right|^2, \qquad \text{for all } z \in \mathbb{R}, \tag{2.61}$$

where $\tilde{C}(\Pi) = \frac{1}{4}\max\left\{\lambda_{\varphi}, 2\beta + \lambda_{\sigma} m_{\sigma}^2\right\}$. In view of (2.61) and (2.44), the estimate for $k = 1$ follows from that of $k = 0$.

Step 3. $k = 2$. Note that

$$\begin{aligned} \left|\partial_{\varphi} W(\Phi;\Pi)\right| &= \left|\lambda_{\varphi}\left(\varphi^2 - 1\right)\varphi + \beta\varphi\sigma^2\right| \leq 2\lambda_{\varphi}\left|(\varphi - sign)\right| + \beta m_{\sigma}\sigma \\ &\leq \max\left\{2\lambda_{\varphi}, \beta m_{\sigma}\right\}\left|\Phi - \mathrm{I}_{\pm}\right| \\ \left|\partial_{\sigma_R} W(\Phi;\Pi)\right| &= \left|\lambda_{\sigma}(\sigma^2 - m_{\sigma}^2)\sigma + \beta\varphi^2\sigma\right| \leq \beta\left|\sigma\right| \leq \beta\left|\Phi - \mathrm{I}_{\pm}\right|, \end{aligned}$$

which together with (2.44) and the estimate for $k = 0$ proves the estimate (2.57) for $k = 2$.

Step 4. $k = n \geq 3$. Each of these estimates follows by differentiating $k - 2$ times the identity $\Phi'' = D_{\Phi} W(\Phi;\Pi)$ and applying a similar argument to that of $k = 2$, using the pointwise boundedness of Φ and of its derivatives (Step 3 of Lemma 2.12), the fact that $W(\Phi,\Pi)$ and its derivatives are all polynomials in terms of φ and σ, and the estimates in (2.57) for all $k \leq n - 2$. □

2.2.3 *Spectral Estimates*

We now show the coercivity result appearing in (2.7).

Lemma 2.16. *Let $\Pi \in \mathcal{O}^0$ and let Φ be a minimizer of $\mathcal{A}_{\Pi}^{1d}$ in $\mathcal{H}$. Then,*

$$\mathcal{Q}(\Phi;\Pi)[V] \geq 0, \qquad \textit{for all } V \in X_1^3, \tag{2.62}$$

with equality if and only if $V \in \ker\left(\mathcal{L}(\Phi;\Pi)\right)$.

Proof. Let $V \in X_1^3$ and write $\mathcal{Q} := \mathcal{Q}(\Phi;\Pi)$ for convenience. Since

$$\left.\frac{d^2}{dh^2}\right|_{h=0} A_{\Pi}^{1d}(\Phi_{\Pi} + hV) = \mathcal{Q}[V], \tag{2.63}$$

and Φ_{Π} is a minimizer of $\mathcal{A}_{\Pi}^{1d}$ on $\mathcal{H}$, the quantity on (2.63) must be non-negative, and (2.62) follows.

The statement $V \in \ker(\mathcal{L}_\Pi) \implies \mathcal{Q}[V] = 0$ follows immediately from the definition of $\mathcal{Q}$. To prove the reverse implication, let $V \in X_1^3$ be such that $\mathcal{Q}[V] = 0$. Since V minimizes $\mathcal{Q}$ by (2.62), it follows that

$$0 = \left.\frac{d}{dh}\right|_{h=0} \mathcal{Q}\left[V + h\Psi\right] = 2\int_{\mathbb{R}} \Psi \cdot \mathcal{L}\left(\Psi;\Pi\right)\left[V\right] dz,$$

for all $\Psi \in C_c^\infty(\mathbb{R};\mathbb{R}^3)$, where the second equality follows by the definition of $\mathcal{Q}$ and the fact that $(F,G) \mapsto \int_{\mathbb{R}} F \cdot \mathcal{L}(\Psi;V)[G]$ is symmetric. As a result, we conclude that $V \in \ker \mathcal{L}(\Phi;\Pi)$. □

Lemma 2.17 (L^2 Spectral Estimate)**.** *Let $\Pi \in \mathcal{O}^0$ and Φ be a minimizer of $\mathcal{A}_\Pi^{1d}$ in $\mathcal{H}$. Then, there exists $\gamma(\Pi) > 0$ such that*

$$\mathcal{Q}(\Phi;\Pi)[V] > \gamma(\Pi)\|V\|_{X_0^3}^2, \qquad \textit{for all } V \in \mathfrak{C}_1(\mathcal{L}(\Phi;\Pi)). \tag{2.64}$$

Proof. For convenience, let $\mathcal{L} := \mathcal{L}(\Phi;\Pi)$ and $\mathcal{Q} := \mathcal{Q}(\Phi;\Pi)$. By the linearity of $\mathcal{L}$, (2.64) is equivalent to

$$\mathcal{Q}[V] > \gamma, \qquad \text{for all } V \in A, \tag{2.65}$$

where

$$A := \left\{V : V \in \mathfrak{C}_1\left(\mathcal{L}(\Phi;\Pi)\right) \text{ and } \|V\|_{X_0^3} = 1\right\}.$$

Or, equivalently,

$$\gamma := \inf_{V \in A} \mathcal{Q}[V] > 0. \tag{2.66}$$

Let $(V_k)_1^\infty \subset A$ be a minimizing sequence for $\mathcal{Q}$ over A. We have that

$$\begin{aligned}\int_{\mathbb{R}} |V_k'|^2 &= \mathcal{Q}\left[V_k\right] - \int_{\mathbb{R}} V_k D_\Phi^2 W(\Phi;\Pi) V_k \\ &\le \mathcal{Q}\left[V_k\right] + 3\|D_\Phi^2 W(\Phi;\Pi)\|_\infty \int_{\mathbb{R}} |V_k|^2 \\ &\le \mathcal{Q}\left[V_k\right] + 3\|D_\Phi^2 W(\Phi;\Pi)\|_\infty,\end{aligned}$$

where

$$\left\|D_\Phi^2 W(\Phi;\Pi)\right\|_\infty := \sup_{\substack{|\alpha|=2 \\ z\in\mathbb{R}}} \left|D_\Phi^\alpha W(\Phi(z);\Pi)\right|,$$

which is finite since $W(\cdot;\Pi)$ is smooth and Φ is pointwise bounded. It follows that $\left(\|V_k'\|_{X_0^3}\right)_{k=1}^{\infty}$ is a bounded sequence, and as a result, so is $\left(\|V_k\|_{X_1^3}\right)_{k=1}^{\infty}$, since $\|V_k\|_{X_0^3} = 1$ for all k. For convenience, we pass to a subsequence for which the norms $\|V_k'\|_{X_0^3}$ converge. Now, for similar reasons to the ones outlined in the proof of Lemma 2.11, there exists a function $V \in X_1^3$ and a subsequence of $(V_k)_1^\infty$, also labeled $(V_k)_1^\infty$, such that

$$V_k \rightharpoonup V \text{ weakly in } X_1^3 \text{ (and thus in } X_0^3) \tag{2.67a}$$

$$V_k' \rightharpoonup V' \text{ weakly in } X_0^3 \tag{2.67b}$$

$$V_k \to V \text{ locally uniformly in } \mathbb{R}. \tag{2.67c}$$

In particular, by the above and the lower semicontinuity of norms,

$$\|V\|_{X_0^3} \le \lim_{k\to\infty} \|V_k\|_{X_0^3} = 1 \qquad \text{and} \qquad \left\|V'\right\|_{X_0^3} \le \lim_{k\to\infty} \left\|V_k'\right\|_{X_0^3}. \tag{2.68a}$$

Let $I_N := [-N, N]$ for every $N > 0$, and define

$$A := D_\Phi W(1,0;\Pi), B(z) := D_\Phi W(\Phi(z);\Pi), \text{ and } C(z) = B(z) - A.$$

For any $N > 0$ we have that

$$\int_{\mathbb{R}} V_k^T B V_k = \int_{I_N} V_k^T B V_k + \int_{I_N^c} V_k^T C V_k + \int_{I_N^c} V_k^T A V_k.$$

Also, by (2.67c), $\lim_{k\to\infty} \int_{I_N} V_k^T B V_k = \int_{I_N} V^T B V$, and thus

$$\begin{aligned}\lim_{k\to\infty} \int_{\mathbb{R}} V_k^T B V_k \ge \int_{\mathbb{R}} V^T B V - \int_{I_N^c} V^T B V \\ - \liminf_{k\to\infty} \int_{I_N^c} V_k^T C V_k + \liminf_{k\to\infty} \int_{I_N^c} V_k^T A V_k.\end{aligned} \tag{2.69}$$

Now, note that $A, B(z), C(z)$ are symmetric for each $z \in \mathbb{R}$, $A \succ 0$ and $B(z) \to A$ as $|z| \to \infty$. Therefore, denoting by $\lambda_{M,\min}$ and $\lambda_{M,\max}$ the minimum and maximum eigenvalues of the 3 by 3 matrix M, respectively, we infer from (2.69) that

$$\begin{aligned}\lim_{k\to\infty}\int_{\mathbb{R}} V_k^T B V_k \geq \int_{\mathbb{R}} V^T B V - \left(\sup_{|z|>N} \lambda_{B(z),\max}\right) \|V\|^2_{L^2(I_N^c)} \\ + \left[\lambda_{A,\min} - \sup_{|z|>N} \lambda_{C(z),\max}\right] \lim_{k\to\infty} \|V_k\|^2_{L^2(I_N^c)}.\end{aligned} \tag{2.70}$$

Additionally, by (2.67c), we have that $\lim_{k\to\infty} \|V_k\|^2_{L^2(I_N^c)} = \lim_{k\to\infty}\left[1 - \|V_k\|^2_{L^2(I_N)}\right] = 1 - \|V\|^2_{L^2(I_N)}$. We also have that $\lim_{N\to\infty} \|V\|^2_{L^2}(I_N^c) = 0$, $\lim_{z\to\infty} \lambda_{B(z),\max} = \lambda_{A,\min}$ and $\lim_{z\to\infty} \lambda_{C(z),\max} = 0$. Thus, taking the limit as $N \to \infty$ in (2.70), we find that

$$\lim_{k\to\infty}\int_{\mathbb{R}} V_k^T B V_k \geq \int_{\mathbb{R}} V^T B V + \lambda_{A,\min}\left(1 - \|V\|^2_{X_0^3}\right). \tag{2.71}$$

Using (2.71) together with (2.68a) while noting that $\lambda_{A,\min} = \lambda_{\min}(\Pi)$, we find that

$$\begin{aligned}\gamma &= \lim_{k\to\infty} \mathcal{Q}[V_k] \\ &= \lim_{k\to\infty}\int_{\mathbb{R}} \left|V_k'\right|^2 + \lim_{k\to\infty}\int_{\mathbb{R}} V_k^T B V_k \\ &\geq \int_{\mathbb{R}} \left(\left|V'\right|^2 + V^T B V\right) + \lambda_{\min}(\Pi)\left(1 - \|V\|^2_{X_0^3}\right) \\ &= \mathcal{Q}[V] + \lambda_{\min}(\Pi)\left(1 - \|V\|^2_{X_0^3}\right).\end{aligned} \tag{2.72}$$

Now, since $V_k \rightharpoonup V$ in X_0^3 and $(V_k)_{k=1}^{\infty} \subset (\ker\mathcal{L})^{\perp}$, we have that $V \in (\ker\mathcal{L})^{\perp}$. As a result, $\mathcal{Q}[V] \geq 0$ with equality if and only if $V = 0$ by Lemma 2.16. The result follows from this observation, (2.72), and the fact that $\|V\|_{X_0^3} \leq 1$. □

The proof of Theorem 2.1 follows as a direct consequence of Lemma 2.17:

Proof of Theorem 2.1. Let $\Pi \in \mathcal{O}^0$ and Φ be a minimizer of $\mathcal{A}_{\Pi}^{1d}$ in $\mathcal{H}$. By Lemma 2.12, $\Phi = (\varphi, e^{i\theta}\sigma)$ for pointwise bounded φ, σ. Therefore, the continuity of $D_{\Phi}^2 W$ implies the existence of $C = C(\Pi) > 0$ such that

$$\sup_{z\in\mathbb{R}} \sup_{\substack{v\in\mathbb{R}^3 \\ \|v\|_2=1}} \left|v^T D_{\Phi}^2 W(\Phi(z);\Pi) v\right| \leq C.$$

In particular, we have that

$$\left|V(z)^T D_\Phi^2 W(\Phi(z);\Pi)V(z)\right| \leq C|V(z)|^2, \qquad \text{for all } z\in\mathbb{R}, V\in X_2^3. \quad (2.73)$$

Let $V\in\mathfrak{C}_1(\mathcal{L}(\Phi;\Pi))$. By Lemma 2.17, there is a $\gamma=\gamma(\Pi)>0$ such that

$$\int_\mathbb{R}\left[\left|V'\right|^2 + V^T D_\Phi^2 W(\Phi;\Pi)V - \gamma\left|V\right|^2\right] > 0.$$

Using this and (2.73),

$$\begin{aligned} 0 &< \int_\mathbb{R}\left[\left|V'\right|^2 + \left(1+\frac{\gamma}{C}\right)V^T D_\Phi^2 W(\Phi;\Pi)V\right] \\ &= \left(1+\frac{\gamma}{C}\right)\mathcal{Q}[V] - \frac{\gamma}{C}\int_\mathbb{R}\left|V'\right|^2, \end{aligned}$$

from which we conclude that

$$\mathcal{Q}[V] \geq \frac{\gamma}{C+\gamma}\int_\mathbb{R}\left|V'\right|^2.$$

The result follows from this last inequality and Lemma 2.17. □

2.2.4 *Extension of Properties of $\Sigma^{N,I}$ to an Open Set*

In what follows, we will denote by X_2^3 the normed space $\left((H^2(\mathbb{R}))^3, \|\cdot\|_{X_2^3}\right)$ as before, and we will write $X_{2,0}^3$ to denote the normed space $\left((H^2(\mathbb{R}))^3, \|\cdot\|_{X_0^3}\right)$. Furthermore, we introduce the Banach spaces

$$\mathcal{X}_k^s := \left\{(\varphi,\sigma_R,\sigma_I)\in X_k^3 : \sigma_R \text{ is even and } \varphi \text{ and } \sigma_I \text{ are odd}\right\}.$$

as subspaces of X_k^3 for each k.

Definition 2.18. Let $\Pi\in\Sigma^{N,I}$ and, in relation to the operators defined in (2.6) and in (2.16), introduce the following shifted operators defined for all $P\in(0,\infty)^4$ and all $V\in\left(H^2(\mathbb{R})\right)^3$:

$$\begin{aligned} \mathcal{F}_\Pi(V,P) &:= \mathcal{F}(\mathfrak{m}(\Pi)+V;\Pi+P), \\ \mathfrak{L}_\Pi(V,P)[\Psi] &:= \mathcal{L}(\mathfrak{m}(\Pi)+V;\Pi+P)[\Psi], \\ \mathcal{L}_\Pi(\Psi) &:= \mathfrak{L}_\Pi(0,0)[\Psi] = -\Psi'' + D_\Phi^2 W(\mathfrak{m}(\Pi);\Pi)\Psi. \end{aligned}$$

With these definitions, we have that $\mathcal{F}_{\Pi}(0,0)=0$ and

$$\mathfrak{m}(\Pi) =: (\varphi_{\Pi},\sigma_{\Pi},0), \text{ where } \varphi_{\Pi} \text{ is odd and } \sigma_{\Pi} \text{ is even by (2.16)}, \tag{2.74a}$$

$$\ker \mathcal{L}_{\Pi} = \operatorname{span}_{\mathbb{R}}\left\{(\varphi'_{\Pi},\sigma'_{\Pi},0),(0,0,\sigma_{\Pi})\right\} \text{ by Theorem 2.5.} \tag{2.74b}$$

In what follows, we use the above properties to justify the application of the Implicit Function Theorem (IFT) to the restriction map $\mathcal{F}^s_{\Pi} := \mathcal{F}_{\Pi}\Big|_{\mathcal{X}^s_2\times(0,\infty)^4} : \mathcal{X}^s_2 \times \mathcal{O}^0_{\Pi} \to \mathcal{X}^s_0$ around the origin. By doing so, we are able to (smoothly) extend the properties (parity, exponential decay and non-degeneracy) of the (unique) minimizers of $\mathcal{A}^{1d}_{\Pi}$ in $\mathcal{H}^s$ for $\Pi \in \Sigma^{N,I}$ to minimizers of $\mathcal{A}^{1d}_{\Pi}$ corresponding to all Π in an open set of parameters containing $\Sigma^{N,I}$. The main result, Proposition 2.21, relies on the fact that (a) $\mathcal{X}^s_2 \subset \mathfrak{C}_2(\mathcal{L}_{\Pi})$ whenever $\Pi \in \Sigma^{N,I}$ by (2.74a) and (2.74b), and (b) $D_V\mathcal{F}^s_{\Pi}(0,0) = \mathfrak{L}_{\Pi}(0,0)\Big|_{\mathcal{X}^s_2} : \mathcal{X}^s_2 \to \mathcal{X}^s_0$ is bijective for each $\Pi \in \Sigma^{N,I}$ (see Corollary 2.20). The latter property in turn follows from the following pivotal result:

Lemma 2.19 ($\mathcal{L}_{\Pi}$ Properties)**.** *Let $\Pi \in \Sigma^{N,I}$. Then, $\mathcal{L}_{\Pi} : X^3_{2,0} \to X^3_0$, where $\mathcal{L}_{\Pi}(V) := -V''+D^2_{\Phi}W(\mathfrak{m}(\Pi);\Pi)V$, is Fredholm. Moreover,* $\operatorname{Im}(\mathcal{L}_{\Pi}) = \ker(\mathcal{L}_{\Pi})^{\perp}$ *and therefore,*

$$\mathcal{L}^{\perp}_{\Pi} : \mathfrak{C}_2(\mathcal{L}_{\Pi}) \to \mathfrak{C}_0(\mathcal{L}_{\Pi}), V \mapsto \mathcal{L}_{\Pi}(V) \textit{ is bijective.} \tag{2.75}$$

Proof. $\mathcal{L}_{\Pi} : X^3_{2,0} \to X^3_0$ is a self-adjoint operator, and therefore it is closed. Furthermore, $\dim(\ker(\mathcal{L}_{\Pi})) = 2$ by Theorem 2.5, and by Lemma 2.17 we have that

$$\exists c>0 \text{ such that } \|\mathcal{L}_{\Pi}(V)\|_{X^3_0} \geq c\|V\|_{X^3_0}, \qquad \text{for all } V \in \mathfrak{C}_2(\mathcal{L}_{\Pi}).$$

This last property, together with e.g., [18] (Theorem V-5.2) or [15] (Theorem 3.3), allow us to conclude that the image of $\mathcal{L}_{\Pi}$ is closed in X^3_0. In particular, we have that $\left(\operatorname{Im}\mathcal{L}^{\perp}_{\Pi}\right)^{\perp} = \operatorname{Im}\mathcal{L}_{\Pi}$. On the other hand, the self-adjointness of $\mathcal{L}_{\Pi}$ implies that $(\operatorname{Im}\mathcal{L}_{\Pi})^{\perp} = \ker\mathcal{L}_{\Pi}$ (see e.g., [18]). Therefore,

$$\operatorname{Im}\mathcal{L}_{\Pi} = (\ker\mathcal{L}_{\Pi})^{\perp}. \tag{2.76}$$

Since $\ker\mathcal{L}_{\Pi}$ is closed (being a finite dimensional subspace of a Hilbert space), we have that $X^3_0 = \ker\mathcal{L}_{\Pi} \oplus (\ker\mathcal{L}_{\Pi})^{\perp}$ and thus, the codimension of Im $\mathcal{L}_\Pi$ in X^3_0 is finite by (2.76). Therefore, $\mathcal{L}_\Pi$ is Fredholm. Lastly, (2.75) follows by the second equality in (2.76) and the definition of $\mathfrak{C}_2(\mathcal{L}_\Pi)$.

Alternative proof that $\mathcal{L}_\Pi$ is Fredholm: If we take into account that

$$\lim_{|z|\to\infty} D^2_\Phi W(\Phi_\Pi(z);\Pi) = D^2_\Phi W((1,0);\Pi) = \mathrm{diag}\left(2\lambda_\varphi, \beta-\lambda_\sigma m_\sigma^2, \beta-\lambda_\sigma m_\sigma^2\right),$$

we may decompose $\mathcal{L}_\Pi$ as $\mathcal{L}_\Pi = L_{0,1} + L_{0,2}$ with

$$L_{0,1}(u,v,w) = \begin{pmatrix} -u'' + 2\lambda_\varphi u \\ -v'' + (\beta-\lambda_\sigma m_\sigma^2)v \\ -w'' + \left(\beta-\lambda_\sigma m_\sigma^2\right)w \end{pmatrix}$$

and

$$L_{0,2}(u,v,w) = \begin{pmatrix} \left[\partial^2_\varphi W(\Phi_\Pi;\Pi) - 2\lambda_\varphi\right]u + \partial_{\sigma_R}\partial_\varphi W(\Phi_\Pi;\Pi)v \\ \partial_{\sigma_R}\partial_\varphi W(\Phi_\Pi;\Pi)u + \left[\partial^2_{\sigma_R} W(\Phi_\Pi;\Pi) - (\beta-\lambda_\sigma m_\sigma^2)\right]v \\ \left[\partial^2_{\sigma_I} W(\Phi_\Pi;\Pi) - \left(\beta-\lambda_\sigma m_\sigma^2\right)\right]w \end{pmatrix}.$$

The spectrum of the operator $L_{0,1}\colon X^3_{2,0}\to X^3_0$ is equal to $[\min(2\lambda_\varphi, \beta - \lambda_\sigma m_\sigma^2), \infty)$ and it is purely essential, while the operator $L_{0,2}\colon X^3_0 \to X^3_0$ is relatively compact with respect to $L_{0,1}$ (compactness is provided via an Arzela-Ascoli argument using the continuous embedding $H^2(\mathbb{R}) \hookrightarrow C^{1,1/2}(\mathbb{R})$). By Weyl's Theorem [18, Theorem 5.35] the essential spectrum of a closed operator is stable under relatively compact perturbations, hence the essential spectrum of $L_{0,1}+L_{0,2}$ is again equal to $[\min(2\lambda_\varphi, \beta-\lambda_\sigma m_\sigma^2),\infty)$. Now observe that $\beta - \lambda_\sigma m_\sigma^2 > 0$ since $\Pi \in \mathcal{O}^0$, and we know that 0 belongs to the spectrum of $\mathcal{L}_\Pi(0,0)$ because of (2.74b); thus $\mathcal{L}_\Pi(0,0)$ is a Fredholm operator and therefore the image of $\mathcal{L}_\Pi$ is closed. In particular, Im $\mathcal{L}_\Pi = (\ker \mathcal{L}_\Pi)^\perp$ (see, for example, [18, Theorem 5.13]). □

Corollary 2.20. *Let $\Pi \in \Sigma^{N,I}$. Then, $\mathcal{L}^s_\Pi := \mathcal{L}_\Pi\Big|_{\mathcal{X}^s_2} : \mathcal{X}^s_2 \to \mathcal{X}^s_0$ is bounded and bijective.*

Proof. Let $\Phi_\Pi = \mathfrak{m}(\Pi)$. The form of $D^2_\Phi W(\Phi_\Pi;\Pi)$, (2.74a), the boundedness of Φ, and (2.74b) imply that

$$\mathcal{X}^s_2 \subset \mathfrak{C}_2(\mathcal{L}_\Pi) \text{ and } \mathcal{X}^s_0 \subset \mathfrak{C}_0(\mathcal{L}_\Pi), \tag{2.77}$$

$$\mathcal{L}_\Pi(V) \in \mathcal{X}^s_0 \text{ for all } V \in \mathcal{X}^s_2, \tag{2.78}$$

$$\|\mathcal{L}_{\Pi}(V)\|_{X_0^3} \leq C(\Pi)\,\|V\|_{X_2^3} \text{ for all } V \in X_2^3. \tag{2.79}$$

Thus $\mathcal{L}^s_{\Pi} : \mathcal{X}^s_2 \to \mathcal{X}^s_0$ defined by $\mathcal{L}^s_{\Pi}(V) = \mathcal{L}_{\Pi}(V) = \mathcal{L}^{\perp}_{\Pi}(V)$ for all $V \in \mathcal{X}^s_2$ is well-defined and bounded.

The injectivity of $\mathcal{L}^s_{\Pi}$ follows from that of $\mathcal{L}^{\perp}_{\Pi}$. To prove that $\mathcal{L}^s_{\Pi}$ is surjective, let $U \in \mathcal{X}^s_0$ and note that (2.75) and (2.77) imply that $\mathcal{L}^{\perp}_{\Pi}(V) = U$ for some $V \in \mathfrak{C}_2(\mathcal{L}_{\Pi})$. Decompose $V = (u_o + u_e, v_o + v_e, w_o + w_e)$, where u_o, v_o, w_o are odd functions and u_e, v_e, w_e are even functions. Then, by the linearity of $\mathcal{L}^{\perp}_{\Pi}$ and the form of $D^2_{\Phi}W(\Phi_{\Pi};\Pi)$, one has that

$$\mathcal{X}^s_0 \ni \mathcal{L}^{\perp}_{\Pi}(V) = \underbrace{\mathcal{L}^{\perp}_{\Pi}(u_e, v_o, w_e)}_{\in (\mathcal{X}^s_0)^{\perp}} + \underbrace{\mathcal{L}^{\perp}_{\Pi}(u_o, v_e, w_o)}_{\in \mathcal{X}^s_0},$$

and thus $\mathcal{L}^{\perp}_{\Pi}(u_o, v_e, w_o) = \mathcal{L}^{\perp}_{\Pi}(V)$. Since $(u_o, v_e, w_o) \in \mathcal{X}^s_2 \subset \mathfrak{C}_2(\mathcal{L}_{\Pi})$ and $\mathcal{L}^{\perp}_{\Pi}$ is injective, we deduce that $V = (u_o, v_e, w_o) \in \mathcal{X}^s_2$. $\square$

Proposition 2.21 (Extension of $\mathfrak{m}$)**.** *There exist open sets $\mathcal{O}^1, \mathcal{O}^2, \mathcal{O}^3 \subset (0,\infty)^4$ and a map $\overline{\mathfrak{m}}$ defined over $\mathcal{O}^1$ such that*

$$\Sigma^{N,I} \subset \mathcal{O}^3 \subset \mathcal{O}^2 \subset \mathcal{O}^1 \subset \mathcal{O}^0, \tag{2.80a}$$

$$\overline{\mathfrak{m}} - (\tanh, 0) \in C^{\infty}\left(\mathcal{O}^1; \mathcal{X}^s_2\right) \ \textit{and}\ \overline{\mathfrak{m}}\Big|_{\Sigma^{N,I}} = \mathfrak{m}, \tag{2.80b}$$

$$\overline{\mathfrak{m}}(\Pi) \textit{ is the unique minimizer of } A^{1d}_{\Pi} \textit{ in } \mathcal{H}^s \textit{ for all } \Pi \in \mathcal{O}^1, \tag{2.80c}$$

$$\ker\left(\mathcal{L}\left(\overline{\mathfrak{m}}(\Pi);\Pi\right)\right) = \mathit{span}_{\mathbb{R}}\left\{\overline{\mathfrak{m}}(\Pi)', i\mathcal{P}_{\sigma}\left(\overline{\mathfrak{m}}(\Pi)\right)\right\} \textit{ for all } \Pi \in \mathcal{O}^2, \tag{2.80d}$$

$$\overline{\mathfrak{m}}(\Pi) = (\varphi_{\Pi}, \sigma_{\Pi}, 0) \in \mathcal{H}^s \textit{ with } \sigma_{\Pi} \textit{ strictly positive for all } \Pi \in \mathcal{O}^3. \tag{2.80e}$$

Proof. Step 1. Existence of $\mathcal{O}^1$ and $\overline{\mathfrak{m}}$.

We have that $F_{\Pi} \in C^{\infty}(X^3_2 \times (0,\infty)^4, X^3_0)$ by Lemma B.1. Also, $\mathfrak{L}_{\Pi}$ is the Fréchet partial derivative of F_{Π} with respect to V. This fact follows since

$$\mathfrak{L}_{\Pi}(V,P)[\Psi] = \frac{d}{dh}\Bigg|_{h=0} \mathcal{F}_{\Pi}(V + h\Psi, P), \qquad \text{for all } P, \Psi \in X^3_2, \Pi \in (0,\infty)^4,$$

meaning that $\mathfrak{L}_{\Pi}$ is the Gateaux derivative of $\mathcal{F}_{\Pi}$ everywhere. By the smoothness of $\mathcal{F}_{\Pi}$, this implies that $\mathfrak{L}_{\Pi}$ is in fact the Fréchet derivative of $\mathcal{F}_{\Pi}$ (see e.g., [4]).

Consider now the restriction map $\mathcal{F}^s_{\Pi} : \mathcal{X}^s_2 \times (0,\infty)^4 \to \mathcal{X}^s_0$, $\mathcal{F}^s_{\Pi}(V,P) = \mathcal{F}_{\Pi}(V,P)$ for all $(V,P) \in \mathcal{X}^s_2 \times (0,\infty)^4$. This map is well defined since the form of $D_\Phi W(\Phi_\Pi;\Pi)$, (2.74b), and (2.74a) imply that $\mathcal{F}_\Pi(V) \in \mathcal{X}_0^s$ for all $V \in \mathcal{X}_2^s$. Also, $\mathcal{F}_\Pi^s \in C^\infty(\mathcal{X}_2^s;\mathcal{X}_0^s)$ by the smoothness of $\mathcal{F}_\Pi$, and $\mathfrak{L}_\Pi^s(V,P): \mathcal{X}_2^s \to \mathcal{X}_0^s$ given by $\mathfrak{L}_\Pi^s(V,P)[\Psi] := \mathfrak{L}_\Pi(V,P)[\Psi]$ for each $\Psi \in \mathcal{X}_2^s$ is the Fréchet derivative of $\mathcal{F}_\Pi^s$ at (V,P) for each $(V,P) \in \mathcal{X}_2^s \times (0,\infty)^4$. Finally, $\mathfrak{L}_\Pi^s(0,0) = \mathcal{L}_\Pi^s$, where $\mathcal{L}_\Pi^s : \mathcal{X}_2^s \to \mathcal{X}_0^s$ is the bijective and bounded linear map from Corollary 2.20. We may therefore invoke the Implicit Function Theorem (see e.g., [4]) to the map $\mathcal{F}_\Pi^s$ around $(V,P)=(0,0)$ to conclude that there exists of open sets $D_\Pi \subset (0,\infty)^4$ and $U_\Pi \subset \mathcal{X}_2^s$ with $(0,0) \in U_\Pi \times D_\Pi$, and a map $\mathcal{M}_\Pi \in C^\infty(D_\Pi;U_\Pi)$ such that

1. $\mathcal{F}_\Pi^s(\mathcal{M}_\Pi(P),P) = 0$ for all $P \in D_\Pi$.

2. If $\mathcal{F}_\Pi^s(V,P) = 0$ and $(V,P) \in U \times D_\Pi$, then $V = \mathcal{M}_\Pi(P)$.

By the definition of $\mathcal{F}_\Pi$, we see that $\mathcal{D}_\Pi := \Pi + D_\Pi$ and $\mathcal{U}_\Pi := \mathfrak{m}(\Pi) + U_\Pi \subset \mathcal{H}$ are neighbourhoods of Π and $\mathfrak{m}(\Pi)$, respectively, such that, together with the map $\mathfrak{m}_\Pi : \mathcal{D}_\Pi \to \mathcal{U}_\Pi$ given by $\mathfrak{m}_\Pi(P) := \mathfrak{m}(\Pi) + \mathcal{M}_\Pi(P-\Pi)$ for each $P \in \mathcal{D}_\Pi$, satisfy:

1. For all $P \in \mathcal{D}_\Pi$, the function $V = \mathfrak{m}_\Pi(P) \in \mathcal{H}$ solves

$$\mathcal{F}(V;P) = -V'' + D_\Phi W(V;P) = 0. \tag{2.81}$$

2. If $(V,P) \in \mathcal{U}_\Pi \times \mathcal{D}_\Pi$ satisfies (2.81), then $V = \mathfrak{m}_\Pi(P)$.

Claim: There exists an open set $\mathfrak{D}_\Pi \subset \mathcal{D}_\Pi \cap \mathcal{O}^0$ such that $\mathfrak{m}_\Pi(P)$ is the unique minimizer of $\mathcal{A}_P^{1d}$ in $\mathcal{H}^s$ for each $P \in \mathfrak{D}_\Pi$.

Proof of claim. $\mathcal{D}_\Pi \cap \mathcal{O}^0$ is a non-empty open set since both $\mathcal{D}_\Pi$ and $\mathcal{O}^0$ are open and contain Π. Also, the existence of a minimizer of $\mathcal{A}_P^{1d}$ in $\mathcal{H}^s$ for each $P \in \mathcal{O}^0$ is guaranteed by Lemma 2.12. Suppose that the statement of the claim does not hold. Then, there are sequences $(\Pi_n)_1^\infty \subset \mathcal{D}_\Pi \cap \mathcal{O}^0$ and $(\Phi_{\Pi_n})_1^\infty \subset \mathcal{H}^s$, with $\Pi_n \to \Pi$ and such that Φ_{Π_n} is a minimizer of $\mathcal{A}_{\Pi_n}^{1d}$ in $\mathcal{H}^s$ and $\mathfrak{m}_\Pi(\Pi_n) \neq \Phi_{\Pi_n}$ for each n. On the other hand, since both $V = \Phi_{\Pi_n}$ and $V = \mathfrak{m}_\Pi(\Pi_n)$ solve

$$-V'' + D_\Phi W\left(V;\Pi_n\right) = 0,$$

for each n, we have that $\Phi_n = \mathfrak{m}_\Pi(\Pi_n)$ unless $\Phi_n \notin \mathcal{U}_\Pi$ by the conclusion from the IFT. Therefore, $\Phi_{\Pi_n} \notin \mathcal{U}_\Pi$ for each n, and thus

$$\|\Phi_{\Pi_n} - \mathfrak{m}_\Pi(\Pi)\|_{X_2^3} > \epsilon, \text{ for some } \epsilon > 0 \text{ and all } n,$$

which contradicts Lemma 2.14 since $\mathfrak{m}_\Pi(\Pi)$ is the unique minimizer of $\mathcal{A}_\Pi^{1d}$ in $\mathcal{H}^s$ by Item 2 of Theorem 2.5. □

The above argument proves that for each $\Pi \in \Sigma^{N,I}$, there exists an open set $\mathfrak{D}_\Pi$ containing Π, and a map $\mathfrak{m}_\Pi : \mathfrak{D}_\Pi \to \mathcal{H}^s$ such that $\mathfrak{m}_\Pi - (\tanh, 0) \in C^\infty(\mathfrak{D}_\Pi, X_2^3)$ and $\mathfrak{m}_\Pi(P)$ is the unique minimizer of $\mathcal{A}_P^{1d}$ in $\mathcal{H}^s$ for each $P \in \mathfrak{D}_\Pi$. In particular, $\mathfrak{m}_\Pi$ agrees with $\mathfrak{m}$ over $\Sigma^{N,I}$, where $\mathfrak{m}$ is the map defined in (2.16). Therefore, letting $\mathcal{O}^1 := \bigcup_{\Pi \in \Sigma^{N,I}} \mathfrak{D}_\Pi$, the map $\overline{\mathfrak{m}} : \mathcal{O}^1 \to \mathcal{H}^s$ given by $\overline{\mathfrak{m}}(P) = \mathfrak{m}_\Pi(P)$ if $P \in \mathfrak{D}_\Pi$ is well-defined and satisfies the conditions outlined in Proposition 2.21.

Step 2. Non-degeneracy: existence of $\mathcal{O}^2$.

Consider the map $L : \mathcal{O}^1 \to L(X_2^3, X_0^3)$ defined by

$$L(P)[V] := \mathcal{L}(\overline{\mathfrak{m}}(P); P)[V] = -V'' + D_\Phi^2 W\left(\overline{\mathfrak{m}}(P); P\right) V, \quad P \in \mathcal{O}^1, V \in X_2^3,$$

so that $\overline{\mathfrak{m}}(P)$ is non-degenerate for P if and only if $\dim(\ker L(P)) = 2$. Fix $V \in X_2^3$ and $\Pi \in \mathcal{O}^1$, and let $P \in \mathcal{O}^1$ be such that $|P - \Pi| < 1$. Then, by the local Lipschitz continuity of $D_\Phi^2 W$, the pointwise boundedness of $\overline{\mathfrak{m}}(P)$ for each $P \in \mathcal{O}^1$, the continuous embedding $H^2(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, and the continuity of $\overline{\mathfrak{m}}$, we have that

$$\begin{aligned}
\|[L(P) - L(\Pi)][V]\|_{X_0^3}^2 &= \int_{\mathbb{R}} \left|\left[D_\Phi^2 W\left(\overline{\mathfrak{m}}(P); P\right) - D_\Phi^2 W\left(\overline{\mathfrak{m}}(\Pi); \Pi\right)\right] V\right|^2 \\
&\le C_0(\Pi) \int_{\mathbb{R}} \left[\left|\overline{\mathfrak{m}}\left(P\right)(z) - \overline{\mathfrak{m}}\left(\Pi\right)(z)\right| + |P - \Pi|\right]^2 |V|^2 \\
&\le C_1(\Pi) \left[\|\overline{\mathfrak{m}}(P) - \overline{\mathfrak{m}}(\Pi)\|_{X_2^3}^2 + |P - \Pi|^2\right] \|V\|_{X_2^3}^2 \\
&\le C_2(\Pi) \left|P - \Pi\right|^2 \|V\|_{X_2^3}^2 ,
\end{aligned} \tag{2.82}$$

for some constants (depending only on Π) $C_0, C_1, C_2 > 0$. As a result,

$$L(P) = L(\Pi) + \mathcal{T}(P, \Pi), \tag{2.83}$$

where $\mathcal{T}(P, \Pi) := L(P) - L(\Pi) \in L(X_2^3, X_0^3)$ and $\|\mathcal{T}(P, \Pi)\| = O\left(|P - \Pi|\right)$.

Let $\Pi \in \Sigma^{N,I}$. Since $\overline{\mathfrak{m}}(\Pi) = \mathfrak{m}(\Pi)$, $L(\Pi)$ is a Fredholm operator by Lemma 2.19 and $\dim(\ker L(\Pi)) = 2$ by Theorem 2.5. This, together with (2.83) and e.g., [18, IV - Theorem 5.22], imply the existence of an open neighbourhood $\Omega_\Pi \subset \mathcal{O}^1$ of Π such that $\dim(\ker L(P)) \leq 2$ whenever $P \in \Omega_\Pi$. In view of (2.9), the specific form of the kernel of $\mathcal{L}(\overline{\mathfrak{m}}(\Pi);\Pi)$ in (2.80d) follows for each $P \in \Omega_\Pi$, and consequently $\mathcal{O}^2 := \bigcup_{\Pi\in\Sigma^{N,I}} \Omega_\Pi$ satisfies (2.80a) and (2.80d).

Step 3. Existence of $\mathcal{O}^3$.

Let $\Pi \in \mathcal{O}^0$ and $\Phi_\Pi = (\varphi_\Pi, \sigma_\Pi, 0)$ be a minimizer of A^{1d}_Π in $\mathcal{H}^s$, whose existence is justified by Lemma 2.12. By Step 2 of the proof of Lemma 2.12, $\sigma_\Pi \in C^\infty_b(\mathbb{R})$ and $\sigma_\Pi(z) \geq 0$ for all $z \in \mathbb{R}$. Suppose that $\sigma_\Pi(z_0) = 0$ for some $z_0 \in \mathbb{R}$. Since σ_Π is bounded by 0 from below, we must have that $\sigma'_\Pi(z_0) = 0$. Therefore, both σ_Π and the zero function solve the initial value problem (cf. second equation in (2.3))

$$\begin{cases} \sigma'' + [\lambda_\sigma(\sigma^2 - m_\sigma^2) + \beta\varphi_\Pi^2]\sigma = 0, & \text{in } \mathbb{R}, \\ \sigma(z_0) = \sigma'(z_0) = 0. \end{cases}$$

As a result, standard ODE results imply that σ_Π is the zero function if it vanishes at any point.

By the above argument, and since σ_Π is known to be strictly positive for each $\Pi \in \Sigma^{N,I}$, the positivity of σ_Π can extended to an open set of parameters $\tilde{\mathcal{O}}^3$ by e.g., the continuity of the quantity $\|\sigma_\Pi\|_{L^2(\mathbb{R})}$ with respect to Π. As a result, the set $\mathcal{O}^3 := \tilde{\mathcal{O}}^2 \cap \mathcal{O}^2$ satisfies (2.80a) and (2.80e). □

Definition 2.22 (Admissible Parameters)**.** The set $\mathcal{O} := \mathcal{O}^3$ from Proposition 2.21 is called the set of ***admissible parameters***. Additionally, we define the restriction map

$$\mathfrak{M} := \overline{\mathfrak{m}}\Big|_{\mathcal{O}} : \mathcal{O} \to \mathcal{H}^s, \tag{2.84}$$

where $\overline{\mathfrak{m}}$ is the map from Proposition 2.21. The image of $\mathfrak{M}$ is called the set of ***one-dimensional profiles***.

Based on Lemma 2.12, Lemma 2.15, and Proposition 2.21, the map $\mathfrak{M}$ has the following properties:

$$\mathfrak{M} \in (\tanh, 0, 0) + C^\infty(\mathcal{O}; \mathcal{X}^s_2), \tag{2.85}$$

$$\mathfrak{M}(\Pi) \in C^{\infty}\left(\mathbb{R};\mathbb{R}^3\right), \tag{2.86}$$

$$[\mathfrak{M}(\Pi)]'' = D_{\Phi}W\left(\mathfrak{M}(\Pi);\Pi\right), \tag{2.87}$$

$$\ker \mathcal{L}(\mathfrak{M}(\Pi);\Pi) = \operatorname{span}_{\mathbb{R}}\left\{\mathfrak{M}(\Pi)', i\mathcal{P}_{\sigma}\left(\mathfrak{M}(\Pi)\right)\right\}, \tag{2.88}$$

$$\left|\frac{d^k}{dz^k}\mathfrak{M}(\Pi)(z)\right| \leq C_k(\Pi)e^{-\frac{3}{4}\sqrt{\lambda_{\min}(\Pi)}|z|}, \tag{2.89}$$

for some $C_k(\Pi) > 0$ and all $\Pi \in \mathcal{O}$.

Note that the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^{\infty}(\mathbb{R})$ implies that

$$\left\|\mathfrak{M}(\Pi) - \mathfrak{M}(\Pi')\right\|_{\infty} \leq C\left\|\mathfrak{M}(\Pi) - \mathfrak{M}(\Pi')\right\|_{X_2^3} \quad \text{for all } \Pi, \Pi' \in \mathcal{O}, \tag{2.90}$$

where $C > 0$ is a constant independent of Π and Π'. Thus by (2.85):

$$\sup_{z\in\mathbb{R}}\left|\mathfrak{M}(\Pi)(z) - \mathfrak{M}(\Pi')(z)\right| \to 0 \text{ whenever } \Pi \to \Pi'. \tag{2.91}$$

Corollary 2.23. *Let $\Pi \in \mathcal{O}$. Then, $\mathcal{L}(\mathfrak{M}(\Pi);\Pi) : X_{2,0}^3 \to X_0^3$ is Fredholm and*

$$\mathcal{L}(\mathfrak{M}(\Pi);\Pi)\Big|_{\mathfrak{C}_2(\mathcal{L}_{\Pi})} : \mathfrak{C}_2\left(\mathcal{L}_{\Pi}\right) \to \mathfrak{C}_0\left(\mathcal{L}_{\Pi}\right) \text{ is bijective.}$$

Proof. Let $\Pi \in \mathcal{O}$. The argument is virtually identical to the one from the proof of Lemma 2.19 since $\mathcal{L}_{\Pi} := \mathcal{L}(\mathfrak{M}(\Pi);\Pi)$ is self-adjoint and $\mathcal{O} \subset \mathcal{O}^0$. The latter justifies the application of Lemma 2.17, which ultimately implies that $\operatorname{Im}\mathcal{L}_{\Pi} = (\ker \mathcal{L}_{\Pi})^{\perp}$. Moreover, $\mathcal{L}_{\Pi}$ is Fredholm as a result of this and the fact that $\ker \mathcal{L}_{\Pi}$ is two-dimensional by (2.88). $\square$

2.2.5 *Properties of 1-dimensional Profiles*

Lemma 2.24 (Inversion of Linearized Operator (part I))**.** *Let $\Pi \in \mathcal{O}$ and $\Phi_{\Pi} := \mathfrak{M}(\Pi)$. Suppose that*[5] *$G \in \mathfrak{C}_0\left(\mathcal{L}(\Phi_{\Pi};\Pi)\right) \cap C^{\infty}\left(\mathbb{R}\right)$ and that there are positive constants $(C_k)_{k=0}^{\infty}$ such that*

$$\left|\frac{d^k G}{dz^k}(z)\right| \leq C_k e^{-\frac{1}{2}\sqrt{\lambda_{\min}(\Pi)}|z|}, \qquad \textit{for all } z \in \mathbb{R} \textit{ and } k \in \mathbb{Z}_+. \tag{2.92}$$

5 Note that (2.92) implies that $G \in X_k^3$ for each $k \in \mathbb{N}$, so that $G \in \mathfrak{C}_k(\mathcal{L})$ for each $k \in \mathbb{N}$.

Then, there exists a unique function $F \in \mathfrak{C}_2\left(\mathcal{L}(\Phi_\Pi;\Pi)\right) \cap C^\infty(\mathbb{R})$ *such that* $\mathcal{L}(\Phi_\Pi;\Pi)[F] = G$. *Moreover, for each* $k \in \mathbb{Z}_+$, *there exists a constant* $D_k(\Pi) > 0$ *such that*

$$\left|\frac{d^k F}{dz^k}(z)\right| \leq D_k(\Pi) e^{-\frac{1}{2}\sqrt{\lambda_{\min}(\Pi)}|z|}, \qquad \textit{for all } z \in \mathbb{R}. \tag{2.93}$$

Proof. For convenience, let $\mathcal{L} := \mathcal{L}(\Phi_\Pi;\Pi)$. The existence and uniqueness of $F \in \mathfrak{C}_2(\mathcal{L})$ such that

$$\mathcal{L}[F] = -F'' + D^2_\Phi W(\Phi_\Pi;\Pi) F = G \tag{2.94}$$

follows from Corollary 2.23. The smoothness of F follows from the fact that $F \in X^3_k$ for each $k \in \mathbb{N}$, which in turn follows from differentiating equation (2.94) repeatedly and by the smoothness and the exponential decay of $W(\Phi_\Pi;\Pi)$ and G.

To prove the exponential decay of F and of its derivatives, note[6] that the form of $D^2_\Phi W$ in (A.3) and the fact that $\varphi^2_\Pi - 1 < 0$ and $\sigma_\Pi \in (0, m_\sigma]$, where $(\varphi_\Pi, \sigma_\Pi, 0) := \Phi_\Pi$, imply that

$$\begin{aligned}
\left|V^T\left[D^2_\Phi W(\Phi_\Pi;\Pi) - D^2_\Phi W((\pm 1,0);\Pi)\right]V\right| = \Big| &\left[3\lambda_\varphi\left(\varphi^2_\Pi - 1\right) + \beta\sigma^2_\Pi\right]V_1^2 + \\
&\left[\beta\left(\varphi^2_\Pi - 1\right) + 3\lambda_\sigma\sigma^2_\Pi\right]V_2^2 + \left[\beta\left(\varphi^2_\Pi - 1\right) + \lambda_\sigma\sigma^2_\Pi\right]V_3^2 + 4\beta\varphi\sigma_\Pi V_1V_2\Big| \\
&\leq \left[3\lambda_\sigma m_\sigma + \beta(m_\sigma + 2)\right]\sigma_\Pi |V|^2.
\end{aligned} \tag{2.95}$$

Therefore, for $K(\Pi) := 3\lambda_\sigma m_\sigma + \beta\left(m_\sigma + 2\right) > 0$ and all $V \in \mathbb{R}^3$:

$$\begin{aligned}
V^T D^2_\Phi W(\Phi_\Pi;\Pi) V &= V^T D^2_\Phi W(1,0;\Pi) V + V^T\left[D^2_\Phi W(\Phi_\Pi;\Pi) - D^2_\Phi W(1,0;\Pi)\right]V \\
&\geq \lambda_{\min}(\Pi)|V|^2 - K(\Pi)\left|\Phi_\Pi \pm (1,0)\right||V|^2,
\end{aligned}$$

where $\lambda_{\min}(\Pi) > 0$ is the smallest eigenvalue of $D^2_\Phi W(1,0;\Pi)$. Next, let $a \in (0,1)$ use the fact that $\Phi_\Pi(z) \to (\pm 1, 0) \in \mathbb{R}\times\mathbb{C}$ as $z \to \pm\infty$ to obtain a $N(\Pi) > 0$ such that for all $V \in \mathbb{R}^3$:

$$V^T D^2_\Phi W(\Phi_\Pi;\Pi) V \geq a\lambda_{\min}(\Pi)|V|^2, \text{ whenever } |z| > N(\Pi, a).$$

6 Alternatively, note that the local Lipschitz continuity the entries of $D^2_\Phi W(\cdot,\cdot;\Pi)$ for each $\Pi \in \mathcal{O}$ implies the existence of $K(\Pi) > 0$ such that the same inequality holds.

Then, using (2.94) and Cauchy-Schwarz, we obtain for each $b \in (0, 2a)$:

$$\begin{aligned}\left(|F|^2\right)'' &= 2\left|F'\right|^2 + 2F \cdot F'' \\ &\geq 2F \cdot \left[D^2_{\Phi} W\left(\Phi_{\Pi}; \Pi\right) F - G\right] \\ &\geq (2a-b)\lambda_{\min}(\Pi)\, |F|^2 - \frac{1}{b\lambda_{\min}(\Pi)}\, |G|^2 \\ &\geq (2a-b)\lambda_{\min}(\Pi)\, |F|^2 - \frac{C_0^2}{b\lambda_{\min}(\Pi)} e^{-\sqrt{\lambda_{\min}(\Pi)}|z|},\end{aligned} \tag{2.96}$$

whenever $|z| > N(\Pi, a)$ and where $C_0 > 0$ is as in (2.92). Choosing a and b such that $2a - b > 1$ and applying Lemma B.7 to the case $z > 0$, and a similar argument for $z < 0$, it follows that there exists $K(\Pi, C_0) > 0$ such that

$$|F(z)| \leq K(\Pi, C_0) e^{-\frac{1}{2}\sqrt{\lambda_{\min}(\Pi)}|z|}, \qquad \text{for all } z \in \mathbb{R}, \tag{2.97}$$

which implies (2.93) for $k = 0$.

On the other hand, differentiating (2.94), we see that

$$\mathcal{L}[V'] = G' - \left[D^2_{\Phi} W(\Phi_{\Pi}; \Pi)\right]' V.$$

Therefore, applying the same argument used to obtain (2.97) with V' in place of V and $G' - \left[D^2_{\Phi} W(\Phi_{\Pi}; \Pi)\right]' V$ in place of G, we obtain the desired exponential decay of $|V'|$.

Similarly, the exponential decay of $|V^{(k)}|$ for $k \geq 2$ follows by an inductive argument on k obtained by differentiating (2.94) repeatedly, together with the exponential decay of the derivatives of G, of $[D^2_{\Phi} W(\Phi_{\Pi}; \Pi)]$, and of $|V^{(m)}|$ for all $m < k$. $\square$

We now verify the smoothness and the exponential decay of the derivatives of the 1-dimensional profiles. We will follow the notation introduced in Section 1.3.1.

Lemma 2.25. *The function* $\Phi : \mathcal{O} \times \mathbb{R} \to \mathbb{R} \times \mathbb{C}$ *defined by* $\Phi(\Pi, z) := \mathfrak{M}(\Pi)(z)$ *for all* $(\Pi, z) \in \mathcal{O} \times \mathbb{R}$, *where* $\mathcal{O}$ *and* $\mathfrak{M}$ *are as in Definition 2.22, is smooth.*

Proof. Step 1: continuity of Φ. Fix $(\Pi, z) \in \mathcal{O}\times\mathbb{R}$. For any $(\Pi_1, z_1) \in \mathcal{O}\times\mathbb{R}$ such that $\|(\Pi, z) - (\Pi_1, z_1)\|_{\mathbb{R}^5} \leq 1$, we have that

$$\begin{aligned}
&\|\Phi(\Pi, z) - \Phi(\Pi_1, z_1)\|_{\mathbb{R}^3} \\
&\qquad = \|\mathfrak{M}(\Pi)(z) - \mathfrak{M}(\Pi_1)(z_1)\|_{\mathbb{R}^3} \\
&\qquad \leq \|\mathfrak{M}(\Pi)(z) - \mathfrak{M}(\Pi)(z_1)\|_{\mathbb{R}^3} + \|\mathfrak{M}(\Pi)(z_1) - \mathfrak{M}(\Pi_1)(z_1)\|_{\mathbb{R}^3} \\
&\qquad \leq C(\Pi)|z - z_1| + C\,\|\mathfrak{M}(\Pi) - \mathfrak{M}(\Pi_1)\|_{X_2^3} \\
&\qquad \leq C(\Pi)\,\|(\Pi - \Pi_1, z - z_1)\|_{\mathbb{R}^5}\,,
\end{aligned}$$

where the second to last inequality follows from the local Lipschitz continuity of $\Phi(\Pi, \cdot)$ and (2.90), while the last inequality follows from the continuity of $\mathfrak{M}$. This proves that Φ is continuous.

Step 2: Existence and continuity of Π *derivatives of* Φ. Since $\mathfrak{M} \in (\tanh, 0, 0)+ C^\infty(\mathcal{O}; X_2^3)$, for $h \in \mathbb{R}$ and $\tilde{\Pi} \in \mathbb{R}^4$, we have that

$$\left\|\mathfrak{M}(\Pi + h\tilde{\Pi}) - \mathfrak{M}(\Pi) - hD\mathfrak{M}(\Pi)[\tilde{\Pi}]\right\|_{X_2^3} = o(|h|).$$

Therefore, by the (continuous) Sobolev embedding $H^2(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$:

$$\begin{aligned}
&\left\|\Phi(\Pi + h\tilde{\Pi}, z) - \Phi(\Pi, z) - hD\mathfrak{M}(\Pi)[\tilde{\Pi}](z)\right\|_{\mathbb{R}^3} \\
&\qquad = \left\|\mathfrak{M}(\Pi + h\tilde{\Pi})(z) - \mathfrak{M}(\Pi)(z) - hD\mathfrak{M}(\Pi)[\tilde{\Pi}](z)\right\|_{\mathbb{R}^3} \\
&\qquad \leq C\left\|\mathfrak{M}(\Pi + h\tilde{\Pi}) - \mathfrak{M}(\Pi) - hD\mathfrak{M}(\Pi)[\tilde{\Pi}]\right\|_{X_2^3} = o(|h|).
\end{aligned}$$

This shows that $D_\Pi\Phi(\Pi, z)$ exists. Also, denoting the i^{th} standard unit vector in $\mathbb{R}^4$ by e_i and setting $\tilde{\Pi} = e_i$, we find that $\partial_{p_i}\Phi(\Pi, z) := D_\Pi\Phi(\Pi, z)\cdot e_i = D\mathfrak{M}(\Pi)[e_i](z)$ for each $(\Pi, z) \in \mathcal{O}\times\mathbb{R}$ and $i = 1, 2, 3, 4$. Using this equality, the continuity of $\partial_{p_i}\Phi$ for each $i \in \{1, 2, 3, 4\}$ can be deduced with a similar argument used to show the continuity of Φ in Step 1. In particular,

$$\begin{aligned}
&\|\partial_{y_i}\Phi(\Pi, z) - \partial_{y_i}\Phi(\Pi_1, z_1)\|_{\mathbb{R}^3} \leq \\
&\|D\mathfrak{M}(\Pi)[e_i](z) - D\mathfrak{M}(\Pi)[e_i](z_1)\|_{\mathbb{R}^3} + \|D\mathfrak{M}(\Pi)[e_i](z_1) - D\mathfrak{M}(\Pi_1)[e_i](z_1)\|_{\mathbb{R}^3}\,,
\end{aligned}$$

and

$$\|D\mathfrak{M}(\Pi)[e_i](z) - D\mathfrak{M}(\Pi)[e_i](z_1)\|_{\mathbb{R}^3} \leq C|z - z_1|$$

by the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow C^{0,1/2}(\mathbb{R})$, and

$$\begin{aligned}
&\|D\mathfrak{M}(\Pi)[e_i](z_1) - D\mathfrak{M}(\Pi_1)[e_i](z_1)\|_{\mathbb{R}^3} \\
&\qquad\qquad \le \sup_{z\in\mathbb{R}} \|D\mathfrak{M}(\Pi)[e_i](z) - D\mathfrak{M}(\Pi_1)[e_i](z)\|_{\mathbb{R}^3} \\
&\qquad\qquad \le \|D\mathfrak{M}(\Pi)[e_i] - D\mathfrak{M}(\Pi_1)[e_i]\|_{X_2^3} \\
&\qquad\qquad \le \|D\mathfrak{M}(\Pi) - D\mathfrak{M}(\Pi_1)\|_{\mathcal{L}(\mathbb{R}^4, X_2^3)} \to 0 \text{ as } \Pi_1 \to \Pi.
\end{aligned}$$

We conclude that $\|\partial_{p_i}\Phi(\Pi, z) - \partial_{p_i}\Phi(\Pi_1, z_1)\|_{\mathbb{R}^3} \to 0$ as $(\Pi_1, z_1) \to (\Pi, z)$. An induction argument following this logic shows that for any multi-index $\alpha = (\alpha_i)_{i=1}^4 \in \mathbb{Z}_+^4$:

$$D_\Pi^\alpha \Phi(\Pi, z) = D^\alpha \mathfrak{M}(\Pi)[\overbrace{e_1, e_1, ..., e_1}^{\alpha_1 \text{ times}}, \overbrace{e_2, e_2, ..., e_2}^{\alpha_2 \text{ times}}, ..., \overbrace{e_4, e_4, ..., e_4}^{\alpha_4 \text{ times}}](z),$$

and thus $D_\Pi^\alpha\Phi$ is continuous and $D_\Pi^\alpha\Phi(\Pi, \cdot) \in X_2^3$.

Step 3: Continuity of $D_\Pi^\alpha \partial_z^k \Phi$. Let $\alpha = (\alpha_i)_{i=1}^4 \in \mathbb{Z}_+^4$ and $k \in \mathbb{Z}_+$. Differentiating both sides of (2.87) with respect to D_Π^α, we find that the continuity of $D_\Pi^\alpha \partial_z^2 \Phi$ follows from that of $D_\Pi^\beta \Phi$ for all $\beta = (\beta_i)_{i=1}^4$ with $\beta_i \le \alpha_i$ for $i = 1, 2, 3, 4$, which follows from step 3. The continuity of $D_\Pi^\alpha \partial_z \Phi$ follows from the continuity of $D_\Pi^\alpha \partial_z^2 \Phi$. For any $k > 2$, repeated differentiation of (2.87) with respect to ∂_z^{k-2} followed by D_Π^α shows that the continuity of $D_\Pi^\alpha \partial_z^k \Phi$ follows from that of that of $D_\Pi^\beta \partial_z^m \Phi$ for all $\beta = (\beta_i)_{i=1}^4$ with $\beta_i \le \alpha_i$ for $i = 1, 2, 3, 4$ and all $m \le k - 2$.

Step 4: Permuting derivatives. Since mixed weak derivatives commute, step 4 shows that weak derivatives of Φ of all orders are continuous. This, together with Lemma C.8 shows that Φ is smooth. □

Lemma 2.26. *Let $\Pi_0 \in \mathcal{O}$ and let Φ be as in Lemma 2.25. There exists an open subset $\mathcal{O}_{\Pi_0} \subset \mathcal{O}$ containing Π_0 such that for every $k \in \mathbb{Z}_+$ and every multi-index α, there exists a constant $C_{\alpha k}(\Pi_0)$ for which*

$$\left| D_\Pi^\alpha \partial_z^k \left[\Phi(\Pi, z) - I_\pm(z)\right]\right| \le C_{\alpha k}(\Pi_0) e^{-\frac{\sqrt{\lambda_{\min}(\Pi)}}{2}|z|} \tag{2.98}$$

holds for all $(\Pi, z) \in \mathcal{O}_{\Pi_0} \times (\mathbb{R}\backslash \{0\})$.

Remark 2.27. As shown in the proof of Lemma 2.26, the exponent on the exponential term can be changed to $-a\lambda_{\min}(\Pi)|z|$ for any $a \in (0, 1)$ for the cases corresponding to $|\alpha| = 0$. Also, a slight modification to the proof shows that the 5^{th} item of Theorem 2.3 holds. The argument in the later case is easier since no uniform lower bounds on $|z|$ (i.e., $N(\Pi_0)$) are needed. The only reason for proving locally uniform bounds as they appear in (2.98) is to justify the passing of a limit in the proof of Lemma 2.29.

Proof. Let $\Pi_0 \in \mathcal{O}$ be arbitrary and $\nu(\Pi, z) := \frac{1}{2}|\Phi(\Pi, z) - I_{\pm}(z)|^2$ as in Lemma 2.15. A direct computation using a Taylor expansion about $(\pm 1, 0)$ shows that there is a positive constant M independent of Π such that

$$\begin{aligned}\nu''(z) &= 2W(\Phi(\Pi, z); \Pi) + D_{\Phi}W(\Phi(\Pi, z); \Pi) \cdot [\Phi(\Pi, z) - I_{\pm}] \\ &\geq \left[4\lambda_{\min}(\Pi) - M|\Pi|\nu^{1/2}(z)\right]\nu(z).\end{aligned}$$

As a result, we have that

$$|\Phi(\Pi, z) - I_{\pm}| \leq \frac{7\sqrt{2}}{4M\,|\Pi|}\lambda_{\min}(\Pi) \implies \nu''(z) \geq \left(\frac{3}{2}\right)^2 \lambda_{\min}(\Pi)\nu(z). \quad (2.99)$$

Let U_{Π_0} be any open bounded subset such that $\Pi_0 \in U_{\Pi_0} \subset \overline{U}_{\Pi_0} \subset \mathcal{O}$ and

$$m_1(U_{\Pi_0}) := \min\left\{\frac{7\sqrt{2}}{4M\,|\Pi|}\lambda_{\min}(\Pi) : \Pi \in \overline{U}_{\Pi_0}\right\}. \quad (2.100)$$

Let $N(\Pi_0) > 0$ be large enough so that

$$|\Phi(\Pi_0, z) - I_{\pm}(z)| < m_1\left(U_{\Pi_0}\right)/2, \qquad \text{whenever } |z| > N(\Pi_0). \quad (2.101)$$

Also, let $\delta(\Pi_0) > 0$ be small enough so that

$$\sup_{z\in\mathbb{R}} |\Phi(\Pi_0, z) - \Phi(\Pi, z)| < m_1\left(U_{\Pi_0}\right)/2, \quad \text{whenever } |\Pi_0 - \Pi| < \delta(\Pi_0). \quad (2.102)$$

The fact that $\delta(\Pi_0)$ exists follows from (2.91). Finally, let $\mathcal{O}_{\Pi_0} := U_{\Pi_0} \cap B_{\delta(\Pi_0)}(\Pi_0)$.

Step 1: $\alpha = 0$. By construction, whenever $\Pi \in \mathcal{O}_{\Pi_0}$ and $|z| > N(\Pi_0)$:

$$|\Phi(\Pi, z) - I_{\pm}(z)| \leq |\Phi(\Pi_0, z) - \Phi(\Pi, z)| + |\Phi(\Pi_0, z) - I_{\pm}(z)| \leq m_1\left(U_{\Pi_0}\right).$$

By (2.99) and (2.100), we thus have that for all $\Pi \in \mathcal{O}_{\Pi_0}$:

$$\nu''(\Pi, z) \geq \left(\frac{3}{2}\right)^2 \lambda_{\min}(\Pi)\nu(\Pi, z), \text{ whenever } |z| > N(\Pi_0),$$

Therefore, by the same logic as the one leading to (2.60), we have that for all $\Pi \in \mathcal{O}_{\Pi_0}$ and all $z \in \mathbb{R}$:

$$|\Phi(\Pi, z) - I_{\pm}(z)| \leq C_0(\Pi_0)e^{-\frac{3}{4}\sqrt{\lambda_{\min}(\Pi)}|z|}, \tag{2.103}$$

where $C_0(\Pi_0) = \max\left\{\sqrt{4 + m_\sigma^2}e^{\frac{3}{4}\lambda_{\min}(\Pi)N(\Pi_0)} : \Pi = (\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in \overline{\mathcal{O}_{\Pi_0}}\right\}$. This proves the result for $k = 0$. The proof for the cases $k \geq 1$ follow exactly as in Steps 2 and 3 of Lemma 2.15.

Step 2: $|\alpha| = m \geq 1$. The analysis here is similar to the one from Lemma 2.24, except that the coefficients of the exponential bounds are taken uniformly on the closure of $\mathcal{O}_{\Pi_0}$.

Recall that for $K(\Pi) = 3\lambda_\sigma m_\sigma + \beta(m_\sigma + 2)$ and all $V \in \mathbb{R}^3$ (see (2.95)):

$$V^T D^2_\Phi W(\Phi_\Pi; \Pi)V \geq \lambda_{\min}(\Pi) |V|^2 - K(\Pi) |\Phi_\Pi - (1, 0)| |V|^2 . \tag{2.104}$$

Let

$$m_2(U_{\Pi_0}) := \frac{1}{5} \cdot \min\left\{\frac{\lambda_{\min}(\Pi)}{K(\Pi)} : \Pi \in \overline{U}\right\}.$$

Then, by possibly making $N(\Pi_0)$ (resp. $\delta(\Pi_0)$) from step 1 larger (resp. smaller), we obtain the estimates (2.101) and (2.102) with $m_1(U_{\Pi_0})$ replaced by $r(\Pi_0) := \min(m_1(U_{\Pi_0}), m_2(U_{\Pi_0}))$. Using these estimates together with (2.104) yields the following inequality that holds for all Π in $\mathcal{O}_{\Pi_0} = U_{\Pi_0} \cap B_{\delta(\Pi_0)}(\Pi_0)$ (same definition as before, but using the possibly smaller $\delta(\Pi_0)$):

$$V^T D^2_\Phi W(\Phi_\Pi; \Pi)V \geq \frac{4\lambda_{\min}(\Pi)}{5} |V|^2 , \text{ whenever } |z| > N(\Pi_0), \text{ for all } V \in \mathbb{R}^3. \tag{2.105}$$

Let $\Pi \in \mathcal{O}_{\Pi_0}$. Assume that (2.98) holds for all α with $|\alpha| < m$ and consider the case $|\alpha| = m$. Differentiating (2.87) with respect to D^α_Π yields

$$\mathcal{L}(\Phi(\Pi, \cdot); \Pi)[F(\Pi, \cdot)] = G(\Pi, \cdot), \tag{2.106}$$

where $F(\Pi, z) := D^{\alpha}_{\Pi}\Phi(\Pi, z)$ and[7]

$$G(\Pi, z) := -\mathcal{D}^{\alpha}_{\Pi}\left[D_{\Phi}W(\Phi;\Pi)\right](\Pi, z) + D^2_{\Phi}W(\Phi(\Pi, z);\Pi)D^{\alpha}_{\Pi}\Phi(\Pi, z).$$

As in Lemma 2.24, we use the estimate (2.105) to arrive at the following inequality, this time holding uniformly for all Π over the open set $\mathcal{O}_{\Pi_0}$ containing Π_0:

$$\begin{aligned}\partial_z^2\left(|F|^2\right) &\geq 2F\cdot\left[D^2_{\Phi}W\left(\Phi_{\Pi};\Pi\right)F - G\right]\\ &\geq \frac{6}{5}\lambda_{\min}(\Pi)\,|F|^2 - \frac{5}{2\lambda_{\min}(\Pi)}\,|G|^2\,, \quad \text{whenever } |z| > N(\Pi_0).\end{aligned} \tag{2.107}$$

Now, write $\Phi = \Phi(\Pi,\cdot) = \binom{\varphi}{\sigma}$ and note that the form of W implies that:

$$\begin{aligned}\left|D^{\alpha}_{\Pi}D_{\Phi}W(\Phi;\Pi)\right|^2 &\leq \begin{cases}\left[\varphi^4 + (1+\varphi^2)\sigma^2 + m_{\sigma}^2(4\lambda_{\sigma}-1)\right]\sigma^2 & \\ \quad + \left[\varphi^2(\varphi+1)^2\right](\varphi-1)^2, & |\alpha| = 1,\\ 4(2m_{\sigma}^2 + \lambda_{\sigma}^2)\sigma^2, & |\alpha| = 2,\\ 12\sigma^2, & |\alpha| = 3,\\ 0, & |\alpha| \geq 4.\end{cases}\\ &\leq C(\Pi)\,|\Phi - I_{\pm}|^2\,, \text{ for all } \alpha,\end{aligned} \tag{2.108}$$

for some continuous function C. Also,

$$|\Phi(\Pi, z)|^2 \leq 1 + m_{\sigma}^2, \text{ for all } (\Pi, z) \in \mathcal{O}\times\mathbb{R}. \tag{2.109}$$

Now, let P be any polynomial on six variables. The bound (2.109), together with the exponential decay of $D^{\gamma}_{\Pi}\Phi$ for all multi-indices γ with $|\gamma| < m$ coming from the induction hypothesis, implies the existence of $D_{\alpha}(\Pi_0)$ (depending on P) such that for all γ with $|\gamma| < m$ and all $(\Pi, z) \in \mathcal{O}_{\Pi_0}\times\mathbb{R}$:

$$\begin{aligned}&\left[|D^{\gamma}_{\Pi}\varphi(\Pi, z)| + |D^{\gamma}_{\Pi}\sigma(\Pi, z)|\right]|P(\varphi(\Pi, z), \sigma(\Pi, z), \Pi)|\\ &\qquad\leq D_{\alpha}(\Pi_0)e^{-\frac{\sqrt{\lambda_{\min}(\Pi)}}{2}|z|}.\end{aligned} \tag{2.110}$$

As a result of (2.108), (2.110) and the continuity of $\lambda_{\min}$, we conclude that

7 We use the notation $\mathcal{D}^{\beta}_{\Pi}$ to mean total derivative. For instance, $\mathcal{D}^{(1,0,0,0)}_{\Pi}W(\Phi;\Pi) = \frac{d}{d\lambda_{\varphi}}W(\Phi;\Pi) = D_{\Phi}W(\Phi;\Pi)\partial_{\lambda_{\varphi}}\Phi + \partial_{\lambda_{\varphi}}W(\Phi;\Pi)$.

there exists $\tilde{D}(\Pi_0) > 0$ such that for all $(\Pi, z) \in \mathcal{O}_{\Pi_0} \times \mathbb{R}$:

$$\frac{5}{2\lambda_{\min(\Pi)}} |G(\Pi, z)|^2 \leq \tilde{D}_\alpha(\Pi_0) e^{-\sqrt{\lambda_{\min}(\Pi)}|z|}. \tag{2.111}$$

Applying Lemma B.7 together with (2.107) and (2.111) to the case $z > 0$, and a similar argument for $z < 0$, it follows that[8] for all $z \in \mathbb{R}$:

$$|D^\alpha_\Pi \Phi(\Pi, z)|^2 \leq K(\Pi_0) e^{-\sqrt{\lambda_{\min}(\Pi)}|z|}, \tag{2.112}$$

where $K(\Pi_0) := \max_{\Pi \in \overline{\mathcal{O}}_{\Pi_0}} \{A(\Pi), B(\Pi)\}$ for

$$\begin{aligned} A(\Pi) &:= |D^\alpha_\Pi \Phi(\Pi, N(\Pi_0))|^2 \, e^{\sqrt{\frac{6}{5}\lambda_{\min}(\Pi)}N(\Pi_0)} + \frac{5\tilde{D}_\alpha(\Pi_0)}{\lambda_{\min}(\Pi)} \\ B(\Pi) &:= \max_{x \in [0, N(\Pi_0)]} |D^\alpha_\Pi \Phi(\Pi, x)|^2 \, e^{\sqrt{\lambda_{\min}(\Pi)}N(\Pi_0)}. \end{aligned}$$

Likewise, differentiating (2.106) k times with respect to z results in

$$\mathcal{L}(\Phi(\Pi, \cdot); \Pi) \left[\partial^k_z D^\alpha_\Pi \Phi(\Pi, \cdot)\right] = \tilde{G}(\Pi, \cdot),$$

where

$$\tilde{G}(\Pi, \cdot) := \partial^k_z G(\Pi, \cdot) - \partial^k_z \left(D^2_\Phi W(\Phi; \Pi) D^\alpha_\Pi \Phi(\Pi, \cdot)\right) + D^2_\Phi W(\Phi(\Pi, \cdot); \Pi) \partial^k_z D^\alpha_\Pi \Phi(\Pi, \cdot).$$

As a result, proceeding again inductively on the value of k, the exponential decay of $\partial^l_z D^\alpha_\Pi \Phi$ for all $l \leq k - 1$, the pointwise boundedness of $|\partial^l_z \Phi|$ for $l \leq k$, and the smoothness of W, imply the existence of $D_{\alpha k}(\Pi_0) > 0$ for which

$$\left|\tilde{G}(\Pi, z)\right| \leq D_{\alpha k}(\Pi_0) e^{-\frac{\sqrt{\lambda_{\min}(\Pi)}}{2}|z|}, \qquad \text{for all } (\Pi, z) \in \mathcal{O}_{\Pi_0} \times \mathbb{R}.$$

Finally, by a similar argument to the one used to arrive at (2.112) from (2.107) and (2.111), we conclude that there exists $C_{\alpha k}(\Pi_0) > 0$ for which

$$\left|D^\alpha_\Pi \partial^k_z \Phi(\Pi, z)\right| \leq C_{\alpha k}(\Pi_0) e^{-\frac{\sqrt{\lambda_{\min}(\Pi)}}{2}|z|}, \text{ for all } (\Pi, z) \in \mathcal{O}_{\Pi_0} \times \mathbb{R}.$$

□

8 Note that we need a decay with exponent bigger than $\sqrt{\lambda_{\min}(\Pi)}/2$ in the end so that we can then repeat the argument for higher derivatives, taking into account that Lemma B.7 requires $a \neq c$.

2.2.6 *Proofs of Theorems 2.3 and 2.4*

Proof of Theorem 2.3. Let $\mathfrak{M}$ be as in (2.84), and let $\Phi : \mathcal{O} \times \mathbb{R} \to \mathbb{R} \times \mathbb{C}$ be given by $\Phi(\Pi, z) = \mathfrak{M}(\Pi)(z)$ for each $(\Pi, z) \in \mathcal{O} \times \mathbb{R}$. Assertions 1, 3 and 4 follow directly from Proposition 2.21, whereas assertion 2 follows from Lemma 2.12 in view of assertion 1. Finally, assertions 5 and 6 follow from Lemma 2.25 and Lemma 2.26, respectively. □

Proof of Theorem 2.4. Step 1: Existence and Smoothness. Consider the map $\mathbb{A} : U \times X_2^3 \to X_0^3$ given by

$$\begin{aligned}\mathbb{A}(y, V) &:= -V'' + D_\Phi^2 W(\Phi_{f(y)}; f(y))V - G(y, \cdot)\\ &= \mathcal{L}\left(\Phi_{f(y)}; f(y)\right)[V] - G(y, \cdot).\end{aligned}$$

For every $y \in U$, Lemma 2.24 implies the existence of a smooth function $V_y \in \mathfrak{C}_2\left(\mathcal{L}\left(\Phi_{f(y)}; f(y)\right)\right)$ such that $\mathbb{A}(y, V_y) = 0$. Define $F : U \times \mathbb{R} \to \mathbb{R} \times \mathbb{C}$ by $F(y, z) = V_y(z)$ for all $(y, z) \in U \times \mathbb{R}$, so that

$$\mathcal{L}\left(\Phi_{f(y)}; f(y)\right)[F(y, \cdot)] = G(y, \cdot), \qquad \text{for all } y \in U, \tag{2.113}$$

thus proving Item 1 of the statement. To show the smoothness of F with respect to y, consider the map $\overline{\mathbb{A}} : U \times X_2^3 \to X_0^3$ given by

$$\overline{\mathbb{A}}(y, V) = \mathbb{A}(y, V) + \mathcal{P}(y)[V],$$

where $\mathcal{P} : U \to L(X_2^3, X_0^3)$ is given by

$$\mathcal{P}(y)[V] := \frac{\left\langle V, \Phi'_{f(y)} \right\rangle_{X_0^3}}{\left\|\Phi'_{f(y)}\right\|_{X_0^3}^2} \Phi'_{f(y)} + \frac{\left\langle V, i\mathcal{P}_\sigma(\Phi_{f(y)}) \right\rangle_{X_0^3}}{\left\|\mathcal{P}_\sigma(\Phi_{f(y)})\right\|_{X_0^3}^2} i\mathcal{P}_\sigma(\Phi_{f(y)}). \tag{2.114}$$

Note that $\mathcal{P}(y)[V_y] = 0$ for each $y \in U$, and thus

$$\overline{\mathbb{A}}(y, V_y) = \mathbb{A}(y, V_y) = 0, \qquad \text{for all } y \in U. \tag{2.115}$$

Also, the map $\overline{\mathbb{A}}$ is smooth, which can be seen by writing it as a composition/sum of smooth maps. Additionally, $D_V\overline{\mathbb{A}}(y, V) = \mathcal{L}(\Phi_{f(y)}; f(y)) + \mathcal{P}(y) \in L(X_2^3, X_0^3)$ is bijective for each $(y, V) \in U \times X_2^3$. To see this, write $\mathcal{L}_{f(y)} := \mathcal{L}(\Phi_{f(y)}; f(y))$ for brevity and note that Theorem 2.5 implies that

$\ker \mathcal{L}_{f(y)} \subset X_m$ for each $m \in \mathbb{Z}_+$ and therefore the following decompositions hold:

$$X_m^3 = \mathfrak{C}_m\left(\mathcal{L}_{f(y)}\right) \bigoplus \ker \mathcal{L}_{f(y)}, \qquad \text{for all } m \in \mathbb{Z}_+.$$

The bijectivity of $D_V\overline{\mathbb{A}}$ then follows by the bijectivity of the maps $\mathcal{P}(y) : \ker \mathcal{L}_{f(y)} \to \ker \mathcal{L}_{f(y)}$ and $\mathcal{L}_{f(y)} : \mathfrak{C}_2\left(\mathcal{L}_{f(y)}\right) \to \mathfrak{C}_0\left(\mathcal{L}_{f(y)}\right)$, which in turn follow from Theorem 2.5 and Corollary 2.20, respectively. A similar argument shows that $\overline{\mathbb{A}}(y,\cdot) : X_2^3 \to X_0^3$ is bijective for each $y \in U$.

Now, Let $y \in U$ be arbitrary. Since $\overline{\mathbb{A}}(y, V_y) = 0$ and $D_V\overline{\mathbb{A}}(y, V_y)$ is bijective and bounded, the Implicit Function Theorem implies the existence of open sets $\mathcal{U}_y \subset U$ and $\Omega_y \subset X_2^3$ with $(y, V_y) \in \mathcal{U}_y \times \Omega_y$, and a smooth map $\mathcal{M} : \mathcal{U}_y \to \Omega_y$ such that

1. $\overline{\mathbb{A}}(y, \mathcal{M}(y)) = 0$ for all $y \in \mathcal{U}_y$.
2. $\overline{\mathbb{A}}(y, V) = 0$ and $(y, V) \in \mathcal{U}_y \times \Omega_y$, then $V = \mathcal{M}(y)$.

By the bijectivity of $\overline{\mathbb{A}}(y,\cdot)$, and given that $\overline{\mathbb{A}}(y, V_y) = 0,$ for all $y \in \mathcal{U}_y$, we find that

$$F(y,\cdot) = V_y = \mathcal{M}(y), \tag{2.116}$$

for all $y \in \mathcal{U}_y$. Since the above argument works for each $y \in U$, we have that (2.116) holds for all $y \in U$ and for some $\mathcal{M} \in C^\infty(U; X_2^3)$. A similar argument to the one in Lemma 2.25 used to show the smoothness of Φ can be used to show the smoothness of F, using the form of $\overline{\mathbb{A}}$ and the smoothness of V_y for each y, of $\mathcal{M}$, of G, and of $\Phi_{f(y)}$ (both as a function of y and as a function of z, which follows by Lemma 2.25 and the chain rule).

Step 2: Exponential decay. We proceed by induction on the value of $|\alpha|$.

Case $|\alpha| = 0$. This case follows directly from property 3 of G together with (2.113) and Lemma 2.24.

Case $|\alpha| = k+1$. Suppose that the claim holds for all multi-indices of order less than $|\alpha| = k+1$. Differentiating (2.113) with respect to D_y^α, we obtain

$$\mathcal{L}\left(\Phi_{f(y)}; f(y)\right)\left[D_y^\alpha F(y,\cdot)\right] = D_y^\alpha G(y,\cdot) - S_\alpha(y,\cdot), \tag{2.117}$$

where S_α is a sum of products of the form $D^\beta\left[D_\Phi^2 W(\Phi_{f(y)(z)}; y)\right] D^\mu F(y,z)$ with $|\beta| + |\mu| = k+1$ and $|\mu| \leq k$. Therefore, as a result of the induction hypothesis, the form of W, and the pointwise-boundedness of Φ and f and of their derivatives, we deduce the existence of functions $C_k : U \to \mathbb{R}_{>0}$ for $k \in \mathbb{Z}_+$ for which the following exponential decay holds:

$$|\partial_z^k S_\alpha(y,z)| \leq C_k(y) e^{-\min\left(\frac{-\sqrt{\lambda_{\min}(\Pi)}}{2}, c\right)|z|}, \quad \text{for all } (y,z) \in U \times \mathbb{R}. \tag{2.118}$$

Using the exponential decay of $D_\Pi^\alpha G$ and (2.118), one may apply the same argument in the proof of Lemma 2.24 to (2.117) to conclude that

$$|\partial_z^k D_y^\alpha F(y,z)| \leq D_{\alpha k}(y) e^{-\min\left(\frac{\sqrt{\lambda_{\min}(\Pi)}}{2}, c\right)|z|}, \quad \text{for all } (y,z) \in U \times \mathbb{R}, \tag{2.119}$$

for all $k \in \mathbb{Z}_+$ and some $D_{\alpha k} : U \to \mathbb{R}_{>0}$. □

2.3 FURTHER PROPERTIES ASSOCIATED WITH A_{Π}^{1d}: THE SHIFTED POTENTIAL

Based on the computations presented in Section 1.1, the presence of a non-zero current carrying field σ with varying phase θ induces a virtual change in the parameter m_σ, an effect which we refer to as a "shift" in the potential W. Effectively, given the parameters $\Pi = (\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in (0,\infty)^4$ fixed from the outset, we will be eventually lead to consider the following modified parameters

$$\Pi_p := \left(\lambda_\varphi, \lambda_\sigma, \sqrt{m_\sigma^2 - \frac{p}{\lambda_\sigma}}, \beta\right), \text{ for some } p \in (-\infty, \lambda_\sigma m_\sigma^2). \tag{2.120}$$

In connection to this effect on the parameters, we want to consider the map μ_Π mentioned in (1.23):

Definition 2.28. Let $\Pi \in \mathcal{O}$ and $I_\Pi := \{p \in \mathbb{R} : \Pi_p \in \mathcal{O}\}$. We denote the energy of the unique minimizer of $\mathcal{A}_{\Pi_p}^{1d}$ in $\mathcal{H}^s$ by $\mu_\Pi(p)$, so that:

$$\mu_\Pi : I_\Pi \to \mathbb{R}_{>0}, \qquad \mu_\Pi(p) := \mathcal{A}_{\Pi_p}^{1d}\left(\mathfrak{M}(\Pi_p)\right), \tag{2.121}$$

where $\mathfrak{M}$ is the map from Definition 2.22.

We have the following results regarding μ_Π:

Lemma 2.29 (Properties of μ_Π)**.** *Let $\Pi \in \mathcal{O}$ and define $(\varphi(z,p), \sigma(z,p), 0) = \Phi(z,p) := \mathfrak{M}(\Pi_p)(z)$ for each $p \in I_\Pi$ and $z \in \mathbb{R}$. Then, for all $p \in I_\Pi$:*

$$\mu_\Pi(p) = \int_{\mathbb{R}} \left\{ [\partial_z \varphi(z,p)]^2 + [\partial_z \sigma(z,p)]^2 \right\} dz, \tag{2.122}$$

$$\mu'_\Pi(p) = \frac{1}{2} \int_{\mathbb{R}} \sigma^2(z,p)\, dz, \tag{2.123}$$

$$\mu''_\Pi(p) = -\mathcal{Q}(\Phi(\cdot,p); \Pi_p)[\partial_p \Phi(\cdot,p)]. \tag{2.124}$$

In particular, $\mu_\Pi(p), \mu'_\Pi(p) > 0$ and $\mu''_\Pi(p) < 0$ for all $p \in I_\Pi$.

Proof. Let $\Pi \in \mathcal{O}$ and $p \in I_\Pi$. Since $\Phi(\cdot,p)$ minimizes $\mathcal{A}^{1d}_{\Pi_p}$, we have that

$$\mu_\Pi(p) = \mathcal{A}^{1d}_{\Pi_p}(\Phi(\cdot,p)) = \int_{\mathbb{R}} \left[\frac{1}{2} |\partial_z \Phi(z,p)|^2 + W(\Phi(z,p); \Pi_p) \right] dz, \tag{2.125}$$

and also

$$\partial_z^2 \Phi(z,p) = D_\Phi W(\Phi(z,p); \Pi_p). \tag{2.126}$$

Applying the equipartition of energy identity (2.44) yields (2.122). On the other hand, let $Q_h(z,p) := \frac{|\partial_z \Phi(z,p+h)|^2 - |\partial_z \Phi(z,p)|^2}{h}$. Then, by the mean value theorem and the exponential decay of $\partial_p \Phi$ given by Lemma 2.26, we have that for small enough $|h|$,

$$|Q_h(z,p)| \le C e^{-a|z|} \qquad \text{for all } z \in \mathbb{R},$$

where $a = \min_{|p^*-p| \le |h|}(\lambda_{\min}(\Pi_{p^*}))$ and $C = \max_{|p^*-p| \le |h|} C(p^*)$. Therefore, by the dominated convergence theorem, we may differentiate with respect to p under the integral sign in (2.125) and obtain

$$\begin{aligned}
\mu'_\Pi(p) &= \int_{\mathbb{R}} \left[\partial_p \partial_z \Phi \cdot \partial_z \Phi + \frac{d}{dp} W(\Phi; \Pi_p) \right] dz \\
&= \int_{\mathbb{R}} \left[-\partial_p \Phi \cdot \partial_z^2 \Phi + D_\Phi W(\Phi; \Pi_p) \cdot \partial_p \Phi + \partial_p W(\Phi; \Pi_p) \right] dz \\
&\overset{(1.14)}{=} \int_{\mathbb{R}} \left[\left(-\partial_z^2 \Phi + D_\Phi W(\Phi; \Pi_p) \right) \cdot \partial_p \Phi + \frac{|\mathcal{P}_\sigma(\Phi)|^2}{2} \right] dz \\
&\overset{(2.126)}{=} \int_{\mathbb{R}} \frac{|\mathcal{P}_\sigma(\Phi)|^2}{2}\, dz.
\end{aligned} \tag{2.127}$$

where the second equality (2.127) follows by applying integration by parts together with the limiting behaviour of the derivatives of Φ.

Also, differentiating (2.126) with respect to p yields

$$\mathcal{L}(\Phi(\cdot,p);\Pi_p)[\partial_p\Phi(\cdot,p)] = -\mathcal{P}_\sigma(\Phi(\cdot,p)). \tag{2.128}$$

Using this last equation and integrating (2.123) under the integral sign (justified by a similar reasoning to that used to prove (2.123)), we have that

$$\begin{aligned}
\mu_\Pi''(p) &= \int_{\mathbb{R}} \mathcal{P}_\sigma(\partial_p\Phi(z,p))\cdot\mathcal{P}_\sigma(\Phi(z,p))\,dz\\
&= \int_{\mathbb{R}} \partial_p\Phi(z,p)\cdot\mathcal{P}_\sigma(\Phi(z,p))\,dz\\
&= -\int_{\mathbb{R}} \partial_p\Phi(z,p)\cdot\mathcal{L}(\Phi(\cdot,p);\Pi_p)[\partial_p\Phi(\cdot,p)](z)\,dz\\
&= -\mathcal{Q}(\Phi(\cdot,p);\Pi_p)[\partial_p\Phi(\cdot,p)],
\end{aligned}$$

which is (2.124). Finally, the fact that $\partial_p\varphi(\cdot,p)$ (resp. $\partial_p\sigma(\cdot,p)$) has the same parity as $\varphi(\cdot,p)$ (resp. $\sigma(\cdot,p)$), implies that $\partial_p\Phi(\cdot,p)\in\mathfrak{C}_1\left(\mathcal{L}(\Phi(\cdot,p);\Pi_p)\right.$. Together with Lemma 2.16, this shows that $\mathcal{Q}(\Phi(\cdot,p);\Pi_p)[\partial_p\Phi(\cdot,p)]>0$, and therefore that $\mu_\Pi''(p)<0$. □

Remark 2.30. Let $\Pi\in\mathcal{O}$. Then, μ_Π is defined for all p such that $\Pi_p\in\mathcal{O}$, or in other words, for all $p\in(p_-,p_+)$ for some $p_-<0<p_+$. However, according to Lemma 2.12, $\mathcal{A}_\Pi$ admits a minimizer in $\mathcal{H}^s$ for each $\Pi\in\mathcal{O}^0$, and we may therefore extend μ_Π to the interval (P_-,P_+), where $P_- := \lambda_\sigma m_\sigma^2 - \min\left(\sqrt{\lambda_\sigma\lambda_\varphi},\beta\right)$ and $P_+ := \lambda_\sigma m_\sigma^2$. Indeed, the first equality corresponds to the simultaneous attainment of the conditions $\lambda_\varphi=\lambda_\sigma[m_\sigma(P_-)]^4$ and $\beta=\lambda_\sigma[m_\sigma(P_-)]^2$ which determine the containment in $\mathcal{O}^0$, while the value of P_+ is determined by the condition that $m_\sigma(p)>0$. Note however, that despite the requirement that $m_\sigma(p)>0$ for physical purposes, we may consider the following extension of μ_Π:

$$\bar\mu_\Pi : (P_-,\infty)\to\mathbb{R}_{\geq 0}, \qquad \text{where} \qquad \bar\mu(p) := \inf_{(\varphi,\sigma)\in\mathcal{H}} \mathcal{A}_{\Pi_p}(\varphi,\sigma),$$

since $\mathcal{A}_{\Pi_p}(\varphi,\sigma)=\int_{\mathbb{R}}\left[\frac{|\varphi'(z)|^2+|\sigma(z)|^2}{2}+W(\varphi,\sigma;\Pi_p)\right]dz$ with

$$W(\varphi,\sigma;\Pi_p) = \frac{\lambda_\varphi}{4}(\varphi^2-1)^2+\frac{\lambda_\sigma}{4}|\sigma|^4+\frac{\beta}{2}\varphi^2|\sigma|^2+\frac{\left(p-\lambda_\sigma m_\sigma^2\right)}{2}|\sigma|^2,$$

which allows one to make sense of $\mathcal{A}_{\Pi_p}$ for $p > \lambda_\sigma m_\sigma^2$ even though Π_p is not an element of $(0,\infty)^4$ in this case.

Two observations follow:

1. If $p \geq P_+ = \lambda_\sigma m_\sigma^2$, it is more energetically favourable to set $\sigma \equiv 0$. As a result, changes in p, and thus in $m_\sigma(p)$, will not alter the optimal value of the energy in this regime and $\bar{\mu}_\Pi(p) = \bar{\mu}_\Pi(P_+)$ for all $p \geq P_+$. Also, consequently, the point $p = P_+$ corresponds to the critical point for current-quenching.

2. For any $p \leq P_-$, we have that $\lambda_\varphi \leq \lambda_\sigma[m_\sigma(p)]^4$, and therefore $W(0, e^{i\theta}m_\sigma(p); \Pi_p) < 0$ for any fixed $\theta \in \mathbb{R}$ (see proof of Lemma 1.1).

The expected behaviour of $\bar{\mu}_\Pi$ is depicted in Figure 2.4.

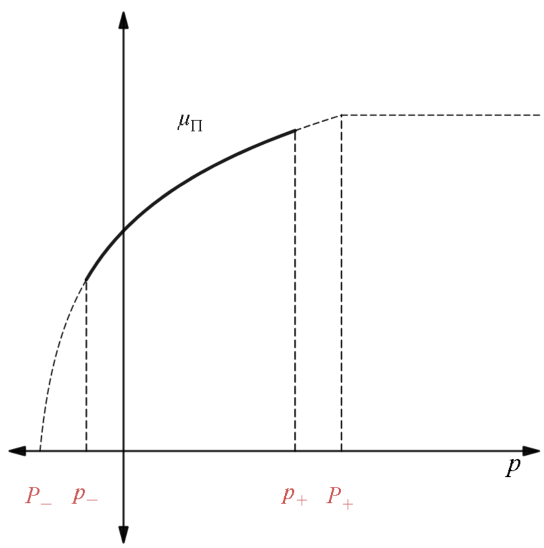


Figure 2.4: Sample graph of $\bar{\mu}_\Pi$ (dotted and solid) and μ_Π (solid only).

3

LAWS OF MOTION

As noted in the introduction and as suggested by the construction of approximate solutions presented in Chapter 4, any realizable configuration described by the superconducting interface model in the scaling regime (1.25) is expected to exhibit a coupling between the interface Γ and the leading order term of the phase, θ, of the current carrying field σ which is supported around the interface. Given an admissible parameter $\Pi \in \mathcal{O}$, this coupling can be expressed as the Euler-Lagrange equations associated with the functional

$$\mathcal{E}_{\Pi}(\Gamma, \theta) := \int_{\Gamma} \mu_{\Pi}\left(\langle \nabla_{\Gamma}\theta, \nabla_{\Gamma}\theta \rangle_m\right) d\lambda, \tag{3.1}$$

acting on timelike manifolds Γ embedded in $[0, T] \times \mathbb{R}^n$ for some $T > 0$ and $n \geq 2$, and on smooth real-valued functions θ defined over such manifolds, where

- $\mu_{\Pi}(x)$ is the energy of the ground states (i.e., minimizers) of $A^{1d}_{\Pi_x}$ over $\mathcal{H}^s$ (see Definition 2.28).
- $\nabla_{\Gamma}\theta$ is the tangential gradient of θ on Γ (see Definition 3.9).
- $\langle \cdot, \cdot \rangle_m$ is the standard Minkowski (pseudo) inner product (see (3.3)).
- $d\lambda$ denotes integration with respect to the area element associated to $\langle \cdot, \cdot \rangle_m$.

In local coordinates (ψ, V), where $V = (0, T) \times U$ for some open subset $U \subset \mathbb{R}^{n-1}$, we may rewrite Equation 3.1 as a functional acting on $\psi \in C^{\infty}(V; \mathbb{R}^{1+n})$ and $\theta \in C^{\infty}(\Gamma; \mathbb{R})$ (see Section 3.2), namely:

$$\Sigma_{\Pi}[\psi, \theta; V] := \int_V \mu_{\Pi}(\langle \nabla_{\Gamma}\theta, \nabla_{\Gamma}\theta \rangle_m (y)) \sqrt{|g(y)|}\, dy. \tag{3.2}$$

where g is the metric on Γ induced by $\langle \cdot, \cdot \rangle_m$ represented in the coordinates (ψ, V) and dy is the Lebesgue measure on $\mathbb{R}^n$.

The fact that the functional in Equation 3.1 (and therefore the one in Equation 3.2) is invariant under reparametrizations of Γ, suggests that the problem of finding the solutions to the Euler Lagrange equations of Equation 3.1 via Equation 3.2 is underdetermined, and a choice of gauge has to be made in order to establish its well-posedness. As a result, we will restrict our attention to the case in which the interface Γ is the graph of a function (aka. the graph gauge condition).

The purpose of this chapter is threefold. First, we introduce a set of coordinates (Fermi coordinates) adapted to the interface Γ in terms of which one may conveniently write (3.1) and carry out the construction of the approximate solutions in Chapter 4. Next, we provide a formal derivation of (3.1)/(3.2). The relevant sections, Section 3.1 and Section 3.2, follow very closely the exposition of [27] and [6], respectively. Lastly, we establish the existence of solutions (Γ^*, θ^*) to the Euler Lagrange equations (aka. laws of motion) associated to Σ_Π (and thus to those of $\mathcal{E}_\Pi$) under the graph gauge assumption on Γ^* in Section 3.4.

3.1 FERMI COORDINATES

We now introduce (under conditions (3.4) and (3.7)) a choice of coordinates adapted to the interface Γ called Fermi coordinates. Introducing Fermi coordinates can be interpreted as a straightening of Γ, as depicted in Figure 3.1, and it is for this reason that we will use them for asymptotic expansions. This is analogous to standard procedures for expansions of elliptic (see e.g., [25], [29], [28]) and parabolic [23] equations , except that we use the Minkowskian rather than the Euclidean unit normal, which is natural for a wave equation (see e.g., [27], [16]) and standard in the physics literature.

We start by introducing some notation and the conditions on Γ under which the existence of Fermi coordinates is guaranteed. To this end, let $\eta = (\eta_{\alpha\beta})_{\alpha,\beta=0}^n = \mathrm{diag}(-1, 1, 1, ..., 1) \in \mathbb{R}^{(1+n)\times(1+n)}$, and

$$\langle v, w \rangle_m := v^T \eta w, \qquad \text{for } v, w \in \mathbb{R}^{1+n}, \tag{3.3}$$

which is the standard Minkowski (pseudo) inner product. We say that

$$\text{a vector } v \in \mathbb{R}^{1+n} \text{ is } \begin{cases} \text{spacelike if} & \langle v, v\rangle_m > 0 \text{ or } v = 0 \\ \text{timelike if} & \langle v, v\rangle_m < 0 \\ \text{lightlike if} & \langle v, v\rangle_m = 0 \text{ and } v \neq 0. \end{cases}$$

Additionally, a subspace $U \subset \mathbb{R}^{1+n}$ is called spacelike if all vectors in U are spacelike, and timelike if $U^\perp := \left\{v \in \mathbb{R}^{1+n} : \langle u, v\rangle_m = 0 \text{ for all } u \in U\right\}$ is spacelike. It is a well-known fact that if $u \in \mathbb{R}^{1+n}$ is timelike and $v \in \mathbb{R}^{1+n}$ is such that $\langle u, v\rangle_m = 0$, then v is spacelike. As a result, if $U \subset \mathbb{R}^{1+n}$ contains a timelike vector, then U is timelike.

Definition 3.1. Let $n \in \mathbb{N}$. A k dimensional manifold $\mathcal{M}$ embedded in $\mathbb{R}^{1+n}$ is ***timelike*** if the tangent space at each point in $\mathcal{M}$ is timelike. Equivalently, $\mathcal{M}$ is timelike if the metric on $\mathcal{M}$ induced by the Minkowski metric has one negative eigenvalue and $k-1$ positive eigenvalues (i.e., if it is Lorentzian[1] with signature $(-++\cdots+)$) everywhere in $\mathcal{M}$.

In what follows, unless otherwise stated, Γ and ν denote:

1. Γ : a smooth, timelike and orientable $n-$dimensional manifold embedded in $\mathbb{R}^{1+n}$. (3.4)
2. ν : a smooth Minkowski unit vector field normal to Γ. That is, ν is smooth and (3.5)

$$\langle \nu(p), \nu(p)\rangle_m \equiv 1 \text{ and } \langle \nu(p), \tau_p\rangle_m = 0 \text{ for all } p \in \Gamma \text{ and } \tau_p \in T_p\Gamma.$$

Also, given any such Γ and ν, we define for each $\delta > 0$ and $U \subset \Gamma$,

$$\mathcal{N}_\delta(U) := \{Y + z\nu(Y) : Y \in U, z \in (-\delta, \delta)\} \subset \mathbb{R}^{1+n}, \tag{3.6}$$

which represents a tubular/normal neighbourhood of radius δ around U.

1 A symmetric $(1+p)\times(1+p)$ matrix A is called ***Lorentzian*** with signature $(1,p)$ or $(-++\cdots+)$ (with p +'s) if it has p positive eigenvalues and 1 negative eigenvalue. The same definition applies to Lorentzian with signature $(p,1)$ or $(+--\cdots-)$ if the roles of negative and positive eigenvalues is reversed.

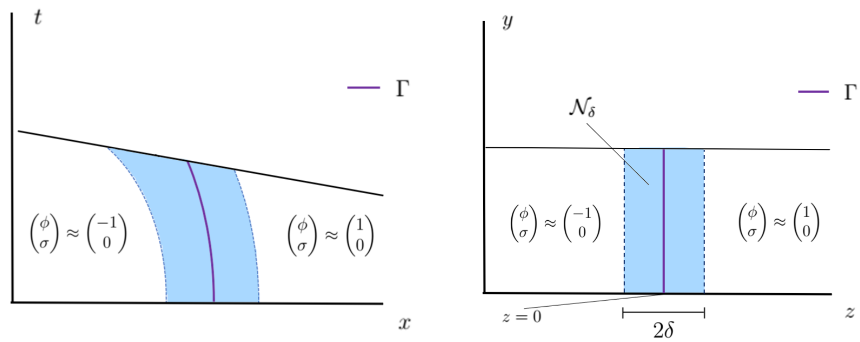


Figure 3.1: Fermi Coordinates.

A necessary condition for defining Fermi coordinates associated to Γ which we will assume from now is that

$$\begin{aligned} \exists \delta > 0 \quad \text{s.t.} \quad &\Gamma \times (-\delta, \delta) \to \mathcal{N}_\delta(\Gamma) \\ &(Y, z) \mapsto (t, x) = Y + z\nu(Y) \qquad \text{is bijective,} \end{aligned} \tag{3.7}$$

a condition that is automatically satisfied when Γ is asymptotically flat or compact, the latter shown in e.g., [16, section 2.4].

Definition 3.2. Let $\delta > 0$ be such that (3.7) holds and (ψ, V) be a smooth coordinate chart for Γ. Due to (3.7), the map

$$\begin{aligned} \Psi : V \times (-\delta, \delta) &\to \Psi\left(V \times (-\delta, \delta)\right) = \mathcal{N}_\delta(\psi(V)) \\ (y, z) &\mapsto (t, x) = \psi(y) + z\nu(\psi(y)) \qquad \text{is bijective,} \end{aligned} \tag{3.8}$$

and the coordinates $(y, z) = (y_0, y_1, ..., y_{n-1}, z) \in V \times (-\delta, \delta) \subset \mathbb{R}^n \times \mathbb{R}$ defined by (3.8) are called ***Minkowskian Fermi coordinates*** or simply Fermi coordinates associated to the local coordinate system (ψ, V) for Γ.

We introduce some additional notation related to the Fermi coordinates described via (3.8). Namely, we will write $\Psi = (\Psi^0, \Psi^1, ..., \Psi^n)$ and $\psi =$

$(\psi^0, \psi^1, ..., \psi^n)$ for the coordinate maps appearing in (3.8). We will also adopt Einstein's summation convention and employ the nomenclature

$$\begin{aligned} &\begin{cases} \alpha, \beta, \gamma, \delta ... & \text{run from } 0 \text{ to } n, \\ a, b, c, d, ... & \text{run from } 0 \text{ to } p, \text{where } 1+p = \dim(\Gamma), \end{cases} \\ &\begin{cases} \partial_a := \partial_{y_a}, & a \in \{0,1,2,...,p\}, \\ \partial_n := \partial_z, & \text{if } \dim(\Gamma) = n, \end{cases} \\ &A^{-1} = (A^{ij})_{i,j=0}^m \text{ for any } A = (A_{ij})_{i,j=0}^m \in \mathbb{R}^{(1+m)\times(1+m)}. \end{aligned} \tag{3.9}$$

For $y \in V$ we define the tensor g given by

$$g_{ab}(y) := \langle \partial_a \psi(y), \partial_b \psi(y) \rangle_m, \qquad a, b = 0, \dots, n-1. \tag{3.10}$$

Thus g represents, in the given parametrization, the metric on Γ induced by the Minkowski metric in $\mathbb{R}^{1+n}$. We will also write

$$(g^{ab}(y)) = \left(g^{-1}(y)\right)_{ab}, \qquad |g(y)| := -\det(g(y)). \tag{3.11}$$

Similarly, for $(y,z) \in V \times (-\delta, \delta)$ we write

$$\mathfrak{g}_{\alpha\beta}(y,z) := \langle \partial_\alpha \Psi(y,z), \partial_\beta \Psi(y,z) \rangle_m, \qquad \alpha, \beta = 0, \dots, n,$$

and

$$(\mathfrak{g}^{\alpha\beta}(y,z)) = \left(\mathfrak{g}^{-1}(y,z)\right)_{\alpha\beta}, \qquad |\mathfrak{g}(y,z)| := -\det(\mathfrak{g}(y,z)).$$

Thus $\mathfrak{g}$ expresses the Minkowski metric on $\mathcal{N}_\delta(\psi(V))$ with respect to the (y,z) coordinates. Now, since ν is a unit normal vector field to $\psi(V) \subset \Gamma$, it follows that $(\mathfrak{g}_{\alpha\beta})$ and thus $(\mathfrak{g}^{\alpha\beta})$ have the form

$$\mathfrak{g}_{\alpha\beta} = \begin{pmatrix} (g_{ab}) + z\,(r_{ab}) & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{pmatrix}, \quad \mathfrak{g}^{\alpha\beta} = \begin{pmatrix} \left(g^{ab}\right) + O(z) & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{pmatrix}, \tag{3.12}$$

where $r_{ab}(y,z) = \langle \partial_a \psi, \partial_b \nu \rangle_m + \langle \partial_b \psi, \partial_a \nu \rangle_m + z \langle \partial_a \nu, \partial_b \nu \rangle_m$, and $F(y,z) = O(z)$ means that there exists a smooth function C such that $|F(y,z)| \leq C(y)|z|$ for all $(y,z) \in V \times (-\delta, \delta)$.

Finally, since $\mathfrak{g}_{\alpha\beta} = \partial_\alpha \Psi^\mu \partial_\beta \Psi^\nu \eta_{\mu\nu}$ and $\det(\eta_{\mu\nu}) = -1$, it is clear that

$$|\mathfrak{g}| = (\det D\Psi)^2 > 0 \qquad \text{and} \qquad |g(y)| = |\mathfrak{g}(y,0)| > 0. \tag{3.13}$$

Remark 3.3. According to the above definition, the following conditions are equivalent

1. $\mathcal{M}$ is timelike
2. $T_p\mathcal{M}$ is timelike for each $p \in \mathcal{M}$.
3. $T_p\mathcal{M}$ contains at least one timelike vector for each $p \in \mathcal{M}$.
4. $(T_p\mathcal{M})^{\perp}$ is spacelike for each $p \in \mathcal{M}$.
5. The metric on $(T_p\mathcal{M})^{\perp}$ induced by the Minkwoski metric is positive definite for each $p \in \mathcal{M}$.

It follows from this that if $\mathcal{M}$ is an orientable timelike hypersurface in $\mathbb{R}^{1+n}$, then $\mathcal{M}$ admits a non-vanishing normal vector field ν in the Minkowski sense which is everywhere spacelike. That is, $\langle \nu, \nu \rangle_m > 0$ and $\langle \nu, \tau \rangle_m = 0$ over $\mathcal{M}$ and for all vector fields τ tangent to $\mathcal{M}$.

Remark 3.4. The Fermi coordinates described above parametrize a neighbourhood of Γ given some coordinates for Γ. As such, Fermi coordinates are not unique, and there is freedom as to how Γ is parametrized.

Remark 3.5. Let Γ be a smooth timelike manifold of dimension n embedded in $\mathbb{R}^{1+n}$ and define

$$\Gamma^t := \Gamma \cap \left\{ (t, x) \in \mathbb{R}^{1+n} : x \in \mathbb{R}^n \right\}, \qquad t \in \mathbb{R}.$$

Then, as shown in [27, Section 3.1], the timelike character of Γ allows one to show that Γ^t is either empty or an $n-1$ dimensional smooth manifold. Moreover, coordinate charts (ψ, V) can be assigned to Γ so that $\psi(y) = (y_0, \gamma(y))$, where $\gamma(y_0, \cdot)$ are coordinates for Γ^{y_0} for each y_0, and the metric on Γ induced by the Minkowski inner product in $\mathbb{R}^{1+n}$ with respect to each of these charts has the form

$$g_{ab}(y) := \langle \partial_a \psi(y), \partial_b \psi(y) \rangle_m = \left(\begin{array}{c|ccc} g_{00}(y) < 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & (g_{ij}(y))_{i,j=1}^{p} \succ 0 & \\ 0 & & & \end{array} \right), \tag{3.14}$$

If Γ is a smooth timelike manifold of dimension $1+p$ with $p \in \{2, 3, ..., n-1\}$, a similar argument can be applied to show that the same result holds, with g_{ij} being a $p \times p$ matrix instead of an $n \times n$ matrix.

3.1.1 *The Wave Operator in Fermi Coordinates*

We will derive an expression for the wave operator $\Box := \partial_{tt} - \Delta_x$ in terms of the Minkowski Fermi coordinates (y, z) from (3.8). To this end, let $\Gamma, \nu, \psi, \Psi, V, \delta$ be as in (3.4),(3.5), and (3.8), and define $V_\delta := V \times (-\delta, \delta)$, so that $\mathcal{N}_\delta(\Gamma) = \Psi(V_\delta)$ is the neighbourhood over which Fermi coordinates are defined. Consider a function $u \in C^\infty\left(\mathbb{R}^{1+n}; \mathbb{R}\right)$ supported in $\mathcal{N}_\delta(\Gamma)$ along with its representation in (y, z) coordinates:

$$v(y, z) := u(t, x), \qquad \text{whenever } (t, x) = \Psi(y, z). \tag{3.15}$$

Upon defining $Du(t,x) := \begin{bmatrix} \partial_t u(t,x) \\ \nabla_x u(t,x) \end{bmatrix}$ and $Dv(y,z) := \begin{bmatrix} \nabla_y v(y,z) \\ \partial_z v(y,z) \end{bmatrix}$, the chain rule implies that

$$Du(t,x) = B(y,z)^{-T}\, Dv(y,z), \qquad (t,x) = \Psi(y,z), \tag{3.16}$$

where $B(y,z) := \begin{bmatrix} \partial_0 \Psi & \partial_1 \Psi & \ldots & \partial_n \Psi \end{bmatrix}(y,z) \in \mathbb{R}^{(1+n)\times(1+n)}$. As a result of (3.16) and the identity $B^T \eta B = \mathfrak{g}$, we find that

$$\langle Du, Du \rangle_\eta (t,x) = Dv(y,z)^T \mathfrak{g}^{-1}(y,z) Dv(y,z), \quad (t,x) = \Psi(y,z). \tag{3.17}$$

Now, let

$$\begin{aligned} I[u] &:= \int_{\Psi(V_\delta)} \left[\left|\nabla_x u(t,x)\right|^2 - \left|\partial_t u(t,x)\right|^2 \right] dt\, dx \\ &= \int_{\Psi(V_\delta)} \langle Du, Du \rangle_\eta (t,x)\, dt\, dx. \end{aligned} \tag{3.18}$$

Using the change of variable formula for integration over $\mathcal{N}_\delta(\Gamma) = \Psi(V_\delta)$, together with (3.18) and (3.17), we obtain

$$I[u] = \int_{V_\delta} \mathfrak{g}^{\alpha\beta}(y,z) \partial_\alpha v(y,z) \partial_\beta v(y,z) \sqrt{|\mathfrak{g}(y,z)|}\, dy\, dz. \tag{3.19}$$

Next, let $\xi \in C_c^\infty(\mathcal{N}; \mathbb{R})$ be arbitrary and define $\tilde{\xi}(y,z) := \xi(t,x)$ whenever $(t,x) = \Psi(y,z)$. Due to (3.18):

$$\frac{1}{2} \frac{d}{dh}\Big|_{h=0} I[u + h\xi] = \int_{\mathcal{N}} \eta^{\alpha\beta} \left(\partial_\alpha u\right) \left(\partial_\beta \xi\right)\, dt\, dx = \int_{\mathcal{N}} (\Box u)\, \xi\, dt\, dx. \tag{3.20}$$

On the other hand, in (y,z) coordinates we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dh}\Big|_{h=0} I[u+h\xi] \\
&\qquad = \frac{1}{2}\frac{d}{dh}\Big|_{h=0} \int_{V_\delta} \mathfrak{g}^{\alpha\beta}\partial_\alpha\left(v+h\tilde{\xi}\right)\partial_\beta\left(v+h\tilde{\xi}\right)\sqrt{|\mathfrak{g}|}\,dy\,dz \\
&\qquad = \int_{V_\delta} \mathfrak{g}^{\alpha\beta}\left(\partial_\beta v\right)\left(\partial_\alpha\tilde{\xi}\right)\sqrt{|\mathfrak{g}|}\,dy\,dz \\
&\qquad = \int_{V_\delta} -\frac{1}{\sqrt{|\mathfrak{g}|}}\partial_\alpha\left(\sqrt{|\mathfrak{g}|}\mathfrak{g}^{\alpha\beta}\partial_\beta v\right)\tilde{\xi}\sqrt{|\mathfrak{g}|}\,dy\,dz.
\end{aligned}
\tag{3.21}
$$

As a result, upon defining

$$
\widetilde{\Box} v(y,z) := -\frac{1}{\sqrt{|\mathfrak{g}(y,z)|}}\partial_\alpha\left(\sqrt{|\mathfrak{g}(y,z)|}\mathfrak{g}^{\alpha\beta}(y,z)\partial_\beta v(y,z)\right),
$$

it follows from (3.20) and (3.21) and the change of variable formula for integration, that

$$
\Box u(t,x) = \widetilde{\Box} v(y,z), \qquad \text{whenever } (t,x) = \Psi(y,z). \tag{3.22}
$$

Thanks to the form of $\mathfrak{g}^{\alpha\beta}$ from (3.12), the operator $\tilde{\Box}$ can be decomposed conveniently as

$$
\widetilde{\Box} = -\partial_z^2 + H_{\Gamma_z}\,\partial_z + \Box_{\Gamma_z}, \tag{3.23}
$$

where, for each $z, z_0 \in (-\delta,\delta)$:

$$
\begin{aligned}
\Box_{\Gamma_z} v(y,z_0) &:= -\frac{1}{\sqrt{|\mathfrak{g}(y,z)|}}\partial_a\left[\sqrt{|\mathfrak{g}(y,z)|}\mathfrak{g}^{ab}(y,z)\partial_b v(y,z_0)\right], \\
H_{\Gamma_z}(y) &:= -\frac{1}{2}\partial_z \ln\left[|\mathfrak{g}(y,z)|\right].
\end{aligned}
\tag{3.24}
$$

Definition 3.6. Let $z_0 \in (-\delta,\delta)$, where $\delta > 0$ is such that (3.7) holds, and

$$
\Gamma_{z_0} := \left\{Y + z\nu(Y) : Y \in \Gamma, z = z_0\right\}. \tag{3.25}
$$

The map $\Box_{\Gamma_z}$ defined on (3.24) acting on functions defined on Γ_z is called the ***wave operator associated to the manifold*** Γ_z. Note that $\Box_{\Gamma_z}$ acts on functions of the y-coordinates for Γ_z. Also, the map $H_{\Gamma_z}(y)$ defined in

(3.24) is called the ***mean curvature in the Minkwoski sense*** of the manifold Γ_z at the point $(t, x) = \Psi(y, z)$, and we will write $H_\Gamma := H_{\Gamma_0}$.

Remark 3.7. As a consequence of the timelike character of Γ and (3.12), $(\mathfrak{g}_{\alpha\beta})$ has n positive eigenvalues and one negative eigenvalue everywhere, which justifies the name and notation for the operator $\Box_{\Gamma_z}$.

Remark 3.8. The operator $\Box_\Gamma := \Box_{\Gamma_0}$ is the Laplace-Beltrami operator on Γ. This operator acting on $C^2(\Gamma; \mathbb{R})$ functions can also be defined in a similar manner to $\Box_{\Gamma_0}$ above for Riemann or pseudo-Riemannian manifolds Γ of arbitrary dimension. It is given in local coordinates (ψ, V) by

$$\Box_\Gamma f := -\frac{1}{\sqrt{|\det g|}}\partial_a\left(\sqrt{|\det g|}g^{ab}\partial_b f\right), \tag{3.26}$$

where g_{ab}, g^{ab}, and g are as in Equation 3.10 and (3.11).

Finally, another definition we will employ later is

Definition 3.9. Let $f : \Gamma \to \mathbb{R}$ be a C^1 function, where Γ is a smooth manifold embedded in $\mathbb{R}^{1+n}$. The tangential gradient of f along Γ is defined as the unique vector field $\nabla_\Gamma f$ on Γ with the property that

$$\langle \nabla_\Gamma f, X\rangle_m = df\,(X)\,, \qquad \text{for all smooth vector fields } X \text{ on } \Gamma, \tag{3.27}$$

where df is the total derivative of f. In local coordinates (ψ, V), using (3.27) we find that $\nabla_\Gamma f(y) = g^{ab}(y)\partial_a \tilde{f}(y)\partial_b\psi(y)$, where $\tilde{f} := f \circ \psi$, or with a slight abuse of notation:

$$\nabla_\Gamma f = g^{ab}\partial_a\psi\partial_b f = \partial^a f\partial_a\psi = \partial_a f\partial^a\psi. \tag{3.28}$$

3.2 FORMAL DERIVATION OF THE REDUCED LAGRANGIAN $E(\Gamma, \theta)$

Let $\Gamma, \nu, \psi, \Psi, V, \delta$ be as in (3.4),(3.5), and (3.8), and define $V_\delta := V \times (-\delta, \delta)$ as before. In this way, (ψ, V) are coordinates for the relatively open subset $\psi(V) \subset \Gamma$, and Fermi coordinates are defined in $\mathcal{N}_\delta(\psi(V)) = \Psi(V_\delta)$.

Consider the action functional $\mathcal{A}_{\Pi,\varepsilon}$ from (1.6) restricted to $\mathcal{N} := \Psi(V_\delta)$. Following a standard practice in the physics literature, we will change to the adapted coordinates for Γ (i.e., the Fermi coordinates $\Psi(y,z)$) and then integrate out the degrees of freedom corresponding to the directions normal to Γ (i.e., in the z direction), thus obtaining an *reduced action/Lagrangian* in the tangential variables (i.e., in the y variables). For this, consider $(\tilde{\phi}, \tilde{\sigma}) \in C^\infty(\mathcal{N}; \mathbb{R} \times \mathbb{C})$ and introduce their representations in Fermi coordinates:

$$\varphi(y,z) := \tilde{\varphi}(t,x), \quad \sigma(y,z) := \tilde{\sigma}(t,x), \qquad \text{whenever } (t,x) = \Psi(y,z).$$

By changing variables and using (3.12):

$$\begin{aligned}
&\mathcal{A}_{\Pi,\varepsilon}[\tilde{\varphi}, \tilde{\sigma}] \\
&\quad = \int_{\Psi(V_\delta)} \left\{ \frac{\varepsilon}{2} \left[\partial^\alpha \tilde{\varphi}\, \partial_\alpha \tilde{\varphi} + (\partial^\alpha \tilde{\sigma}, \partial_\alpha \tilde{\sigma}) \right] + \frac{1}{\varepsilon} W(\tilde{\varphi}, \tilde{\sigma}; \Pi) \right\} dx\, dt \\
&\quad = \varepsilon \int_{V_\delta} \left[\frac{(\partial_z \varphi)^2 + |\partial_z \sigma|^2}{2} + \frac{\mathfrak{g}^{ab}}{2} \left(\partial_a \varphi \partial_b \varphi + (\partial_a \sigma, \partial_b \sigma) \right) \right] \sqrt{|\mathfrak{g}|}\, dz\, dy \\
&\qquad\quad + \frac{1}{\varepsilon} \int_{V_\delta} \left[W(\varphi, \sigma; \Pi) \right] \sqrt{|\mathfrak{g}|}\, dz\, dy.
\end{aligned} \tag{3.29}$$

Guided by the form of the 1D solutions with variable phase from (1.12), we posit that (φ, σ) are of the form

$$\varphi(y,z) = v(y,\zeta), \qquad \sigma(y,z) = e^{i\theta(y)/\varepsilon} w(y,\zeta), \qquad \zeta := z/\varepsilon.$$

where v, w and θ are smooth and real-valued with all derivatives of $(v - sign, w)$ decaying exponentially to 0 as $|\zeta| \to \infty$. Then, changing variables again, the quantity in (3.29) may be written as

$$\begin{aligned}
&\int_{V_{\delta/\varepsilon}} \left[\frac{(\partial_\zeta v)^2 + (\partial_\zeta w)^2}{2} + \frac{1}{2} \mathfrak{g}^{ab}(y, \varepsilon\zeta)\ (\partial_a \theta\ \partial_b \theta)\ w^2 + W(v, w; \Pi) \right] \sqrt{|\mathfrak{g}(y, \varepsilon\zeta)|}\, d\zeta dy \\
&\qquad + \frac{\varepsilon}{2} \int_{V_{\delta/\varepsilon}} \mathfrak{g}^{ab}(y, \varepsilon\zeta) \left(\partial_a v \partial_b v + \partial_a w \partial_b w \right) \sqrt{|\mathfrak{g}(y, \varepsilon\zeta)|}\ d\zeta\ dy.
\end{aligned}$$

Noting that $\mathfrak{g}^{ab}(y, \varepsilon\zeta) = g^{ab}(y) + O(\varepsilon\zeta)$ and using the exponential decay of $W(v, w; \Pi)$ and of the derivatives of v and w, while recalling the definition of Π_p from (1.14), we may rewrite the above expression as

$$\mathcal{A}_{\Pi,\epsilon}[\tilde{\varphi}, \tilde{\sigma}] = \int_V \left[\int_{-\delta/\varepsilon}^{\delta/\varepsilon} \frac{(\partial_\zeta v)^2 + (\partial_\zeta w)^2}{2} + W\left(v, w; \Pi_{\rho(y)}\right)\ d\zeta \right]\ \sqrt{|g(y)|}\ dy + O(\varepsilon),$$

where the symbol $O(\varepsilon)$ is used to denote a quantity bounded in absolute value by $C\varepsilon$ as $\varepsilon \to 0^+$, and $\rho(y) \in \mathbb{R}$ is the given by (see Definition 3.9):

$$\rho(y) := g^{ab}(y)\,\partial_a\theta(y)\,\partial_b\theta(y) = \langle \nabla_\Gamma \theta, \nabla_\Gamma \theta \rangle_m\,.$$

Formally, the resulting action in the limit $\varepsilon \to 0^+$ thus becomes

$$\mathcal{A}_\Pi[v, w, \theta, \Gamma] := \int_V \mathcal{A}^{1d}_{\Pi_{\rho(y)}}\left(v(y,\cdot), w(y,\cdot)\right)\ \sqrt{|g(y)|}\ dy, \tag{3.30}$$

where $\mathcal{A}^{1d}_\Pi$ is given by Equation 2.4.

Now, consider the set of admissible parameters $\mathcal{O}$ from Theorem 2.3. The family of critical points of (3.30) that we consider arises from the fact that for any fixed y such that $\Pi_{\rho(y)} \in \mathcal{O}$, we can minimize $\mathcal{A}^{1d}_{\Pi_{\rho(y)}}$ by setting

$$(v(y,\cdot), w(y,\cdot)) = \Phi(\Pi_{\rho(y)}, \cdot), \tag{3.31}$$

where Φ is the map from Theorem 2.3. The choice (3.31) implies that (v, w) depends only on Π_ρ, and $\mathcal{A}_\Pi$ in (3.30) therefore reduces to the following functional acting solely on the triplet (Π, Γ, θ):

$$\Sigma_\Pi[\psi, \theta; V] := \int_V \mu_\Pi(\rho(y))\sqrt{|g(y)|}\ dy, \qquad \rho = g^{ab}\partial_a\theta\,\partial_b\theta, \tag{3.32}$$

where μ_Π is defined in (2.121). Given that (3.32) is independent of the choice of coordinates for $\psi(V) \subset \Gamma$, we conclude that the limiting functional of interest associated with $\mathcal{A}_{\Pi,\varepsilon}$ from (1.6) in the limit as $\varepsilon \to 0^+$ is

$$\mathcal{E}_\Pi(\Gamma, \theta) := \int_\Gamma \mu_\Pi(\langle \nabla_\Gamma\theta, \nabla_\Gamma\theta \rangle_m)\, d\lambda, \tag{3.33}$$

where $d\lambda$ represents integration with respect to $\langle\cdot,\cdot\rangle_m$ in the sense of (3.32).

The above non-rigorous argument thus predicts that Γ and θ should satisfy the Euler-Lagrange equations associated to (3.33). We will see in Chapter 4 another argument which suggests why this is a necessary condition on Γ and θ for constructing approximate solutions to (1.7) which near Γ are of the form (3.34) below.

Remark 3.10. In relation to Equation 1.7, and motivated by the above discussion and by the form (1.12) of the simple explicit solutions found in Chapter 2, the approximate solutions we will build are roughly of the form

$$\begin{pmatrix} \varphi \\ \sigma \end{pmatrix}(y,z) = \begin{pmatrix} v\left(y, \frac{z}{\varepsilon}\right) + \cdots \\ e^{i\theta(y)/\varepsilon} w\left(y, \frac{z}{\varepsilon}\right) + \cdots \end{pmatrix}, \tag{3.34}$$

where (v, w) are given by (3.31).

Remark 3.11. The above analysis applies to the case where of Γ is a manifold embedded in $\mathbb{R}^{1+n}$ of codimension $k = 1$. In Section 5.2, we present an argument similar to the one above for the neutral superconducting strings case, which corresponds to the case $k = 2$. Motivated by these heuristics, we will consider Γ of arbitrary codimension k, together with a reduced action functional of the form (3.33), where μ_Π is a function which is strictly positive, increasing, and concave down on an open interval containing 0.

3.3 EQUIVALENT FORMS OF THE LAWS OF MOTION

Motivated by the heuristics from the previous section, we turn our attention to the critical points of the functional Σ_Π defined in Equation 3.2 (and therefore of $\mathcal{E}_\Pi$ from Equation 3.1). We begin by establishing the equivalence of different forms in which one can state the relevant Euler-Lagrange equations (aka laws of motion or equations of motion in this context).

Lemma 3.12 (Different Forms of Laws of Motion)**.** *The Euler-Lagrange equations of $\mathcal{E}$ in Equation 3.2, defined for C^2 manifolds Γ of dimension $1+p$ embedded in $(1+n)$ dimensional Minkowski space, and for $\theta \in C^2(\Gamma;\mathbb{R})$, are*

$$\begin{cases} \vec{H}_\Gamma = -f_1(\rho)\vec{I\!I}(\nabla_\Gamma\theta, \nabla_\Gamma\theta) \\ \Box_\Gamma\theta = \frac{1}{2} f_2(\rho)\left\langle \nabla_\Gamma\rho, \nabla_\Gamma\theta\right\rangle_m \end{cases} \quad \text{with } \rho := \left\langle \nabla_\Gamma\theta, \nabla_\Gamma\theta\right\rangle_m, \tag{3.35}$$

where $\vec{H}_\Gamma, \vec{I\!I}, \Box_\Gamma, \nabla_\Gamma f$ are the mean curvature vector of Γ (see (3.55)), the second fundamental form of Γ (see (3.56)), the Laplace-Beltrami operator defined on Γ (see (3.26)), and $\nabla_\Gamma f$ is the tangential gradient of $f \in C^1(\Gamma;\mathbb{R})$ along Γ (see (3.28)), respectively.

Moreover, in terms of any local coordinates (ψ, V) for Γ, the restriction of (3.35) to $\psi(V) \subset \Gamma$ can be written as

$$\begin{cases} \partial_a \left(L(D\psi, D\theta) C_1^{ab}(D\psi, D\theta) \partial_b \psi \right) = 0 \\ \partial_a \left[L(D\psi, D\theta) f_1(\rho) g^{ab} \partial_b \theta \right] = 0, \end{cases} \tag{3.36}$$

or, alternatively,

$$\begin{cases} C_1^{ab}(D\psi, D\theta) \left[\partial_{ab}\psi - \langle \partial^c \psi, \partial_{ab}\psi \rangle_m \partial_c \psi \right] = 0 \\ C_2^{ab}(D\psi, D\theta) \left[\partial_{ab}\theta - \langle \partial^c \psi, \partial_{ab}\psi \rangle_m \partial_c \theta \right] = 0, \end{cases} \tag{3.37}$$

where

$$\begin{aligned} g_{ab}(D\psi) &:= \langle \partial_a \psi, \partial_b \psi \rangle_m \\ \rho(D\psi, D\theta) &:= g^{ab}(D\psi) \partial_a \theta \partial_b \theta \\ L(D\psi, D\theta) &:= \mu_\Pi(\rho(D\psi, D\theta)) \sqrt{|g(D\psi)|} \\ C_j^{ab}(D\psi, D\theta) &:= g^{ab}(D\psi) + f_j(\rho(D\psi, D\theta)) g^{ac}(D\psi) g^{bd}(D\psi) \partial_c \theta \partial_d \theta, \\ f_1(x) &:= -2 \frac{\mu_\Pi'(x)}{\mu_\Pi(x)}, \qquad f_2(x) = 2 \frac{\mu_\Pi''(x)}{\mu'(x)} \end{aligned} \tag{3.38}$$

for all $a, b \in \{0, 1, ..., p\}$, $j \in \{1, 2\}$, and $x \in I_\Pi$ (see Definition 2.28).

Proof. We first show that (3.36) are the Euler Lagrange equations of (3.2). Then, we show that (3.36) is equivalent to (3.37) and finally that (3.37) is equivalent to (3.35).

We will drop the explicit dependency of the functions above on $D\psi$ and $D\theta$. As before, we denote by $g = (g_{ab})_{a,b=0}^p$ the metric induced by $\langle \cdot, \cdot \rangle_m$ on Γ, by g^{ab} the $(a, b)^{th}$ entry of the inverse of g, and $|g| := -\det(g)$. Additionally, we lower and raise indices using g and its inverse, so that e.g., $\partial^a \theta := g^{ab} \partial_b \theta$ and thus $g_{ab} \partial^b \theta = \partial_a \theta$. Also, we will use the following identities:

$$\partial_a |g| = |g| \, g^{bc} \left(\partial_a g_{bc} \right) \tag{3.39}$$

$$\partial_a g^{bc} = -g^{bd} g^{ce} \left(\partial_a g_{de} \right) \tag{3.40}$$

$$\frac{1}{\sqrt{|g|}} \partial_a \left(\sqrt{|g|} g^{ab} \right) = -g^{ac} g^{bd} \langle \partial_{ac} \psi, \partial_d \psi \rangle_m . \tag{3.41}$$

Equation (3.39) follows from Jacobi's formula for matrix valued functions:

$$\frac{d}{dt} \det A(t) = [\det A(t)] \operatorname{tr} \left[A(t)^{-1} A'(t) \right].$$

Equation (3.40) follows by writing $g^{ab}g_{bc} = \delta^a_c$ and using the product rule. Equation (3.41) follows directly from the other two equations and the identity $g_{ab} = g_{ba} = \langle \partial_a \psi, \partial_b \psi \rangle_m$.

Step 1. (3.36) are the Euler Lagrange equations of (3.2). let $\tilde{\theta}$ and $\tilde{\psi}$ be compactly supported smooth functions of the same character as that of θ and ψ, respectively, and define the following functions over $\mathbb{R}$:

$$l_1(h) := \Sigma_\Pi[\psi + h\tilde{\psi}, \theta; V] \qquad l_2(h) := \Sigma_\Pi[\psi, \theta + h\tilde{\theta}; V].$$

In particular,

$$\begin{aligned} l_1(h) &= \int_V \mu_\Pi \left(\tilde{g}^{ab} \partial_a \theta \partial_b \theta \right) \sqrt{|\tilde{g}|}\, dy \\ l_2(h) &= \int_V \mu_\Pi \left(g^{ab} \partial_a (\theta + h\tilde{\theta}) \partial_b (\theta + h\tilde{\theta}) \right) \sqrt{|g|}\, dy, \end{aligned}$$

where $\tilde{g}_{ab} = \left\langle \partial_a(\psi + h\tilde{\psi}), \partial_b(\psi + h\tilde{\psi}) \right\rangle_m$.

It follows from the definition of $\tilde{g}$ that the identities (3.39)-(3.41) hold with ∂_a replaced by $\frac{d}{dh}\Big|_{h=0}$. Using this fact, a direct computation shows that

$$\begin{aligned} l_1'(0) &= \int_V \left(\mu_\Pi(\rho) g^{ab} \langle \partial_a \psi, \partial_b \tilde{\psi} \rangle_m - 2\mu_\Pi'(\rho) g^{ac} g^{bd} \partial_a \theta \partial_b \theta \langle \partial_c \psi, \partial_d \tilde{\psi} \rangle_m \right) \sqrt{|g|}\, dy \\ l_2'(0) &= \int_V 2\mu_\Pi'(\rho) \sqrt{|g|} g^{ab} \partial_b \theta \partial_a \tilde{\theta}\, dy. \end{aligned}$$

As a result, the Euler Lagrange equations corresponding to (3.2) are:

$$\partial_a \left(\mu_\Pi(\rho) \sqrt{|g|}\, g^{ab} \partial_b \psi - 2\mu_\Pi'(\rho) \sqrt{|g|}\, g^{ab} \partial_b \theta \nabla_\Gamma \theta \right) = 0. \tag{3.42}$$

$$\partial_a \left(\mu_\Pi'(\rho) \sqrt{|g|} g^{ab} \partial_b \theta \right) = 0 \tag{3.43}$$

We can write (3.42) and (3.43) more concisely as (3.36) by rearranging the expressions in the parenthesis and using the definitions in (3.38).

Step 2. (3.36) is equivalent to (3.37). The positivity of μ_Π and $\sqrt{|g|}$ imply that $L := \mu_\Pi(\rho)\sqrt{|g|}$ is everywhere positive. Using the product rule and dividing by L in (3.43), we find that (3.43) is equivalent to

$$\Box_\Gamma\theta = \frac{\mu''_\Pi(\rho)}{\mu'_\Pi(\rho)} g^{ab}(\partial_a\rho)\partial_b\theta = \frac{f_2(\rho)}{2}\left(\partial^a\rho\right)\partial_a\theta, \tag{3.44}$$

where $\Box_\Gamma$ is defined in Equation 3.26. Alternatively, we may expand both sides of (3.44) using (3.26), (3.28), (3.40), and (3.41) as follows

$$\begin{aligned}\Box_\Gamma\theta &= -g^{ab}\partial_{ab}\theta - \frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}g^{ab}\right)\partial_b\theta \\ &= -g^{ab}\partial_{ab}\theta + g^{ab}g^{cd}\left\langle\partial_{ab}\psi,\partial_d\psi\right\rangle\partial_c\theta \\ &= -g^{ab}\left(\partial_{ab}\theta - \left\langle\partial_{ab}\psi,\nabla_\Gamma\theta\right\rangle_m\right),\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}\left(\partial^a\rho\right)\partial_a\theta = g^{ab}\partial_b\rho\partial_a\theta &= g^{ab}\partial_b\left(g^{cd}\partial_c\theta\partial_d\theta\right)\partial_a\theta \\ &= (\partial_b g^{cd})\partial_c\theta\partial_d\theta\partial^b\theta + 2\partial^a\theta\partial^b\theta\partial_{ab}\theta \\ &= 2\partial^a\theta\partial^b\theta\left(\partial_{ab}\theta - \left\langle\partial_{ab}\psi,\nabla_\Gamma\theta\right\rangle_m\right).\end{aligned} \tag{3.46}$$

Based on (3.44)-(3.46), we conclude that (3.43) is equivalent to

$$C_2^{ab}(D\psi, D\theta)\Bigg(\partial_{ab}\theta - \left\langle\partial_{ab}\psi,\partial^c\psi\right\rangle_m\partial_c\theta\Bigg) = 0. \tag{3.47}$$

On the other hand, upon expanding and dividing by L, equation (3.42) can be expressed as

$$\begin{aligned}&-\Box_\Gamma\psi - f_1(\rho)\left[\frac{1}{2}(\partial_a\rho)\partial^a\psi - \partial^a\theta\partial_a\left(\nabla_\Gamma\theta\right)\right] \\ &\qquad - f_1(\rho)\left[\Box_\Gamma\theta - \frac{f_2(\rho)}{2}(\partial_a\rho)\partial^a\theta\right]\nabla_\Gamma\theta = 0,\end{aligned} \tag{3.48}$$

or, using (3.44) as

$$\Box_\Gamma\psi + f_1(\rho)\left[\frac{1}{2}(\partial_a\rho)\partial^a\psi - \partial^a\theta\partial_a\left(\nabla_\Gamma\theta\right)\right] = 0. \tag{3.49}$$

Also, just as in (3.45)

$$-\Box_\Gamma\psi = g^{ab}\partial_{ab}\psi - g^{ab}\left\langle\partial_{ab}\psi,\partial^c\psi\right\rangle_m\partial_c\psi. \tag{3.50}$$

and

$$\begin{aligned}
\partial^a\theta\partial_a\left(\nabla_\Gamma\theta\right) &= \partial^a\theta\partial_a\left(g^{bc}\partial_b\theta\partial_c\psi\right)\\
&= \partial^a\theta\left[-\partial^b\theta\partial^c\psi\partial_a g_{bc} + \partial^b\psi\partial_{ab}\theta + \partial^c\theta\partial_{ac}\psi\right] \qquad \text{(by (3.40))}\\
&= -\partial^a\theta\partial^b\theta\partial^c\psi\left(\langle\partial_{ab}\psi,\partial_c\psi\rangle_m + \langle\partial_b\psi,\partial_{ac}\psi\rangle_m\right)\\
&\qquad + \left(\partial^a\psi\partial^b\theta\right)\partial_{ab}\theta + \left(\partial^a\theta\partial^b\theta\right)\partial_{ab}\psi\\
&= \frac{1}{2}(\partial_a\rho)\partial^a\psi - \partial^a\theta\partial^b\theta\left\langle\partial_{ab}\psi,\partial^c\psi\right\rangle_m\partial_c\psi\\
&\qquad + (\partial^a\theta\partial^b\theta)\partial_{ab}\psi,
\end{aligned} \tag{3.51}$$

where the last identity follows from a nearly identical computation as (3.46) used to conclude that $(\partial^a\rho)\partial_a\psi = 2\partial^a\theta\left(\partial_{ab}\theta - \partial^c\theta\langle\partial_{bc}\psi,\partial_a\psi\rangle_m\right)\partial^b\psi$. Using (3.49) together with (3.50) and (3.51), we can rewrite (3.42) as

$$\left(g^{ab} + f_1(\rho)\partial^a\theta\partial^b\theta\right)\partial_{ab}\psi - \left\langle\left(g^{ab} + f_1(\rho)\partial^a\theta\partial^b\theta\right)\partial_{ab}\psi,\partial^c\psi\right\rangle_m\partial_c\psi = 0,$$

which is simply

$$C_1^{ab}\left(\partial_{ab}\psi - \langle\partial_{ab}\psi,\partial_c\psi\rangle_m\partial^c\psi\right) = 0. \tag{3.52}$$

Step 3. (3.37) *is equivalent to* (3.35). Note that (the restriction to $\psi(V)$ of) the second equation of (3.35) is equivalent to (3.44), and therefore to the second equation of (3.37). As for the remaining equations, first note that (3.52) is equivalent to

$$\left\langle C_1^{ab}\partial_{ab}\psi,\nu\right\rangle_m = 0, \quad \text{for all unit vector fields } \nu \text{ normal (with respect to } \langle\cdot,\cdot\rangle_m\text{) to } \psi(V)\subset\Gamma. \tag{3.53}$$

Indeed, since $\langle\partial^c\psi,\partial_d\psi\rangle_m = g^{ca}\langle\partial_a\psi,\partial_d\psi\rangle_m = \delta^c_d$, it follows that

$$\left\langle\left(\partial_{ab}\psi - \langle\partial_{ab}\psi,\partial_c\psi\rangle_m\partial^c\psi\right),\partial_d\psi\right\rangle_m = 0 \qquad d = 0,1,...,p. \tag{3.54}$$

Consequently, $C_1^{ab}\left(\partial_{ab}\psi - \langle\partial_{ab}\psi,\partial_c\psi\rangle_m\partial^c\psi\right)$ is everywhere orthogonal to Γ. As a result, and since $\partial^c\psi(y) = g^{ac}\partial_a\psi(y) \in T_{\psi(y)}\Gamma$ for each y, the equivalence between (3.52) and (3.53) follows.

To see the equivalence of (3.53) and the first equation in (3.35), and analogously to the procedure from Section 3.1, we parametrize a neighbourhood of $\psi(V) \subset \Gamma$. To this end, let $\{\nu_i\}_{i=1}^k$ be a list of smooth maps defined over Γ such that $\{\nu_i(y)\}_{i=1}^k$ forms an orthonormal basis for $\left(T_{\psi(y)}\Gamma\right)^\perp$ at each $\psi(y) \in \Gamma$, and consider the following map parametrizing a neighbourhood of Γ in $\mathbb{R}^{1+n}$:

$$\Psi : V \times (-\delta, \delta)^k \subset \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$$
$$\Psi(y, z) = \psi(y) + \sum_{i=1}^k z^i \nu_i(y), \qquad z = (z^1, ..., z^k) \in (-\delta, \delta)^k,$$

for small enough $\delta > 0$. As before, we will use Greek letters for indices that run from 0 to n and Latin letters for those that run from 0 to p. Also, we write $\partial_\alpha = \partial_a = \partial_{y_a}$ whenever $\alpha = a \in \{0, ..., p\}$, and $\partial_{(1+p)+j} = \partial_{z_j}$ for each $j \in \{1, ..., k\}$. Additionally, we write $\mathfrak{g}_{\alpha\beta} := \langle \partial_\alpha \Psi, \partial_\beta \Psi \rangle_m$ (resp. $g_{ab} = \langle \partial_a \psi, \partial_b \psi \rangle_m = \mathfrak{g}_{ab}|_{z=0}$) to represent the metric induced by $\langle \cdot, \cdot \rangle_m$ on the image of Ψ (resp. on Γ) in these coordinates, and denote by $(\mathfrak{g}^{\alpha\beta})_{\alpha,\beta=0}^n$ (resp. $(g^{ab})_{a,b=0}^p$) its inverse.

In this notation, the Minkowskian mean curvature vector of Γ at $\psi(y)$ is given in terms of the coordinates Ψ by (see e.g., [17, Section 7.5]):

$$\begin{aligned} \vec{H}_\Gamma(y) &:= -\frac{1}{2} \sum_{\alpha > p} \left[\left(\mathfrak{g}^{ab} \partial_\alpha \mathfrak{g}_{ab} \right) \partial_a \Psi \right] \Big|_{(y,0)} \\ &= \mathfrak{g}^{ab}(y, 0) \vec{\mathrm{II}}(\partial_a \psi, \partial_b \psi)(y), \end{aligned} \tag{3.55}$$

where $\vec{\mathrm{II}}(\cdot, \cdot)(y) : T_{\psi(y)}\Gamma \times T_{\psi(y)}\Gamma \to \left(T_{\psi(y)}\Gamma\right)^\perp$ is the second fundamental form of Γ at $\psi(y)$. $\vec{\mathrm{II}}$, in turn, is defined in the coordinates ψ by its action on the basis vectors of $T_{\psi(y)}\Gamma$, given by the formula

$$\vec{\mathrm{II}}(\partial_a \psi, \partial_b \psi)(y) = \sum_{\alpha > p} \Gamma^\alpha_{ab}(y, 0) \partial_\alpha \Psi(y, 0), \tag{3.56}$$

where $\Gamma^\mu_{\alpha\beta}$ denote the Christoffel symbols (Γ here is not to be confused with the manifold):

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} \mathfrak{g}^{\mu\delta} \left(\partial_\alpha \mathfrak{g}_{\beta\delta} + \partial_\beta \mathfrak{g}_{\alpha\delta} - \partial_\delta \mathfrak{g}_{\alpha\beta} \right), \qquad \alpha, \beta, \mu = 0, 1, ..., n.$$

A direct computation using the properties of the basis $\{\nu_i\}_{i=1}^k$, shows that

$$(\mathfrak{g}_{\alpha\beta}) = \begin{pmatrix} (g_{ab}) + O(|z|) & O(|z|) \\ O(|z|) & I_k \end{pmatrix}, \tag{3.57}$$

where I_k is the identity matrix in $\mathbb{R}^{k\times k}$. Therefore, for each $a, b \in \{0, ..., p\}$ and $\alpha > p$:

$$\begin{aligned} \partial_\alpha \mathfrak{g}_{ab}(y,0) &= \partial_\alpha\Big|_{(y,0)} \left\langle \partial_a\psi + \sum_i z^i \partial_a v_i, \partial_b\psi + \sum_i z^i \partial_b v_i \right\rangle_m \\ &= \sum_{i=1}^k \partial_\alpha\Big|_{(y,0)} \left[z^i \left(\langle \partial_a\psi, \partial_b\nu_i\rangle_m + \langle \partial_b\psi, \partial_a\nu_i\rangle_m \right)\right] \\ &= \left(\langle \partial_a\psi, \partial_b\nu_\alpha\rangle_m + \langle \partial_b\psi, \partial_a\nu_\alpha\rangle_m \right) \\ &= -2\,\langle \partial_{ab}\psi, \nu_\alpha\rangle_m\,, \end{aligned}$$

where we have used the identity

$$\langle \partial_a\psi, \partial_b\nu\rangle_m = -\,\langle \partial_{ab}\psi, \nu\rangle_m\,, \tag{3.58}$$

which holds for every smooth (Minkowski) normal vector field ν to Γ and follows from differentiating both sides of the identity $\langle \partial_a\psi, \nu\rangle_m \equiv 0$ with respect to ∂_b.

As a result, for all $\alpha > p$ and $a, b \in \{0, ..., p\}$:

$$\begin{aligned} \Gamma^\alpha_{ab}(y,0) &= \frac{1}{2} g^{\alpha\alpha} \left(\partial_\alpha \mathfrak{g}_{b\alpha} + \partial_b \mathfrak{g}_{a\alpha} - \partial_\alpha \mathfrak{g}_{ab} \right)\Big|_{(y,0)} \\ &= -\frac{1}{2} \partial_\alpha \mathfrak{g}_{ab}\Big|_{(y,0)} \\ &= \langle \partial_{ab}\psi(y), \nu_\alpha(y)\rangle_m\,. \end{aligned} \tag{3.59}$$

Finally, $\partial_\alpha\Psi = \nu_\alpha$ for each $\alpha > p$, and therefore, by (3.59), the formulae (3.56) and (3.55) may be written in terms of the coordinates ψ and the normal vector fields $\{\nu_i\}_{i=1}^k$ as

$$\vec{\mathrm{II}}(\partial_a\psi, \partial_b\psi)(y) = \sum_{i=1}^k \langle \partial_{ab}\psi(y), \nu_i(y)\rangle_m\, \nu_i(y), \tag{3.60}$$

$$\vec{H}_\Gamma(y) = g^{ab}(y) \sum_{i=1}^k \langle \partial_{ab}\psi(y), \nu_i(y)\rangle_m\, \nu_i(y). \tag{3.61}$$

Consequently, using (3.60) we have

$$\vec{\mathrm{II}}(\nabla_\Gamma\theta, \nabla_\Gamma\theta) = \mathrm{II}(\partial^a\theta\partial_a\psi, \partial^b\theta\partial_b\psi) = \partial^a\theta\partial^b\theta\sum_{i=1}^{k}\langle\partial_{ab}\psi, \nu_i\rangle_m \nu_i,$$

from which we readily obtain, thanks to (3.61),

$$\vec{H}_\Gamma = -f_1(\rho)\vec{\mathrm{II}}(\nabla_\Gamma\theta, \nabla_\Gamma\theta) \iff (3.53) \text{ holds.}$$

□

Remark 3.13. There is a simple relationship between the expressions (3.61) and (3.24) when Γ is of codimension 1. In this particular case, denoting by ν a Minkowski normal vector field to Γ with $\langle\nu,\nu\rangle_m \equiv 1$ on Γ, then

$$\vec{H}_\Gamma = H_\Gamma\,\nu. \tag{3.62}$$

To see this, note that

$$\partial_z\mathfrak{g}_{\alpha\beta}(y,z) = \begin{cases}\langle\partial_\alpha\psi, \partial_\beta\nu\rangle_m + \langle\partial_\beta\psi, \partial_\alpha\nu\rangle_m + 2z\,\langle\partial_\alpha\nu, \partial_\beta\nu\rangle_m & \alpha,\beta\in\{0,...,n-1\}\,,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, using (3.39), (3.57) with $k=1$ (i.e., (3.12)), and (3.58):

$$\begin{aligned}\partial_z\Big|_{(y,0)}|\mathfrak{g}| &= |g(y)|\,\mathfrak{g}^{\alpha\beta}(y,0)\partial_z\Big|_{(y,0)}\mathfrak{g}_{\alpha\beta}\\ &= -2|g(y)|g^{ab}(y)\,\langle\partial_{ab}\psi(y), \nu(y)\rangle_m\,.\end{aligned}$$

Therefore,

$$H_\Gamma(y) = -\frac{1}{2}\partial_z\Big|_{(y,0)}\ln[|\mathfrak{g}|] = g^{ab}(y)\,\langle\partial_{ab}\psi(y), \nu(y)\rangle_m\,. \tag{3.63}$$

Comparing this expression for H_Γ with (3.61) shows (3.62).

Another useful expression that applies to the case where Γ has co-dimension 1 and follows from the above discussion is the following equivalent form of the first equation in (3.35):

$$H_\Gamma = -f_1(\rho)\partial^a\theta\partial^b\theta\,\langle\partial_{ab}\psi, \nu\rangle_m = -\frac{1}{2}f_1(\rho)\partial_z\mathfrak{g}^{ab}(y,0)\partial_a\theta\partial_b\theta. \tag{3.64}$$

3.4 EXISTENCE THEOREMS

We now present some existence results for (3.35), the Euler Lagrange equations of the functional $\Sigma_\Pi[\cdot,\cdot,V]$ from Equation 3.2, for given initial conditions on Γ and θ. In order to express this problem as a standard PDE initial value problem, we choose local coordinates for Γ, $(\psi,(0,T)\times V)$, where $V\subset\mathbb{R}^p$ is open. In addition, we will restrict our attention to the cases where Π_ρ is guaranteed to lie in the set $\mathcal{O}$ from Theorem 2.3, and thus impose the condition that $|\rho|<\rho_{\max}$ for some $\rho_{\max}>0$ throughout. As a result, and thanks to Lemma 3.12, we may translate the problem of interest over $\psi(V)\subset\Gamma$ as the following system of PDEs on the unknowns ψ and θ:

$$
\begin{cases}
C_1^{ab}(D\psi,D\theta)\left[\partial_{ab}\psi-\left\langle g^{cd}(D\psi)\partial_d\psi,\partial_{ab}\psi\right\rangle_m\partial_c\psi\right]=0, & \text{in } (0,T)\times V\\
C_2^{ab}(D\psi,D\theta)\left[\partial_{ab}\theta-\left\langle g^{cd}(D\psi)\partial_d\psi,\partial_{ab}\psi\right\rangle_m\partial_c\theta\right]=0, & \text{in } (0,T)\times V\\
|\rho|<\rho_{max}, & \text{in } (0,T)\times V\\
(\psi,\partial_0\psi)=(p_\psi,q_\psi), & \text{on } \{0\}\times V\\
(\theta,\partial_0\theta)=(p_\theta,q_\theta), & \text{on } \{0\}\times V.
\end{cases}
\tag{3.65}
$$

where the functions $p_\psi,p_\theta,q_\psi,q_\theta$ are given and $\rho_{\max}>0$ is fixed. We further require that p_ψ and q_ψ have the intended geometrical meaning, p_ψ being coordinates of $\psi(\{0\}\times V)\subset\Gamma_0$, and q_ψ representing e.g., a timelike vector field normal to $\psi(\{0\}\times V)$.

Given the inherent geometric character of the problem described by (3.65), and more specifically the fact that the choice of coordinates for Γ is not unique[2], the problem (3.65) is underdetermined. In order to alleviate this shortcoming, we impose constraints on the type of coordinates ψ for Γ (i.e., make a choice of gauge)[3]. In particular, we will employ the so called graph gauge condition on Γ, meaning that we posit that Γ is the graph of a function. More precisely, we will assume that Γ can be parametrized by ψ, where

$$\psi(y)=(y,\gamma(y)),\qquad y=(y_0,\cdots,y_p)\in[0,T]\times\mathbb{R}^p,\tag{3.66}$$

2 Write (3.37) as $D[\psi,\theta]=0$ for some differential operator D. Let (ψ,θ) be such that $D[\psi,\theta]=0$ and $f:(0,T)\times V\to(0,T)\times V$ be a diffeomorphism. It follows that $D[\psi_f,\theta_f]=0$ where $\psi_f=\psi\circ f$ and $\theta_f=\theta\circ f$. Moreover, (ψ_f,θ_f) solve (3.65). for any such f which satisfies e.g., $f(y_0,\bar{y})=(y_0,\bar{y})$ whenever $y_0<T/2$.

3 Another possible choice of gauge is given by the type of coordinates mentioned in Remark 3.5, which does not impose an additional geometric restriction on Γ apart from the timelike condition. This contrasts to our choice of gauge, which discards e.g., manifolds Γ whose time slices are closed. The graph gauge, however, proved to be advantageous for the purposes of writing (3.65) as a symmetric hyperbolic initial value problem.

for some $p \in \mathbb{N}_{<n}$ and some $\gamma \in C^2([0,T] \times \mathbb{R}^p; \mathbb{R}^{n-p})$. In this way, $1+p$ is the dimension of Γ and $k := n-p$ its codimension in $\mathbb{R}^{1+n}$. We will use the following slight abuse of notation to express everything in terms of the unknowns γ and θ: $C_1^{ab} = C_1^{ab}(D\gamma, D\theta) := C_1^{ab}(D\psi, D\theta)$, $g = g(D\gamma, D\theta) := g(D\psi, D\theta)$, etc. More precisely, we will write

$$\begin{aligned} g_{ab}(D\gamma) &:= \tilde{\eta}_{ab} + (\partial_a\gamma)^T \partial_b\gamma, \qquad \tilde{\eta} := \text{ Minkowskian metric in } \mathbb{R}^{1+p} \\ g^{ab}(D\gamma) &:= \left(g^{-1}(D\gamma)\right)^{ab} \\ \rho(D\gamma, D\theta) &:= g^{ab}(D\gamma)\partial_a\theta\partial_b\theta \\ C_j^{ab}(D\gamma, D\theta) &:= g^{ab}(D\gamma) + f_j(\rho(D\gamma, D\theta))g^{ac}(D\gamma)g^{bd}(D\gamma)\partial_c\theta\partial_d\theta. \end{aligned} \tag{3.67}$$

Note that if (3.66) holds, the first $1+n$ equations in (3.37) become

$$C_1^{ab}\begin{pmatrix} 0_{(1+p)\times 1} \\ \partial_{ab}\gamma \end{pmatrix} - \begin{pmatrix} \left\langle \partial^0\gamma, C_1^{ab}\partial_{ab}\gamma \right\rangle \\ \vdots \\ \left\langle \partial^p\gamma, C_1^{ab}\partial_{ab}\gamma \right\rangle \\ \left\langle \partial^c\gamma, C_1^{ab}\partial_{ab}\gamma \right\rangle \partial_c\gamma \end{pmatrix} = 0, \qquad \text{in } (0,T) \times V. \tag{3.68}$$

Thus a necessary and sufficient condition for (ψ, θ) to solve (3.65) under the graph gauge condition (3.66) for initial conditions of the form

$$p_\psi(y) = (0, y, p_\gamma(y)) \quad \text{ and } \quad q_\psi(y) = (\overbrace{1, 0, ..., 0}^{\in\mathbb{R}^p}, q_\gamma(y)), \quad \text{ for all } y \in \mathbb{R}^p,$$

is that (γ, θ) solve the following system, which we will consider hereafter

$$\begin{cases} C_1^{ab}(D\gamma, D\theta)\partial_{ab}\gamma = 0, & \text{in } (0,T) \times \mathbb{R}^p \\ C_2^{ab}(D\gamma, D\theta)\left[\partial_{ab}\theta - \left\langle g^{dc}(D\gamma)\partial_d\theta\partial_c\gamma, \partial_{ab}\gamma \right\rangle\right] = 0, & \text{in } (0,T) \times \mathbb{R}^p \\ |\rho| < \rho_{\max}, & \text{in } (0,T) \times \mathbb{R}^p \\ (\gamma, \partial_0\gamma) = (p_\gamma, q_\gamma), & \text{on } \{0\} \times \mathbb{R}^p \\ (\theta, \partial_0\theta) = (p_\theta, q_\theta), & \text{on } \{0\} \times \mathbb{R}^p, \end{cases} \tag{3.69}$$

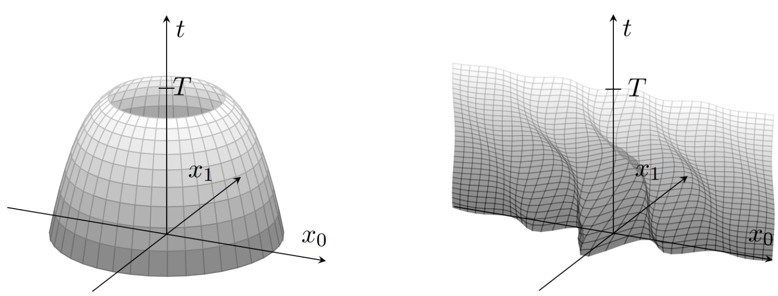
for given $\rho_{\max} > 0$, $\mathbb{R}^{n-p}$-valued functions p_γ, q_γ and real valued functions p_θ, q_θ defined on $\mathbb{R}^p$.

We present two theorems addressing the well-posedness of (3.69) (and therefore of (3.65) under the condition that ψ is of the form (3.66)), namely Theorem 3.16 and Theorem 3.22. In both cases, we establish the existence of $T > 0$ for which the problem (3.69) admits a unique solution $\gamma : [0,T] \times \mathbb{R}^p \to \mathbb{R}^{n-p}$ provided that the initial data enforce a smallness condition on ρ and a Lorentzian condition over Γ_0 on either: (a) the metric g in Theorem 3.16 (see (3.73)) or (b) the tensors C_1 and C_2 appearing in (3.69) in Theorem 3.22 (see (3.92)) (a generally stronger condition). Theorem 3.16 applies only to the special case $p = 1$, whereas Theorem 3.22 applies to any $p \geq 1$. In both cases, the solutions exhibit the following regularity provided that the initial data on γ and θ are of type H^{s+1}:

$$\theta, \gamma^i \in C_b^{s-[p/2]}\left([0,T] \times \mathbb{R}^p\right), \qquad i \in \{1, ..., n-p\}\,, \tag{3.70}$$

where $[x] :=$ the integer part of $x \in \mathbb{R}$. The idea behind the proofs of these theorems consists in exploiting the structure of the equations in Equation 3.65 to express them as a first order symmetric hyperbolic system of PDEs whose well-posedness is revealed thanks to the results from [19].

Remark 3.14. In [34], the case $n = 2$ was considered under the additional assumption that Γ (necessarily of dimension $(1+1)$) is radially symmetric and the leading order term of the phase of the current carrier field σ (i.e., θ) is an integer multiple of $\arg(x_0 + ix_1)$, where x_0 and x_1 are the space coordinates. In this case, the system of equations (3.35)(or equivalently (3.36) or (3.37)) reduces to an ODE in terms of a coordinate function $R(t)$ of Γ, where $R(t)$ = radius of the circle $\Gamma^t := \Gamma \cap \{(t,x) : x \in \mathbb{R}\}$. As a result, the well-posedness of the problem of finding admissible pairs (Γ, θ) in this case is argued on the grounds of standard results from ODE theory. Figure 3.2 shows sample graphical representations of Γ associated to this case and to the case treated in Theorem 3.16 and Theorem 3.22 for $n = 2$.



(a) Sample illustration of the type of interface Γ for the radially symmetric case studied in [34] ($n = 2$).

(b) Sample illustration of the type of interface Γ whose existence is guaranteed by Theorem 3.16 and Theorem 3.22 for $n = 2$.

Figure 3.2: Sample interfaces for the case $n = 2$.

Remark 3.15. In view of the hyperbolic character of (3.69), the results of Theorem 3.16 and Theorem 3.22 should be extendible to manifolds Γ with more general geometries. Notably, this includes the case in which the "initial" submanifold Γ_0 is compact, as in the case of closed interfaces/closed strings. In essence, due to the compactness of Γ_0, the gluing of solutions corresponding to each open set in a finite cover of Γ_0 should be possible thanks to the properties of existence, uniqueness and finite speed of propagation associated with Equation 3.35. A thorough treatment of this procedure and of more general results as they apply to the (hyperbolic) minimal surface equation $\vec{H}_\Gamma \equiv 0$ is presented in [21].

3.4.1 *The Case* dim(Γ)=*2 Under the Graph Gauge*

Let $k \in \mathbb{N}$ and define

$$\begin{aligned} G_k(x,y) &:= \begin{pmatrix} -1+|x|^2 & x\cdot y \\ x\cdot y & 1+|y|^2 \end{pmatrix}, \\ Q_k(x,y,v,w) &:= (v\ w)\, G_k(x,y)^{-1} \begin{pmatrix} v \\ w \end{pmatrix}, \end{aligned} \tag{3.71}$$

for all $x, y \in \mathbb{R}^k$ for which $\det(G_k(x,y)) \neq 0$ and for all $v, w \in \mathbb{R}$. It follows that if Γ is parametrized by a map ψ of the form (3.66), then $g_{ab} = (G_{n-1}(\partial_0\gamma, \partial_1\gamma))_{ab}$ and $\rho = Q_{n-1}(\partial_0\gamma, \partial_1\gamma, \partial_0\theta, \partial_1\theta)$, and the timelike condition on Γ implies that $\det\left(G_{n-1}(\partial_0\gamma, \partial_1\gamma)\right) < 0$ over Γ.

Theorem 3.16. *Let $n, s \in \mathbb{N}_{\geq 2}, \rho_{\max} > 0$ and*

$$\delta_\rho := \min\left\{\left\{x \in \mathbb{R} : [1 + xf_1(x)]\,[1 + xf_2(x)] = 0\right\} \cup \{\rho_{\max}\}\right\} > 0, \tag{3.72}$$

where f_1 and f_2 are defined in (3.38). Fix $\delta \in (0, \delta_\rho)$ and suppose $(p_\gamma, q_\gamma) \in H^{s+1}(\mathbb{R};\mathbb{R}^{n-1}) \times H^s(\mathbb{R};\mathbb{R}^{n-1})$ and $(p_\theta, q_\theta) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$ are such that, for all $x \in \mathbb{R}$,

$$\begin{cases} |\rho(0,x)| = \left|Q_{n-1}(q_\gamma(x), p'_\gamma(x), q_\theta(x), p'_\theta(x))\right| < \delta_p - \delta \\ \det\left[G_{n-1}(q_\gamma(x), p'_\gamma(x))\right] < -\delta. \end{cases} \tag{3.73}$$

Then, there exists $T > 0$ such that the problem (3.65) with

$$p_\psi(x) = (0, x, p_\gamma(x)) \qquad \textit{and} \qquad q_\psi(x) = (1, 0, q_\gamma(x)) \quad \textit{for all } x \in \mathbb{R}$$

admits a unique solution (ψ,θ)*, where* ψ *is of the form* (3.66) *and* γ *and* θ *satisfy*

$$\theta, \gamma^i \in C_b^s([0,T]\times\mathbb{R}), \qquad i \in \{1,...,n-1\}.$$

Furthermore, for all $y \in [0,T]\times\mathbb{R}$*:*

$$|\rho(y)| < \delta_\rho \text{ and } |g|(y) = -\det\left[G_{n-1}(\partial_0\gamma(y), \partial_1\gamma(y))\right] > \delta/2. \tag{3.74}$$

Remark 3.17. The fact that δ_ρ in (3.72) is positive follows from the fact that f_1 and f_2 are negative and continuous. Additionally, the second condition on (3.74) implies that Γ is timelike since (g_{ab}) is a $(2,0)$ symmetric tensor[4].

Proof of Theorem 3.16: Let $n \in \mathbb{N}_{\geq 2}$ and $k := n-1$. We will find sufficient conditions under which a first order system of PDEs equivalent to (3.69) is hyperbolic (shortly defined). Having established these conditions, the resulting system will be symmetrized, allowing us to employ the results of [19] for proving the existence of solutions as a final step. To this end, let

$$\Phi := \begin{pmatrix} \partial_0\gamma \\ \partial_1\gamma \\ \partial_0\theta \\ \partial_1\theta \end{pmatrix} \qquad \text{and} \qquad \Phi_\gamma = \begin{pmatrix} \partial_0\gamma \\ \partial_1\gamma \end{pmatrix}, \tag{3.75}$$

and note that, under condition (3.66), we have that

$$\begin{aligned}
g_{ab} = G_k(\Phi_\gamma) &= \begin{pmatrix} -1+|\partial_0\gamma|^2 & \partial_0\gamma\cdot\partial_1\gamma \\ \partial_0\gamma\cdot\partial_1\gamma & 1+|\partial_1\gamma|^2 \end{pmatrix}, \\
|g| = -\det(G_k(\Phi_\gamma)) &= |\partial_1\gamma|^2 + 1 - |\partial_0\gamma|^2 + (\partial_0\gamma\cdot\partial_1\gamma)^2 - |\partial_0\gamma|^2|\partial_1\gamma|^2, \\
g^{ab} = (G_k(\Phi_\gamma))^{-1} &= \frac{1}{|g|}\begin{pmatrix} -(1+|\partial_1\gamma|^2) & \partial_0\gamma\cdot\partial_1\gamma \\ \partial_0\gamma\cdot\partial_1\gamma & 1-|\partial_0\gamma|^2 \end{pmatrix}, \\
\rho = Q_k(\Phi) &= g^{ab}\partial_a\theta\partial_b\theta.
\end{aligned}$$

Presuming that Γ is timelike, the following conditions must hold:

$$\begin{aligned}
C_1^{00}(\Phi) &= g^{00} + f_1(\rho)\left(\partial^0\theta\right)^2 < 0 \\
C_2^{00}(\Phi) &= g^{00} + f_2(\rho)\left(\partial^0\theta\right)^2 < 0.
\end{aligned} \tag{3.76}$$

4 Recall that $|g|$ denotes the negative of the determinant of g.

Indeed, f_1 and f_2 are negative functions, and $\det(G_k(\Phi_\gamma)) < 0$ if Γ is timelike, and thus $g^{00} = -g_{11}/|g| = (1 + |\partial_1\gamma|^2)/\det(G_k(\Phi_\gamma)) < 0$. As a result, under the timelike condition on the graph of γ, which we will see is guaranteed for some non-zero finite time due to the conditions imposed by the initial data, the PDE system in (3.69) is equivalent to

$$\begin{cases} \partial_{00}\gamma = 2c_1^{01}(\Phi)\partial_{01}\gamma + c_1^{11}(\Phi)\partial_{11}\gamma \\ \partial_{00}\theta = 2c_2^{01}(\Phi)\partial_{01}\theta + c_2^{11}(\Phi)\partial_{11}\theta + 2\left[m^{01}(\Phi)\right]^T \partial_{01}\gamma + \left[m^{11}(\Phi)\right]^T \partial_{11}\gamma, \end{cases} \tag{3.77}$$

where, for all $a, b \in \{0,1\}$ and $j \in \{1,2\}$:

$$\begin{aligned} c_j^{ab}(\Phi) &:= -\left(C_j^{00}(\Phi)\right)^{-1} C_j^{ab}(\Phi) \\ m^{ab}(\Phi) &:= (c_1^{ab}(\Phi) - c_2^{ab}(\Phi))\partial^c\theta\partial_c\gamma, \end{aligned} \tag{3.78}$$

We may therefore rewrite (3.69) for the case of interest in matrix form as:

$$\partial_0\Phi + A^1(\Phi)\partial_1\Phi = 0 \qquad \text{in } (0,T)\times\mathbb{R}, \tag{3.79}$$

where, denoting by I_l the $l \times l$ identity matrix and by $0_{l\times m}$ the $l \times m$ zero matrix for $l, m \in \mathbb{N}$:

$$A^1(\Phi) := -\begin{pmatrix} 2c_1^{01}(\Phi)I_k & c_1^{11}(\Phi)I_k & 0_{k\times1} & 0_{k\times1} \\ I_k & 0_k & 0_{k\times1} & 0_{k\times1} \\ 2\left(m^{01}(\Phi)\right)^T & \left(m^{11}(\Phi)\right)^T & 2c_2^{01}(\Phi) & c_2^{11}(\Phi) \\ 0_{1\times k} & 0_{1\times k} & 1 & 0 \end{pmatrix}. \tag{3.80}$$

Step 1. Hyperbolicity. We say that the system (3.79) is (uniformly) hyperbolic if $A^1(\Phi)$ is diagonalizable and all its eigenvalues are real over $(0,T)\times\mathbb{R}$. The necessary and sufficient conditions for the hyperbolicity of Equation 3.79 are provided by the following lemma, whose proof can be found in Appendix C:

Lemma 3.18. *Let $k \in \mathbb{N}, v \in \mathbb{R}^{k\times1}$, and $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Then, the matrix*

$$A := \begin{pmatrix} 2a_1 I_{k\times k} & b_1 I_{k\times k} & 0_{k\times1} & 0_{k\times1} \\ I_{k\times k} & 0_{k\times k} & 0_{k\times1} & 0_{k\times1} \\ 2(a_1 - a_2)v^T & (b_1 - b_2)v^T & 2a_2 & b_2 \\ 0_{1\times k} & 0_{1\times k} & 1 & 0 \end{pmatrix} \in \mathbb{R}^{2(k+1)\times2(k+1)}, \tag{3.81}$$

is diagonalizable and has only real eigenvalues if

$$\Delta_i := a_i^2 + b_i > 0, \qquad \textit{for } i = 1, 2. \tag{3.82}$$

Furthermore, whenever (C.3) is satisfied, a set of linearly independent vectors of A is given by (C.5)-(C.6).

The application of Lemma C.2 to the matrix $A^1(\Phi)$, upon setting $a_i = c_i^{01}(\Phi), b_i = c_i^{11}(\Phi)$ for $i = 1, 2$ and $v = \partial^c\theta\partial_c\gamma$, shows that the system (3.79) is hyperbolic if and only if

$$\Delta_i(\Phi) := \left[c_i^{01}(\Phi)\right]^2 + c_i^{11}(\Phi) = -\det\left(c_i^{ab}(\Phi)\right) > 0 \qquad i = 1, 2. \tag{3.83}$$

This last condition can be expressed as a condition on ρ as follows:

$$\begin{aligned}
\Delta_i(\Phi) &= \left(C_i^{00}\right)^{-2}\left[\left[g^{01} + f_i(\rho)\,\partial^0\theta\partial^1\theta\right]^2 - (g^{00} + f_i(\rho)\,(\partial^0\theta)^2)(g^{11} + f_i(\partial^1\theta)^2)\right] \\
&= \left(C_i^{00}\right)^{-2}\left[\left(g^{01}\right)^2 - g^{00}g^{11} - f_i(\rho)\left[g^{00}\left(\partial^1\theta\right)^2 - 2g^{01}\partial^0\theta\partial^1\theta + g^{11}(\partial^0\theta)^2\right]\right] \\
&= \left(C_i^{00}\right)^{-2}\left[-\det g^{ab} - \left(\det g^{ab}\right) f_i(\rho)\,\rho\right] \\
&= \frac{\left(C_i^{00}(\Phi)\right)^{-2}}{|g(\Phi_\gamma)|}\left[1 + f_i(\rho(\Phi))\rho(\Phi)\right],
\end{aligned} \tag{3.84}$$

where we have used the fact that

$$\rho = g^{00}\left(\partial_0\theta\right)^2 + 2g^{01}(\partial_0\theta)(\partial_1\theta) + g^{11}(\partial_1\theta)^2 = \nabla\theta^T g^{-1}\nabla\theta$$

$$\left(g^{00}, g^{01}, g^{11}\right) = \frac{1}{|g|}\left(-g_{11}, g_{01}, -g_{00}\right).$$

Therefore, condition (3.83) is equivalent to

$$1 + f_i(\rho(\Phi))\rho(\Phi) > 0 \qquad i = 1, 2,$$

and altogether we may conclude that

$$\begin{gathered}
\Gamma \text{ is timelike and } 1 + f_i(Q_k(\Phi)) \cdot Q_k(\Phi) > 0 \text{ for } i = 1, 2 \\
\Longrightarrow \ (3.79) \text{ is hyperbolic.}
\end{gathered} \tag{3.85}$$

Based on (3.85), we define sets on which Φ must take its values to ensure the hyperbolicity of (3.79) and the timelike character of Γ. Namely, let δ_ρ be as in (3.72), and for each[5] $\epsilon \in [0, \min\{1, \delta_\rho\})$ define

$$V_\epsilon^k := \big\{ u \in \mathbb{R}^{2(k+1)} \,\big|\, Q_k(u) \in (-\rho_{\max} + \epsilon, \delta_\rho - \epsilon), \\ \det(G_k(u_1, u_2, ..., u_{2k})) < -\epsilon \big\}. \quad (3.86)$$

Then, each set $V_\epsilon^k \subset \mathbb{R}^{2(k+1)}$ is non-empty and open, since it is the intersection of two open sets containing the origin. Also, it follows from (3.85) and the definition of δ_ρ that the system (3.79) is hyperbolic and the graph of γ (i.e., Γ) is timelike whenever $\mathrm{Im}(\Phi) \subset V_\epsilon^k \subset V_0^k$ for any $\epsilon \in [0, \delta_\rho)$.

Step 2. Symmetrization of (3.79). In what follows, we will adopt the following definition of a symmetric hyperbolic system that will suit our needs to invoke the results from [19]:

Definition 3.19. A first order system of the form

$$\sum_{j=0}^{m} S^j(\Phi)\partial_j \Phi = F(\Phi),$$

where $F : \mathbb{R}^{1+m} \to \mathbb{R}^{1+m}$ and $S^j : \mathbb{R}^{1+m} \to \mathbb{R}^{(1+m)\times(1+m)}$ for each $j \in \{0, 1, ..., m\}$, will be called ***symmetric hyperbolic at*** $\Phi \in \mathbb{R}^{1+m}$ if $S^j(\Phi)$ is symmetric for all $j \in \{0, 1, ..., n\}$ and $S^0(\Phi)$ is positive definite.

We will "symmetrize" the system (3.79) by using the diagonalizability of $A^1(\Phi)$ whenever $\Phi \in V_0^k$. To this end, let $\epsilon \in [0, \delta_\rho)$. Then, for each $u = \big(u_1, \ldots, u_{2(k+1)}\big) \in V_\epsilon^k \subset V_0^k$ we may write

$$A^1(u) = P(u)\Lambda(u)P(u)^{-1},$$

where the columns of $P(u)$ are any set of $2(k+1)$ linearly independent eigenvectors of $A^1(u)$, e.g., of the form (C.5) and (C.6) with $a_i = c_i^{01}(\Phi)$, $b_i = c_i^{11}(\Phi)$ and $v = \partial^c\theta\partial_c\gamma$. As an example, for the case $k = 1$, we can set

$$P(u) = \begin{pmatrix} \lambda_1^+(u) & \lambda_1^-(u) & 0 & 0 \\ 1 & 1 & 0 & 0 \\ a(u)\lambda_1^+(u) & a(u)\lambda_1^-(u) & \lambda_2^+(u) & \lambda_2^-(u) \\ a(u) & a(u) & 1 & 1 \end{pmatrix},$$

5 We take $\min\{1, \delta_\rho\}$ instead of δ_ρ so that $0 \in V_\varepsilon^k$ independently of ε and δ_ρ.

where $a(u) = (u_1\ u_2)\,[G_1(u_1,u_2)]^{-1}\begin{pmatrix}u_3\\u_4\end{pmatrix}$.

Letting $Q(u) := [P(u)]^{-1}$ and $S^0(u) := Q^T(u)Q(u)$, and multiplying both sides of equation (3.79) by $S^0(\Phi)$ while assuming that $\operatorname{Im}\Phi \subset V_0^k$, we obtain the following symmetric hyperbolic system equivalent to (3.79):

$$S^0(\Phi)\partial_0\Phi + S^1(\Phi)\partial_1\Phi = 0, \qquad \text{where} \qquad S^1(\Phi) := Q(\Phi)^T\Lambda(\Phi)Q(\Phi). \tag{3.87}$$

Note that the entries of $P(u)$ are rational functions of the entries of u, and thus so are those of $S^0(u)$ and $S^1(u)$. Therefore, since the closure of V_ϵ^k is strictly contained in V_0^k for each $\epsilon \in (0,\delta_\rho)$, the following assertions hold all $\varepsilon \in (0,\delta_\rho)$:

$$\begin{cases} S^0, S^1 \in C_b^\infty(V_\epsilon^k;\mathbb{R}^{2(k+1)\times 2(k+1)}) \\ S^0, S^1 \text{ are symmetric on } V_\epsilon^k \\ S^0(u) \succeq \delta_\epsilon I \text{ for some } \delta_\epsilon > 0 \text{ and all } u \in V_\epsilon^k \end{cases}. \tag{3.88}$$

Step 3. Existence of Solutions to (3.87). Let $\epsilon \in (0,\delta_\rho)$ and consider the system

$$\begin{cases} S^0(\Phi)\partial_0\Phi + S^1(\Phi)\partial_1\Phi = 0 & \text{in } [0,T]\times\mathbb{R} \\ \Phi \in V_\epsilon^k & \text{in } [0,T]\times\mathbb{R} \\ \Phi(0,\cdot) = \Phi_0 & \text{in } \mathbb{R}, \end{cases} \tag{3.89}$$

for fixed initial conditions $\Phi_0 \in H^s(\mathbb{R};V_\epsilon^k)$, where V_ϵ^k is defined in (3.86).

We will apply the results from [19] to obtain solutions of (3.89). To accomplish this, and in reference to the results found therein, we employ Theorem IV to verify the conditions of Theorem II in the context of (3.89).

First, let $M > 1$ be arbitrary and $r > 0$ be small enough so that

$$\Omega_r := \left\{ w \in \mathbb{R}^{2(k+1)} : |w - \Phi_0(x)| < r \text{ for some } x \in \mathbb{R} \right\} \subset V^k_{\frac{\varepsilon}{M}}. \tag{3.90}$$

The Sobolev embedding $H^s(\mathbb{R};\mathbb{R}^{2(k+1)}) \hookrightarrow C_b^{s-1}(\mathbb{R};\mathbb{R}^{2(k+1)})$ for $s \geq 1$ implies that $R > 0$ can be chosen small enough so that

$$D := \left\{ f \in H^s\left(\mathbb{R};\mathbb{R}^{2(k+1)}\right) : \|f - \Phi_0\|_{H^s\left(\mathbb{R};\mathbb{R}^{2(k+1)}\right)} < R \right\} \subset C_b^{s-1}(\mathbb{R};\Omega_r).$$

With this definition of D, and as a result of (3.88) and (3.90), conditions (4.2)-(4.9) and (4.11) of Theorem II of [19] are satisfied, and we conclude that there is $T > 0$ and a unique solution Φ of (3.89) defined on $[0, T'] \times \mathbb{R}$ where $0 < T' \leq T$ and $\Phi(t, \cdot) := \tilde{\Phi}(t)$, where

$$\begin{cases} \tilde{\Phi} \in C\left((0,T'); D\right) \cap C^1\left((0,T'); H^{s-1}(\mathbb{R}; \mathbb{R}^{2(k+1)})\right) \\ \tilde{\Phi}(0) = \Phi_0 \end{cases} . \tag{3.91}$$

To conclude, suppose we are given initial conditions

$$(p_\gamma, q_\gamma) \in H^{s+1}(\mathbb{R}; \mathbb{R}^k) \times H^s(\mathbb{R}; \mathbb{R}^k) \text{ and } (p_\theta, q_\theta) \in H^{s+1}(\mathbb{R}; \mathbb{R}) \times H^s(\mathbb{R}; \mathbb{R})$$

such that (3.73) holds. Then, $(q_\gamma, p'_\gamma, q_\theta, p'_\theta) \in H^s(\mathbb{R}; V^k_\epsilon)$ for some $\varepsilon > 0$, and the above argument yields a solution $\Phi = (\partial_0\gamma, \partial_1\gamma, \partial_0\theta, \partial_1\theta)$ to (3.89) taking values in $\Omega_r \subset V^k_0$ for some sufficiently small $r > 0$, which ensures that conditions (3.74) hold. In terms of regularity, the components of Φ obey (3.70) due to (3.91) and Lemma C.7. As a result, we obtain a unique solution (ψ, θ) to (3.65) with ψ of the form (3.66) with all the properties stated. □

Remark 3.20. The reason why Theorem 3.16 is limited to the case $p = 1$ is that the system (3.79) corresponding to a general value of $p \in \mathbb{N}$:

$$\partial_0\Phi + \sum_{k=1}^{p} A^j(\Phi)\partial_j\Phi = 0$$

cannot be symmetrized as in Step 2 of the proof unless all $A^j(\Phi)$'s are simultaneously diagonalizable, that is, unless there is a matrix $P(\Phi)$ such that $A^j(\Phi)P(\Phi) = P(\Phi)D^j(\Phi)$ for some diagonal matrix $D^j(\Phi)$ for each $j \in \{1, 2, ..., p\}$ (or, e.g., if each $A^j(\Phi)$ is diagonalizable and the pairwise products of the $A^j(\Phi)$'s commute). This is automatically the case whenever $A^1(\Phi)$ is diagonalizable in the $p = 1$ case, but does not appear to be the case for $p > 1$.

3.4.2 *The General Case Under the Graph Gauge*

We now reduce the system (3.37) under the graph gauge condition on Γ to a symmetric system equivalent to (3.69) (as opposed to just symmetrizable as was the case in Theorem 3.16). This procedure allows one to treat the case where Γ is of arbitrary dimension $1+p \geq 2$. A useful definition we will use from now on is the following:

Definition 3.21. Let $p \in \mathbb{N}$. A symmetric $(1+p) \times (1+p)$ matrix A is called ***canonical Lorentzian*** if $A_{00} < 0$ and $\bar{A}$ is positive definite, where $\bar{A}$ is the $p \times p$ submatrix of A obtained by removing the first row and the first column from A. Moreover, for any $\varepsilon \geq 0$, define

$$\mathcal{C}_{p,\varepsilon} := \left\{ A \in M_{(1+p)\times(1+p)} : A \text{ is symmetric}, A_{00} < -\varepsilon, \bar{A} \succeq \varepsilon I_p \right\}.$$

Using [31, Lemmas 8.3 and 8.5], we know that if A is canonical Lorentzian, then it is Lorentzian. Moreover, if $g \in \mathcal{C}_{p,\varepsilon}$ for any $\varepsilon \geq 0$, then g^{-1} is also canonical Lorentzian.

Theorem 3.22. *Let $\rho_{\max} > 0, (k,p) \in \mathbb{N}^2$, and s be any integer such that $s \geq 2 + [p/2]$, where $[x]$ denotes the integer part of $x \in \mathbb{R}$. Furthermore, let $(p_\gamma, q_\gamma) \in H^{s+1}(\mathbb{R}^p; \mathbb{R}^k) \times H^s(\mathbb{R}^p; \mathbb{R}^k)$ and $(p_\theta, q_\theta) \in H^{s+1}(\mathbb{R}^p) \times H^s(\mathbb{R}^p)$ be such that the following conditions hold uniformly over $\mathbb{R}^p$ for some $\delta > 0$:*

$$\begin{aligned} &g^{00}(q_\psi, D_x p_\psi) < -\delta, \\ &\overline{C}_i(q_\psi, D_x p_\psi, q_\theta, D_x p_\theta) \succ \delta I_p, \qquad \text{for } i = 1, 2, \\ &|\rho(q_\psi, D_x p_\psi, q_\theta, D_x p_\theta)| < \rho_{\max} - \delta, \end{aligned} \tag{3.92}$$

where C_1 and C_2 are defined in (3.38), and $\overline{C}_1, \overline{C}_2$ are their respective restricted counterparts:

$$\overline{C}_l^{ij} := C_l^{ij} \qquad \text{for } i, j = 1, 2, \ldots, p \text{ and } l = 1, 2.$$

Then, there exists $T > 0$ such that the problem (3.65) with

$$p_\psi(x) := (0, x, p_\gamma(x)) \qquad \text{and} \qquad q_\psi(x) := (1, 0, q_\gamma(x)) \quad \text{for all } x \in \mathbb{R}^p$$

admits a unique solution (ψ, θ) of the form (3.66), where γ and θ satisfy

$$\theta, \gamma^i \in C_b^{s-[p/2]}\left([0,T] \times \mathbb{R}^p\right), \qquad i \in \{1, ..., n-p\}.$$

Furthermore, for all $y \in [0,T] \times \mathbb{R}^p$*:*

$$\begin{aligned} & g(D\psi), C_1(D\psi, D\theta), C_2(D\psi, D\theta) \in \mathcal{C}_{n,\delta/2}, \\ & |\rho(D\psi, D\theta)| < \rho_{\max} - c, \qquad \text{for some } c \in (0, \rho_{\max}). \end{aligned} \tag{3.93}$$

Remark 3.23. Based on Lemma 3.12, the conclusions from Theorem 3.16 and Theorem 3.22 can be phrased as: under the conditions listed, there is $T > 0$ and a solution (Γ, θ) to (3.35), where Γ is the (timelike) graph of the map γ, $\Pi_{\rho(y)} \in \mathcal{O}$ over Γ, and θ and all the components of γ belong to $C_b^{s-[p/2]}([0,T] \times \mathbb{R}^p)$.

Remark 3.24. The greater level of generality of Theorem 3.22 as compared to that of Theorem 3.16 comes at the expense of stricter conditions on the initial data as compared to those of Theorem 3.16. In particular, compare (3.73) to (3.92) when $p = 1$. Under the timelike condition, (3.73) is equivalent to requiring that $1+\rho f_i(\rho) > 0$ for $i = 1$ and 2 (see (3.83)). On the other hand, when $p = 1$, the conditions (3.92) imply that $|g| > 0$, $C_i^{00}(\Phi) < 0$ and $C_i^{11}(\Phi) > 0$, resulting in $1 + \rho f_i(\rho) = -|g| \det \left(C_i^{ab}\right) > 0$. However, since

$$C_i^{11} = \frac{1}{|g|} \left[-g_{00} \left[1 + f_i(\rho)\rho\right] + f_i(\rho) \left(\partial_0 \theta\right)^2 \right],$$

C_i^{11} can be negative, even if $1 + f_i(\rho)\rho > 0$.

Proof of Theorem 3.22. Consider equation (3.48) and refrain from using (3.44) to simplify the expression therein. Instead, use the computations leading to (3.47) and (3.52) to rewrite (3.42) as[6]

$$M_1^{ab}(D\psi, D\theta)\partial_{ab}\psi + M_2^{ab}(D\psi, D\theta)\partial_{ab}\theta = 0, \tag{3.94}$$

where

$$\begin{aligned} M_2^{ab}(D\psi, D\theta) &= f_1(\rho(D\psi, D\theta)) C_2^{ab}(D\psi, D\theta) \nabla_\Gamma \theta \\ M_1^{ab}(D\psi, D\theta) &= C_1^{ab}(D\psi, D\theta) I_{1+p+k} \end{aligned}$$

6 Note that another way to obtain (3.94) is to use the second equation of (3.68) to obtain

$$C_2^{ab}\partial_{ab}\theta - C_2^{ab} \langle \nabla_\Gamma \theta, \partial_{ab}\psi \rangle_m = 0 \implies M_2^{ab}\partial_{ab}\theta - \left\langle M_2^{ab}, \partial_{ab}\psi \right\rangle_m \nabla_\Gamma \theta = 0.$$

Then, (3.94) results from adding $M_2^{ab}\partial_{ab}\theta - \left\langle M_2^{ab}, \partial_{ab}\psi \right\rangle_m \nabla_\Gamma \theta$ to the first equation of (3.37). However,we stress the fact that (3.94) had essentially shown up earlier in the derivations from Lemma 3.12, which further suggests that it is natural to use this particular "redundant" form of the equations.

$$-\left(C_1^{ab}(D\psi, D\theta)\partial^c\psi\,(\partial_c\psi)^T + \nabla_\Gamma\theta\left(M_2^{ab}(D\psi, D\theta)\right)^T\right)\eta.$$

Using (3.94) and multiplying the second equation of (3.37) by $|f_1(\rho)| = -f_1(\rho)$, we arrive at the following system equivalent to (3.37):

$$\begin{cases} M_1^{ab}(D\psi, D\theta)\partial_{ab}\psi + M_2^{ab}(D\psi, D\theta)\partial_{ab}\theta = 0 \\ |f_1(\rho(D\psi, D\theta))|C_2^{ab}(D\psi, D\theta)\partial_{ab}\theta + \left\langle M_2^{ab}(D\psi, D\theta), \partial_{ab}\psi\right\rangle_m = 0 \end{cases},$$

which, under the graph gauge condition (3.66), is equivalent to

$$\begin{cases} m_1^{ab}(D\gamma, D\theta)\partial_{ab}\gamma + m_2^{ab}(D\gamma, D\theta)\partial_{ab}\theta = 0 \\ c_2^{ab}(D\gamma, D\theta)\partial_{ab}\theta + \left\langle m_2^{ab}(D\gamma, D\theta), \partial_{ab}\gamma\right\rangle = 0 \end{cases}, \tag{3.95}$$

for

$$\begin{aligned} c_1^{ab}(D\gamma, D\theta) &= C_1^{ab}(D\psi, D\theta) \\ c_2^{ab}(D\gamma, D\theta) &= |f_1(\rho(D\psi, D\theta))|C_2^{ab}(D\psi, D\theta) \\ m_2^{ab}(D\gamma, D\theta) &= -c_2^{ab}(D\gamma, D\theta)\,(\partial^c\theta\partial_c\gamma) \\ m_1^{ab}(D\gamma, D\theta) &= c_1^{ab}(D\gamma, D\theta)I_k - (\partial^c\theta\partial_c\gamma)\left(m_2^{ab}(D\gamma, D\theta)\right)^T \\ &= c_1^{ab}(D\gamma, D\theta)I_k + c_2^{ab}(D\gamma, D\theta)\,(\partial^c\theta\partial_c\gamma)\,(\partial^c\theta\partial_c\gamma)^T. \end{aligned}$$

The main appeal of (3.95) is that its associated first order system, (3.97), can be shown to be symmetric hyperbolic for Φ taking values on the open sets (3.100). This allows us to apply the results from [19] by a similar logic to the one used in the proof of Theorem 3.16. To see this, let

$$\Phi = \begin{pmatrix} \partial_0\gamma \\ \vdots \\ \partial_p\gamma \\ \partial_0\theta \\ \vdots \\ \partial_p\theta \end{pmatrix} \qquad \text{and} \qquad v := \partial^a\theta\partial_a\gamma, \tag{3.96}$$

and define the (symmetric) $(0,p)$ tensors

$$\bar{c}_k^{ij} := c_k^{ij} \qquad \text{and} \qquad \bar{m}_k^{ij} := m_k^{ij}, \qquad \text{for } k = 1,2 \text{ and } i,j = 1,2,\cdots,p.$$

With these definitions, (3.95) can be expressed as

$$S^0(\Phi)\partial_0\Phi + \sum_{j=1}^{p} S^j(\Phi)\partial_j\Phi = 0, \tag{3.97}$$

where

$$S^0(\Phi) := \begin{pmatrix} -\left(c_1^{00}I_k + c_2^{00}vv^T\right) & 0_{k\times pk} & c_2^{00}v & 0_{k\times p} \\ 0_{pk\times k} & \overline{c}_1\otimes I_k + \overline{c}_2\otimes vv^T & 0_{kp\times 1} & -\overline{c}_2\otimes v \\ c_2^{00}v^T & 0_{1\times kp} & -c_2^{00} & 0_{1\times p} \\ 0_{p\times k} & -\overline{c}_2\otimes v^T & 0_{n\times 1} & \overline{c}_2 \end{pmatrix}, \tag{3.98}$$

$$S^j(\Phi) := -\begin{pmatrix} 2m_1^{0j} & m_1^{1j} & \dots & m_1^{pj} & 2m_2^{0j} & m_2^{1j} & \dots & m_2^{pj} \\ m_1^{1j} & & & & m_2^{1j} & & & \\ \vdots & & 0_{kp} & & \vdots & & 0_{kp\times p} & \\ m_1^{pj} & & & & m_2^{pj} & & & \\ 2(m_2^{0j})^T & \left(m_2^{1j}\right)^T & \dots & \left(m_2^{pj}\right)^T & 2c_2^{0j} & c_2^{1j} & \dots & c_2^{pj} \\ \left(m_2^{1j}\right)^T & & & & c_2^{1j} & & & \\ \vdots & & 0_{p\times pk} & & \vdots & & 0_p & \\ \left(m_2^{pj}\right)^T & & & & c_2^{pj} & & & \end{pmatrix},$$

for $j = 1, 2, ..., p$. Observe that the matrices $S^a(\Phi)$ for $a = 0, 1, \dots, p$ are symmetric. On the other hand, it follows from Lemma C.3, that

$$S^0(\Phi(y)) \succ 0 \quad \text{if} \quad \left(c_i^{00}(\Phi(y)) < 0 \text{ and } \overline{c}_i(\Phi(y)) \succ 0 \text{ for } i = 1, 2\right). \tag{3.99}$$

Now, for each $\lambda \in (0, \min(1, \rho_{\max}))$, define

$$V_\lambda^k := \{u \in \mathbb{R}^{(1+k)(1+p)} : g^{00}(u) < -\lambda, \overline{C}_i(u) \succ \lambda I_p, \text{ for } i = 1, 2, \\ |\rho(u)| < \rho_{\max} - \lambda\}. \tag{3.100}$$

The sets V_λ^k are non-empty and include a neighbourhood of the origin since, for all $a, b \in \{0, 1, ..., p\}$ and $i \in \{1, 2\}$,

$$C_i^{ab}\Big|_{D\gamma=0, D\theta=0} = g^{ab}\Big|_{D\gamma=0} = \eta^{ab} \qquad \text{and} \qquad \rho\Big|_{D\gamma=0, D\theta=0} = 0,$$

and the functions C_i^{ab}, g^{ab} and ρ are continuous. Furthermore, note that $g^{00} < -\lambda \implies C_i^{00} = g^{00} + f_i(\rho)\left(\partial^0\theta\right)^2 < -\lambda$ and that $\overline{C}_i \succeq \lambda I_p \implies \bar{g}^{-1} \succeq \lambda I_p$ for $i = 1, 2$, since

$$\bar{C}_m = \bar{g}^{-1} + f_m(\rho) v_\theta v_\theta^T, \qquad \text{where } v_\theta^i = \partial_i \theta.$$

As a result, $u \in V_\lambda^k$ implies that $g^{-1}(u), C_1(u), C_2(u) \in \mathcal{C}_{p,\lambda}$. The remaining part of the argument goes exactly as for the system (3.89) making use of (3.99) and the continuous embedding $H^s(\mathbb{R}^p;\mathbb{R}) \hookrightarrow C^{s-1-[p/2]}(\mathbb{R}^p;\mathbb{R})$. This argument yields, for each $\lambda \in (0, \min(1, \rho_{\max}))$ and initial conditions $\Phi_0 \in H^s(\mathbb{R}^p; V_{2\lambda}^k)$, a solution to

$$\begin{cases} S^a(\Phi)\partial_a \Phi = 0 & \text{in } (0,T) \times \mathbb{R}^p \\ \Phi \in V_\lambda^k & \text{in } (0,T) \times \mathbb{R} \\ \Phi(0,\cdot) = \Phi_0 & \text{in } \mathbb{R}, \end{cases} \tag{3.101}$$

for some $T > 0$, where V_λ^k is defined in (3.100). The solution Φ is such that $\Phi(t,\cdot) = \tilde{\Phi}(t)$ for each $t \in [0,T]$, where $\tilde{\Phi}$ has the following regularity

$$\begin{cases} \tilde{\Phi} \in C\left([0,T]\,; H^s(\mathbb{R}^p; V_\lambda^k)\right) \cap C^1\left([0,T]\,; H^{s-1}(\mathbb{R}^p; \mathbb{R}^{(1+p)(1+k)})\right) \\ \tilde{\Phi}(0) = \Phi_0 \end{cases}, \tag{3.102}$$

which in turn implies (3.70) by Lemma C.7. $\square$

4

APPROXIMATE SOLUTIONS

We henceforth adopt the following assumption:

Assumption 4.1. *Throughout this chapter, $n \geq \mathbb{N}_{\geq 2}$ denotes the number of space dimensions and Π is a fixed element of $\mathcal{O}$, where $\mathcal{O}$ is the set of admissible parameters defined in Definition 2.22. Also, $\gamma, \theta \in C_b^\infty\left([0,T] \times \mathbb{R}^{n-1}\right)$ is a fixed pair of smooth solutions to Equation 3.69 coming from Theorem 3.22*[1] *(e.g, any (γ, θ) coming out of Theorem 3.22 corresponding to smooth initial data with compact support), and Γ is the (timelike) graph of γ. In particular, (Γ, θ) is a solution to the laws of motion, Equation 3.35, and there are constants $c \in (0, \rho_{\max})$ and $d > 0$ such that the following statements*[2] *hold uniformly over Γ:*

$$
\begin{aligned}
|\rho| = |\langle \nabla_\Gamma \theta, \nabla_\Gamma \theta \rangle_m| &< \rho_{\max} - c, \\
C_1^{ab}(D\gamma, D\theta), C_2^{ab}(D\gamma, D\theta) &\in \mathcal{C}_{n,d}.
\end{aligned}
$$

Furthermore, $\delta > 0$ is any number satisfying Equation 3.7 for the choice of Γ above.

Given any tuplet $(\Pi, \Gamma, \theta, \delta)$ as described in Assumption 4.1, we construct a sequence of increasingly "better" approximate solutions to Equation 1.8 of the form (1.25), where $(y, z) \in \mathbb{R}^{n-1} \times (-\delta, \delta)$ are Fermi coordinates defined in a normal neighbourhood $\mathcal{N}_\delta$ of the interface Γ (see Section 3.1 for relevant definitions). The main result is stated in terms of the ***error of approximation*** of $\tilde{\Phi} \in C^2(\mathcal{N}_\delta; \mathbb{R} \times \mathbb{C})$ at (t, x), defined as

$$
S_\varepsilon^R[\tilde{\Phi}](t,x) := \Box \tilde{\Phi}(t,x) + \frac{1}{\varepsilon^2} D_\Phi W\left(\tilde{\Phi}(t,x); \Pi\right), \tag{4.1}
$$

1 Except for the construction of the initial approximation, where the second condition can be weakened, as explained later. Also, many of the arguments in the proof of Theorem 4.2 apply/may be slightly adapted to the case where (Γ, θ) are solutions to Equation 3.35 satisfying milder hypotheses (e.g., results of lower regularity based on slight modifications on the hypotheses to Theorem 3.16 or Theorem 3.22).

2 See Definition 3.21, Definition 3.9 and Equation 3.3, (3.67) for relevant definitions.

whose name is motivated by the fact that $S^R_\varepsilon[\Phi] \equiv 0$ is precisely Equation 1.8.

Most of the construction is performed in terms of Fermi coordinates as defined in Definition 3.2. Borrowing the notation therein, we let $(y, z) \in \Lambda \times (-\delta, \delta)$, be Fermi coordinates associated to a local coordinate system (ψ, Λ) for Γ, and Ψ be as in Equation 3.8. As such, $\mathcal{N}_\delta := \Psi(\Lambda \times (-\delta, \delta))$ is a normal neighbourhood around Γ of radius $\delta > 0$. For convenience, we also introduce the dilated coordinates (y, ζ) where y is as before and $\zeta := z/\varepsilon$. Based on the expression for the wave operator in Fermi coordinates from (3.23), the formula for the error (4.1) in (y, ζ) coordinates is given by

$$\begin{aligned} S^F_\varepsilon\left[\Phi\right](y,\zeta) := \frac{1}{\varepsilon^2}\left[-\partial^2_\zeta \Phi(y,\zeta) + D_\Phi W(\Phi(y,\zeta);\Pi)\right] \\ + \frac{1}{\varepsilon} H_{\Gamma_{\varepsilon\zeta}}(y)\partial_\zeta \Phi(y,\zeta) + \Box_{\Gamma_{\varepsilon\zeta}}\Phi(y,\zeta), \end{aligned} \tag{4.2}$$

defined for all (y, ζ) in the set

$$\mathcal{D}_{\delta,\varepsilon} := \Lambda \times \left(-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon}\right).$$

In this way,

$$S^R_\varepsilon[\tilde{\Phi}](t,x) \;=\; S^F_\varepsilon\left[\Phi\right](y,\zeta), \quad \text{whenever } (t,x) \;=\; \Psi(y,z) \text{ and } \zeta \;=\; z/\epsilon.$$

We introduce additional notation to describe the way in which the approximate solutions will be built. Let $h \in C^\infty(\Gamma;\mathbb{R})$ (as an abuse of notation of $h \in C^\infty(\Lambda;\mathbb{R})$) and define

$$\zeta_h(y,\zeta) := \zeta - h(y) \quad \text{ and } \quad \mathcal{D}_{\delta,\varepsilon,h} := \Lambda \times \left(h(y) - \delta/\epsilon, h(y) + \delta/\epsilon\right), \tag{4.3}$$

so that $(y, \zeta_h) \in \mathcal{D}_{\delta,\varepsilon} \iff (y, \zeta) \in \mathcal{D}_{\delta,\varepsilon,h}$. Also, for $\Phi = (\varphi, \sigma) \in C^\infty(\mathcal{D}_{\delta,\varepsilon}; \mathbb{R} \times \mathbb{C})$ define

$$\Phi_h : \mathcal{D}_{\delta,\varepsilon,h} \to \mathbb{R}\times\mathbb{C}, \qquad \Phi_h(y,\zeta) := \Phi(y,\zeta_h(y,\zeta)).$$

Using this notation, we will write, for instance

$$\begin{aligned} \partial_\zeta \Phi_h(y,\zeta) &= \left(\partial_\zeta \Phi\right)_h(y,\zeta) = \partial_\zeta \Phi(y,\zeta_h(y,\zeta)) \\ \partial_a \Phi_h(y,\zeta) &= \left(\partial_a \Phi\right)_h(y,\zeta) - \left[\partial_a h(y)\right]\left(\partial_\zeta \Phi\right)_h(y,\zeta) \end{aligned}$$

$$\partial_{ab}\Phi_h(y,\zeta) = \left(\partial_{ab}\Phi\right)_h(y,\zeta) - \left(\partial_a h(y)\right)\left(\partial_{b\zeta}\Phi\right)_h(y,\zeta) - \left(\partial_{ab}h(y)\right)\left(\partial_\zeta\Phi\right)_h(y,\zeta)$$
$$- \left(\partial_b h(y)\right)\left(\partial_{a\zeta}\Phi\right)_h(y,\zeta) + \left(\partial_a h(y)\right)\left(\partial_b h(y)\right)\left(\partial_\zeta^2\Phi\right)_h(y,\zeta).$$

The main result is:

Theorem 4.2. *Let n, Π, Γ, θ, and δ be as in Assumption 4.1. Then, for every $k \in \mathbb{N}$, there exists $\varepsilon > 0$ and functions $h_k, \alpha_k \in C_b^\infty(\Gamma;\mathbb{R})$ and $V_k \in C_b^\infty\left(\mathcal{D}_{\delta/2,\varepsilon};\mathbb{R}\times\mathbb{C}\right)$ defined on*

$$\mathcal{D}_{\delta/2,\varepsilon} := \left\{(y,s) : y \in \Gamma, s \in (-\delta/2\varepsilon, \delta/2\varepsilon)\right\},$$

such that for

$$\theta_k(y,\zeta) := \theta(y) + \varepsilon\alpha_k(y) + \varepsilon^2\zeta\left\langle\nabla_\Gamma\theta, \nabla_\Gamma h_k\right\rangle(y), \tag{4.4}$$

the function

$$\mathcal{Y}_k := \left(e^{i\frac{\theta_k}{\varepsilon}} \ltimes V_k\right)_{h_k} \tag{4.5}$$

satisfies the following estimates for each α and j:

$$\left|D_y^\alpha\partial_\zeta^j\left(e^{-i\theta_k/\varepsilon} \ltimes S_\varepsilon^F\left[\mathcal{Y}_k\right]\right)(y,\zeta)\right| \le C_{\alpha jk}(1+|\zeta|^{k+2})\varepsilon^k e^{-a(\Pi)|\zeta|}, \tag{4.6}$$

$$\left|D_y^\alpha h_k(y)\right|, \left|D_y^\alpha\alpha_k(y)\right| \le C_{\alpha k}\varepsilon, \tag{4.7}$$

for $a(\Pi) := \dfrac{1}{2}\min\limits_{|p|\le\rho_{\max}-c}\lambda_{\min}(\Pi_p)$ and some constants $C_{\alpha jk}, C_{\alpha k} > 0$ independent of ε, where $\lambda_{\min}(\Pi)$ is the minimum eigenvalue of $D_\Phi^2 W(\pm1,0;\Pi)$ and c is the constant appearing in Assumption 4.1.

Moreover, if we define the remainder $\Phi_{k,r}(y,\zeta)$ by setting

$$V_k(y,\zeta) = \Phi\left(\Pi_{\rho(y)},\zeta\right) + \varepsilon\Phi_{k,r}(y,\zeta),$$

where Φ is the map described in Theorem 2.3, $\rho = \left\langle\nabla_\Gamma\theta, \nabla_\Gamma\theta\right\rangle_m$, and Π_p is as defined in Equation 2.120, then $\Phi_{k,r}$ satisfies

$$|D_y^\alpha\partial_\zeta^j\Phi_{k,r}(y,\zeta)| \le C_{\alpha jk}(1+|\zeta|)e^{-a(\Pi)|\zeta|},$$

for some constant $C_{\alpha jk}$ independent of ε.

Remark 4.3. The constants $C_{\alpha jk}, C_{\alpha k} > 0$ appearing above all depend solely on Π and on the size of the derivatives of θ and γ.

Remark 4.4. The factor $e^{-i(\theta/\varepsilon)}$ appearing in Equation 4.6 is employed to control the effects of the rapid oscillations associated with the leading order phase, θ/ε. Such term is not necessary if no tangential derivatives are taken, i.e., one has that

$$\left|\partial_\zeta^j S_\varepsilon^F\left[\mathcal{Y}_k\right](y,\zeta)\right| \leq C_{jk}\left(1+|\zeta|^{k+2}\right)\varepsilon^k e^{-a(\Pi)|\zeta|}.$$

Remark 4.5. The conclusion of Theorem 4.2 can be extended to the case where Γ is compact by an argument using partitions of unity and the uniqueness of solutions to Equation 4.12 and Equation 4.41 under prescribed initial conditions. The uniqueness of Equation 4.12 follows from Theorem 2.4, while the uniqueness of Equation 4.41 with given initial conditions follows from e.g., [31, Corollary 7.10].

Remark 4.6. When defining the functions V_k from Theorem 4.2, it is first assumed that their domain is $\Gamma \times \mathbb{R}$ as opposed to $D_{\delta/2,\varepsilon} = \Gamma \times (-\delta/\varepsilon, \delta/\varepsilon)$. The corresponding counterparts of these functions in (t,x) coordinates are not necessarily well-defined over $[0,T] \times \mathbb{R}^n$, as their domain of definition might contain points outside the largest possible domain of definition of Fermi coordinates associated to Γ. It is nevertheless justified to carry out this construction, and later multiply these functions by an appropriate cut-off function which takes care of what takes place outside the domain of definition of the Fermi coordinates (see Section 4.3).

4.1 NOTATION

Before proving Theorem 4.2, we list a series of definitions and notation that will be used repeatedly. First, the symbol ρ will always be used to refer to the function (see Definition 3.9):

$$\rho : \Gamma \to \mathbb{R}, \qquad \rho(y) = \langle \nabla_\Gamma \theta, \nabla_\Gamma \theta \rangle_m (y).$$

Additionally, we will write

$$\langle \cdot, \cdot \rangle_{\Gamma_z} := \langle \nabla_{\Gamma_z} \cdot, \nabla_{\Gamma_z} \cdot \rangle_m \text{ and } \langle \cdot, \cdot \rangle_{\Gamma} := \langle \cdot, \cdot \rangle_{\Gamma_0}. \tag{4.8}$$

More explicitly, for functions F and G defined for all $(y,s)\in\Gamma\times\mathbb{R}$, we have

$$\begin{aligned}\langle F,G\rangle_{\Gamma_z}(y,s) &= \left\langle \nabla_{\Gamma_z}\Big|_y F(\cdot,s), \nabla_{\Gamma_z}\Big|_y G(\cdot,s)\right\rangle_m \\ &= \mathfrak{g}^{ab}(y,z)\partial_a F(y,s)\partial_b G(y,s),\end{aligned}$$

in which we distinguish the variable $z\in(-\delta,\delta)$, used to indicate the manifold Γ_z over which the operators are being defined, from the variable s on which the input functions depend. In relation to the notation for the y-dependent shift on (4.3), we have e.g.,

$$\begin{aligned}\langle F_h,G_h\rangle_{\Gamma_z}(y,\zeta) &= \mathfrak{g}^{ab}(y,z)\partial_a F_h(y,\zeta)\partial_b G_h(y,\zeta) \\ &= \mathfrak{g}^{ab}(y,z)\left[(\partial_a F)_h-(\partial_a h)(\partial_\zeta F)_h\right]\left[(\partial_b G)_h-(\partial_b h)(\partial_\zeta G)_h\right](y,\zeta) \\ &= \left[\langle F,G\rangle_{\Gamma_z}-\langle h,G\rangle_{\Gamma_z}\partial_\zeta F-\langle F,h\rangle_{\Gamma_z}\partial_\zeta G+\langle h,h\rangle_{\Gamma_z}(\partial_\zeta F)(\partial_\zeta G)\right](y,\zeta_h),\end{aligned}$$

meaning that all the functions involving F,G and their derivatives are evaluated at $(y,\zeta-h(y))$, h is evaluated at y, and $\mathfrak{g}$ is evaluated at (y,z).

Also, we will make use of the following notation from Section 3.1 for objects and operators defined for $z,z_0\in(-\delta,\delta), s\in\mathbb{R}$, and any C^2 function v defined over $\Gamma\times\mathbb{R}$:

$$\begin{aligned}\Gamma_{z_0} &:= \{Y+z\nu(Y): Y\in\Gamma, z=z_0\}, \\ \Box_{\Gamma_{z_0}}v(y,s) &:= -\frac{1}{\sqrt{|\mathfrak{g}(y,z_0)|}}\frac{\partial}{\partial_{y_a}}\left[\sqrt{|\mathfrak{g}(y,z_0)|}\mathfrak{g}^{ab}(y,z_0)\partial_b v(y,s)\right], \\ H_{\Gamma_{z_0}}(y) &:= -\frac{1}{2}\partial_z\Big|_{(y,z_0)}\ln|\mathfrak{g}|.\end{aligned} \tag{4.9}$$

The Taylor series expansions about $z=0$ for $\Box_{\Gamma_z}, H_{\Gamma_z}$ and $\langle\cdot,\cdot\rangle_{\Gamma_z}$ defined in (4.9) and (4.8) will be central to our discussion. As a result, and in connection to this aspect, we introduce some additional notation for convenience. In particular, for $B^b(y,z) := -\frac{1}{\sqrt{|\mathfrak{g}(y,z)|}}\partial_a\left[\mathfrak{g}^{ab}(y,z)\sqrt{|\mathfrak{g}(y,z)|}\right]$ and functions F,G of the variables (y,ζ), we define:

$$\begin{aligned}\mathrm{I}_{\Box F}(y,\zeta) &:= -g^{ab}(y)\partial_{ab}F(y,\zeta)-B^a(y,0)\partial_a F(y,\zeta) \\ \mathrm{II}_{\Box F}(y,\zeta) &:= -\partial_z\mathfrak{g}^{ab}(y,0)\partial_{ab}F(y,\zeta)-\partial_z B^a(y,0)\partial_a F(y,\zeta) \\ \mathrm{III}_{\Box F}(y,\zeta) &:= -\partial_z^2\mathfrak{g}^{ab}(y,0)\partial_{ab}F(y,\zeta)-\partial_z^2 B^a(y,0)\partial_a F(y,\zeta).\end{aligned}$$

Similarly,

$$\begin{aligned}
I_{\langle F,G\rangle}(y,\zeta) &:= g^{ab}(y)\left(\partial_a\Big|_{(y,\zeta)}F\right)\left(\partial_a\Big|_{(y,\zeta)}G\right)\\
\mathrm{II}_{\langle F,G\rangle}(y,\zeta) &:= \partial_z\mathfrak{g}^{ab}(y,0)\left(\partial_a\Big|_{(y,\zeta)}F\right)\left(\partial_a\Big|_{(y,\zeta)}G\right)\\
\mathrm{III}_{\langle F,G\rangle}(y,\zeta) &:= \partial_z^2\mathfrak{g}^{ab}(y,0)\left(\partial_a\Big|_{(y,\zeta)}F\right)\left(\partial_a\Big|_{(y,\zeta)}G\right),
\end{aligned}$$

and (see Equation 3.24)

$$\mathrm{I}_\Gamma := H_{\Gamma_0} = H_\Gamma, \qquad \mathrm{II}_\Gamma := \partial_z\Big|_{z=0}H_{\Gamma_z}, \qquad \mathrm{III}_\Gamma := \partial_z^2\Big|_{z=0}H_{\Gamma_z}.$$

In this way, we have that

$$\begin{aligned}
\Box_{\Gamma_z}F &= \mathrm{I}_{\Box F} + z\mathrm{II}_{\Box F} + \frac{z^2}{2}\mathrm{III}_{\Box F} + \ldots\\
\langle F,G\rangle_{\Gamma_z} &= I_{\langle F,G\rangle} + z\mathrm{II}_{\langle F,G\rangle} + \frac{z^2}{2}\mathrm{III}_{\langle F,G\rangle} + \ldots\\
H_{\Gamma_z} &= \mathrm{I}_\Gamma + z\mathrm{II}_\Gamma + \frac{z^2}{2}\mathrm{III}_\Gamma + \ldots,
\end{aligned} \tag{4.10}$$

and $I_{\langle\theta,\theta\rangle} = \rho$ and $\mathrm{I}_{\Box f} = \Box_{\Gamma_0}f = \Box_\Gamma f$ for any $f \in C^2(\Gamma)$ (see Remark 3.8). We remark that, in all of the expressions in (4.10), the expansion applies to $\mathfrak{g}(y,z)$ and $B^b(y,z)$, and not to the functions F and G, even if they depend on z through $\zeta = z/\varepsilon$.

Also, we define the following map which will play a fundamental role in our discussion:

$$\Phi_0[\rho] : \mathbb{R}^{1+p}\times\mathbb{R} \to \mathbb{R}\times\mathbb{C}, \qquad \Phi_0[\rho](y,\zeta) := \Phi\left(\Pi_{\rho(y)},\zeta\right), \tag{4.11}$$

where Π_p and Φ are as defined in Equation 2.120 and Theorem 2.3, respectively. The proof of Theorem 4.2 involves solving a series of linear problems related to $\Phi_0[\rho]$. In connection to this idea, a useful definition that we will reference repeatedly from now on is:

Definition 4.7. Given $F : \Gamma\times\mathbb{R} \to \mathbb{R}\times\mathbb{C}$, consider the problem (see Equation 1.45):

$$\begin{cases} \mathcal{L}\left(\Phi_0[\rho](y,\cdot);\Pi_{\rho(y)}\right)[V(y,\cdot)] = F(y,\cdot) \\ V(y,\cdot)\in\mathfrak{C}_2\left(\mathcal{L}\left(\Phi_0[\rho](y,\cdot);\Pi_{\rho(y)}\right)\right) \end{cases} \qquad \forall y\in\Gamma, \tag{4.12}$$

and the set

$$\mathfrak{G}_\perp := \left\{G : \Gamma \times \mathbb{R} \to \mathbb{R} \times \mathbb{C} : G(y,\cdot) \in \mathfrak{C}_0\left(\mathcal{L}\left(\Phi_0[\rho](y,\cdot);\Pi_{\rho(y)}\right)\right) \text{ for all } y \in \Gamma\right\}.$$

Additionally, for a map $G \in \mathfrak{G}_\perp$, define the following operator whose existence is justified by Theorem 2.4:

$$\mathfrak{S}[G] = \text{ the unique solution to (4.12) corresponding to } F = G.$$

We list some useful properties of the map $\Phi_0[\rho]$. In particular, an immediate consequence of Theorem 2.3 is that for each $y \in \Gamma$:

$$\Phi_0[\rho](y,\cdot) \in \mathcal{H}^s, \tag{4.13a}$$

$$\mathcal{F}\left(\Phi_0[\rho](y,\cdot);\Pi_{\rho(y)}\right)(\zeta) = 0 \text{ for all } \zeta \in \mathbb{R}, \tag{4.13b}$$

$$\ker \mathcal{L}\left(\Phi_0[\rho](y,\cdot);\Pi_{\rho(y)}\right) = \operatorname{span}_{\mathbb{R}}\left\{\mathfrak{N}_1[\rho](y,\cdot), \mathfrak{N}_2[\rho](y,\cdot)\right\}, \tag{4.13c}$$

where

$$\mathfrak{N}_1[\rho] := \partial_\zeta \Phi_0[\rho] \qquad \text{and} \qquad \mathfrak{N}_2[\rho] := \mathcal{P}_\sigma\left(i\Phi_0[\rho]\right).$$

Also, the smoothness of Γ and θ from Assumption 4.1 implies that ρ is smooth, which in turn implies the smoothness of $\Phi_0[\rho]$. Moreover, Equation 2.98 implies that for all $\zeta \in \mathbb{R}\setminus\{0\}$:

$$\left|D_y^\alpha \partial_\zeta^l\left[\Phi_0[\rho](y,\zeta) - I_\pm(\zeta)\right]\right| \le c_{\alpha l}\left(\Pi_{\rho(y)}\right) e^{-\frac{\sqrt{\lambda_{\min}\left(\Pi_{\rho(y)}\right)}}{2}|\zeta|}, \tag{4.14}$$

for each $l \in \mathbb{Z}_+$ and each multi-index α, where $\lambda_{\min}\left(\Pi_{\rho(y)}\right) > 0$ is the minimium eigenvalue of $D_\Phi^2 W\left(\pm 1, 0; \Pi_{\rho(y)}\right)$ and $c_{\alpha l}(\Pi_{\rho(y)}) > 0$. Therefore, for $C_{\alpha l}(\Pi) := \max_{p \in \overline{\operatorname{Im}(\rho)}} c_{\alpha l}\left(\Pi_p\right)$ and $a(\Pi) := \dfrac{1}{2}\min_{p \in \overline{\operatorname{Im}(\rho)}} \lambda_{\min}(\Pi_p)$, we have

$$\left|D_y^\alpha \partial_\zeta^l\left[\Phi_0[\rho](y,\zeta) - I_\pm(\zeta)\right]\right| \le C_{\alpha l}\left(\Pi\right) e^{-a(\Pi)|\zeta|}, \tag{4.15}$$

for all $(y,\zeta) \in V \times (\mathbb{R}\setminus\{0\})$.

Finally, for $h \in C^2(\Gamma;\mathbb{R})$ and $\Phi \in C^2\left(\mathcal{D}_{\delta,\varepsilon};\mathbb{R}\times\mathbb{C}\right)$ and $\alpha \in C^2\left(\mathcal{D}_{\delta,\varepsilon};\mathbb{R}\right)$, we define

$$S_\varepsilon^F\left[\Phi,\alpha,h\right](y,\zeta) := S_\varepsilon^F\left[\left(e^{i\alpha} \ltimes \Phi\right)_h\right](y,\zeta), \qquad (y,\zeta) \in \mathcal{D}_{\delta,\varepsilon,h}, \tag{4.16}$$

where we have used the notation for $\ltimes$ introduced in (1.46).

4.2 PROOF OF *Theorem 4.2*

The proof of Theorem 4.2 is based on induction argument on the value of $k \geq 1$, where all the approximations are of the form (4.5). For the base case, it is shown that the hypotheses on (Γ, θ) guarantee the existence of the maps $\Phi_1 := \mathfrak{S}[-G_1]$ and $\Phi_2 := \mathfrak{S}[-G_2]$ with G_1 from (4.26) and G_2 from (4.33), and that the conclusion of the theorem for $k = 1$ is satisfied with $V_1 = \Phi_0[\rho] + \varepsilon\Phi_1 + \varepsilon^2\Phi_2$, $\alpha_1 = 0$ and $h_1 = 0$. Subsequently, given an approximation $\mathcal{Y}_k$, a better approximation $\mathcal{Y}_{k+1}$ is constructed in two steps:

Step 1. (α_k, h_k) are modified according to the recursive relation

$$\begin{cases} \alpha_{k+1} = \alpha_k + \varepsilon^k \alpha \\ h_{k+1} = h_k + \varepsilon^k h \\ \alpha_1 = 0, h_1 = 0, \end{cases} \tag{4.17}$$

where α and h are functions which satisfy a coupled system of linear wave equations (see Equation 4.41) whose coefficients depend on all the functions that make up $\mathcal{Y}_k$ as well as their derivatives up to second order. As we will see, the equations solved by α and h represent an orthogonality condition needed in Step 2 below to improve the accuracy of the approximation by at least one order of ε. One may interpret α as a perturbation to the phase of the $\mathbb{C}$-valued component of $\mathcal{Y}_k$, and h as a perturbation to the location of the zero set of the real component of $\mathcal{Y}_k$, whose counterparts in the case of the leading order term of $\mathcal{Y}_k$ in terms of powers of ε (i.e., for $e^{i\theta/\varepsilon} \times \Phi_0[\rho]$) are θ and Γ, respectively. As a result, the coupling between these two maps is to be expected in view of the coupling between θ and Γ according to the base case or the heuristics from Chapter 3. Finally, the term $\varepsilon^2\zeta I_{\langle\theta, h_{k+1}\rangle}$ is included in the phase to cancel out terms in the error that arise due to the nonlinear relationship between Γ and θ.

Step 2. The function V_k is modified according to

$$\begin{cases} V_k = \Phi\left(\Pi_{\rho_k(y)}, \zeta\right) + \varepsilon P_k(y,\zeta) \\ \rho_k = \langle \nabla_\Gamma(\theta + \varepsilon\alpha_k), \nabla_\Gamma(\theta+\varepsilon\alpha_k)\rangle_m \,, \end{cases} \tag{4.18}$$

where P_k changes according to the recursive relation

$$\begin{cases} P_{k+1} = P_k + \varepsilon^{k+1} P \\ P_1 = \Phi_1 + \varepsilon\Phi_2, \end{cases} \tag{4.19}$$

for a function P whose existence is guaranteed by the orthogonality condition mentioned in Step 1. In particular, $P = \mathfrak{S}[G]$, for some $G \in \mathfrak{G}_\perp$ which depends on the functions that make up $\mathcal{Y}_k, h, \alpha$ and their derivatives up to second order.

4.2.1 *Base Case*

In this section, we treat the $k = 1$ case in Theorem 4.2. For this case, it suffices to consider the following expression for the error of approximation for C^2 functions η (real valued) and V ($\mathbb{R}\times\mathbb{C}$ valued) defined over $\mathcal{D}_{\delta,\varepsilon}$:

$$\begin{aligned} e^{-i\eta/\varepsilon} \ltimes S^F_\varepsilon\left[e^{i\eta/\varepsilon}\ltimes V\right] &= \frac{1}{\varepsilon^2}\left[-\partial_\zeta^2 V + \langle \eta,\eta\rangle_{\Gamma_z}\mathcal{P}_\sigma(V) + D_\Phi W(V;\Pi)\right] \\ &+ \frac{1}{\varepsilon}\left[H_{\Gamma_z}\partial_\zeta V + (\Box_{\Gamma_z}\eta)\,\mathcal{P}_\sigma(iV) - 2\langle \eta, \mathcal{P}_\sigma(iV)\rangle_{\Gamma_z} - 2\frac{\partial_\zeta\eta}{\varepsilon^2}\mathcal{P}_\sigma(i\partial_\zeta V)\right] \\ &+ \Box_{\Gamma_z}V + \left(\frac{\partial_\zeta\eta}{\varepsilon^2}\right)^2\mathcal{P}_\sigma(V) + H_{\Gamma_z}\left(\frac{\partial_\zeta\eta}{\varepsilon^2}\right)\mathcal{P}_\sigma(iV) - \frac{\partial_\zeta^2\eta}{\varepsilon^3}\mathcal{P}_\sigma(iV), \end{aligned} \tag{4.20}$$

where $z = \varepsilon\zeta$. This expression is found by substituting $\Phi = e^{i\eta/\varepsilon}\ltimes V$ into Equation 4.2 and gathering terms based on powers of ε, together with the implicit assumption that $\partial_\zeta\eta = O(\varepsilon^2)$ which will be justified in Section 4.2.2 (in particular, as a result of the appropriateness of the ansatz in Equation D.2).

Step 1. In order to “eliminate” the term of order $1/\varepsilon^2$ in (4.20), it is enough to assume that η depends only on y and that V depends on y only through $\rho(y)$, so long as $\operatorname{Im}(\rho) \subset I_\Pi$. In particular, this is the only step in the construction that may be carried out even if (Γ,θ) is not a solution to the equations of motion, Equation 3.35. To see this, set $\eta(y,\zeta) = \theta(y)$ and $V_{-1} = \Phi_0[\rho]$ and plug them into (4.20) to obtain

$$\begin{aligned}
&e^{-i\theta/\varepsilon} \ltimes S_\varepsilon^F\left[V_{-1}, \theta/\varepsilon, 0\right] = e^{-i\theta/\varepsilon} \ltimes S_\varepsilon^F\left[e^{i\theta/\varepsilon} \ltimes \Phi_0[\rho]\right] \\
&= \frac{1}{\varepsilon^2}\left[-\partial_\zeta^2\Phi_0[\rho] + \rho\mathcal{P}_\sigma(\Phi_0[\rho]) + D_\Phi W(\Phi_0[\rho];\Pi)\right] \\
&+ \frac{1}{\varepsilon}\left[H_{\Gamma_{\varepsilon\zeta}}\partial_\zeta\Phi_0[\rho] + \left(\Box_{\Gamma_{\varepsilon\zeta}}\theta\right)\mathcal{P}_\sigma\left(i\Phi_0[\rho]\right) - 2\left\langle\theta, \mathcal{P}_\sigma\left(i\Phi_0[\rho]\right)\right\rangle_{\Gamma_{\varepsilon\zeta}}\right] + \Box_{\Gamma_{\varepsilon\zeta}}\Phi_0[\rho] \\
&= \frac{1}{\varepsilon^2}\mathcal{F}\left(\Phi_0[\rho](y,\cdot);\Pi_{\rho(y)}\right)(\zeta) + \frac{1}{\varepsilon}R(y,\zeta) \\
&= \frac{1}{\varepsilon}R(y,\zeta),
\end{aligned} \tag{4.21}$$

where

$$\begin{aligned}
R(y,\zeta) := {} & \zeta\partial_z\mathfrak{g}^{ab}(y,\zeta^*)\partial_a\theta(y)\partial_b\theta(y)\mathcal{P}_\sigma\left(\Phi_0[\rho](y,\zeta)\right) + H_{\Gamma_{\varepsilon\zeta}}(y)\mathfrak{N}_1[\rho](y,\zeta) \\
& + \Box_{\Gamma_{\varepsilon\zeta}}\theta(y)\mathfrak{N}_2[\rho](y,\zeta) - 2\left\langle\theta, \mathcal{P}_\sigma\left(i\Phi_0[\rho]\right)\right\rangle_{\Gamma_{\varepsilon\zeta}} + \varepsilon\Box_{\Gamma_{\varepsilon\zeta}}\Phi_0[\rho],
\end{aligned}$$

for some ζ^* between 0 and $\varepsilon\zeta$. As a result, (4.6) with $k = -1$ follows from the above expression for the error, (4.15), and the boundedness and regularity of all the terms depending solely on $y \in \Gamma$.

Remark 4.8. In general, when performing expansions in powers of ε as in (4.21) of the more general expression

$$e^{-\theta_k/\varepsilon} \ltimes S_\varepsilon^F[V_k, \theta_k, h_k], \tag{4.22}$$

with $V_k = \sum_{m=0}^k \varepsilon^m\Phi_m$, the leading order term (i.e., the sum of the terms multiplying the lowest power of ε, which will be shown to be ε^k) in (4.22) that remains after the appropriate choices of functions Φ_m, θ_k and h_k, is a combination of functions of Φ_m for $m \leq k$ and their derivatives, multiplied by functions of y (which are uniformly bounded over Γ) and/or by ζ^m for $m \in \{1, ..., k+2\}$. In particular, the ζ^{k+2} term arises due to the remainder of the Taylor series expansion of $\langle\theta_k, \theta_k\rangle_{\Gamma_z}\mathcal{P}_\sigma(i\Phi_0[\rho])$ around $z = 0$. This is the reason for the term $|\zeta|^{k+2}$ appearing in (4.6).

Step 2. Consider now the ansatz

$$\mathcal{Y}_0(y,\zeta) = e^{i\theta(y)/\varepsilon} \ltimes V_0(y,\zeta), \qquad \text{with } V_0 = \Phi_0[\rho] + \varepsilon\Phi_1, \tag{4.23}$$

where Φ_1 is to be determined and $\Phi_0[\rho]$ is as in Step 1. We claim that, as long as (Γ, θ) is any solution to the equations of motion (3.35), Φ_1 can be chosen to obtain

$$e^{-i\theta/\varepsilon} \ltimes S_\varepsilon^F[\mathcal{Y}_0] = e^{-i\theta/\varepsilon} \ltimes S_\varepsilon^F[V_0, \theta/\varepsilon, 0] = O(1). \tag{4.24}$$

Indeed, note that for (4.23), we have that

$$\begin{aligned}
&e^{-i\theta(y)/\varepsilon} \ltimes S_\varepsilon^F[V_0, \theta/\varepsilon, 0](y, \zeta) \\
&= \frac{1}{\varepsilon^2}\left[-\partial_\zeta^2 \Phi + \rho(y)\mathcal{P}_\sigma(\Phi) + D_\Phi W(\Phi; \Pi)\right] \\
&\quad + \frac{1}{\varepsilon}\left[\zeta \mathrm{II}_{\langle\theta,\theta\rangle}\mathcal{P}_\sigma(\Phi) + \mathrm{I}_\Gamma \partial_\zeta \Phi + \mathrm{I}_{\Box\theta}\mathcal{P}_\sigma(i\Phi) - 2I_{\langle\theta,\mathcal{P}_\sigma(i\Phi)\rangle}\right] + O(1) \\
&= \frac{1}{\varepsilon^2}\left[\mathcal{F}(\Phi_0[\rho](y,\cdot); \Pi_{\rho(y)})(\zeta)\right] \\
&\quad + \frac{1}{\varepsilon}\left\{\mathcal{L}\left(\Phi_0[\rho](y,\cdot); \Pi_{\rho(y)}\right)[\Phi_1(y,\cdot)](\zeta) + G_1[\Gamma,\theta](y,\zeta)\right\} + O(1),
\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}
G_1[\Gamma,\theta](y,\zeta) :=& \zeta \mathrm{II}_{\langle\theta,\theta\rangle}(y)\mathcal{P}_\sigma(\Phi_0[\rho](y,\zeta)) + \mathrm{I}_\Gamma(y)\mathfrak{N}_1[\rho](y,\zeta) \\
&+ \mathrm{I}_{\Box\theta}(y)\mathfrak{N}_2[\rho](y,\zeta) - 2\underbrace{I_{\langle\theta,\mathcal{P}_\sigma(i\Phi_0[\rho])\rangle}(y,\zeta)}_{I_{\langle\theta,\rho\rangle}(y)\mathcal{P}_\sigma(i\partial_p\Phi_0[\rho](y,\zeta))}.
\end{aligned} \tag{4.26}$$

The $O(\varepsilon^{-2})$ term in (4.25) vanishes due to (4.13b). Additionally, Theorem 2.4 guarantees the existence of $\Phi_1 \in \mathfrak{C}_2\left(\mathcal{L}\left(\Phi_0[\rho](y,\cdot); \Pi_{\rho(y)}\right)\right)$ so that the $O(\varepsilon^{-1})$ term in this same expression vanishes as long as $G_1[\Gamma,\theta] \in \mathfrak{G}_\perp$. As a consequence of (4.13c) and (4.15), the above condition amounts to satisfying the following two identities defined over Γ:

$$\begin{aligned}
\int_{\mathbb{R}} G_1[\Gamma,\theta](y,\zeta) \cdot \mathfrak{N}_1[\rho](y,\zeta)\, d\zeta = 0 \\
\int_{\mathbb{R}} G_1[\Gamma,\theta](y,\zeta) \cdot \mathfrak{N}_2[\rho](y,\zeta)\, d\zeta = 0,
\end{aligned} \tag{4.27}$$

which may alternatively and respectively be written as

$$\mathrm{I}_\Gamma = \frac{\mu_\Pi'(\rho)}{\mu_\Pi(\rho)} \cdot \mathrm{II}_{\langle\theta,\theta\rangle} \quad \text{and} \quad \mathrm{I}_{\Box\theta} = \frac{\mu_\Pi''(\rho)}{\mu_\Pi'(\rho)} \cdot I_{\langle\rho,\theta\rangle}, \qquad \text{over } \Gamma. \tag{4.28}$$

Comparing the equations in (4.28) to Equation 3.64 and Equation 3.44, we recognize that the orthogonality conditions (4.27) are equivalent to the condition that (θ, Γ) solve the equations of motion, Equation 3.35. As a result, (4.27) hold in view of Assumption 4.1, and we are led to make the following choice for Φ_1:

$$\Phi_1 = \mathfrak{S}[-G_1[\Gamma, \theta]]. \tag{4.29}$$

Now, (4.15) and the form of G_1 imply that $D_y^\nu \partial_\zeta^j G_1(y, \zeta) \leq c_{\nu j} e^{-a(\Pi)\zeta}$, where $c_{\nu j}$ depends on Π and the size of the derivatives of θ and γ. As a result, due to Theorem 2.4, we find that for all j and ν, there exists a constant $C_{\nu j} > 0$ depending on the same quantities such that

$$\left| D_y^\nu \partial_\zeta^j \Phi_1(y, \zeta) \right| \leq C_{\nu j} \left(1 + |\zeta|\right) e^{-a(\Pi)|\zeta|}. \tag{4.30}$$

This bound, together with (4.15) and the particular form of the $O(1)$ term in the right hand side of (4.25) (see Remark 4.8)[3] results in (4.6) for $k = 0$.

Remark 4.9. For each $y \in \Gamma$, write $\Phi_0[\rho](y, \cdot) =: (\varphi_0, \sigma_0, 0)$. It follows from (4.13a) that φ_0 is odd and σ_0 is even. As a result, $G_1[\Gamma, \theta](y, \cdot) = (u, v, w)$, where u, w are even and v is odd. Consequently, since the operator $\mathcal{L}\left(\Phi_0[\rho](y, \cdot); \Pi_{\rho(y)}\right)$ preserves the parity of its argument, it follows that $\Phi_1 = (\varphi_1, v_1, w_1)$, where $\varphi_1(y, \cdot), w_1(y, \cdot)$ are even, and $v_1(y, \cdot)$ is odd for each $y \in \Gamma$.

Step 3 (Case $k = 1$). Let $\Phi_0[\rho]$ be as in (4.11) and (Γ, θ) and Φ_1 be as in (4.29). Consider now the ansatz

$$\mathcal{Y}_1(y, \zeta) = e^{i\theta(y)/\varepsilon} \ltimes V_1(y, \zeta),$$

where

$$V_1 = \Phi_0[\rho](y, \zeta) + \varepsilon \Phi_1(y, \zeta) + \varepsilon^2 \Phi_2(y, \zeta). \tag{4.31}$$

We have that

$$\begin{aligned} e^{-i\theta(y,\zeta)/\varepsilon} \ltimes S_\varepsilon^F \left[\mathcal{Y}_1\right](y, \zeta) = \\ \mathcal{L}\left(\Phi_0[\rho]\,(y, \cdot)\,; \Pi_{\rho(y)}\right) \left[\Phi_2\,(y, \cdot)\right](\zeta) + G_2(y, \zeta) + O(\varepsilon), \end{aligned} \tag{4.32}$$

3 In particular, that the fact that $|\zeta|^2$ appears in (4.6) is due to the presence of term $\zeta^2 \partial_z \mathfrak{g}^{ab}(y, \zeta^*) \mathcal{P}_\sigma(\Phi_0[\rho])$ for some ζ^* between 0 and $\varepsilon\zeta$ in the right hand side of (4.25).

where

$$\begin{aligned}G_2 :=& \frac{1}{2}\mathrm{III}_{\langle\theta,\theta\rangle}\zeta^2\mathcal{P}_\sigma\left(\Phi_0[\rho]\right) + \mathrm{II}_\Gamma\zeta\partial_\zeta\Phi_0[\rho] - 2\mathrm{II}_{\langle\theta,\rho\rangle}\mathcal{P}_\sigma\left(i\zeta\partial_p\Phi_0[\rho]\right)\\ &+ \mathrm{II}_{\Box\theta}\mathcal{P}_\sigma\left(i\zeta\Phi_0[\rho]\right)\mathrm{I}_{\Box\Phi_0[\rho]} + \mathrm{II}_{\langle\theta,\theta\rangle}\mathcal{P}_\sigma\left(\zeta\Phi_1\right) + \mathrm{I}_\Gamma\partial_\zeta\Phi_1 - 2I_{\langle\theta,\mathcal{P}_\sigma(i\Phi_1)\rangle}\\ &+ \mathrm{I}_{\Box\theta}\mathcal{P}_\sigma\left(i\Phi_1\right) + \frac{1}{2}D^3_\Phi W\left(\Phi_0[\rho];\Pi\right)\left[\Phi_1,\Phi_1\right].\end{aligned} \tag{4.33}$$

In view of Remark 4.9, and the parity of the entries of $\Phi_0[\rho](y,\cdot)$ for each $y \in \Gamma$, we have that $G_2 \in \mathfrak{G}_\perp$. Therefore, according to Theorem 2.4, we may make the following choice for Φ_2:

$$\Phi_2 = \mathfrak{S}[-G_2], \tag{4.34}$$

which results in

$$e^{-i\theta/\varepsilon} \ltimes S^F_\varepsilon\left[e^{i\theta/\varepsilon} \ltimes \left(\Phi_0[\rho] + \varepsilon\Phi_1 + \varepsilon^2\Phi_2\right)\right](y,\zeta) = O(\varepsilon). \tag{4.35}$$

In terms of the decay as $\zeta \to \infty$, note that Φ_2 satisfies the same type of bound as that of Φ_1 in (4.30) with $|\zeta|$ replaced by $|\zeta|^2$ for the same reasons leading to (4.30) and by (4.30) itself. The decay in (4.6) for $k = 1$ then follows by that of $\Phi_0[\rho], \Phi_1$ and Φ_2 and Remark 4.8.

4.2.2 *Induction Step*

We prove that for each $k \geq 1$, there exist functions α_k, h_k, P_k such that

$$S^F_\varepsilon\left[\Phi_0[\rho_k] + \varepsilon P_k, \left(\frac{\theta}{\varepsilon} + \alpha_k + \varepsilon\zeta I_{\langle\theta,h_k\rangle}\right), h_k\right](y,\zeta) = O(\varepsilon^k), \tag{4.36}$$

where α_k, h_k and P_k evolve according to the recursive relations

$$\begin{aligned}\alpha_{k+1} &= \alpha_k + \varepsilon^k\alpha,\\ h_{k+1} &= h_k + \varepsilon^k h,\\ P_{k+1} &= P_k + \varepsilon^{k+1}P.\end{aligned} \tag{4.37}$$

Moreover, the resulting functions exhibit the asymptotic behaviour as $\zeta \to \infty$ and as $\varepsilon \to 0$ listed in the statement of Theorem 4.2.

For convenience, we introduce the following auxiliary functions

$$\begin{aligned} \theta_k(y,\zeta) &= \theta(y) + \varepsilon\alpha_k(y) + \varepsilon^2\zeta I_{\langle\theta,h_k\rangle}(y) \\ V_k(y,\zeta) &= \Phi_0[\rho_k](y,\zeta) + \varepsilon P_k(y,\zeta), \end{aligned} \tag{4.38}$$

with ρ_k as in (D.1).

The base case $k = 1$ has been proven, using

$$\alpha_1 \equiv h_1 \equiv 0, \qquad \text{and} \qquad P_1 = \Phi_1 + \varepsilon\Phi_2,$$

where,

$$\Phi_1 = \mathfrak{S}[-G_1[\Gamma,\theta]], \qquad G_1[\Gamma,\theta] \text{ as defined in (4.26)} \tag{4.39a}$$

$$\Phi_2 = \mathfrak{S}[-G_2], \qquad G_2 \text{ as defined in (4.33).} \tag{4.39b}$$

Now, let $k \geq 1$ and suppose that the claim holds for all $m \leq k$. We claim that there exist α, h, and P such that (4.36) holds for $k+1$. The proof of the induction step follows a similar philosophy as Step 2 above. In particular, the expressions from Step 2 obtained by expressing the error in terms of powers of ε suggests that a natural choice of Γ and θ is that which enforces the orthogonal condition (4.27) necessary for solving a linear problem of the form (4.12). Formally, introducing h and α corresponds to performing a perturbation to the interface Γ and to the leading order phase θ, respectively. It is therefore expected that, in order to reduce the order of the error by a factor of ε, the functions h and α are to be chosen as the solutions to the system of equations representing the linearization about (Γ,θ) of the equations satisfied by Γ and θ (i.e., the laws of motion, (3.35)). These choices for h and α in turn lead to the natural choice of the function P (i.e., the perturbation to $\Phi_0[\rho]$) as the solution to a linear problem of the form (4.12).

An abridged version of the computations involved in the procedure described above is presented below. A more detailed account can be found in Appendix D. The main idea behind these computations is to express $S_\varepsilon^F[V_{k+1}, \theta_{k+1}/\varepsilon, h_{k+1}]$ (error from the $k+1$ step) in terms of the error from the k step plus some remaining terms which, due to the choice of ansatz[4], turn out to be of order ε^k as well. Subsequently, it is proven that

4 In particular, modifying ρ according to the formula for ρ_k in (D.1) and adding the $\varepsilon^2\zeta I_{\langle\theta,h_k\rangle}$ term to the phase as in (D.2) is essential to cancel out error terms of order ε^{k-1} that would otherwise be present in $S_\varepsilon^F[V_{k+1}, \theta_{k+1}/\varepsilon, h_{k+1}]$.

the resulting term of order ε^k vanishes upon an appropriate choice of h, α and P. An important remark regarding the computational aspects involved in this process is that the definition of $S_\varepsilon^F[V,\eta,h]$ from (4.2) implies that the approximation $\left(e^{i\eta} \ltimes V\right)_h$ is a function that has been shifted in the ζ coordinate by h, whereas the interface Γ with respect to which the Fermi coordinates are defined is held fixed[5]. Effectively, by introducing the shift function h the point $(y,0)$ is shifted to $(y,h(y))$ for each $y \in \Gamma$, which may be interpreted as applying a normal deformation to Γ of $h(y)$ units at each $y \in \Gamma$. In this way, a shift by h_k on the approximation at step k corresponds to a deformation by h_k to Γ. Therefore, in order to use the induction hypothesis that $S_\varepsilon^F[V_k,\theta_k/\varepsilon,h_k] = O(\varepsilon^k)$ in the calculation of $S_\varepsilon^F[V_{k+1},\theta_{k+1}/\varepsilon,h_{k+1}]$, one needs to ensure that the k^{th} approximation is shifted by the corresponding deformation of Γ of h_k. Since all the computations at the $(k+1)^{th}$ step are made with respect to the shifted variable $\zeta_{k+1} := \zeta - h_{k+1}$, the above entails that the desired form of the error at the $(k+1)^{th}$ step can be written more precisely as

$$
\begin{aligned}
&e^{-i\theta_{k+1}(y,\zeta_{k+1})/\varepsilon} \ltimes S_\varepsilon^F\left[(\mathcal{Y}_{k+1})_{h_{k+1}}\right](y,\zeta) \\
&= \varepsilon^k\left[\mathcal{L}\left(\Phi_0[\rho](y,\cdot);\Pi_{\rho(y)}\right)[P(y,\cdot)](\zeta_{k+1}) + \mathcal{G}(y,\zeta_{k+1})\right] + O\left(\varepsilon^{k+1}\right),
\end{aligned}
\tag{4.40}
$$

where $\mathcal{G}(y,\zeta) = e^{-i\theta_k(y,\zeta)/\varepsilon} \ltimes E_k(y,\zeta) + G[h,\alpha](y,\zeta)$, with E_k being the portion of order ε^k of

$$
S_\varepsilon^F\left[(\mathcal{Y}_k)_{h_k}\right]\Big|_{z=\varepsilon(\zeta_{k+1}+h_k)}(y,\zeta-\varepsilon^k h),
$$

which represents the error from the k^{th} step evaluated at (y,ζ_{k+1}), and some function $G[h,\alpha]$ that ensures the fulfillment of the orthogonality condition $\mathcal{G} \in \mathfrak{G}_\perp$. More specifically,

$$
\begin{aligned}
G[h,\alpha](y,\zeta) :=& \left[g^{ab}\partial_\zeta\Phi_0 + 2\left(\partial^a\theta\partial^b\theta\right)\mathcal{P}_\sigma(\zeta\Phi_0)\right]\partial_{ab}h \\
&- \left[g^{ab}\mathcal{P}_\sigma(i\Phi_0) + 2\left(\partial^a\theta\partial^b\theta\right)\mathcal{P}_\sigma(i\partial_p\Phi_0)\right]\partial_{ab}\alpha \\
&+ \mathfrak{D}[h,\alpha](y,\zeta),
\end{aligned}
$$

5 We have opted for this definition of the error, but we could also shift the interface Γ around which we are defining the Fermi coordinates.

where $\mathfrak{D}$ is given by

$$
\begin{aligned}
&\mathfrak{D}\left[h,\alpha\right](y,\zeta) := \\
&2g^{ab}\partial_a\theta\left[\left(\partial_b g^{cd}\right)\partial_c\theta + g^{cd}\partial_{bc}\theta\right]\left[\left(\partial_d h\right)\zeta\mathcal{P}_\sigma\left(\Phi_0[\rho]\right) - 2\left(\partial_d\alpha\right)\mathcal{P}_\sigma\left(i\partial_\rho\Phi_0[\rho]\right)\right] \\
&-2\left[\beta\mathrm{II}_{\langle\theta,\theta\rangle} + hg^{ab}\partial_a\theta\partial_b\left(\mathrm{II}_{\langle\theta,\theta\rangle}\right)\right]\mathcal{P}_\sigma\left(i\partial_\rho\Phi_0[\rho]\right) + \beta\mathrm{I}_\Gamma\mathcal{P}_\sigma\left(i\Phi_0[\rho]\right) \\
&+B^b\left[-\left(\partial_b h\right)\partial_\zeta\Phi_0[\rho] + \left(\partial_b\alpha\right)\mathcal{P}_\sigma\left(i\Phi_0[\rho]\right)\right] \\
&+\left(h\mathrm{II}_{\langle\theta,\theta\rangle} + 2I_{\langle\theta,\alpha\rangle}\right)\Big\{\mathcal{P}_\sigma\left(\Phi_1\right) + D^3_\Phi W\left(\Phi_0[\rho];\Pi\right)\left[\Phi_1,\partial_\rho\Phi_0[\rho]\right] \\
&+\left(\mathrm{I}_{\Box\theta}\right)\mathcal{P}_\sigma\left(i\partial_\rho\Phi_0[\rho]\right) - I_{\langle\theta,\rho\rangle}\mathcal{P}_\sigma\left(i\partial^2_\rho\Phi_0[\rho]\right) + \mathrm{I}_\Gamma\partial_{\zeta p}\Phi_0[\rho]\Big\} \\
&+h\left[\mathrm{II}_{\Box\theta}\mathcal{P}_\sigma\left(i\Phi_0[\rho]\right) - 2\mathrm{II}_{\langle\theta,\rho\rangle}\mathcal{P}_\sigma\left(i\partial_\rho\Phi_0[\rho]\right) + \mathrm{III}_{\langle\theta,\theta\rangle}\mathcal{P}_\sigma\left(\zeta\Phi_0[\rho]\right) + \mathrm{II}_\Gamma\partial_\zeta\Phi_0[\rho]\right] \\
&+2\mathrm{II}_{\langle\theta,h\rangle}\mathcal{P}_\sigma\left(i\zeta\partial_\zeta\Phi_0[\rho]\right) + 2I_{\langle\rho,h\rangle}\partial_{p\zeta}\Phi_0[\rho] + 2\mathrm{II}_{\langle\theta,\alpha\rangle}\mathcal{P}_\sigma\left(\zeta\Phi_0[\rho]\right) \\
&-2I_{\langle\rho,\alpha\rangle}\mathcal{P}_\sigma\left(i\partial_\rho\Phi_0[\rho]\right).
\end{aligned}
$$

As a result of (4.13c), the condition $\mathcal{G}\in\mathfrak{G}_\perp$ translates to asking that (h,α) satisfies the following linear system of PDEs:

$$
\begin{aligned}
C_1^{ab}\partial_{ab}h(y) + \mathfrak{D}_1[h,\alpha](y) &= F_1(y) \\
C_2^{ab}\partial_{ab}\alpha(y) + \mathfrak{D}_2[h,\alpha](y) &= F_2(y),
\end{aligned}
\tag{4.41}
$$

where $C_j^{ab} := g^{ab} + f_j(\rho)\partial^a\theta\partial^b\theta$ for $f_j(\rho) := (-1)^k 2\mu_\Pi^{(k)}(\rho)/\mu_\Pi^{(k-1)}(\rho)$, as in (3.38), and

$$
\begin{aligned}
\mathfrak{D}_1[h,\alpha] :=& \frac{1}{\mu_\Pi(\rho)}\left\langle\mathfrak{D}[h,\alpha](y,\cdot),\mathfrak{N}_1[\rho](y,\cdot)\right\rangle_{L^2(\mathbb{R})}, \\
\mathfrak{D}_2[h,\alpha] :=& -\frac{1}{2\mu'_\Pi(\rho)}\left\langle\mathfrak{D}[h,\alpha](y,\cdot),\mathfrak{N}_2[\rho](y,\cdot)\right\rangle_{L^2(\mathbb{R})}, \\
F_1(y) :=& -\frac{1}{\mu_\Pi(\rho)}\left\langle E_k(y,\cdot),\mathfrak{N}_1[\rho](y,\cdot)\right\rangle_{L^2(\mathbb{R})}, \\
F_2(y) :=& \frac{1}{2\mu'_\Pi(\rho)}\left\langle E_k(y,\cdot),\mathfrak{N}_2[\rho](y,\cdot)\right\rangle_{L^2(\mathbb{R})}.
\end{aligned}
$$

As shown in the proof of Lemma D.1, the system in (4.41) may be expressed as a linear symmetric hyperbolic system of PDEs (see Definition 3.19). It is at this point that we are required to make the additional assumptions on the tensors C_1^{ab} and C_2^{ab} listed as part of the hypotheses of Theorem 4.2. Under such conditions, it follows from standard results on hyperbolic systems of PDEs (see e.g., [31, Corollary 7.10]) that (4.41) admits a unique solution

$h, \alpha \in C_b^\infty(\Gamma)$ with zero initial conditions. As a result, the following choices for h, α, P are justified:

$$\begin{cases} (h,\alpha) \text{ is the unique solution to (4.41) with } (h,\alpha,\partial_0 h, \partial_0\alpha)\Big|_{y_0=0} \equiv (0,0,0,0) \\ P = -\mathfrak{S}\left[G[h,\alpha]\right], \end{cases}$$

making the resulting functions V_{k+1}, h_{k+1}, and θ_{k+1} defined according to (4.37) and (D.2) satisfy

$$e^{-i\theta_{k+1}(y,\zeta)/\varepsilon} \ltimes S_\varepsilon^F \left[V_{k+1}, \theta_{k+1}/\varepsilon, h_{k+1}\right](y,\zeta) = O(\varepsilon^{k+1}).$$

Finally, the decay in (4.6) follows from: 1) the decay of $G[h,\alpha]$ in Equation 4.40, which in turn follow by the boundedness of h and α, the exponential decay of Φ_0 and of its derivatives away from Γ, and the decay of the term $e^{-i\theta_k/\varepsilon} \ltimes E_k$ coming from the induction hypothesis; and 2) by Remark 4.8.

4.2.3 *Restricting the Domain of Approximate Solutions*

Let $\mathcal{Y}_k := \left(e^{\frac{\theta_k}{\varepsilon}} \ltimes V_k\right)_{h_k}$ be any approximate solution obtained using the induction argument from the previous section. Also, just as before, let Ψ be Fermi coordinates associated with Γ, and $\mathcal{N}_\delta$ be domain of definition of Ψ (see Equation 3.8). The construction of V_k implies that $\mathcal{Y}_k(y,\zeta)$ is defined over $(y,\zeta) \in \Gamma \times \mathbb{R}$. However, since $\mathcal{N}_\delta$ might be strictly contained in $[0,T] \times \mathbb{R}^n$, the function $\mathcal{Y}_k$ in general does not induce a well-defined function over $[0,T] \times \mathbb{R}^n$. To account for this, we think of the function $\Phi_0[\rho]$ as being defined over $\mathcal{D}_{\delta,\varepsilon} = \Gamma \times (-\delta/\varepsilon, \delta/\varepsilon)$, and, since subsequent approximations apply a shift on the ζ variable of $h_k(y)$ for each $y \in \Gamma$, we would like to think of the corresponding domain of definition of $\mathcal{Y}_k$ as $\mathcal{D}_{\delta,\varepsilon,h_k} = \left\{(y,\zeta) : y \in \Gamma, \zeta \in \left(-\frac{\delta}{\varepsilon} + h_k(y), \frac{\delta}{\varepsilon} + h_k(y)\right)\right\}$. Any such domain contains $\mathcal{D}_{\delta/2,\varepsilon}$ for all small enough $\varepsilon > 0$ due to the boundedness of h_k, and therefore, $\mathcal{Y}_k$ induces a well-defined function on $\mathcal{N}_{\delta/2}$ so long as $\varepsilon > 0$ is small enough. As a slight abuse of notation, we will denote the resulting function, which is only defined on $\mathcal{N}_{\delta/2}$, by $\mathcal{Y}_k$ from now on.

This concludes the proof of Theorem 4.2.

4.3 EXTENDED APPROXIMATE SOLUTIONS

We may extend the definition of the function $\mathcal{Y}_k$ from the previous section to $[0,T]\times\mathbb{R}^n$. To this end, recall that Γ divides the space $[0,T]\times\mathbb{R}^n$ into two disjoint components Ω_- and Ω_+. Suppose that the Minkowski normal vector field $\nu(y)$ points in the direction of Ω_+ for each $y\in\Gamma$, and define

$$\mathbb{I}(t,x) := \begin{cases} -1, & (t,x)\in\mathcal{O}^- \\ +1, & (t,x)\in\mathcal{O}^+. \end{cases}$$

Furthermore, write $(y,\varepsilon\zeta)=\Psi^{-1}(t,x)$ (as before), and introduce a cut-off function $\chi\in C^\infty\left([0,T]\times\mathbb{R}^n;[0,1]\right)$ with the property that

$$\chi\left(t,x\right)=1 \qquad \text{for } |\zeta|\leq\frac{\delta}{4\varepsilon}, \qquad \text{and} \qquad \chi\left(t,x\right)=0 \qquad \text{for } |\zeta|>\frac{\delta}{2\varepsilon}.$$

We define the k^{th} approximate solution $\mathcal{Y}_k^*\in C^\infty([0,T]\times\mathbb{R}^n;\mathbb{R}\times\mathbb{C})$ as

$$\mathcal{Y}_k^*(t,x) := \begin{cases} \chi\left(t,x\right)\mathcal{Y}_k\left(y,\zeta\right)+\left[1-\chi\left(t,x\right)\right]\mathbb{I}(t,x), & \text{if } (y,\zeta)=\Psi^{-1}(t,x)\in\mathcal{N}_{\delta/2} \\ \mathbb{I}(t,x), & \text{otherwise,} \end{cases} \tag{4.42}$$

and a result of Theorem 4.2, we immediately obtain:

Corollary 4.10. *Let $k\geq1$. The definition of $\mathcal{Y}_k^*$ from Equation 4.42 implies that*

$$\Box\mathcal{Y}_k^*+\frac{1}{\varepsilon^2}D_\Phi W\left(\mathcal{Y}_k^*;\Pi\right)=0 \qquad \text{in } \left([0,T]\times\mathbb{R}^n\right)\setminus\mathcal{N}_{\delta/2},$$

and for every j and β, there exists smooth positive constants C_{kj} and $C_{kj\beta}$ depending on Π and on the size of the derivatives of θ,γ, such that for all small enough $0<\varepsilon\ll1$:

$$\left|\partial_\zeta^j S_\varepsilon^F\left[\mathcal{Y}_k^*\right](y,\zeta)\right|\leq C_{kj}\varepsilon^k(1+|\zeta|^{k+2})e^{-c(\Pi)\frac{|z|}{\varepsilon}} \qquad \text{in } \mathcal{N}_{\delta/2},$$

$$\left|\partial_\zeta^j D_y^\beta\left[e^{-i\theta_k(y,\zeta)/\varepsilon}\ltimes S_\varepsilon^F\left[\mathcal{Y}_k^*\right]\right](y,\zeta)\right|\leq C_{kj\beta}\varepsilon^k(1+|\zeta|^{k+2})e^{-c(\Pi)\frac{|z|}{\varepsilon}} \qquad \text{in } \mathcal{N}_{\delta/2},$$

for some constant $c(\Pi)>0$ independent of ε.

5

CONCLUDING REMARKS AND FUTURE WORK

We conclude by highlighting two possible ways in which the results of this thesis can be extended.

5.1 EXACT SOLUTIONS

Given an approximate solution $\mathcal{Y}^* = (\varphi^*, \sigma^*) : [0,T] \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{C}$ as in Section 4.3, we would like to find a perturbation map $\tilde{\Phi}$ which makes $\mathcal{Y}^* + \tilde{\Phi}$ an exact solution to the superconducting interface system, Equation 1.7, over $[0,T] \times \mathbb{R}^n$. More precisely, we seek a map $\tilde{\Phi}$ such that

$$\begin{cases} \Phi := \mathcal{Y}^* + \tilde{\Phi} \text{ solves Equation 1.7 on } [0,T] \times \mathbb{R}^n \\ \tilde{\Phi} = O(\varepsilon^p) \text{ in some norm for some } p > 0. \end{cases} \tag{5.1}$$

This type of result is obtained in [27] for the case where $\sigma \equiv 0$ and Γ has zero Minkowskian mean curvature (i.e., is minimal) using energy estimates for the perturbation map both near and far away from Γ. The strategy to obtain these estimates relies on the existence of approximate solutions of arbitrary precision in the sense of Theorem 4.2, and on the fact that Γ has zero Minkowskian mean curvature. Unfortunately, the fact that Γ is generally not minimal according to (3.35) and, most critically, the presence of the rapid oscillations introduced by the leading order phase of σ^* near Γ (i.e., the term θ/ε in each approximation in Theorem 4.2), render the direct application of such strategies inefficacious when trying to obtain similar estimates for the perturbation $\tilde{\Phi}$ in the case of Equation 1.7 under the requirement that Φ exhibits the scaling (1.25). We believe that there should be a way to incorporate the relationship between Γ and θ dictated by (3.35) to resolve such intricacies, but this is yet to be verified.

On the other hand, we also remark that the desired result above was obtained in [34], although following a different approach to that of [27]. The approach in this case is closer in nature to the spirit of Lyapunov-Schmidt reduction and requires only the construction of the approximation corresponding to $k = 0$ in Theorem 4.2. In essence, it is argued via the implicit function theorem that there exists a unique function $h \in C^1(\Gamma)$ and a perturbation Φ solving (5.1) such that[1], in our notation, $\Phi_h(y, \cdot) \in \ker\left(\mathcal{L}\left((\Phi_0[\rho])_h(y,\cdot);\Pi_{\rho(y)}\right)\right)^{\perp}$ for each $y \in \Gamma$. It is then shown that Φ and h are of the desired size as long as their corresponding initial conditions are small enough. This step is accomplished through energy estimates which are involved due to the fact that both the size of perturbation Φ and of the shift h have to be established simultaneously. To note is that, due to the assumptions made in therein, the kernel of $\mathcal{L}\left((\Phi_0[\rho])_h(y,\cdot);\Pi_{\rho(y)}\right)$ is one dimensional for each $y \in \Gamma$, which goes hand in hand with the fact that only the function h is needed to satisfy the above condition on Φ_h. In our case, the fact that the corresponding kernel is two dimensional for each $y \in \Gamma$ means that two functions h and a α defined over Γ are required to yield the analogous condition, imposing significant technical difficulties which make the computations less tractable but could nevertheless be explored further.

5.2 THE NEUTRAL SUPERCONDUCTING COSMIC STRING CASE

Consider the functional in Equation 1.1 with $q_\sigma = 0$ and complex-valued φ (i.e., the neutral superconducting string case). We now present some heuristics for this case analogous to those presented in Section 3.2 for the superconducting interface model. In doing so, we highlight the similarities between the two models, and some directions in which the ideas presented in this thesis could prove useful to obtain a better understanding of the superconducting string model.

To begin, we proceed as in Section 3.2, this time labelling A_φ as $A = A_0 dt + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$, so that the relevant action functional with a scaling parameter $\varepsilon > 0$ in this case reduces to

1 The function h minimizes the distance the L^2 distance between the exact solution and $(\Phi_0[\rho])_h$.

$$\begin{aligned}&\mathcal{A}_\varepsilon^{s,N}[\varphi,\sigma,A;\Pi]\\&:=\int_0^T\int_{\mathbb{R}^3}\left[\frac{\varepsilon}{2}\left(D_\alpha\varphi,D^\alpha\varphi\right)+\frac{\varepsilon}{2}\left(\partial_\alpha\sigma,\partial^\alpha\sigma\right)+\frac{\varepsilon^3}{4}F^{\alpha\beta}F_{\alpha\beta}+\frac{1}{\varepsilon}W\left(\varphi,\sigma;\Pi\right)\right]dx\,dt,\end{aligned} \tag{5.2}$$

where $D_\alpha := \partial_\alpha - iA$ and $F_{\alpha\beta} := (dA)_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$ for $\alpha, \beta \in \{0, 1, 2, 3\}$.

We would like to investigate the existence of solutions to the Euler-Lagrange equations of $\mathcal{A}_\varepsilon^{s,N}$ for which

- φ has a zero set along a timelike codimension 2 manifold embedded in $\mathbb{R}^{1+3}$ (a trajectory/worldsheet of a curve that represents the cosmic string).
- σ is non-zero, and concentrated around the zero set of φ.

We proceed as in Section 3.2, looking for static simple solutions first. To this end, introduce the notation $\Box_A := -D^\alpha D_\alpha = D_0^2 - D_1^2 - D_2^2 - D_3^2$, $\nabla_A := (D_0, D_1, D_2, D_3)$, and $\Delta_A := \sum_{k=1}^3 D_k^2$, and note that the Euler Lagrange equations of $\mathcal{A}_\varepsilon^{s,N}$ in (5.2) are

$$\begin{aligned}\Box_A\varphi+\frac{1}{\varepsilon^2}D_\varphi W(\varphi,\sigma;\Pi)&=0\\ \Box\sigma+\frac{1}{\varepsilon^2}D_\sigma W(\varphi,\sigma;\Pi)&=0\\ \varepsilon^2\partial^\alpha F_{\alpha\beta}+(i\varphi,D_\beta\varphi)&=0,\qquad\beta=0,1,2,3.\end{aligned} \tag{5.3}$$

Let $\bar{x} = (x^0, x^1, x^2, x^3)$ be the standard coordinates in $\mathbb{R}^{1+3}$ and write $t = x^0$, $x = x^1$, and $z = (x^2, x^3)$. We look for solutions to (5.3) of the form

$$\begin{aligned}\varphi(\bar{x})&=\varphi_0(z/\varepsilon)\\ \sigma(\bar{x})&=e^{i(a_0t+a_1x)/\varepsilon}\sigma_0(z/\varepsilon)\\ A(\bar{x})&=A_2(\bar{x})dx^2+A_3(\bar{x})dx^3=\frac{1}{\epsilon}\left[\mathcal{A}_2(z/\varepsilon)dx^2+\mathcal{A}_3(z/\varepsilon)dx^3\right].\end{aligned} \tag{5.4}$$

Plugging the ansatz (5.4) into (5.3), results in the equations

$$\begin{aligned}-\Delta_A\varphi_0+D_\varphi W\left(\varphi_0,\sigma_0;\Pi_\rho\right)&=0\\ -\Delta\sigma_0+D_\sigma W\left(\varphi_0,\sigma_0;\Pi_\rho\right)&=0\\ \partial_jF_{jk}+(i\varphi_0,D_j\varphi_0)&=0,\quad(j,k)\in\{(2,3),(3,2)\},\end{aligned} \tag{5.5}$$

where $\rho = -a_0^2 + a_1^2$. The equations appearing in (5.5) are the Euler-Lagrange equations of the functional

$$\mathcal{A}^{2d}_{\Pi_\rho}\left[\varphi, \sigma, A\right] := \int_{\mathbb{R}^2} \left[\frac{1}{2}\left(\left|\nabla_A \varphi\right|^2 + \left|\nabla \sigma\right|^2\right) + \frac{1}{2}F_{12}^2 + W\left(\varphi, \sigma; \Pi_\rho\right)\right] dz^1 dz^2,$$

defined over a suitable functional space $\mathcal{K}$ of functions $\varphi, \sigma : \mathbb{R}^2 \to \mathbb{C}$ and $A = A_1 dz^1 + A_2 dz^2$ with $(A_1, A_2) : \mathbb{R}^2 \to \mathbb{R}^2$, which we state formally for the sake of concreteness as

$$\begin{aligned}\mathcal{K} := \Big\{&(\varphi, \sigma, A) :\ \varphi \in \dot{H}^1(\mathbb{R}^2; \mathbb{C}), \sigma \in H^1(\mathbb{R}^2; \mathbb{C}), (\partial_1 A_2 - \partial_2 A_1) \in L^2(\mathbb{R}^2; \mathbb{R}),\\ &|\varphi(x)| \to 1 \text{ as } |x| \to \infty,\ \text{the winding number of } \varphi \text{ at } \infty \text{ is } 1\Big\}.\end{aligned}$$

The minimizers of $\mathcal{A}^{2d}_{\Pi_\rho}$ over $\mathcal{K}$ (aka $2D$ profiles) solve the elliptic system Equation 5.5, and thus induce solutions to Equation 5.3 according to Equation 5.4. The zero set of φ in each of these minimizers is a straight vortex filament which represents a straight cosmic string, analogously to how the zero set of φ in the minimizers of $\mathcal{A}^{1d}_{\Pi}$ over $\mathcal{H}$ (see Equation 2.4 and Equation 2.2) is a hyperplane that represents a flat interface. Just as in the case of the superconducting interface model, one expects to find solutions to (5.3) which behave locally like these $2D$ profiles and describe the scenario in which cosmic strings with more complex geometries are present. To the best of our knowledge, a study of the properties of such minimizers that mimics the one presented in Chapter 2 (see in particular Theorem 2.1, Theorem 2.3 and Theorem 2.4) does not exist in the literature.

On the other hand, we note the similarities between the laws of motion presented in Chapter 3 and their (formally derived) counterpart for the superconducting cosmic string case. To this end, we proceed as in Section 3.2. In particular, let (Λ, ψ) be a local chart for Γ, and ν^1 and ν^2 be two vector fields everywhere normal to Γ such that $\left\{\nu^j(p)\right\}_{j=1}^2$ is a basis for $(T_p\Gamma)^\perp$ and $\left\langle \nu^i(p), \nu^j(p)\right\rangle_m = \delta^{ij}$ for $i, j = 1, 2$ at each $p \in \Gamma$. Furthermore, let $\delta > 0$ be small enough so that the map (Fermi coordinates)

$$\begin{aligned}\Psi :&\Lambda \times B_\delta(0) \to \mathcal{N}_\delta := \Psi\left(\Lambda \times B_\delta(0)\right) \subset \mathbb{R}^{1+3}\\ (t, x) = \Psi(y, z) &= \psi(y) + z_1 \nu^1(y) + z_2 \nu^2(y), \qquad y = (y_0, y_1), z = (z_1, z_2),\end{aligned}$$

is bijective, where $B_\delta(0)$ is the ball of radius δ in $\mathbb{R}^2$ centered at the origin. As in Section 3.1, we introduce the metrics on Γ, g_{ab}, and on $\mathcal{N}_\delta$, $\mathfrak{g}_{\alpha\beta}$, induced by the Minkowski metric:

$$\begin{aligned} \mathfrak{g}_{\alpha\beta}(y,z) &:= \langle \partial_\alpha \Psi(y,z), \partial_\beta \Psi(y,z)\rangle_m, && \alpha,\beta = 0,1,2,3. \\ g_{ab}(y) &:= \langle \partial_a \psi(y), \partial_b \psi(y)\rangle_m, && a,b = 0,1. \end{aligned}$$

A direct computation using the definition of ν^1 and ν^2 shows that

$$\mathfrak{g}_{\alpha\beta}(y,z) = \begin{pmatrix} g_{ab}(y) + O(|z|) & O(|z|) \\ O(|z|) & I_2 \end{pmatrix}.$$

Now, let

$$(\varphi, \sigma, A)(t,x) = (\tilde{\varphi}, \tilde{\sigma}, \tilde{A})(y,z) \qquad \text{whenever } (t,x) = \Psi(y,z),$$

and assume that all the functions are supported within $\mathcal{N}_\delta$. Then, the corresponding action functional becomes

$$\begin{aligned} &\mathcal{A}_\varepsilon^{s,N}\left[\tilde{\varphi}, \tilde{\sigma}, \tilde{A}; \Pi\right] := \\ &\quad \int_\Lambda \int_{\mathbb{R}^2} \left[\frac{1}{2}\mathfrak{g}^{\alpha\beta}(y,z)(D_\alpha\varphi, D_\beta\varphi) + \frac{1}{2}\mathfrak{g}^{\alpha\beta}(y,z)\left(\partial_\alpha\sigma, \partial_\beta\sigma\right)\right. \\ &\quad \left. + \frac{\varepsilon^2}{4}\mathfrak{g}^{\alpha\gamma}(y,z)\mathfrak{g}^{\beta\mu}(y,z)F_{\gamma\mu}F_{\alpha\beta} + \frac{1}{\varepsilon^2}W\left(\varphi, \sigma; \Pi\right)\right]\sqrt{|\mathfrak{g}(y,z)|}\,dy\,dz. \end{aligned}$$

Now, suppose further that

$$\begin{aligned} &\varphi(y,z) = v(y,\zeta), \\ &\sigma(y,z) = e^{i\theta(y)/\varepsilon} w(y,\zeta), \\ &A(y,z) = \mathbf{A}_0(y,\zeta)dy^0 + \mathbf{A}_1(y,\zeta)dy^1 + \frac{1}{\epsilon}\mathbf{A}_2(y,\zeta)d\zeta^1 + \frac{1}{\epsilon}\mathbf{A}_3(y,\zeta)d\zeta^2 \qquad \zeta := z/\varepsilon, \end{aligned}$$

for some complex-valued function v and real-valued functions $w, \mathbf{A}_\alpha$, all defined on $\Lambda \times B_{\delta/\varepsilon}(0)$, and a real-valued function θ defined on Λ. Under this ansatz, we have that

$$\begin{aligned} &\mathcal{A}_\varepsilon^{s,N}\left[\tilde{\varphi}, \tilde{\sigma}, \tilde{A}; \Pi\right] \\ &\propto \int_\Lambda \left\{ \int_{B_{\delta/\varepsilon}(0)} \left[\frac{1}{2}\sum_{j=1}^{2}\left(\left|D_{\zeta_j} v\right|^2 + \left(\partial_{\zeta_j} w\right)^2\right) + \frac{1}{2}\left(\partial_{\zeta_1}\mathbf{A}_3 - \partial_{\zeta_2}\mathbf{A}_2\right)^2\right.\right. \end{aligned}$$

$$+ \frac{1}{2} g^{ab} \left(\partial_{y_a}\theta\right)(\partial_{y_b}\theta)\, w^2 + W\left(v, w; \Pi\right) \Big] \, d\zeta_1 \, d\zeta_2 \Big\} \sqrt{|g(y)|}\, dy + O\left(\varepsilon\right)$$

$$= \int_\Lambda \left\{ \mathcal{A}^{2d}_{\Pi_{\rho(y)}}[v(y,\cdot), w(y,\cdot), A(y,\cdot)] \right\} \sqrt{|g(y)|}\, dy + O\left(\varepsilon\right),$$

where $\mathcal{A}^{2d}_{\Pi_{\rho(y)}}[v(y,\cdot), w(y,\cdot), A(y,\cdot)]$ is the integral with respect to ζ_1 and ζ_2 in the above expression. As such, and similarly to the superconducting interface case (cf. (3.2)), to leading order, the action above reduces to

$$\Sigma^s[\psi, \theta; V] := \int_V \mu^s_\Pi(\rho(y)) \sqrt{|g(y)|}\; dy, \qquad \rho = g^{ab}\partial_a\theta\, \partial_b\theta,$$

where

$$\mu^s_\Pi(\rho(y)) := \inf_{(\varphi,\sigma,A)\in\mathcal{K}} \mathcal{A}^{2d}_{\Pi_{\rho(y)}}[\varphi, \sigma, A].$$

As a result, the dynamic laws governing the admissible string-current pairs are identical to Equation 3.35, with μ_Π replaced by μ^s_Π. In particular, the results of Theorem 3.22 apply to the superconducting string laws of motion, as long as the function μ^s_Π is positive, increasing and concave down as μ_Π. These properties of μ^s_Π are also yet to be verified.

A

PROPERTIES OF THE POTENTIAL W

Let W denote the potential whose action on $(\varphi, \sigma, \Pi) \in \mathbb{R} \times \mathbb{C} \times (0, \infty)^4$ is

$$W(\varphi, \sigma; \Pi) := \frac{\lambda_\varphi}{4}\left(\varphi^2 - 1\right)^2 + \frac{\lambda_\sigma}{4}\left(|\sigma|^2 - 2m_\sigma^2\right)|\sigma|^2 + \frac{\beta}{2}\varphi^2|\sigma|^2 . \quad \text{(A.1)}$$

A.1 SOME USEFUL FORMULAS INVOLVING W

Let $\sigma = \sigma_1 + i\sigma_2$ and $\partial_0 := \partial_\varphi, \partial_1 := \partial_{\sigma_1}$ and $\partial_2 := \partial_{\sigma_2}$. Also, for convenience, write $W := W(\varphi, \sigma; \Pi)$. A direct computation shows that

$$\begin{aligned} \partial_0 W &= \left[\lambda_\varphi\left(\varphi^2 - 1\right) + \beta|\sigma|^2\right]\varphi \\ \partial_i W &= \left[\lambda_\sigma\left(|\sigma|^2 - m_\sigma^2\right) + \beta\varphi^2\right]\sigma_i, \qquad \text{for } i = 1, 2, \end{aligned} \quad \text{(A.2)}$$

and

$$\begin{aligned} \partial_{00}W &= \lambda_\varphi\left(3\varphi^2 - 1\right) + \beta|\sigma|^2, & \partial_{10}W &= \partial_{01}W = 2\beta\varphi\sigma_1 \\ \partial_{11}W &= \lambda_\sigma\left(3\sigma_1^2 + \sigma_2^2 - m_\sigma^2\right) + \beta\varphi^2, & \partial_{20}W &= \partial_{02}W = 2\beta\varphi\sigma_2 \\ \partial_{22}W &= \lambda_\sigma\left(\sigma_1^2 + 3\sigma_2^2 - m_\sigma^2\right) + \beta\varphi^2, & \partial_{21}W &= 2\lambda_\sigma\sigma_1\sigma_2. \end{aligned} \quad \text{(A.3)}$$

Therefore,

$$D_\Phi^2 W(\pm 1, 0; \Pi) = \operatorname{diag}(2\lambda_\varphi, \beta - \lambda_\sigma m_\sigma^2, \beta - \lambda_\sigma m_\sigma^2), \quad \text{(A.4)}$$

from which it follows that the minimum eigenvalue of $D_\Phi^2 W(\pm 1, 0; \Pi)$ is

$$\lambda_{\min}(\Pi) := \min\left(2\lambda_\varphi, \beta - \lambda_\sigma m_\sigma^2\right). \quad \text{(A.5)}$$

The Taylor expansion for W about $(\varphi, \sigma) = (x, 0)$ with $x \in \{-1, 1\}$, is

$$\begin{aligned} W(\Phi; \Pi) &= \frac{1}{2}\Phi^T D^2_\Phi W((x,0);\Pi)\Phi + x\lambda_\varphi (\varphi - x)^3 + x\beta(\varphi - x)|\sigma|^2 \\ &\quad + \frac{\lambda_\varphi}{4}(\varphi - x)^4 + \frac{\lambda_\sigma}{4}|\sigma|^4 + \frac{\beta}{2}(\varphi - x)^2 |\sigma|^2 . \\ &\geq \left[\frac{1}{2}\lambda_{\min}(\Pi) - (\lambda_\varphi + \beta)\|(\varphi - x, |\sigma|)\|_2\right] \|(\varphi - x, |\sigma|)\|_2^2 . \end{aligned} \tag{A.6}$$

Finally, we have that

$$\begin{aligned} W(\varphi,\sigma;\Pi) &= \frac{\lambda_\varphi}{4}\left(\varphi^2 - 1\right)^2 + \left[\frac{\lambda_\sigma}{4}\left(|\sigma|^2 - 2m_\sigma^2\right) + \frac{\beta}{2}\varphi^2\right]|\sigma|^2 \\ &\geq \frac{\lambda_\varphi}{4}\left(\varphi^2 - 1\right)^2 + \frac{1}{4}\left[\min(\lambda_\sigma, 2\beta)\left(\varphi^2 + |\sigma|^2\right) - 2m_\sigma^2\lambda_\sigma\right]|\sigma|^2 . \end{aligned} \tag{A.7}$$

A.2 PROOFS OF *Lemma 1.1* AND *Lemma 1.2*

Proof of Lemma 1.1. The two desired conditions on Π are

$$D^2_\Phi W(\pm 1, 0; \Pi) \text{ is positive definite.} \tag{A.8a}$$

$$W(\varphi, \sigma; \Pi) \geq 0, \text{ with equality only if } (|\varphi|, \sigma) = (1, 0). \tag{A.8b}$$

It follows from (A.4) that the condition (A.8a) is equivalent to

$$\eta := \beta - \lambda_\sigma m_\sigma^2 > 0. \tag{A.9}$$

Also, note that $W(\cdot, \cdot; \Pi)$ is smooth, $W(\varphi, \sigma; \Pi) \to \infty$ as $|(\varphi, \sigma)| \to \infty$, and $W(\pm 1, 0; \Pi) = 0$ independently of Π. Therefore, denoting by $C_W(\Pi)$ the set of critical points of $W(\cdot; \Pi)$, condition (A.8b) is equivalent to

$$W(\varphi, \sigma; \Pi) > 0 \text{ for all } (\varphi, \sigma) \in C_W(\Pi) \setminus \{(\pm 1, 0)\} . \tag{A.10}$$

We claim that

$$\eta > 0 \implies \left((\text{A.10}) \iff \lambda_\varphi > \lambda_\sigma m_\sigma^4\right), \tag{A.11}$$

which in turn implies that

$$(\text{A.8a}) \text{ and } (\text{A.8b}) \iff \Pi \in \mathcal{O}^0,$$

where $\mathcal{O}^0 := \left\{\Pi = (\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in (0,\infty)^4 : \beta > \lambda_\sigma m_\sigma^2, \lambda_\varphi > \lambda_\sigma m_\sigma^4\right\}$.

To prove (A.11), suppose that $\eta > 0$ and $(\varphi, \sigma) \in C_W(\Pi)$. Using (A.2), we find that

$$\begin{aligned} \varphi = 0 &\implies \sigma \in \{0\} \cup \left\{e^{i\theta} m_\sigma : \theta \in \mathbb{R}\right\} \\ \sigma = 0 &\implies \varphi \in \{-1, 1\}. \end{aligned} \tag{A.12}$$

On the other hand, if $\varphi \neq 0$ and $\sigma \neq 0$, then

$$\begin{aligned} \lambda_\varphi \left(\varphi^2 - 1\right) + \beta \left|\sigma\right|^2 &= 0 \\ \lambda_\sigma \left(\left|\sigma\right|^2 - m_\sigma^2\right) + \beta\varphi^2 &= 0, \end{aligned} \tag{A.13}$$

from which we infer that

$$\mu\varphi^2 = \lambda_\sigma \alpha \qquad \text{and} \qquad \mu \left|\sigma\right|^2 = \lambda_\varphi \eta, \tag{A.14}$$

where $\alpha := \beta m_\sigma^2 - \lambda_\varphi$ and $\mu := \beta^2 - \lambda_\varphi \lambda_\sigma$. Since $\varphi, \sigma \neq 0$ and $\eta > 0$ in this case, both μ and α must be positive, and thus

$$(\varphi, |\sigma|) = \left(\pm\sqrt{\lambda_\sigma \alpha/\mu}, \sqrt{\lambda_\varphi \eta/\mu}\right). \tag{A.15}$$

Given that (A.12) and (A.15) describe $C_W(\Pi) \setminus \{(\pm 1, 0)\}$ completely, the statement (A.11) is a consequence of the following identities

$$\begin{aligned} W(0, e^{i\theta} m_\sigma; \Pi) &= \frac{1}{4}\left(\lambda_\varphi - \lambda_\sigma m_\sigma^4\right), && \text{for all } \theta \in \mathbb{R} \\ W(0,0;\Pi) &= \frac{\lambda_\varphi}{4}, \\ W\left(\sqrt{\lambda_\sigma \frac{\alpha}{\mu}}, e^{i\theta}\sqrt{\lambda_\varphi \frac{\eta}{\mu}}; \Pi\right) &= \frac{\lambda_\varphi \eta^2}{4\mu}, && \text{for all } \theta \in \mathbb{R}. \end{aligned}$$

The last equality follows from the fact that if (A.15) holds, then

$$\begin{aligned} (\varphi^2 - 1)^2 &= \left(\frac{\beta\eta}{\mu}\right)^2, \\ \left(\left|\sigma\right|^2 - 2m_\sigma^2\right)\left|\sigma\right|^2 &= -\frac{\lambda_\varphi \eta}{\mu^2}\left(\beta\alpha + m_\sigma^2 \mu\right) \\ \left|\sigma\right|^2 \varphi^2 &= \frac{\lambda_\varphi \lambda_\sigma \alpha \eta}{\mu^2}, \end{aligned}$$

and therefore,

$$W(\varphi,\sigma;\Pi)=\frac{\lambda_\varphi\eta}{4\mu^2}\left[\beta^2\eta-\lambda_\sigma\left(m_\sigma^2\mu-\beta\alpha\right)\right]=\frac{\lambda_\varphi\eta}{4\mu^2}\left[\beta^2\eta-\lambda_\sigma(\lambda_\varphi\eta)\right]=\frac{\lambda_\varphi\eta^2}{4\mu}.$$

□

Proof of Lemma 1.2. Let $\Pi\in\mathcal{O}^0$ and $a\in(0,1)$, and define

$$r(\Pi,a):=\min\left(1,a\frac{\lambda_{\min}(\Pi)}{2(\lambda_\varphi+\beta)}\right)\text{ and }R(\Pi):=\sqrt{\frac{2m_\sigma^2\lambda_\sigma+\lambda_\varphi}{\min(\lambda_\sigma,2\beta)}},\tag{A.16}$$

which are both positive since $\lambda_{\min}(\Pi)>0$ due to Lemma 1.1.

Near $(\varphi,\sigma)=(\pm1,0)$, (A.6) implies that

$$W(\varphi,\sigma;\Pi)\geq\frac{(1-a)}{2}\lambda_{\min}(\Pi)\left\|(\varphi\mp1,|\sigma|)\right\|_2^2,\tag{A.17}$$

whenever (φ,σ) is in the set

$$\mathcal{O}^{\pm}_{near,a}(\Pi):=\left\{(\varphi,\sigma)\in\mathbb{R}\times\mathbb{C}:\left\|(\varphi\mp1,|\sigma|)\right\|_2<r(\Pi,a)\right\}.$$

Now, the definition of $r(\Pi,a)$ implies that $\varphi\in(-1,1)$ and thus $(\varphi\pm1)^2\in(0,4)$ whenever $(\varphi,\sigma)\in\mathcal{O}^{\pm}_{near,a}(\Pi)$. Using this and setting $a=1/2$, we conclude that

$$W(\varphi,\sigma;\Pi)\geq\frac{1}{16}\lambda_{\min}(\Pi)\left\|\left(\varphi^2-1,|\sigma|\right)\right\|_2^2,\text{ for }(\varphi,\sigma)\in\mathcal{O}^{\pm}_{near,\frac{1}{2}}.\tag{A.18}$$

Far from $(\varphi,\sigma)=(0,0)$, we use (A.7) and (A.16) to conclude that

$$W(\varphi,\sigma;\Pi)\geq\frac{\lambda_\varphi}{4}\left\|(\varphi^2-1,|\sigma|)\right\|_2^2,\text{ for }(\varphi,\sigma)\in\mathcal{O}_{far}(\Pi),\tag{A.19}$$

where

$$\mathcal{O}_{far}(\Pi):=\left\{(\varphi,\sigma)\in\mathbb{R}\times\mathbb{C}:\left\|(\varphi,\sigma)\right\|_2>R(\Pi)\right\}.$$

Now, consider the compact set

$$\mathcal{K}(\Pi):=(\mathbb{R}\times\mathbb{C})\setminus\left(\mathcal{O}^+_{near}\cup\mathcal{O}^-_{near}\cup\mathcal{O}_{far}\right)$$

and the continuous function

$$c_{\mathcal{K}}(\Pi) := \frac{\min\limits_{(\varphi,\sigma)\in\mathcal{K}(\Pi)} W(\varphi,\sigma;\Pi)}{\max\limits_{(\varphi,\sigma)\in\mathcal{K}(\Pi)} \|(\varphi^2-1,|\sigma|)\|_2^2}.$$

Putting together (A.18) and (A.19) with the definitions of $\mathcal{K}$ and $c_{\mathcal{K}}$, we conclude that

$$W(\varphi,\sigma;\Pi) \geq c(\Pi)\left[\left(\varphi^2-1\right)^2+|\sigma|^2\right] \qquad \text{for all } (\varphi,\sigma)\in\mathbb{R}\times\mathbb{C},$$

where $c : \mathcal{O}^0 \to (0,\infty)$ is the continuous function

$$\Pi = (\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \mapsto \min\left\{\frac{\lambda_{\min}(\Pi)}{16}, \frac{\lambda_\varphi}{4}, c_{\mathcal{K}}(\Pi)\right\}.$$

$\square$

B

1D ANALYSIS: AUXILIARY RESULTS

B.1 SMOOTHNESS OF THE MAP F FROM *Equation* 2.6

In this section, we provide a proof that the map from Equation 2.6 is smooth in the sense of Fréchet differentiability (see e.g., [4] for relevant definitions). It is included solely for completeness and follows readily from Sobolev embedding theorems and the definitions of the Fréchet derivative. Throughout this section, we use the notation in Section 1.3.1 and define

$$Z := X_2^3 \times \mathbb{R}^4. \tag{B.1}$$

The main result of this section is the following lemma, whose proof follows directly from the proofs of Lemmas B.5 and B.6 below:

Lemma B.1. *The map* $\mathcal{F} : Z \to X_0^3$*, given by*

$$\mathcal{F}(\Phi, \Pi) = -\Phi'' + D_\Phi W(\Phi; \Pi),$$

belongs to $C^\infty(Z; X_0^3)$*.*

In what follows, we denote by $\mathcal{L}_n(Z, X_2)$ the space of continuous $n-$linear maps from Z to X_2 for each $n \in \mathbb{N}$, equipped with the norm

$$\|T\|_{\mathcal{L}_n(Z,X_2)} := \sup_{h \in (B^Z(0,1))^n} \|T(h)\|_{X_2} = \sup_{h_1,h_2,...,h_n \in B^Z(0,1)} \|T(h_1, h_2, ..., h_n)\|_{X_2}.$$

Also, we write $\mathcal{L}_0(Z, X_2) = X_2$.

Lemma B.2. *Let* $n, m \in \{0, 1, 2, ...\}$*. Given any two maps* $F : Z \to \mathcal{L}_n(Z, X_2)$ *and* $G : Z \to \mathcal{L}_m(Z, X_2)$ *or* $G : Z \to \mathcal{L}_m(Z, \mathbb{R})$*, define the product of* F *and* G *as the map* $FG : Z \to \mathcal{L}_{n+m}(Z, X_2)$ *given by*

$$(FG)(u)[k, l] = F(u)[k]G(u)[l], \qquad (u, k, l) \in Z \times Z^n \times Z^m.$$

Then, if F and G are differentiable so is FG.

Proof. We will show that if $n, m \neq 0$, then the Fréchet derivative of FG, $d(FG) : Z \to \mathcal{L}_{n+m+1}(Z, X_2)$, exists and is given by the following formula which holds for all $(u, k, l) \in Z \times Z^n \times Z^m$:

$$d(FG)(u)[h,k,l] = dF(u)[h,k]G(u)[l] + F(u)[k]dG(u)[h,l]. \tag{B.2}$$

A slight modification of the argument can be used to show that

$$\begin{aligned} d(FG)(u)[h,k] &= dF(u)[k,h]G(u) + F(u)[k]dG(u)[h], && \text{if } m = 0, n \neq 0, \\ d(FG)(u)[h,k] &= dF(u)[h]G(u)[k] + F(u)dG(u)[h,k], && \text{if } m \neq 0, n = 0 \\ d(FG)(u)[h] &= dF(u)[h]G(u) + F(u)dG(u)[h], && \text{if } m, n = 0. \end{aligned}$$

To show (B.2), let $k = (k_1, k_2, ..., k_n) \in Z^n$ and $l = (l_1, l_2, ..., l_m) \in Z^m$ be such that $k_i, l_j \in B^Z(0,1)$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$. It suffices to show that

$$\begin{aligned} \| FG(u+h)[k,l] - FG(u)[k,l] - dF(u)[h,k]G(u)[l] \\ - F(u)[k]dG(u)[h,l] \|_{X_2} = o\left(\|h\|_Z\right). \end{aligned} \tag{B.3}$$

To this end, write the left hand side of (B.3) as

$$\begin{aligned} LHS = \| & F(u+h)[k]\left[G(u+h) - G(u) - dG(u)[h]\right][l] \\ & + \left[G(u) + dG(u)[h]\right][l]\left[F(u+h) - F(u) - dF(u)[h]\right][k] \\ & + dG(u)[h,l]dF(u)[h,k]\|_{X_2}, \end{aligned} \tag{B.4}$$

and let $h \in Z$ be such that $\|h\|_Z \leq 1$. We have that,

$$\begin{aligned} & \| F(u+h)[k]\left[G(u+h) - G(u) - dG(u)[h]\right][l] \|_{X_2} \\ & \quad \leq C \|F(u+h)[k]\|_{X_2} \|\left[G(u+h) - G(u) - dG(u)[h]\right][l]\|_{X_2} \\ & \quad \leq C_u \|\left[G(u+h) - G(u) - dG(u)[h]\right][l]\|_{X_2} \\ & \quad \leq C_u \|G(u+h) - G(u) - dG(u)[h]\|_{X_2} = o\left(\|h\|_Z\right), \end{aligned} \tag{B.5}$$

Which follows from the fact that $\|fg\|_{X_2} \leq C \|f\|_{X_2} \|g\|_{X_2}$ for some $C > 0$ independent of f and g, by the continuity of $F(u)$ for each $u \in Z$ and by the definition of dG. Similarly, by the continuity of $G(u)$ and of $dG(u)$ for each $u \in Z$,

$$\begin{aligned} &\|[G(u) + dG(u)[h]]\,[l]\,[F(u+h) - F(u) - dF(u)[h]]\,[k]\|_{X_2} \\ &\quad \le C\,\|[G(u) + dG(u)[h]]\,[l]\|_{X_2}\,\|[F(u+h) - F(u) - dF(u)[h]]\,[k]\|_{X_2} \\ &\quad \le C_u\,(1 + \|h\|_Z)\,\|[F(u+h) - F(u) - dF(u)[h]]\,[k]\|_{X_2} = o\left(\|h\|_Z\right). \end{aligned} \tag{B.6}$$

Finally,

$$\begin{aligned} \|dG(u)[h,l]dF(u)[h,k]\|_{X_2} &\le \|dG(u)[h,l]\|_{X_2}\,\|dF(u)[h,k]\|_{X_2} \\ &\le C_u\,\|h\|_Z^2 = o\left(\|h\|_Z\right). \end{aligned} \tag{B.7}$$

The triangle inequality, along with (B.4) and (B.5), (B.6), and (B.7), implies that $d(FG)$ exists and is given by (B.2).

To conclude, we claim that $d(FG)(u) \in \mathcal{L}_{n+m+1}(Z, X_2)$ for each $u \in Z$. Indeed, note that the $m+n+1$-linearity of $d(FG)(u)$ follows easily from the linearity properties of $G(u), F(u), dF(u)$ and $dG(u)$. On the other hand, let $B^Z(0,1) = \{z \in Z : \|z\|_Z \le 1\}$ and $B = \left(B^Z(0,1)\right)^{1+m+n}$, then, for each $u \in Z$:

$$\begin{aligned} &\sup_{(h,k,l)\in B} \|dF(u)[h,k]G(u)[l]\|_{X_2} \\ &\qquad \le C \sup_{(h,k,l)\in B} \|dF(u)[h,k]\|_{X_2}\,\|G(u)[l]\|_{X_2} \\ &\qquad \le C \sup_{(h,k,l)\in B} \|dF(u)[h,k]\|_{X_2} \cdot \sup_{(h,k,l)\in B} \|G(u)[l]\|_{X_2} \\ &\qquad \le C\,\|dF(u)\|_{\mathcal{L}_{m+1}(Z,X_2)}\,\|G(u)\|_{\mathcal{L}_n(Z,X_2)} < \infty, \end{aligned} \tag{B.8}$$

and similarly,

$$\sup_{(h,k,l)\in B} \|F(u)[k]dG(u)[h,l]\|_{X_2} < \infty. \tag{B.9}$$

Putting together (B.8) and (B.9), we see that $d(FG)(u)$ is bounded for each $u \in Z$. □

Corollary B.3. *Let $F \in C^\infty(Z, X_2)$ and $G \in C^\infty(Z, X_2)$ or $G \in C^\infty(Z, \mathbb{R})$. Then, the map*

$$FG : Z \to X_2, \ \textit{given by } FG(u) = F(u)G(u), \ \textit{for all } u \in Z,$$

is smooth (i.e., $FG \in C^\infty(Z, X_2)$).

Proof. Since the sum of two smooth functions is smooth [4], the result follows by applying Lemma B.2 repeatedly starting with $n = m = 0$. □

Lemma B.4. *Let $n \in \mathbb{N}$ and $F^j \in C^\infty(Z, X_2)$ for $j = 1, 2, ..., n$. Then, the map $F : Z \to X_2^n, F(x) = \big(F^1(x), F^2(x), ..., F^n(x)\big)$ belongs to $C^\infty(Z, X_2^n)$.*

Proof. Direct computation. □

Lemma B.5. *The map $D_\Phi W : Z \to X_2^3$, defined by*

$$D_\Phi W(\varphi, \sigma_R, \sigma_I, \lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) = \begin{pmatrix} \lambda_\varphi\left(\varphi^2 - 1\right)\varphi + \beta\varphi|\sigma|^2 \\ \lambda_\sigma(|\sigma|^2 - m_\sigma^2)\sigma_R + \beta\varphi^2\sigma_R \\ \lambda_\sigma(|\sigma|^2 - m_\sigma^2)\sigma_I + \beta\varphi^2\sigma_I \end{pmatrix},$$

belongs to $C^\infty(Z, X_2^3)$ and therefore to $C^\infty(Z, X_0^3)$.

Proof. Consider the following maps from Z to X_2, and label an arbitrary element of Z as $U = (U^1, U^2, ..., U^7)$:

$$F_j : Z \to X_2, F_j(U) = U^j, \text{ for each } j = 1, 2, ..., 7.$$

The maps F_j are all in $C^\infty(Z, X_2)$ for $j = 1, 2, 3$ and in $C^\infty(Z, \mathbb{R})$ for $j = 4, 5, 6, 7$. Also, the maps $\partial_\varphi W, \partial_{\sigma_R} W, \partial_{\sigma_I} W : Z \to X_2$ can be expressed as sums of terms, each one being a product of some of the F_j's, which appear in such a way that Corollary B.3 can be applied to each of them. We conclude that the maps $\partial_\varphi W, \partial_{\sigma_R} W, \partial_{\sigma_I} W : Z \to X_2$ belong to $C^\infty(Z, X_2)$. This fact together with Lemma B.4 allows one to conclude that $D_\Phi W \in C^\infty(Z, X_2^3)$. □

Lemma B.6. *The map $G : Z \to X_0^3$, defined by*

$$G(\Phi, \Pi) = -\Phi'',$$

belongs to $C^\infty(Z, X_0^3)$.

Proof. Since G is linear on Φ and independent of Π, we have that $dG(\Phi, \Pi)[(\Psi, \tilde{\Pi})] = G(\Psi, \tilde{\Pi})$ for all $(\Phi, \Pi), (\Psi, \tilde{\Pi}) \in Z$ and $d^k G \equiv 0$ for all $k \geq 2$. □

B.2 ADDITIONAL RESULTS

Lemma B.7 (Exponential Decay). *Let $v : \mathbb{R} \to \mathbb{R}$ be a non-negative function such that*

$$\begin{aligned} &v(z) \to 0, && \text{as } z \to \infty, \\ &v''(z) \geq a^2 v(z) - De^{-cz}, && \text{for all } z \geq l, \end{aligned}$$

for some constants $a, c, l > 0$, and $D \geq 0$, where $a > c$. Then, there exists a constant $C > 0$ such that

$$v(z) \leq \begin{cases} Ce^{-az}, & \text{if } D = 0 \\ Ce^{-cz}, & \text{if } D \neq 0, \end{cases} \qquad \text{for all } z \geq 0.$$

Proof. We apply a similar argument to that of Proposition 2.4 from section 2.3 of [3]. Let $L > l$ and w be the solution to $w'' = a^2 w - De^{-cz}$ with the Dirichlet conditions $w(l) = v(l)$ and $w(L) = v(L)$. Then,

$$v(z) \leq w(z), \qquad \text{for all } z \in (l, L). \tag{B.10}$$

To see this, let $f := v - w$ and suppose on the contrary that

$$f(z) = v(z) - w(z) > 0, \text{ for some } z \in (l, L). \tag{B.11}$$

Since $f(l) = f(L) = 0$, condition (B.11) would imply that f attains a maximum value over $[l, L]$ at some $z^* \in (l, L)$ and therefore $f''(z^*) < 0$. However, the hypothesis on v implies that

$$f'' = v'' - w'' \geq a^2 [v - w] > 0, \text{ over } (l, L),$$

leading to a contradiction and proving that (B.10) has to hold. Now, a direct computation shows that

$$\begin{aligned} w(z) = &\left[v(l) - Ee^{-cl}\right] \frac{\sinh[a(L - z)]}{\sinh[a(L - l)]} \\ &+ \left[v(L) - Ee^{-cL}\right] \frac{\sinh[a(z - l)]}{\sinh[a(L - l)]} + Ee^{-cz}, \text{ for all } z \in \mathbb{R}, \end{aligned}$$

where $E = D/(a^2 - c^2) \geq 0$. As a result, using (B.10) and the fact that $v(z) \to 0$ as $z \to \infty$, we deduce that in the limit $L \to \infty$ we have that

$$v(z) \leq \left[v(l) - Ee^{-cl}\right] e^{al} e^{-az} + Ee^{-cz}, \qquad \text{for all } z \in [l, \infty).$$

In particular, if $D = 0$, we have that $E = 0$ and thus

$$v(z) \leq \left(\max_{x\in[0,l]} v(x)\right) e^{a(l-z)}, \qquad \text{for all } z \geq 0.$$

Also, if $D > 0$:

$$v(z) \leq Ce^{-cz}, \qquad \text{for all } z \geq 0, \tag{B.12}$$

where $C := \max\left(v(l)e^{al} + E, \left(\max_{z\in[0,l]} v(z)\right) e^{cl}\right)$. □

Remark B.8. If $a = c$ and $D > 0$, we have that

$$\begin{aligned} w(z) &= \left[v(l) - \frac{D}{2a} l e^{-al}\right] \frac{\sinh\left[a\left(L-z\right)\right]}{\sinh\left[a\left(L-l\right)\right]} \\ &\quad + \left[v(L) - \frac{D}{2a} L e^{-aL}\right] \frac{\sinh\left[a\left(z-l\right)\right]}{\sinh\left[a\left(L-l\right)\right]} + \frac{D}{2a} z e^{-az}, \text{ for all } z \in [l, L], \end{aligned}$$

from which we deduce that

$$v(z) \leq \left(v(l)e^{al} + \frac{D}{2a} z\right) e^{-az}, \qquad z \in [l, \infty)$$

and thus

$$v(z) \leq C(1+z)e^{-az} \qquad \text{for all } z \geq 0,$$

where $C := \max\left(\left(\max_{z\in[0,l]} v(z)\right) e^{al}, \frac{D}{2a}\right)$.

Lemma B.9. *$\mathcal{H} \simeq (\tanh, 0) + H^1(\mathbb{R}; \mathbb{R} \times \mathbb{C})$, where $\mathcal{H}$ is defined in (2.2).*

Proof. Denote $\tilde{\mathcal{H}} := (\tanh, 0) + H^1(\mathbb{R}; \mathbb{R} \times \mathbb{C})$.

($\subset$): Let $(\varphi, \sigma) \in \mathcal{H}$. Then, $\sigma \in H^1(\mathbb{R}; \mathbb{C})$ and it suffices to show that $\varphi - \tanh \in H^1(\mathbb{R})$. To this end, note that the limiting behaviour of φ implies that $\varphi(x) + \tanh(x) \to \pm 2$ as $x \to \pm\infty$, and we may therefore fix

$M > 0$ such that $|\varphi(x) + \tanh(x)| > 1$ for all $|x| > M$. For any such M, we have that

$$\begin{aligned}\int_{|x|>M} (\varphi - \tanh)^2 &\leq \int_{|x|>M} (\varphi^2 - \tanh^2)^2 \\ &\leq 2\left[\int_{|x|>M} \left(\varphi^2 - 1\right)^2 + \int_{|x|>M} \left(\tanh^2 - 1\right)^2\right] < \infty.\end{aligned}$$

On the other hand, the Sobolev embedding $\dot{H}^1(\mathbb{R}) \hookrightarrow C^{0,1/2}(\mathbb{R})$ and the fact that $\varphi \in \dot{H}^1(\mathbb{R})$ implies that

$$\int_{|x|<M} (\varphi - \tanh)^2 < \infty.$$

Therefore, $(\varphi - \tanh) \in L^2(\mathbb{R})$. Finally, since the first derivatives of φ and of tanh are in $L^2(\mathbb{R})$, we conclude that $(\varphi - \tanh) \in H^1(\mathbb{R})$.

($\supset$): Let $(\varphi, \sigma) = (\tanh + \tilde{\varphi}, \sigma) \in \tilde{\mathcal{H}}$, so that $\tilde{\varphi} \in H^1(\mathbb{R})$. We want to show that $\varphi(x) \to \pm 1$ as $x \to \pm\infty$ and $(\varphi^2 - 1) \in L^2(\mathbb{R})$. Thanks to the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow C^{0,1/2}(\mathbb{R})$, we know that $\tilde{\varphi}$ is bounded and $\tilde{\varphi}(x) \to 0$ as $|x| \to \infty$. Therefore, using the fact that $\tanh(x) \to \pm 1$ as $x \to \pm\infty$, we readily conclude the desired limiting behaviour of φ. Also,

$$\begin{aligned}\varphi^2 - 1 &= \tilde{\varphi}^2 + 2\tilde{\varphi}\tanh + \tanh^2 - 1 \\ &= (\tilde{\varphi} + 2\tanh)\,\tilde{\varphi} + \left(\tanh^2 - 1\right) \in L^2(\mathbb{R}),\end{aligned}$$

since $(\tilde{\varphi} + 2\tanh)$ is bounded and $\tilde{\varphi}$ and $\left(\tanh^2 - 1\right)$ are in $L^2(\mathbb{R})$.

$\square$

C

LAWS OF MOTION: AUXILIARY RESULTS

C.1 MATRIX ALGEBRA

In what follows, we will repeatedly make use of the following notation for convenience:

Definition C.1. Let $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. The Kronecker product of A and B is defined as the block matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix} \in \mathbb{R}^{mp\times nq}. \tag{C.1}$$

We refer the reader to [13, Section 3.6 and Problem 10.39] for a list of many useful properties of the Kronecker product.

Lemma C.2. *Let $k \in \mathbb{N}, v \in \mathbb{R}^{k\times 1}$, and $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Then, the matrix*

$$A := \begin{pmatrix} 2a_1 I_{k\times k} & b_1 I_{k\times k} & 0_{k\times 1} & 0_{k\times 1} \\ I_{k\times k} & 0_{k\times k} & 0_{k\times 1} & 0_{k\times 1} \\ 2\,(a_1 - a_2)\,v^T & (b_1 - b_2)\,v^T & 2a_2 & b_2 \\ 0_{1\times k} & 0_{1\times k} & 1 & 0 \end{pmatrix} \in \mathbb{R}^{2(k+1)\times 2(k+1)}, \tag{C.2}$$

is diagonalizable and has only real eigenvalues if

$$\Delta_i := a_i^2 + b_i > 0, \qquad \text{for } i = 1, 2. \tag{C.3}$$

Furthermore, whenever (C.3) is satisfied, a set of linearly independent vectors of A is given by (C.5)-(C.6).

Proof. We write the matrix (C.2) as

$$A = \begin{pmatrix} U \otimes I_{k\times k} & 0_{2k\times 2} \\ V & W \end{pmatrix},$$

where

$$U = \begin{pmatrix} 2a_1 & b_1 \\ 1 & 0 \end{pmatrix}, V = \begin{pmatrix} 2\left(a_1 - a_2\right) v^T & \left(b_1 - b_2\right) v^T \\ 0_{1\times k} & 0_{1\times k} \end{pmatrix} \text{ and } W = \begin{pmatrix} 2a_2 & b_2 \\ 1 & 0 \end{pmatrix}.$$

Using the properties of the Kronecker product $\otimes$ listed in [13, Section 3.6 and Problem 10.39] and of block triangular matrices, we obtain:

$$\begin{aligned} \det\left(A - \lambda I_{2(k+1)\times 2(k+1)}\right) &= \det(U \otimes I_{k\times k} - \lambda I_{2k\times 2k}) \det(W - \lambda I_{2\times 2}) \\ &= \det((U - \lambda I_2) \otimes I_{k\times k}) \det(W - \lambda I_{2\times 2}) \\ &= \det(U - \lambda I_2)^k \det(W - \lambda I_2) \\ &= \left[\lambda^2 - 2a_1\lambda - b_1\right]^k \left[\lambda^2 - 2a_2\lambda - b_2\right] \\ &= (\lambda - \lambda_1^+)^k (\lambda - \lambda_1^-)^k (\lambda - \lambda_2^-)(\lambda - \lambda_2^+), \end{aligned}$$

where

$$\lambda_i^\pm(\Phi) = a_i \pm \sqrt{a_i^2 + b_i}, \qquad \text{for } i = 1, 2.$$

The matrix $U(\Phi) \otimes I_k$ has eigenvectors (each of which is a shifted version of the previous one):

$$u_{1,1}^\pm := \begin{pmatrix} \lambda_1^\pm \\ 0_{(k-1)\times 1} \\ 1 \\ 0_{(k-1)\times 1} \end{pmatrix}, u_{1,2}^\pm := \begin{pmatrix} 0 \\ \lambda_1^\pm \\ 0_{(k-1)\times 1} \\ 1 \\ 0_{(k-2)\times 1} \end{pmatrix}, \ldots, u_{1,k}^\pm := \begin{pmatrix} 0_{(k-1)\times 1} \\ \lambda_1^\pm \\ 0_{(k-1)\times 1} \\ 1 \end{pmatrix}, \tag{C.4}$$

which are linearly independent as long as $\lambda_1^- \neq \lambda_1^+$ (if $k = 1$, by the above formula means that $u_{1,1}^\pm = \binom{\lambda_1^\pm}{1}$). Additionally, due to the relationship of the entries of V with those of U and W, a computation shows that the following is a set of $2k$ linearly independent vectors of A as long as $\lambda_1^- \neq \lambda_1^+$:

$$v_{1,j}^{\pm} := \begin{pmatrix} u_{1,j}^{\pm} \\ \lambda_1^{\pm} v^j \\ v^j \end{pmatrix}, \qquad j \in \{1, 2, ..., k\}. \tag{C.5}$$

On the other hand, the vectors

$$v_2^{\pm} := \begin{pmatrix} 0_{2k\times 1} \\ \lambda_2^{\pm} \\ 1 \end{pmatrix} \tag{C.6}$$

are also eigenvectors of A which are linearly independent as long as $\lambda_2^- \neq \lambda_2^+$. Therefore, equations (C.5) and (C.6) show that the conditions

$$\lambda_1^- \neq \lambda_1^+ \text{ and } \lambda_2^- \neq \lambda_2^+ \tag{C.7}$$

are sufficient for A to have $2k+2$ linearly independent eigenvectors, and thus to be diagonalizable. The conclusion of the theorem follows since the conditions $a_i^2 + b_i > 0$ for $i = 1, 2$ ensure that (C.7) hold and that the eigenvalues of A are real. $\square$

Lemma C.3. *The matrix $S^0(\Phi)$ appearing in (C.8) is positive definite whenever $c_i^{00}(\Phi) < 0$ and $\overline{c}_i(\Phi) \succ 0$ for $i = 1, 2$.*

Proof. Write the matrix $S^0(\Phi)$ from Equation 3.98 as:

$$S^0(\Phi) = \underbrace{\begin{pmatrix} -c_1^{00} I_k & 0_{k\times kn} & 0_{k\times 1} & 0_{k\times n} \\ 0_{kn\times k} & \overline{c}_1 \otimes I_k & 0_{kn\times 1} & 0_{kn\times n} \\ 0_{1\times k} & 0_{1\times kn} & 0 & 0_{1\times n} \\ 0_{n\times k} & 0_{n\times kn} & 0_{n\times 1} & 0_{n\times n} \end{pmatrix}}_{B_0(\Phi)} + \underbrace{(-c_2^{00}) \begin{pmatrix} vv^T & 0_{k\times kn} & -v & 0_{k\times n} \\ 0_{kn\times k} & 0_{kn\times kn} & 0_{kn\times 1} & 0_{kn\times n} \\ -v^T & 0_{1\times kn} & 1 & 0_{1\times n} \\ 0_{n\times k} & 0_{n\times kn} & 0_{n\times 1} & 0_{n\times n} \end{pmatrix}}_{B_1(\Phi)}$$

$$+ \underbrace{\begin{pmatrix} 0_{k\times k} & 0_{k\times kn} & 0_{k\times 1} & 0_{k\times n} \\ 0_{kn\times k} & \overline{c}_2 \otimes vv^T & 0_{kn\times 1} & -\overline{c}_2 \otimes v \\ 0_{1\times k} & 0_{1\times kn} & 0 & 0_{1\times n} \\ 0_{n\times k} & -\overline{c}_2 \otimes v^T & 0_{n\times 1} & \overline{c}_2 \end{pmatrix}}_{B_2(\Phi)}.$$

Let $k, n \in \mathbb{N}$ and $u \in \mathbb{R}^{(k+1)(n+1)}$ be an arbitrary non-zero vector and denote by e_j the j^{th} standard basis vector of $\mathbb{R}^{(k+1)(n+1)}$. Write $u = v_0 + v_1 + v_2$ where $v_0 = \sum_{i=1}^{kn+1} u^i e_i$, $v_1 = u^{kn+2} e_{kn+2}$, and $v_2 = \sum_{i=kn+3}^{(k+1)(n+1)} u^i e_i$.

Case 1: if $v_0 \neq 0$, then

$$\begin{aligned} u^T S^0(\Phi) u &= u^T B_0(\Phi) u + u^T B_1(\Phi) u + u^T B_2(\Phi) u \\ &= v_0^T B_0(\Phi) v_0 + u^T B_1(\Phi) u + u^T B_2(\Phi) u > 0, \end{aligned}$$

which follows since $v_0^T B_0(\Phi) v_0 > 0$ whenever $c_1^{00}(\Phi) < 0$ and $\bar{c}_1(\Phi) \succ 0$ (given that, for any symmetric square matrix C, we have that $C \otimes I_k \succ 0$ if and only if $C \succ 0$), and since $B_1(\Phi) + B_2(\Phi)$ is positive semi-definite by Lemma C.5.

Case 2: if $v_0 = 0$, then

$$\begin{aligned} u^T S^0(\Phi) u &= u^T B_0(\Phi) u + u^T B_1(\Phi) u + u^T B_2(\Phi) u \\ &= v_1^T B_0(\Phi) v_1 + v_2^T B_2(\Phi) v_2 \\ &= -c_2^{00} \left(u^{kn+2}\right)^2 + v_2^T \bar{c}_2 v_2 > 0. \end{aligned}$$

The second line follows from the first because of the structure of the matrices $B_0(\Phi)$, $B_1(\Phi)$ and $B_2(\Phi)$. □

Lemma C.4. *Let $m, n, \in \mathbb{N}$, $A \in \mathbb{R}^{m \times m}$ and $v \in \mathbb{R}^{n \times 1}$, then*

$$(A \otimes v)\, A = A^2 \otimes v.$$

Proof. Let $i, j \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$. Then, denoting the j^{th} row (resp. column) of a matrix M by M^j (resp. M_j) and (correspondingly) its $(i, j)^{th}$ entry by M_j^i, we have that:

$$\begin{aligned} [(A \otimes v) A]_j^{(i-1)n+k} &= (A \otimes v)^{(i-1)n+k} A_j \\ &= \left(A^i v_k\right) A_j \\ &= \left(A^i A_j\right) v_k \\ &= \left(A^2\right)_j^i v_k \\ &= \left(A^2 \otimes v\right)_j^{(i-1)n+k}. \end{aligned}$$

□

Lemma C.5. *The matrices* $B_1(\Phi)$ *and* $B_2(\Phi)$ *appearing in* (C.8) *are positive semi-definite if* $c_i^{00}(\Phi) < 0$ *and* $\bar{c}_i(\Phi) \succ 0$ *for* $i = 1, 2$.

Proof. $B_1(\Phi) \succeq 0$ under the given assumptions, as can be seen by writing

$$B_1(\Phi) = -c_2^{00}(\Phi) w w^T, \qquad \text{where} \qquad w = \begin{pmatrix} v \\ 0_{kn\times 1} \\ -1 \\ 0_{n\times 1} \end{pmatrix}.$$

On the other hand, let $u \in \mathbb{R}^{(k+1)(n+1)}$ be an arbitrary non-zero vector, $p = \left(u_{k+1}, u_{k+2}, \ldots, u_{k(n+1)}\right) \in \mathbb{R}^{kn}$, and $q = \left(u_{k(n+1)+2}, u_{k(n+1)+3}, \ldots, u_{(k+1)(n+1)}\right) \in \mathbb{R}^n$. Furthermore, let $\bar{c} \in \mathbb{R}^n$ be the square root of $\bar{c}_2$ (i.e., $\bar{c}_2 = \bar{c}^2$). We have that

$$\begin{aligned} u^T B_2(\Phi) u &= p^T \left(\bar{c}_2 \otimes v v^T\right) p - p^T \left(\bar{c}_2 \otimes v\right) q - q^T \left(\bar{c}_2 \otimes v^T\right) p + q^T \bar{c}_2 q \\ &= p^T \left(\bar{c} \otimes v\right) \left(\bar{c} \otimes v^T\right) p - 2p^T (\bar{c}_2 \otimes v) q + q^T \bar{c}_2 q \\ &= \left[\left(\bar{c} \otimes v^T\right) p\right]^T \left[\left(\bar{c} \otimes v^T\right) p\right] - 2p^T (\bar{c} \otimes v) \bar{c} q + q^T \bar{c}_2 q \\ &= \left[\left(\bar{c} \otimes v^T\right) p - \bar{c} q\right]^T \left[\left(\bar{c} \otimes v^T\right) p - \bar{c} q\right] \geq 0, \end{aligned}$$

where we have used the fact that $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ for all matrices A, B, C, D of dimensions for which the products in the second identity are defined (see, e.g., [13]). Additionally, we have used the fact that $\bar{c}$ is symmetric throughout and Lemma C.4 in going from the second to the third line. □

C.2 REGULARITY OF SOLUTIONS TO THE LAWS OF MOTION

Lemma C.6. *Let* $p, r \in \mathbb{N}$, $T > 0$, *and*

$$\mathfrak{u} \in C\left([0,T]; H^s\left(\mathbb{R}^p; \mathbb{R}\right)\right) \cap C^1\left([0,T]; H^{s-1}\left(\mathbb{R}^p; \mathbb{R}\right)\right), \tag{C.8}$$

for some integer $s \geq r + 1 + [p/2]$.

Define $S_T := [0,T] \times \mathbb{R}^p$ *and* $u : S_T \to \mathbb{R}$ *via* $u(t,x) := \mathfrak{u}(t)(x)$ *for each* $(t,x) \in S_T$. *Then,*

$$D_x^\alpha \partial_t^m u \in C_b\left(S_T; \mathbb{R}\right) \qquad \text{for } m = 0, 1 \text{ and } |\alpha| \leq r - m. \tag{C.9}$$

Proof. Since $D_x^\alpha \partial_t^m u(t,x) = D_x^\alpha \partial_t^m \mathfrak{u}(t)(x)$ for all $(t,x) \in S_T$, we have

$$\begin{aligned}|D_x^\alpha \partial_t^m u(t,x) - D_x^\alpha \partial_t^m u(\tau,y)| &\le |D_x^\alpha \partial_t^m \mathfrak{u}(t)(x) - D_x^\alpha \partial_t^m \mathfrak{u}(t)(y)| \\ &\quad + |(D_x^\alpha \partial_t^m \mathfrak{u}(t) - D_x^\alpha \partial_t^m \mathfrak{u}(\tau))(y)| . \end{aligned} \tag{C.10}$$

Also, $\partial_t^m \mathfrak{u}(t) \in H^{s-m}(\mathbb{R}^p) \implies D_x^\alpha \partial_t^m \mathfrak{u}(t) \in H^{s-m-|\alpha|}(\mathbb{R}^p)$. Therefore, by the Sobolev embedding $H^k(\mathbb{R}^p;\mathbb{R}) \hookrightarrow C_b^{k-1-[p/2]}(\mathbb{R}^p;\mathbb{R})$ (see e.g., [19, Lemma 2.1], [7, Corollary 9.13]), we find that the condition $s \ge |\alpha| + m + 1 + [p/2]$ ensures that $D_x^\alpha \partial_t^m \mathfrak{u}(t)$ is continuous and bounded for each $t \in (0,T)$, and therefore

$$|D_x^\alpha \partial_t^m \mathfrak{u}(t)(x) - D_x^\alpha \partial_t^m \mathfrak{u}(t)(y)| \to 0 \text{ as } x \to y.$$

Additionally, under this condition on s, we have the following estimate for the second term on the right hand side of (C.10):

$$\begin{aligned}\sup_{y\in\mathbb{R}^p} |(D_x^\alpha \partial_t^m \mathfrak{u}(t) - D_x^\alpha \partial_t^m \mathfrak{u}(\tau))(y)| &\le C \left\| D_x^\alpha \partial_t^m \mathfrak{u}(t) - D_x^\alpha \partial_t^m \mathfrak{u}(\tau) \right\|_{H^{1+[p/2]}} \\ &\le C \left\| \partial_t^m \mathfrak{u}(t) - \partial_t^m \mathfrak{u}(\tau) \right\|_{H^{s-m}} ,\end{aligned}$$

since $1 + [p/2] + |\alpha| \le s - m$. Therefore, using the continuity of $\partial_t^m \mathfrak{u}$, we conclude that the second term on the right hand side of (C.10) tends to 0 as $t \to \tau$. On the other hand, the boundedness of $D_x^\alpha \partial_t^m u$ follows by the boundedness of $D_x^\alpha \partial_t^m u(t,\cdot)$ for each $y \in [0,T]$. Finally, note that since $s \ge r + 1 + [p/2]$ by assumption, then $|\alpha| \le r - m$ implies that $s \ge |\alpha| + m + 1 + [p/2]$, and thus (C.9) follows. □

Lemma C.7 (Regularity of Solutions)**.** *Let $r \in \mathbb{N}_{\ge 2}$, $s \ge r + [p/2]$ and (γ, θ) be a solution to (3.69) arising from Theorem 3.16 or Theorem 3.22 such that $\gamma(t,x) = \bar{\gamma}(t)(x)$ and $\theta(t,x) = \bar{\theta}(t)(x)$ for all $(t,x) \in [0,T] \times \mathbb{R}^p$, where*

$$\partial_a \bar{\gamma}^i, \partial_a \bar{\theta} \in S, \qquad (a,i) \in \{0,1,...,p\} \times \{1,...,k\},$$

with

$$S := C([0,T]; H^s(\mathbb{R}^p)) \cap C^1\left([0,T]; H^{s-1}(\mathbb{R}^p)\right).$$

Then, $\theta, \gamma^j \in C_b^r([0,T] \times \mathbb{R}^p)$ for $i = 1,2,...,k$ and $a = 0,...,p$.

Proof. In what follows, a, b denote indices running from 0 to p. Also, $S_T := [0,T] \times \mathbb{R}^p$ and we write $(t,x) = (t, x_1 ..., x_p) \in S_T$, $\partial_0 := \partial_t, \partial_{y_i} := \partial_{x_i}$, and $D = (\partial_0, \partial_1, ..., \partial_p)$.

Let $i \in \{1, ..., k\}$. Using $s \geq r + [p/2]$ together with Lemma C.6 and the relationship between (γ, θ) and $(\bar{\gamma}, \bar{\theta})$, we see that for $a = 0, ..., p$:

$$D_x^\alpha \partial_0^m \partial_a \gamma^i, D_x^\alpha \partial_0^m \partial_a \theta \in C_b\left(S_T; \mathbb{R}\right), \quad |\alpha| \leq r - 1 - m \text{ for } m = 0, 1. \tag{C.11}$$

In particular, all weak derivatives of $\partial_a \gamma^i$ and $\partial_a \theta$ of order less than or equal to $r-1$ with 1 or less time derivatives have a continuous and bounded representative for each $a \in \{0, 1..., p\}$. Also, it follows from (C.11) that $\partial_a \gamma^i$ and $\partial_a \theta$ are C_b^1 functions, and we conclude that the (continuous) functions θ and γ^i are also bounded using the identity

$$|f(t,x)| \leq \int_0^T |\partial_t f(\tau, x)| \, dt + |f(0,x)|, \qquad t \in [0,T],$$

for $f = \theta, \gamma^i$, and the fact that the regularity of the initial conditions on θ and γ^i imply that $\theta(0, \cdot)$ and $\gamma^i(0, \cdot)$ are bounded.

Let c_1^{ab} and c_2^{ab} be as in (3.78). It follows from the conclusion of Theorem 3.16 or Theorem 3.22 that c_1^{ab} and c_2^{ab} are bounded and have bounded derivatives up to certain order whenever the same conditions are met by $D\gamma^i$ and $D\theta$. Now, write

$$\partial_{00}\gamma = \sum_{(a,b) \neq (0,0)} c_1^{ab}(D\gamma, D\theta) \partial_{ab}\gamma, \tag{C.12a}$$

$$\partial_{00}\theta = -c_2^{ab}(D\gamma, D\theta) \left\langle \partial^c \theta \partial_c \gamma, \partial_{ab}\gamma \right\rangle + \sum_{(a,b) \neq (0,0)} c_1^{ab}(D\gamma, D\theta) \partial_{ab}\theta. \tag{C.12b}$$

Let $j \in \{1, ..., r-2\}$ and μ be any multiindex of size $|\mu| \leq r - 2 - j$. We argue using strong induction on j. In particular, starting with $j = 1$, differentiate (C.12a) with respect to $D_x^\mu \partial_0^j$ to find that the continuity and the boundedness of the weak derivative $D_x^\mu \partial_0^{j+2} \gamma$ follows from (C.11). Using this, the regularity of $D_x^\mu \partial_0^{j+2} \theta$ can be deduced by also differentiating (C.12b) with respect to $D_x^\mu \partial_0^j$. Thus all weak derivatives of order less than or equal to r with $j+2$ or less time derivatives of γ^i and θ are C_b functions for each a. Working our way upwards to $j = r-2$, we conclude that all weak derivatives of γ^i and θ of order 0 to r are bounded and continuous, and thus $\gamma^i, \theta \in C_b^r(S_T)$ for $a = 0, 1, ..., p$ by Lemma C.8. □

Lemma C.8. *Let $k, n \in \mathbb{N}$ and $f \in C(\mathbb{R}^n)$. Suppose that f has continuous weak derivatives of all orders up to and including k. Then, $f \in C^k(\mathbb{R}^n)$.*

Proof. Case $k = 1$.

Step 1: Let φ be a standard mollifer e.g.,

$$\varphi(x) = \begin{cases} C\exp\left(\frac{-1}{1-|x|^2}\right) & x \in B_1(0) \\ 0 & \text{else} \end{cases},$$

and $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(x/\varepsilon)$ for each $\varepsilon > 0$ and $x \in \mathbb{R}^n$. Denote by w_i the weak derivative of f with respect to the x_i variable for each $i \in \{1, 2, ..., n\}$, and define $g_\varepsilon := g * \varphi_\varepsilon$ for any continuous function g. Note that

1. $\partial_i f_\varepsilon = (w_i)_\varepsilon$.
2. If $g \in C(\mathbb{R}^n)$, then $g_\varepsilon \to g$ uniformly on compact sets.
3. $\left(f_{1/k}\right)_{k=1}^\infty$ is Cauchy in $C^1(K)$ for every compact set $K \subset \mathbb{R}^n$.

To see why the first item is true, note that

$$\partial_i f_\varepsilon = f * \partial_i \varphi_\varepsilon = w_i * \varphi_\varepsilon,$$

since w_i is the weak derivative of f with respect to x_i, and $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$.

To see why the second item is true, let $K \subset \mathbb{R}^n$ be compact and pick $\varepsilon, R > 0$ so that $\operatorname{supp}(\varphi_\varepsilon) \subset K \subset B_R(0)$. Let $x \in K$ and write

$$\begin{aligned} f(x) - f_\varepsilon(x) &= \int_{\mathbb{R}^n} [f(x) - f(y)]\,\varphi_\varepsilon(x - y)\,dy \\ &= \int_{B_R(0)} [f(x) - f(x - \varepsilon y)]\,\varphi(y)\,dy, \end{aligned}$$

from which it follows that (since $\int_{B_R(0)} \varphi = 1$),

$$\sup_{x \in K} |f(x) - f_\varepsilon(x)| \leq \sup_{\substack{x \in K \\ y \in B_R(0)}} |f(x) - f(x - \varepsilon y)|.$$

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x,y) := f(x) - f(x-\varepsilon y)$. Then, F is continuous and we have that

$$\lim_{\varepsilon \to 0} \sup_{\substack{x \in K \\ y \in B_R(0)}} |f(x) - f(x-\varepsilon y)| = \lim_{\varepsilon \to 0} \max_{\substack{x \in K \\ y \in \overline{B_R(0)}}} |F(x, \varepsilon y)| = F(x,0) = 0.$$

To see why the third item is true, simply write

$$\begin{aligned} \left\| f_{1/n} - f_{1/m} \right\|_{C^1(K)} = \sup_{x \in K} \left| f_{1/n}(x) - f_{1/m}(x) \right| \\ + \sum_{i=1}^{n} \sup_{x \in K} \left| (w_i)_{1/n}(x) - (w_i)_{1/m}(x) \right|, \end{aligned}$$

where we have used item 2 to write the second term in the right hand side. The result then follows by the second item, which implies that $\left((w_i)_{1/k}\right)_{k=1}^{\infty}$ and $\left(f_{1/k}\right)_{k=1}^{\infty}$ are Cauchy in $C(K)$.

Step 2: conclusion. By the first step, there exists a function $F \in C^1(K)$ such that $f_{1/n} \to F$ and $(w_i)_{1/n} \to \partial_i F$ uniformly on K for each $i \in \{1, 2, ..., n\}$. By item 1 and the uniqueness of limits, we conclude that $F = f$ and $\partial_i F = \partial_i f = w_i$, for each $i \in \{1, 2, ..., n\}$, and thus $f \in C^1(K)$. Since the above argument holds for every compact set $K \subset \mathbb{R}^n$, we conclude that $f \in C^1(\mathbb{R}^n)$.

Cases $k > 1$: suppose that the statement holds for some $k \geq 1$. Let f be a function whose weak derivatives of order less than or equal to $k+1$ are continuous. By the induction hypothesis, $f \in C^k(\mathbb{R}^n)$. As a result, we have that $D^\alpha f \in C(\mathbb{R}^n)$ for any multiindex α with $|\alpha| = k$. This, together with the fact that the weak derivatives of f of order $k+1$ are continuous, implies that $D^\alpha f \in C^1(\mathbb{R}^n)$ by the argument for the case $k = 1$. Since the choice of α was arbitrary, we conclude that $f \in C^{k+1}(\mathbb{R}^n)$. $\square$

D

APPROXIMATE SOLUTIONS: AUXILIARY RESULTS

D.1 INDUCTION STEP COMPUTATIONS

Let (Γ, θ) be any C^2 solution to Equation 3.35 and $k \geq 1$. As before, let

$$\Phi_0[\rho](y,\zeta) := \Phi\left(\Pi_{\rho(y)}, \zeta\right), \text{ and } \rho_k(y) := \langle \theta + \varepsilon\alpha_k, \theta + \varepsilon\alpha_k \rangle_{\Gamma_{\varepsilon h_k}}(y), \tag{D.1}$$

where Π_p and Φ are as in Equation 2.120 and Theorem 2.3, respectively. Define

$$\mathcal{Y}_k := e^{i\theta_k/\varepsilon} \ltimes V_k \qquad \text{and} \qquad \zeta_k := \zeta - h_k,$$

with

$$\begin{aligned} \theta_k(y,\zeta) &:= \theta(y) + \varepsilon\alpha_k(y) + \varepsilon^2 \zeta I_{\langle \theta, h_k \rangle}(y) \\ V_k(y,\zeta) &:= \Phi_0[\rho_k](y,\zeta) + \varepsilon P_k(y,\zeta), \end{aligned} \tag{D.2}$$

where α_k, h_k, and P_k change according to

$$\begin{cases} \alpha_{k+1} = \alpha_k + \varepsilon^k \alpha \\ h_{k+1} = h_k + \varepsilon^k h \\ P_{k+1} = P_k + \varepsilon^{k+1} P \\ \alpha_1 = 0, h_1 = 0, P_1 = \Phi_1 + \varepsilon\Phi_2, \end{cases} \tag{D.3}$$

and,

$$\begin{aligned} \Phi_1 &= \mathfrak{S}[-G_1[\Gamma,\theta]], && G_1[\Gamma,\theta] \text{ as defined in } (4.26), \\ \Phi_2 &= \mathfrak{S}[-G_2], && G_2 \text{ as defined in } (4.33). \end{aligned}$$

Note that

$$\begin{aligned}
\theta_{k+1} &= \theta_k + \varepsilon^{k+1}(\alpha + \varepsilon\zeta\beta), && \text{where } \beta = I_{\langle\theta,h\rangle},\\
\rho_{k+1} &= \rho_k + \varepsilon^{k+1}R_\rho + O(\varepsilon^{k+3}), && \text{where } R_\rho = h\mathrm{II}_{\langle\theta,\theta\rangle} + 2I_{\langle\theta,\alpha\rangle},\\
\Phi_0[\rho_{k+1}] &= \Phi_0[\rho_k] + \varepsilon^{k+1}R_\rho\partial_\rho\Phi_0[\rho] + O(\varepsilon^{k+3}),\\
V_{k+1} &= V_k + \varepsilon^{k+1}R_\rho\partial_\rho\Phi_0[\rho] + \varepsilon^{k+2}P + O(\varepsilon^{k+3}),\\
\mathcal{Y}_{k+1} &= e^{i\varepsilon^k C} \ltimes \mathcal{Y}_k + \varepsilon^{k+1}e^{i\theta_{k+1}/\varepsilon} \ltimes (R_\rho\partial_\rho\Phi_0[\rho] + \varepsilon P) + O\left(\varepsilon^{k+3}\right),\\
\zeta_{k+1} &= \zeta - h_k - \varepsilon^k h = \zeta_k + \varepsilon^k h.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\rho_k &= \rho + \varepsilon h_k \mathrm{II}_{\langle\theta,\theta\rangle} + 2\varepsilon I_{\langle\theta,\alpha_k\rangle} + O(\varepsilon^2) = \rho + O(\varepsilon^2)\\
V_k &= \Phi_0[\rho] + \varepsilon\Phi_1 + O(\varepsilon^2)\\
\partial_\zeta^m\mathcal{Y}_k &= e^{i\theta_k/\varepsilon} \ltimes \left(\partial_\zeta^m\Phi_0[\rho] + \varepsilon\partial_\zeta^m\Phi_1\right) + O(\varepsilon^2)\\
\partial_a\mathcal{Y}_k &= e^{i\theta_k/\varepsilon} \ltimes \left(\partial_a V_k + \frac{\partial_b\theta}{\varepsilon}\mathcal{P}_\sigma(i\Phi_0[\rho]) + (\partial_b\theta)\,\mathcal{P}_\sigma(i\Phi_1)\right) + O(\varepsilon^2)\\
&= e^{i\theta_k/\varepsilon} \ltimes \left(\frac{\partial_a\theta}{\varepsilon}\mathcal{P}_\sigma(i\Phi_0[\rho]) + (\partial_a\rho)\partial_\rho\Phi_0[\rho] + (\partial_a\theta)\,\mathcal{P}_\sigma(i\Phi_1) + \varepsilon\partial_a\Phi_1\right) + O(\varepsilon^2)\\
\partial_{\zeta a}\mathcal{Y}_k &= e^{i\theta_k/\varepsilon} \ltimes \left(\frac{\partial_a\theta}{\varepsilon}\mathcal{P}_\sigma\left(i\partial_\zeta\Phi_0[\rho]\right) + (\partial_a\theta)\,\mathcal{P}_\sigma(i\partial_\zeta\Phi_1) + (\partial_a\rho)\,\partial_{\zeta p}\Phi_0[\rho]\right) + O(\varepsilon)\\
\partial_\zeta\theta_k &= \varepsilon^2 I_{\langle\theta,h_k\rangle} = O(\varepsilon^3)\\
\partial_a\theta_k &= \partial_a\theta + \varepsilon\partial_a\alpha_k + \varepsilon^2\zeta\partial_a I_{\langle\theta,h_k\rangle}\\
\partial_\zeta\partial_a\theta_k &= \varepsilon^2\partial_a I_{\langle\theta,h_k\rangle} = O\left(\varepsilon^3\right).
\end{aligned}$$

Using the above identities, we obtain the following relations:

$$\begin{aligned}
&e^{-i\theta_{k+1}(y,\zeta_{k+1})/\varepsilon} \ltimes D_\Phi W\left(\mathcal{Y}_{k+1}(y,\zeta_{k+1});\Pi\right)\\
&\quad= \Big[e^{-\theta_k/\varepsilon} \ltimes D_\Phi W(\mathcal{Y}_k;\Pi) + \varepsilon^{k+1}R_\rho D_\Phi^2 W\left(\Phi_0[\rho];\Pi\right)\partial_\rho\Phi_0[\rho]\\
&\quad+ \varepsilon^{k+2}R_\rho D_\Phi^3 W\left(\Phi_0[\rho];\Pi\right)[\Phi_1,\partial_\rho\Phi_0[\rho]] + \varepsilon^{k+2}D_\Phi^2 W\left(\Phi_0[\rho];\Pi\right)P\Big]\Big|_{(y,\zeta_{k+1})} + O(\varepsilon^{k+3}).\\
&e^{-i\theta_{k+1}(y,\zeta_{k+1})} \ltimes \partial_\zeta^2\left[(\mathcal{Y}_{k+1})_{h_{k+1}}\right]\\
&\quad= \left(e^{-i\theta_k/\varepsilon} \ltimes \partial_\zeta^2\mathcal{Y}_k\right)_{h_{k+1}} + \varepsilon^{k+1}\left[2\beta\mathcal{P}_\sigma\left(i\partial_\zeta\Phi_0[\rho]\right) + R_\rho\partial_\zeta^2\partial_\rho\Phi_0[\rho]\right]_{h_{k+1}}\\
&\qquad+ \varepsilon^{k+2}\left[2\beta\mathcal{P}_\sigma\left(i\partial_\zeta\Phi_1\right) + \partial_\zeta^2 P\right]_{h_{k+1}} + O\left(\varepsilon^{k+3}\right).
\end{aligned}$$

$$
\begin{aligned}
&e^{-i\theta_{k+1}(y,\zeta_{k+1})} \ltimes H_{\Gamma_z}(y)\partial_\zeta \left[(\mathcal{Y}_{k+1})_{h_{k+1}}\right](y,\zeta) \\
&\qquad = H_{\Gamma_{\varepsilon(\zeta_{k+1}+h_k)}} \left[e^{-i\theta_k/\varepsilon} \ltimes \partial_\zeta \mathcal{Y}_k\right](y,\zeta_{k+1}) \\
&\qquad + \varepsilon^{k+1}\left[h\mathrm{I\!I}_\Gamma \partial_\zeta \Phi_0[\rho] + \mathrm{I}_\Gamma \beta \mathcal{P}_\sigma\left(i\Phi_0[\rho]\right) + R_\rho \mathrm{I}_\Gamma \partial_{p\zeta}\Phi_0[\rho]\right]_{h_{k+1}} + O\left(\varepsilon^{k+2}\right).
\end{aligned} \tag{D.5}
$$

$$
\begin{aligned}
&e^{-i\theta_{k+1}(y,\zeta_{k+1})/\varepsilon} \ltimes \Box_{\Gamma_z}\left[\left(e^{i\theta_{k+1}/\varepsilon} \ltimes V_k\right)_{h_{k+1}}\right](y,\zeta) = \\
&\quad e^{-i\theta_k(y,\zeta_{k+1})/\varepsilon} \ltimes \left[-g^{ab}(y,z)\left(\partial_{ab}\left[(\mathcal{Y}_k)_{h_k}\right]\right)_{\varepsilon^k h} - B^b(y,z)\left(\partial_b\left[(\mathcal{Y}_k)_{h_k}\right]\right)_{\varepsilon^k h}\right] \\
&\quad + \varepsilon^{k-1}\left[2\beta\mathcal{P}_\sigma\left(i\partial_\zeta \Phi_0[\rho]\right) + 2I_{\langle\theta,\alpha\rangle}\mathcal{P}_\sigma\left(\Phi_0[\rho]\right)\right] \\
&\quad + \varepsilon^k\left[-\left(\Box_\Gamma h\right)\partial_\zeta \Phi_0[\rho] + \left(\Box_\Gamma \alpha\right)\mathcal{P}_\sigma\left(i\Phi_0[\rho]\right) + 2I_{\langle\theta,\beta\rangle}\mathcal{P}_\sigma\left(\zeta\Phi_0[\rho]\right)\right] \\
&\quad + \varepsilon^k\left[2\mathrm{I\!I}_{\langle\theta,h\rangle}\mathcal{P}_\sigma\left(i\zeta\partial_\zeta\Phi_0[\rho]\right) + 2I_{\langle\rho,h\rangle}\partial_{p\zeta}\Phi_0[\rho]\right] \\
&\quad + \varepsilon^k\left[2\mathrm{I\!I}_{\langle\theta,\alpha\rangle}\mathcal{P}_\sigma\left(\zeta\Phi_0[\rho]\right) - 2I_{\langle\rho,\alpha\rangle}\mathcal{P}_\sigma\left(i\partial_\rho\Phi_0[\rho]\right)\right] \\
&\quad + \varepsilon^k\left[2\beta\mathcal{P}_\sigma\left(i\partial_\zeta\Phi_1\right) + 2I_{\langle\theta,\alpha\rangle}\mathcal{P}_\sigma\left(\Phi_1\right)\right] + O\left(\varepsilon^{k+1}\right).
\end{aligned} \tag{D.6}
$$

Additionally,

$$
\begin{aligned}
\mathfrak{g}^{ab}(y,z) &= \mathfrak{g}^{ab}\left(y,\varepsilon\left(\zeta_{k+1}+h_k\right)\right) + \varepsilon^{k+1}h\,\partial_z\mathfrak{g}^{ab}(y,0) \\
&\qquad + \varepsilon^{k+2}h\left(\zeta_{k+1}+h_k\right)\partial_z^2\mathfrak{g}^{ab}(y,0) + O\left(\varepsilon^{k+3}\right), \\
B^b(y,z) &= B^b\left(y,\varepsilon\left(\zeta_{k+1}+h_k\right)\right) + \varepsilon^{k+1}h\,\partial_z B^b(y,0) + O\left(\varepsilon^{k+2}\right), \\
H_{\Gamma_z}(y) &= H_{\Gamma_{\varepsilon(\zeta_{k+1}+h_k)}} + \varepsilon^{k+1}h\mathrm{I\!I}_\Gamma + O\left(\varepsilon^{2k+2}\right).
\end{aligned} \tag{D.7}
$$

It follows that (D.5) and (D.6) may be rewritten as

$$
\begin{aligned}
&e^{-i\theta_{k+1}(y,\zeta_{k+1})} \ltimes H_{\Gamma_z}(y)\partial_\zeta \left[(\mathcal{Y}_{k+1})_{h_{k+1}}\right](y,\zeta) \\
&\qquad = H_{\Gamma_{\varepsilon(\zeta_{k+1}+h_k)}} \left[e^{-i\theta_k/\varepsilon} \ltimes \partial_\zeta \mathcal{Y}_k\right](y,\zeta_{k+1}) \\
&\qquad\qquad + \varepsilon^{k+1}\left[h\mathrm{I\!I}_\Gamma \partial_\zeta \Phi_0[\rho] + \mathrm{I}_\Gamma \beta \mathcal{P}_\sigma\left(i\Phi_0[\rho]\right) + R_\rho \mathrm{I}_\Gamma \partial_{p\zeta}\Phi_0[\rho]\right]_{h_{k+1}} + O\left(\varepsilon^{k+2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&e^{-i\theta_{k+1}(y,\zeta_{k+1})/\varepsilon} \ltimes \Box_{\Gamma_z}\left[\left(e^{i\theta_{k+1}/\varepsilon} \ltimes \left(V_k + \varepsilon^{k+1}R_\rho\partial_\rho\Phi_0[\rho] + \varepsilon^{k+2}P\right)\right)_{h_{k+1}}\right](y,\zeta) = \\
&\qquad\qquad e^{-i\theta_k(y,\zeta_{k+1})/\varepsilon} \ltimes \Box_{\Gamma_{\varepsilon(\zeta_{k+1}+h_k)}}\left[(\mathcal{Y}_k)_{h_k}\right](y,\zeta_{k+1})
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big\{ \varepsilon^{k-1} \left[2\beta \mathcal{P}_\sigma \left(i\partial_\zeta \Phi_0[\rho] \right) + R_\rho \mathcal{P}_\sigma \left(\Phi_0[\rho] \right) + R_\rho \rho \mathcal{P}_\sigma \left(\partial_\rho \Phi_0[\rho] \right) \right] \\
&+ \varepsilon^k h \left[\mathrm{II}_{\langle\theta,\theta\rangle} \mathcal{P}_\sigma \left(\Phi_1 \right) + \mathrm{II}_{\square\theta} \mathcal{P}_\sigma \left(i\Phi_0[\rho] \right) - 2\mathrm{II}_{\langle\theta,\rho\rangle} \mathcal{P}_\sigma \left(i\partial_\rho \Phi_0[\rho] \right) \right] \\
&+ \varepsilon^k h \mathrm{III}_{\langle\theta,\theta\rangle} \mathcal{P}_\sigma \left(\zeta \Phi_0[\rho] \right) \\
&+ \varepsilon^k \left[R_\rho \mathrm{I}_{\square\theta} \mathcal{P}_\sigma \left(i\partial_\rho \Phi_0[\rho] \right) - R_\rho I_{\langle\theta,\rho\rangle} \mathcal{P}_\sigma \left(i\partial_\rho^2 \Phi_0[\rho] \right) \right] \\
&+ \varepsilon^k \left[-2I_{\langle\theta,R_\rho\rangle} \mathcal{P}_\sigma \left(i\partial_\rho \Phi_0[\rho] \right) + \rho \mathcal{P}_\sigma(P) \right] \\
&+ \varepsilon^k \left[-\left(\square_\Gamma h \right) \partial_\zeta \Phi_0[\rho] + \left(\square_\Gamma \alpha \right) \mathcal{P}_\sigma \left(i\Phi_0[\rho] \right) + 2I_{\langle\theta,\beta\rangle} \mathcal{P}_\sigma \left(\zeta \Phi_0[\rho] \right) \right] \\
&+ \varepsilon^k \left[2\mathrm{II}_{\langle\theta,h\rangle} \mathcal{P}_\sigma \left(i\zeta \partial_\zeta \Phi_0[\rho] \right) + 2I_{\langle\rho,h\rangle} \partial_{p\zeta} \Phi_0[\rho] \right] \\
&+ \varepsilon^k \left[2\mathrm{II}_{\langle\theta,\alpha\rangle} \mathcal{P}_\sigma \left(\zeta \Phi_0[\rho] \right) - 2I_{\langle\rho,\alpha\rangle} \mathcal{P}_\sigma \left(i\partial_\rho \Phi_0[\rho] \right) \right] \\
&+ \varepsilon^k \left[2\beta \mathcal{P}_\sigma \left(i\partial_\zeta \Phi_1 \right) + 2I_{\langle\theta,\alpha\rangle} \mathcal{P}_\sigma \left(\Phi_1 \right) \right] \Big\}_{h_{k+1}} (y,\zeta) + O\left(\varepsilon^{k+1} \right).
\end{aligned}
$$

Putting all the above formulas together:

$$
\begin{aligned}
&e^{-i\theta_{k+1}(y,\zeta_{k+1})/\varepsilon} \ltimes S_\varepsilon^{(y,\zeta)} \left[\left(\mathcal{Y}_{k+1} \right)_{h_{k+1}} \right] (y,\zeta) \\
&= e^{-i\theta_{k+1}(y,\zeta_{k+1})/\varepsilon} \ltimes \left\{ \frac{1}{\varepsilon^2} D_\Phi W \left(\left(\mathcal{Y}_{k+1} \right)_{h_{k+1}}; \Pi \right) + \left[-\frac{1}{\varepsilon^2} \partial_\zeta^2 + \frac{1}{\varepsilon} H_{\Gamma_z} \partial_\zeta + \square_{\Gamma_z} \right] \left(\mathcal{Y}_{k+1} \right)_{h_{k+1}} \right\} (y,\zeta) \\
&= e^{-i\theta_k(y,\zeta_{k+1})/\varepsilon} \ltimes \Big\{ \frac{1}{\varepsilon^2} \left[-\partial_\zeta^2 \left(\mathcal{Y}_k \right)_{h_k} + D_\Phi W \left(\left(\mathcal{Y}_k \right)_{h_k}; \Pi \right) \right] + \frac{1}{\varepsilon} H_{\Gamma_{\varepsilon(\zeta_{k+1}+h_k)}} \partial_\zeta \left(\mathcal{Y}_k \right)_{h_k} \\
&\qquad\qquad + \square_{\Gamma_{\varepsilon(\zeta_{k+1}+h_k)}} \left[\left(\mathcal{Y}_k \right)_{h_k} \right] \Big\}_{\varepsilon^k h} (y,\zeta) \\
&+ \Big\{ \varepsilon^{k-1} R_\rho \left[-\partial_\zeta^2 \partial_\rho \Phi_0[\rho] + \rho \mathcal{P}_\sigma \left(\partial_\rho \Phi_0[\rho] \right) + D_\Phi^2 W \left(\Phi_0[\rho]; \Pi \right) \partial_\rho \Phi_0[\rho] + \mathcal{P}_\sigma \left(\Phi_0[\rho] \right) \right] \\
&+ \varepsilon^k \left[-2\beta \mathcal{P}_\sigma \left(i\partial_\zeta \Phi_1 \right) - \partial_\zeta^2 P + h \mathrm{II}_\Gamma \partial_\zeta \Phi_0[\rho] + \mathrm{I}_\Gamma \beta \mathcal{P}_\sigma \left(i\Phi_0[\rho] \right) + R_\rho \mathrm{I}_\Gamma \partial_{\zeta p} \Phi_0[\rho] \right] \\
&\qquad + \varepsilon^k \left\{ R_\rho D_\Phi^3 W \left(\Phi_0[\rho]; \Pi \right) \left[\Phi_1, \partial_\rho \Phi_0[\rho] \right] + D_\Phi^2 W \left(\Phi_0[\rho]; \Pi \right) P \right\} \\
&+ \varepsilon^k h \left[\mathrm{II}_{\langle\theta,\theta\rangle} \mathcal{P}_\sigma \left(\Phi_1 \right) + \mathrm{II}_{\square\theta} \mathcal{P}_\sigma \left(i\Phi_0[\rho] \right) - 2\mathrm{II}_{\langle\theta,\rho\rangle} \mathcal{P}_\sigma \left(i\partial_\rho \Phi_0[\rho] \right) + \mathrm{III}_{\langle\theta,\theta\rangle} \mathcal{P}_\sigma \left(\zeta \Phi_0[\rho] \right) \right] \\
&+ \varepsilon^k \left[R_\rho \left(\mathrm{I}_{\square\theta} \right) \mathcal{P}_\sigma \left(i\partial_\rho \Phi_0[\rho] \right) - R_\rho I_{\langle\theta,\rho\rangle} \mathcal{P}_\sigma \left(i\partial_\rho^2 \Phi_0[\rho] \right) - 2I_{\langle\theta,R_\rho\rangle} \mathcal{P}_\sigma \left(i\partial_\rho \Phi_0[\rho] \right) + \rho \mathcal{P}_\sigma(P) \right] \\
&\qquad + \varepsilon^k \left[2\mathrm{II}_{\langle\theta,h\rangle} \mathcal{P}_\sigma \left(i\zeta \partial_\zeta \Phi_0[\rho] \right) + 2I_{\langle\rho,h\rangle} \partial_{p\zeta} \Phi_0[\rho] \right] \\
&\qquad + \varepsilon^k \left[2\mathrm{II}_{\langle\theta,\alpha\rangle} \mathcal{P}_\sigma \left(\zeta \Phi_0[\rho] \right) - 2I_{\langle\rho,\alpha\rangle} \mathcal{P}_\sigma \left(i\partial_\rho \Phi_0[\rho] \right) \right] \\
&\qquad\qquad + \varepsilon^k \left[2\beta \mathcal{P}_\sigma \left(i\partial_\zeta \Phi_1 \right) + 2I_{\langle\theta,\alpha\rangle} \mathcal{P}_\sigma \left(\Phi_1 \right) \right] \Big\}_{h_{k+1}} (y,\zeta).
\end{aligned}
$$

Cleaning up:

$$
\begin{aligned}
&e^{-i\theta_{k+1}(y,\zeta_{k+1})/\varepsilon} \ltimes S_\varepsilon^F \left[\left(\mathcal{Y}_{k+1} \right)_{h_{k+1}} \right] (y,\zeta) \\
&= e^{-i\theta_k(y,\zeta_{k+1})/\varepsilon} \ltimes S_\varepsilon^F \left[\left(\mathcal{Y}_k \right)_{h_k} \right] \Big|_{z=\varepsilon(\zeta_{k+1}+h_k)} (y, \zeta - \varepsilon^k h)
\end{aligned}
$$

$$
\begin{aligned}
&+ \varepsilon^k \mathcal{L}\left(\Phi_0[\rho](y,\cdot);\Pi_{\rho(y)}\right)[P(y,\cdot)](\zeta_{k+1}) \\
&+ \varepsilon^k \Big\{ -\left(\Box_\Gamma h\right)\partial_\zeta \Phi_0[\rho] + \left(\Box_\Gamma \alpha\right)\mathcal{P}_\sigma\left(i\Phi_0[\rho]\right) + 2I_{\langle\theta,\beta\rangle}\mathcal{P}_\sigma\left(\zeta\Phi_0[\rho]\right) \\
&- 2I_{\langle\theta,R_\rho\rangle}\mathcal{P}_\sigma\left(i\partial_\rho\Phi_0[\rho]\right) + \beta \mathrm{I}_\Gamma \mathcal{P}_\sigma\left(i\Phi_0[\rho]\right) + R_\rho\left[\mathcal{P}_\sigma\left(\Phi_1\right) + \left(\mathrm{I}_{\Box\theta}\right)\mathcal{P}_\sigma\left(i\partial_\rho\Phi_0[\rho]\right)\right] \\
&+ R_\rho\left\{D^3_\Phi W\left(\Phi_0[\rho];\Pi\right)\left[\Phi_1,\partial_\rho\Phi_0[\rho]\right] - I_{\langle\theta,\rho\rangle}\mathcal{P}_\sigma\left(i\partial_\rho^2\Phi_0[\rho]\right) + \mathrm{I}_\Gamma\partial_{\zeta p}\Phi_0[\rho]\right\} \\
&+ h\left[\mathrm{II}_{\Box\theta}\mathcal{P}_\sigma\left(i\Phi_0[\rho]\right) - 2\mathrm{II}_{\langle\theta,\rho\rangle}\mathcal{P}_\sigma\left(i\partial_\rho\Phi_0[\rho]\right) + \mathrm{III}_{\langle\theta,\theta\rangle}\mathcal{P}_\sigma\left(\zeta\Phi_0[\rho]\right) + \mathrm{II}_\Gamma\partial_\zeta\Phi_0[\rho]\right] \\
&+ 2\mathrm{II}_{\langle\theta,h\rangle}\mathcal{P}_\sigma\left(i\zeta\partial_\zeta\Phi_0[\rho]\right) + 2I_{\langle\rho,h\rangle}\partial_{p\zeta}\Phi_0[\rho] + 2\mathrm{II}_{\langle\theta,\alpha\rangle}\mathcal{P}_\sigma\left(\zeta\Phi_0[\rho]\right) \\
&- 2I_{\langle\rho,\alpha\rangle}\mathcal{P}_\sigma\left(i\partial_\rho\Phi_0[\rho]\right)\Big\}_{h_{k+1}}(y,\zeta) + O\left(\varepsilon^{k+1}\right).
\end{aligned}
$$

Finally, note that

$$
\begin{aligned}
I_{\langle\theta,\beta\rangle} &= \left(\partial^a\theta\partial^b\theta\right)\partial_{ab}h + X^d\partial_d h \\
I_{\langle\theta,R_\rho\rangle} &= 2\left(\partial^a\theta\partial^b\theta\right)\partial_{ab}\alpha + 2X^d\partial_d\alpha + \beta\mathrm{II}_{\langle\theta,\theta\rangle} + hg^{ab}\partial_a\theta\partial_b\left(\mathrm{II}_{\langle\theta,\theta\rangle}\right),
\end{aligned}
$$

where $X^d := g^{ab}\partial_a\theta\left[\left(\partial_b g^{cd}\right)\partial_c\theta + g^{cd}\partial_{bc}\theta\right]$. Using this, in summary we have that

$$
\begin{aligned}
&e^{-i\theta_{k+1}(y,\zeta_{k+1})/\varepsilon} \ltimes S^F_\varepsilon\left[\left(\mathcal{Y}_{k+1}\right)_{h_{k+1}}\right](y,\zeta) \\
&= \varepsilon^k\left[\mathcal{L}\left(\Phi_0[\rho](y,\cdot);\Pi_{\rho(y)}\right)[P(y,\cdot)]\Big|_{(y,\zeta_{k+1})} + G\left[h,\alpha\right](y,\zeta_{k+1})\right] + O\left(\varepsilon^{k+1}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
G\left[h,\alpha\right](y,\zeta) := &\left[g^{ab}\partial_\zeta\Phi_0 + 2\left(\partial^a\theta\partial^b\theta\right)\mathcal{P}_\sigma\left(\zeta\Phi_0\right)\right]\partial_{ab}h \\
&- \left[g^{ab}\mathcal{P}_\sigma\left(i\Phi_0\right) + 2\left(\partial^a\theta\partial^b\theta\right)\mathcal{P}_\sigma\left(i\partial_p\Phi_0\right)\right]\partial_{ab}\alpha \\
&+ E_k(y,\zeta) + \mathfrak{D}\left[h,\alpha\right](y,\zeta),
\end{aligned}
$$

with $E_k(y,\zeta)$ being the portion of order ε^k of

$$
e^{-i\theta_k(y,\zeta_{k+1})/\varepsilon} \ltimes S^F_\varepsilon\left[\left(\mathcal{Y}_k\right)_{h_k}\right]\Big|_{z=\varepsilon(\zeta_{k+1}+h_k)}(y,\zeta-\varepsilon^k h), \tag{D.8}
$$

and

$$
\begin{aligned}
&\mathfrak{D}\left[h,\alpha\right](y,\zeta) := 2X^d\left(\partial_d h\right)\mathcal{P}_\sigma\left(\zeta\Phi_0[\rho]\right) \\
&- \left[4X^d\left(\partial_d\alpha\right) + 2\beta\mathrm{II}_{\langle\theta,\theta\rangle} + 2hg^{ab}\partial_a\theta\partial_b\left(\mathrm{II}_{\langle\theta,\theta\rangle}\right)\right]\mathcal{P}_\sigma\left(i\partial_\rho\Phi_0[\rho]\right)
\end{aligned}
$$

$$
\begin{aligned}
&- B^b\left(\partial_b h\right)\partial_\zeta \Phi_0[\rho] + B^b\left(\partial_b \alpha\right)\mathcal{P}_\sigma\left(i\Phi_0[\rho]\right) + \beta \mathrm{I}_\Gamma \mathcal{P}_\sigma\left(i\Phi_0[\rho]\right) \\
&+ R_\rho \Big\{ \mathcal{P}_\sigma\left(\Phi_1\right) + D^3_\Phi W\left(\Phi_0[\rho];\Pi\right)\left[\Phi_1, \partial_\rho \Phi_0[\rho]\right] \\
&+ \left(\mathrm{I}_{\Box\theta}\right)\mathcal{P}_\sigma\left(i\partial_\rho \Phi_0[\rho]\right) - I_{\langle\theta,\rho\rangle}\mathcal{P}_\sigma\left(i\partial^2_\rho \Phi_0[\rho]\right) + \mathrm{I}_\Gamma \partial_{\zeta p}\Phi_0[\rho] \Big\} \\
&+ h \Big\{ \mathrm{II}_{\Box\theta}\mathcal{P}_\sigma\left(i\Phi_0[\rho]\right) - 2\mathrm{II}_{\langle\theta,\rho\rangle}\mathcal{P}_\sigma\left(i\partial_\rho \Phi_0[\rho]\right) + \mathrm{III}_{\langle\theta,\theta\rangle}\mathcal{P}_\sigma\left(\zeta\Phi_0[\rho]\right) \\
&+ \mathrm{II}_\Gamma \partial_\zeta \Phi_0[\rho] \Big\} + 2\mathrm{II}_{\langle\theta,h\rangle}\mathcal{P}_\sigma\left(i\zeta\partial_\zeta \Phi_0[\rho]\right) + 2I_{\langle\rho,h\rangle}\partial_{p\zeta}\Phi_0[\rho] \\
&+ 2\mathrm{II}_{\langle\theta,\alpha\rangle}\mathcal{P}_\sigma\left(\zeta\Phi_0[\rho]\right) - 2I_{\langle\rho,\alpha\rangle}\mathcal{P}_\sigma\left(i\partial_\rho \Phi_0[\rho]\right).
\end{aligned}
$$

For the sake of clarity, we write (D.8) more explicitly:

$$
\begin{aligned}
&S^F_\varepsilon\left[\left(e^{i\theta_k/\varepsilon} \ltimes V_k\right)_{h_k}\right]\Big|_{z=\varepsilon(\zeta_{k+1}+h_k)}(y,\zeta-\varepsilon^k h) = \\
&\frac{1}{\varepsilon^2}\left[-\partial^2_\zeta\left(e^{i\theta_k/\varepsilon} \ltimes V_k\right)_{h_k}(y,\zeta-\varepsilon^k h) + D_\Phi W\left(\left(e^{i\theta_k/\varepsilon} \ltimes V_k\right)_{h_k}(y,\zeta-\varepsilon^k h);\Pi\right)\right] \\
&\qquad + \frac{1}{\varepsilon}H_{\Gamma_{\varepsilon(\zeta_{k+1}+h_k)}}\left[\partial_\zeta\left(e^{i\theta_k/\varepsilon} \ltimes V_k\right)_{h_k}\right](y,\zeta-\varepsilon^k h) \\
&\qquad\qquad + \Box_{\Gamma_{\varepsilon(\zeta_{k+1}+h_k)}}\left(e^{i\theta_k/\varepsilon} \ltimes V_k\right)_{h_k}(y,\zeta-\varepsilon^k h).
\end{aligned}
$$

In the above expression:

$$
\begin{aligned}
&D_\Phi W\left(\left(e^{i\theta_k/\varepsilon} \ltimes V_k\right)_{h_k}(y,\zeta-\varepsilon^k h);\Pi\right) \\
&\quad = D_\Phi W\left(e^{i\theta_k(\zeta-h_k-\varepsilon^k h)/\varepsilon} \ltimes V_k(y,\zeta-h_k-\varepsilon^k h);\Pi\right) \\
&\quad = D_\Phi W\left(\left(e^{i\theta_k/\varepsilon} \ltimes V_k\right)_{h_{k+1}}(y,\zeta);\Pi\right). \\
&\Box_{\Gamma_{\varepsilon(\zeta_{k+1}+h_k)}}\left(e^{i\theta_k/\varepsilon} \ltimes V_k\right)_{h_k}(y,\zeta-\varepsilon^k h) \\
&\quad = \left[-\frac{1}{\sqrt{-|\mathfrak{g}(y,z)|}}\partial_a\left[\sqrt{-|\mathfrak{g}(y,z)|}\,\mathfrak{g}^{ab}(y,z)\partial_b\left(e^{i\theta_k/\varepsilon} \ltimes V_k\right)_{h_k}(y,\zeta)\right]\right]\Big|_{z=\varepsilon(\zeta_{k+1}+h_k)}.
\end{aligned}
$$

D.2 SOLUTIONS TO *Equation 4.41*

Lemma D.1. *Let $p \in \mathbb{N}_{\geq 1}$ and $T > 0$, and consider the following linear system for the functions $h, \alpha : [0,T) \times \mathbb{R}^p \to \mathbb{R}$:*

$$\begin{cases} C_1^{ab}\partial_{ab}h + M_1^a\partial_a h + m_1 h + N_1^a \partial_a \alpha + n_1 \alpha = F_1, & \textit{in } (0,T)\times\mathbb{R}^p \\ C_2^{ab}\partial_{ab}\alpha + M_2^a\partial_a h + m_2 h + N_2^a \partial_a \alpha + n_2 \alpha = F_2, & \textit{in } (0,T)\times\mathbb{R}^p \\ (h, \alpha, \partial_{y_0} h, \partial_{y_0}\alpha) = 0, & \textit{on } \{0\}\times\mathbb{R}^p, \end{cases} \tag{D.9}$$

where C_j^{ab} are given by (3.67), and M_k^a, N_k^a, m_k, n_k for $k = 1,2$ and $a = 0, 1, \cdots, p$ are given functions.

Suppose that for all $k \in \{1,2\}$, $a, b \in \{0,1,2,...,p\}$, and $i, j \in \{1,2,...,p\}$:

1. *$M_k^i, N_k^i, m_k, n_k, F_k, C_k^{ab} \in C_b^\infty([0,T]\times\mathbb{R}^p)$.*
2. *There is a constant $c > 0$ such that $C_k^{00} < -c$ and $\overline{C}_k \succ cI$ uniformly over $[0,T]\times\mathbb{R}^p$. Here $\overline{C}_k$ is the $\mathbb{R}^{p\times p}$-valued map given by $\overline{C}_k^{ij} := C_k^{ij}$ (i.e., just as in Theorem 3.22).*

Then, Equation D.9 accepts a unique solution (h, α) of class $C_b^\infty([0,T)\times\mathbb{R}^p)$.

Proof. Let

$$\begin{aligned} V &:= \begin{bmatrix} h & \alpha & \nabla h & \nabla\alpha \end{bmatrix}^T, \\ \left(\bar{C}_k\right)_{ij} &:= C^{ij}, \qquad\qquad i,j \in \{0,\cdots,p\} \\ \mathcal{C}_{k,j} &:= \begin{pmatrix} 2C_k^{0j} & C_k^{1j} & \cdots & C_k^{pj} \\ C_k^{1j} & & & \\ \vdots & & 0 & \\ C_k^{pj}. & & & \end{pmatrix} \end{aligned}$$

Then, the system (D.9) can be expressed as the first order system

$$\begin{cases} S^a\partial_a V + S^{p+1}V = F, & \text{over } [0,T)\times\mathbb{R}^p \\ V(0,x) = 0, & x \in \mathbb{R}^p, \end{cases} \tag{D.10}$$

where a runs from 0 to p, and

$$S^0 := \operatorname{diag}\left\{1, 1, -C_1^{00}, \bar{C}_1, -C_2^{00}, \bar{C}_2\right\},$$

$$S^a := \operatorname{diag}\{0, 0, -\mathcal{C}_{1,a}, -\mathcal{C}_{2,a}\}, \qquad a = 0, 1, \cdots, p$$

$$S^{p+1} := -\begin{pmatrix} 0 & 0 & e_1^T & 0_{1\times p} \\ 0 & 0 & 0_{1\times p} & e_1^T \\ m_1 & n_1 & M_1^T & N_1^T \\ & & 0_{p\times 2(p+2)} & \\ m_2 & n_2 & M_2^T & N_2^T \\ & & 0_{p\times 2(p+2)} & \end{pmatrix},$$

$$F := \begin{bmatrix} 0_{1\times 2} & F_1 & 0_{1\times p} & F_2 & 0_{1\times p} \end{bmatrix}^T.$$

The conditions on C_k^{ab} listed in the hypotheses of the theorem imply that S^0 is symmetric and uniformly positive definite. Moreover, S^α is everywhere symmetric for all $\alpha = 0, 1, ..., p$ and uniformly bounded as a result of the uniform boundedness of C_k^{ab} for $k = 1, 2$ and $a, b = 0, 1, ..., p$. The existence of a unique smooth solution (h, α) to (D.9) then follows from the proof of [31, Corollary 7.10].

The boundedness of (h, α) and of their derivatives follows from the uniqueness/finite speed of propagation and energy estimates associated to (D.10) (see e.g., [31, Proposition 7.7 and Lemma 7.5]). In particular, [31, Proposition 7.7] guarantees the existence of a number s_0, which depends on the lower bound on S^0 and the upper bounds on S^i for $i = 1, 2, ..., p$, such that if F in (D.10) vanishes in the set

$$C_{r,x_0} := \{(t, x) \in [0, T] \times \mathbb{R}^p : x \in B_{r-s_0 t}(x_0)\}, \quad (x_0 \in \mathbb{R}^p),$$

then any C^1 solution V of (D.10) vanishes in C_{r,x_0} as well. Consequently, we say that the system (D.10) has finite speed of propagation no bigger than s_0.

Fix one such admissible value s_0 for (D.10) and $x_0 \in \mathbb{R}^p$. Also, let $r > Ts_0$, and let $\phi_r \in C_c^\infty(\mathbb{R}^n)$ be such that $\phi_r(x) = 1$ for $x \in B_r(x_0)$ and $\phi_r(x) = 0$

for $x \in \mathbb{R}^p \backslash B_{2r}(x_0)$. Define $F_r(t,x) := \phi_r(x)F(t,x)$ and consider the unique solution $V_r = \begin{bmatrix} h_r & \alpha_r & \nabla h_r & \nabla \alpha_r \end{bmatrix}^T$ to the problem

$$\begin{cases} S^a \partial_a V + S^{p+1} V = F_r, & \text{over } (0,T) \times \mathbb{R}^p \\ V(0,x) = 0, & x \in \mathbb{R}^p, \end{cases} \tag{D.11}$$

which is known to exist due to [31, Theorem 7.9 or Corollary 7.10]. Since F_r and F coincide in C_r, it follows by the finite speed of propagation of (D.10) that the difference $V - V_r$ vanishes in C_r. Additionally, for each $l \in \mathbb{N}$, [31, Lemma 7.5] implies the existence of a constant k_l, depending on the size of the coefficients S^a (and of their derivatives) and T, such that for $m = 0, 1$:

$$\sup_{t \in [0,T]} \|(\partial_t^m h_r(t,\cdot), \partial_t^m \alpha_r(t,\cdot))\|_{H^l(\mathbb{R}^p)} \leq k_l \int_0^T \|F_r(t,\cdot)\|_{H^l(\mathbb{R}^p)} \, dt. \tag{D.12}$$

As a result, since $F \in C_b^\infty([0,T] \times \mathbb{R}^p)$ and F_r has compact support, we have that

$$\sup_{t \in [0,T]} \|(\partial_t^m h_r(t,\cdot), \partial_t^m \alpha_r(t,\cdot))\|_{H^l(\mathbb{R}^p)} < K_l \qquad l \in \mathbb{N},\ m = 0, 1, \tag{D.13}$$

for some K_l which depends on the size of S^a, F (and of their derivatives) and on T, but is independent of x_0. On the other hand, using the fact that $C_1^{00}, C_2^{00} < -\delta$ uniformly on $[0,T] \times \mathbb{R}^p$, we may rewrite (D.11) as

$$-\partial_t^2 h_r = L_1[V_r] + \frac{(F_r)_1}{\left|C_1^{00}\right|} \quad \text{and} \quad -\partial_t^2 \alpha_r = L_2[V_r] + \frac{(F_r)_2}{\left|C_1^{00}\right|}, \tag{D.14}$$

where

$$L_j[V_r] := -\frac{1}{\left|C_j^{00}\right|} \left(M_j^a \partial_a h_r + m_j h_r + N_j^a \partial_a \alpha_r + n_j \alpha_r \right) \qquad j = 1, 2.$$

Consequently, thanks to the Sobolev embedding $H^l(\mathbb{R}^p; \mathbb{R}) \hookrightarrow C_b^k(\mathbb{R}^p; \mathbb{R})$ for $l \geq k + [p/2] + 2$, the bound (D.13) with $l \geq [p/2] + 4$ (i.e., the case $k = 2$) together with Equation D.14 and the boundedness of the coefficients therein, it follows that $\partial_t^2 h_r$ and $\partial_t^2 \alpha_r$ are bounded with bounds independent of x_0. For any other $k \in \mathbb{N}$, proceeding inductively in a similar fashion using (D.12) with $l \geq [p/2] + 2 + k$, the boundedness of the coefficients in Equation D.14, and successive differentiation of Equation D.14, yield $h_r, \alpha_r \in C_b^k([0,T] \times \mathbb{R}^p)$ with bounds independent of x_0. Finally, recall that $h = h_r$ and $\alpha = \alpha_r$ in C_{r,x_0}, and that $x_0 \in \mathbb{R}^p$ was fixed but arbitrary. Therefore, since any $(t,x) \in [0,T] \times \mathbb{R}^p$ lies on C_{r,x_0} for some $x_0 \in \mathbb{R}^p$ and the bounds mentioned above are independent of the choice of x_0, the above argument implies that $h, \alpha \in C_b^\infty([0,T] \times \mathbb{R}^p)$. $\square$

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