ON BREUIL'S LATTICE CONJECTURE FOR GL2

by

hymn chan

A thesis submitted in conformity with
the requirements for the degree of
Doctor of Philosophy

Graduate Department of Mathematics
University of Toronto

ABSTRACT

On Breuil's Lattice Conjecture for GL2

Hymn Chan

Doctor of Philosophy

Graduate Department of Mathematics

University of Toronto

2026

We prove Breuil's lattice conjecture for higher Hodge–Tate weights in the case of $\mathrm{GL}_2(K)$ where K is an unramified extension of $\mathbb{Q}_p$. More precisely, under some genericity conditions, we show that the lattice inside a locally algebraic type induced by the completed cohomology of a $U(2)$-arithmetic manifold depends only on the Galois representation at the fixed place above p for arbitrary Hodge–Tate weights, which are small relative to p. We further prove that the patched modules of all lattices inside the locally algebraic types with irreducible cosocle are cyclic.

One key input of the paper is a structure theorem for mod p representations of $\mathrm{GL}_2(\mathcal{O}_K)$, which are residually multiplicity free and of finite length. Another input is an explicit computation of universal framed Galois deformation rings, which parameterize potentially crystalline lifts with fixed tame inertial types and higher Hodge–Tate weights.

To my parents

ACKNOWLEDGEMENTS

First and foremost, I want to thank Florian Herzig; I could not imagine having a better supervisor. I thank him for introducing me to the p-adic Langlands world and suggesting this problem to me. Throughout my PhD, he has been patient, kind, and supportive, answering many of my questions, providing many valuable insights for my research, and pointing out some mistakes in the earlier drafts.

I want to thank Heejong Lee, Chol Park, and Yitong Wang for many helpful discussions and teaching me many things related to the p-adic Langlands Program. I also thank Yitong Wang for careful reading of an early draft. I want to thank Daniel Le for a helpful insight on the Breuil–Mézard Conjecture.

I thank all my friends in Toronto, too many to name individually, but especially Evan Ng and Joseph Teh. Their presence makes me feel at home away from home and makes my PhD journey much more enjoyable. I appreciate all their prayers and friendships.

Finally, I would like to thank my family. I thank my parents for letting me study abroad for a decade and always supporting me in pursuing mathematics; it would not be possible without their support.

Soli Deo Gloria

CONTENTS

1 INTRODUCTION

1.1 INTRODUCTION

Introduction to the Langlands Program

In this section, we will give a brief overview of the background and motivations behind the main results. In many places, we will omit the technicalities and not define some terms precisely. The main references are [Gee22], [Bre10], [Eme14], [Cal23], and [EGH25].

Let F be a number field, a finite extension of $\mathbb{Q}$, and fix an algebraic closure $\overline{F}$ of F. If v is a place in F, we write F_v for the completion of F at v. For any field L with separable closure $\overline{L}$, we write G_L for the Galois group $\mathrm{Gal}(\overline{L}/L)$. Our coefficient field E is a finite extension of $\mathbb{Q}_p$ which we assume to be sufficiently large, with residue field $\mathbb{F}$, a uniformizer ϖ, and the ring of integers $\mathcal{O}$.

Understanding the absolute Galois group G_F is central to many problems in number theory, such as Fermat's Last Theorem. An important result in the early twentieth century is the class field theory, which gives an isomorphism between the maximal abelian quotient of G_F, denoted as G_F^{ab} and $\mathbb{A}_F^\times/F^\times F_\infty^\circ$, where $\mathbb{A}_F$ is the adeles of F and F_∞° is the connected component of the identity in $\prod_{v|\infty} F_v$. Moreover, we have the local class field theory as follows. Let K be a finite extension of $\mathbb{Q}_p$ with a uniformizer π and the residue field k; and let I_K be the inertia group of G_K. We have the following short exact sequence:

$$1 \to I_K \to G_K \to G_k \cong \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z},$$

where $\hat{\mathbb{Z}}$ is generated by the Frobenius element. The Weil group W_K is defined as the inverse image of $\mathbb{Z}$ inside $\hat{\mathbb{Z}}$, which is dense inside G_K. The local class field theory gives an isomorphism between W^{ab}, the maximal abelian quotient of W_K, and $K^\times$, mapping the geometric Frobenius Frob^{-1} to the uniformizer π. (For more details, one can refer to [CF10].)

Historically, the global class field theory was first proved, and the local class field theory was deduced from it. However, a more natural approach is to deduce the global class field theory from the local class field theory, using certain local-global compatibility.

The Langlands Program seeks to extend the result to the non-abelian case. Roughly speaking, the Langlands Program posits a correspondence between

$$\{\text{Galois rep } r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)\} \leftrightarrow \{\text{cuspidal automorphic rep } \pi \text{ of } \mathrm{GL}_n(\mathbb{A}_F^\infty)\}.$$

(Rep is short for representation.) We will not give a precise definition of cuspidal automorphic representations, except to comment that they are smooth functions on some double quotients of $\mathrm{GL}_n(\mathbb{A}_F)$, which generalize cusp forms, as cusp forms are cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$.

When $n = 1$, any representation of G_F is a character, hence it factors through its maximal abelian quotient G_F^{ab}. Therefore, by the global class field theory, it can be viewed as a character of $\mathbb{A}_F^\times / F^\times F_\infty^\circ$, which is an automorphic representation of $\mathrm{GL}_1(\mathbb{A}_F^\infty)$.

However, not all Galois representations come from geometry; hence, they do not correspond to cuspidal automorphic representations. Therefore, we should restrict the Galois representations in the correspondence to those arising from geometry.

Definition 1.1. A continuous representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is geometric if it is unramified outside a finite set of primes of F and for all places $v|\ell$, $r|_{G_{F_v}}$ is de Rham.

We see a bifurcation between $\ell \neq p$ and $\ell = p$, and the case where $\ell = p$ is special for the following reason. Fix $v|p$, the wild inertia group inside G_{F_v} is a pro-p group, while $\mathrm{GL}_n(\mathbb{Q}_\ell)$ has the ℓ-adic topology. When $\ell \neq p$ the two topologies are not compatible, $r_v := r|_{G_{F_v}}$ has finite image in $\mathrm{GL}_n(\mathbb{Q}_\ell)$. However, when $\ell = p$, this is no longer the case. Indeed, for nearly every elliptic curve with good reduction at p, the Galois representation associated to its Tate modules at p is infinitely ramified. Instead, we have the notion of "crystalline" to capture the case where the curves have good reduction at p, and the notion of "de Rham" to capture the case where the Galois representations arise from geometry [Tsu99].

Conjecture 1.2. *[Fontaine–Mazur–Langlands conjecture][Gee22, Conjectures 2.33, 2.34] Any n-dimensional continuous semisimple geometric Galois representation $G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is automorphic, meaning that it arises from cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_F^\infty)$, with the property that $\mathrm{WD}(r|_{F_v}) \cong \mathrm{rec}_{F_v}(\pi_v \det^{(1-n)/2})$ for each finite place v of F.*

Here, rec_v is the local Langlands correspondence map and $\mathrm{WD}(r_v)$ is the Weil–Deligne representation associated to r_v as explained in Theorem 1.3 and the discussion before. The last part captures the local-global compatibility expected for the Langlands Program.

By the works of Kowittz [Kot92], Clozel [Clo91] and many others (see [Cal23, §9.1] for a more comprehensive review), in many cases, one can attach a Galois representation $r_{\pi,\ell} : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ to a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F^\infty)$, such that $\mathrm{WD}(r_\pi|_{G_{F_v}}) \cong \mathrm{rec}_{F_v}(\pi_v \det^{(1-n)/2})$ for each place v. In the other direction, one would like to prove certain modularity/ automorphy results, that given a geometric Galois representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, there exists a cuspidal automorphic representation π, such that $r \sim r_{\pi,\ell}$.

The famous proof of Fermat's Last Theorem [Wil95] [TW95] fits nicely within this framework and can be seen as a special case. Given an elliptic curve E defined over F, one obtains a Galois representation $r_{E,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ from the Galois action on the ℓ-adic Tate module. On the other hand, by the work of Deligne [Del71], Deligne–Serre [DS74] and Eichler–Shimura, one can also attach a Galois representation $r_{f,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ to a modular eigenform f. The proof of Fermat's Last Theorem hinges on showing that every semistable elliptic curve E, $r_{E,\ell}$ is modular, meaning that there exists an eigenform f with a prime ℓ such that $r_{E,\ell} \sim r_{f,\ell}$.

At risk of oversimplification, proving an automorphy result for $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ involves two steps:

1. Showing that there exists an automorphic representation π, such that $\bar{r} \sim \bar{r}_{\pi,\ell}$. Here, $\bar{r}$ and $\bar{r}_{\pi,\ell}$ is the mod ℓ reduction of a G_F-invariant integral lattice inside the Galois representation. (The reduction is independent of the choice of the lattice).

2. Automorphy lifting: showing that if step 1 holds, then there exists a cuspidal automorphic representation π', such that $r \sim r_{\pi',\ell}$.

In the direction of 1 is Serre's Conjecture [Ser87], which conjectured that for any odd and irreducible $\bar{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, $\bar{r}$ is modular. Moreover, Serre gives an explicit recipe for the minimal level and minimal weight of the modular eigenform f giving rise to $r \sim r_{f,\ell}$. This was proved by Khare and Wintenberger [KW09a] [KW09b]. One can easily generalize the automorphy part of the conjecture for $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$, but it is unclear at first sight how to generalize the prediction for the minimal weight. (The minimal level has to do with the conductor). The correct generalization of the minimal weight is a set of irreducible representations of $\prod_{v|p} \mathrm{GL}_n(\mathcal{O}_{F_v})$ over $\overline{\mathbb{F}}_p$. (We will come back to this in Section 1.1).

Automorphy lifting and Galois deformation ring

In order to prove automorphy lifting (step 2), one would study the deformation of $\bar{r}$. Fix $\bar{r} : G_F \to \mathrm{GL}_n(\mathbb{F})$, and we assume $\bar{r}$ to be absolutely irreducible for simplicity. Let $\mathcal{C}_{\mathcal{O}}$ be the category of complete local Noetherian $\mathcal{O}$-algebra A with maximal ideal $\mathfrak{m}_A$ such that $A/\mathfrak{m}_A \cong \mathbb{F}$. We consider the functor $\mathcal{C}_{\mathcal{O}} \to \mathrm{Set}$ which sends $(A, \mathfrak{m}_A) \mapsto r_A : G_F \to \mathrm{GL}_n(A)$ such that $\bar{r}_A \cong \bar{r}$. This functor is representable by $r^{\mathrm{univ}} : G_F \to \mathrm{GL}_n(R_{\bar{r}}^{\mathrm{univ}})$, and we refer to $R_{\bar{r}}^{\mathrm{univ}}$ as the universal deformation ring. (We ignored the difference between framed deformation rings and deformation rings.) On the automorphic side, we have the

Hecke algebra $\mathbb{T}$, which is generated by double coset operators and acts on the space of automorphic forms. We consider its localization $\mathbb{T}_{\mathfrak{m}}$ at the maximal ideal $\mathfrak{m}$ corresponding to $\bar{r}$. Proving the automorphy lifting result amounts to proving a "$R^{\mathrm{univ}}_{\bar{r}} = \mathbb{T}_{\mathfrak{m}}$" theorem. (At least after inverting p, but we can prove that they are isomorphic as $\mathcal{O}$-algebras in some cases; see the discussion after Theorem 1.8).

One key step in proving automorphy lifting is to allow more ramification on the Galois representation away from p, which corresponds to patching together automorphic forms of varying level, to "smooth out" the singularities of $R^{\mathrm{univ}}_{\bar{r}}$. One then applies the pigeonhole principle in a non-canonical way. (For more details, one can refer to [Gee22, §5].) Building on the construction of [TW95], [Dia97], [Kis09b], one can rephrase this process as a patching functor M_∞ [EGS15], which is discussed in Chapter 4.

One can analogously define local deformation rings for $\bar{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$. Kisin reformulates $R^{\mathrm{univ}}_{\bar{r}}$ as a power series ring over the completed tensor product of local deformation rings [Kis09b]. From this perspective, all the singularities of $R^{\mathrm{univ}}_{\bar{r}}$ come from local Galois deformation rings at bad places, especially those dividing p [LLHLM23, §1.1]. It is crucial to show that the local deformation spaces for $\bar{r}|_{G_{F_v}}$ where $v|p$ are smooth [Kis09a]. This leads to the formulation of the Breuil–Mézard Conjecture [BM02] (cf. Lemma 3.8). The conjecture predicts the underlying cycles of local Galois deformation rings with fixed inertial type τ (introduced below) and Hodge–Tate weights λ in terms of mod p representations associated to λ and τ. This is the motivation behind the result in Chapter 3.

Using the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ as an input, Kisin was able to prove the Breuil–Mézard Conjecture and hence the Fontaine–Mazur Conjecture (Conjecture 1.2) for $\mathrm{GL}_2(\mathbb{Q}_p)$ (with some mild restrictions) [Kis09a]. This result suggests that the p-adic Langlands Program plays an important role in the (global) Langlands Program. In the following, I will explain the p-adic Langlands Program.

Similar to the class field theory, we have the local Langlands correspondence for GL_n. Let $K/\mathbb{Q}_p$ be a finite extension. A Weil–Deligne representation is a pair (ρ, N), where ρ is a representation (with discrete topology) of the Weil group W_K and N is a nilpotent endomorphism. When $\ell \neq p$, by the Grothendiek monodromy theorem ([ST68, appendix]), one can recover the representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ from its associated Weil–Deligne representation because of the incompatibility between the topologies discussed above. When $\ell = p$, Fontaine associates a Weil–Deligne representation $WD(\rho)$ to a Galois representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ [Fon94]. However, unlike the $\ell \neq p$ case, $WD(\rho)$ does not determine ρ. Indeed, ρ carries an extra piece of information which is not captured in the Weil–Deligne representation, namely the filtration given by the jump in the Hodge–Tate weights, which can be seen as a tuple in $(\mathbb{Z}^n)^f$.

We have the following local Langlands correspondence for GL_n due to Harris–Taylor [HT01] and independently, Henniert [Hen00].

Theorem 1.3. *[local Langlands correspondence for* GL_n*] We have a* 1*-to-*1 *correspondence between*

$$\{n\text{-dim Weil–Deligne rep of } W_K\} \xrightarrow[\sim]{\text{rec}} \{\text{irreducible smooth representation of } \mathrm{GL}_n(K)\}$$

where the representations on both sides are over $\mathbb{C}$*.*

Moreover, this correspondence has certain properties that determine the correspondence uniquely, and it is compatible with the local class field theory. When $n = 1$, it recovers the local class field theory.

As the Weil–Deligne representation has discrete topology, one can reformulate the local Langlands correspondence as between

$$\{\text{cont. rep } \rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)\} \underset{\sim}{\to} \{\mathbb{Q}_\ell\text{-linear locally constant irred. rep.of } \mathrm{GL}_n(K)/\overline{\mathbb{Q}}_\ell \text{ v.s.}\}$$

where $\ell \neq p$. (Here cont. is short for continuous, irred. is short for irreducible, and v.s. is short for vector space.)

Introduction to the p -adic and mod p Langlands Program

It is natural to try to find a similar correspondence when $\ell = p$. As discussed before, we need a way to recover the Hodge–Tate weights of ρ, which can be achieved by having unitary Banach space representations. The p-adic Langlands Program is looking for a correspondence between

$$\{\text{cont. rep } \rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)\} \underset{\pi}{\to} \{\text{certain unitary Banach space rep of } \mathrm{GL}_n(K)/E \text{ v.s.}\}$$

If we reduce modulo p on both sides, then we expect a mod p Langlands correspondence:

$$\{\text{cont. rep } \bar{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)\} \underset{\pi}{\to} \{\text{admissible rep of } \mathrm{GL}_n(K)/\mathbb{F}\}.$$

The p-adic and mod p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ was first discovered by Breuil [Bre03b] [Bre03a], and was proved by Colmez [Col10]. The correspondence is 1-to-1 in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$. However, when $K/\mathbb{Q}_p$ is a nontrivial unramified extension, Breuil and Paškūnas prove that there are too many smooth admissible mod p representations of $\mathrm{GL}_2(K)$ to have a naive correspondence with Galois representations $G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ [BP12].

The weight part of Serre's conjecture can be rephrased in the framework of the mod p Langlands Program. Recall that the socle of a smooth representation is the direct sum of all its irreducible subrepresentations. The notion of minimal weight in Serre's Conjecture was generalized to a set of modular Serre weights $W(\bar{r}) := \{\otimes_{v|p}\sigma_v : \sigma_v \in W(\bar{r}|_{F_v})\}$ where

$W(\bar{r}|_{F_v})$ is conjectured to depend only on $\bar{r}|_{I_{F_v}}$ [BDJ10]. Then, one expects $W(\overline{\rho})$ to be given by $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\,\pi(\overline{\rho})$, where π is the conjectural mod p Langlands correspondence map above.

Using a global method, one can construct the candidate for the mod p Langlands correspondence $\pi(\overline{\rho})$. As a first step, one would verify if its socle is given by $W(\overline{\rho})$. To approach this, Breuil suggested two conjectures [Bre14], a lattice conjecture Conjecture 1.5 and one on multiplicity one at the Iwahori level, closely related to Theorem 1.9. (See [EGS15, § 1.1] for more details). We generalize results on $\pi(\overline{\rho})$ in Chapter 7, relying on results concerning mod p representation of $\mathrm{GL}_2(\mathcal{O}_K)$ in Chapter 2.

1.2 BREUIL'S LATTICE CONJECTURE

One can study the p-adic Langlands Program using a global setting, analogous to how the local class field theory was first deduced from the global class field theory historically. Emerton proved that the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is realized in the completed cohomology of modular curves [Eme11]. We fix some tame level (compact subgroup K^p of $\mathrm{GL}_2(\mathbb{A}^p_{\mathbb{Q}})$). For any compact subgroup K_p of $\mathrm{GL}_2(\mathbb{Q}_p)$, we obtain a modular curve $Y(K_pK^p)(\mathbb{C})$. The completed cohomology is given by

$$\widetilde{H}^i(K^p) := \varprojlim_n \varinjlim_{K_p} H^i(Y(K^pK_P)(\mathbb{C}), \mathcal{O}/\varpi^n)$$

where the direct limit is taken over all compact subgroups K_p of $\mathrm{GL}_2(\mathbb{Q}_p)$.

This is a much larger space than the classical cohomology space for modular curves. One advantage of the completed cohomology is that it carries a $\mathrm{GL}_2(\mathbb{Q}_p)$ action, which commutes with the Hecke action. Moreover, it has an action of $G_{\mathbb{Q}}$. Emerton proved the following result.

Theorem 1.4. *[Eme11, Theorem 6.2.1] Let $r : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ be an odd, continuous representation, with some mild restriction such that the patching method applies. For some tame level K^p, we have*

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(r, \widetilde{H}^1(K^p)_{\mathfrak{m}}[\frac{1}{p}]) = \pi(r_p)$$

where $\mathfrak{m}$ is the maximal ideal of the Hecke algebra corresponding to $\bar{r}$ and π is the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

When $F/\mathbb{Q}$ is nontrivial, the natural generalization of modular curves are Shimura curves or 0-dimensional arithmetic manifolds associated to unitary groups $U(2)$ that are compact at infinity. We assume p to be inert in F. We now fix the tame level K^p and omit it from the notation. For simplicity, we assume that $\bar{r}$ is absolutely irreducible. Motivated by local-global compatibility, one would expect $\pi(r_p)$ to be $\widetilde{H}^i_{\mathfrak{m}}[\mathfrak{p}]$ where $\mathfrak{p}$ is the prime ideal corresponding to r (cf. [CEG+18]). However, $\widetilde{H}^i_{\mathfrak{m}}[\mathfrak{p}]$ is a global object, and it is unclear whether it is purely local, that it only depends on $r|_{G_{F_p}}$, which we will denote by r_p.

As discussed above (cf. Conjecture 1.2), if r corresponds to the automorphic representation π, then we have r_ℓ corresponding to π_ℓ for $\ell \neq p$ by the local-global compatibility. Let V be the algebraic representation corresponding to the Hodge–Tate weights of r_p. It is expected that r_p can be recovered from π_p and V. Let $(\pi_p \otimes_E V)^\circ := (\pi_p \otimes_E V) \cap \widetilde{H}^i_{\mathfrak{m}}$, which is a $\mathrm{GL}_2(F_p)$-invariant $\mathcal{O}$-lattice inside $\pi_p \otimes_E V$. By completing with respect to this lattice, we obtain a unitary Banach space $\widehat{\pi_p \otimes V}$ such that $\widehat{\pi_p \otimes V} \hookrightarrow \widetilde{H}^i_{\mathfrak{m}}[\frac{1}{p}]$. As $\bar{r}$ is absolutely irreducible, r can be conjugated to a $\mathcal{O}$-lattice r°, then one would expect $(\pi_p \otimes_E V(\lambda - \eta))^\circ$ to be uniquely determined by $r^\circ|_{G_{F_p}}$. Indeed, one would expect the p-adic Langlands correspondence to hold even at the integral level.

Breuil suggested a lattice conjecture [Bre14], upon proving which will provide evidence for such a claim. On the Galois side, an inertial type τ is a continuous representation of I_K with an open kernel and can be extended to G_K. An inertial type τ associated to a Galois representation ρ measures how far ρ fails to be crystalline. In particular, ρ is crystalline if and only if $\tau = 1$. By the inertial local Langlands correspondence in the appendix of [BM02] by Henniart, an inertial type τ corresponds to a type $\sigma(\tau)$, which is a representation of $\mathrm{GL}_2(\mathcal{O}_K)$ over E. The type $\sigma(\tau)$ determines the Bernstein component of π_p. We have the following Breuil's Lattice Conjecture.

Conjecture 1.5 (Breuil's Lattice Conjecture)**.** *The* $\mathrm{GL}_2(\mathcal{O}_{F_p})$*-invariant* $\mathcal{O}$*-lattice*

$$\sigma^\circ(\lambda,\tau) := ((\sigma(\tau) \otimes V(\lambda - \eta))) \cap \widetilde{H}^i_{\mathfrak{m}}[\mathfrak{p}]$$

is uniquely determined by $r^\circ|_{G_{F_p}}$.

Breuil showed that there are many homothety classes of lattices in $\sigma(\tau) \otimes V(\lambda - \eta)$ [Bre14, Théorème 1.1]. When the local Galois representation is Barsotti–Tate (the Hodge–Tate weights are $(1,0)$ at all embeddings [Bre00]) and of principal series type, in the setting of Shimura curves, Breuil then conjectured that $\sigma^\circ(\lambda,\tau)$ is determined by the Dieudonné module associated to r_p [Bre14, Conjecture 1.2]. Under a mild genericity condition on $\bar{r}|_{F_p}$ and the usual Taylor–Wiles conditions, Emerton, Gee and Savitt proved Breuil's Lattice Conjecture for tame inertial type in the Barsotti–Tate case [EGS15, Theorem B].

1.3 SUMMARY OF RESULTS

In this paper, our main theorem generalizes Conjecture 1.5 to higher Hodge–Tate weights, as predicted by [EGS15].

Theorem 1.6. *(Theorem 5.13) Fix $n \geq 1$. If the gap between the Hodge–Tate weights of r_p is between 1 and n, $\overline{r}_p$ and τ are sufficiently generic (linearly dependent on n), then the lattice $\sigma^\circ(\lambda, \tau)$ depends only on r_p.*

In Theorem 5.13, we give an explicit formula for the lattice in terms of the Breuil–Kisin modules associated to r_p. In order to allow the Hodge–Tate weights to vary at different embeddings and avoid parity issues, we use arithmetic manifolds associated with unitary groups, instead of an inner form of GL_2.

Representation theory result

Given a Serre weight $\kappa \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$, we can find a unique, up to homothety, $\mathrm{GL}_2(\mathcal{O}_{F_p})$-invariant $\mathcal{O}$-lattice inside $\sigma(\lambda, \tau)$, with cosocle κ, which we label as σ_κ. Inspired by the approach of [EGS15], we study the cosocle filtration of σ_κ, and of its reduction $\overline{\sigma}_\kappa$. We call an irreducible representation of $\mathrm{GL}_2(\mathcal{O}_K)$ over $\mathbb{F}$ a *Serre weight*; equivalently, it is an irreducible representation of $\Gamma := \mathrm{GL}_2(k)$ over $\mathbb{F}$. Putting an edge between two Serre weights for which there exists a nontrivial extension, we form an extension graph [LMS22] (with the idea coming from [LLHLM20]). Let $\mathrm{Inj}_\Gamma \sigma$ be the injective envelope of σ in the category of representations of $\mathrm{GL}_2(k)$ over $\mathbb{F}$. Assuming σ, τ are Serre weights, with τ an irreducible subquotient of $\mathrm{Inj}_\Gamma \sigma$, Breuil and Paškūnas showed in [BP12, Corollary 3.12]. that there is a unique representation $I(\sigma, \tau)$ with socle σ and cosocle τ, which is multiplicity free and whose cosocle filtration is given by the extension graph between τ and σ. However, these injective envelopes in the category of representation of $\mathrm{GL}_2(k)$ over $\mathbb{F}$ are too small for our purposes. Let K_1 be the first congruence subgroup of $\mathrm{GL}_2(\mathcal{O}_K)$, Z be the centre of $\mathrm{GL}_2(\mathcal{O}_K)$, and let $Z_1 := Z \cap K_1$. We have the Iwasawa algebra $\mathbb{F}[\![K_1/Z_1]\!]$ which is local with maximal ideal $\mathfrak{m}_{K_1}$. We abuse notation and denote the ideal generated by the image of $\mathfrak{m}_{K_1}$ under the natural inclusion $\mathbb{F}[\![K_1/Z_1]\!] \hookrightarrow \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]$ also as $\mathfrak{m}_{K_1}$. Then $\mathbb{F}[\Gamma] = \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]/\mathfrak{m}_{K_1}$. Instead of representations of $\mathrm{GL}_2(k)$ over $\mathbb{F}$, we consider representations of $\mathrm{GL}_2(\mathcal{O}_K)$ over $\mathbb{F}$ killed by $\mathfrak{m}_{K_1}^n$ for some fixed positive integer n. Let $\mathrm{Inj}\, \sigma$ be the injective envelope of σ in the category of smooth representations of $\mathrm{GL}_2(\mathcal{O}_K)$ over $\mathbb{F}$. We generalize the results in [BP12, § 3-4]($n = 1$) and [HW18, § 2]($n = 2$), and obtain the following theorem. (The notion of m-generic Serre weight is defined in Definition 2.3.)

Theorem 1.7. *(Theorem 2.10) Assuming σ, τ are Serre weights, which are $(2n-1)$-generic, and $\tau \in \mathrm{JH}((\mathrm{Inj}\, \sigma)[\mathfrak{m}_{K_1}^n])$, there is a unique multiplicity-free representation $I(\sigma, \tau)$ of $\mathrm{Inj}\, \sigma$ with cosocle τ. Moreover, the cosocle filtration of $I(\sigma, \tau)$ is determined by the extension graph between τ and σ. In particular, if $\tau \in \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$, then $I(\sigma, \tau)$ recovers the Γ-representation defined in [BP12, Corollary 3.12].*

This theorem not only allows us to deduce the submodule structure of σ_κ, but also allows us to deduce that certain subquotients of σ_κ are Γ-representations.

Galois deformation ring result

Another key input for the proof of the lattice conjecture is the notion of a patching functor, which was first developed in [EGS15]. We let R_∞ be a suitable power series ring over $R^{\square}_{\overline{r}_p}$, the universal framed deformation ring of $\overline{r}_p$. A patching functor M_∞ is a functor from the category of finitely generated $\mathcal{O}$-modules with a continuous $\mathrm{GL}_2(\mathcal{O}_{F_p})$-action to the category of coherent sheaves over R_∞ satisfying some natural properties. A fundamental property of a patching functor is that

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_{F_p})}(\sigma_\kappa, \widetilde{H}[\mathfrak{p}]) = (M_\infty(\sigma_\kappa)/\mathfrak{p})^\vee \tag{1.1}$$

for a prime ideal $\mathfrak{p} \subseteq R_\infty$ corresponding to r_p. Let $R^{\lambda,\tau}_{\overline{\rho}}$ (resp. $R^{\leq\lambda,\tau,\mathrm{reg}}_{\overline{\rho}}$) be the Galois deformation ring which parametrizes potentially crystalline lifts of $\overline{\rho}$ with Hodge–Tate weights λ (resp. regular $\lambda' \leq \lambda$) and with inertial type τ. As the action of R_∞ on $M_\infty(\sigma_\kappa)$ factors through $R_\infty \otimes_{R^{\square}_{\overline{r}_p}} R^{\lambda,\tau}_{\overline{r}_p}$, we need to compute the Galois deformation rings $R^{\lambda,\tau}_{\overline{r}_p}$ with explicit genericity bounds. (The results in [LLHLM23] are insufficient for our purpose, as the genericity bound is not explicit.)

We generalize the result in [BHH+23, § 4], which is based on the method developed in [LLHLM18] and [LLHLM23]. Let $\lambda_j := (\ell_j, 0)$ for some positive integers ℓ_j for each j and let $n := \max\{\ell_j\}$. Let $W(\overline{\rho})$ denote the set of modular Serre weights of $\overline{\rho}$ defined in [BDJ10]. We compute some explicit height and monodromy conditions and deduce that

$$R^{\leq\lambda,\tau,\mathrm{reg}}_{\overline{\rho}}[\![X_1, \dots, X_{2f}]\!] \cong (R/\sum_j I^{(j)})[\![Y_1, \dots, Y_4]\!],$$

where R is a certain power series ring over $\mathcal{O}$, and $I^{(j)}$ is generated by a set of equations that are explicit modulo p^n and reg denotes the quotient that kills all components of non-maximal dimension. On the other hand, from these explicit equations we can deduce that $p^{2n+1} \in H$, where H is the ideal used in Elkik's approximation. With these two calculations, we deduce the following theorem.

Theorem 1.8. *(Corollary 3.30) Assume that $\overline{\rho}$ is $(4n+1)$-generic and the tame type τ is $(2n+1)$-generic. If $W(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\lambda, \tau)) \neq \emptyset$, then*

$$R^{\lambda,\tau}_{\overline{\rho}} \cong \mathcal{O}[\![(x_j, y_j)_{j=1}^m, Z_1, \dots, Z_{f-m+4}]\!]/(x_j y_j - p)_{1\leq j\leq m}$$

for some positive integer m. Recall that m is determined by $2^m = |W(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\lambda,\tau))|$. *In particular,* $R_{\overline{\rho}}^{\lambda,\tau}$ *is a normal domain and a complete intersection ring. Moreover, the special fibre* $\overline{R}_{\overline{\rho}}^{\lambda,\tau}$ *is reduced, and every component of the special fibre is formally smooth over* $\mathbb{F}$ *and can be identified explicitly with* $W(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$ *via the isomorphism above.*

This explicit description of the Galois deformation rings for higher Hodge–Tate weights was only previously known for $\lambda \leq (3,0)$ [BHH+23, § 4], [Wan23, § 4], and may be of independent interest. We showed that these Galois deformation rings are complete intersection rings, generalizing the result in [HP19, Theorem 1.1], although with a more restrictive bound on the Hodge–Tate weights. We can thus deduce that certain global Galois deformation rings is p-torsion free, and the "$R_{\overline{r}}^{\mathrm{univ}} = \mathbb{T}$" theorem holds without inverting p (see [HP19, § 8]). Moreover, the property of complete intersection may have applications to derived Galois deformation rings [GV18].

Inducting on the distance in the extension graph, we deduce that

$$M_\infty(\sigma_\kappa) = \varpi(\kappa,\kappa')M_\infty(\sigma_{\kappa'})$$

where $\varpi(\kappa,\kappa')$ is given by a certain element in $R_{\overline{r}_p}^{\lambda,\tau}$. Since any lattice σ° inside $\sigma(\lambda,\tau)$ can be written as $\sum_{\kappa \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))} p^{v(\kappa)}\sigma_\kappa$ such that $p^{v(\kappa)}\sigma_\kappa \hookrightarrow \sigma^\circ$ is saturated. We conclude the lattice conjecture using Equation (1.1).

Cyclicity of patched modules

In this paper, we also show that certain patched modules are cyclic, which is closely related to proving a multiplicity one result at the Iwahori level and a conjecture of Dembélé (appendix of [Bre14]).

Theorem 1.9. *(Theorem 6.2) Under some mild genericity conditions (dependent linearly on the gap of the Hodge–Tate weights) on* $\overline{r}_p$ *and* τ*, given a minimal unramified patching functor* M_∞*,* $M_\infty(\sigma_\kappa)$ *is a cyclic module over its scheme-theoretic support.*

Its scheme-theoretic support is irreducible by Theorem 1.8, and hence it is sufficient to show that $M_\infty(\overline{\sigma}_\kappa)$ is a cyclic R_∞-module by Nakayama's lemma. Since the patching functor is an exact functor, by Theorem 1.7, we can show that $M_\infty(\overline{\sigma}_\kappa) = M_\infty(W)$ where W is a subquotient of $\overline{\sigma}_\kappa$ and is isomorphic to a quotient of a lattice of $\sigma(\tau')$ for another tame type τ'. We therefore deduce our theorem from the analogous result proven in the potentially Barsotti–Tate case in [EGS15, Theorem 10.1.1].

Candidate for the mod p Langlands Correspondence

Now let F be a totally real number field in which p is unramified. Fix a place v lying above p. Let D be a quaternion algebra with centre F, which splits at exactly one infinite place. Fix U^v a compact open subgroup of $D \otimes_F \mathbb{A}_{F,f}^v$. Given a compact open subgroup U of $(D \otimes_F \mathbb{A}_{F,f})^\times$, we let X_U be the associated smooth projective Shimura curve over F. Letting U_v run over compact open subgroups of $(D \otimes_F F_v)^\times \cong \mathrm{GL}_2(F_v)$, we consider

$$\pi(\overline{\rho}) := \varinjlim_{U_v} \mathrm{Hom}_{G_F}(\overline{r}, H^1_{\text{ét}}(X_{U^v U_v} \times_F \overline{F}, \mathbb{F})),$$

which is a smooth admissible representation of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$. A priori, $\pi(\overline{\rho})$ depends on $\overline{r}$ rather than $\overline{\rho} := \overline{r}|_{G_{F_v}}$, as it is the global candidate to correspond to $\overline{\rho}$ under the conjectural mod p Langlands correspondence, but we will abuse notation and write it as $\pi(\overline{\rho})$. By [LMS22, Theorem 1.1], [HW18, Theorem 1.1] and [Le19, Theorem 1.1], we deduce that $\pi^{K_1} = \pi[\mathfrak{m}_{K_1}]$ is the maximal Γ-representation with socle given by $W(\overline{\rho})$ (same as Theorem 1.10) and is multiplicity-free. We have the following result regarding π, which generalizes the result above.

Theorem 1.10. *(Corollary 7.3) Under a genericity condition on σ depending on n, $\pi[\mathfrak{m}_{K_1}^n]$ is the largest multiplicity-free representation of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ with socle $\bigoplus_{\sigma \in W(\overline{\rho})} \sigma$, which is killed by $\mathfrak{m}_{K_1}^n$.*

Again, Theorem 1.7 plays an important role in the proof, as it allows us to reduce the statement to the case where $n = 2$, which was previously proven in [BHH+23, Theorem 8.4.2] and [HW22, Corollary 8.13].

1.4 NOTATION

If F is a field, we write $G_F := \mathrm{Gal}(\overline{F}/F)$. If F is a number field and v is a finite place in F, we write F_v for the completion of F at v, and we denote the ring of integers of F_v as $\mathcal{O}_{F_v}$ with residue field k_v. We fix an embedding $\overline{F} \hookrightarrow \overline{F}_v$, which allows us to identify the decomposition group of F at v with G_{F_v}. If $K/\mathbb{Q}_p$ is finite, we write I_K for the inertial subgroup of G_K. We normalize the Artin's reciprocity map so that the uniformizers are mapped to geometric Frobenius elements. We will always have $p > 3$.

We assume E to be a finite extension over $\mathbb{Q}_p$, which is sufficiently large, in particular, E contains all embeddings of F. We will take E to be unramified in Chapter 6. We denote $\mathbb{F}$ for the residue field of E and ϖ for the uniformizer of E and $\mathcal{O}$ the ring of integers of E. We use $[x]$ to denote the Technüller lift of x. We write ε (resp. ω) for the p-adic (resp. mod p) cyclotomic character. We write ω_f for the Serre's fundamental character of level f.

If F is a p-adic field and V is a de Rham representation of G_F over E, then for each embedding $\kappa\colon F \hookrightarrow E$, we have $\mathrm{HT}_\kappa(V)$, the multiset of Hodge–Tate weights of V with respect to κ. We take the normalization such that $\mathrm{HT}_\kappa(\varepsilon) = \{1\}$ for all embeddings κ.

Let K be an unramified extension of $\mathbb{Q}_p$ of degree f with residue field k. We fix an embedding $\sigma_0\colon k \hookrightarrow \mathbb{F}$. If φ is the arithmetic Frobenius and we let $\sigma_j := \sigma_0 \circ \varphi^j$, then we can identify $\mathcal{J} := \mathrm{Hom}(k, \mathbb{F})$ with $(\mathbb{Z}/f\mathbb{Z})$ via $\sigma_j \leftrightarrow j$.

If V is a finite-dimensional representation of a group G over $\mathcal{O}$, then we denote by $\overline{V}$ the reduction modulo ϖ of the semi-simplification of a G-stable $\mathcal{O}$-lattice in V. For readability, we write $\overline{\sigma}(\lambda, \tau)$ instead of $\overline{\sigma(\lambda, \tau)}$, $\overline{\sigma}_\kappa$ instead of $\overline{\sigma_\kappa}$ etc. If R is a ring (for example, $\mathbb{F}[\![G]\!]$) and M is a left R-module, we denote by $\mathrm{soc}(M)$ (resp. $\mathrm{cosoc}(M)$) for the socle (resp. cosocle) of M. (See [HW22, Definition A.3].) We can then define the socle and cosocle filtration of M inductively. If M is of finite length, we denote by $\mathrm{JH}(M)$ the Jordan–Hölder factors (i.e., the multiset of the composition factors). In the case where M is a finite representation of G, this is the set of Jordan–Hölder factors of M in the usual sense. If σ is a simple R-module and M is a finite length R-module, and we denote the multiplicity of σ in M as the number of times σ appears in $\mathrm{JH}(M)$.

If $s \in S_2$ is a permutation, we let $\mathrm{sgn}(s) \in \{\pm 1\}$ be the signature of s.

We write $\sigma^\vee$ for the Pontryagin dual $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\sigma, E/\mathcal{O})$ and V^d for the Schikhof dual $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(V, \mathcal{O})$.

2 REPRESENTATION THEORY RESULTS

2.1 NOTATION AND BACKGROUND

First, we recall some notation and results concerning the extension graph for GL_2 from [BHH+23, § 2]. Let K be a fixed finite unramified extension of $\mathbb{Q}_p$ of degree f, with $\mathcal{O}_K$ its ring of integers and k is the residue field. We consider the group scheme GL_2 defined over $\mathbb{Z}$, let $T \subseteq GL_2$ be the diagonal maximal torus and Z its centre. We write R for the set of roots of (GL_2, T), W for its Weyl group, with the longest element $\mathfrak{w}$. Let G_0 be the algebraic group $\mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}_p} GL_{2/\mathcal{O}_K}$ with T_0 the diagonal maximal torus and the centre Z_0. Let $\underline{G}$ be the base change $G_0 \times_{\mathbb{Z}_p} \mathcal{O}$, and similarly define $\underline{T}$ and $\underline{Z}$. Let $\underline{R}$ denote the set of roots of $(\underline{G}, \underline{T})$.

There is a natural isomorphism $\underline{G} \cong \prod_{\mathcal{J}} GL_{2/\mathcal{O}}$, and analogously for $\underline{T}$, $\underline{Z}, \underline{R}$. We identify $X^*(\underline{T})$ with $(\mathbb{Z}^2)^{\mathcal{J}}$, and so for $\mu \in X^*(\underline{T})$, we write correspondingly $\mu = (\mu_j)_{j\in\mathcal{J}}$. We let $\eta_j \in X^*(\underline{T})$ be $(1,0)$ in the jth coordinate and 0 otherwise. We write $\eta_J := \sum_{j\in J} \eta_j$ for all $J \subseteq \mathcal{J}$. We define $\eta := \eta_{\mathcal{J}}$. Let $\alpha_j \in \underline{R}$ be $(1,-1)$ in the jth coordinate and 0 otherwise. The set of positive roots of $\underline{G}$ with respect to the upper triangular Borel in each embedding is given by $\underline{R}^+ = \{\alpha_j : 0 \leq j \leq f-1\}$. We have the following definitions;

$$\text{dominant weights: } X^*_+(\underline{T}) := \{\lambda \in X^*(\underline{T}) : 0 \leq \langle \lambda, \alpha^\vee \rangle \text{ for all } \alpha \in \underline{R}^+\}.$$

$$p\text{-restricted weights: } X_1(\underline{T}) := \{\lambda \in X^*_+(\underline{T}) : 0 \leq \langle \lambda, \alpha^\vee \rangle \leq p-1 \text{ for all } \alpha \in \underline{R}^+\}.$$

$$\text{regular wights: } X_{\mathrm{reg}}(\underline{T}) := \{\lambda \in X^*_+(\underline{T}) : 0 \leq \langle \lambda, \alpha^\vee \rangle < p-1 \text{ for all } \alpha \in \underline{R}^+\}.$$

$$X^0(\underline{T}) := \{\lambda \in X^*_+(\underline{T}) : \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in \underline{R}^+\}.$$

The lowest alcove is defined as $\underline{C}_0 := \{\lambda \in X^*(\underline{T}) \otimes \mathbb{R} : 0 < \langle \lambda + \eta, \alpha^\vee \rangle < p \text{ for all } \alpha \in \underline{R}^+\}$. Let $\underline{W}$ be the affine Weyl group of $(\underline{G}, \underline{T})$. We can identify $\underline{W}$ with $\prod_{j\in\mathcal{J}} W$, which acts on $X^*(\underline{T}) \cong (\mathbb{Z}^2)^{\mathcal{J}}$ in a compatible manner. Let $\underline{W}_a \cong \Lambda_R \rtimes \underline{W}$ be the affine Weyl group, where Λ_R is the root lattice. And let $\widetilde{W} \cong X^*(\underline{T}) \rtimes \underline{W}$ be the extended Weyl group. For $\lambda \in X^*(\underline{T})$, we denote t_λ its image in $\widetilde{W}$. We have p-dot action of $\widetilde{W}$ on $X^*(\underline{T})$, defined as follows: if $\widetilde{w} = wt_v \in \widetilde{W}$ and $\mu \in X^*(\underline{T})$, then $\widetilde{w} \cdot \mu := w(\mu + \eta + pv) - \eta$.

Let Ω be the stabilizer of the lowest alcove C_0 in $\widetilde{W}$. One checks that $\widetilde{W} = \underline{W}_a \rtimes \Omega$ and Ω is the subgroup of $\widetilde{W}$ generated by $X^0(\underline{T})$ and $\{1, \mathfrak{w} t_{-(1,0)}\}$.

A Serre weight of $\underline{G}_0 \times_{\mathbb{Z}_p} \mathbb{F}_p$, or simply a Serre weight if it is clear from the context, is an isomorphism class of an absolutely irreducible representation of $\underline{G}_0(\mathbb{F}_p) = \mathrm{GL}_2(k)$ over $\mathbb{F}$. If $\lambda \in X_1(\underline{T})$, we write $L(\lambda)$ for the irreducible algebraic representation of $\underline{G} \times_{\mathcal{O}} \mathbb{F}$ of highest weight λ, and $F(\lambda)$ for the restriction of $L(\lambda)$ to the group $\underline{G}_0(\mathbb{F}_p)$. We define an automorphism π on $X^*(\underline{T})$ by $\pi(\mu)_j := \mu_{j-1}$. The map $\lambda \mapsto F(\lambda)$ induces a bijection between $X_1(\underline{T})/(p-\pi)X^0(\underline{T})$ and the set of Serre weights of $\underline{G}_0 \times_{\mathbb{Z}_p} \mathbb{F}_p$. A Serre weight σ is regular if $\sigma \cong F(\lambda)$ with $\lambda \in X_{\mathrm{reg}}(\underline{T})$.

Let $\Lambda_W := X^*(\underline{T})/X^0(\underline{T})$ denote the weight lattice of $(\mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}_p} \mathrm{SL}_{2/\mathcal{O}_K}) \times_{\mathbb{Z}_p} \mathcal{O}$. We identify Λ_W with $\mathbb{Z}^f$ as above. Given $\mu \in X^*(\underline{T})$, we define

$$\Lambda_W^\mu := \{\omega \in \Lambda_W \colon 0 \leq \langle \overline{\mu} + \omega, \alpha^\vee \rangle < p-1 \text{ for all } \alpha \in \underline{R}^+\},$$

where $\overline{\mu}$ denotes the image of μ in Λ_W. The set Λ_W^μ is called the extension graph associated to μ. Moreover, we define $\Sigma \subseteq \Lambda_W$ as the set $\{\overline{\eta}_J \colon J \subseteq \mathcal{J}\}$.

We construct a map $\mathfrak{t}'_\mu \colon X^*(\underline{T}) \to X^*(\underline{T})/(p-\pi)X^0(\underline{T})$ as follows: given $\omega' \in X^*(\underline{T})$, there is a unique $\widetilde{\omega}' \in \Omega \cap t_{-\pi^{-1}(\omega')}\underline{W}_a$. We set

$$\mathfrak{t}'_\mu(\omega') := \widetilde{\omega}' \cdot (\mu + \omega') \bmod (p-\pi)X^0(\underline{T}).$$

This map factors through $X^*(\underline{T})/X^0(\underline{T}) \cong \Lambda_W$ by the definition. Therefore, we have an induced map

$$\mathfrak{t}_\mu \colon \Lambda_W^\mu \to X_{\mathrm{reg}}(\underline{T})/(p-\pi)X^0(\underline{T}).$$

This map gives a bijection between Λ_W^μ and regular Serre weights with central character $\mu|_{\underline{Z}_0(\mathbb{F}_p)}$.

We have the following "change of origin" formula for the map $\mathfrak{t}_\mu$. For $\omega \in \Lambda_W^\mu$, we take a lift $\omega' \in X^*(\underline{T})$. Then we define w_ω as the unique image of $\widetilde{\omega}'$ under the map $\Omega \cap t_{-\pi^{-1}(\omega')}\underline{W}_a \to \underline{W}$ as above. It can be easily checked that w_ω does not depend on the choice of the lift.

Lemma 2.1. *([BHH+23, Lemma 2.4]) Let $\omega \in \Lambda_W^\mu$ and let $\lambda \in X^*(\underline{T})$ be such that $\mathfrak{t}_\mu(\omega) \equiv \lambda$ mod $(p-\pi)X^0(\underline{T})$. Then $w_\omega^{-1}(\beta) + \omega \in \Lambda_W^\mu$ and $\mathfrak{t}_\lambda(\beta) = \mathfrak{t}_\mu(w_\omega^{-1}(\beta) + \omega)$ for all $\beta \in \Lambda_W^\lambda$. Equivalently $\mathfrak{t}_\mu(\omega') = \mathfrak{t}_\lambda(w_\omega(\omega' - \omega))$ for all $\omega' \in \Lambda_W^\mu$.*

Following [BHH+23, Remark 2.4.7], we see that the change of origin map is a graph automorphism.

Let K_1 be the first principal congruence subgroup, i.e., the kernel of the mod p reduction morphism $\mathrm{GL}_2(\mathcal{O}_K) \twoheadrightarrow \mathrm{GL}_2(k)$. Let Z be the centre of $\mathrm{GL}_2(K)$ and let $Z_1 := Z \cap K_1$. We have an Iwasawa algebra $\mathbb{F}[\![K_1/Z_1]\!]$ and we denote its maximal ideal by $\mathfrak{m}_{K_1}$. Abusing notation, we denote the ideal generated by the image of $\mathfrak{m}_{K_1}$ under the natural inclusion $\mathbb{F}[\![K_1/Z_1]\!] \hookrightarrow \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_k)/Z_1]\!]$ as $\mathfrak{m}_{K_1}$. Let $\Gamma := \mathrm{GL}_2(k)$, then $\mathbb{F}[\![\Gamma]\!] = \mathbb{F}[\![K_1/Z_1]\!]/\mathfrak{m}_{K_1}$. We now begin to develop some terminology for the $\mathfrak{m}_{K_1}^n$-torsion representation for some small n.

Definition 2.2. Given a Serre weight σ, by inflation, we consider it as an admissible smooth $\mathbb{F}$-representation of $\mathrm{GL}_2(\mathcal{O}_K)/Z_1$. Then we define $\mathrm{Proj}\,\sigma^\vee$ (respectively $\mathrm{Proj}\,\sigma^\vee$) as the projective (respectively injective) envelope of $\sigma^\vee$ in the category of psuedo-compact $\mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z]\!]$-modules. Let $\mathrm{Inj}\,\sigma$ (respectively $\mathrm{Proj}\,\sigma$) be the algebraic dual of $\mathrm{Proj}\,\sigma^\vee$ (respectively $\mathrm{Inj}\,\sigma^\vee$). Define $\mathrm{Inj}_n\,\sigma$ (respectively $\mathrm{Proj}_n\,\sigma$) to be $(\mathrm{Inj}_{K/Z_1}\,\sigma)[\mathfrak{m}_{K_1}^n]$ (respectively $(\mathrm{Proj}_{K/Z_1}\,\sigma)/\mathfrak{m}_{K_1}^n$). Noted that $\mathrm{Inj}_1\,\sigma = \mathrm{Inj}_\Gamma\,\sigma$, the injective envelope in the category of admissible smooth representation of Γ. We further define $V^0 := 0$ and $V^n := V[\mathfrak{m}_{K_1}^n]$ for positive integers n. Fix a Serre weight σ, then a Serre weight τ is called an n-weight (with respect to σ) if τ is a subquotient of $\mathrm{Inj}_n\,\sigma$ but not of $\mathrm{Inj}_{n-1}\,\sigma$.

Definition 2.3. Given $\tau = F(\mathfrak{t}_\mu(\omega))$, we let $\widetilde{\tau} := F(\mathfrak{t}_\mu(\widetilde{\omega}))$ such that $\widetilde{\omega}_j = 2\lfloor \frac{\omega_j}{2} \rfloor$ if $\omega_j \geq 0$ and $\widetilde{\omega}_j = 2\lceil \frac{\omega_j}{2} \rceil$ if $\omega_j \leq 0$. Note that $|\widetilde{\omega}_j| = 2\lfloor \frac{|\omega_j|}{2} \rfloor$. Given $\sigma = F(\mu)$, let

$$\Delta^k(\sigma) := \{F(\mathfrak{t}_\mu(\omega)) \colon \omega \in \mathbb{Z}^f, \omega_j \in 2\mathbb{Z} \text{ for all } j \text{ and } \sum_j \frac{|\omega_j|}{2} = k\}.$$

We say $\mu \in \underline{C_0}$ is N-deep if $N < \langle \mu + \eta, \alpha \rangle < p - N$ for all $\alpha \in \underline{R}^+$ and $F(\mu)$ is N-generic if μ is N-deep.

All the 1-weights, i.e., subquotients of $\mathrm{Inj}_\Gamma\,\sigma$ are described in the following lemma.

Lemma 2.4. *[HW22, Lemma 6.2.1] Suppose that $F(\mu)$ is 0-generic. The set of Jordan–Hölder factors of $\mathrm{Inj}_1\,F(\mu)$ is given by $\{F(\mathfrak{t}_\mu((a_j)_j)) \colon a_j \in \{0, \pm 1\}$ for all $j \in \mathcal{J}\}$.*

Lemma 2.5. *Suppose σ is a $(2n-1)$-generic Serre weight.*

$$\mathrm{Inj}_n\,\sigma / \mathrm{Inj}_{n-1}\,\sigma \cong \bigoplus_{i=0}^{n-1} \bigoplus_{\delta \in \Delta^i(\sigma)} (\mathrm{Inj}_1\,\delta)^{\oplus k_i},$$

for all $k_i \in \mathbb{Z}_{>0}$ with $k_{n-1} = 1$.

Proof. Consider the dual (*cf.* [Alp86, Proposition 18.4])

$$\mathfrak{m}_{K_1}^{n-1}\mathrm{Proj}_{K/Z_1}\sigma^\vee / \mathfrak{m}_{K_1}^n \mathrm{Proj}_{K/Z_1}\sigma^\vee \cong (\mathfrak{m}_{K_1}^{n-1}/\mathfrak{m}_{K_1}^n) \otimes_{\mathbb{F}} \mathrm{Proj}_1\,\sigma^\vee.$$

For $p > 3$, the group K_1/Z_1 is uniform, so the ring $\mathbb{F}[\![K_1/Z_1]\!]$ is a polynomial ring. Therefore,

$$\mathfrak{m}_{K_1}^{n-1}/\mathfrak{m}_{K_1}^n \cong \mathrm{Sym}^{n-1}(\mathfrak{m}_{K_1}/\mathfrak{m}_{K_1}^2).$$

Moreover,

$$\mathfrak{m}_{K_1}/\mathfrak{m}_{K_1}^2 \cong \oplus_{j=0}^{f-1} F((1,-1)^{(j)}),$$

and

$$F(a,b) \otimes F((1,-1)^{(j)}) \cong F((a_i+\delta_{ij}, b_i-\delta_{ij})_i) \oplus F((a_i,b_i)_i) \oplus F((a_i-\delta_{ij}, b_i+\delta_{ij})_i),$$

for all $2 \le a_j - b_j \le p-3$, which is always satisfied because of the genericity condition. (Here $(1,-1)^{(j)}$ denotes the weight vector, which is $(1,-1)$ in the jth coordinate and 0 otherwise, and δ_{ij} is the Kronecker delta.) The result then follows. □

Lemma 2.6. *Let $\sigma = F(\mu)$ be a $(2n-1)$-generic Serre weight and $\tau = F(\mathfrak{t}_\mu(\omega))$ be a k-weight, where $k \le n$. Then, $\tau \in \mathrm{Inj}_1 \widetilde{\tau}$, where $\widetilde{\tau} \in \Delta^{k-1}(\sigma)$. In particular, τ is a k-weight if and only $\sum_j \lfloor \frac{|\omega_j|}{2} \rfloor = k-1$. In such a case, τ is $2(n-k)$-generic and hence a regular Serre weight.*

Proof. If τ is a k-weight with respect to σ, then by Lemma 2.5, $\tau \in \mathrm{JH}(\mathrm{Inj}_1 \theta)$ for some $\theta \in \Delta^{k-1}(\sigma)$ and $\tau \notin \mathrm{JH}(\mathrm{Inj}_1 \theta')$ for all $\theta' \in \Delta^{k'}(\sigma)$ with $k' < k-1$.

Suppose $\tau \in \mathrm{Inj}_1 \theta$ where $\theta =: F(\mathfrak{t}_\mu(\xi)) \in \Delta^{k-1}(\sigma)$. Then $\xi_j \in 2\mathbb{Z}$ for all j and $\sum_j |\frac{\xi_j}{2}| = k-1$. We will show that $\xi = \widetilde{\omega}$. Assume for the sake of contradiction that there exists some j, $\xi_j \neq \widetilde{\omega}_j$, then as $\xi_j, \widetilde{\omega}_j \in 2\mathbb{Z}$, by the definition of $\widetilde{\omega}$, $\xi_j \neq \omega_j$. Therefore, we must have $|\xi_j| < |\widetilde{\omega}_j| \le |\omega_j|$ or $|\widetilde{\omega}_j| \le |\omega_j| \le |\xi_j|$. By Lemma 2.4, as $\tau \in \mathrm{JH}(\mathrm{Inj}_1 \theta)$, $|\xi_j - \omega_j| = 1$; hence the first scenario is impossible as $\xi_j, \widetilde{\omega}_j \in 2\mathbb{Z}$. If the second scenario holds, the $|\widetilde{\omega}_j| = |\xi_j| - 2$, and hence $\widetilde{\tau} \in \Delta^{k'}(\sigma)$, for some $k' < k$, also a contradiction. For the last part, since σ is a $2(n-1)$-generic Serre weight, $2n-2 < \langle \mu, \alpha_i^\vee \rangle < p-2n$. If $\tau = F(\mathfrak{t}_\mu(\omega))$ is a k-weight, then for all i,

$$2n-2k-1 < \langle \mu, \alpha_i^\vee \rangle - |\omega_i| \le \langle \mu+\omega, \alpha_i^\vee \rangle \le \langle \mu, \alpha_i^\vee \rangle + |\omega_i| < p-2n+2k-1.$$

Therefore, $\mathfrak{t}_\mu(\omega) \in \Lambda_W^\lambda$, and τ is a regular Serre weight. □

From now on, we will assume that $\sigma = F(\mu)$ and that all Serre weights are regular.

Definition 2.7. Given $\omega, \omega' \in \Lambda_W^\mu$, we write $\omega \le \omega'$, if for each j, $0 \le \omega_j \le \omega'_j$ or $0 \ge \omega_j \ge \omega'_j$. Suppose $\kappa = F(\mathfrak{t}_\mu(\omega)), \kappa' = F(\mathfrak{t}_\mu(\omega')), \kappa'' = F(\mathfrak{t}_\mu(\omega''))$ with $\omega, \omega', \omega'' \in \Lambda_W^\mu$. We write $\kappa' - \kappa \le \kappa'' - \kappa$ if we have $\omega' - \omega \le \omega'' - \omega$ in the above sense. Note that $\kappa' - \kappa \le \kappa'' - \kappa$ is equivalent to $\kappa' - \kappa'' \le \kappa - \kappa''$. We simply write $\kappa' \le \kappa''$ if $\kappa = F(\mu)$. Because of the bijection between Λ_W^μ and regular Serre weights with central character $\mu|_{\underline{Z}_0(\mathbb{F}_p)}$, given $\kappa = F(\mathfrak{t}_\mu(\omega))$, we sometimes simply write $\{\kappa' \le \kappa\}$ to denote the set $\{\kappa' = F(\mathfrak{t}_\mu(\omega')) \colon \omega' \le \omega\}$.

Fix $\tau = F(\mathfrak{t}_\mu(\omega))$ a regular Serre weight. We define the following:

$$\Omega_k^\tau := \{F(\mathfrak{t}_\mu(\omega')) \colon F(\mathfrak{t}_\mu(\omega')) \leq \tau \text{ and } \sum_j \left\lfloor \frac{|\omega'_j|}{2} \right\rfloor = k\}.$$

$${}^0\Omega_k^\tau := \{F(\mathfrak{t}_\mu(\omega')) \colon F(\mathfrak{t}_\mu(\omega')) \leq \tau, \omega'_j \in 2\mathbb{Z} \text{ for all } j \text{ and } \sum_j \frac{|\omega'_j|}{2} = k\} \subseteq \Omega_k^\tau.$$

Moreover, given $\kappa = F(\mathfrak{t}_\mu(\nu)) \in \Delta^m(\sigma)$ for some m. Let

$$(\nu_+)_k := \nu_k + \epsilon(\omega_k - \nu_k),$$

where $\epsilon(x) = \operatorname{sgn}(x)$ if $x \neq 0$ and $\epsilon(0) = 0$. Define $\kappa_+ := F(\mathfrak{t}_\mu(\nu_+))$. If $\kappa \leq \tau$, then $\kappa_+ \leq \tau$. The condition that $\xi \in {}^0\Omega_k^\tau$ is equivalent to $\xi \in \Delta^k(\sigma)$ and $\xi \leq \tau$.

Moreover, we define $\omega^{(i)}$ to be the element such that $\omega_k^{(i)} = \omega_k$ for some $k \neq i$ and $\omega_i^{(i)} = 0$. Define $\tau^{(i)} := F(\mathfrak{t}_\mu(\omega^{(i)}))$. We further define $\delta_i^{\epsilon_i}(\sigma) := F(\mathfrak{t}_\mu(2\epsilon_i\overline{\eta}_i))$ and $\mu_i^{\epsilon_i}(\sigma) := F(\mathfrak{t}_\mu(\epsilon_i\overline{\eta}_i))$.

Lemma 2.8. *Suppose σ, τ, τ' are regular Serre weights and τ, τ' are subquotients of* $\operatorname{Inj}_1 \sigma'$. *Then τ' occurs as a subquotient in $I(\sigma, \tau)$, if and only if $\tau' - \sigma \leq \tau - \sigma$.*

Proof. Suppose $\sigma = F(\mathfrak{t}_\mu(\gamma))$, $\tau := F(\mathfrak{t}_\mu(\omega))$, $\tau' := F(\mathfrak{t}_\mu(\omega'))$. We apply the change of origin formula Lemma 2.1, and send $F(\mathfrak{t}_\mu(\omega)) \mapsto F(\mathfrak{t}_\mu(\omega - \gamma))$. By Lemma 2.5, for all $\tau, \tau' \in \operatorname{JH}(\operatorname{Inj}_1 \kappa)$, we must have $(\omega - \gamma)_j, (\omega' - \gamma)_j \in \{-1, 0, 1\}$ for all j. In [BP12, Corollary 4.11], the condition for τ' to occur as a subquotient in $I(\kappa, \tau)$ is given by $\mathcal{S}(\lambda') \subseteq \mathcal{S}(\lambda)$ and λ, λ' being compatible. In our notation, $\mathcal{S}(\lambda) = \{j \colon \omega_j \neq 0\}$ and $\mathcal{S}(\lambda') = \{j \colon \omega'_j \neq 0\}$. Moreover, λ, λ' is compatible if and only if $\operatorname{sgn}(\omega_j) = \operatorname{sgn}(\omega'_j)$ when $\omega_j, \omega'_j \neq 0$. Therefore, the condition that $\mathcal{S}(\lambda') \subseteq \mathcal{S}(\lambda)$ and λ, λ' are compatible is equivalent to $\omega - \gamma \leq \omega' - \gamma$. □

Lemma 2.9. *For any Serre weight σ and any $\tau \in \operatorname{JH}(\operatorname{Inj}_n \sigma)$, there exists a subrepresentation V of $\operatorname{Inj}_n \sigma$ such that $\operatorname{cosoc}(V) = \tau$ and $[V : \sigma] = 1$ (hence σ occurs in V as a sub-object).*

Proof. The proof goes exactly as in [HW22, Lemma 2.22] with $\operatorname{Inj}_{\tilde{\Gamma}} \sigma$ (respectively, $\operatorname{Proj}_{\tilde{\Gamma}} \sigma$) replaced by $\operatorname{Inj}_n \sigma$ (respectively $\operatorname{Proj}_n \sigma$). □

2.2 MAIN RESULT

From now on, we will assume $n, m \in \mathbb{Z}_{\geq 1}$.

Theorem 2.10. *Let $\sigma = F(\mu)$ be a $(2n-1)$-generic Serre weight. Assume that V is a subrepresentation of $\mathrm{Inj}_n\,\sigma$ with irreducible cosocle τ and $[V : \sigma] = 1$. If τ is an m-weight, then V is multiplicity free, $\mathfrak{m}_{K_1}^m$-torsion (that is, $m = n$), and uniquely determined by σ, τ up to scalar multiplication. Moreover, for $0 \le k \le m-1$, we have*

$$V^{k+1}/V^k \cong \bigoplus_{\nu \in {}^0\Omega_k^\tau} I(\nu, \nu_+).$$

By Lemma 2.5, m is the smallest positive integer such that $\tau \in \mathrm{JH}(\mathrm{Inj}_m(\sigma))$.

As before, we denote such a representation by $I(\sigma,\tau)$. When τ is a 1-weight, then $I(\sigma,\tau)$ is a Γ-representation and coincides with the definition in [BP12, Corollary 3.12], and when τ is a 2-weight, $I(\sigma,\tau)$ also coincides with the definition in [HW22, Theorem 2.30]. This theorem is a generalization of [HW22, Theorem 2.23].

To elucidate the theorem, we have the following lemma as a remark.

Lemma 2.11. *Given $\nu \in {}^0\Omega_k^\tau$ for some $k < n$. Assume that κ is a regular Serre weight.*

(i) $\kappa - \widetilde{\kappa} \le \widetilde{\kappa}_+ - \widetilde{\kappa}$ if and only if $\kappa \le \tau$.

(ii) $\kappa \in \mathrm{JH}(I(\nu,\nu_+))$ if and only if $\nu = \widetilde{\kappa}$ and $\kappa \le \tau$. In this case, κ is a k-weight.

(iii) Assume that Theorem 2.10 holds, the Jordan–Hölder factors of V are exactly those κ where $\kappa \le \tau$.

(iv) Given $\nu' \in {}^0\Omega_{k'}^\tau$, $I(\nu,\nu_+)$ and $I(\nu',\nu'_+)$ do not share a common Jordan–Hölder factor if $\nu \ne \nu'$.

Proof. We can assume $\kappa = F(\mathfrak{t}_\mu(\omega'))$ and $\nu = F(\mathfrak{t}_\mu(\alpha))$.

(i) Assume $\omega' \le \omega$, then for each j, $0 \le \omega'_j \le \omega_j$ or $0 \ge \omega'_j \ge \omega_j$. If it is the former case, then it follows from the definition that $0 \le \widetilde{\omega}'_j \le \omega'_j \le \widetilde{\omega}'_{+j}$, and analogously for the latter case. Conversely, assume $\omega' - \widetilde{\omega}' \le \widetilde{\omega}'_+ - \widetilde{\omega}'$. For each j, if $0 \le 2\lfloor \frac{\omega'_j}{2} \rfloor \le \omega'_j \le \omega'_j + \epsilon(\omega_j - \omega'_j)$, then $0 \le \omega'_j \le \omega_j$. And the same result holds if $\le$ is replaced by $\ge$ and the floor function is replaced by the ceiling function.

(ii) If $\kappa \in \mathrm{JH}(I(\nu,\nu_+))$, by Lemma 2.8, $\kappa - \nu \le \nu_+ - \nu$, and hence $|\alpha_j| \le |\omega'_j| \le |\alpha_{+j}|$. As ν, ν_+ are $k+1$-weights, by Lemma 2.6, so is κ. Moreover, then by Lemma 2.6, we deduce that $\nu = \widetilde{\kappa}$. By (i), we deduce that $\omega' \le \omega$. The converse follows from (i) and Lemma 2.8.

(iii) Given $\kappa \in \mathrm{JH}(V)$, then $\kappa \in \mathrm{JH}(I(\nu,\nu_+))$ for some $\nu \in {}^0\Omega_k^\tau$, and by (ii), $\nu \cong \widetilde{\kappa}$ and $\kappa \le \tau$. Conversely, if $\kappa \le \tau$, by (ii), $\kappa \in \mathrm{JH}(I(\widetilde{\kappa},\widetilde{\kappa}_+))$ and the result follows by (ii).

(iv) It follows from (ii). □

These three corollaries follow immediately from the theorem.

Corollary 2.12. *Let V be a subrepresentation of $\operatorname{Inj}_n \sigma^{\oplus s}$ for some $s \geq 1$. Then for any irreducible Serre weight τ, we have $[V : \sigma] \geq [V : \tau]$. Moreover, if $\operatorname{cosoc}(V)$ is isomorphic to $\tau^{\oplus r}$ for some $(2n-1)$-generic Serre weight τ and some $r \geq 1$, then $[V : \sigma] = [V : \tau]$.*

Proof. It follows verbatim from [HW22, Corollary 2.3]. Since $\operatorname{soc}(V)$ has the form $\sigma^{\oplus s'}$ for some $s' \leq s$, we can construct a finite filtration of V such that each graded piece has socle isomorphic to σ and σ occurs only once there. Hence, we reduce it to the situation where $\operatorname{soc}(V) = \sigma$ and $[V : \sigma] = 1$, and the result follows from Theorem 2.10. The second assertion follows by duality. □

Corollary 2.13. *Assume that σ is $(2n-1)$-generic, and $\theta, \theta', \tau \in \operatorname{JH}(\operatorname{Inj}_n \sigma)$. Then θ' is a subquotient of $I(\theta, \tau)$ if and only if $\theta' - \theta \leq \tau - \theta$. Furthermore, if $\theta = F(\mathfrak{t}_\mu(\omega'))$ and $\tau = F(\mathfrak{t}_\mu(\omega))$, then the graded pieces of its socle filtration are given by:*

$$I(\theta, \tau)_k \cong \bigoplus_{\omega' - \omega \leq \omega'' - \omega, \sum_j |\omega''_j - \omega_j| = k} F(\mathfrak{t}_\mu(\omega'')).$$

In other words, the socle filtration coincides with the filtration given by the distance from the socle θ in the extension graph.

Proof. As in Lemma 2.8, by the change of origin map $F(\mathfrak{t}_\mu(\omega)) \mapsto F(\mathfrak{t}_\mu(\omega - \gamma))$, we change the origin to $\mathfrak{t}_\mu(\gamma)$. The proof follows the same way as in [HW22, Corollary 2.35], which reformulates the theorem using [BP12, Corollary 4.11], which is reinterpreted in light of Lemma 2.8 and Lemma 2.11. □

Remark 2.14. Intuitively, the theorem is saying that the Jordan Hölder factors of $I(\sigma, \tau)$ are given by the points in the extension graph within (including the boundary of) the rectangle with opposite corners given by σ and τ. Moreover, the socle filtration of $I(\sigma, \tau)$ is given by the distance from σ in the extension graph.

Comparing the results when $n = 1$ and $n = 2$ with [BP12] and [HW18], our genericity condition is higher by 1, because it is needed for the induction argument in Proposition 2.30.

The general strategy of the proof is as follows. We first need to handle the case where $n = 3$ and τ is right next to σ Proposition 2.20. In general, we first show that $\operatorname{soc}(V/V^1)$ are exactly those Serre weights inside the rectangle which are two apart from the origin (*cf.* Lemma 2.17, Proposition 2.24). (It is empty if τ is a 1-weight.) We need to make sure that these Serre weights appear with multiplicity one, in particular, they are not in $\operatorname{JH}(V/V^{n-1})$ Proposition 2.22. If $m < n$, then we apply the induction hypothesis to V/V^1 to deduce that V/V^1 is $\mathfrak{m}_{K_1}^{m-1}$-torsion Proposition 2.23 and finish the proof.

If $n = m$, the second step is to deduce that $\operatorname{JH}(V)$ is as conjectured by the theorem up to multiplicity Proposition 2.25. We deduce that $\operatorname{JH}(V/V^1)$ is correct using our induction hypothesis. Then, using certain Jordan–Hölder factors of V^2, we deduce that $\mathrm{JH}(V^1)$ is correct. In the example below for $n = 3$, the green dots denote $\mathrm{soc}(V/V^1)$ and the green rectangle is given by V/V^1. Then we know that $x, y \in \mathrm{JH}(V^2)$, and hence $\mathrm{JH}(V^2)$ contains and is indeed given by the points in the orange rectangle.

The third step is to prove that V is multiplicity free Proposition 2.28. As V^{n-1} is multiplicity free by assumption, it suffices to show that for V/V^{n-1}. We do so by considering certain quotients of V and applying the induction hypothesis. In the example below, the quotient is given by the blue rectangle.

Finally, we show the uniqueness of V by showing that the dimension of the extension between a subrepresentation of V and the quotient by this subrepresentation is one Proposition 2.30. Instead, we replace the subrepresentation (resp. the quotient) by its quotient (resp. its subrepresentation). By the induction hypothesis, we conclude that the extension of the subquotients has dimension one. In the example below, the subrepresentation (resp. quotient) is given by the red rectangle (resp. green rectangle), and the subquotients we replace them with are the two vertical line segments in the middle.

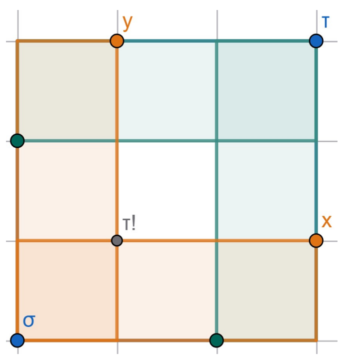

Figure 2.1: step 2

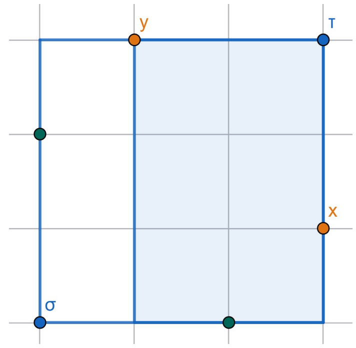

Figure 2.2: Step 3

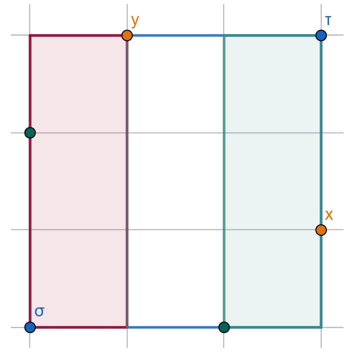

Figure 2.3: Step 4

Figure 2.4: Example when $m = n = 3$

2.3 PRELIMINARY LEMMAS

Before we prove the theorem, we first prove the following lemmas.

Lemma 2.15. *Let V be a $\mathfrak{m}_{K_1}^n$-torsion representation.*

1. $\mathrm{soc}(V^k/V^{k-1}) = \mathrm{soc}(V/V^{k-1})$ *and* $\mathrm{cosoc}(V^k) = \mathrm{cosoc}(V^k/V^{k-1})$ *for all k.*

2. *Let T be a proper subrepresentation of V^{n-1}. In addition, assume that* $\mathrm{soc}(V/V^{n-1}) =: \theta$ *is irreducible, and* $\theta \not\subseteq \mathrm{soc}(V)$, *and* $\mathrm{Ext}^1_{K/Z_1}(\theta, \sigma') = 0$ *for all* $\sigma' \in \mathrm{JH}(T)$. *Then* $\mathrm{soc}(V^{n-1}/T) = \mathrm{soc}(V/T)$.

Proof. (i) From the exact sequence $0 \to V^k/V^{k-1} \to V/V^{k-1} \to V/V^k \to 0$, we have $\mathrm{soc}(V^k/V^{k-1}) \hookrightarrow \mathrm{soc}(V/V^{k-1})$. Conversely, if $\sigma \subseteq \mathrm{soc}(V/V^{k-1})$ is a Serre weight, then $\sigma \subseteq (V/V^{k-1})[\mathfrak{m}_{K_1}] = V^k/V^{k-1}$. Therefore, $\mathrm{soc}(V^k/V^{k-1}) = \mathrm{soc}(V/V^{k-1})$.
For the cosocle case, we apply the same argument dually, noticing that the $\mathfrak{m}_{K_1}$-torsion quotient of V^k is V^k/V^{k-1}.

(ii) Since $\mathrm{soc}(V/T) \hookrightarrow \mathrm{soc}(V^{n-1}/T) \oplus \mathrm{soc}(V/V^{n-1})$, suppose to the contrary that $T \neq V^{n-1}$ but $\theta \subseteq \mathrm{soc}(V/T)$. Then let $\pi_T = V \to V/T$ be the quotient map, and consider the subrepresentation $\pi_T^{-1}(\theta)$. By construction, all the Jordan–Hölder factors of $\pi_T^{-1}(\theta)$ except θ are in $\mathrm{JH}(T)$. Since $\mathrm{Ext}^1_{K/Z_1}(\theta, \sigma') = 0$ for all $\sigma' \in \mathrm{JH}(T)$, by dévissage, we deduce that $\mathrm{Ext}^1_{K/Z_1}(\theta, T) = 0$. Therefore, since $\pi_T^{-1}(\theta)$ is a subrepresentation of V, $\theta \subseteq \mathrm{soc}(\pi_T^{-1}(\theta)) \subseteq \mathrm{soc}(V)$, a contradiction. □

Lemma 2.16. *Suppose V as in Theorem 2.10 with $m = n$, then $\mathrm{soc}(V/V^{n-1}) \cong \widetilde{\tau}$ is irreducible.*

Proof. By Lemma 2.5, there is an embedding, $V/V^{n-1} \hookrightarrow \oplus_{i=0}^{n-1} \oplus_{\theta \in \Delta^i(\sigma)} (\mathrm{Inj}_1 \theta)^{\oplus k_i}$ where $k_n = 1$. Since $\mathrm{cosoc}(V) = \tau$ is an n-weight, by Lemma 2.6, we must have $V/V^{n-1} \hookrightarrow \mathrm{Inj}_1 \widetilde{\tau}$. □

Lemma 2.17. *Assume that $\sigma = F(\mu)$ and $\theta = F(\mathfrak{t}_\mu(\omega))$ is an n-weight. Let V be a subrepresentation of $\mathrm{Inj}_n \sigma$ with $[V : \sigma] = 1$ and $\mathrm{soc}(V/V^{n-1}) = \theta$. We have $\mathrm{soc}(V/V^1) \hookrightarrow \bigoplus_{|\omega_i|>1} \delta_i^{\mathrm{sgn}(\omega_i)}(\sigma)$.*

Proof. By Lemma 2.5

$$V^2/V^1 \hookrightarrow (\mathrm{Inj}_1 \sigma)^{\oplus k_0} \oplus_{\delta' \in \Delta(\sigma)} \mathrm{Inj}_1 \delta'. \tag{2.1}$$

Hence, as $[V^2/V^1 : \sigma] = 0$, $\mathrm{soc}(V^2/V^1) \hookrightarrow \oplus_{(i,\epsilon_i)} \delta_i^{\epsilon_i}(\sigma)$ where $i \in \mathcal{J}$ and $\epsilon_i \in \{\pm\}$. By Lemma 2.15, we have $\mathrm{soc}(V/V^1) = \mathrm{soc}(V^2/V^1)$. If $\delta_i^{\epsilon_i}(\sigma) \subseteq \mathrm{soc}(V/V^1)$, we can find a subquotient V' of V/V^1, such that $\mathrm{soc}(V') = \delta_i^{\epsilon_i}(\sigma)$ and $\mathrm{cosoc}(V') = \theta$. Then $\theta \in \mathrm{JH}(\mathrm{Inj}_{n-1} \delta_i^{\epsilon_i}(\sigma))$. Therefore, $\sum_{j \neq i} \frac{|\omega_j|}{2} + |\frac{\omega_i}{2} - \epsilon_i 1| \leq n - 2$. Since $\sum_j \frac{|\omega_j|}{2} = n - 1$, we conclude that $\omega_i \geq 2$ and $\epsilon_i = \mathrm{sgn}(\omega_i)$. □

Assuming $m < n$, we will prove by contradiction that $V/V^{n-1} = 0$. We now show that it is sufficient to disprove the case where V/V^{n-1} is irreducible.

Lemma 2.18. *Suppose V as in Theorem 2.10. Assume $\theta \subseteq \mathrm{soc}(V/V^{n-1})$ for some Serre weight θ, then V contains a subrepresentation V' with $V'/V'^{n-1} \cong I(\theta, \tau)$, as well as a subrepresentation V'' with $V''/V''^{n-1} \cong \theta$.*

Proof. Assume $V/V^{n-1} \neq 0$. By Lemma 2.5, since $[V : \sigma] = 1$,

$$V/V^{n-1} \hookrightarrow \bigoplus_{i=1}^{n} \bigoplus_{\theta \in \Delta^i(\sigma)} (\mathrm{Inj}_1 \theta)^{\oplus k_i}. \tag{2.2}$$

For τ a m-weight, with $m < n$, τ may occur in distinct $\mathrm{Inj}_1\,\theta'$ for some $\theta' \in \bigcup_{s=1}^{n-1} \Delta^s(\sigma)$. By assumption, Equation (2.2) induces a nonzero map $\pi_\theta\colon V/V^{n-1} \to \mathrm{Inj}_1\,\theta$ when composed with the natural projection to $\mathrm{Inj}_1\,\theta$. We call the image C_θ. If $[C_\theta\colon \theta] = 1$, then we are done by [BP12, Corollary 3.12]. Otherwise, we dualize C_θ, such that $\mathrm{soc}(C_\theta^\vee) = \tau^\vee$ and $\mathrm{cosoc}(C_\theta^\vee) = \theta^\vee$. Then we can find a quotient $\widetilde{V'}$ in $C_\theta^\vee/\tau^\vee$ with socle $\tau^\vee$, and hence $[\widetilde{V'}\colon \tau^\vee] < [C_\theta^\vee\colon \tau^\vee]$. By repeating the process, we eventually find a quotient $\widetilde{V}$ of $C_\theta^\vee$, with $\mathrm{soc}(\widetilde{V}) = \tau^\vee$, $\mathrm{cosoc}(\widetilde{V}) = \theta^\vee$ and $[\widetilde{V}\colon \tau^\vee] = 1$. Then, by [HW22, Corollary 2.3], $\widetilde{V}^\vee \cong I(\theta,\tau)$ and is a subrepresentation of C_θ. Furthermore, let $\pi\colon V \to V/V^{n-1}$ be the projection map, then $V' := \pi^{-1}(I(\theta,\tau))$ is a subrepresentation of V. Moreover, $V'^{n-1} = V' \cap V^{n-1}$. Therefore, $V'/V'^{n-1} \cong I(\theta,\tau)$. For the last part, since C_θ contains θ as a subrepresentation, $\pi_\theta^{-1}(\theta)$ is a subrepresentation of V and is the V'' we are looking for using the same argument as above. □

Lemma 2.19. *Assume that Theorem 2.10 holds for $\mathfrak{m}_{K_1}^{n-1}$-torsion representations. Suppose V is a subrepresentation of $\mathrm{Inj}_n\,\sigma$ as in Theorem 2.10. If $\sigma' \in \mathrm{JH}(V)$ and σ' is a k-weight for some $k < n$, then $I(\sigma,\sigma')$ is a subrepresentation of V^k and σ' is a k-weight.*

Proof. We can find a subrepresentation V' of V with cosocle σ'. Since the theorem holds for all $m < n$; therefore, $V' \cong I(\sigma,\sigma')$, a $\mathfrak{m}_{K_1}^k$-torsion representation. Therefore, V' is a subrepresentation of V^k. Moreover, this implies $\sigma' \in \mathrm{JH}(V^k)$. □

2.4 PROOF OF THE MAIN THEOREM

Proof. We will prove by induction on lexicographic order (n,m). When $n = 1$, it is given by [BP12, Corollary 3.12] and when $n = 2$, it is given by [HW22, Theorem 2.23]. Since the following propositions, where the case $n < 3$ applies, can be deduced from Theorem 2.10, we will assume without loss of generality that $n \geq 3$.

When we apply the induction hypothesis in the case where $(m,n) = (a,b)$, we write Theorem 2.10$[(a,b)]$, Corollary 2.12$[b]$ or Corollary 2.13$[b]$ (since Theorem 2.10 shows that $a = b$ in such a case).

First, we prove the theorem in the case where $m < n$.

Proposition 2.20. Suppose V as in Theorem 2.10 with $n = 3$ and $\tau = \mu_i^\epsilon(\sigma)$ for some $\epsilon \in \{\pm\}$, then $V \cong I(\sigma,\tau)$, a Γ-representation, as predicted in the theorem.

Proof. Suppose that $V/V^2 \neq 0$. By Lemma 2.18, it suffices to consider the subrepresentation V' of V with $\mathrm{soc}(V'/V'^2) =: \theta$ irreducible and to show that such a subrepresentation does not exist. If $\theta = F(\mathfrak{t}_\mu(\xi)) \in \Delta^2(\sigma)$, there exists i with $|\xi_i| \geq 4$, or there exists $i \neq j$ with $|\xi_i|, |\xi_j| \geq 2$. By Lemma 2.4, $\tau \notin \mathrm{JH}(\mathrm{Inj}_1\,\theta)$. As $[V\colon \sigma] = 1$, $\theta \neq \sigma$. Therefore, we can assume $\theta = \delta_j^{\epsilon'}(\sigma) \in \Delta^1(\sigma)$ for some $j \in \mathcal{J}$ and $\epsilon' \in \{\pm\}$, then we must have $|2\epsilon'\delta_{jk} - \epsilon\delta_{ik}| \le 1$ for all k. Therefore, we must have $j = i$ and $\epsilon' = \epsilon$.

By Lemma 2.17, $\mathrm{soc}(V/V^1) = \mathrm{soc}(V^2/V^1) \hookrightarrow \bigoplus_{\theta' \in \Delta^1(\sigma)} \theta'$. If $\delta_j^{\epsilon_j}(\sigma) \subseteq \mathrm{soc}(V/V^1)$, then we can form a quotient $\widetilde{V}$ of V/V^1 with socle $\delta_j^{\epsilon_j}(\sigma)$. Then $\mathrm{soc}(\widetilde{V}/\widetilde{V}^1) \subseteq \mathrm{soc}(V/V^2) \cong \theta$. Therefore, by Lemma 2.5 $\theta \in \Delta^1(\delta_j^{\epsilon_j}(\sigma))$ or $\theta = \delta_j^{\epsilon_j}(\sigma)$. The former is impossible, so $(j, \epsilon_j) = (i, \epsilon)$ and $\theta' \cong \theta$. As the theorem holds for $n = 2$, by Lemma 2.19, if $\tau \in \mathrm{JH}(V^2)$, then $\tau \in \mathrm{JH}(V^1)$; hence $[V^2/V^1 \colon \tau] = 0$. By [HW22, Corollary 2.26], V^2 is multiplicity free; hence $[V^2/V^1 \colon \theta] = 1$. Therefore,

$$[V^2/V^1 \colon \theta] = 1 > [V^2/V^1 \colon \tau] = 0.$$

On the other hand, applying [HW22, Corollary 2.3] to V/V^2,

$$[V/V^2 \colon \theta] \ge [V/V^2 \colon \tau].$$

Therefore, we have

$$[V/V^1 \colon \theta] > [V/V^1 \colon \tau].$$

As V/V^1 has socle $\theta' \cong \theta$ and cosocle τ, this contradicts [HW22, Corollary 2.26]. Therefore, we conclude that $V/V^2 = 0$. □

Lemma 2.21. *Suppose V as in Theorem 2.10 with $m < n = 3$, then* $\mathrm{JH}(\mathrm{soc}(V/V^2)) \subseteq \Delta^2(\sigma)$.

Proof. By Equation (2.2), it suffices to show that there does not exist $\theta \in \mathrm{JH}(\mathrm{soc}(V/V^2)) \cap \Delta^1(\sigma)$. Assume for the sake of contradiction that such θ exists. Let $\pi \colon V \to V/V^2$ be the projection map, then $V' := \pi^{-1}(\theta)$ is a subrepresentation of V with $V'/V'^2 \cong \theta$, and it suffices to prove such a representation does not exist. Without loss of generality, we assume $V = V'$.

Suppose $\theta = \delta_i^{\epsilon}(\sigma) \in \Delta(\sigma)$ for some $\epsilon \in \{\pm\}$. Assume $\theta' \in \mathrm{JH}(\mathrm{soc}(V/V^1))$. Then we can find a quotient $\widetilde{V}$ of V/V^1 with socle θ'. Therefore, $\mathrm{soc}(\widetilde{V}/\widetilde{V}^1) \subseteq \mathrm{soc}(V/V^2) \cong \theta$ and hence by Lemma 2.5, $\theta \subseteq \Delta^1(\theta')$ or $\theta \cong \theta'$. The former is impossible; therefore, we have $\theta \cong \theta'$.

As $\mathrm{soc}(V^2/V^1) \cong \theta$, V^2 contains a subrepresentation with cosocle θ. By Corollary 2.13 [2] (*cf.* [HW22, Corollary 2.28]), such a subrepresentation has socle filtration

$$\sigma — \mu_i^{\epsilon}(\sigma) — \theta.$$

Applying Lemma 2.15 with $T = \sigma$, we have

$$\mathrm{soc}(V/\sigma) = \mathrm{soc}(V^2/\sigma) = \mathrm{soc}(V^1/\sigma) = \mu_i^{\epsilon}(\sigma).$$

Note that $V/V^2 \cong \theta$; therefore, $[V\colon \mu_i^\epsilon(\sigma)] = [V^2\colon \mu_i^\epsilon(\sigma)] = 1$ and $[V\colon \theta] = 2$. However, as $\theta = \mu_i^\epsilon(\mu_i^\epsilon(\sigma))$, V/σ contradicts Proposition 2.20. □

Proposition 2.22. Assume that Theorem 2.10 holds for all pairs $< (n, m)$. Suppose V as in Theorem 2.10 with $m < n$. If $V/V^{n-1} \neq 0$, then all the Serre weights of $\mathrm{soc}(V/V^{n-1})$ are in $\Delta^{n-1}(\sigma)$ and $\mathrm{soc}(V/V^{n-1})$ is multiplicity free.

Proof. When $n = 3$, By Lemma 2.21, $\mathrm{JH}(\mathrm{soc}(V/V^2)) \subseteq \Delta^2(\sigma)$.

For general $n > 3$, suppose to the contrary that $\mathrm{soc}(V/V^{n-1}) \not\subseteq \Delta^{n-1}(\sigma)$, then by Lemma 2.5, there exists $\theta \subseteq \mathrm{soc}(V/V^{n-1})$ such that $\theta \in \Delta^k(\sigma)$ for some $k < n-1$, in particular, θ is not an n-weight. Similar to the proof of Lemma 2.18, for each such θ, we can find a subrepresentation V' with $V'/V'^{n-1} = \theta$. It is enough to show that such a representation does not exist. Therefore, we reduce to the case where $V/V^{n-1} \cong \theta$ is irreducible and $\theta \in \Delta^k(\sigma)$ for some $k < n-1$. Let $\theta =: F(\mathfrak{t}_\mu(\xi))$.

By Lemma 2.15, $\mathrm{soc}(V/V^{n-2}) = \mathrm{soc}(V^{n-1}/V^{n-2})$; hence by the induction hypothesis $\mathrm{soc}(V/V^{n-2}) \subseteq \Delta^{n-2}(\sigma)$. Pick any $\theta' = F(\mathfrak{t}_\mu(\xi')) \subseteq \mathrm{soc}(V/V^{n-2})$, then $\sum_j \frac{|\xi'_j|}{2} = n-2$. Therefore, V/V^{n-2} contains a quotient $\widetilde{V}$ with $\mathrm{soc}(\widetilde{V}) = \theta'$ and $\mathrm{cosoc}(\widetilde{V}) = \theta$. As $\widetilde{V}^1 \subseteq V^{n-1}/V^{n-2}$, $\mathrm{soc}(\widetilde{V}/\widetilde{V}^1) \subseteq \mathrm{soc}(V/V^{n-1}) = \theta$, which is therefore an equality. Therefore, by Lemma 2.5, $\theta \cong \theta'$ or $\theta \in \Delta^1(\theta')$. We deduce that $k = n-2$ if $\theta \cong \theta'$ and $k = n-1$ or $n-3$ if $\theta \in \Delta^1(\theta')$. As we assume $\theta \notin \Delta^{n-1}(\sigma)$, $k = n-2$ or $n-3$.

Now we show that $\theta \hookrightarrow \mathrm{soc}(V/V^{\ell-1})$ for some $\ell \leq n-2$. If $k = n-2$, then $\theta' \cong \theta$ and we are done. If $k = n-3$, $\theta \in \Delta^1(\theta')$, and this is only possible if $\theta \leq \theta'$. The subrepresentation in V^{n-1} with cosocle θ' (as $\theta' \subseteq \mathrm{soc}(V^{n-1}/V^{n-2})$) is isomorphic to $I(\sigma, \theta')$ by Theorem 2.10[$(n-1, n-2)$]. By Theorem 2.10[$(n-1, n-2)$], $\theta \subseteq \mathrm{soc}(I(\sigma,\theta')^{n-2}/I(\sigma,\theta')^{n-3})$; hence $\theta \hookrightarrow \mathrm{soc}(V^{n-2}/V^{n-3})$. By Lemma 2.15, $\mathrm{soc}(V/V^{n-3}) = \mathrm{soc}(V^{n-2}/V^{n-3})$; therefore, this finishes the proof of the claim.

By Theorem 2.10[$(n-1, n-2)$], there is a unique subrepresentation V' of V^{n-1} with cosocle θ. Pick a $\delta_i^{\epsilon_i}(\sigma) \subseteq \mathrm{soc}(V'/V'^1) \subseteq \mathrm{soc}(V^{n-1}/V^1) = \mathrm{soc}(V/V^1)$. We claim $\theta \ncong \delta_i^{\epsilon_i}(\sigma)$. Otherwise, we must have $n = 4$. By the discussion above, $\theta' = \delta_i^\epsilon(\theta) \in \Delta^2(\sigma)$ for some $\epsilon \in \{\pm\}$. We consider the subrepresentation V' with cosocle θ' in V^3. As $\mu_i^\epsilon(\theta) - \theta \leq \delta_i^\epsilon(\theta) - \theta$, by applying Corollary 2.13[2] to V' and observing that $V'/V'^2 \cong \theta'$, we see that $\mu_i^\epsilon(\theta) \in \mathrm{JH}(V'^2) \subseteq \mathrm{JH}(V^2)$. On the other hand, V/V^2, admits a quotient with socle θ', which is $(2n-5)$-generic, and cosocle θ, which is isomorphic to $I(\theta', \theta)$ by Theorem 2.10[$(n-2, 2)$], call it $\widetilde{V}$. Since $\mu_i^\epsilon(\theta) - \delta_i^\epsilon(\theta) \leq \theta - \delta_i^\epsilon(\theta)$, by applying Corollary 2.13 [2] to $\widetilde{V}$ and observing that $\widetilde{V}/\widetilde{V}^1 \cong \theta$, we deduce that $\widetilde{V}^1$ contains $\mu_i^\epsilon(\theta)$ as a subquotient. In particular, $\mu_i^\epsilon(\theta) \in \mathrm{JH}(V^3/V^2)$. Then

$$\begin{aligned}
[V^3\colon \mu_i^\epsilon(\theta)] &= [V^2\colon \mu_i^\epsilon(\theta)] + [V^3/V^2\colon \mu_i^\epsilon(\theta)] \\
&= 2.
\end{aligned}$$

However, V^3 is multiplicity free by Corollary 2.12[3] (we assume $n = 4$ here), which is a contradiction.

Therefore, $\delta_i^{\epsilon_i}(\sigma) \not\cong \theta$. Furthermore, V^{n-1} is multiplicity free by Corollary 2.12[$n-1$]. Therefore $[V\colon \delta_i^{\epsilon_i}(\sigma)] = 1$. Similar to the proof in Lemma 2.21, we have a (unique up to scalar) nonzero map $f\colon V \twoheadrightarrow V/V^1 \to \mathrm{Inj}_{n-1}\delta_i^{\epsilon_i}(\sigma)$. Then we claim that $[f(V)\colon \theta] = 2$. Assume $[f(V)\colon \theta] \leq 1$, then $[\ker(f)\colon \theta] \geq 1$. As f is nonzero, $\ker(f) \subsetneq \mathrm{Rad}(f) = V^{n-1}$, which is $\mathfrak{m}^{n-1}$-torsion. Then by Theorem 2.10[$(n-1, k+1)$], $\ker(f)$ contains a subrepresentation isomorphic to $I(\sigma,\theta)$. Moreover, by Corollary 2.13[$k+1$], as $\delta_i^{\epsilon}(\sigma) \leq \theta$, $\delta_i^{\epsilon_i}(\sigma)$ is a subquotient of $I(\sigma,\theta) \subseteq \ker(f)$. However, this contradicts f being nonzero, as $[V\colon \delta_i^{\epsilon_i}(\sigma)] = 1$. Therefore, $[f(V)\colon \theta] = 2$. On the other hand, $\mathrm{soc}(f(V)) = \delta_i^{\epsilon_i}(\sigma)$, which is $(2n-3)$-generic, and $[f(V)\colon \delta_i^{\epsilon_i}(\sigma)] \leq [V\colon \delta_i^{\epsilon_i}(\sigma)] = [V^{n-1}\colon \delta_i^{\epsilon_i}(\sigma)] = 1$. Therefore, applying Corollary 2.12[$n-1$] to $f(V)$, we obtain $[f(V)\colon \theta] \leq [f(V)\colon \delta_i^{\epsilon_i}(\sigma)] = 1$, which is a contradiction. This finishes the proof.

The statement on multiplicity free follows from the first assertion and Lemma 2.5 that $k_n = 1$. □

Proposition 2.23. Fix a pair (n, m) with $m < n$. Assume that Theorem 2.10 holds for all pairs $(n', m') < (n, m)$. Then the theorem holds for (n, m).

Proof. It is enough to show that $V/V^{n-1} = 0$ and then by the induction hypothesis, i.e., Theorem 2.10[$(n-1, m)$], we can conclude the result. Assume for the sake of contradiction that $V/V^{n-1} \neq 0$, then by Lemma 2.18, it suffices to disprove the case where $V/V^{n-1} \cong I(\theta,\tau)$. By Proposition 2.22, $\theta \in \Delta^{n-1}(\sigma)$. Let $\theta =: F(\mathfrak{t}_\mu(\xi))$. As $\tau \in \mathrm{JH}(\mathrm{Inj}_1\theta)$ and $\xi_j \in 2\mathbb{Z}$ for all j, we have $|\xi_j| > 1 \iff \omega_j \neq 0$ and $\mathrm{sgn}(\xi_j) = \mathrm{sgn}(\omega_j) =: \epsilon_j$ in this case.

Furthermore, as $\mathrm{soc}(V/V^{n-1})$ is an n-weight, we can apply Lemma 2.17 and deduce that

$$\mathrm{soc}(V/V^1) \hookrightarrow \bigoplus_{|\xi_i|>1} \delta_i^{\mathrm{sgn}(\xi_i)}(\sigma) \cong \bigoplus_{\omega_i \neq 0} \delta_i^{\mathrm{sgn}(\omega_i)}(\sigma).$$

We will show that if $|\omega_i| = 1$, then $\delta_i^{\epsilon_i}(\sigma) \not\hookrightarrow \mathrm{soc}(V/V^1)$. Assume for the sake of contradiction that there exists an i with $|\omega_i| = 1$ and $\delta_i^{\epsilon_i}(\sigma) \hookrightarrow \mathrm{soc}(V/V^1)$. Then V/V^1 admits a subquotient with socle $\delta_i^{\epsilon_i}(\sigma)$, which is $(2n-3)$-generic, and cosocle θ, which is isomorphic to $I(\delta_i^{\epsilon_i}(\sigma),\theta)$ by Theorem 2.10[$(n-1, n-1)$]. Then as $|\omega_i| = 1$, $\mu_i^{\epsilon_i}(\sigma) - \delta_i^{\epsilon_i}(\sigma) \leq \tau - \delta_i^{\epsilon_i}(\sigma)$. By Corollary 2.13[$n-1$], we deduce that $\mu_i^{\epsilon_i}(\sigma) \in \mathrm{JH}(I(\delta_i^{\epsilon_i}(\sigma),\theta))$. Since $I(\delta_i^{\epsilon_i}(\sigma),\theta)/(I(\delta_i^{\epsilon_i}(\sigma),\theta))^{n-2} = \theta$; therefore $\mu_i^{\epsilon_i}(\sigma) \in \mathrm{JH}((I(\delta_i^{\epsilon_i}(\sigma),\theta))^{n-2}) \subseteq \mathrm{JH}(V^{n-1}/V^1)$. On the other hand, V^2 admits a subrepresentation with cosocle $\delta_i^{\epsilon_i}(\sigma)$, which is isomorphic to $I(\sigma, \delta_i^{\epsilon_i}(\sigma))$ by Theorem 2.10 [$(2,2)$]. By Corollary 2.13[2] (*cf.* [HW22, Corollary 2.28]), we deduce that $\mu_i^{\epsilon_i}(\sigma) \in \mathrm{JH}((I(\sigma,\delta_i^{\epsilon_i}(\sigma)))^1) \subseteq \mathrm{JH}(V^1)$. Therefore, $[V^{n-1}\colon \mu_i^{\epsilon_i}(\sigma)] = [V^{n-1}/V^1\colon \mu_i^{\epsilon_i}(\sigma)] + [V^1\colon \mu_i^{\epsilon_i}(\sigma)] \geq 2$. By Corollary 2.12[$n-1$], V^{n-1} is multiplicity free, a contradiction. Therefore, $\operatorname{soc}(V/V^1) \hookrightarrow \bigoplus_{|\omega_i|>1} \delta_i^{\operatorname{sgn}(\omega_i)}(\sigma)$. As a result, we have an induced map $g\colon V/V^1 \hookrightarrow \bigoplus_{|\omega_i|>1} \operatorname{Inj}_{n-1} \delta_i^{\epsilon_i}(\sigma)$.

By Proposition 2.22, we know that then $\theta \in \Delta^{n-1}(\sigma)$. Since $n \geq 3$, by Lemma 2.4 $\delta_i^{\epsilon_i}(\sigma) \notin \mathrm{JH}(\operatorname{Inj}_1 \theta) \supset \mathrm{JH}(V/V^{n-1})$ for all (i, ϵ_i). By Corollary 2.12[$n-1$], V^{n-1} is multiplicity free. Therefore, $[V\colon \delta_i^{\epsilon_i}] = 1$. Therefore, the projection of the image of g to each $\operatorname{Inj}_{n-1} \delta_i^{\epsilon_i}(\sigma)$ is $I(\delta_i^{\epsilon_i}(\sigma), \tau)$ or 0, by Theorem 2.10[$(n-1, m-1)$], noting that $\delta_i^{\epsilon_i}(\sigma)$ is $(2n-3)$-generic. Therefore, g factors through $V/V^1 \hookrightarrow \bigoplus_{|\omega_i|>1} I(\delta_i^{\epsilon_i}(\sigma), \tau)$. As $\sum_j \lfloor \frac{|\omega_j|}{2} \rfloor = m$, and $\epsilon_i = \operatorname{sgn}(\omega_i)$ for $\omega_i = 0$, $\sum_j \lfloor \frac{|\omega_j - \epsilon_i 2\delta_{ij}|}{2} \rfloor = m-1$. Therefore, each $I(\delta_i^{\epsilon_i}(\sigma), \tau)$ is $\mathfrak{m}_{K_1}^{m-1}$-torsion, so is V/V^1. It follows that V is m-torsion, and hence $V/V^{n-1} = 0$, a contradiction. $\square$

It remains to prove by induction for the case where $m = n$. From now on, we write ϵ_i for $\operatorname{sgn}(\omega_i)$ when $\omega_i \neq 0$ and n_i for $\lfloor \frac{|\omega_i|}{2} \rfloor$.

Proposition 2.24. Assume that Theorem 2.10 holds for all pairs $(m', n') < (n, n)$. Suppose V as in Theorem 2.10 with $m = n$. Then

$$\operatorname{soc}(V/V^1) \cong \bigoplus_{|\omega_i|>1} \delta_i^{\operatorname{sgn}(\omega_i)}(\sigma).$$

Proof. As $n = m$, by Lemma 2.16, we have $\operatorname{soc}(V/V^{n-1}) = \widetilde{\tau}$, an n-weight. By Lemma 2.17, we have $\operatorname{soc}(V/V^1) \hookrightarrow \oplus_{|\omega_i|>1} \delta_i^{\operatorname{sgn}(\omega_i)}(\sigma)$. We will now prove that we have an injection in the other direction. Let $\pi\colon V \to V/V^{n-1}$ be the projection map, then $V' := \pi^{-1}(\widetilde{\tau})$ is a subrepresentation of V. As $V'/V'^1 \subseteq V/V^1$, it suffices to show that $\delta_i^{\operatorname{sgn}(\omega_i)}(\sigma) \hookrightarrow \operatorname{soc}(V'/V'^1)$ for all i with $|\omega_i| > 1$. Therefore, we can assume without loss of generality that $V/V^{n-1} \cong \tau$.

If we have a unique i with $\lfloor \frac{|\omega_i|}{2} \rfloor = n$, then as $\operatorname{soc}(V/V^1) \neq \varnothing$, we must have $\operatorname{soc}(V/V^1) = \delta_i^{\epsilon_i}(\sigma)$. Assume that there exist $i \neq j$, with $|\omega_i|, |\omega_j| > 1$. Assume for the sake of contradiction that there exists a i such that $|\omega_i| > 1$, but $\delta_i^{\epsilon_i}(\sigma) \not\subseteq \operatorname{soc}(V/V^1)$. When $n = 3$, then $V/V^2 \cong F(\mathfrak{t}_\mu(\epsilon_i 2\overline{\eta}_i + \epsilon_j 2\overline{\eta}_j))$ for some $i \neq j$. By Lemma 2.17, we know that $\operatorname{soc}(V/V^1) \hookrightarrow \delta_i^{\epsilon_i}(\sigma) \oplus \delta_j^{\epsilon_j}(\sigma)$. Assume for the sake of contradiction that $\operatorname{soc}(V/V^1) \cong \delta_i^{\epsilon_i}(\sigma)$ or $\delta_j^{\epsilon_j}(\sigma)$. Without loss of generality, assume $\operatorname{soc}(V/V^1) = \delta_i^{\epsilon_i}(\sigma)$. Then as V/V^1 is a $\mathfrak{m}_{K_1}^2$-torsion representation with socle $\delta_i^{\epsilon_i}(\sigma)$, cosocle τ, with $[V/V^1\colon \delta_i^{\epsilon_i}(\sigma)] = [V^2/V^1\colon \delta_i^{\epsilon_i}(\sigma)] = 1$, as V^2 is multiplicity free by Corollary 2.12[2]. Therefore, applying Theorem 2.10[$(2,2)$] to V/V^1, we can conclude that $V/V^1 \cong I(\delta_i^{\epsilon_i}(\sigma), \tau)$. In particular, V/V^1 has socle filtration

$$\delta_i^{\epsilon_i}(\sigma) — F(\mathfrak{t}_\mu(\epsilon_i 2\overline{\eta}_i + \epsilon_j \overline{\eta}_j)) — F(\mathfrak{t}_\mu(\epsilon_i 2\overline{\eta}_i + \epsilon_j 2\overline{\eta}_j)).$$

Therefore, we deduce that $\mathrm{cosoc}(V^2) = \mathrm{cosoc}(V^2/V^1) \cong F(\mathfrak{t}_\mu(\epsilon_i 2\overline{\eta}_i + \epsilon_j\overline{\eta}_j))$. Therefore, by Theorem 2.10[(2, 2)], we have

$$V^2 \cong I(\sigma, F(\mathfrak{t}_\mu(\epsilon_i 2\overline{\eta}_i + \epsilon_j\overline{\eta}_j))).$$

In particular, as $\epsilon_i\overline{\eta}_i + \epsilon_j\overline{\eta}_j \leq \epsilon_i 2\overline{\eta}_i + \epsilon_j\overline{\eta}_j$ for all k, by Corollary 2.13[2] on V^2, we have $\mu_j^{\epsilon_j}(\sigma) = F(\mathfrak{t}_\mu(\epsilon_j\overline{\eta}_j)) \in \mathrm{JH}(V)$. Therefore, we can find a quotient $\widetilde{V}$ of V with socle $\mu_j^{\epsilon_j}(\sigma)$. As τ is a 2-weight with respect to $\mu_j^{\epsilon_j}(\sigma)$, by Proposition 2.23, $\widetilde{V} \cong I(\mu_j^{\epsilon_j}(\sigma), \tau)$. As $\epsilon_j 2\overline{\eta}_j - \epsilon_j\overline{\eta}_j \leq \epsilon_i 2\overline{\eta}_i + \epsilon_j 2\overline{\eta}_j - \epsilon_j\overline{\eta}_j$ for all k, by Corollary 2.13[2], $\delta_j^{\epsilon_j}(\sigma) \in \mathrm{JH}(I(\mu_j^{\epsilon_j}(\sigma), \tau)) \subseteq \mathrm{JH}(V)$, a contradiction.

Now assume $n > 3$. As $\mathrm{soc}(V/V^1) \neq 0$, There exists a (j, ϵ_j) with $j \neq i$ such that $\delta_j^{\epsilon_j}(\sigma) \subseteq \mathrm{soc}(V/V^1)$. We can find a quotient of V/V^1 with socle $\delta_j^{\epsilon_j}(\sigma)$, which is $(2n-3)$-generic, and we call it W_{δ_j}. Since $\epsilon_j 2\overline{\eta}_j, \epsilon_i 2\overline{\eta}_i \leq \omega$, $\epsilon_j 2\overline{\eta}_j + \epsilon_i 2\overline{\eta}_i \leq \omega$. Hence, by Corollary 2.13[$n-1$], $F(\mathfrak{t}_\mu(\epsilon_j 2\overline{\eta}_j + \epsilon_i 2\overline{\eta}_i))$ is a subquotient of $I(\delta_j^{\epsilon_j}(\sigma), \tau)$. As $V/V^{n-1} \cong \tau$, $F(\mathfrak{t}_\mu(\epsilon_j 2\overline{\eta}_j + \epsilon_i 2\overline{\eta}_i))$ is a subquotient of V^{n-1}. Then V^{n-1} admits a subrepresentation V' with cosocle $F(\mathfrak{t}_\mu(\epsilon_j 2\overline{\eta}_j + \epsilon_i 2\overline{\eta}_i))$. By Theorem 2.10[$(n-1, 3)$], $V' \cong I(\sigma, F(\mathfrak{t}_\mu(\epsilon_j 2\overline{\eta}_j + \epsilon_i 2\overline{\eta}_i)))$. Again, as $\delta_j^{\epsilon_j}(\sigma) \leq F(\mathfrak{t}_\mu(\epsilon_j 2\overline{\eta}_j + \epsilon_i 2\overline{\eta}_i))$ Corollary 2.13[3] implies that $\delta_j^{\epsilon_j}(\sigma) \subseteq \mathrm{soc}(V'/V'^1) \subseteq (V/V^1)$, a contradiction. ∎

Proposition 2.25. Assume that Theorem 2.10 holds for all pairs $< (n, n)$. Suppose V as in Theorem 2.10 with $m = n$. Then the Jordan–Hölder factors of V are exactly as described in Theorem 2.10 up to multiplicity. In other words, $F(\mathfrak{t}_\mu(\omega')) \in \mathrm{JH}(V) \iff \omega' \leq \omega$.

Proof. By Proposition 2.24, we have $\mathrm{soc}(V/V^1) \cong \bigoplus_{|\omega_i|>1} \delta_i^{\epsilon_i}(\sigma)$. For each $\delta_i^{\epsilon_i}(\sigma) \subseteq \mathrm{soc}(V/V^1)$, we consider the quotient $\widetilde{W}_i$ of V/V^1 with socle $\delta_i^{\epsilon_i}(\sigma)$, which is $(2n-3)$-generic. Moreover, by Corollary 2.12[$n-1$], V^{n-1} is multiplicity free. Moreover, by Proposition 2.22, $\mathrm{soc}(V/V^{n-1}) \cong \widetilde{\tau}$ and $\delta_i^{\epsilon_i}(\sigma) \notin \mathrm{Inj}_1 \widetilde{\tau}$. Therefore, $[V : \delta_i^{\epsilon_i}(\sigma)] = 1$ for all such $\delta_i^{\epsilon_i}(\sigma)$. Then, since $\widetilde{W}_i$ is $\mathfrak{m}_{K_1}^{n-1}$-torsion, we can apply Theorem 2.10[$(n-1, n-1)$] to each $\widetilde{W}_i$ and show that $\widetilde{W}_i \cong I(\delta_i^{\epsilon_i}(\sigma), \tau)$. As $\widetilde{W}_i$ is $\mathfrak{m}_{K_1}^{n-1}$-torsion, we can apply Corollary 2.13 to $\widetilde{W}_i$, and deduce that

$$\begin{aligned} \bigcup_i \mathrm{JH}(\widetilde{W}_i) &= \{F(\mathfrak{t}_\mu(\omega')) \colon \epsilon_i 2\overline{\eta}_i \leq \omega' \,\&\, \omega' \leq \omega\} \\ &= \{F(\mathfrak{t}_\mu(\omega')) \colon \omega' \leq \omega \text{ and there exists an } i \text{ s.t } |\omega'_i| > 1\}. \end{aligned} \tag{2.3}$$

In particular, $F(\mathfrak{t}_\mu(\epsilon_i 2\overline{\eta}_i))_+ \in \mathrm{JH}(V)$ for all i such that $|\omega_i| > 1$. Fix one of such i. By Theorem 2.10[$(n, 2)$], $I(\sigma, F(\mathfrak{t}_\mu(\epsilon_i 2\overline{\eta}_i))_+)$ is $\mathfrak{m}_{K_1}^2$-torsion; therefore, it is a subrepresentation of V^2. Moreover, as $\sigma_+ = F(\mathfrak{t}_\mu(\sum_{\omega_j \neq 0} \epsilon_j\overline{\eta}_j)) \leq F(\mathfrak{t}_\mu(\epsilon_i 2\overline{\eta}_i))_+$, we can also deduce that

$\sigma_+ \in \mathrm{JH}(V^1)$. From this we see that if $\omega' \le \omega$ and $|\omega'_j| \le 1$, then $F(\mathrm{t}_\mu(\omega')) \in \mathrm{JH}(V^1)$. Therefore, if $\omega' \le \omega$, then $F(\mathrm{t}_\mu(\omega')) \in \mathrm{JH}(V)$.

Now we prove the converse that $F(\mathrm{t}_\mu(\omega')) \in \mathrm{JH}(V)$ then $\omega' \le \omega$. By the argument above, we have a map $f\colon V/V^1 \to \bigoplus_{|\omega_i|>1} I(\delta_i^{\epsilon_i}(\sigma), \tau)$. As $\mathrm{soc}(V/V^1) \cong \bigoplus_{|\omega_i|>1} \delta_i^{\epsilon_i}(\sigma)$, f is injective on the socles, f is injective. Therefore, if $F(\mathrm{t}_\mu(\omega')) \in \mathrm{JH}(V/V^1)$, by Equation (2.3), $\omega' \le \omega$. We claim that if $\sigma' \in \mathrm{JH}(V^1) \setminus \mathrm{JH}(I(\sigma, \sigma_+))$ and $\tau' \in \mathrm{JH}(V/V^1)$, $\mathrm{Ext}^1_{K/Z_1}(\tau', \sigma') = 0$. Write $\tau' := F(\mathrm{t}_\mu(\omega''))$ and $\sigma' := F(\mathrm{t}_\mu(\omega'))$. If $\tau' \in \mathrm{JH}(V/V^2)$, then by Lemma 2.6, $\sum_j \left|\frac{\lfloor \omega''_j \rfloor}{2}\right| \ge 2$ and $\sum_j \left|\frac{\lfloor \omega'_j \rfloor}{2}\right| = 0$. Therefore, there exists a j with $|\omega''_j - \omega'_j| \ge 2$, or there exists $i \ne j$ with $\omega''_i \ne \omega'_i$ and $\omega''_i \ne \omega'_i$. Therefore, by [BHH+23, Lemma 2.4.6], $\mathrm{Ext}^1_{K/Z_1}(\tau', \sigma') = 0$. If $\tau' \in \mathrm{JH}(V^2/V^1)$, then we can apply [HW22, Lemma 2.2.1], noting that $\lambda_!(\sigma) \le \sigma_+$ ($\lambda_!$ is defined in [HW22]), and deduce that if $\sigma' \notin \mathrm{JH}(I(\sigma, \sigma_+))$, then $\mathrm{Ext}^1_{K_1/Z_1}(\tau', \sigma') = 0$. This proves the claim. Therefore, by dévissage, if $\sigma' \notin \mathrm{JH}(I(\sigma, \sigma_+))$,

$$\mathrm{Ext}^1_{K/Z_1}(V/V^1, \sigma') = 0.$$

Consequently, $\mathrm{Hom}_{K/Z_1}(V^1, \sigma') = \mathrm{Hom}_{K/Z_1}(V, \sigma')$. However, as V has an n-weight as its cosocle; therefore $\mathrm{Hom}_{K/Z_1}(V, \sigma') = 0$ for any σ' as above and so $\mathrm{Hom}_{K/Z_1}(V^1, \sigma') = 0$. We deduce that $\mathrm{JH}(V^1) = \mathrm{JH}(I(\sigma, \sigma_+))$. Therefore, we conclude the result. The second assertion follows from Lemma 2.11. ☐

Lemma 2.26. *Assume that Theorem 2.10 holds for all pairs $(n', m') < (n, n)$. Suppose V as in Theorem 2.10 with $m = n$. Then for all $0 \le k < n-1$,*

$$V^{k+1}/V^k \cong \bigoplus_{\xi \in {}^0\Omega_k^\tau} I(F(\mathrm{t}_\mu(\xi)), F(\mathrm{t}_\mu(\xi_+))).$$

Proof. Given $\theta \in \mathrm{soc}(V^{k+1}/V^k)$ for some $0 \le k < n-1$. Then, as the theorem holds for all $m < n$, by Lemma 2.19, θ is a $k+1$ weight. By Lemma 2.5, we deduce that $\theta \in \Delta^k(\sigma)$. By the remark in Definition 2.7 and Proposition 2.25, we conclude that $\theta \in {}^0\Omega_k^\tau$. By definition $\theta_+ \le \omega$; hence by Proposition 2.25, we deduce that $\theta_+ \in \mathrm{JH}(V)$. Moreover, by Lemma 2.8, $\theta_+ \in \mathrm{JH}(\mathrm{Inj}_1 \theta)$; hence $I(F(\mathrm{t}_\mu(\xi)), F(\mathrm{t}_\mu(\xi_+))) \hookrightarrow V^{k+1}/V^k$. Therefore, we have $\bigoplus_{\xi \in {}^0\Omega_k^\tau} I(F(\mathrm{t}_\mu(\xi)), F(\mathrm{t}_\mu(\xi_+))) \hookrightarrow V^{k+1}/V^k$. By Corollary 2.12[$n-1$], V^{n-1} is multiplicity free, so is V^{k+1}/V^k for all $0 \le k < n-1$. As $\bigoplus_{\xi \in {}^0\Omega_k^\tau} I(F(\mathrm{t}_\mu(\xi)), F(\mathrm{t}_\mu(\xi_+)))$ and V^{k+1}/V^k have the same Jordan–Hölder factors by Proposition 2.25 and both are multiplicity free, they are isomorphic. ☐

Proposition 2.27. Assume that Theorem 2.10 holds for all pairs $< (n,n)$. Suppose V as in Theorem 2.10 with $m = n$. If $|\omega_i| > 1$, then $\tau^{(i)} \in \mathrm{JH}(V^{n-1})$ (*cf.* Definition 2.7), and $I(\sigma, \tau^{(i)})$ is isomorphic to a proper subrepresentation of V^{n-1}. Moreover,

$$\mathrm{soc}(V/I(\sigma,\tau^{(i)})) = \mu_i^{\epsilon_i}(\sigma).$$

Proof. Let $n_i = \frac{|\omega_i|}{2} \geq 1$. By definition, $\tau^{(i)}$ is a $(n-n_i)$-weight and $\tau^{(i)} \leq \tau$. By Proposition 2.25, V has a subrepresentation with cosocle $\tau^{(i)}$. By Theorem 2.10[$(n-1, n-n_{\mathrm{i}})$], this subrepresentation is isomorphic to $I(\sigma,\tau^{(i)})$ and is a subrepresentation of V^{n-1}. As $|\omega_i| > 1$, by Lemma 2.26, $\mu_i^{\epsilon_i}(\sigma) \in \mathrm{JH}(V^1)$. However, applying Corollary 2.13[$(n-n_{\mathrm{i}})$] to $I(\sigma,\tau^{(i)})$, we deduce that for any $F(\mathrm{t}_\mu(\omega')) \in \mathrm{JH}(I(\sigma,\tau^{(i)}))$, $\omega_i' = 0$. Therefore, $\mu_i^{\epsilon_i}(\sigma) \notin \mathrm{JH}(I(\sigma,\tau^{(i)}))$ and hence

$$I(\sigma,\tau^{(i)}) \subsetneq V^{n-1}.$$

By Lemma 2.16, $\mathrm{soc}(V/V^{n-1}) \cong \widetilde{\tau}$ is an n-weight, $\widetilde{\tau} \not\subseteq \mathrm{soc}(V)$. Furthermore, for any $F(\mathrm{t}_\mu(\omega')) \in \mathrm{JH}(I(\sigma,\tau^{(i)}))$, we just showed that $\omega_i' = 0$. Since $|\omega_i| > 1$, by [BHH+23, Lemma 2.4.6], we deduce that

$$\mathrm{Ext}^1_{K_1/Z_1}(\widetilde{\tau}, F(\mathrm{t}_\mu(\omega'))) = 0.$$

Therefore, all the assumptions of Lemma 2.15(ii) are satisfied, and we deduce that $\mathrm{soc}(V/I(\sigma,\tau^{(i)})) = \mathrm{soc}(V^{n-1}/I(\sigma,\tau^{(i)}))$. Now, we claim that

$$\mathrm{soc}(V^{n-1}/I(\sigma,\tau^{(i)})) = \mu_i^{\epsilon_i}(\sigma).$$

As $\mu_i^{\epsilon_i}(\sigma) \in \mathrm{JH}(V^1)$, we have a (unique up to scalar) nonzero map

$$f\colon V^{n-1} \to \mathrm{Inj}_{n-1}\,\mu_i^{\epsilon_i}(\sigma).$$

The claim is equivalent to $\ker(f) \cong I(\sigma,\tau^{(i)})$.

First, I will show that $I(\sigma,\tau^{(i)})$ is a subrepresentation of $\ker(f)$. It suffices to show $\tau^{(i)} \in \mathrm{JH}(\ker(f))$, since then $\ker(f)$ admits a subrepresentation with socle σ and cosocle $\tau^{(i)}$. As $[\ker(f)\colon \sigma] \leq [V\colon \sigma] = 1$, by Theorem 2.10[$(n-1, n-n_{\mathrm{i}})$], such a representation is isomorphic to $I(\sigma,\tau^{(i)})$. Assume for the sake of contradiction that $\tau^{(i)} \notin \mathrm{JH}(\ker(f))$, then $\tau^{(i)} \in \mathrm{JH}(\mathrm{Im}(f))$. As V^{n-1} is multiplicity free by Theorem 2.10[$n-1$], $[\mathrm{Im}(f)\colon \mu_i^{\epsilon_i}(\sigma)] = 1$. Therefore, $\mathrm{Im}(f)$ admits a subrepresentation with cosocle $\tau^{(i)}$, which is isomorphic to $I(\mu_i^{\epsilon_i}(\sigma),\tau^{(i)})$ by Theorem 2.10[$(n-1, n-n_{\mathrm{i}})$]. Since $0 - \epsilon_i\overline{\eta}_i \leq \omega^{(i)} - \epsilon_i\overline{\eta}_i$ for all k, from the theorem, we further deduce that σ is a subquotient of $I(\mu_i^{\epsilon_i}(\sigma),\tau^{(i)}) \subseteq \mathrm{Im}(f)$.

However, as V^{n-1} is multiplicity free, $[\ker(f)\colon \sigma] = 0$. Therefore, $\ker(f) = 0$ and f is injective. However, this is a contradiction as $\mathrm{soc}(\mathrm{Im}(f)) = \mu_i^{\epsilon_i}(\sigma) \neq \sigma$.

Conversely, assume that we have some $F(\mathrm{t}_\mu(\omega')) \in \mathrm{JH}(\ker(f) \setminus I(\sigma, \tau^{(i)})) \subseteq \mathrm{JH}(V^{n-1})$. Then by Proposition 2.25, $\omega' \leq \omega$, if $\omega'_i \neq 0$, then we must have $\omega' - \epsilon_i\overline{\eta}_i \leq \omega - \epsilon_i\overline{\eta}_i$. By Corollary 2.13[$n-1$], we have $\mu_i^{\epsilon_i}(\sigma) \in \mathrm{JH}(I(\sigma, F(\mathrm{t}_\mu(\omega'))))$. Therefore, $\mu_i^{\epsilon_i}(\sigma) \in \mathrm{JH}(\ker(f))$, which is a contradiction as $[V\colon \mu_i^{\epsilon_i}(\sigma)] = 1$, and f is nonzero. Therefore, $\omega'_i = 0$ and $\omega' \leq \omega^{(i)}$. Hence, $F(\mathrm{t}_\mu(\omega'))$ is a subquotient of $I(\sigma, \tau^{(i)})$ by Corollary 2.13[$n - n_i$]. □

Proposition 2.28. Assume that Theorem 2.10 holds for all pairs $< (n, n)$. Suppose V as in Theorem 2.10 with $m = n$, then V is multiplicity free. Moreover, for all $0 \leq k \leq n-1$, V^{k+1}/V^k is exactly as described in Theorem 2.10.

Proof. By Corollary 2.12[$n-1$], V^{n-1} is multiplicity free. By Proposition 2.22, $\mathrm{soc}(V/V^{n-1}) \cong \widetilde{\tau}$. If we show that $[V/V^{n-1}\colon \widetilde{\tau}] = 1$, then by Theorem 2.10[$(1,1)$], $V/V^{n-1} \cong I(\widetilde{\tau}, \tau)$, which is multiplicity free. As the theorem holds for all $m < n$; therefore, by Lemma 2.19, V^{n-1} and V/V^{n-1} do not share common Jordan–Hölder factors. Therefore, V is multiplicity free. Moreover, as $V/V^{n-1} \cong I(\widetilde{\tau}, \tau)$, together with Lemma 2.26, we can conclude the second assertion.

Assume $|\omega_i| > 1$ for some fixed i. By Proposition 2.27, $I(\sigma, \tau^{(i)})$ is a subrepresentation of V, and $V/I(\sigma, \tau^{(i)}) =: \widetilde{W}_i$ has socle $\mu_i^{\epsilon_i}(\sigma) = F(\mathrm{t}_\mu(\epsilon_i\overline{\eta}_i))\delta_i^{\epsilon_i}(\sigma)$, which is $(2n-3)$-generic. Then $\widetilde{\tau} = F(\mathrm{t}_\mu(\sum_j 2\lfloor \frac{\omega_j}{2} \rfloor \overline{\eta}_j))$ is an $n-1$-weight with respect to $\mu_i^{\epsilon_i}(\sigma)$. Therefore, applying Corollary 2.12[$n-1$] to $\widetilde{W}_i$, we conclude that $[\widetilde{W}_i\colon \theta] \leq [\widetilde{W}_i\colon \mu_i^{\epsilon_i}(\sigma)]$. On the other hand, $\mu_i^{\epsilon_i}(\sigma)$ is a 1-weight, by Lemma 2.19, $[V/V^{n-1}\colon \mu_i^{\epsilon_i}(\sigma)] = 0$. Furthermore, V^{n-1} is multiplicity free, so $[V\colon \mu_i^{\epsilon_i}(\sigma)] = 1$. Therefore, $[\widetilde{W}_i\colon \mu_i^{\epsilon_i}(\sigma)] = 1$, and hence $[\widetilde{W}_i\colon \theta] \leq 1$. As $I(\sigma, \tau^{(i)}) \subsetneq V^{n-1}$, V/V^{n-1} is a quotient of $\widetilde{W}_i$, $[V/V^{n-1}\colon \theta] \leq [\widetilde{W}_i\colon \theta] \leq 1$. On the other hand, $\theta = \mathrm{soc}(V/V^{n-1})$. This finishes the proof. □

Lemma 2.29. *Assume that Theorem 2.10 holds for all pairs $< (n, n)$. Suppose V as in Theorem 2.10 with $m = n$. Assume $\tau' := F(\mathrm{t}_\mu(\omega')) \in \mathrm{JH}(V)$ with $0 \leq \omega'_i < \widetilde{\omega}_i$ or $0 \geq \omega'_i > \widetilde{\omega}_i$ and $\omega'_j = \omega_j$ for all $j \neq i$. Assume $|\omega_i - \omega'_i| < 2(n-1)$. Then $I(\sigma, \tau')$ is a subrepresentation of V and*

$$V/I(\sigma, \tau') \cong I(F(\mathrm{t}_\mu((\omega'_i + \epsilon_i 1)\overline{\eta}_i), \tau).$$

Proof. Let $\ell = \sum_j \lfloor \frac{|\omega'_j|}{2} \rfloor < n-1$ and $\ell' = \lfloor \frac{|\omega_i - \omega'_i|}{2} \rfloor < n-1$. By Proposition 2.28, V is multiplicity free. As $\tau' \leq \tau$, by Proposition 2.25, V admits a unique subrepresentation W^i with cosocle τ'. By Theorem 2.10[(n, ℓ)], $W^i \cong I(\sigma, \tau')$. Similarly, V admits a quotient $\widetilde{W}^i$ with socle $F(\mathrm{t}_\mu((\omega'_i + \epsilon_i 1)\overline{\eta}_i))$ which is $(2n - 2\ell - 3)$-generic. By Theorem 2.10[(n, ℓ')], $\widetilde{W}^i \cong I(F(\mathrm{t}_\mu((\omega'_i + \epsilon_i 1)\overline{\eta}_i)), \tau)$.

As $\omega' \leq \omega$, $\mathrm{sgn}(\omega_i') = \epsilon_i$ if $\omega_i' \neq 0$. Given $F(\mathrm{t}_\mu(\omega')) \in \mathrm{JH}(V)$, by applying Corollary 2.13$[(n,\ell)]$ and respectively $[(n,\ell')]$ to $I(\sigma,\tau')$ and respectively $I(F(\mathrm{t}_\mu((\omega_i'+\epsilon(\omega_i))\overline{\eta}_i)),\tau)$, we deduce that

$$F(\mathrm{t}_\mu(\omega'')) \in \begin{cases} \mathrm{JH}(I(\sigma,\tau')) \text{ if and only if } |\omega_i''| \leq |\omega_i'| \\ \mathrm{JH}(I(F(\mathrm{t}_\mu((\omega_i'+\epsilon(\omega_i)1)\overline{\eta}_i)),\tau)) \text{ if and only if } |\omega_i''| \geq |\omega_i'|+1. \end{cases} \tag{2.4}$$

Consider the map $f\colon V \to \mathrm{Inj}_n F(\mathrm{t}_\mu((\omega_i'+\epsilon(\omega_i)1)\overline{\eta}_i))$, as V is multiplicity free, so is the image of f. Moreover, $\mathrm{cosoc}(f(V)) \cong \tau$. By Theorem 2.10$[(n,\ell')]$, we deduce that $f(V) \cong I(F(\mathrm{t}_\mu((\omega_i'+\epsilon(\omega_i)1)\overline{\eta}_i)),\tau)$. It follows from Equation (2.4) that $\tau' \in \mathrm{JH}(\ker(f))$. Therefore, $\ker(f)$ admits a subrepresentation with cosocle τ'. By Theorem 2.10$[(n,\ell)]$, such a representation is isomorphic to $I(\sigma,\tau')$. On the other hand, by Equation (2.4), we have $\mathrm{JH}(\ker(f)) = \mathrm{JH}(I(\sigma,\tau'))$. Therefore, $\ker(f) \cong I(\sigma,\tau')$. □

Proposition 2.30. Assume Theorem 2.10 holds for all pairs $< (n,n)$. Suppose V as in Theorem 2.10 with $m = n$, then V is uniquely determined by σ and τ up to multiplication by scalar.

Proof. Pick an i such that $|\omega_i| > 1$. By Proposition 2.25, $F(\mathrm{t}_\mu(\omega_k - \epsilon_i(\widetilde{\omega}_i - \omega_i + 1)\overline{\eta}_i))$, $F(\mathrm{t}_\mu(\widetilde{\omega}_i\overline{\eta}_i))$ are in $\mathrm{JH}(V)$. Therefore, the assumption of Lemma 2.29 holds and we deduce that V admits $W^i\colon \cong I(\sigma, F(\mathrm{t}_\mu(\omega_k - \epsilon_i(\widetilde{\omega}_i - \omega_i + 1)\overline{\eta}_i)))$ as a subrepresentation, and $\widetilde{W}^i := I(F(\mathrm{t}_\mu(\widetilde{\omega}_i\overline{\eta}_i)),\tau)$ as a quotient; and $V/W^i = \widetilde{W}^i$.

Hence, V represents a nontrivial class in $\mathrm{Ext}^1_{K/Z_1}(\widetilde{W}^i, W^i)$. Therefore, to prove the uniqueness of this representation, it is sufficient to show

$$\dim_{\mathbb{F}}(\mathrm{Ext}^1_{K/Z_1}(\widetilde{W}^i, W^i)) \leq 1.$$

First, we reduce to the case where $\widetilde{\omega}_i = \omega_i$. Assume $|\omega_i| = |\widetilde{\omega}_i| + 1$. By the proof of Lemma 2.29, we see that if $F(\mathrm{t}_\mu(\beta')) \in \mathrm{JH}(V)$, $F(\mathrm{t}_\mu(\beta')) \in \mathrm{JH}(W^i)$ if and only if $|\beta_i'| < |\widetilde{\omega}_i|$. $\widetilde{W}^i$ admits a quotient $\widetilde{W}'$ with socle $F(\mathrm{t}_\mu(\omega_i\overline{\eta}_i))$, which is $(2n-1-|\omega_i|)$-generic. As $|\omega_i| = |\widetilde{\omega}_i| + 1$, we can apply Corollary 2.13$[n - |\frac{\widetilde{\omega}_i}{2}|]$ to $I(F(\mathrm{t}_\mu(\omega_i\overline{\eta}_i)),\tau)$ and deduce that $F(\mathrm{t}_\mu(\omega')) \in \mathrm{JH}(I(F(\mathrm{t}_\mu(\omega_i\overline{\eta}_i)),\tau))$ only if $\omega_i' = \omega_i = \widetilde{\omega}_i + \epsilon_i 1$. Therefore, if $\tau' \in \mathrm{JH}(I(F(\mathrm{t}_\mu(\omega_i\overline{\eta}_i)),\tau))$ and $\sigma' \in \mathrm{JH}(W^i)$, then by [BHH+23, Lemma 2.4.6], $\mathrm{Ext}^1_{K/Z_1}(\tau',\sigma') = 0$. Therefore, by dévissage,

$$\mathrm{Ext}^1_{K/Z_1}(I(F(\mathrm{t}_\mu(\omega_i\overline{\eta}_i)),\tau), W^i) = 0.$$

Moreover, by applying Proposition 2.27 to $\widetilde{W}^i \cong I(F(\mathrm{t}_\mu(\widetilde{\omega}_i\overline{\eta}_i)), \tau)$ with $n = \sum_{j\neq i} \frac{|\omega_j|}{2}$ here and the multiplicity free condition, we have the following short exact sequence:

$$0 \to I(F(\mathrm{t}_\mu(\widetilde{\omega}_i\overline{\eta}_i)), F(\mathrm{t}_\mu(\omega_k - \epsilon_i\overline{\eta}_j))) \to \widetilde{W}^i \to I(F(\mathrm{t}_\mu(\omega_i\overline{\eta}_i), \tau) \to 0. \tag{2.5}$$

Applying the $\mathrm{Hom}_{K/Z_1}(—, W^i)$ functor to Equation (2.5) and observing that the Jordan–Hölder factors of W^i and $\widetilde{W}^i$ are disjoint, we deduce that the first 3 terms vanish and we have

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{K/Z_1}(I(F(\mathrm{t}_\mu(\omega_i\overline{\eta}_i)), \tau), W^i) \to \mathrm{Ext}^1_{K/Z_1}(\widetilde{W}^i, W^i) \\ \to \mathrm{Ext}^1_{K/Z_1}(I(F(\mathrm{t}_\mu(\widetilde{\omega}_i\overline{\eta}_i)), F(\mathrm{t}_\mu(\omega_k - \epsilon_i\overline{\eta}_j))), W^i). \end{aligned}$$

As $\mathrm{Ext}^1_{K/Z_1}(I(F(\mathrm{t}_\mu(\omega_i\overline{\eta}_i)), \tau), W^i) = 0$,

$$\dim_{\mathbb{F}}(\mathrm{Ext}^1_{K/Z_1}(\widetilde{W}^i, W^i)) \leq \dim_{\mathbb{F}}(\mathrm{Ext}^1_{K/Z_1}(I(F(\mathrm{t}_\mu(\widetilde{\omega}_i\overline{\eta}_i)), F(\mathrm{t}_\mu(\omega_k - \epsilon_i\overline{\eta}_j))), W^i)).$$

Hence, it is sufficient to show that the latter is 1. Therefore, we can assume $\widetilde{\omega}_i = \omega_i$.

By Proposition 2.27, we know that $I(\sigma, \tau^{(i)})$ is a subrepresentation of V^{n-1}. Applying Corollary 2.12$[n - \frac{\omega_i}{2}]$ to $I(\sigma, \tau^{(i)})$, we deduce that $F(\mathrm{t}_\mu(\omega')) \in \mathrm{JH}(I(\sigma, \tau^{(i)}))$ only if $\omega'_i = 0$. Recall from the second paragraph that $F(\mathrm{t}_\mu(\beta')) \in \mathrm{JH}(\widetilde{W}^{(i)})$ only if $\beta'_i = \omega_i$, in particular $|\beta_i| \geq 2$. By [BHH+23, Lemma 2.4.6], if $\sigma' \in I(\sigma, \tau^{(i)})$ and $\tau' \in \widetilde{W}^{(i)}$, we have $\mathrm{Ext}^1_{K/Z_1}(\tau', \sigma') = 0$. Therefore, by dévissage,

$$\mathrm{Ext}^1_{K/Z_1}(\widetilde{W}^{(i)}, I(\sigma, \tau^{(i)})) = 0.$$

Again, by the result of Proposition 2.27 for $V = W^i$, we have a short exact sequence:

$$0 \to I(\sigma, \tau^{(i)}) \to W^i \to I(\mu_i^{\epsilon_i}(\sigma), F(\mathrm{t}_\mu(\omega - \epsilon_i\overline{\eta}_))) \to 0. \tag{2.6}$$

Therefore, we again apply the functor $\mathrm{Hom}_{K/Z_1}(\widetilde{W}^i, —)$ to Equation (2.6), and observe that the Jordan–Hölder factors of W^i and $\widetilde{W}^i$ are disjoint, and hence the first 3 terms vanish, and we have an exact sequence

$$0 \to \mathrm{Ext}^1_{K/Z_1}(\widetilde{W}^i, I(\sigma, \tau^{(i)})) \to \mathrm{Ext}^1_{K/Z_1}(\widetilde{W}^i, W^i) \to \mathrm{Ext}^1_{K/Z_1}(\widetilde{W}^i, I(\mu_i^{\epsilon_i}(\sigma), F(\mathrm{t}_\mu(\omega - \epsilon_i\overline{\eta}_i)))).$$

As $\mathrm{Ext}^1_{K/Z_1}(\widetilde{W}^i, I(\sigma, \tau^{(i)})) = 0$,

$$\dim_{\mathbb{F}}(\mathrm{Ext}^1_{K/Z_1}(\widetilde{W}^i, W^i)) \leq \dim_{\mathbb{F}}(\mathrm{Ext}^1_{K/Z_1}(\widetilde{W}^i, I(\mu_i^{\epsilon_i}(\sigma), F(\mathrm{t}_\mu(\omega - \epsilon_i\overline{\eta}_i)))))).$$

As $\omega_i \in 2\mathbb{Z}$, $\sum_k \lfloor \frac{|\omega_k - \epsilon_i \delta_{ik}|}{2} \rfloor = n-1$, τ is an $n-1$ weight with respect to $\mu_i^{\epsilon_i}(\sigma)$ according to Lemma 2.19. Therefore, by Theorem 2.10[$(n, n-1)$], there is a unique m_{K/Z_1}^{n-1}-torsion representation with socle $\mu_i^{\epsilon_i}(\sigma)$, which is $(2n-2)$-generic, and τ, and so

$$\dim_{\mathbb{F}}(\mathrm{Ext}^1_{K/Z_1}(\widetilde{W}^i, I(\mu_i^{\epsilon_i}(\sigma), F(\mathrm{t}_\mu(\omega_k - \epsilon_i \overline{\eta}_i)))))) = 1. \qquad \square$$

$\square$

Corollary 2.31. *If V has cosocle τ, $[V : \tau] = 1$, τ is $(2n+1)$-generic and $\mathrm{JH}(V) \subseteq \mathrm{JH}(\mathrm{Proj}_n \tau)$, then $\mathrm{Proj}_n \tau \twoheadrightarrow V$, in particular V is $\mathfrak{m}_{K_1}^n$-torsion.*

Proof. Assume that this does not hold, let M be the counterexample with minimal length. Then by definition, $V/\mathrm{soc}(V)$ has cosocle τ and $\mathrm{length}(V/\mathrm{soc}(V)) < \mathrm{length}(V)$; therefore, $V/\mathrm{soc}(V)$ is $\mathfrak{m}_{K_1}^n$-torsion. We have

$$0 \to \mathrm{soc}(V) \to V \to V/\mathrm{soc}(V) \to 0$$

and $\mathrm{soc}(V)$ is semisimple and therefore K_1-invariant. Therefore, V is $\mathfrak{m}_{K_1}^{n+1}$-torsion. By the dual version of Corollary 2.12[$n+1$], we deduce that V is multiplicity free. We have $V \hookrightarrow \bigoplus_{\sigma \subseteq \mathrm{soc}(V)} \mathrm{Inj}\, \sigma$, which by Theorem 2.10, factors through $I(\sigma, \tau)$. Since $\sigma \in \mathrm{JH}(V) \subseteq \mathrm{JH}(\mathrm{Proj}_n \tau)$, $I(\sigma, \tau)$ is $\mathfrak{m}_{K_1}^n$-torsion, so is V. $\square$

3 GALOIS DEFORMATION RINGS

In this section, our goal is to use "local models" to compute the Galois deformation ring $R_{\overline{\rho}}^{\lambda,\tau}$ for sufficiently generic $\overline{\rho}$. On the one hand, we will compute the ring R_{poly}, which approximates $R_{\overline{\rho}}^{\leq(\ell_j,0)_j,\tau}$, up an explicit tail. On the other hand, we will calculate the integer k such that p^k lies in a certain ideal. We can then carry out Elkik's approximation and compute $R_{\overline{\rho}}^{\leq(\ell_j,0)_j,\tau}$ with an explicit genericity condition. For any Hodge–Tate weights λ and tame inertial type τ under some explicit genericity conditions, we will give an explicit description of $R_{\overline{\rho}}^{\lambda,\tau}$. In particular, we show such a ring to be a normal domain and a complete intersection ring. Moreover, we give a bijection between the irreducible components of the special fibre of $R_{\overline{\rho}}^{\lambda,\tau}$ and $W(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$. We follow the approach and notation of [BHH+23] with modification from [Wan23] for the non-semisimple cases, which in turn, use the methods and results of [LLHLM18], [LLHL19], [LLHLM20] and [LLHLM23].

3.1 NOTATIONS

An inertial type is a representation of I_K with open kernel which can be extended to G_K. Given $\lambda \in X^*_+(\underline{T})$, we can define the λ-admissible set relative to the Bruhat order, $\mathrm{Adm}^\vee(t_\lambda)$, which will be described explicitly below. Given $\widetilde{w} \in \underline{\widetilde{W}}^\vee$, we can associate $\widetilde{w}^* \in \underline{\widetilde{W}}$, such that $((st_\mu)^*)_j = t_{\mu_{f-1-j}} s^{-1}_{f-1-j}$. Let $\lambda = (\lambda'_{j,1}, \lambda_{j,2})$ with $\lambda_{j,1} \geq \lambda_{j,2}$. we write $\lambda' \leq \lambda$ if for all j, $\lambda_{j,1} + \lambda_{j,2} = \lambda'_{j,1} + \lambda'_{j,2}$ and $\lambda_{j,1} \geq \lambda'_{j,1} \geq \lambda_{j,2}$. It can be shown that

$$\mathrm{Adm}^\vee(t_\lambda) = \{\widetilde{w} \colon \widetilde{w}_{f-1-j} = t_{\lambda'_j} \text{ or } \mathfrak{w}t_{\lambda'_j}, \text{ with } \lambda'_j \leq \lambda_j \text{ and } \widetilde{w}_{f-1-j} \neq \mathfrak{w}t_{(\lambda_{j,2},\lambda_{j,1})}\}.$$

Given $(s,\mu) \in \underline{W} \times X^*(\underline{T})$, we can associate a tame inertial type $\tau(s,\mu)$. (For more details, see [BHH+23, § 2.3]) We say that τ is N-generic for some $N \in \mathbb{Z}_{\geq 0}$ if $\tau \cong \tau(s,\mu+\eta)$ for some μ which is N-deep in $\underline{C_0}$.

We let $\overline{\rho} \colon G_K \to \mathrm{GL}_n(\mathbb{F})$ be a Galois representation. We say $\overline{\rho}$ is N-generic if $\overline{\rho}^{ss}|_{I_K} \cong \overline{\tau}(s,\mu)$ for some $s \in \underline{W}$ and $\mu - \eta \in X^*(\underline{T})$ which is N-deep in $\underline{C_0}$. Let ρ be a twodimensional de Rham representation of G_K over $\overline{\mathbb{Q}}_p$, with regular Hodge–Tate weights. If there is a unique $\lambda = (\lambda_{\kappa,i}) \in (\mathbb{Z}^2)^f$ such that for each $\sigma_i \colon K \hookrightarrow \overline{\mathbb{Q}}_p$,

$$HT_i(\rho) = \{\lambda_{i,1}, \lambda_{i,2}\},$$

with $\lambda_{i,1} > \lambda_{i,2}$, then we say ρ is regular of Hodge type λ. For two Hodge–Tate weights λ, λ', we write $\lambda' \geq \lambda$ if for all j, $\lambda_{j,1} + \lambda_{j,2} = \lambda'_{j,1} + \lambda'_{j,2}$ and $\lambda'_{j,1} \geq \lambda_{j,1} \geq 0$. We normalize Hodge-–Tate weights so that ε has Hodge–Tate weight 1 at every embedding.

Let $R^{\square}_{\overline{\rho}}$ be the local $\mathcal{O}$-algebra parameterizing framed deformation of $\overline{\rho}$. For each dominant weight $\lambda \in X^*_+(\underline{T})$, let $R^{\lambda,\tau}_{\overline{\rho}}$ (resp. $R^{\leq\lambda,\tau}_{\overline{\rho}}$) be the maximal reduced, $\mathcal{O}$-flat quotient of $R^{\square}_{\overline{\rho}}$, which parametrizes potentially crystalline lifts of ρ with Hodge–Tate weights λ (resp. $\lambda' \leq \lambda$) and tame inertial type τ (its existence follows from [Kis08]). We define $R^{\sigma}_{\overline{\rho}}$ to be the reduced, p-torsion free quotient of $R^{\square}_{\overline{\rho}}$ corresponding to the crystalline deformation of Hodge type σ.

Given a tame inertial type τ, by the inertial local Langlands correspondence given in the appendix of [BM02], we have a finite-dimensional irreducible E-representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathcal{O}_K)$, which by extending scalar is defined over E. We write $\overline{\sigma}(\tau)$ for the mod p semisimplification of $\sigma(\tau)$. Then the action of $\mathrm{GL}_2(\mathcal{O}_K)$ on $\overline{\sigma}(\tau)$ factors through $\mathrm{GL}_2(k)$, so that the Jordan–Hölder factors of $\overline{\sigma}(\tau)$ are Serre weights. More precisely, the Jordan–Hölder factors of $\overline{\sigma}(\tau)$ are described as follows.

Proposition 3.1. [BHH+23, Prop. 2.4.3] Suppose $\tau = \tau(sw^{-1}, \mu - sw^{-1}(\nu))$ for some $(s,\mu),(w,\nu) \in \underline{W} \times X^*(\underline{T})$ such that $\mu - sw^{-1}(\nu) - \eta$ is 1-deep in $\underline{C}_0$. If $\nu \in \eta + \Lambda_R$, then

$$\mathrm{JH}\left(\overline{\sigma}(\tau)\right) = \left\{ F(\mathfrak{t}_{\mu-\eta}(sw^{-1}(\omega - \overline{\nu}))) \colon \omega \in \Sigma \right\}.$$

We let $V(\lambda)$ be the irreducible algebraic representation with the highest weight λ. We write $V(\lambda-\eta) := \otimes_{E,j} V(\lambda_j - \eta)^{(j)}$ and $\sigma(\lambda,\tau) := \sigma(\tau) \otimes_{E,j} V(\lambda_j - \eta)^{(j)}$ for the $\mathrm{GL}_2(\mathcal{O}_K)$ representation over E. We write $\overline{\sigma}(\lambda,\tau)$ for the mod p semisimplification of $\sigma(\lambda,\tau)$.

Lemma 3.2. *Assume $\tau = \tau(sw^{-1}, \mu - sw^{-1}(\nu))$ is N-generic, for $N \geq 1$ then for all $\sigma \in \mathrm{JH}(\overline{\sigma}(\tau))$, σ is $N-1$-generic (cf. Definition 2.3). If $\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$ where $\lambda \leq (\ell_j, 0)$, then σ is $N-\ell$-generic.*

Proof. As τ is N-generic, by Proposition 3.1, we let $\sigma = F(\mathfrak{t}_{\mu-\eta}(sw^{-1}(\omega - \overline{\nu})))$ for some $\omega \in \Sigma$. We have

$$N < \mu - sw^{-1}(\nu) - \eta < p - N.$$

Since

$$\langle \mathfrak{t}_{\mu-\eta}(sw^{-1}(\omega - \overline{\nu})), \alpha_j^\vee \rangle = \langle \mu - \eta + sw^{-1}(\overline{\nu}), \alpha_j^\vee \rangle \pm 1 \text{ (for all } j),$$

$$N - 1 < \langle \mathfrak{t}_{\mu-\eta}(sw^{-1}(\omega - \overline{\nu})), \alpha^\vee \rangle < p - N + 1.$$

Therefore, σ is $(N-1)$-generic. We can deduce the last assertion from the fact that

$$L(a,b)\otimes_{\mathbb{F}} L(m-1,n) = L(a+m-1,b+n)\oplus L(a+m-2,b+n+1)\oplus\cdots\oplus L(a+n,b+m-1). \tag{3.1}$$

□

3.2 KISIN MODULES

We will use without explanation the notation of [BHH+23, § 3] and [Wan23, § 3]. Let R be a p-adically complete Noetherian local $\mathcal{O}$-algebra and $h\in\mathbb{Z}_{\geq 0}$. We denote the category of Kisin modules over R of $E(u')$-height $\leq h$ and type τ by $Y^{[0,h],\tau}(R)$ as in [LLHLM20, Definition 3.1.3]. Given an eigenbasis β for $\mathfrak{M}\in Y^{[0,h],\tau}(R)$ (*cf.* [LLHLM20, Definition 3.1.6]), we have a matrix $A^{j}_{\mathfrak{M},\beta}$. Given a dominant weight λ, we can then define a subcategory of Kisin modules of height $\leq\lambda$, denoted by $Y^{\leq\lambda,\tau}(R)\subseteq Y^{[0,h],\tau}(R)$. Let $I(\mathbb{F})$ be the Iwahori subgroup of $\mathrm{GL}_2(\mathbb{F}[\![v]\!])$ consisting of the matrices which are upper triangular modulo v. Then $\overline{\mathfrak{M}}$ has shape $\widetilde{w}$ if $A^{j}_{\mathfrak{M},\beta}\in I(\mathbb{F})\widetilde{w}_j I(\mathbb{F})$ for any choice of eigenbasis β, we have for each $0\leq j\leq f-1$. In order to account for non-semisimple Galois representation, we need to use $\widetilde{w}$-gauge basis [Wan23, Definition 3.1] instead of Gauge basis defined in [LLHL19, Definition 3.2.23]. As noted in [Wan23], $\overline{\mathfrak{M}}$ has a unique shape, but it could have $\widetilde{w}$-gauge for many choices of $\widetilde{w}$.

Example 3.3. (*cf.* [Wan23, Example 3.3]) Let $\alpha,\beta\in\mathbb{F}^{\times}$ and $a\in\mathbb{F}$. We list the gauges and shapes of some matrices in $\mathrm{GL}_2(\mathbb{F}(\!(v)\!))$ that will be considered in Lemma 3.11.

	Matrix	One choice of gauge	Shape
$m>n$	$\begin{pmatrix}\alpha v^m & 0\\ a v^m & \beta v^n\end{pmatrix}$	$\mathfrak{t}_{(m,n)}$-gauge	$\mathfrak{t}_{(m,n)}$
$m\leq n$	$\begin{pmatrix}\alpha v^m & 0\\ a v^m & \beta v^n\end{pmatrix}$	$\mathfrak{t}_{(m,n)}$-gauge	$\mathfrak{t}_{(m,n)}$ if $a=0$ $\mathfrak{w}\mathfrak{t}_{(m,n)}$ if $a\neq 0$
$m>n$	$\begin{pmatrix}0 & \beta v^n\\ \alpha v^m & a v^n\end{pmatrix}$	$\mathfrak{w}\mathfrak{t}_{(m,n)}$-gauge	$\mathfrak{w}\mathfrak{t}_{(m,n)}$ if $a=0$ $\mathfrak{t}_{(m,n)}$ if $a\neq 0$
$m\leq n$	$\begin{pmatrix}0 & \beta v^n\\ \alpha v^m & a v^n\end{pmatrix}$	$\mathfrak{w}\mathfrak{t}_{(m,n)}$-gauge	$\mathfrak{w}\mathfrak{t}_{(m,n)}$

Let $\mathcal{O}_{\mathcal{E},K}$ be the p-adic completion of $W(k)[\![v]\!][1/v]$. We write $\Phi\,\mathrm{Mod}^{\text{ét}}(R)$ for the category of étale φ-modules over $\mathcal{O}_{\mathcal{E},K}\widehat{\otimes}R$. We have an equivalence of categories $\mathbb{V}^*_K\colon\Phi\,\mathrm{Mod}^{\text{ét}}(R)\xrightarrow{\sim}\mathrm{Rep}_{G_{K_\infty}}(R)$. By post-composing it with the functor $\epsilon_\tau\colon Y^{[0,h],\tau}(R)\to\Phi\,\mathrm{Mod}^{\text{ét}}(R)$ (*cf.* [LLHLM23, § 5.4]), we have a functor $T^*_{dd}\colon Y^{[0,h],\tau}(R)\to\mathrm{Rep}_{G_{K_\infty}}(R)$.

Fix $(\ell_1, \dots, \ell_f) \in \mathbb{Z}_+^f$, and let $\ell = \max\{\ell_j\}$. We fix a Galois representation $\overline{\rho}\colon G_K \to \mathrm{GL}_2(\mathbb{F})$ and $(s, \mu) \in \underline{W} \times X^*(\underline{T})$ such that $\overline{\rho}^{ss}|_{I_K} \cong \overline{\tau}(s, \mu)$ (here ss denotes the semisimplification of $\overline{\rho}$), where

1. $s_j = \mathfrak{w}$ precisely when $j = 0$ and $\overline{\rho}$ is irreducible;
2. $\mu - \eta$ is N-deep in $\underline{C_0}$.

Twisting $\overline{\rho}$ with a power of ω_f if necessary, we further assume that $\mu_j = (r_j + 1, 0) \in \mathbb{Z}^2$ with $N < r_j + 1 < p - N$ for all j so that

$$\overline{\rho}|_{I_k} \cong \begin{cases} \begin{pmatrix} \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j} & * \\ 0 & 1 \end{pmatrix} & \text{if } \overline{\rho} \text{ is reducible;} \\ \begin{pmatrix} \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j} & 0 \\ 0 & \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^{j+f}} \end{pmatrix} & \text{if } \overline{\rho} \text{ is irreducible.} \end{cases} \tag{3.2}$$

(The pair (s, μ) is not uniquely determined by $\overline{\rho}|_{I_K}$, however if $\overline{\rho}$ is $(N+1)$-generic, then by choosing an appropriate choice of s, 1., 2. always hold [LLHL19, Propositions 2.2.15, 2.2.16].) We will assume $N \geq 4\ell$ in the following.

Let $\overline{\mathcal{M}}$ be the étale φ-module over $k((v)) \otimes_{\mathbb{F}_p} \mathbb{F}$ such that $\mathbb{V}_K^*(\overline{\mathcal{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$. By [Le19], we have a decomposition $\overline{\mathcal{M}} \cong \bigoplus_{i \in \mathcal{J}} \overline{\mathcal{M}}^{(j)}$ with $\overline{\mathcal{M}}^{(j)} = F((v))e_1^{(j)} \oplus F((v))e_2^{(j)}$ such that the matrices of the Frobenius map $\phi_{\overline{\mathcal{M}}}^{(j)}\colon \overline{\mathcal{M}}^{(j)} \to \overline{\mathcal{M}}^{(j+1)}$ with respect to the basis $\{(e_1^{(j)}, e_2^{(j)})\}$ have the following form

$$\mathrm{Mat}(\phi_{\overline{\mathcal{M}}}^{(f-1-j)}) = \begin{cases} \begin{pmatrix} \alpha_j v^{r_j+1} & 0 \\ \alpha_j \gamma_{f-1-j} v^{r_j+1} & \beta_j \end{pmatrix} & \text{if } \overline{\rho} \text{ is reducible;} \\ \begin{pmatrix} \alpha_j v^{r_j+1} & 0 \\ 0 & \beta_j \end{pmatrix} & \text{if } \overline{\rho} \text{ is irreducible and } j \neq 0; \\ \begin{pmatrix} 0 & -\beta_j \\ \alpha_j v^{r_j+1} & 0 \end{pmatrix} & \text{if } \overline{\rho} \text{ is irreducible and } j = 0. \end{cases} \tag{3.3}$$

where $\alpha_j, \beta_j \in \mathbb{F}^\times$ and $\gamma_j \in \mathbb{F}$. When $\overline{\rho}$ is irreducible, we define $\gamma_j = 0$ for all j. From now on, we fix a choice of $\alpha_j, \beta_j, \gamma_j$. Note that $\overline{\rho}$ is semisimple if and only if $\gamma_j = 0$ for all j.

Proposition 3.4. [Le19, Proposition 3.2], [DL21, Proposition 3.5] Given a Galois representation $\overline{\rho}\colon G_K \to \mathrm{GL}_2(\mathbb{F})$, there is a set of Serre weights $W(\overline{\rho})$ associated to $\overline{\rho}$, described as follows:

$$W(\overline{\rho}) = \left\{ F(\mathfrak{t}_{\mu-\eta}(b_0, \dots, b_{f-1})\colon b_j \in \{0, \mathrm{sgn}(s_j)\} \text{ if } \gamma_{f-1-j} = 0 \text{ and } b_j = 0 \text{ if } \gamma_{f-1-j} \neq 0 \right\}.$$

Alternatively, $W(\overline{\rho}) = \{\sigma_J | J \subseteq J_{\overline{\rho}}\}$, and we can associate $\sigma \mapsto J_\sigma = \{j \colon b_j \neq 0\}$.

For $\widetilde{w} \in \mathrm{Adm}^\vee(\mathfrak{t}_\lambda)$, where $\widetilde{w}^* = \mathfrak{t}_\nu w$ for some unique $(w, \nu) \in \underline{W} \times X^*(\underline{T})$, we associate type

$$\tau_{\widetilde{w}} := \tau(sw^{-1}, \mu - sw^{-1}(\nu))$$

with a lowest alcove representation $(s(\tau), \mu(\tau)) = (sw^{-1}, \mu - sw^{-1}(\nu) - \eta)$, in particular $\tau_{\widetilde{w}}$ is $(N-1)$-generic. Explicitly, $s(\tau)_j = w_j^{-1}$ except when $j = 0$ and $\overline{\rho}$ is irreducible, in which case, we have $s(\tau)_0 = \mathfrak{w} w_0^{-1}$. We have

$$\mu(\tau)_j + \eta_j = \begin{cases} (r_j + 1 - m, -n) & \text{if } (\mathfrak{t}_{\nu_j} w_j, s_j) = (t_{(m,n)}, 1) \text{ or } (t_{(m,n)}\mathfrak{w}, \mathfrak{w}); \\ (r_j + 1 - n, -m) & \text{if } (\mathfrak{t}_{\nu_j} w_j, s_j) = (t_{(m,n)}\mathfrak{w}, 1) \text{ or } (t_{(m,n)}, \mathfrak{w}). \end{cases} \tag{3.4}$$

Definition 3.5. Given a dominant weight λ, we define

$$X(\overline{\rho}, \lambda) := \{\widetilde{w} \in \mathrm{Adm}^\vee(\mathfrak{t}_\lambda) \colon \mathrm{JH}(\overline{\sigma}(\tau_{\widetilde{w}}, \lambda) \cap W(\overline{\rho}) \neq \varnothing\}.$$

Lemma 3.6. *If $\lambda = (\lambda_{j,1}, \lambda_{j,2})$ is a dominant weight,*

$$X(\overline{\rho}, \lambda) = \{\widetilde{w} \in \mathrm{Adm}^\vee(\mathfrak{t}_\lambda) \colon \widetilde{w}_{f-1-j} \neq \mathfrak{t}_{(\lambda_{j,2}, \lambda_{j,1})} \text{ if } \gamma_{f-1-j} \neq 0\}.$$

Proof. The proof is similar to [Wan23, § 4]. By Proposition 3.1 and Equation (3.1), we can deduce that $F(\mathfrak{t}_{\mu-\eta}(b_0, \ldots, b_{f-1})) \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau_{\widetilde{w}}))$ if and only if $b_j \in$

$$\{\mathrm{sgn}(s_j)\,\mathrm{sgn}(w_j)(r-1) + \lambda_{j,1} + \lambda_{j,2} + 1 - 2k \colon k \in \mathbb{Z}, \lambda_{j,2} < k \leq \lambda_{j,1}, r \in \{0, 1\}\}. \tag{3.5}$$

Assume $\widetilde{w}_{f-1-j} = \omega_j \mathfrak{t}_{(a,b)}$. If $\lambda_{j,2} < a \leq \lambda_{j,1}$, by taking $r = 1$ and $k = \frac{(1+\mathrm{sgn}(s_j)\,\mathrm{sgn}(w_j)1)}{2}(\lambda_{j,1} + \lambda_{j,2} + 1) - \mathrm{sgn}(s_j)\,\mathrm{sgn}(w_j)a$, we deduce that 0 is in Equation (3.5). If $a = \lambda_{j,2}$, then 0 is not in Equation (3.5), but $\mathrm{sgn}(s_j)$ is if $\mathrm{sgn}(\omega_j) = -1$, $r = 0$ and $k = \frac{1}{2}((\mathrm{sgn}(s_j) + 1)(\lambda_{j,1}) + (1 - \mathrm{sgn}(s_j))\lambda_{j,2}))$. These are all the possibilities for $\widetilde{w}_{f-1-j}$. □

Definition 3.7. Given $\widetilde{w} \in \mathrm{Adm}^\vee(\mathfrak{t}_{(\ell_j,0)_j})$. Let $X(\widetilde{w}, \lambda)$ be the set of all regular Hodge–Tate weights λ', such that $\lambda' \leq \lambda$ and $\widetilde{w} \in X(\overline{\rho}, \lambda')$. And let $S(\widetilde{w}, \lambda)$ be the cardinality of $X(\widetilde{w}, \lambda)$. We define

$$S(\widetilde{w}_j) = \begin{cases} \min(m, n) + 1 & \text{if } \widetilde{w}_j = \mathfrak{t}_{(m,n)} \text{ with } m > n \text{ or } (\gamma_j = 0 \text{ and } m < n); \\ \min(m, n) & \text{if } \widetilde{w}_j = \mathfrak{t}_{(m,n)} \text{ with } (m < n \text{ and } \gamma_j \neq 0) \text{ or } m = n; \\ \min(m, n+1) & \text{if } \widetilde{w}_j = \mathfrak{m}\mathfrak{t}_{(m,n)}. \end{cases}$$

By Lemma 3.6, we can deduce that

$$S(\widetilde{w},\lambda)=\prod_j \max\{0, S(\widetilde{w}_{f-1-j})-\lambda_{2,j}\}. \tag{3.6}$$

We now recall the results for the geometric Breuil–Mézard Conjecture for GL_2. Following the notations from [EG14]: Given a closed subscheme $\mathcal{Z}$ of $\mathcal{X}$, the cycles $Z(\mathcal{Z}) := \sum_{\mathfrak{a}} e(\mathcal{Z},\mathfrak{a})$ are well-defined, where $e(\mathcal{Z},\mathfrak{a})$ is the Hilbert–Samuel multiplicity of $\mathcal{Z}$ at $\mathfrak{a}$ and the sum is over the points of $\mathcal{X}$ with the same dimension as $\mathcal{Z}$.

Lemma 3.8. *Fix τ which is $(2n+2)$-generic and $\lambda \le (\ell_j, 0)$ with $\ell_j \le n$. Let $a_\sigma(\lambda,\tau) \in \{0,1\}$ such that $\overline{\sigma}^{\mathrm{ss}}(\lambda,\tau) = \sum a_\sigma(\lambda,\tau)\sigma$ where the sum is over all Serre weights. Given a Serre weight σ, let $C_\sigma := Z(\operatorname{Spec} R^{\sigma}_{\overline{\rho}})$. We have the following equality of cycles:*

$$Z(\operatorname{Spec} R^{\lambda,\tau}_{\overline{\rho}}) = \sum_{\sigma\in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))} a_\sigma(\lambda,\tau) C_\sigma.$$

Proof. This can be deduced from [FH25, Theorem 1.3.1 (1)], where we take G in the theorem as $G = \operatorname{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_2$. Specifically, there exists cycles $\mathcal{Z}(\sigma)$ such that

$$[\mathcal{X}^{\lambda,\tau}_{2,\mathbb{F}_p}] = \sum_{\sigma\in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))} a_\sigma(\lambda,\tau)\mathcal{Z}(\sigma),$$

where $[\cdot]$ denotes the cycle class and $\mathcal{X}^{\lambda,\tau}_{2,\mathbb{F}}$ is the special fibre of the Emerton–Gee stack which parametrizes 2-dimensional potentially crystalline representations of G_K with Hodge–Tate weights λ and inertial type τ. Moreover, as pointed out by Daniel Le, by comparing the result with [CEG$^+$18, Theorem 1.5] (cf.[FH25, § 1.4]), we know $C_{F(\lambda)} = [\mathcal{X}^{\lambda,\mathrm{triv}}|_{\mathbb{F}_p}]$. By the discussion of [EG23, § 8.3], we can recover the version of geometric Breuil–Mézard conjecture in terms of algebraic cycles, as in [EG14]. □

Lemma 3.9. *Assume $\lambda = (\lambda_j)_{j\in\mathcal{J}} \in X^*(T^\vee)^{\mathcal{J}}$ satisfies $\lambda_j \le (\ell_j, 0)$. Given that τ is $(2\ell+2)$-generic. Then $R^{\lambda,\tau}_{\overline{\rho}} \neq 0$ if and only if $\tau = \tau_{\widetilde{w}}$ with $\widetilde{w} \in X(\overline{\rho},\lambda)$. For each fixed 4ℓ-generic tame type $\tau_{\widetilde{w}}$, there are $S(\widetilde{w},\lambda)$ regular Hodge–Tate weights $\lambda' \le \lambda$, such that $\overline{\rho}$ admits a potentially crystalline lift ρ of inertial type τ with $\mathrm{HT}_j(\rho) = \lambda'_j$ for all j.*

Proof. The first statement follows from Breuil–Mézard conjecture in Lemma 3.8. By [EG14, Theorem 5.4.4], the mod ϖ_E-fibre of the deformation space $\overline{X}(\lambda,\tau_{\widetilde{w}})$ is the union of ϖ_E-fibres $\overline{X}(\overline{\sigma})$ where σ runs over the Jordan–Hölder factors of $\overline{\sigma}(\lambda,\tau_{\widetilde{w}})$. Therefore, $R^{\lambda,\tau_{\widetilde{w}}}_{\overline{\rho}} \neq 0$ if and only if there exists $\overline{\sigma} \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$ with $X^{\overline{\sigma}}_{\overline{\rho}} \neq 0$. Moreover, $\overline{X}(\overline{\sigma})$ is nonempty if and only if $\overline{\sigma} \in W(\overline{\rho})$, by [GLS14, Theorem A] (*cf.* [EGS15, Theorem 7.1.1]). The last statement follows from the first one, together with Equation (3.6). □

Remark 3.10. We assume τ to be $(2\ell+1)$-generic a priori, however if $R_{\overline{\rho}}^{\lambda,\tau} \neq 0$, by Lemma 3.9 and Lemma 3.6, we deduce that τ is actually 3ℓ-generic

Lemma 3.11. *Let $\widetilde{w} \in X(\overline{\rho}, \lambda)$. Up to isomorphism there exists a unique Kisin module $\overline{\mathfrak{M}} \in Y^{\leq \lambda, \tau_{\widetilde{w}}}(\mathbb{F}) \subseteq Y^{\leq (\ell_j, 0)_j, \tau_{\widetilde{w}}}$ such that $T^*_{dd}(\overline{\mathfrak{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$*

Proof. If $\overline{\rho}$ is irreducible, the proof goes exactly as in [BHH+23, Lemma 4.1.1], provided that $\ell < \langle \mu(\tau)_j + \eta_j, \alpha_j^\vee \rangle < p - \ell - 1$. If $\overline{\rho}$ is reducible, the proof goes exactly the same way as in [Wan23, Lemma 4.1], we will simply comment on the changes required. Define a Kisin module $\overline{\mathfrak{M}}$ over $\mathbb{F}$ of type $\tau_{\widetilde{w}}$ by imposing the matrix of the partial Frobenius map to be $\overline{A}^{(f-1-j)} = \mathrm{Mat}(\phi_{\mathcal{M}}^{(f-1-j)}) v^{-(\mu(\tau)_j + \eta_j)} \dot{s}(\tau)_j$, where $\mathrm{Mat}(\phi_{\mathcal{M}}^{(f-1-j)})$ and $\mu(\tau)_j + \eta_j$ are computed in Equation (3.3) and Equation (3.4) respectively. Therefore, we have

$$\overline{A}^{(f-1-j)} = \begin{cases} \begin{pmatrix} \alpha_j v^m & 0 \\ \alpha_j \gamma_{f-1-j} v^m & \beta_j v^n \end{pmatrix} & \text{if } \widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}; \\ \begin{pmatrix} 0 & \alpha_j v^n \\ \beta_j v^m & \alpha_j \gamma_{f-1-j} v^n \end{pmatrix} & \text{if } \widetilde{w}_{f-1-j} = \mathfrak{w}\mathfrak{t}_{(m,n)}. \end{cases} \tag{3.7}$$

In general, $\overline{\mathfrak{M}}$ has $\widetilde{w}$-gauge basis, but may not have shape $\widetilde{w}$. If $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$ with $m \leq n$, then Example 3.3 shows that $\overline{A}^{(f-1-j)}$ has shape contained $\mathrm{Adm}^\vee(\mathfrak{t}_{(n,m)})$ if and only if $\gamma_{f-1-j} = 0$. By Lemma 3.6, we deduce that $\widetilde{w} \in X(\overline{\rho}, \lambda)$ if and only if $\overline{A}^{(f-1-j)}$ has shape contained $\mathrm{Adm}^\vee(\mathfrak{t}_{\lambda_j})$ for all j. Therefore, $\overline{\mathfrak{M}} \in Y^{\leq \lambda, \tau_{\widetilde{w}}}(\mathbb{F}) \subseteq Y^{\leq (\ell_j, 0))_j, \tau_{\widetilde{w}}}(\mathbb{F})$ for all $\lambda \in X(\widetilde{w}, (\ell_j, 0)_j)$ by [CL18, Proposition 5.4]. The rest of the proof goes through the same way given our genericity assumption. □

3.3 GALOIS DEFORMATION RING

Given a complete Noetherian local $\mathcal{O}$-algebra R with residue field $\mathbb{F}$ and $(\ell_1, \ldots, \ell_f)$ a f-tuple of positive integers, we define $D_{\overline{\mathfrak{M}}, \overline{\beta}}^{\leq (\ell_j, 0)_j, \tau}(R)$ to be the groupoid of the triplet $(\mathfrak{M}, \beta, \jmath)$, where $\mathfrak{M} \in Y^{\leq (\ell_j, 0)_j, \tau_{\widetilde{w}}}(R)$, β a $\widetilde{w}$-gauge basis of $\mathfrak{M}$ and $\jmath\colon \mathfrak{M} \otimes_R \mathbb{F} \xrightarrow{\sim} \overline{\mathfrak{M}}$ sending β to $\overline{\beta}$. Then for any $(\mathfrak{M}, \beta, \jmath) \in D_{\overline{\mathfrak{M}}, \overline{\beta}}^{\leq (\ell_j, 0)_j, \tau}(R)$, we have a corresponding matrix $A^{(f-1-j)}$ such that $A^{(f-1-j)} \mod m_R \equiv \overline{A}^{(f-1-j)}$. We will compute $A^{(f-1-j)}$ using the monodromy and height conditions as in [BHH+23, Proposition 4.2.1], *cf.* [LLHLM18, Proposition 4.18].

If $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$, i.e., $\overline{A}^{(f-1-j)} = \begin{pmatrix} \alpha_j v^m & 0 \\ \alpha_j \gamma_{f-1-j} v^m & \beta_j v^n \end{pmatrix}$. Then

$$A^{(f-1-j)} = \begin{pmatrix} \sum_{0\le i\le m} a_i^{(j)}(v+p)^i & \sum_{0\le i\le n-1} b_i^{(j)}(v+p)^i \\ v(\sum_{0\le i\le m-1} c_i^{(j)}(v+p)^i) & \sum_{0\le i\le n} d_i^{(j)}(v+p)^i \end{pmatrix}.$$

Given Shape $\widetilde{w}_{f-1-j} = \mathfrak{w}\mathfrak{t}_{(m,n)}$, i.e., $\overline{A}^{(f-1-j)} = \begin{pmatrix} 0 & \beta_j v^n \\ \alpha_j v^m & \alpha_j \gamma_{f-1-j} v^n \end{pmatrix}$. Then

$$A^{(f-1-j)} = \begin{pmatrix} \sum_{0\le i\le m-1} a_i^{(j)}(v+p)^i & \sum_{0\le i\le n} b_i^{(j)}(v+p)^i \\ v(\sum_{0\le i\le m-1} c_i^{(j)}(v+p)^i) & \sum_{0\le i\le n} d_i^{(j)}(v+p)^i \end{pmatrix}.$$

We will suppress the superscript j when it is clear from the context. Recall that the finite height condition is given by

$$\det A^{(f-1-j)} \in R^\times (v+p)^{\ell_{f-1-j}} \text{ for all } j.$$

For $0 \le k \le \ell_{f-1-j} - 1$ the kth height condition is given by

$$H(k) = \sum_{i+j=k} (a_i d_j + p b_j c_i) - \sum_{i+j=k-1} b_j c_i.$$

Since $\overline{\rho}$ is $N = 4\ell$-generic, if $R_{\overline{\rho}}^{\lambda,\tau} \neq 0$, by Remark 3.10, τ is 3ℓ-generic. We can then apply [BHH+23, Proposition 3.19] with $h = \ell$ and obtain the monodromy condition given as follows:

$$\left(\frac{d}{dv}\right)^t \bigg|_{v=-p} \left\{ \left[v\frac{d}{dv} A^{(f-1-j)} - A^{(f-1-j)} \begin{pmatrix} \mathfrak{a} & 0 \\ 0 & 0 \end{pmatrix} \right] (v+p)^h (A^{(f-1-j)})^{-1} \right\} + O(p^{N-h-t}) \tag{3.8}$$

for all $0 \le t \le h-2$, $0 \le j \le f-1$. As $\det A^{(f-1-j)} \in R^\times (v+p)^{\ell_{f-1-j}}$, Equation (3.8) is 0 for $0 \le t \le h - \ell_{f-1-j}$. Therefore, using the Leibniz rule, we can reduce it to the following equation:

$$\left(\frac{d}{dv}\right)^t \bigg|_{v=-p} \left\{ \left[v\frac{d}{dv} A^{(f-1-j)} - A^{(f-1-j)} \begin{pmatrix} \mathfrak{a} & 0 \\ 0 & 0 \end{pmatrix} \right] (v+p)^{\ell_{f-1-j}} (A^{(f-1-j)})^{-1} \right\} + O(p^{N-\ell_{f-1-j}-t})$$

for all $0 \le t \le \ell_{f-1-j} - 1$, $0 \le j \le f-1$. Here $O(p^{N-\ell_{f-1-j}-t}) =: O(p^{u_j - t})$ is a specific but inexplicit element of $p^{N-\ell_{f-1-j}-t} M_2(R)$ and

$$\mathfrak{a} \equiv -\langle (ws^{-1}(\mu) - \nu)_j, \alpha_j^\vee \rangle \pmod{p}. \tag{3.9}$$

For $0 \le k \le \ell_{f-1-j} - 2$, we label the entry (s,t) of the kth monodromy as $A(k,s,t)$. Then we have the following.

$$\begin{aligned}
A(k,1,1) =& k!\Big\{\sum_{i+j=k+1} -p(ia_id_j - jb_jc_i) + \sum_{i+j=k}\left[(i-\mathfrak{a})a_id_j + 2pjb_jc_i\right] + \sum_{i+j=k-1} jb_jc_i\Big\} + O(p^{u_j-k});\\
A(k,1,2) =& k!\Big\{\sum_{i+j=k}(\mathfrak{a}+j-i)a_ib_j + p\sum_{i+j=k+1}(i-j)a_ib_j\Big\} + O(p^{u_j-k});\\
A(k,2,1) =& k!\Big\{\sum_{i+j=k+1} p^2(i-j)c_id_j + \sum_{i+j=k} p(\mathfrak{a}+2j-2i+1)c_id_j + \sum_{i+j=k-1} -(\mathfrak{a}-i+j-1)c_id_j\Big\}\\
&+ O(p^{u_j-k});\\
A(k,2,2) =& k!\Big\{\sum_{i+j=k+1} -p(pic_ib_j + ja_id_j) + \sum_{i+j=k}\left[ja_id_j - p(\mathfrak{a}-2i-1)c_ib_j\right]\\
&+ \sum_{i+j=k-1}(\mathfrak{a}-i-1)c_ib_j\Big\} + O(p^{u_j-k}).
\end{aligned}$$

Let $M(-1,s,t) = 0$ and $\widetilde{A}(k,i,j) = A(k,i,j) - O(p^{u_j-k})$. Define $M_k(s,t)$ for $0 \le k \le \ell_{f-1-j} - 2$, $1 \le s,t \le 2$ recursively as follows:

$$M_k(1,1) = \left(\frac{\widetilde{A}(k,1,1)}{k!} + \mathfrak{a}H_k + M(k-1,1,1)\right)/p;\, M_k(1,2) = \frac{\widetilde{A}(k,1,2)}{k!};$$
$$M_k(2,1) = \left(\frac{\widetilde{A}(k,2,1))}{k!} + M(k-1,2,1)\right)/p;\, M_k(2,2) = \left(\frac{\widetilde{A}(k,2,2)}{k!} + M(k-1,2,2)\right)/p.$$

Then for $0 \le k \le \ell_{f-1-j} - 2$, we have

$$\begin{aligned}
M_k(1,1) &= \sum_{i+j=k}(\mathfrak{a}+j)c_ib_j - \sum_{i+j=k+1} ia_id_j + jpc_ib_j;\\
M_k(1,2) &= \sum_{i+j=k}(\mathfrak{a}+j-i)a_ib_j + p\sum_{i+j=k+1}(i-j)a_ib_j;\\
M_k(2,1) &= \sum_{i+j=k}(\mathfrak{a}-i+j-1)c_id_j + p\sum_{i+j=k+1}(i-j)c_id_j;\\
M_k(2,2) &= \sum_{i+j=k}(i+1-\mathfrak{a})b_jc_i - \sum_{i+j=k+1} ja_id_j + ipc_ib_j.
\end{aligned} \tag{3.10}$$

Definition 3.12. Let $R = \widehat{\otimes}_j R^{(j)}$ where $R^{(j)}$ is defined in Table 3.1 and Table 3.2. Let $I^{(j),\le(\ell_{f-1-j},0)}$ be the ideal of R generated by the equations given by the height conditions

$H(k)$, $0 \le k \le \ell_{f-1-j} - 1$. And we let $R^{\le(\ell_{f-1-j},0)_j,\tau}_{\mathfrak{M},\overline{\beta}}$ be the maximal reduced p-flat quotient of $\widehat{\otimes} R^{(j)} / I^{(j),\le(\ell_{f-1-j},0)}$.

Let $I^{(j),\nabla}$ be the ideal generated by the monodromy condition $A(k,s,t)$ for $0 \le k \le \ell_{f-1-j} - 2$, $1 \le s,t \le 2$. Let $R^{\le(\ell_{f-1-j},0)_j,\tau,\nabla}_{\mathfrak{M},\overline{\beta}}$ be the maximal, reduced $\mathcal{O}$-flat quotient of the ring $R / \sum_j (I^{(j),\le(\ell_{f-1-j},0)} + I^{(j),\nabla})$.

We further define $R^{\le(\ell_j,0)_j,\tau,\mathrm{reg}}_{\overline{\rho}}$ as the quotient of $R^{\le(\ell_j,0)_j,\tau}_{\overline{\rho}}$ such that each component is of maximal dimension, i.e., $R^{\le(\ell_j,0)_j,\tau,\mathrm{reg}}_{\overline{\rho}} = R^{\le(\ell,0)_j,\tau}_{\overline{\rho}} / (\cap_i \mathfrak{p}_i)$, where the intersection is over $\mathfrak{p}_i$, such that $R^{\le(\ell_j,0)_j,\tau}_{\overline{\rho}} / \mathfrak{p}_i$ is of maximal dimension. We define $R^{\le(\ell_j,0)_j,\tau,\nabla,\mathrm{reg}}_{\mathfrak{M},\overline{\beta}}$ as the quotient such that every component is of the same maximal dimension analogously.

By [BBH+24, Corollary 1.8], $R^{\le(\ell_j,0)_j,\tau,\mathrm{reg}}_{\overline{\rho}}$ corresponds to the quotient of $R^{\le(\ell_j,0)_j,\tau}_{\overline{\rho}}$ consisting of only regular Hodge–Tate weights, since the component with irregular Hodge–Tate weights has a positive codimension (*cf.* Lemma 3.22).

As in [LLHLM18, 5], we have an isomorphism

$$R^{\le(\ell_j,0)_j,\tau,\mathrm{reg}}_{\overline{\rho}}[\![X_1,\dots,X_{2f}]\!] \cong R^{\le(\ell_j,0)_j,\tau,\nabla,\mathrm{reg}}_{\mathfrak{M},\overline{\beta}}[\![Y_1,\dots,Y_4]\!].$$

We will compute the generators of $I^{\mathrm{reg}}_\infty := \ker(R \twoheadrightarrow R^{\le(\ell_j,0)_j,\tau,\nabla,\mathrm{reg}}_{\mathfrak{M},\overline{\beta}})$ and show that $I^{\mathrm{reg}}_\infty = \sum_j I^{(j),\mathrm{reg}}$ where $I^{(j),\mathrm{reg}}$ is given in Table 3.1 and Table 3.2. In general, $I^{(j),\mathrm{reg}}$ is not an ideal of $R^{(j)}$, as $O(p^{u_j-k})$ is an element of $M_2(R)$ rather than $M_2(R^{(j)})$.

Since $M_k(2,2) + M_k(1,1) = -(k+1)H(k+1)$, $(I^{(j),\le(\ell_{f-1-j},0)} + I^{(j),\nabla}, p^{N-2\ell_{f-1-j}+1})$ is generated by $H(0)$ and $M_k(s,t)$ for $0 \le k \le \ell_{f-1-j} - 2$, $1 \le s,t \le 2$. We will find the solutions to equations arising from $H(0)$ and $M_k(s,t)$ for $0 \le k \le \ell_{f-1-j} - 2$, $1 \le s,t \le 2$. If ρ is of Hodge–Tate weight (m,n), then $\rho \otimes \epsilon^k$ is of Hodge–Tate weight $(m+k, n+k)$. On the representation side, this corresponds to twisting $\sigma(\tau)$ by $(N_{k/\mathbb{F}_p} \circ \det)^k$. Assume $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$ (respectively $\mathfrak{w}\mathfrak{t}_{(m,n)}$), we let $\widetilde{w}_{f-1-j} + (k,k) = \mathfrak{t}_{(m+k,n+k)}$ (respectively $\mathfrak{w}\mathfrak{t}_{(m+k,n+k)}$). Hence, $\tau_{\widetilde{w}} \otimes \epsilon^k = \tau_{\widetilde{w}+(k,k)}$. Moreover, we have

$$R^{(m,n),\tau_{\widetilde{w}}}_{\overline{\rho}} \hookrightarrow R^{(m+k,n+k)\tau_{\widetilde{w}+(k,k)}}_{\overline{\rho}\otimes\omega^k}.$$

Therefore, in order to compute the monodromy conditions arising from the Galois deformation space $R^{\le(\ell_j,0)_j,\tau_{\widetilde{w}}}$, we can instead consider the monodromy conditions from $R^{\le(\ell_j+2k,0)_j,\tau_{\widetilde{w}+(k,k)}}$, which has more variables. If we relabel the solutions to the height and monodromy equations for $\widetilde{w}_{f-1-j}$ as $a_k = \boldsymbol{a}_{-m+k}$, $b_k = \boldsymbol{b}_{-n+1+k}$, $c_k = \boldsymbol{c}_{-m+1+k}$, $d_k = \boldsymbol{d}_{-n+k}$, we expect them to be the same as the solutions $\{\boldsymbol{a}'_k, \boldsymbol{b}'_k, \boldsymbol{c}'_k, \boldsymbol{d}'_k\}$ (relabelled analogously) for $\widetilde{w}_{f-1-j} + (k,k)$ when both are well-defined. (We will use the superscript $*$ to indicate that it is a unit.) We will show this is the case below. Moreover, by cancelling the extra variables introduced, we can compute the Galois deformation space of higher Hodge–Tate weights.

Assume $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$ **with** $m \geq n$ **or** $\mathfrak{wt}_{(m,n)}$ **with** $m > n$. Denote $M_k^a(s,t)$ the monodromy equations for $\rho \otimes \epsilon^a$. By induction, we can relate $M^{m-n}_{3m-n-1-k}(s,t)$ with $M^0_{m+n-1-k}(s,t)$ as follows. We use the notation $i+j =^* (k-1)/k$ to denote $i+j = k-1$ if $\widetilde{w} = \mathfrak{t}_{(m,n)}$ and $i+j=k$ if $\widetilde{w} = \mathfrak{wt}_{(m,n)}$. Similarly, we use $j \geq^* n/(n+1)$ to denote $j \geq n$ if $\widetilde{w} = \mathfrak{t}_{(m,n)}$ and $j \geq n+1$ if $\widetilde{w} = \mathfrak{wt}_{(m,n)}$.

$$\begin{aligned}
M^{m-n}_{3m-n-1-k}(1,1) &= M^0_{m+n-1-k}(1,1) + \sum_{\substack{i+j=^*k-1/k,\\ i\geq m \text{ or}\\ j\geq^* n/n+1}} (\mathfrak{a}+m-1-j)\boldsymbol{c}_{-i}\boldsymbol{b}_{-j} \\
&- \sum_{\substack{i+j=^*k/k-1,\\ i\geq^* m+1/m \text{ or}\\ j\geq n+1}} (2m-n-i)\boldsymbol{a}_{-i}\boldsymbol{d}_{-j} + \sum_{\substack{i+j=^*k-2/k-1,\\ i\geq m \text{ or}\\ j\geq^* n/n+1}} (m-1-j)p\boldsymbol{c}_{-i}\boldsymbol{b}_{-j}, \\
M^{m-n}_{3m-n-1-k}(1,2) &= M^0_{m+n-1-k}(1,2) + \\
&\sum_{\substack{i+j=k,\\ i\geq^* m+1/m \text{ or}\\ j\geq^* n/n+1}} (\mathfrak{a}+n-m-1-j+i)\boldsymbol{a}_{-i}\boldsymbol{b}_{-j} - \sum_{\substack{i+j=k-1,\\ i\geq^* m+1/m \text{ or}\\ j\geq^* n/n+1}} p(m-n+1-i+j)\boldsymbol{a}_{-i}\boldsymbol{b}_{-j}, \\
M^{m-n}_{3m-n-1-k}(2,1) &= M^0_{m+n-1-k}(2,1) + \\
&\sum_{\substack{i+j=k,\\ i\geq m \text{ or}\\ j\geq n+1}} (\mathfrak{a}-m+n+i-j)\boldsymbol{c}_{-i}\boldsymbol{d}_{-j} + \sum_{\substack{i+j=k-1,\\ i\geq m \text{ or}\\ j\geq n+1}} p(n-m-1-i+j)\boldsymbol{c}_{-i}\boldsymbol{d}_{-j}, \\
M^{m-n}_{3m-n-1-k}(2,2) &= M^0_{m+n-1-k}(2,2) + \\
&\sum_{\substack{i+j=^*k-1/k,\\ i\geq m \text{ or}\\ j\geq^* n/n+1}} (m-i-\mathfrak{a})\boldsymbol{c}_{-i}\boldsymbol{b}_{-j} - \sum_{\substack{i+j=^*k/k-1,\\ i\geq^* m+1/m \text{ or}\\ j\geq n+1}} (m-j)\boldsymbol{a}_{-i}\boldsymbol{d}_{-j} + \sum_{\substack{i+j=^*k-2/k-1,\\ i\geq m \text{ or}\\ j\geq^* n/n+1}} (m-1-i)p\boldsymbol{c}_{-i}\boldsymbol{b}_{-j}.
\end{aligned} \tag{3.11}$$

The inequalities for i and j come from the fact that they are introduced as extra variables. Let $I^{(j),\text{extra}}$ be the ideal of $R^{(j)}$ generated by the extra terms on the right-hand side of (3.11). If we can find $\boldsymbol{a}_{-k}, \boldsymbol{b}_{-k}, \boldsymbol{c}_{-k}, \boldsymbol{d}_{-k}$ for $0 \leq k \leq m+n-1$ and $1 \leq i,j \leq 2$ from $M^{m-n}_{3m-n-1-k}(s,t)$ where $0 \leq k \leq m+n-1, \leq s,t \leq 2$, then they are also a solution to $M^0_{m+n-1-k}(s,t)$ for $0 \leq k \leq m+n-1, \leq s,t \leq 2$ up to modulo by the ideal $I^{(j),\text{extra}}$. As $j \geq 0$ and $i+j = k, k-1$ or $k-2$, we must have $i \leq k$.

Assume $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$, note that $\overline{\boldsymbol{a}_0^*} = \alpha_j$, $\overline{\boldsymbol{d}_0^*} = \beta_j$ and $\overline{\boldsymbol{b}_0} = \gamma_{f-1-j}\alpha_j$ when $n \geq 1$. In particular, if $\gamma_{f-1-j} \neq 0$, then $\boldsymbol{b}_0 \neq 0$. Also, we have $\ell_{f-1-j} = m+n$.

Theorem 3.13. *Assume* $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$,

$$(I^{(j),\leq(\ell_{f-1-j},0)} + I^{(j),\nabla}, p^{N-2\ell_{f-1-j}+1}) = (I^{(j)}_{poly}, p^{N-2\ell_{f-1-j}+1}),$$

where $I^{(j)}_{poly}$ *is given by row 5 of Table 3.1 without the* $O(p^{k_j})$ *term.*

Proof. We first deal with the case where with $m \geq n$. The value of $\mathfrak{a}$ follows from Equation (3.9) and Equation (3.4). As $p > 2l$, $\mathfrak{a} \pm k \not\equiv 0 \mod p$, so $\mathfrak{a} \pm k$ is a unit for all $0 \leq k \leq \ell_{f-1-j}$. Moreover, the monodromy equations are given by $M^k(j,s,t)$ up to modulo $p^{N-2\ell_{f-1-j}+1}$. We first prove that the solution to $M^{m-n}_{3m-n-1-k}(s,t)$ for $0 \leq k \leq m+n-1$, $\leq s,t \leq 2$ is $\boldsymbol{a}_{-k}, \boldsymbol{b}_{-k}, \boldsymbol{c}_{-k}, \boldsymbol{d}_{-k}$ as given by the $I^{(j),\mathrm{reg}}$ row without the $O(p^{k_j})$ tail in Table 3.1, up to modulo $I^{(j),\mathrm{extra}}$. For $i = 0$, it is just the definition. We proceed by induction that given we have verified $\boldsymbol{a}_{-k}$, $\boldsymbol{d}_{-k}$, $\boldsymbol{b}_{-k}$ and $\boldsymbol{c}_{-k}$ for $k \leq i-1$, we can deduce $\boldsymbol{a}_{-i}$ and $\boldsymbol{d}_{-i}$ from $M^{m-n}_{3m-n-1-i}(1,1)$ and $M^{m-n}_{3m-n-1-i}(2,2)$. From the combinations of the indices, we deduce that $\boldsymbol{a}_{-i}$ and $\boldsymbol{d}_{-i}$ are the only indeterminate. From $M^{m-n}_{3m-n-1-i}(1,1)$, we have

$$\begin{aligned}&(2m-n-i)\boldsymbol{a}_{-i}\boldsymbol{d}_0^* + (2m-n)\boldsymbol{a}_0^*\boldsymbol{d}_{-i}\\ &= \sum_{1\leq j\leq i}(\mathfrak{a}+m-j)\boldsymbol{c}_{-i+j}\boldsymbol{b}_{-j+1} - \sum_{1\leq j\leq i-1}\left[(2m-n-i+j)\boldsymbol{a}_{-i+j}\boldsymbol{d}_{-j} + (m-j)p\boldsymbol{c}_{-i+j+1}\boldsymbol{b}_{-j+1}\right].\end{aligned}$$

And from $M^{m-n}_{3m-n-i}(2,2)$, we have

$$\begin{aligned}m\boldsymbol{a}_{-i}\boldsymbol{d}_0^* + (m-i)\boldsymbol{a}_0^*\boldsymbol{d}_{-i} = &\sum_{1\leq j\leq i}(2m-n-i+j-\mathfrak{a})\boldsymbol{c}_{-i+j}\boldsymbol{b}_{-j+1}\\ &- \sum_{1\leq j\leq i-1}\left[(m-j)\boldsymbol{a}_{-i+j}\boldsymbol{d}_{-j} + (2m-n-i+j)p\boldsymbol{c}_{-i+j+1}\boldsymbol{b}_{-j+1}\right].\end{aligned}$$

Hence,

$$\boldsymbol{a}_{-i} = \frac{-1}{i\boldsymbol{d}_0^*}\left[\sum_{1\leq j\leq i}(\mathfrak{a}-m+n-j)\boldsymbol{c}_{-i+j}\boldsymbol{b}_{1-j} + \sum_{1\leq j\leq i-1}(m-n+j)p\boldsymbol{c}_{1-i+j}\boldsymbol{b}_{1-j} + (i-j)\boldsymbol{a}_{-i+j}\boldsymbol{d}_{-j}\right], \tag{3.12}$$

$$\boldsymbol{d}_{-i} = \frac{1}{i\boldsymbol{a}_0^*}\left[\sum_{1\leq j\leq i}(\mathfrak{a}+i-j-m+n)\boldsymbol{c}_{-i+j}\boldsymbol{b}_{1-j} - \sum_{1\leq j\leq i-1}(i-j-m+n)p\boldsymbol{c}_{1-i+j}\boldsymbol{b}_{1-j} + j\boldsymbol{a}_{-i+j}\boldsymbol{d}_{-j}\right].$$

By a simple calculation, we can show the following:

Lemma 3.14. *Assume* $\boldsymbol{a}_{-i+j}, \boldsymbol{b}_{1-j}, \boldsymbol{c}_{1-i+j}, \boldsymbol{d}_{-j}$ *for* $0 < i, j, i-j$ *are as given in the* $I^{(j),\mathrm{reg}}$ *row without the* $O(p^{k_j})$ *tail in Table 3.1, then*

$$i\boldsymbol{a}_{-i}\boldsymbol{d}_0^* = -(\mathfrak{a}-m+n-1)\boldsymbol{c}_{1-i}\boldsymbol{b}_0,$$

$$i\boldsymbol{a}_0^*\boldsymbol{d}_{-i} = (\mathfrak{a}-m+n)\boldsymbol{c}_0\boldsymbol{b}_{1-i},$$

$$\boldsymbol{a}_{-i+j}\boldsymbol{d}_{-j} = \frac{-(\mathfrak{a}-m+n)(\mathfrak{a}-m+n-1)\boldsymbol{b}_0\boldsymbol{c}_0\boldsymbol{b}_{1-j}\boldsymbol{c}_{1-i+j}}{\boldsymbol{a}_0^*\boldsymbol{d}_0^*(i-j)j}.$$

Therefore, the right-hand side of Equation (3.12) is

$$\frac{-(\mathfrak{a}-m+n-1)\boldsymbol{c}_{1-i}\boldsymbol{b}_0}{i\boldsymbol{d}_0^*} + \sum_{1\le j\le i-1}\left[(\mathfrak{a}-m+n-1-j)\boldsymbol{c}_{-i+j+1}\boldsymbol{b}_{-j} + (i-j)\boldsymbol{a}_{-i+j}\boldsymbol{d}_{-j} + (m-n+j)p\boldsymbol{c}_{1-i+j}\boldsymbol{b}_{1-j}\right].$$

From the expression for $\boldsymbol{b}_{-j}$ and Lemma 3.14, we have the terms in the summand cancelling each other out, and $\boldsymbol{a}_{-i}$ is indeed as in the $I^{(j),\mathrm{reg}}$ row without the $O(p^{k_j})$ tail in Table 3.1. The proof for $\boldsymbol{d}_{-i}$ is analogous.

We now show that given the solutions $\boldsymbol{a}_{-k}$, $\boldsymbol{d}_{-k}$ for $k \le i$ and $\boldsymbol{b}_{-k}$, $\boldsymbol{c}_{-k}$ for $k \le i-1$, we can deduce $\boldsymbol{b}_{-i}$ and $\boldsymbol{c}_{-i}$ from $M^{m-n}_{3m-n-1-i}(1,2)$ and $M^{m-n}_{3m-n-1-i}(2,1)$, respectively.

$$\boldsymbol{b}_{-i} = \frac{-1}{(\mathfrak{a}-m+n-1-i)\boldsymbol{a}_0^*} \left\{\sum_{0\le j\le i-1}\left[(\mathfrak{a}-m+n-1+i-2j)\boldsymbol{a}_{-i+j}\boldsymbol{b}_{-j} + p(m-n+2-i+2j)\boldsymbol{a}_{-i+1+j}\boldsymbol{b}_{-j}\right]\right\}; \tag{3.13}$$

$$\boldsymbol{c}_{-i} = \frac{-1}{(\mathfrak{a}-m+n+i)\boldsymbol{d}_0^*} \left\{\sum_{0\le j\le i-1}\left[(\mathfrak{a}-m+n-i+2j)\boldsymbol{c}_{-j}\boldsymbol{d}_{-i+j} + p(m-n-2+i-2j)\boldsymbol{c}_{-j}\boldsymbol{d}_{1-i+j}\right]\right\}.$$

Lemma 3.15. *Assume* $\boldsymbol{a}_{-i+j}, \boldsymbol{b}_{-j}, \boldsymbol{c}_{-j}, \boldsymbol{d}_{-i+j}$ *for* $0 \le j \le i$ *are as given in the* $I^{(j),\mathrm{reg}}$ *row without the* $O(p^{k_j})$ *tail in Table 3.1. Let*

$$T_j = (\mathfrak{a}-m+n-1+i-2j)\boldsymbol{a}_{-i+j}\boldsymbol{b}_{-j} + p(m-n+2-i+2j)\boldsymbol{a}_{+1-i+j}\boldsymbol{b}_{-j},$$

$$R_j = \frac{-j}{i}(\mathfrak{a}-m+n-1-j)\boldsymbol{a}_{-i+j}\boldsymbol{b}_{-j}.$$

Then we have for $0 \le j \le i-1$, $T_j + R_j = R_{j+1}$. *Similarly, let*

$$T'_j = (\mathfrak{a} - m + n - i + 2j)\boldsymbol{c}_{-j}\boldsymbol{d}_{-i+j} + p(m - n - 2 + i - 2j)\boldsymbol{c}_{-j}\boldsymbol{d}_{1-i+j},$$

$$R'_j = \frac{-j}{i}(\mathfrak{a} - m + n + j)\boldsymbol{c}_{-j}\boldsymbol{d}_{-i+j}.$$

Then we have that $T'_j + R'_j = R'_{j+1}$ *for* $0 \le j \le i-1$.

By Lemma 3.15 and the fact that $R_0 = 0$, the right-hand side of Equation (3.13) is

$$\frac{-1}{(\mathfrak{a} - m + n - 1 - i)\boldsymbol{a}_0^*} \sum_{0 \le j \le i} T_j = \frac{-1}{(\mathfrak{a} - m + n - 1 - i)\boldsymbol{a}_0^*} R_i,$$

which is precisely the conjectured $\boldsymbol{b}_{-i}$ in the $I^{(j),\mathrm{reg}}$ row without the $O(p^{k_j})$ tail in Table 3.1, again by Lemma 3.15. The proof for $\boldsymbol{c}_{1-n-i}$ goes exactly the same way, using T'_j, R'_j instead of T_j, R_j. Therefore, we finish the induction step and prove that the conjectured solution to $M^{m-n}_{3m-n-1-k}(1,1)$ for $0 \le k \le m+n-1$ for all i, j are $\{\boldsymbol{a}_{-k}, \boldsymbol{b}_{-k}, \boldsymbol{c}_{-k}, \boldsymbol{d}_{-k}\}_{0 \le k \le m+n-1}$.

Now by (3.11), we know that the solutions, which is given by the $I^{(j),\mathrm{reg}}$ row without the $O(p^{k_j})$ tail in Table 3.1, are also solutions to the monodromy equations $M_k(i,j)$ modulo I^{extra}. We claim that the term with $*$ when $m > n$ (resp. terms with † when $m = n$) in Table 3.1 without the $O(p^{k_j})$ tail, generates I^{extra}. On the one hand, by (3.11)

$$M^{m-n}_{3m-2n-1}(1,2) - (\mathfrak{a} - m - 1)\boldsymbol{a}_0^*\boldsymbol{b}_{-n} = M^0_{m-1}(1,2),$$

and $\boldsymbol{a}_0^*$ is a unit, from the formula for $\boldsymbol{b}_{-n}$ we deduce that the term with $*$ when $m > n$ (resp. first term with † when $m = n$), without the $O(p^{k_j})$ tail, in Table 3.1 is contained in I^{extra}. Furthermore, for $m = n$, by (3.11), $M^{m-n}_{3m-2n-1}(2,1) - (\mathfrak{a} + m)\boldsymbol{d}_0^*\boldsymbol{c}_{-n} = M^0_{m-1}(1,2)$, we deduce that the terms with † without the $O(p^{k_j})$ tail is contained in I^{extra} in this case. On the other hand, all the terms that generate I^{extra} are divisible by $\boldsymbol{a}_{-j}$ where $j \ge m+1$, $\boldsymbol{b}_{-j}$ where $j \ge n$, $\boldsymbol{c}_{-j}$ where $j \ge m$ or $\boldsymbol{d}_{-j}$ where $j \ge n+1$. These are all computed to be according to the $I^{(j),\mathrm{reg}}$ row without the $O(p^{k_j})$ tail in Table 3.1, hence they are all divisible by

$$\prod_{m \ge j \ge 1} \left(\frac{(\mathfrak{a} - m + n)(\mathfrak{a} - m + n - 1)\boldsymbol{b}_0\boldsymbol{c}_0}{\boldsymbol{a}_0^*\boldsymbol{d}_0^*} - (m - n - j)jp \right). \tag{3.14}$$

Furthermore, all $\boldsymbol{a}_{-j}, \boldsymbol{d}_{-j}$ with $j > 0$ and all $\boldsymbol{b}_{-j}$ are divisible by $\boldsymbol{b}_0$; and all $\boldsymbol{a}_{-j}, \boldsymbol{d}_{-j}$ with $j > 0$, and all $\boldsymbol{c}_{-j}$ are divisible by $\boldsymbol{c}_0$. Therefore, all the generators of I^{extra} are divisible by the term with $*$ when $m > n$ (resp. terms with † when $m = n$) in Table 3.1 without the $O(p^{k_j})$ tail. Therefore, $(I^{(j),\nabla}, p^{N-2\ell_{f-1-j}+1})$ is generated by the terms in $I^{(j),\mathrm{reg}}$ in Table 3.1 without the $O(p^{k_j})$ tail, except the term with $*$ replaced by two terms with † if $m = n$.

Since $-(k+1)H(k+1) = M_k(2,2) + M_k(1,1)$ for $0 \le k \le m+n-1$. To finish the proof, we substitute the conjectured solution in the equation $H(0) = \boldsymbol{a}_{-m}\boldsymbol{d}_{-n} + p\boldsymbol{b}_{-n+1}\boldsymbol{c}_{-m+1}$, and a direct calculation shows that it is divisible by the term with $*$ when $m > n$ (resp. terms with $\dagger$ when $m = n$), without the $O(p^{k_j})$ tail, in Table 3.1. □

Lemma 3.16. *Theorem 3.13 holds also for $m < n$.*

Proof. We will show reduce it to the case in Theorem 3.13 where we swap a_j with d_j, c_j with $-b_j$, and $\mathfrak{a}$ with $-\mathfrak{a}+1$. The value of $\mathfrak{a}$ follows from Equation (3.9) and Equation (3.4). Assume $m < n$, let A be the $A^{(f-1-j)}$ for $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$ and A be the $A^{(f-1-j)}$ for $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(n,m)}$. Also, let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then

$$\mathrm{inv}(A) := \begin{pmatrix} 0 & -\frac{1}{v} \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ -v & 0 \end{pmatrix} = \begin{pmatrix} \sum_{0\le i\le m} \boldsymbol{d}_{-i}^{(j)}(v+p)^i & \sum_{0\le i\le n-1} -\boldsymbol{c}_{-i}^{(j)}(v+p)^i \\ v(\sum_{0\le i\le m-1} -\boldsymbol{b}_{-i}^{(j)}(v+p)^i) & \sum_{0\le i\le n} \boldsymbol{a}_{-i}^{(j)}(v+p)^i \end{pmatrix}.$$

The monodromy equation is given by

$$\left(\frac{d}{dv}\right)^t\bigg|_{v=-p} \left\{\left[v\frac{d}{dv}A - A\begin{pmatrix} \mathfrak{a} & 0 \\ 0 & 0 \end{pmatrix}\right](A)^{\mathrm{adj}}\right\} + O(p^{N-\ell_{f-1-j}-t}) \tag{3.15}$$

for all $0 \le t \le 1$, $0 \le j \le f-1$, where adj stands for adjugate.
We apply inv to Equation (3.15), after simplification, we have the following

$$\left(\frac{d}{dv}\right)^t\bigg|_{v=-p} \left\{\left[v\frac{d}{dv}\mathrm{inv}(A) - \begin{pmatrix} 0 & \frac{1}{v}\gamma \\ -v\beta & 0 \end{pmatrix} - \mathrm{inv}(A)\begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{a} \end{pmatrix}\right](\mathrm{inv}(A))^{\mathrm{adj}}\right\} + O(p^{N-\ell_{f-1-j}-t}).$$

Since $\mathrm{inv}(A)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathrm{inv}(A) = \begin{pmatrix} 0 & \frac{1}{v}\gamma \\ -v\beta & 0 \end{pmatrix}$, The leading term up to modulo $(v+p)^\ell$ is equivalent to

$$\left(\frac{d}{dv}\right)^t\bigg|_{v=-p} \left\{\left[v\frac{d}{dv}\mathrm{inv}(A) - \mathrm{inv}(A)\begin{pmatrix} 1-\mathfrak{a} & 0 \\ 0 & 0 \end{pmatrix}\right]\mathrm{inv}(A)^{\mathrm{adj}}\right\}.$$

Therefore, we can apply Theorem 3.13 to $\mathrm{inv}(A)$. □

Now assume $\widetilde{w}_{f-1-j} = \mathfrak{w}\mathfrak{t}_{(m,n)}$. Similar to the case where $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$, note that $\overline{\boldsymbol{b}_0^*} = \alpha$, $\overline{\boldsymbol{c}_0^*}$ and $\overline{\boldsymbol{d}_0} = \gamma$. In particular, if $\gamma_{f-1-j} \neq 0$, then $\boldsymbol{d}_0 \neq 0$.

Theorem 3.17. *Assume $\widetilde{w}_{f-1-j} = \mathfrak{w}\mathfrak{t}_{(m,n)}$,*

$$(I^{(j),\le(\ell_{f-1-j},0)} + I^{(j),\nabla}, p^{N-2\ell_{f-1-j}+1}) = (I^{(j)}_{poly}, p^{N-2\ell_{f-1-j}+1}),$$

where $I^{(j)}_{poly}$ *is given by row 5 of Table 3.2 without the* p^{k_j} *tail.*

Proof. We first deal with the case where $m \geq n$. The value of $\mathfrak{a}$ follows from Equation (3.9) and Equation (3.4). As $p > 2\ell_{f-1-j}$, we have $\mathfrak{a} \pm k \not\equiv 0 \mod p$, so $\mathfrak{a} \pm k$ is a unit for all $0 \leq k \leq \ell_{f-1-j}$. We first prove that the solution to $M^{m-n}_{3m-n-1-k}(s,t)$ for $0 \leq k \leq m+n-1$, $\leq s,t \leq 2$ is $\boldsymbol{a}_{-k}, \boldsymbol{b}_{-k}, \boldsymbol{c}_{-k}, \boldsymbol{d}_{-k}$ as conjectured by the equations in Table 3.2 without the $O(p^{k_j})$ term. For $k=0$, it is just the definition. We then proceed as in the case where $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$. We will prove by induction. If we have verified $\boldsymbol{a}_{-k}$, $\boldsymbol{d}_{-k}$, $\boldsymbol{b}_{-k}$ and $\boldsymbol{c}_{-k}$ for $k \leq i-1$, we will deduce $\boldsymbol{b}_{-i}$ and $\boldsymbol{c}_{-i}$ from $M^{m-n}_{3m-n-1-i}(1,1)$ and $M^{m-n}_{3m-n-1-i}(2,2)$. From $M^{m-n}_{3m-n-i}(1,1)$, we have

$$\begin{aligned}&(\mathfrak{a}+m-i)\boldsymbol{b}_{-i}\boldsymbol{c}_0^* + (\mathfrak{a}+m)\boldsymbol{b}_0^*\boldsymbol{c}_{-i}\\ &= -\sum_{1\leq j\leq i-1}(\mathfrak{a}+m-j)\boldsymbol{c}_{-i+j}\boldsymbol{b}_{-j} + \sum_{0\leq j\leq i-1}(2m-n-i+j)\boldsymbol{a}_{1-i+j}\boldsymbol{d}_{-j} + (m-j)p\boldsymbol{c}_{-i+j}\boldsymbol{b}_{1-j}.\end{aligned}$$

And from $M^{m-n}_{3m-n-i}(2,2)$, we have

$$\begin{aligned}(2m-n-\mathfrak{a})\boldsymbol{b}_{-i}\boldsymbol{c}_0^* + (2m-n-\mathfrak{a}-i)\boldsymbol{b}_0^*\boldsymbol{c}_{-i} &= -\sum_{1\leq j\leq i-1}(2m-n-i+j+\mathfrak{a})\boldsymbol{c}_{-i+j}\boldsymbol{b}_{-j}\\ &+ \sum_{0\leq j\leq i-1}(m-j)\boldsymbol{a}_{1-i+j}\boldsymbol{d}_{-j} + (2m-n-i+j)p\boldsymbol{c}_{1-i+j}\boldsymbol{b}_{-j}.\end{aligned}$$

Hence,

$$\begin{aligned}\boldsymbol{b}_{-i} = \frac{-1}{i\boldsymbol{c}_0^*}\Bigg(&\sum_{0\leq j\leq i-2}(j+1)\boldsymbol{c}_{1-i+j}\boldsymbol{b}_{-1-j}\\ &- \sum_{0\leq j\leq i-1}\left[(\mathfrak{a}+i-j-m+n)\boldsymbol{a}_{1-i+j}\boldsymbol{d}_{-j} + (\mathfrak{a}+j)p\boldsymbol{c}_{1-i+j}\boldsymbol{b}_{-j}\right]\Bigg).\end{aligned} \tag{3.16}$$

$$\boldsymbol{c}_{-i} = \frac{-1}{i\boldsymbol{b}_0^*}\left(\sum_{1\leq j\leq i-1}(i-j)\boldsymbol{c}_{-i+j}\boldsymbol{b}_{-j} + \sum_{0\leq j\leq i-1}\left[(\mathfrak{a}-j-m+n)\boldsymbol{a}_{1-i+j}\boldsymbol{d}_{-j} + (\mathfrak{a}-i+j)p\boldsymbol{c}_{1-i+j}\boldsymbol{b}_{-j}\right]\right).$$

Lemma 3.18. *Assume* $\boldsymbol{a}_{1-i+j}, b_{m-j}, \boldsymbol{c}_{1-i+j}, \boldsymbol{d}_{-j}$ *are as given in the* $I^{(j),\mathrm{reg}}$ *row without the* $O(p^{k_j})$ *tail in Table 3.2, then for* $j, i-j-1 \geq 0$*, we have equalities:*

$$\begin{aligned}(j+1)\boldsymbol{b}_{-j-1}\boldsymbol{c}_{1-i+j} &= p(\mathfrak{a}+j)\boldsymbol{b}_{-j}\boldsymbol{c}_{1-i+j} + (\mathfrak{a}-m+n+i-j)\boldsymbol{a}_{1-i+j}\boldsymbol{d}_{-j};\\ -(i-j)\boldsymbol{b}_{-j}\boldsymbol{c}_{-i+j} &= p(\mathfrak{a}-i+j)\boldsymbol{b}_{-j}\boldsymbol{c}_{-i+j+1} + (\mathfrak{a}-m+n-j)\boldsymbol{a}_{-i+j+1}\boldsymbol{d}_{-j}\end{aligned}$$

By Lemma 3.18, the right-hand side of Equation (3.16) all cancel out except the term $(\mathfrak{a}+i-1)p\boldsymbol{c}_0^*b_{-i+1}+(\mathfrak{a}+1-m+n)\boldsymbol{a}_0\boldsymbol{d}_{-i+1}$, which is equal to $\boldsymbol{b}_{-i}$ again by Lemma 3.18. we can similarly verify that $\boldsymbol{c}_{-i}$ is as in the $I^{(j),\mathrm{reg}}$ row without the $O(p^{k_j})$ tail in Table 3.2.

We now show that given solutions $\boldsymbol{b}_{-k}$ and $\boldsymbol{c}_{-k}$ for $k \leq i$ and $\boldsymbol{a}_{-k}$ and $\boldsymbol{d}_{-k}$ for $k \leq i-1$, we can deduce $\boldsymbol{a}_{-i}$ and $\boldsymbol{d}_{-i}$ from $M^{m-n}_{3m-n-1-i}(1,2)$ and $M^{m-n}_{3m-n-1-i}(2,1)$ respectively that:

$$\begin{aligned}\boldsymbol{a}_{-i} =& \frac{-1}{(\mathfrak{a}-m+n+1+i)\boldsymbol{b}_0^*} \\ &\Big(\sum_{0\leq j\leq i-1}(\mathfrak{a}-m+n+1-i+2j)\boldsymbol{a}_{-j}\boldsymbol{b}_{-i+j}+p(m-n-2+i-2j)\boldsymbol{a}_{-j}\boldsymbol{b}_{1-i+j}\Big);\end{aligned} \tag{3.17}$$

$$\boldsymbol{d}_{-i} = \frac{-1}{(\mathfrak{a}-m+n-i)\boldsymbol{c}_0^*}\Big(\sum_{0\leq j\leq i-1}(\mathfrak{a}-m+n+i-2j)\boldsymbol{c}_{-i+j}\boldsymbol{d}_{-j}+p(m-n-i+2j)\boldsymbol{c}_{1-i+j}\boldsymbol{d}_{-j}\Big).$$

Lemma 3.19. *Assume $\boldsymbol{a}_{+j}, \boldsymbol{b}_{-i+j}, \boldsymbol{c}_{+j}, \boldsymbol{d}_{-i+j}$ for $0 \leq j \leq i$ are as given in the $I^{(j),\mathrm{reg}}$ row without the $O(p^{k_j})$ tail in Table 3.2. Let*

$$T_j = (\mathfrak{a}-m+n+1-i+2j)\boldsymbol{a}_{-j}\boldsymbol{b}_{-i+j}+p(m-n-2+i-2j)\boldsymbol{a}_{-1-j}\boldsymbol{b}_{+1-i+j},$$

$$R_j = \frac{-j}{i}(\mathfrak{a}-m+n+1+j)\boldsymbol{a}_{-j}\boldsymbol{b}_{-i+j}.$$

Then we have for $0 \leq j \leq i-1$, $T_j + R_j = R_{j+1}$. Similarly, let

$$T'_j = (\mathfrak{a}-m+n+i-2j)\boldsymbol{c}_{-i+j}\boldsymbol{d}_{-j}+p(m-n-i+2j)\boldsymbol{c}_{1-i+j}\boldsymbol{d}_{-j},$$

$$R'_j = \frac{-j}{i}(\mathfrak{a}-m+n-j)\boldsymbol{c}_{-i+j}\boldsymbol{d}_{-j}.$$

Then, for $0 \leq j \leq i-1$, $T'_j + R'_j = R'_j$.

By Lemma 3.19 and the fact that $R_0 = 0$, the right-hand side of Equation (3.17) is

$$\frac{-1}{(\mathfrak{a}-m+n-1-i)\boldsymbol{a}_0^*}\sum_{0\leq j\leq i}T_j = \frac{-R_i}{(\mathfrak{a}-m+n-1-i)\boldsymbol{a}_0^*},$$

which is precisely the conjectured $\boldsymbol{a}_{-i}$ in the $I^{(j),\mathrm{reg}}$ row in Table 3.2 without the $O(p^{k_j})$ tail, again by Lemma 3.19. The proof for $\boldsymbol{d}_{-i}$ goes exactly the same way, using T'_j, R'_j instead of T_j, R_j. Therefore, we finished the induction step and proved that the solution to $M^{m-n}_{3m-n-1-k}(s,t)$ for $0 \leq k \leq m+n-1$, $\leq s,t \leq 2$ is as conjectured in the $I^{(j),\mathrm{reg}}$ row in Table 3.2 for all $0 \leq k \leq m+n$ and all i, j.

As in the case of $\mathfrak{t}_{(m,n)}$, by (3.11), we know that the solutions, given in the $I^{(j),\mathrm{reg}}$ row in Table 3.2 without the $O(p^{k_j})$ tail, satisfy the monodromy equations up to modulo I^{extra}. By Equation (3.11), we have

$$M^{m-n}_{3m-2n-2}(1,1) = M^{0}_{m-2}(1,1) + (\mathfrak{a}+m-n-2)\boldsymbol{c}_0^*\boldsymbol{b}_{-n-1} + \delta_{m,n+1}(\mathfrak{a}+n)\boldsymbol{c}_{-n-1}\boldsymbol{b}_0^*;$$
$$M^{m-n}_{3m-2n-2}(2,2) = M^{0}_{m-2}(2,2) + (m-\mathfrak{a})\boldsymbol{c}_0^*\boldsymbol{b}_{-n-1} - \delta_{m,n+1}\mathfrak{a}\boldsymbol{c}_{-n-1}\boldsymbol{b}_0^*.$$

As $\boldsymbol{b}_0^*, \boldsymbol{c}_0^*$ are units, we deduce that $\boldsymbol{b}_{-n-1} \in I^{\mathrm{extra}}$, and hence the terms with $*$ in Table 3.2, without the term $O(p^{k_j})$, are in I^{extra}, following the description of $\boldsymbol{b}_{-n-1}$ according to Table 3.2. Conversely, by (3.11), all the generators of I^{extra} are divisible by $\boldsymbol{a}_{-j}$ or $\boldsymbol{c}_{-j}$ where $j \geq m$ or $\boldsymbol{b}_{-j}$ or $\boldsymbol{d}_{-j}$ where $j \geq n+1$. By the computation above, they in turn are divisible by terms with $*$ in Table 3.2 without the $O(p^{k_j})$ tail. We then verify that if we substitute the conjectured solution in $H(0)$, it is divisible by the terms with $*$ in Table 3.2 without the $O(p^{k_j})$ tail, which is straightforward. □

Lemma 3.20. *Theorem 3.17 holds for $m < n$.*

Proof. It goes exactly the same as the proof of Lemma 3.16 □

Definition 3.21. Let $R^{(j)}_{\mathrm{poly}}$ be the polynomial ring generated over $\mathcal{O}$ as the variables generating $R^{(j)}$ in the 4th row of Table 3.1 and Table 3.2. We define $R_{\mathrm{poly}} := \otimes_{\mathcal{O},j} R^{(j)}_{\mathrm{poly}}$. We let $I^{(j)}$ be defined by the elements in the row $I^{(j),\mathrm{reg}}$ in Table 3.1 and Table 3.2, where the term with $*$ is replaced by the terms with † if $\widetilde{w} = \mathfrak{t}_{(m,m)}$. We define $I^{(j)}_{\mathrm{poly}}$ as the ideal of $R^{(j)}_{\mathrm{poly}}$ generated in the same way but without the $O(p^{k_j})$ tail.

Lemma 3.22. *In the case where $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,m)}$, $R^{(j)}_{\mathrm{poly}}/(I^{(j)}_{\mathrm{poly}}, \boldsymbol{b}_0) = R^{(j)}_{\mathrm{poly}}/(I^{(j)}_{\mathrm{poly}}, \boldsymbol{c}_0) = \mathcal{O}[x_{11}, x_{22}]$.*

Proof. If $\boldsymbol{b}_0 = 0$, by row for $I^{(j)}$ of Table 3.1, $\boldsymbol{c}_0 = 0$. Moreover, by the $I^{(j),\mathrm{reg}}$ row without the $O(p^{k_j})$ tail in Table 3.1, $a_i, b_i, c_i, d_i = 0$ except for a_m, d_m. By symmetry, the same happens if $\boldsymbol{c}_0 = 0$. □

Definition 3.23. The ideal $I^{(j),\mathrm{reg}}$ is given in Table 3.1 and Table 3.2. We let $I^{(j),\mathrm{reg}}_{\mathrm{poly}}$ be the ideal of $R^{(j)}_{\mathrm{poly}}$ generated in the same way but without the tail $O(p^{k_j})$. We let $I^{\mathrm{reg}}_{\mathrm{poly}} := \sum_j I^{(j),\mathrm{reg}}_{\mathrm{poly}}$.

Corollary 3.24. *We have $p^{\ell_{f-1-j}} \in H^{(j)} + I^{(j),\mathrm{reg}}_{\mathrm{poly}}$.*

Proof. We use G to denote the term with $*$ in row for $I^{(j),\mathrm{reg}}$ of Table 3.1 and Table 3.2 without the term $O(p^{k_j})$. Note that $H^{(j)} + I^{(j)}_{\mathrm{poly}}$ contains G and the partial derivatives of G. If $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$ let $x = \frac{b_0}{a_0^*}$ and $y = \frac{d_0}{c_0^*}$. If $m > n$ and $\gamma_{f-1-j} = 0$, then $G(x,y) = x\prod_{i=1}^{n}(xy - \alpha_i)$. Otherwise, $G(x,y) = \prod_{i=1}^{n}(xy - \alpha_i)$. If $\widetilde{w}_{f-1-j} = \mathfrak{wt}_{(m,n)}$, we let $x = \frac{a_0}{b_0^*}$, $y = \frac{d_0}{c_0^*}$, then $G(x,y) = \prod_{i=1}^{n+1}(xy - \alpha_i)$. In any case, α_i are distinct and.

$$v_p(\alpha_i) = 1 \text{ for all } i \text{ and } v_p(\alpha_i - \alpha_j) = 1 \text{ for all } i \neq j. \tag{3.18}$$

(Here, v_p denotes the p-adic valuation, and we define $v_p(0) = \infty$.) We first consider the case where $G(x,y) = \prod_{i=1}^{n+1}(xy - \alpha_i)$ (respectively, $\prod_{i=1}^{n}(xy - \alpha_i)$). If we let $xy = z$, then G is a polynomial in z of degree $n+1$ (respectively n) and $x\frac{\partial G}{\partial x} = z\frac{\partial G}{\partial z}$. We will show below that p^{2n+1}(resp. p^{2n-1}) $\in (G(z), z\frac{\partial G}{\partial z})$ which will imply that $p^{\ell_{f-1-j}} \in (G, x\frac{\partial G}{\partial x})$.

Given $G = z^{n+1} + a_n z^n + \cdots + a_0$ with roots satisfying Equation (3.18). Let $p_{-1} = \frac{\partial G}{\partial z} = (n+1)z^n + n a_n z^{n-1} + \cdots + a_1$. For $-1 \leq i \leq n-1$, given $p_i = b_{i,n}z^n + \cdots + b_{i,0}$, we obtain $p_{i+1} = b_{i,n}G - zp_i$.

Lemma 3.25. $v_p(b_{i,k}) \geq i + 1 + n - k$.

Proof. We will prove this by inducting on i. For $1 \leq k \leq n$, $b_{-1,k} = (k+1)a_{k+1} = (k+1)(-1)^k \sum_{i_1 < \cdots < i_{n-k}} \alpha_{i_1} \ldots \alpha_{i_{n-k}}$, where the sum is over any k-tuple, and $b_{-1,0} = 0$, the lemma holds for $i = -1$. Assume that it is true for i, then

$$v_p(b_{i+1,0}) = v_p(b_{i,n}a_0) = v_p(b_{i,n}) + v_p(a_0) \geq i + n + 2.$$

For $k > 0$,

$$\begin{aligned} v_p(b_{i+1,k}) &= v_p(b_{i,n}a_k + b_{i,k-1}) \\ &\geq \max\{v_p(b_{i,n}) + v_p(a_k), v_p(b_{i,k-1})\} \\ &\geq i + n + 2 - k. \end{aligned}$$

□

Lemma 3.26. *The determinant D of the matrix given by the coefficients of the polynomials p_i is determined as follows:*

$$D := \begin{vmatrix} b_{0,0} & \cdots & b_{0,n} \\ \vdots & \ddots & \vdots \\ b_{n,0} & \cdots & b_{n,n} \end{vmatrix} = \pm \prod_{i \neq j} (\alpha_i - \alpha_j) \prod_i \alpha_i.$$

Proof. First, we show that D is equal to $\operatorname{res}(G, z\frac{\partial G}{\partial z})$. Recall that $\operatorname{res}(G, z\frac{\partial G}{\partial z}) =$

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & n+1 & 0 & \cdots & 0 \\ a_n & 1 & \cdots & & na_n & n+1 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & 1 & 0 & a_1 & \cdots & n+1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & a_1 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & a_0 & 0 & \cdots & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_n & 1 & \cdots & & b_{0,n} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & \ddots & & \vdots \\ a_0 & a_1 & \cdots & 1 & b_{0,0} & b_{0,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & a_1 & 0 & \cdots & 0 & b_{0,1} \\ 0 & 0 & \cdots & a_0 & 0 & \cdots & 0 & b_{0,0} \end{vmatrix}.$$

Then the process of producing p_{i+1} from p_i is equivalent to recursively subtracting the $2n+2-i-k$th column by multiples $n+1-k$th column, for all $0 \leq k \leq n-i$, to reduce it to an upper triangular matrix. Therefore, by applying the column reduction corresponding to generating p_2 to p_{n+1}, we obtain

$$\begin{vmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & \ddots & \vdots \\ a_0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & a_n & b_{n,n} & \cdots & b_{0,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_0 & b_{n,0} & \cdots & b_{0,0} \end{vmatrix}.$$

Therefore, we show that D is equal to $\operatorname{res}(G, z\frac{\partial G}{\partial z})$. On the other hand,

$$\begin{aligned} \operatorname{res}(G, z\frac{\partial G}{\partial z}) &= \operatorname{res}(G, \frac{\partial G}{\partial z}) \operatorname{res}(G, z) \\ &= (-1)^{\frac{n(n-1)}{2}} \operatorname{Disc}(G)(a_0) \\ &= a_0 \prod_{i \neq j} (\alpha_i - \alpha_j). \end{aligned}$$ □

In particular, assume that $v_p(\alpha_i) = 1$ for all i, and $v_p(\alpha_i - \alpha_j) = 1$ for all $i \neq j$, then $v_p(D) = (n+1)^2$.

Now we would like to calculate D in another way. We would like to perform row reduction to reduce the matrix to a shape such that exactly one entry on each row and each column is nonzero (since the determinant is nonzero). As row reduction corresponds to adding scalar multiple of polynomials together, the nonzero entry appearing on the first column after the reduction, corresponding to the scalar term, is in $(G(z), z\frac{\partial G}{\partial z})$.

As we know that D is nonzero, not all $b_{j,n}$ are 0, so we can find j one such that $v_p(b_{j,n})$ is the smallest (and finite), let $x_n := b_{j,n}$. Then $\frac{b_{k,n}}{b_{j,n}} \in \mathcal{O}$ for all k. Then we can subtract the kth row by $\frac{b_{k,n}}{b_{j,n}} \times j$th row, corresponding to $p_k - \frac{b_{k,n}}{b_{j,n}} \times p_j$. We then perform row reductions recursively. We set $b^0_{s,t} = b_{s,t}$. After the ith row reduction, we relabel the (s,t)th entry as $b^i_{s,t}$. We choose a j such that $v_p(b^i_{j,n-i})$ is the lowest and $b_{j,n-k} \neq x_{n-k}$ for $k < i$ (it is possible as the determinant is nonzero), and define $x_{n-i} := b_{j,n-i}$. By repeating the process, we reduce the matrix to a shape such that exactly one entry on each row and each column is nonzero. Moreover, all nonzero entries are given by $x_i = b^n_{\sigma(i),i}$, for some $\sigma \in \mathfrak{S}_{n+1}$.

Lemma 3.27. *If $x_i = b^n_{j,i}$, $v_p(x_i) = j + 1 + n - i$.*

Proof. We consider the $(n+1) \times (n+1)$ matrix given by the lower bound of the valuation v_p of the entry of $b_{s,t}$:

$n+1$	n	$\cdots$	1
$n+2$	$n+1$	$\cdots$	2
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$2n+1$	$2n$	$\cdots$	$n+1$

As $b^{i+1}_{s,t} = b^i_{s,t} - \frac{b_{s,n-i}}{b_{j,n-i}} b^i_{j,t}$ for some j, $v_p(b^{i+1}_{s,t}) \geq v_p(b^i_{s,t})$. Therefore, the grid remains unchanged if we replace $b_{s,t}$ by $b^i_{s,t}$ for any i. On the other hand, $D = \prod x_i$, hence $\sum v_p(x_i) = v_p(D) = (n+1)^2$. It is a simple calculation to show that if we choose one element from each row and column, then they add up to exactly $(n+1)^2$. Therefore, the inequality $v_p(x_i) = v_p(b_{j,i}) \geq j + 1 + n - i$ must be an equality. □

Therefore, we deduce $v_p(x_0) \leq 2n+1$. Since $x_0 \in (G(z), z\frac{\partial G}{\partial z})$, we finish the proof for the first case where $G(x,y) = \prod_{i=1}^{n+1}(xy - \alpha_i)$ or $\prod_{i=1}^{n}(xy - \alpha_i)$.

Now assume $G(x,y) = x\prod_{i=1}^{n}(xy - \alpha_i)$, with α_i satisfying Equation (3.18). As in the other case, we will show that $p^{2n} \in (G, \frac{\partial G}{\partial x})$. Let $F = yG$, and $xy = z$, then $F = z\prod_{i=1}^{n}(z - \alpha_i)$, then F is a polynomial of degree $n+1$ in z. Moreover, $\frac{\partial G}{\partial x} = \frac{\partial F}{\partial z}$. Let $p_1 = \frac{\partial F}{\partial z}$, and given p_i, we obtain $p_{i+1} = (b_{i,n})F - z(p_i)$ where $b_{i,k}$ is the coefficient of z^k in p_i. Again, we consider the determinant of a $n+1 \times n+1$ matrix

$$D' = \left|(p_{n+1}) \quad \cdots \quad (p_1)\right|,$$

where the column with (p_i) means that the entries are given by the coefficients of p_i in the descending power of z. We can apply the same argument about column reduction, except that the resultant matrix is now an $(2n+1)\times(2n+1)$ matrix, with the last column fixed in the column reduction. We obtain

$$\begin{aligned} D' &= \pm \operatorname{res}(F, F') \\ &= \pm \operatorname{Disc}(F) \\ &= \pm \prod_{i\neq j}(\alpha_i - \alpha_j)(\prod_{i=1}^{n} -\alpha_i^2). \end{aligned}$$

Therefore, $v_p(D') = n^2 + n$.

We then similarly consider the $n+1\times n+1$ grid given by the lower bound of the valuation v_p of the entry of $b_{s,t}$, except in this case, all are shifted by 1:

$2n$	$2n-1$	$\cdots$	n
$2n-1$	$2n$	$\cdots$	$n-1$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$n-1$	$\cdots$	0

Again, the same kind of simple calculation shows that if we choose one element from each row and column, then they add up to exactly n^2+n. Therefore, by the same argument as above, we show that $p^{2n} \in (F, \frac{\partial F}{\partial z}) = (G, \frac{\partial G}{\partial x})$. □

Definition 3.28. We let $I_\infty = \ker(R \twoheadrightarrow R^{\leq(\ell_j,0)_j,\tau,\nabla}_{\overline{\mathfrak{M}},\overline{\beta}})$ and $I_\infty^{\mathrm{reg}} = \ker(R \twoheadrightarrow R^{\leq(\ell_j,0)_j,\tau,\nabla,\mathrm{reg}}_{\overline{\mathfrak{M}},\overline{\beta}})$. Fix $\lambda = (\lambda_{j,1}, \lambda_{j,2})_j \leq (\ell_j, 0)_j$.

Theorem 3.29. *Assume $\overline{\rho}$ is of the form in Equation (3.2) and is 4ℓ-generic and τ is 2ℓ-generic where $\ell = \max\{\ell_j\}$. If $W(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}((\ell_j,0),\tau)) \neq \emptyset$, we have an isomorphism*

$$R^{\leq(\ell_j,0)_j,\tau,\mathrm{reg}}_{\overline{\rho}}[\![X_1,\ldots,X_{2f}]\!] \cong (R/\sum_j I^{(j),\mathrm{reg}})[\![Y_1,\ldots,Y_4]\!].$$

The irreducible components of $\operatorname{Spec} R^{\leq(\ell_j,0)_j,\tau,\mathrm{reg}}_{\overline{\rho}}$ are given by $\operatorname{Spec} R^{\lambda,\tau}_{\overline{\rho}}$ where $\lambda_j \leq (\ell_j,0)_j$ and are regular for all j and $\mathrm{JH}(\overline{\sigma}(\lambda,\tau)) \cap W(\overline{\rho}) \neq \emptyset$. Moreover, via the isomorphism above, the kernel of the natural isomorphism $R^{\leq(\ell_j,0)_j,\tau,\mathrm{reg}}_{\overline{\rho}}[\![X_1,\ldots,X_{2f}]\!] \to R^{\lambda,\tau,\mathrm{reg}}_{\overline{\rho}}[\![X_1,\ldots,X_{2f}]\!]$ is given by $\mathfrak{p}^\lambda := \sum_j \mathfrak{p}^{(j),\lambda_{f-1-j}}$ where $\mathfrak{p}^{(j),\lambda_{f-1-j}}$ is given by row 6 in Table 3.1 and Table 3.2.

Proof. We follow the proof of [BHH⁺23, Proposition 4.2.1]. Instead of $h=3$, we allow $h=\ell$. Moreover, to account for non-semisimple $\overline{\rho}$, we follow the proof of [Wan23, Theorem 4.2]. We use $\widetilde{w}$-gauge bases instead of gauge bases. If $W(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}((\ell_j,0),\tau)) \neq \emptyset$, by Lemma 3.9, $R^{\leq(\ell_j,0),\tau,\mathrm{reg}}_{\overline{\rho}} \neq 0$. Moreover, by Lemma 3.9, we have $\tau = \tau_{\widetilde{w}}$ for some

$\tau_{\widetilde{w}} \in X(\overline{\rho}, (\ell_j, 0))$. We modify the definition of $D^{\leq(\ell_j,0)_j,\tau_{\widetilde{w}}}_{\overline{\mathfrak{M}},\overline{\beta}}(R)$ by requiring β to be a $\widetilde{w}$-gauge basis instead of a basis. Then for any $(\mathfrak{M}, \beta, \jmath) \in D^{\leq(\ell_j,0)_j,\tau_{\widetilde{w}}}_{\overline{\mathfrak{M}},\overline{\beta}}(R)$, we have a corresponding matrix $A^{(f-1-j)}$ where $A^{(f-1-j)} \mod m_R \equiv \overline{A}^{(f-1-j)}$. Note that $A^{(f-1-j)}$ may have a $\widetilde{w}$-gauge basis, but may not have shape $\widetilde{w}$ (*cf.* Example 3.3).

By the same argument as in [Wan23, Theorem 4.2], we have an isomorphism

$$R^{\leq(\ell_j,0)_j,\tau_{\widetilde{w}}}_{\overline{\rho}}[\![x_1,\ldots,x_{2f}]\!] \cong R^{\leq(\ell_j,0)_j,\tau_{\widetilde{w}},\nabla}_{\overline{\mathfrak{M}},\overline{\beta}}[\![Y_1,\ldots Y_4]\!]$$

Recalls that

$$I^{\mathrm{reg}}_{\infty} = \ker(R \twoheadrightarrow R^{\leq(\ell_j,0)_j,\tau_{\widetilde{w}},\nabla,\mathrm{reg}}_{\overline{\mathfrak{M}},\overline{\beta}}).$$

And we will show that $I^{\mathrm{reg}}_{\infty} = \sum_j I^{(j),\mathrm{reg}}$, where $I^{(j),\mathrm{reg}}_{\mathrm{poly}}$ is defined in Definition 3.23. By construction, $I^{(j)} \subseteq (I^{(j),\leq(\ell_j,0)}, I^{(j),\nabla}) \subseteq \ker(R \twoheadrightarrow R^{\leq(\ell_j,0)_j,\tau_{\widetilde{w}},\nabla}_{\overline{\mathfrak{M}},\overline{\beta}})$. Since $R^{\leq(\ell_j,0)_j,\tau_{\widetilde{w}},\nabla,\mathrm{reg}}_{\overline{\mathfrak{M}},\overline{\beta}}$ is the quotient of $R^{\leq(\ell_j,0)_j,\tau_{\widetilde{w}},\nabla}_{\overline{\mathfrak{M}},\overline{\beta}}$ for which every component has the maximal dimension, by Lemma 3.22, we deduce that all the equations generating $I^{\mathrm{reg}}_{\infty}$ do not contain $\boldsymbol{b}_0$ or $\boldsymbol{c}_0$ as a factor when $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,m)}$. It follows that $I^{\mathrm{reg}}_{\mathrm{poly}} := \sum_j I^{(j),\mathrm{reg}}_{\mathrm{poly}} \subseteq (I^{\mathrm{reg}}_{\infty}, p^{N-2\ell+1})$. From Corollary 3.24, we know that $p^{\ell}_j \in H^{(j)} + I^{(j),\mathrm{reg}}_{\mathrm{poly}}$. Since $N - (2\ell - 1) > 2 \times \ell$, by applying Elkik's approximation ([Elk73, Lemme 1]) in the same way as in [BHH[+]23, Proposition 4.2.1], we obtain an $\mathcal{O}$-algebra homomorphism, $\widetilde{\phi}^{(j),\mathrm{reg}} \colon R^{(j)}_{\mathrm{poly}}/I^{(j),\mathrm{reg}}_{\mathrm{poly}} \to R/I^{\mathrm{reg}}_{\infty}$ such that $\widetilde{\phi}^{(j),\mathrm{reg}}$ agrees with the natural map modulo $p^{N-3\ell+1}$. We let $\widetilde{\phi}^{\mathrm{reg}} := \otimes_j \widetilde{\phi}^{(j),\mathrm{reg}}$. As $N > 4\ell - 1$, we have the following surjection:

$$\widetilde{\phi}^{\mathrm{reg}} \colon R/I^{\mathrm{reg}}_{\mathrm{poly}} \twoheadrightarrow R^{\leq(\ell_j,0)_j,\tau_{\widetilde{w}},\nabla,\mathrm{reg}}_{\overline{\mathfrak{M}},\overline{\beta}}. \tag{3.19}$$

We will show that $\widetilde{\phi}^{\mathrm{reg}}$ is an isomorphism. Note that $\left(\frac{(\mathfrak{a}-m+n)(\mathfrak{a}-m+n-1)\boldsymbol{b}_0\boldsymbol{c}_0}{\boldsymbol{a}^*_0\boldsymbol{d}^*_0} - (m-n-k)kp\right)$ and $\left(\frac{(\mathfrak{a}-m+n)(\mathfrak{a}-m+n+1)\boldsymbol{a}_0\boldsymbol{d}_0}{\boldsymbol{b}^*_0\boldsymbol{c}^*_0} - (\mathfrak{a}-m+n-k)(\mathfrak{a}+k)p\right)$ are irreducible for all k by [BHH[+]23, Lemma 3.3.1]. Therefore, $R^{(j)}/I^{(j,\mathrm{reg})}_{\mathrm{poly}}$ is reduced, $\mathcal{O}$-flat, with $S(\widetilde{w}_{f-1-j})$ (*cf.* Definition 3.7) irreducible components which are geometrically integral and of relative dimension 3 over $\mathcal{O}$. By [Cal18, Lemma 2.6] and [BLGHT11, Lemma 3.3], $R/I^{\mathrm{reg}}_{\mathrm{poly}}$ is reduced, $\mathcal{O}$-flat with $S(\tau_{\widetilde{w}})$ irreducible components, each with dimension $3f$ over $\mathcal{O}$. Hence, to show $\widetilde{\phi}^{\mathrm{reg}}$ is an isomorphism, it remains to show that $R^{\leq(\ell_j,0)_j,\tau,\nabla}_{\overline{\mathfrak{M}},\overline{\beta}}$, equivalently, $R^{\leq(\ell_j,0)_j,\tau,\nabla}_{\overline{\rho}}$ has at least $S(\tau_{\widetilde{w}})$ components, which follows from Lemma 3.9. Therefore, $\widetilde{\phi}^{\mathrm{reg}}$ is an isomorphism and induces the natural map modulo $p^{N-3\ell+1}$, we show that $(I^{\mathrm{reg}}_{\mathrm{poly}}, p^{N-3\ell+1}) = (I^{\mathrm{reg}}_{\infty}, p^{N-3\ell+1})$.

By the same argument as in [BHH+23, Lemma 4.2.4], we can show that there exists an automorphism of local $\mathcal{O}$-algebra $\psi\colon R \to R$ such that

$$\begin{array}{ccc} R & \xrightarrow[\sim]{\psi} & R \\ \downarrow & & \downarrow \\ R/I^{\mathrm{reg}}_{\mathrm{poly}} & \xrightarrow[\sim]{\widetilde{\phi}^{\mathrm{reg}}} & R/I^{\mathrm{reg}}_{\infty} \end{array}.$$

commutes and such that ψ induces the identity modulo $p^{N-3\ell+1}$. Hence, ψ identifies $I^{\mathrm{reg}}_{\mathrm{poly}}$ with $I^{\mathrm{reg}}_{\infty}$, and $I^{\mathrm{reg}}_{\infty} = \sum_j I^{(j),\mathrm{reg}}$. Moreover, it follows that $\mathfrak{p}^{\lambda}$ are distinct minimal primes containing I_∞. As Equation (3.19) is an isomorphism, we have the irreducible components of $R_{\overline{\rho}}^{\leq(\ell_j,0)_j,\tau_{\widetilde{w}}}$ in bijection with the set λ such that $\mathrm{JH}(\overline{\sigma}(\lambda,\tau_{\widetilde{w}})) \cap W(\overline{\rho}) \neq 0$. As in [BHH+23, Proposition 4.2.1], this is given explicitly by sending a component $\mathcal{C}$ to the labelled Hodge–Tate weights of the framed deformation corresponding to any closed point of the generic fibre of $\mathcal{C}$. Hence, the components are given by $R_{\overline{\rho}}^{\lambda,\tau_{\widetilde{w}}}$ for some $\lambda \leq (\ell_j,0)_j$.

It remains to identify the components. We consider the kernel of the composition

$$\phi_\lambda\colon R \twoheadrightarrow R/I^{\mathrm{reg}}_{\infty} \cong R_{\overline{\mathfrak{M}},\overline{\beta}}^{\leq(\ell_j,0)_j,\tau,\nabla} \twoheadrightarrow R_{\overline{\mathfrak{M}},\overline{\beta}}^{\leq\lambda,\tau,\nabla}.$$

By Lemma 3.9 and the discussion above, we know that $\ker(\phi_\lambda)$ is of the form $\cap_{\lambda'\in X}\mathfrak{p}^{\lambda'}$ for some subset $X \subseteq X(\tau_{\widetilde{w}},(\ell_j,0)_j)$ of cardinality $S(\tau_{\widetilde{w}},\lambda)$ (*cf.* Definition 3.7). We would like to show that $X = X(\tau_{\widetilde{w}},\lambda)$. We will show that $\lambda'_{f-1-j} \leq \lambda_{f-1-j}$ for all $\lambda' \in X$. If $\lambda_{f-1-j} = (\ell_j,0)$ then there is nothing to prove. Otherwise, because of the finite height condition, $A^{(f-1-j)}$ is divisible by $(v+p)^{\lambda_{f-1-j,2}}$ we conclude that $a_k^{(j)}, b_k^{(j)}, c_k^{(j)}, d_k^{(j)} = 0$ for all $0 \leq k < \lambda_{f-1-j,2}$. Assume $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$. As in (3.11), we obtain the equation that appears in the case where the weight is $(\ell - 2\lambda_{f-1-j,2}, 0)$ and $\widetilde{w}_{f-1-j} = \mathfrak{t}_{m-\lambda_{f-1-j,2},n-\lambda_{f-1-j,2}}$. More precisely, we have

$$\boldsymbol{x}_0^{(j)} \prod_{n-1\geq j\geq \lambda_{f-1-j,2}} \left(\frac{(\mathfrak{a}-m+n)(\mathfrak{a}-m+n-1)\boldsymbol{b}_0^{(j)}\boldsymbol{c}_0^{(j)}}{\boldsymbol{a}_0^{(j)*}\boldsymbol{d}_0^{(j)*}} - (m-j)(n-j)p \right) + O(p^{N-3\ell+1}) \in \mathfrak{p}^{\lambda'} \tag{3.20}$$

for all $\lambda' \in X$; where $\boldsymbol{x}_0^{(j)} = \boldsymbol{b}_0^{(j)}$ if $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$ with $m > n$; 1 if $m = n$ or $\alpha_j = 0$ and $\boldsymbol{c}_0^{(j)}$ if $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$ with $m < n$. Assume for the sake of contradiction that $\lambda'_{f-1-j} > \lambda_{f-1-j}$ for some $\lambda' \in X$ and $0 \leq j \leq f-1$. Then $\mathfrak{p}^{(j),\lambda'_{f-1-j}} =$

$$\left(\frac{(\mathfrak{a} - m + n)(\mathfrak{a} - m + n - 1)\boldsymbol{b}_0^{(j)}\boldsymbol{c}_0^{(j)}}{\boldsymbol{a}_0^{(j)*}\boldsymbol{d}_0^{(j)*}} - (m - \lambda'_{f-1-j,2})(n - \lambda'_{f-1-j,2})p + O(p^{N-3\ell+1}) \right) + I^{(j),\mathrm{reg}}. \tag{3.21}$$

As $\lambda'_{f-1-j} > \lambda_{f-1-j}$, $\lambda'_{f-1-j,2} < \lambda_{f-1-j,2}$. Considering Equation (3.20) and Equation (3.21), since $N \geq 4l$, $\mathfrak{a} - m + n, a - m + n - 1, m - n + j, \boldsymbol{a}_0^*, \boldsymbol{d}_0^*$ are units for all $\ell \geq m, n, j \geq 0$, we deduce that $p^k \in \mathfrak{p}^{\lambda'}$ for some k, and therefore $p \in \mathfrak{p}^{\lambda'}$, which is a contradiction. The proof for the case where $\widetilde{w}_{f-1-j} = \mathfrak{w}\mathfrak{t}_{(m,n)}$ is analogous. Therefore, this completes the proof. □

We write $\mathcal{W} = \{j \in \mathcal{J} \colon F(\mathfrak{t}_{\mu-\eta}(0,\ldots,0,\mathrm{sgn}(s_j),0\ldots,0)) \in W(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\lambda,\tau))\}$ (cf. Proposition 3.4). Then $m = |\mathcal{W}|$ is the positive integer such that $2^m = |W(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\lambda,\tau))|$. Given a Serre weight $\sigma = F(\mathfrak{t}_{\mu-\eta}(b_j)) \in W(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$ where $b_j \in \{0, \mathrm{sgn}(s_j)\}$ (cf. Proposition 3.4), we define

$$z(\sigma)_j = \begin{cases} x_j & \text{if } b_j = 0, \\ y_j & \text{if } b_j = \mathrm{sgn}(s_j). \end{cases}$$

And we define $\widetilde{z}(\sigma)_j$ analogously, with x_j, y_j swapped

Corollary 3.30. *Assume that $\overline{\rho}$, up to twisting by a power of ω_f, is of the form in Equation (3.2) and 4ℓ-generic, λ are Hodge–Tate weights with $0 < \lambda_{j,1} - \lambda_{j,2} \leq \ell$ and τ is a 2ℓ-generic inertial type. If $W(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\lambda,\tau)) \neq \emptyset$, then*

$$R_{\overline{\rho}}^{\lambda,\tau} \cong \mathcal{O}[\![(x_j, y_j)_{j\in\mathcal{K}}, Z_1, \ldots, Z_{f-m+4}]\!]/(x_jy_j - p)_{j\in\mathcal{W}},$$

where (x_j)(resp. (y_j)) corresponds to $(\boldsymbol{b}_0^{(j)}) \in R$ (resp. $(\boldsymbol{c}_0^{(j)}) \in R$) if $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$ and $(\boldsymbol{a}_0^{(j)}) \in R$ (resp. $(\boldsymbol{d}_0^{(j)}) \in R$) if $\widetilde{w}_{f-1-j} = \mathfrak{w}\mathfrak{t}_{(m,n)}$. In particular, $R_{\overline{\rho}}^{\lambda,\tau}$ is a normal domain and a complete intersection ring. Moreover, the special fibre $\overline{R}_{\overline{\rho}}^{\lambda,\tau}$ is reduced.

Furthermore, the irreducible components of the special fibre of $\overline{R}_{\overline{\rho}}^{\lambda,\tau}$ is given by

$$\overline{R}_{\overline{\rho}}^{\sigma} = R_{\overline{\rho}}^{\lambda,\tau}/(z(\sigma)_{j\in\mathcal{W}}) \cong \mathbb{F}[\![(\widetilde{z}_j)_{j\in\mathcal{W}}, \ldots, Z_1, \ldots, Z_{f-m-4}]\!]$$

for all $\sigma \in W(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$, and $R_{\overline{\rho}}^{\lambda,\tau}$ in the middle term is identified via the isomorphism above. In particular, all the irreducible components are formally smooth over $\mathbb{F}$.

Proof. We will first reduce it to the case where $\lambda_{j,2} > 0$ such that $\lambda_j \leq (\ell_j, 0)$ for all j where ℓ_j is some positive integer. By [GHS18, Lemma 5.1.6], this can always be achieved via twisting by a crystalline character ψ where $\overline{\psi}|_{I_K}$ is a power of ω_f.

By Lemma 3.9, if $W(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\lambda,\tau)) \neq \emptyset$, then $R_{\overline{\rho}}^{\lambda,\tau} \neq 0$. By Theorem 3.29, we know that

$$R_{\overline{\rho}}^{\lambda,\tau}[\![X_1,\dots,X_{2f}]\!] \cong (\mathcal{O}[\![(x'_j, y'_j, u_j, v_j, (Z_j^k)_{k=1}^{M_j}))_{j=1}^f]\!]/(\mathfrak{p}^\lambda, I_\infty^{\mathrm{reg}}))[\![Y_1,\dots,Y_4]\!]$$

for some positive integer M_j. Here x'_j, y'_j corresponds to $\frac{\boldsymbol{b}_0^{(j)}}{\boldsymbol{a}_0^{*(j)}}, \frac{\boldsymbol{c}_0^{(j)}}{\boldsymbol{d}_0^{*(j)}} \in R^{(j)}$ (resp. $\frac{\boldsymbol{a}_0^{(j)}}{\boldsymbol{b}_0^{*(j)}}, \frac{\boldsymbol{d}_0^{(j)}}{\boldsymbol{c}_0^{*(j)}} \in R^{(j)}$) and u_j, v_j corresponds to $x_{11}^{(j)}, x_{22}^{(j)}$ (resp. $x_{12}^{(j)}, x_{21}^{(j)}$) if $\widetilde{w} = \mathfrak{t}_{(m,n)}$ (resp. if $\widetilde{w} = \mathfrak{w}\mathfrak{t}_{(m,n)}$), and Z_j^k corresponds to the rest of $\boldsymbol{a}_{-i}^{(j)}, \boldsymbol{b}_{-i}^{(j)}, \boldsymbol{c}_{-i}^{(j)}, \boldsymbol{d}_{-i}^{(j)}$ for some $j > 1$. Note that $\mathfrak{p}^{(j),\lambda_{f-1-j}} + I^{(j),\mathrm{reg}}$ are generated by equations of the form $Z_j + \beta_j + \gamma_j p$, where β_j is divisible by x'_j, y'_j, u_j or v_j and $W_j - \alpha_j^* p$ where $W_j = x'_j, y'_j$ or $x'_j y'_j$ and α_j^* is a unit. Let $A = \mathcal{O}[\![(x_j, y_j, u_j, v_j)_{j=1}^f, (Z_j)_{1\leq i\leq M, i\neq j}]\!]$ with maximal ideal $\mathfrak{m}$, then $Z_j + \beta_j + \gamma_j p \in A[\![Z_j]\!]$. By the Weierstrass preparation theorem, we have $Z_j + \beta_j + \gamma_j p = uf(Z_j)$ where u is a unit in $A[\![Z_j]\!]$ and f is a distinguished polynomial in $A[t]$ of degree k. Reducing modulo $\mathfrak{m}$, we have $Z_j \equiv \overline{u} Z_j^k \mod \mathfrak{m}$. Therefore, we must have $k = 1$, and $(Z_j + \beta_j + \gamma_j p) = (Z_j + \delta_j)$ in $A[\![Z_j]\!]$ with $\delta_j \in \mathfrak{m}$. We can therefore eliminate the variables Z_j. Similarly, we can eliminate W_j if $W_j = x'_j, y'_j$. If $W_j = x_j y_j$, then let $x_j = x'_j$ and $y_j = y'_j(\alpha_j^*)^{-1}$, and hence we have $x_j y_j = p$. By [Ham75, Theorem 4], if R, S are quasi-local, then $R[\![x]\!] \cong S[\![x]\!]$ implies $R \cong S$; hence we can cancel the variables $X_1, \dots X_{2f}$ with $\{u_j, v_j\}_{j=1}^f$. By definition, $R_{\overline{\rho}}^{\lambda,\tau}$ is reduced and irreducible by Theorem 3.29, and hence a domain. By [BHH+23, Lemma 3.3.1], $x_j y_j - p$ are irreducible; therefore $(x_j y_j - p)_{j=1}^N$ is a regular sequence and $R_{\overline{\rho}}^{\lambda,\tau}$ is a complete intersection ring. Therefore, it is Cohen–Macaulay and s_k holds for all k. Given a height-1 prime $\mathfrak{p}$, if $\varpi \notin \mathfrak{p}$, then by the description of $R_{\overline{\rho}}^{\lambda,\tau}$ in Theorem 3.29, $R_{\overline{\rho}}^{\lambda,\tau}$ is nonsingular at $\mathfrak{p}$. Moreover, $R_{\overline{\rho}}^{\lambda,\tau}[\frac{1}{p}]$ is regular, by [Kis08, Theorem 3.3.8]. Therefore, R_1 is satisfied and $R_{\overline{\rho}}^{\lambda,\tau}$ is normal. The last statement follows from taking the modulo ϖ that

$$\overline{R}_{\overline{\rho}}^{\lambda,\tau} \cong \mathbb{F}[\![(x_i, y_i)_{i=1}^m, z_1, \dots, z_{f-m+4}]\!]/(x_i y_i)_{i=1}^m = \prod_{\substack{1\leq i\leq m \\ \widetilde{x}_i = x_i \text{ or } y_i}} \mathbb{F}[\![\widetilde{x}_1, \dots \widetilde{x}_m, z_1, \dots, z_{f-m+4}]\!].$$

For the identification of components, we closely follow [LLHLM20, § 3.6]. We have the same canonical diagram as [LLHLM20, Diagram (3.9)] with appropriate modification, for example, the rank of $\mathfrak{M}$ is 2 instead of 3, we have λ instead of η etc. In particular, [LLHLM20, Corollary 3.6.3] is still valid. We have

$$\overline{R}_{\overline{\rho}}^{\lambda,\tau}[\![X_1,\dots,X_{2f}]\!] \cong \bigotimes_j R^{(j)}/(p^{(j),\lambda_{f-1-j}},p)[\![Y_1,\dots,Y_4]\!].$$

Therefore, it suffices to match the components of $\bigotimes_j R^{(j)}/(p^{(j),\lambda_{f-1-j}},p)$ with $W(\overline{\rho})\cap \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$. Notice that $x_j=0$ (resp. $y_j=0$) if and only if $x'_j=0$ (resp. $y'_j=0$) in the notation above. As explained in [LLHLM20, § 3.6], given a Serre weight $\sigma\in W(\overline{\rho})\cap\mathrm{JH}(\overline{\sigma}(\lambda,\tau))$, we first find a minimal type τ', such that $W(\overline{\rho})\cap\mathrm{JH}(\overline{\sigma}(\lambda,\tau'))=\{\sigma\}$, then by Lemma 3.8, $\overline{R}_{\overline{\rho}}^{\lambda,\tau}=\overline{R}^{\sigma}$. Using the same calculation as in Lemma 3.6 and Lemma 3.9, assume $\lambda_j=(m,n)$ and $\sigma=F(\mathfrak{t}_{\mu-\eta}(b_j))$ where $b_j\in\{0,\mathrm{sgn}(s_j)\}$, we see that $\tau'=\tau_{\widetilde{w}}$ where $\widetilde{w}'_{f-1-j}=\mathfrak{t}_{(\mathfrak{m},\mathfrak{n})}$ if $b_j=0$ and $\widetilde{w}'_{f-1-j}=\mathfrak{t}_{(\mathfrak{n},\mathfrak{m})}$ if $b_j=\mathrm{sgn}(s_j)$. In this case,

$$R^{(j)}/(p^{(j),\lambda_{f-1-j}},\varpi)\cong\mathbb{F}[\![z^{(j)},\boldsymbol{a}_0^{*(j)},\boldsymbol{d}_0^{*(j)}]\!],$$

where $(z^{(j)})=(y'_j)=(\boldsymbol{c}_0^{(j)})$ if $b_j=0$ and $(z_j)=(x'_j)=(\boldsymbol{b}_0^{(j)})$ if $b_j\neq 0$. We now carry out a similar calculation as in [LLHLM20, § 3.6]. By Lemma 3.9, we can assume $\tau=\tau_{\widetilde{w}}$ with $\widetilde{w}'=\widetilde{z}\widetilde{w}$. Assume $\widetilde{w}_{f-1-j}=\mathfrak{w}\mathfrak{t}_{(a,b)}$. If $\widetilde{w}'_{f-1-j}=\mathfrak{t}_{(m,n)}$ (i.e., $b_j=0$), then $\widetilde{z}_{f-1-j}=\mathfrak{w}\mathfrak{t}_{(a-n,b-m)}$. Note that $A^{(f-1-j)}$ now has entries in characteristic p. By Theorem 3.13 and Theorem 3.29, $A^{(j)'}=\begin{pmatrix} v^m\boldsymbol{a}_0^{*'} & 0\\ v^m\boldsymbol{c}'_0 & v^n\boldsymbol{d}_0^{*'}\end{pmatrix}$. Similarly, by Theorem 3.17 and Theorem 3.29, $A^{(f-1-j)} \mod \boldsymbol{a}_0=\begin{pmatrix}0 & v^b\boldsymbol{b}_0^*\\ v^a\boldsymbol{c}_0^* & v^b\boldsymbol{d}_0\end{pmatrix}$. (Here $'$ is used to denote those in $A'^{(f-1-j)}$ given by $\widetilde{w}'$ and we omit the index (j) for legibility.) By setting

$$\begin{pmatrix}0 & v^b\boldsymbol{b}_0^*\\ v^a\boldsymbol{c}_0^* & v^b\boldsymbol{d}_0\end{pmatrix}=\begin{pmatrix} v^m\boldsymbol{a}_0^{*'} & 0\\ v^m x'_{21} & v^n\boldsymbol{d}_0^{*'}\end{pmatrix}\begin{pmatrix}0 & v^{b-m}\\ v^{a-n} & 0\end{pmatrix},$$

we deduce the following identification:

$$\boldsymbol{b}_0^*=\boldsymbol{a}_0^{*\prime}\ \ \boldsymbol{c}_0^*=\boldsymbol{d}_0^{*\prime}\ \ \boldsymbol{c}'_0=\boldsymbol{d}_0.$$

Similarly, If $\widetilde{w}'_{f-1-j} = \mathfrak{t}_{(n,m)}$ (i.e., $b_j \neq 0$), then $\widetilde{z}_{f-1-j} = \mathfrak{w}\mathfrak{t}_{(a-m,b-n)}$ and $A^{(f-1-j)'} = \begin{pmatrix} v^n \boldsymbol{a}_0^{*'} & v^{m-1} x'_{12} \\ 0 & v^m \boldsymbol{d}_0^{*'} \end{pmatrix}$ and $A^{(f-1-j)} \mod \boldsymbol{d}_0 = \begin{pmatrix} v^{a-1} \boldsymbol{a}_0 & v^b \boldsymbol{b}_0^* \\ v^a \boldsymbol{c}_0^* & 0 \end{pmatrix}$. By setting

$$\begin{pmatrix} v^{a-1} \boldsymbol{a}_0 & v^b \boldsymbol{b}_0^* \\ v^a \boldsymbol{c}_0^* & 0 \end{pmatrix} = \begin{pmatrix} v^n \boldsymbol{a}_0^{*'} & v^{m-1} \boldsymbol{b}'_0 \\ 0 & v^m \boldsymbol{d}_0^{*'} \end{pmatrix} \begin{pmatrix} 0 & v^{b-n} \\ v^{a-m} & 0 \end{pmatrix},$$

we deduce the following identification:

$$\boldsymbol{b}_0^* = \boldsymbol{a}_0^{*\prime} \;\; \boldsymbol{c}_0^* = \boldsymbol{d}_0^{*\prime} \;\; \boldsymbol{b}'_0 = \boldsymbol{a}_0.$$

By the same argument, it is straightforward to see that when $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(a,b)}$, $\boldsymbol{a}_0^{*\prime} = \boldsymbol{a}_0^*$, $\boldsymbol{d}_0^{*\prime} = \boldsymbol{d}_0^*$, $\boldsymbol{b}'_0 = \boldsymbol{b}_0 / \boldsymbol{c}'_0 = \boldsymbol{c}_0$ where appropriate. Therefore, the last statement follows. □

Table 3.1: $\widetilde{w}_{f-1-j} = \mathfrak{t}_{(m,n)}$

$\overline{A}^{(f-1-j)}$	$\begin{pmatrix} \alpha_j v^m & 0 \\ \alpha_j \gamma_{f-1-j} v^m & \beta_j v^n \end{pmatrix}$	
shape	$\gamma_{f-1-j} \neq 0, m \leq n$	$\mathfrak{w}\mathfrak{t}_{(m,n)}$
	otherwise	$\mathfrak{t}_{(m,n)}$
$A^{(f-1-j)}$	$\begin{pmatrix} \sum_{0\leq i\leq m} \boldsymbol{a}_{-(m-i)}(v+p)^i & \sum_{0\leq i\leq n-1} \boldsymbol{b}_{-(n-1-i)}(v+p)^i \\ v(\sum_{0\leq i\leq m-1} \boldsymbol{c}_{-(m-1-i)}(v+p)^i) & \sum_{0\leq i\leq n} \boldsymbol{d}_{-(n-i)}(v+p)^i \end{pmatrix}$	
$R^{(j)}$	$\mathcal{O}[\![x_{11}, x_{12}x_{22}, x_{21}, (\boldsymbol{a}_{-k})_{k=1}^{m}, (\boldsymbol{b}_{-k})_{k=0}^{n-1}, (\boldsymbol{c}_{-k})_{k=1}^{m-1}, (\boldsymbol{d}_{-k})_{k=1}^{n}]\!]$	
$I^{(j),\mathrm{reg}}$	For $0 \leq k \leq m$, $\boldsymbol{a}_{-k} - \frac{(-1)^k \boldsymbol{a}_0}{k! \prod_{i=0}^{k-1}(\mathfrak{a}'+i)} \prod_{i=0}^{k-1}(Z + i(m-n-i)p) + O(p^{k_j})$, For $0 \leq k \leq n-1$, $\boldsymbol{b}_{-k} - \frac{\boldsymbol{b}_0}{k! \prod_{i=0}^{k-1}(\mathfrak{a}'-i)} \prod_{i=1}^{k}(Z - i(m-n+i)p) + O(p^{k_j})$, For $0 \leq k \leq m-1$, $\boldsymbol{c}_{-k} - \frac{(-1)^k \boldsymbol{c}_0}{k! \prod_{i=0}^{k-1}(\mathfrak{a}'+i-1)} \prod_{i=1}^{k}(Z - i(n-m+i)p) + O(p^{k_j})$, For $0 \leq k \leq n$, $\boldsymbol{d}_{-k} - \frac{\boldsymbol{d}_0}{k! \prod_{i=0}^{k-1}(\mathfrak{a}'-1-i)} \prod_{i=0}^{k-1}(Z + i(n-m-i)p) + O(p^{k_j})$, $(\boldsymbol{x}_0 + O(p^{k_j})) \prod_{n\geq j\geq 1} \left(Z - (m-n+j)jp + O(p^{k_j})\right)^*$	
$I^{(j)}$	$m = n$	$\boldsymbol{b}_0 + O(p^{k_j})) \prod_{n\geq j\geq 1}(Z - (m-n+j)jp + O(p^{k_j}))^\dagger$ $\boldsymbol{c}_0 + O(p^{k_j})) \prod_{n\geq j\geq 1}(Z - (m-n+j)jp + O(p^{k_j}))^\dagger$
$\mathfrak{p}^{(j),\lambda_{f-1-j}}$	$\lambda_{f-1-j} = (m,n), n < m$	$I^{(j),\mathrm{reg}} + \boldsymbol{b}_0 + O(p^{k_j})$
	otherwise	$I^{(j),\mathrm{reg}} + (Z - (m-\lambda_{f-1-j,2})(n-\lambda_{f-1-j,2})p + O(p^{k_j}))$
$z(\sigma)_j$	$b_j = 0$	$\boldsymbol{b}_0$
	$b_j = \mathrm{sgn}(b_j)$	$\boldsymbol{c}_0$

Here, we define $Z := \frac{(\mathfrak{a}')(\mathfrak{a}'-1)\boldsymbol{b}_0\boldsymbol{c}_0}{\boldsymbol{a}_0^*\boldsymbol{d}_0^*}$ and $\boldsymbol{x}_0 = \boldsymbol{b}_0$ if $m > n$, $\boldsymbol{c}_0$ if $m < n$ and 1 if $m = n$. Recall that $O(p^{k_j})$ denotes a specific but inexplicit element in $p^{N-3\ell_{f-1-j}+1}M_2(R)$, it depends on the whole tuple $\widetilde{w}$, not just $\widetilde{w}_{f-1-j}$. Moreover, $\mathfrak{a}' = \mathfrak{a} - m + n$, $\mathfrak{a} \in \mathbb{Z}_{(p)}$ and $\mathfrak{a} \equiv -\langle s_j^{-1}(\mu_j) - (m,n), \alpha_j^\vee \rangle \equiv -\mathrm{sgn}(s_j)(r_j+1) + (m-n) \mod p$. For readability, we remove the superscript (j). Furthermore, $x_{11} = \boldsymbol{a}_0^* - [\overline{\boldsymbol{a}}_0^*]$, $x_{12} = \boldsymbol{b}_0$, $x_{21} = \boldsymbol{c}_0 - [\overline{\boldsymbol{c}}_0]$ if $\gamma_{f-1-j} \neq 0$, otherwise $x_{21} = \boldsymbol{c}_0$, and $x_{22} = \boldsymbol{d}_0^* - [\overline{\boldsymbol{d}}_0^*]$. Here, $\sigma = F(\mathfrak{t}_{\mu-\eta}(b_j))$ where $b_j \in \{0, \mathrm{sgn}(s_j)\}$ as in Proposition 3.4

Table 3.2: $\widetilde{w}_{f-1-j} = \mathfrak{w}\mathfrak{t}_{(m,n)}$

$\overline{A}^{(f-1-j)}$	$\begin{pmatrix} 0 & \alpha_j v^n \\ \beta_j v^m & \alpha_j \gamma_{f-1-j} v^n \end{pmatrix}$	
shape	$\gamma_{f-1-j} \neq 0, m > n$	$\mathfrak{t}_{(m,n)}$
	otherwise	$\mathfrak{w}\mathfrak{t}_{(m,n)}$
$A^{(f-1-j)}$	$\begin{pmatrix} \sum_{0\le i\le m-1} \boldsymbol{a}_{-(m-i)}(v+p)^i & \sum_{0\le i\le n} \boldsymbol{b}_{-(n-i)}(v+p)^i \\ v(\sum_{0\le i\le m-1} \boldsymbol{c}_{-(m-1-i)}(v+p)^i) & \sum_{0\le i\le n} \boldsymbol{d}_{-(n-i)}(v+p)^i \end{pmatrix}$	
$R^{(j)}$	$\mathcal{O}[\![x_{11}, x_{12}x_{22}, x_{21}, (\boldsymbol{a}_{-k})_{k=1}^{m-1}, (\boldsymbol{b}_{-k})_{k=0}^{n}, (\boldsymbol{c}_{-k})_{k=1}^{m-1}, (\boldsymbol{d}_{-k})_{k=1}^{n-1}]\!]$	
$I^{(j),\mathrm{reg}}$	For $0 \le k \le m-1$, $\boldsymbol{a}_{-k} - \frac{(-1)^k \boldsymbol{a}_0}{k!\prod_{i=0}^{k-1}(\mathfrak{a}'+i)} \prod_{i=1}^{k} (Z + (\mathfrak{a}-i)(\mathfrak{a}'+i)p) + O(p^{k_j})$, For $0 \le k \le n$, $\boldsymbol{b}_{-k} - \frac{\boldsymbol{b}_0}{k!\prod_{i=0}^{k-1}(\mathfrak{a}'-i)} \prod_{i=0}^{k-1} (Z + (\mathfrak{a}+i)(\mathfrak{a}'-i)p) + O(p^{k_j})$, For $0 \le k \le m-1$, $\boldsymbol{c}_{-k} - \frac{(-1)^k \boldsymbol{c}_0}{k!\prod_{i=0}^{k-1}(\mathfrak{a}'+1+i)} \prod_{i=1}^{k} (Z + (\mathfrak{a}-i)(\mathfrak{a}'+i)p) + O(p^{k_j})$, For $0 \le k \le n$, $\boldsymbol{d}_{-k} - \frac{\boldsymbol{d}_0}{k!\prod_{i=0}^{k-1}(\mathfrak{a}'+1-i)} \prod_{i=0}^{k-1} (Z + (\mathfrak{a}+i)(\mathfrak{a}'-i)p) + O(p^{k_j})$, $\prod_{0\le k\le n-1} \Big(Z - (\mathfrak{a}'-k)(\mathfrak{a}+k)p + O(p^{k_j})\Big)^*$	
$\mathfrak{p}^{(j),\lambda_{f-1-j}}$	$I^{(j),\mathrm{reg}} + \Big(Z - (\mathfrak{a}-m+\lambda_{f-1-j,2})(\mathfrak{a}+n-\lambda_{f-1-j,2})p + O(p^{k_j})\Big)$	
$z(\sigma)_j$	$b_j = 0$	$\boldsymbol{a}_0$
	$b_j = \mathrm{sgn}(b_j)$	$\boldsymbol{d}_0$

Here we let $Z := \frac{(\mathfrak{a}')(\mathfrak{a}'+1)\boldsymbol{a}_0\boldsymbol{d}_0}{\boldsymbol{b}_0^*\boldsymbol{c}_0^*}$ and $O(p^{k_j})$ denotes a specific but inexplicit element in $p^{N-3\ell_{f-1-j}+1}M_2(R)$, it depends on the whole tuple $\widetilde{w}$, not just $\widetilde{w}_{f-1-j}$. Moreover, $\mathfrak{a}' = \mathfrak{a} - m + n$, $\mathfrak{a} \in \mathbb{Z}_{(p)}$ and $\mathfrak{a} \equiv -\langle \mathfrak{w}s_j^{-1}(\mu_j) - (m,n), \alpha_j^\vee \rangle \equiv \mathrm{sgn}(s_j)(r_j+1) + (m-n)$ mod p. For readability, we remove the superscript (j). Furthermore, $x_{11} = \boldsymbol{a}_0$, $x_{12} = \boldsymbol{b}_0^* - [\overline{\boldsymbol{b}}_0^*]$, $x_{21} = \boldsymbol{c}_0^* - [\overline{\boldsymbol{c}}_0^*]$, and if $\gamma_{f-1-j} \neq 0$, $x_{22} = \boldsymbol{d}_0 - [\overline{\boldsymbol{d}}_0]$, otherwise $x_{22} = \boldsymbol{d}_0$. Here, $\sigma = F(\mathfrak{t}_{\mu-\eta}(b_j))$ where $b_j \in \{0, \mathrm{sgn}(s_j)\}$ as in Proposition 3.4

4 PATCHING FUNCTOR

4.1 ABSTRACT PATCHING FUNCTOR

Let $\mathcal{S}$ be a finite set. For each v, we fix a local field L_v with the residue field k_v, such that if $\mathrm{char}(k_v) = p$, then L_v is a finite extension of $\mathbb{Q}_p$. For each v, we will let K_v be a compact open subgroup of $\mathrm{GL}_2(F_v)$, and let $K := \prod_{v\in\mathcal{S}} K_v$. We let $\overline{\rho}_v \colon G_{L_v} \to \mathrm{GL}_2(\mathbb{F})$. We define $\mathcal{C}$ to be the category of finitely generated $\mathcal{O}$-algebra with a continuous action of K and let C' be a Serre subcategory of C.

We define

$$R_\infty := \widehat{\bigotimes}_{v\in\mathcal{S}} R^{\square}_{\overline{\rho}_v}[\![x_1, x_2, \dots, x_h]\!]$$

for some positive integer $h \geq |S| - 1$, with maximal ideal $\mathfrak{m}_\infty$. If $\mathrm{char}(k_v) \neq p$, let $R^{\overline{\sigma}_v}_{\overline{\rho}_v} = R^{\square}_{\overline{\rho}_v}$ and $R^{\lambda_v,\tau_v}_{\overline{\rho}_v} = R^{\tau_v}_{\overline{\rho}_v}$. We let

$$R_\infty(\lambda, \tau) := R_\infty \widehat{\bigotimes}_{R^{\square}_{\overline{\rho}_v}} R^{\lambda_v,\tau_v}_{\overline{\rho}_v}; \quad R^{\overline{\sigma}}_\infty := R_\infty \widehat{\bigotimes}_{v\in\mathcal{S}} R^{\overline{\sigma}_v}_{\overline{\rho}_v}. \tag{4.1}$$

We write $X_\infty := \mathrm{Spf}\, R_\infty$ and analogously $X_\infty(\tau) := \mathrm{Spf}\, R_\infty(\eta, \tau)$, $X_\infty(\lambda, \tau) := \mathrm{Spf}\, R_\infty(\lambda, \tau)$, $X_\infty(\overline{\sigma}) := \mathrm{Spf}\, R^{\overline{\sigma}}_\infty$ and we write $\overline{X}_\infty(\overline{\sigma})$ for the special fibre of the space $X_\infty(\overline{\sigma})$.

Following the notion in [EGS15, § 6], a patching functor M_∞ is a nonzero covariant exact functor from C' to the category of coherent sheaves over $\mathrm{Spf}\, R_\infty$ with the following properties. If $\tau = (\tau_v)_{v\in\mathcal{S}}$ is a collection of tame inertial types, then $\sigma(\tau_v)$ is a representation of $\mathrm{GL}_2(\mathcal{O}_{L_v})$ over $\mathcal{O}$ corresponding to τ_v by the local Langlands correspondence. We fix $\sigma^\circ(\tau_v)$ a $\mathcal{O}$-lattice in $\sigma(\tau_v)$, and we write $\sigma^\circ(\tau) := \otimes_{v\in\mathcal{S}}\sigma^\circ(\tau_v)$. Then, we have the following:

1. $M_\infty(\sigma^\circ(\tau))$ is p-torsion free and is a maximal Cohen–Macaulay sheaf on $X_\infty(\tau)$.
2. For all $\overline{\sigma} \in \mathrm{JH}(\overline{\sigma}(\tau))$, $M_\infty(\overline{\sigma})$ is a maximal Cohen–Macaulay sheaf on $\overline{X}_\infty(\overline{\sigma})$.

We say that M_∞ is a minimal patching functor if the locally free sheaf $M_\infty(\sigma^\circ(\tau))[\frac{1}{p}]$ has rank at most one on each connected component. We say a patching functor is unramified if the coefficient field is unramified over $\mathbb{Q}_p$.

4.2 GLOBAL SETUP

For the global setup, we will construct a minimal patching functor by unitary groups, following closely [CEG+18] and [LLHLM24]. Let F be a CM field with maximal totally real subfield F^+. We call a place in F^+ split (resp. inert) if it splits (resp. is inert) in F. We denote S_p for the set of primes of F^+ lying above p. Let Σ be the set of primes of F^+ away from p where $\bar{r}$ ramifies.

Let $\mathcal{O}_{F^+}, \mathcal{O}_{F_V^+}$ and $\mathcal{O}_{F_w}$ denote the ring of integers of F^+, F_v^+ and F_w respectively, where v is a place of F^+ and w a place of F. Let $G_{/F^+}$ be a reductive group which is an outer form for GL_2 such that

1. $G_{/F}$ is an inner form of GL_2;
2. $G_{/F^+}(F_v^+) \cong U_n(\mathbb{R})$ for all $v|\infty$;
3. $G_{/F^+}$ is quasisplit at all inert finite places.

By [EGH13, § 7.1], G admits a reductive model $\mathcal{G}$ over $\mathcal{O}_{F^+}[1/N]$ for some $p \nmid N$ and an isomorphism $\iota\colon \mathcal{G}_{/\mathcal{O}_F[1/N]} \to \mathrm{GL}_{2/\mathcal{O}_F[1/N]}$ which specializes to $\iota_w\colon \mathcal{G}(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathcal{G}(\mathcal{O}_{F_w}) \xrightarrow{\iota} \mathrm{GL}_n(\mathcal{O}_{F_w})$ for all split finite places w in F prime to N where $w|_{F^+} = v$. Let $U = U^p U_p \leqslant G(\mathbb{A}_{F^+}^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$ where $\mathcal{G}(\mathcal{O}_{F^+,p}) := \prod_{v|p} \mathcal{G}(\mathcal{O}_{F_v^+})$ be a compact open subgroup. If W is a finitely generated $\mathcal{O}$-module endowed with a continuous action of $U_\Sigma := \prod_{v\in\sigma} U_v$, we define the space of automorphic forms on G of level U with coefficients in W to be the $\mathcal{O}$-module $S(U, W) :=$

$$\{f\colon \text{continuous map } G(F^+)\backslash G(\mathbb{A}_{\mathbb{F}^+}^\infty) \to W | f(gu) = u_\Sigma^{-1} f(g) \text{ for all } g \in G(\mathbb{A}_{F^+}^\infty), u \in U\}.$$

We say U is unramified at v, if $U = \mathcal{G}(\mathcal{O}_{F_v^+})U^v$. Let $\mathcal{S}$ be the set of finite split places of F^+, which is composed of $S_p \sqcup \Sigma$ and all place v such that U is not unramified. Let $\mathcal{P}_U$ be the set of finite places w such that $v := w|_{F^+}$ is split in F, and does not divide any primes in $\mathcal{S}$, nor any prime dividing N. For any subset $\mathcal{P} \subseteq \mathcal{P}_U$ of finite complement that is closed under complex conjugation, we define

$$\mathbb{T}_\mathcal{P} = \mathcal{O}[T_w^{(i)}, w \in \mathcal{P}, 0 \le i \le 2] \tag{4.2}$$

to be the universal Hecke algebra on $\mathcal{P}$. Then, $S(U, W)$ is endowed with an action of $\mathbb{T}_\mathcal{P}$ that $T_w^{(i)}$ acts by the double coset operator

$$\iota_w^{-1}\left[\mathrm{GL}_2(\mathcal{O}_{F_w})\begin{pmatrix}\varpi_w \mathrm{Id}_i & 0 \\ 0 & \varpi_w \mathrm{Id}_{n-i}\end{pmatrix}\mathrm{GL}_2(\mathcal{O}_{F_w})\right].$$

Definition 4.1. [LLHLM18, Definition 7.1] Let $\bar{r}\colon G_F \to \mathrm{GL}_2(\mathbb{F})$ be a continuous Galois representation. Let $\mathfrak{m} \subseteq \mathbb{T}_{\mathcal{P}}$ for some $\mathcal{P} \subseteq \mathcal{P}_U$ corresponding to the kernel of the system of eigenvalues $\bar{\alpha}\colon \mathbb{T}_{\mathcal{P}} \to \mathbb{F}$ such that

$$\det(1 - \bar{r}^{\vee}(\mathrm{Frob}_w)X) = \sum_{j=0}^{2}(-1)^j(\mathbf{N}_{F/\mathbb{Q}}(w))^{\binom{j}{2}}\bar{\alpha}(T_w^{(j)})X^j$$

for all $w \in \mathcal{P}$. Then we say $\bar{r}$ is automorphic if there exists a compact open subgroup $U \leqslant G(\mathbb{A}_{F^+}^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$, a finite $\mathcal{O}$-module W endowed with a continuous action of U_Σ and a cofinite subset $\mathcal{P} \subseteq \mathcal{P}_U$ such that

$$S(U,W)_{\mathfrak{m}} \neq 0$$

Given a continuous absolutely irreducible representation $\bar{r}\colon G_F \to \mathrm{GL}_2(\mathbb{F})$, we will assume it satisfies the following properties:

Properties 4.2.

1. p is unramified in F^+ and every place of F^+ dividing p splits in F.
2. F/F^+ is unramified at all finite places, and hence $[F^+ : Q]$ is even.
3. $\overline{F}^{\ker \mathrm{ad}\bar{r}|_{G_F}}$ does not contain $F(\zeta_p)$.
4. $\bar{r}|_{G_{F(\zeta_p)}}$ is absolutely irreducible.
5. $\bar{r}$ is automorphic as in Definition 4.1.
6. $\bar{r}$ is ramified only at split places and with minimal ramification in the sense of [CHT08, Definition 2.4.14].
7. $\bar{r}(G_F)$ contains $\mathrm{GL}_2(\mathbb{F})$ with $|\mathbb{F}| > 6$.
8. $\bar{r}(G_{F(\zeta_p)})$ is adequate (in the sense of [Tho12, Definition 2.3]).
9. For all places $\tilde{v}|p$ of F, $\bar{r}|_{G_{F_{\tilde{v}}}}$, up to twisting by a crystalline character, satisfies Equation (3.2) for some N_v.

By [CEG+18, § 2.3], we can find a finite place $v_1 \notin \mathcal{S}$ such that

1. v_1 splits as $w_1 w_1^c$ in F;
2. v_1 does not split completely in $F(\zeta_p)$, i.e $(\mathbf{N}v_1) \neq 1(\mathrm{mod}\ p)$;

3. $\bar{r}|_{G_{F^+_{v_1}}}$ is unramified and the ratio between the eigenvalues of $\bar{r}(\mathrm{Frob}_{F_{v_1}})$ is not equal to $(\mathrm{N}v_1)^{\pm 1}$ or 1.

Remark 4.3. By [Sho16, § 5], this ensures that $R^{\square}_{\bar{r}_{\tilde{v}_1}}$ is formally smooth over $W(\mathbb{F})$.

Definition 4.4. We construct a compact subgroup $U = \prod_v U_v$ of $G(\mathbb{A}^{\infty}_{F^+})$ such that each U_v is a compact subgroup of $G(F^+_v)$ with the following properties:

1. $U_v = \iota_{\tilde{v}}^{-1}\mathcal{G}(\mathcal{O}_{F_{\tilde{v}}})$ if v is a split place in F and $v \neq v_1$.
2. U_v is a hyperspecial maximal compact subgroup of $\mathcal{G}(F^+_v)$ if v is inert in F.
3. U_{v_1} is the preimage of the upper triangular matrices under

$$\mathcal{G}(\mathcal{O}_{F^+_{v_1}}) \xrightarrow[\sim]{\iota_{v_1}} \mathrm{GL}_2(\mathcal{O}_{F_{w_1}}) \xrightarrow{\bmod p} \mathrm{GL}_2(k_{w_1}).$$

Because of the condition at v_1, U is sufficiently small in the sense that for some $v \in F^+$, the projection of U to $G(F^+_v)$ does not contain nontrivial element of finite order (cf. the discussion in [GK14, § 3.1.2]). Hence for any $\mathcal{O}$-algebra A and $\mathcal{O}$-module W, we have

$$S(U, W \otimes_{\mathcal{O}} A) \cong S(U, W) \otimes_{\mathcal{O}} A.$$

For each $v \in \Sigma$, we fix an inertial type τ_v which is the restriction to the inertia of a minimally ramified lift of $\bar{r}|_{G_{F^+_v}}$. By the inertial local Langlands correspondence, we have a finite-dimensional $\mathrm{GL}_2(\mathcal{O}_{F^+_v})$-representation $\sigma(\tau_v^\vee)$ over $\mathcal{O}$ corresponding to $\tau_v^\vee$, and we fix a $\mathcal{O}$-lattice $\sigma(\tau_v^\vee)^\circ \subseteq \sigma(\tau_v^\vee)$. Let $W_\Sigma = \bigotimes_{v\in\Sigma}(\sigma(\tau_v^\vee)^\circ \circ \iota_v^{-1})$. Let V be any finite $\mathbb{F}$-module with continuous $\prod_{s\in S_p} G(\mathcal{O}_{F^+_v})$-action. Then, the patching functor $V \mapsto M_\infty(V \otimes W_\Sigma)$ is a patching functor for $\mathcal{S} = S_p$. We write $U_p = \prod_{v\in S_p} U_v$ and $U = U_p U^p$. We define

$$S(U^p, W) := \varinjlim_{U_p} S(U^p U_p, W) \text{ and } \widetilde{S}(U^p, W) := \varprojlim_{n} S(U^p, W/\varpi^n),$$

where the subgroups $U_p \leqslant \prod_{v\in S_p} \mathcal{G}(\mathcal{O}_{F^+_v})$ run over all compact open neighbourhoods of 1. Similarly, we define

$$S(U^v, W) := \varinjlim_{U_v} S(U^v U_v, W) \text{ and } \widetilde{S}(U^v, W) := \varprojlim_{n} S(U^v, W/\varpi^n).$$

Let $w_1 \in F$ be such that $w_1|_{F^+} = v_1$. We define

$$\mathbb{T}'_{\mathcal{P}} = \mathbb{T}_{\mathcal{P}}[T^{(1)}_{w_1}, T^{(2)}_{w_1}].$$

Then, $S(U,W), S(U^{\mathcal{S}},W), \widetilde{S}(U^{\mathcal{S}},W)$ is endowed with an action of $\mathbb{T}'_{\mathcal{P}}$ such that $T_{v_1}^{(i)}$ act by the double coset operator:

$$\left[U_{v_1}\iota_{w_1}^{-1}\begin{pmatrix}\varpi_{w_1}\mathrm{Id}_i & 0\\ 0 & \mathrm{Id}_{n-i}\end{pmatrix}U_{v_1}\right].$$

Label the eigenvalues of $\bar{r}(\mathrm{Frob}_{w_1})$ as δ_1,δ_2. Let $\mathfrak{m}'$ be the maximal ideal of $\mathbb{T}'_{\mathcal{P}}$ generated by $\mathfrak{m}$ and the elements $T_{w_1}^{(1)}-\delta_1, T_{w_1}^{(2)}-(\mathbf{N}v_1)^{-1}\delta_1\delta_2$.

we define $M_{\mathfrak{m}'} := M\otimes(\mathbb{T}'_{\mathcal{P}})_{\mathfrak{m}'}$. In particular, $\widetilde{S}(U^v,W)_{\mathfrak{m}'} := \varprojlim_n \varinjlim_{U_v} S(U^vU_v,W)_{\mathfrak{m}'}$.

Proposition 4.5. Given Galois representation $\bar{r}\colon G_F\to \mathrm{GL}_2(\mathbb{F})$ satisfying Properties 4.2. There exists a minimal patching functor M_∞ for $\mathcal{S}=S_p$, $L_v=F_v^+\cong F_{\tilde{v}}$, $\bar{\rho}_v=\bar{r}|_{F_v^+}\cong\bar{r}|_{F_{\tilde{v}}}$ and $K_v=U_v$ as above.

Proof. This follows from [LLHLM24, Lemma 5.5.4], which in turn uses the idea in [CEG+18], [EGS15], [GK14] among many others. For $\bar{\sigma}\in\mathrm{JH}(\bar{\sigma}(\tau_{S_p}))$, $M_\infty(\bar{\sigma})$ is a priori a maximal Cohen–Macaulay sheaf on $\overline{X}_\infty(\tau_{S_p})$. By [EGS15, Proposition 3.5.1], we can find a tame type τ such that $\mathrm{JH}(\bar{\sigma}(\tau))\cap W(\bar{r})=\bar{\sigma}$. Then, by [EGS15, Theorem 7.2.1(2),(4)], the scheme-theoretic support of $M_\infty(\bar{\sigma})$ is exactly $\overline{X}_\infty(\tau)=\overline{X}_\infty(\bar{\sigma})$. (*cf.* [EGS15, § B.1]) ☐

We can further construct a (not necessarily minimal) patching functor for any $\mathcal{S}\subseteq S_p$. For each $v\in S_p\setminus\mathcal{S}$, we fix a tame inertial type τ_v and regular weights $\lambda_v\in(\mathbb{Z}^2)^{f_{v'}}$ ($(\lambda_v)_{j,1}-(\lambda_v)_{j,2}>0$). For the purpose of Chapter 6, we will also construct a minimal patching functor for $\mathcal{S}$. In the minimal case, we will choose for each λ_v, a generic tame inertial type τ_v such that $W(\bar{r}_v)\cap\mathrm{JH}(\lambda_v,\tau_v)$ is a singleton, which can be arranged using Equation (3.6). We then fix a $\mathrm{GL}_2(\mathcal{O}_{F_{v'}^+})$-invariant lattice $\sigma^0(\lambda_v,\tau_v)$ in $\sigma(\tau_v)^\vee\otimes V(\lambda_v-\eta)$, which can be assumed to be a free $\mathcal{O}$-module. Then $(\sigma^0(\lambda_v,\tau_v))^d$ is a lattice inside $\sigma(\lambda_v,\tau_v)$. We define $\sigma^{\mathcal{S}} := \otimes_{v'\in S_p\setminus\{v\}}(\sigma^0(\lambda_{v'},\tau_{v'})\circ\iota_{v'}^{-1})$. We simply write σ^v if $\mathcal{S}=\{v\}$. We then obtain a patching functor

$$M_\infty^{\sigma^{\mathcal{S}}}: V\mapsto M_\infty(V\otimes(\sigma^{\mathcal{S}})^d),$$

which we also denote as M_∞ when there is no ambiguity.

In the process of constructing a patching functor, we patch together the space of modular forms and obtain $S_\infty=\mathcal{O}[\![y_1,\dots y_{4(|S_p|+1)},z_1,\dots z_{h+|F^+:\mathbb{Q}|}]\!]$ with $\mathfrak{a}_\infty$ generated by all the z_i,y_i [CEG+18, § 2.8]. For V a finitely generated $\mathcal{O}$-module with an action of $\mathrm{GL}_2(\mathcal{O}_{F_v^+})$, $M_\infty^{\sigma^v}(V)$ is a finitely generated module over S_∞. We can relate the patched modules, the spaces of completed cohomology, and the classical algebraic automorphic forms as follows.

Proposition 4.6. Let V, W be a finitely generated free $\mathcal{O}$-module with a continuous U_p action. Then,

$$\mathrm{Hom}_{\mathcal{O}[\![U_p]\!]}(V, \widetilde{S}(U^p, W)_{\mathfrak{m}'}) \cong S(U_p U^p, W \otimes V^d)_{\mathfrak{m}'}.$$

Now assume that V is a finitely generated free $\mathcal{O}$-module with a continuous U_v action, then

$$(M_\infty^{\sigma^v}(V)/a_\infty)^\vee \cong \mathrm{Hom}_{\mathcal{O}[\![U_v]\!]}(V, \widetilde{S}(U^v, W_\Sigma \otimes \sigma^v)_{\mathfrak{m}'}) \cong S(U, W_\Sigma \otimes \sigma^v \otimes V^d)_{\mathfrak{m}'}.$$

Proof. By [Emeo6, Proposition 3.2.4], we have

$$\mathrm{Hom}_{\mathcal{O}[\![U_p]\!]}(V, \widetilde{S}(U^p, W)_{\mathfrak{m}'})[\frac{1}{p}] = \mathrm{Hom}_{U_p}(V, \widetilde{S}(U^p, W)_{\mathfrak{m}'}^{\mathrm{l.alg}})[\frac{1}{p}] \cong S(U_p U^p, W \otimes V^d)_{\mathfrak{m}'}[\frac{1}{p}]$$

where l. alg denotes the subspace given by all the locally algebraic representations.

Moreover, by [Emeo6, Corollary 2.25], $\widetilde{S}(U^p, W)_{\mathfrak{m}'}$ is the same as $\widehat{S}(U^p, W)_{\mathfrak{m}'}$, the ϖ-adic completion of $S(U^p, W)_{\mathfrak{m}'}$. As each $S(U_p U^p, W)_{\mathfrak{m}'}$ is ϖ-torsion free for U sufficiently small, so is $\widetilde{S}(U^p W)_{\mathfrak{m}'}$.

Note that the spectral sequence in [Emeo6, Theorem 2.15] is general, and one can obtain a statement analogous to [Emeo6, Corollary 2.25] for $\widetilde{S}(U^v, W)$, and the second isomorphism in the second statement follows the same argument as the first statement.

Let V be a free $\mathcal{O}$-module with a U_p-action. By the construction of the patching functor ([LLHLM23, Appendix A]),

$$M_\infty(V) = \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![U_p]\!]}(M_\infty, V^\vee)^\vee.$$

By [LLHLM23, Equation A.4], we have

$$\begin{aligned} M_\infty(V)/\mathfrak{a}_\infty &\cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![U_p]\!]}(M_\infty/\mathfrak{a}_\infty, V^\vee)^\vee \\ &\cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![U_p]\!]}(\varprojlim_{K_p \subseteq U_p, n} S(K_p U^p, W_\Sigma/\varpi^n)^\vee_{\mathfrak{m}'}, V^\vee)^\vee. \end{aligned}$$

As $E/\mathcal{O}$ and W_Σ/ϖ^n are discrete, and any continuous map to discrete objects has an open kernel,

$$\begin{aligned} \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![U_p]\!]}(\varprojlim_{K_p \subseteq U_p, n} S(K_p U^p, W_\Sigma/\varpi^n)^\vee_{\mathfrak{m}'}, V^\vee) &\cong \varprojlim_n \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![U_p]\!]}((\varinjlim_{K_p \subseteq U_p} S(K_p U^p, W_\Sigma/\varpi^n)_{\mathfrak{m}'})^\vee, V^\vee) \\ &\cong \varprojlim_n \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![U_p]\!]}(V, \varinjlim_{K_p \subseteq U_p} S(K_p U^p, W_\Sigma/\varpi^n)_{\mathfrak{m}'}) \\ &\cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![U_p]\!]}(V, \varprojlim_n \varinjlim_{K_p \subseteq U_p} S(K_p U^p, W_\Sigma/\varpi^n)_{\mathfrak{m}'}), \end{aligned}$$

where the second isomorphism comes from the Pontrjagin duality and the third comes from the property of limit. Therefore,

$$M_\infty^{\sigma^v}(V)/\mathfrak{a}_\infty = \mathrm{Hom}_{\mathcal{O}[\![U_p]\!]}(V \otimes (\sigma^v)^d, \widetilde{S}(U^p, W_\Sigma))^\vee. \qquad \square$$

4.3 AUTOMORPHY LIFTING

Theorem 4.7. *Given a continuous Galois representation $r\colon G_F \to \mathrm{GL}_2(E)$ with the following properties:*

1. *$\bar{r}$ satisfies Properties 4.2;*
2. *r is unramified almost everywhere and satisfies $r^c = r^\vee \epsilon^{-1}$;*
3. *For all places $v \in S_p$, $r|_{G_{F_{\tilde{v}}}}$ is potentially crystalline with Hodge–Tate weights λ_v and with $2\ell_v$-generic tame inertial type τ_v, where $\ell_v = \max\{(\lambda_v)_{j,1} - (\lambda_v)_{j,2}\}$ and $4\ell_v \leq N_v$.*
4. *$\bar{r} \cong \bar{r}_\iota(\pi)$ for a regular conjugate self-dual cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ with infinitesimal character $\lambda - \eta$ such that $\otimes_{v \in S_p}\sigma(\tau_v)$ is a K-type for $\otimes_{S_p}\pi_v$, where $r_\iota(\pi)$ is the continuous representation attached to π by [BLGG13, Theorem 2.1.2].*

Then there exists a RACSDC representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $r \otimes_E \overline{\mathbb{Q}}_p \cong r_\iota(\pi)$.

Proof. By Corollary 3.30, we know that $R_{\bar{r}_{\tilde{v}}}^{\lambda_{\tilde{v}},\tau_{\tilde{v}}}$ is a domain. Therefore, the automorphy lifting result follows from applying the usual Taylor–Wiles method. $\square$

Remark 4.8. It is possible to prove the theorem with the results in the existing literature. As explained in Lemma 3.8, we know that Breuil–Mézard conjecture holds for GL_2 by [FH25, Theorem 1.3.1]. Then by [GK14, Lemma 4.3.9] (*cf.* [EG14, Lemma 5.5.1]), we deduce that the support of $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ meets every irreducible component of Spec $R^{loc}[1/p]$, and hence the automorphy lifting theorem holds.

5 BREUIL'S LATTICE CONJECTURE

5.1 STRUCTURE OF LATTICES

Recall that we define $\sigma(\lambda,\tau) := \sigma(\tau)\otimes V(\lambda-\eta)$ and write $\overline{\sigma}(\lambda,\tau)$ for the mod p reduction of a $\mathrm{GL}_2(\mathcal{O}_K)$-invariant $\mathcal{O}$-lattice inside $\sigma(\lambda,\tau)$. Replacing E with an extension, we will assume that $\sigma(\tau)$ is defined over E.

Lemma 5.1. *Given a Serre weight $\kappa\in\mathrm{JH}(\overline{\sigma}(\lambda,\tau))$, there exists a unique $\mathcal{O}$-lattice up to homothety in $\sigma(\lambda,\tau)$, which we denote as σ_κ, such that the cosocle of σ_κ is precisely $\kappa\in\mathrm{JH}(\overline{\sigma}(\lambda,\tau))$.*

Remark 5.2. The lower index notation agreed with the convention in [EGS15], but was opposite to [LLHLM20].

Proof. Since $\mathrm{JH}(\overline{\sigma}(\lambda,\tau)) = \mathrm{JH}(\overline{\sigma}(\tau)\otimes_{\mathbb{F},j} L(\lambda^{(j)}-\eta))$, by Equation (3.1), $\sigma(\lambda,\tau)$ is residual multiplicity free. To show that $\sigma(\lambda,\tau)$ is an irreducible representation, we prove by induction that if V is a smooth irreducible representation over $\mathrm{GL}_2(\mathcal{O}_K)$, and $V(\lambda)$ is an irreducible algebraic representation with the highest weight λ, then $V\otimes V(\lambda)$ is an irreducible representation of $\mathrm{GL}_2(\mathcal{O}_K)$. Since $\mathrm{GL}_2(\mathcal{O}_K)$ is an open subgroup, we can consider the representation of the corresponding Lie algebra $\mathfrak{g}$. The associated $V(\lambda)$ is an irreducible $\mathfrak{g}$-representation. As $\mathfrak{g}$ acts trivially on V, $V\otimes V(\lambda)\cong\oplus_{i-1}^{n}V(\lambda)$ as $\mathfrak{g}$-representation. Then, any $\mathfrak{g}$-subrepresentation W of $V\otimes V(\lambda)$ is of the form $V'\otimes V(\lambda)$, where V' is a subspace of V. Then $V' = \mathrm{Hom}_{\mathfrak{g}-mod}(V(\lambda),W)$, and it naturally has a K-action; therefore, it is a K-representation. However, this implies $V'=0$ or V, as V is irreducible. Therefore, the result follows from [EGS15, Lemma 4.1.1] □

Lemma 5.3. *Let $\ell_j := \lambda_{j,1}-\lambda_{j,2}$ and $\ell := \max_j\{\ell_j\}$. Further assume that τ is 3ℓ-generic. Then for all $\kappa\in\mathrm{JH}(\overline{\sigma}(\lambda,\tau))$, $\overline{\sigma}_\kappa$ is $\mathfrak{m}_{K_1}^{\ell}$-torsion.*

Proof. When $n\ell=1$, it follows from the definition, hence we can assume $\ell\geq 2$. For $\sigma\in\mathrm{JH}(\overline{\sigma}(\tau))$, as τ is 3ℓ-generic, by Proposition 3.1, σ is a 3ℓ-generic Serre weight. By [BP12, Lemma 3.2] $\mathrm{JH}(\mathrm{Proj}_1\,\sigma)$ is given by points in the hypercube of length 3 with centre σ in the extension graph. By Lemma 2.5, we know that $\mathrm{JH}(\mathrm{Proj}_n\,\sigma)$ is given recursively by adding two points in all directions to the ones from $\mathrm{JH}(\mathrm{Proj}_{n-1}\,\sigma)$ in the extension graph. On the other hand, by Equation (3.1) and Proposition 3.1, $\mathrm{JH}(\overline{\sigma}(\lambda,\tau))$ is given by the points in a hypercuboid of length $2\ell+1$ in the extension graph. Therefore, we can deduce that $\mathrm{JH}(\overline{\sigma}_\kappa) \subseteq \mathrm{JH}(\mathrm{Proj}_n \kappa)$. Since τ is 3ℓ-generic, and $3\ell \geq 2\ell + 2$, for all $\kappa \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$, κ is 2ℓ-generic by Lemma 3.2. By Corollary 2.31, we deduce that $\overline{\sigma}_\kappa$ is $\mathfrak{m}_{K_1}^{\ell}$-torsion. □

Under our genericity conditions, by Lemma 3.9, $R_{\overline{r}_v}^{\lambda,\tau} = 0$ if and only if $\mathrm{JH}(\overline{\sigma}(\lambda, \tau)) \cap W(\overline{r}_v) = \emptyset$. Without loss of generality, we assume this is the case and fix $\kappa_\circ$ such that $\kappa_\circ \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau)) \cap W(\overline{r}_v)$. For $\kappa \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$, $\kappa \neq \kappa_\circ$, we fix a lattice σ_κ of $\sigma(\lambda, \tau)$ such that σ_κ has cosocle κ and we have a saturated inclusion $\sigma_{\kappa_\circ} \hookrightarrow \sigma_\kappa$. We write $\overline{\sigma}_\kappa$ for its reduction modulo p. Given a lattice σ°, we define $\epsilon_\kappa(\sigma^\circ)$ to be the minimum integer such that $p^{\epsilon_\kappa(\sigma^\circ)}\sigma_\kappa \hookrightarrow \sigma^\circ$ is saturated. We can therefore reinterpret and generalize the result of [EGS15, § 5.2.2] using Corollary 2.13, and obtain the following lemma:

Lemma 5.4. *Assume $\ell = \max_j\{\lambda_{j,1} - \lambda_{j,2}\}$ and τ is 3ℓ-generic. Given Serre weights $\delta, \kappa, \kappa' \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$, then*

$$\epsilon_\delta(\sigma_\kappa) + \epsilon_\kappa(\sigma_{\kappa'}) \geq \epsilon_\delta(\sigma_{\kappa'}),$$

with equality if and only if $\overline{\sigma}_{\kappa'}$ contains a subquotient with socle δ and cosocle κ, which is equivalent to $\kappa - \delta \leq \kappa' - \delta$ (Definition 2.7).

Proof. The inequality follows from the definition of ϵ_δ. We can compare the inclusion

$$p^{\epsilon_\delta(\sigma_{\kappa'})}\sigma_\delta \subseteq \sigma_{\kappa'}, \qquad p^{\epsilon_\delta(\sigma_\kappa)+\epsilon_\kappa(\sigma_{\kappa'})}\sigma_\delta \subseteq p^{\epsilon_\kappa(\sigma_{\kappa'})}\sigma_\kappa \subseteq \sigma_{\kappa'}.$$

Therefore, $\overline{\sigma}_{\kappa'}$ contains a subquotient with socle δ and cosocle κ if and only if

$$\epsilon_\delta(\sigma_\kappa) + \epsilon_\kappa(\sigma_{\kappa'}) < \epsilon_\delta(\sigma_{\kappa'}) + 1.$$

As $\overline{\sigma}_\kappa$ is $\mathfrak{m}_{K_1}^{\ell}$-torsion by Lemma 5.3, we can apply the results from section 2. In particular, recall from Theorem 2.10, $I(\delta, \kappa')$ is the unique multiplicity-free representation with socle δ and cosocle κ'. Moreover, $\overline{\sigma}_{\kappa'}$ contains a subquotient with socle δ and cosocle κ if and only if κ is a subquotient of $I(\delta, \kappa')$. By Corollary 2.13, this is equivalent to $\kappa - \delta \leq \kappa' - \delta$. □

Remark 5.5. We can give the description of the socle of $\overline{\sigma}_\kappa$ as follows. By Equation (3.1), $\mathrm{JH}(\overline{\sigma}(\lambda, \tau))$ is given by a hypercuboid in the extension graph. Using Corollary 2.13, the socle of $\overline{\sigma}_\kappa$ is the sum of the corners in the hypercuboid, which are different from κ in all f dimensions. For instance, if κ is at the corner of the hypercuboid, then the socle of $\overline{\sigma}_\kappa$ is the opposite corner. In general, the socle is not irreducible.

We have the following proposition analogous to [LLHLM20, Theorem 4.1.9]

Proposition 5.6. Assume $\max_j\{\lambda_{j,1} - \lambda_{j,2}\} = \ell$ and τ is 3ℓ-generic. Given Serre weights $\kappa, \kappa' \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$ such that κ and κ' are distance one apart in the extension graph. Assume that $\sigma_\kappa \hookrightarrow \sigma_{\kappa'}$ is saturated (i.e., $\epsilon_\kappa(\sigma_{\kappa'}) = 0$), then $\epsilon_{\kappa'}(\sigma_\kappa) = 1$. Moreover, for any $\delta \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$, we have $\delta \in \mathrm{JH}(\mathrm{Coker}(\sigma_\kappa \hookrightarrow \sigma_{\kappa'}))$ if and only if $\kappa' - \kappa \leq \delta - \kappa$. Conversely, $\delta \in \mathrm{JH}(\mathrm{Coker}(p\sigma_{\kappa'} \hookrightarrow \sigma_\kappa))$ if and only if $\kappa - \kappa' \leq \delta - \kappa'$.

Proof. We follow the argument of [LLHLM20, Proposition 4.3.7]. As τ is 3ℓ-generic, all $\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$ is 2ℓ-generic. By Proposition 3.4, we can find a (non-semisimple unless $f=1$) local Galois representation $\overline{\rho}$ such that $W(\overline{\rho}) = \{\kappa,\kappa'\}$. By [GK14, Theorem A.4], we can find an imaginary CM field F with a maximal real subfield F^+ and an RACSDC automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\overline{r}_{p,\iota}(\pi)$ satisfies Properties 4.2 and for each place $v|p$ in F^+, there is a place $\widetilde{v}$ of F lying over v such that $\overline{r}_{p,\iota}(\pi)|_{G_{F_{\widetilde{v}}}}$ is isomorphic to an unramified twist of $\overline{\rho}$. Then, we can obtain a patching functor M'_∞ for $\mathcal{S} = \{v\}$ as in Proposition 4.5. By the axiom of the patching functor, $M'_\infty(\sigma_\kappa), M'_\infty(\sigma_{\kappa'})$ are p-torsion free and maximal Cohen–Macaulay. Similarly, $M'_\infty(\kappa), M'_\infty(\kappa')$ are maximal Cohen–Macaulay over $\overline{R}'_\infty(\kappa)$ and $\overline{R}'_\infty(\kappa')$ respectively.

By Lemma 3.8, the mod ϖ-fibre of the deformation space $\overline{X}'_\infty(\lambda,\tau)$ is the union of the ϖ-fibres $\overline{X}'_\infty(\kappa)$ and $\overline{X}'_\infty(\kappa')$. Recall that $R^{\overline{\kappa}}_{\overline{\rho}}$ is reduced, p-torsion free quotient of $R^{\square}_{\overline{\rho}}$ corresponding to the crystalline deformation of Hodge type $\overline{\kappa}$. By Proposition 3.1 (*cf.* [EGS15, Proposition 3.5.2]), we can find another tame type τ' such that $\mathrm{JH}(\overline{\sigma}(\tau') \cap W(\overline{\rho}) = \{\kappa,\kappa'\}$. By [EGS15, Theorem 7.2.1], we have that $R^{\tau'}_{\overline{\rho}}$ is isomorphic to

$$\mathcal{O}[\![x,y,z_1,z_2]\!]/(xy-p).$$

Again by Lemma 3.8, the mod ϖ-fibre of the deformation space $\overline{X}'_\infty(\tau')$ is the union of the ϖ-fibres $\overline{X}'_\infty(\kappa)$ and $\overline{X}'_\infty(\kappa')$. By Lemma 3.8, the quotient map $R^{\square}_{\overline{\rho}} \twoheadrightarrow R^{\kappa}_{\overline{\rho}}$ factors through $R^{\tau'}_{\overline{\rho}}$. If we let $\mathfrak{p}(\kappa) = \ker(R^{\tau'}_{\overline{\rho}} \twoheadrightarrow R^{\kappa}_{\overline{\rho}})$, then $\mathfrak{p}(\kappa) = (x)$ or (y). Without loss of generality, we can assume $\mathfrak{p}(\kappa) = (x)$, $\mathfrak{p}(\kappa') = (y)$. We fix a chain of saturated inclusions of lattices $p^k\sigma_{\kappa'} \subseteq \sigma_\kappa \subseteq \sigma_{\kappa'}$.

Fixing an isomorphism $R'_\infty(\tau') \cong \mathcal{O}[\![x,y,z_1,\dots z_k]\!]/(xy-p)$, we let $S \subseteq R'_\infty(\tau')$ be the sub-ring $\mathbb{Z}_p[\![x,y,z_1,\dots z_k]\!]/(xy-p)$. Then we let $\mathfrak{p}_s(\kappa) \subseteq S$ be the preimage of $\mathfrak{p}(\kappa)R'_\infty(\tau')$; hence $\mathfrak{p}_S(\kappa) = (x)$. If M is a maximal Cohen–Macaulay $R'_\infty(\tau')$ module, it is also maximal Cohen–Macaulay over S. Let $C := \mathrm{Coker}(p^k\sigma_{\kappa'} + p\sigma_\kappa \hookrightarrow \sigma_\kappa)$. As C is annihilated by p, the scheme theoretic support of C in $\mathrm{Spec}\, S$ is contained in $\mathrm{Spec}\, S/pS$ and hence generically reduced. Moreover, as $\sigma(\lambda,\tau)$ is residually multiplicity free, C does not contain κ' as a Jordan–Hölder factor (can be seen by descent to unramified coefficients). Using maximal Cohen–Macaulay property as explained in [LLHLM20, Lemma 3.6.2], we conclude that $\mathrm{Supp}_S\, M'_\infty(C) = \mathrm{Supp}_S\, M'_\infty(\kappa) = \mathrm{Spec}(S/(x))$.

Therefore, x annihilates $M'_\infty(C)$, and we have

$$xM'_\infty(\sigma_\kappa) \subseteq M'_\infty(p^k\sigma_{\kappa'} + p\sigma_\kappa).$$

Multiplying both sides by y and noticing that $xy = p$, we then divide both sides by p and obtain

$$M'_\infty(\sigma_\kappa) \subseteq yM'_\infty(p^{k-1}\sigma_{\kappa'} + \sigma_\kappa).$$

Assume for the sake of contradiction that $k > 1$, we consider the image under the composition of the reduction modulo ϖ map and the projection map $M'_\infty(\sigma_\kappa) \twoheadrightarrow M'_\infty(\kappa)$, and deduce that $M'_\infty(\kappa)$ is killed by x. However, $M'_\infty(\kappa)$ is a free module over R'^{κ}_∞, which is a power series ring. We have a contradiction.

We now determine $\mathrm{JH}(\mathrm{Coker}(\sigma_\kappa \hookrightarrow \sigma_{\kappa'}))$. Assume $\delta \in \mathrm{JH}(\mathrm{Coker}(\sigma_\kappa \hookrightarrow \sigma_{\kappa'}))$. Similar to the proof of [EGS15, Theorem 5.2.4], we compare the two inclusions:

$$p^{\epsilon_\delta(\sigma_\kappa)}\sigma_\delta \subseteq \sigma_\kappa \subseteq \sigma_{\kappa'}, \qquad p^{\epsilon_\delta(\sigma_{\kappa'})}\sigma_\delta \subseteq \sigma_{\kappa'}.$$

We see that $\delta \in \mathrm{JH}(\mathrm{Coker}(\sigma_\kappa \hookrightarrow \sigma_{\kappa'}))$ if and only if $\epsilon_\delta(\sigma_\kappa) > \epsilon_\delta(\sigma_{\kappa'})$, which is equivalent to not having $\kappa - \delta \leq \kappa' - \delta$ by Lemma 5.4. This is only possible if $\kappa' - \delta \leq \kappa - \delta$, as κ' and κ are distance one apart. This is also equivalent to $\kappa' - \kappa \leq \delta - \kappa$, by the remark in Definition 2.7. The last statement follows analogously from comparing the two inclusions:

$$p^{\epsilon_\delta(\sigma_\kappa)}\sigma_\delta \subseteq \sigma_\kappa, \qquad p^{\epsilon_\delta(\sigma_{\kappa'})+1}\sigma_\delta \subseteq p\sigma_{\kappa'} \subseteq \sigma_\kappa.$$ □

5.2 RELATIONS OF PATCHED MODULES OF LATTICES

We will now apply Corollary 3.30 to r_v. In particular, from now on, we will assume that, for all v, up to twisting by a power of ω_f, $\overline{r}_v$ is of the form in Equation (3.2) and 4ℓ-generic. Moreover, we assume that r_v is potentially crystalline with Hodge–Tate weights λ with $\ell := \lambda_{j,1} - \lambda_{j,2} > 0$ and τ_v is a 2ℓ-generic inertial type, where $\ell = \max_j\{\ell_v\}$.

Fix $\kappa_\circ \in W(\overline{r}) \cap \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$, by Lemma 3.8, we have $R^{\lambda,\tau}_{\overline{r}} \twoheadrightarrow \overline{R}^{\kappa_\circ}_{\overline{r}}$, and the kernel $\mathfrak{p}(\kappa_\circ)$ is given by $(z(\kappa_\circ)_j)_{j\in\mathcal{K}_v}$ in Corollary 3.30.

Definition 5.7. We define (*cf.* Proposition 3.4)

$$\widetilde{s}_j = \begin{cases} \mathrm{sgn}(s_j) & \text{if } j \in \mathcal{K}_v \text{ (i.e., } F(\mathfrak{t}_{\mu-\eta}(0,\ldots,\mathrm{sgn}(s_j),0)) \in W(\overline{r}_v)); \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we define $\widetilde{z}(\kappa_\circ)_j = p/z(\kappa_\circ)_j$. (That is $\widetilde{z}(\kappa_\circ)_j = y_j$ if $z(\kappa_\circ)_j = x_j$, and vice versa.)

Given a patching functor M_∞ be $M_\infty^{\sigma^v}$ constructed in Chapter 4. Given Serre weights $\kappa, \kappa' \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$ with $\kappa = F(\mathfrak{t}_{\mu-\eta}(\alpha))$ and $\kappa' = F(\mathfrak{t}_{\mu-\eta}(\alpha'))$ are distance 1 apart in the extension graph, with $\alpha_i = \alpha'_i$ for all $i \neq j$. Assume further that $\sigma_\kappa \hookrightarrow \sigma_{\kappa'}$ is saturated. Define $\varpi(\kappa,\kappa') \in R^{\lambda,\tau}_{\bar{r}_v}$ as follows,

$$\varpi(\kappa,\kappa') := \begin{cases} 1 & \text{if } ((\alpha')_j < (\alpha)_j \le \min\{0,(\widetilde{s})_j\}) \text{ or } (\max\{0,(\widetilde{s})_j\} \le (\alpha)_j < (\alpha')_j); \\ y_j & \text{if } (\alpha)_j = 0 \text{ and } (\alpha')_j = (\widetilde{s})_j \neq 0; \\ x_j & \text{if } \alpha = (\widetilde{s})_j \neq 0 \text{ and } (\alpha')_j = 0; \\ p & \text{if } ((\alpha)_j < (\alpha')_j \le \min\{0,(\widetilde{s})_j\}) \text{ or } (\max\{0,(\widetilde{s})_j\} \le (\alpha')_j < (\alpha)_j). \end{cases} \tag{5.1}$$

We define $\varpi'(\kappa,\kappa')$ analogously by swapping 1 with p and x_j with y_j.

Remark 5.8. The first condition should be understood as that κ' is "further away from the sets of modular Serre weights" than κ.

Proposition 5.9. We have equalities,

$$\varpi(\kappa,\kappa')M_\infty(\sigma_{\kappa'}) = M_\infty(\sigma_\kappa);$$

$$\varpi'(\kappa,\kappa')M_\infty(\sigma_\kappa) = pM_\infty(\sigma_{\kappa'}).$$

Proof. The proof goes the same way as in [EGS15, Proposition 8.1.1]. We deduce from Proposition 5.6 that the cokernel of $\sigma_\kappa \hookrightarrow \sigma_{\kappa'}$ is a successive extension of the weights $\delta \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$ where $\delta = F(\mathfrak{t}_{\mu-\eta}(\beta))$ with $\alpha' - \alpha \le \beta - \alpha$.

By Lemma 5.4, we deduce that this cokernel is annihilated by p. Then the cokernel of $M_\infty(\sigma_\kappa) \hookrightarrow M_\infty(\sigma_{\kappa'})$ is given by a successive extension of patched modules $M_\infty(\delta)$ and is annihilated by p. Similar to the proof of Lemma 5.4, using Corollary 3.30, we can fix an isomorphism

$$R_\infty(\lambda,\tau) \cong \mathcal{O}[\![(x_j,y_j)_{j\in\mathcal{K}}, z_1,\dots z_k]\!]/(x_jy_j - p)_{j\in\mathcal{K}}$$

for some $\mathcal{K}$. We let $S \subseteq R_\infty(\lambda,\tau)$ be the sub-ring $\mathbb{Z}_p[\![(x_j,y_j)_{j\in\mathcal{K}}, z_1,\dots z_k]\!]/(x_jy_j - p)_{j\in\mathcal{K}}$. Let $\mathfrak{p}_S(\sigma) = \sum_v \mathfrak{p}_S(\sigma_v)$ where $\mathfrak{p}_S(\sigma_v) = (z_j(\sigma_v))_{j\in\mathcal{K}_v}$. Then $\mathrm{Supp}_S\, M_\infty(\sigma)$ is annihilated by $\mathfrak{p}_S(\sigma)$.

Therefore, the scheme-theoretical support of $M_\infty(\sigma_{\kappa'}/\sigma_\kappa)$ in $\mathrm{Spec}\, S$ is contained in $\mathrm{Spec}\, S/pS$. Therefore, it is generically reduced, and the underlying reduced subscheme is equal to

$$\bigcup_{\substack{\delta\in W(\bar{r}_v,\lambda,\tau) \\ \delta=F(\mathfrak{t}_{\mu-\eta}(\beta)):\, \alpha'-\alpha\le\beta-\alpha}} \mathrm{Supp}_S\, M_\infty(\delta).$$

In the first case of Equation (5.1), we see that the scheme-theoretic support of the cokernel in S is trivial. In the second case of Equation (5.1), we must have $\delta = F(\mathfrak{t}_{\mu-\eta}(\beta))$ with $\beta_j = \mathrm{sgn}(s_j)$, and hence the cokernel is annihilated by y_j, by Corollary 3.30. Analogously, in the third case of Equation (5.1), the cokernel is annihilated by x_j. Finally, in the fourth case, by Proposition 5.6, the cokernel is annihilated by p. Therefore,

$$\varpi(\kappa,\kappa')M_\infty(\sigma_{\kappa'}) \subseteq M_\infty(\sigma_\kappa). \tag{5.2}$$

Similarly, the cokernel of $pM_\infty(\sigma_{\kappa'}) \hookrightarrow M_\infty(\sigma_\kappa)$ is annihilated by $\varpi'(\kappa,\kappa')$, and hence

$$\varpi'(\kappa,\kappa')M_\infty(\sigma_\kappa) \subseteq pM_\infty(\sigma_{\kappa'}). \tag{5.3}$$

Note that $\varpi(\kappa,\kappa')\varpi'(\kappa,\kappa') = p$. Multiplying Equation (5.2) by $\varpi'(\kappa,\kappa')$, we obtain that

$$pM_\infty(\sigma_{\kappa'}) \subseteq \varpi'(\kappa,\kappa')M_\infty(\sigma_\kappa).$$

Combining with Equation (5.3), we deduce that this is indeed an equality. As $M_\infty(\sigma_\kappa), M_\infty(\sigma_{\kappa'})$ are p-torsion free, the second equality is obtained by multiplying by $\varpi(\kappa,\kappa')$ and dividing by p. □

Assume $\kappa_\circ = F(\mathfrak{t}_{\mu-\eta}(b)) \in W(\bar{r}_v,\lambda,\tau)$, where $b_j \in \{0, \mathrm{sgn}(s_j)\}$ is given by Proposition 3.4 applying to $W(\bar{r}_v)$. For $\kappa \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$ if $\kappa = F(\mathfrak{t}_{\mu-\eta}(\alpha))$, define

$$\varpi_j(\kappa) := \begin{cases} 1 & \text{if } \alpha_j < b_j = \min(0,\widetilde{s}_j) \text{ or } \alpha_j > b_j = \max\{0,\widetilde{s}_j\}; \\ x_j & \text{if } b_j = \widetilde{s}_j < 0 \le \alpha_j \ \text{ or } b_j = \widetilde{s}_j > 0 \ge \alpha_j; \\ y_j & \text{if } b_j = 0 < \widetilde{s}_j \le \alpha_j \ \text{ or } b_j = 0 > \widetilde{s}_j \ge \alpha_j \end{cases}$$

We define $\varpi(\kappa) := \prod_{1\le j\le f_v} \varpi_j(\kappa) \in R^{\lambda,\tau}_{\bar{r}_v}$.

Proposition 5.10. For $\kappa \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$, we have an equality:

$$M_\infty(\sigma_{\kappa_\circ}) = \varpi(\kappa)M_\infty(\sigma_\kappa).$$

Remark 5.11. The last two conditions capture the case for which there is a modular Serre weight between κ and $\kappa_\circ$ when we consider the projection to the jth coordinate of the extension graph.

Proof. We prove by induction on the distance between κ and $\kappa_\circ$ in the extension graph. Using the extension graph, we fix a sequence $\kappa_0 = \kappa_\circ, \cdots, \kappa_f = \kappa$, such that κ_{j-1} and κ_j differ only in the jth direction of the extension graph. In particular, if $\kappa_{j-1} = F(\mathfrak{t}_\mu(\alpha^{j-1}))$, then $\alpha^{j-1}_j = b_j$, since we did not change the jth coordinate in the first $j-1$ steps. Moreover, $\varpi(\kappa) = \prod_{j=1}^{f} \varpi(\kappa_j,\kappa_{j-1})$. Therefore, it suffices to show that $\varpi_j(\kappa) = \varpi(\kappa_{j-1},\kappa_j)$. If the distance between κ_j and κ_{j-1} is 1, then by Proposition 5.9, $\varpi_j(\kappa) = \varpi(\kappa_{j-1},\kappa_j)$.

Assume that it holds for distance $n-1 \geq 1$, then let κ'_j be the Serre weight on the line segment in the extension graph between κ_{j-1} and κ_j which is distance 1 away from κ_j and $n-1$ away from κ_{j-1}. By the induction hypothesis $\varpi_j(\kappa') = \varpi(\kappa_{j-1}, \kappa'_j)$. Moreover, as $\kappa' - \kappa_\circ \leq \kappa - \kappa_\circ$, by Lemma 5.4, we deduce that $\epsilon_{\kappa'}(\sigma_\kappa) = 0$, that is, $\sigma_{\kappa'} \hookrightarrow \sigma_\kappa$. As $n \geq 2$, the jth coordinate of κ_j is not in $\{0, \widetilde{s}_j\}$, and κ_j is further away than κ'_j from 0 in the jth coordinate, we are in the first case of Equation (5.1), by Proposition 5.9, $\varpi(\kappa'_j, \kappa_j) = 1$. Therefore, $\varpi_j(\kappa) = \varpi(\kappa_{j-1}, \kappa'_j)\varpi(\kappa'_j, \kappa_j) = \varpi(\kappa_{j-1}, \kappa_j)$. □

5.3 BREUIL'S LATTICE CONJECTURE

Suppose $r\colon G_F \to \mathrm{GL}_2(E)$ is a Galois representation attached to an eigenform in $S(U, W)_{\mathfrak{m}}$ We assume that $\bar{r}$ satisfies Properties 4.2, minimally ramified only at primes $v \nmid p$. In particular, $\bar{r}$ corresponds to $\mathfrak{m} \subseteq \mathbb{T}_{\mathcal{P}}$, which is non-Eisenstein. As $\bar{r}$ is absolutely irreducible, we can conjugate r such that it takes value in $\mathcal{O}$. We assume that for $v \in S_p$, r_v is potentially crystalline with regular Hodge-Tate weights λ_v and with inertial type τ_v. Fix a $v \in S_p$, and write λ for λ_v and τ for τ_v. We further assume that $\max_j\{\lambda_{j,1} - \lambda_{j,2}\} = \ell$ and τ is 4ℓ-generic. We fix σ^v as in Proposition 5.10. Let $\mathbb{T}'(U, W)_{\mathfrak{m}'}$ be the image of the universal Hecke algebra $\mathbb{T}'_{\mathcal{P}}$ in $\mathrm{End}(S(U, W)_{\mathfrak{m}'})$.We write $\mathfrak{p}$ for the kernel of the system of Hecke eigenvalues $\alpha\colon \mathbb{T}'(U, W)_{\mathfrak{m}'} \to E$ associated to r, i.e.; α satisfies

$$\det(r^\vee(\mathrm{Frob}_w)X) = \sum_{j=0}^{2} (-1)^j (\mathbf{N}_{F/\mathbb{Q}}(w))^{\binom{j}{2}} \alpha(T_w^{(j)}) X^j$$

for all $w \in \mathcal{P}$.

Proposition 5.12. We have the following isomorphism,

$$(\widetilde{S}(U^v, \sigma^v \otimes_{\mathcal{O}} W_\Sigma)_{\mathfrak{m}'} \otimes_{\mathcal{O}} E)^{\mathrm{l.alg}}[\mathfrak{p}] = \pi_v \otimes V(\lambda - \eta),$$

where π_v corresponds to the Weil–Deligne representation associated to r_v by the local Langlands correspondence. (More precisely, $\mathrm{WD}(r_v)^{F-ss} = \mathrm{rec}(\pi_v \otimes |\det|^{\frac{-1}{2}})$.)

Proof. We recall the proof from [CEG+18, Theorem 4.35]. By [Eme06, Proposition 3.2.4], we deduce that the locally algebraic vectors are precisely the algebraic automorphic forms of that weight. Together with the classical local-global compatibility result [Car14, Theorem 1.1], we deduce that $\pi_v \otimes V(\lambda - \eta)$ appears in $(\widetilde{S}(U^v, \sigma^v \otimes_{\mathcal{O}} W_\Sigma)_{\mathfrak{m}'} \otimes_{\mathcal{O}} E)^{\mathrm{l.alg}}[\mathfrak{p}]$ with some multiplicity. By our construction and [Lab11, Théorèmes 5.4, 5.9], we deduce that it appears with multiplicity one. □

As $\sigma(\tau)$ determines the Berstein centre of π_v, we have

$$\sigma(\tau) \otimes V(\lambda-\eta) \hookrightarrow \widetilde{S}(U^v, \sigma^v \otimes_{\mathcal{O}} W_\Sigma)_{\mathfrak{m}'}[\mathfrak{p}] \otimes_{\mathcal{O}} E.$$

Therefore, the completed cohomology with integral coefficient determines the following $\mathrm{GL}_2(\mathcal{O}_{F_v^+})$-invariant $\mathcal{O}$-lattice inside $\sigma(\lambda,\tau)$,

$$\sigma^\circ(\lambda,\tau) := \sigma(\lambda,\tau) \cap \widetilde{S}(U^v, \sigma^v \otimes_{\mathcal{O}} W_\Sigma)_{\mathfrak{m}'}[\mathfrak{p}].$$

Let R_Σ^{univ} be the universal deformation ring for the deformations of $\bar{r}$ which are unramified outside Σ. As r is a Galois representation attached to an eigenform $S(U,W)_{\mathfrak{m}'}$, we have a Galois representation $r^{\mathrm{mod}}\colon G_F \to \mathrm{GL}_2(\mathbb{T}'(U,W)_{\mathfrak{m}'})$, with $\bar{r}^{\mathrm{mod}} \cong \bar{r}$. This induces a map $R_\Sigma^{\mathrm{univ}} \to \mathbb{T}'(U,W)_{\mathfrak{m}'}$. The composite map $R_\infty \to R_\Sigma^{\mathrm{univ}} \to \mathbb{T}'(U,W)_{\mathfrak{m}'}$, by the local-global compatibility, further induces a map $h\colon R_\infty(\lambda,\tau) \to \mathbb{T}'(U,W)_{\mathfrak{m}'}$. We define $\varpi_{\mathfrak{p}}(\kappa)$ as the image of $\varpi(\kappa)$ under $\mathfrak{p}\circ h$. As $\varpi(\kappa) \in R_{\bar{r}_v}^{\lambda,\tau}$, the image of h coincides with the image of the natural map $R_{\bar{r}_v}^{\lambda,\tau} \to R_\Sigma^{\mathrm{univ}} \to \mathbb{T}'(U,W)_{\mathfrak{m}'}$. Therefore, $\varpi_{\mathfrak{p}}(\kappa)$ only depends on r_v. We have the following version of Breuil's lattice conjecture:

Theorem 5.13. *Up to homothety, $\sigma^\circ(\lambda,\tau)$ is equal to*

$$\sum_{\kappa\in \mathrm{JH}(\lambda,\tau)} \varpi_{\mathfrak{p}}(\kappa)\sigma_\kappa.$$

Proof. We follow the proof of [EGS15, Theorem 8.2.1]. Since we only consider the lattice up to homothety, without loss of generality, we assume that $\sigma_{\kappa_\circ} \hookrightarrow \sigma^\circ(\lambda,\tau)$ is saturated. By our normalization before Lemma 5.3, we have $\sigma_{\kappa_\circ} \hookrightarrow \sigma_\kappa$. Therefore, we can apply [EGS15, Proposition 4.1.4], and deduce that $\sigma^\circ(\lambda,\tau) = \sum_{\kappa\in\mathrm{JH}(\lambda,\tau)} p^{v(\kappa)}\sigma_\kappa$ for some $p^{v(\kappa)}$ an element in $\mathcal{O}$ with valuation $v(\kappa)$ such that $p^{v(\kappa)}\sigma_\kappa \hookrightarrow \sigma^\circ(\lambda,\tau)$ is saturated. By Proposition 4.6 (cf. the proof of [LLHLM20, Theorem 5.3.5]), we have

$$(\mathrm{Hom}_{U_v}(\sigma_\kappa, \sigma^\circ(\lambda,\tau)))^\vee = (M_\infty(\sigma_\kappa)/\mathfrak{a}_\infty)/\mathfrak{p}.$$

By Proposition 5.10, we deduce that

$$\mathrm{Hom}_{U_v}(\sigma_\kappa, \sigma^\circ(\lambda,\tau)) = \varpi_{\mathfrak{p}}(\kappa)\,\mathrm{Hom}_{U_v}(\sigma_{\kappa_\circ}, \sigma^\circ(\lambda,\tau)).$$

Therefore, by the uniqueness of the gauges [EGS15, Proposition 4.1.4], we show that $\varpi_{\mathfrak{p}}(\kappa)$ has the same valuation as $p^{v(\kappa)}$ and finish the proof. □

Remark 5.14. One should be able to prove an analogous result for Shimura curves for higher parallel Hodge–Tate weight under the same genericity condition. (The requirement of parallel Hodge–Tate weights comes from the parity condition on quaternion algebras, which does not exist for unitary groups.) One can construct a minimal unramified patching M_∞ using quaternion algebras following [EGS15, § 6.2]. Since the proof relies on Proposition 5.10 which compares $M_\infty(\sigma_\kappa)$ and $M_\infty(\sigma_{\kappa'})$, which, in turn, relies on the result on the Galois deformation rings in Corollary 3.30, and results on the mod p representations of $\mathrm{GL}_2(\mathcal{O}_K)$ in Theorem 2.10, it is independent of the construction of the patching functor.

Remark 5.15. Breuil's original lattice conjecture in [Bre14, Conjecture 1.2] is stated for $\sigma^\circ(\tau) := \sigma(\tau) \cap \varinjlim_{U_v} H^1(U_v U^v, \mathcal{O})_{\mathfrak{m}}$. By [Eme06, Corollary 2.2.25], the ϖ-adic completion of $\varinjlim_{U_v} H^1(U_v U^v, \mathcal{O})$ is the same as $\widetilde{H}^1(U^v, \mathcal{O})$. Therefore, $\sigma^\circ(\tau) = \sigma(\tau) \cap \widetilde{H}^1(U^v, \mathcal{O})$. Hence, the formulation here is the natural analog for higher Hodge–Tate weights.

Remark 5.16. If we compare Theorem 5.13 with [EGS15, Theorem 8.2], $\sigma_{\kappa_\circ}$ plays a similar role as $\sigma_{\imath(\varnothing)}$ in [EGS15, Proposition 8.1.1]. We do not claim that X_j, Y_j coincide with the ones in [EGS15, Theorem 7.1.1], as we have taken a different normalization, and the Galois deformation ring is computed by strongly divisible modules in [EGS15] and by Breuil–Kisin modules here.

6 CYCLICITY OF PATCHED MODULES

In this chapter, we assume that the minimal patching functor is minimal and unramified. Let $\mathcal{S}$ be a subset of S_p. If there is a CM field F with maximal real subfield F^+ with $\bar{r}\colon G_F \to \mathrm{GL}_2(\mathbb{F})$ satisfying Properties 4.2, such that $L_v = F_v$ and $\overline{\rho}_v = \bar{r}|_{G_{F^+_{\bar{v}}}}$ for all $v \in \mathcal{S}$. Then by Proposition 4.5 and the discussion that follows, there exists a minimal patching functor for $\{\overline{\rho}_v\}_{v\in\mathcal{S}}$.

By extending the coefficients, we can assume that the lattice inside $\sigma(\lambda_v, \tau_v)$ is defined over $\mathcal{O} = W(\mathbb{F})$. As before, we let $\ell_v = \max_j\{(\lambda_v)_{j,1} - (\lambda_v)_{j,2}\}$ and assume that τ_v is $3\ell_v$-generic. In particular, all the Jordan–Hölder factors of $\overline{\sigma}(\lambda_v, \tau_v)$ is $3\ell_v$ generic. We write $\sigma(\lambda_{\mathcal{S}}, \tau_{\mathcal{S}}) := \otimes_{v\in\mathcal{S}}\sigma(\lambda_v, \tau_v)$ and write $\overline{\sigma}(\lambda_{\mathcal{S}}, \tau_{\mathcal{S}})$ for the mod p reduction of a $\prod_{v\in\mathcal{S}} \mathrm{GL}_2(\mathcal{O}_{F^+_{\bar{v}}})$-invariant $\mathcal{O}$-lattice inside $\sigma(\lambda_{\mathcal{S}}, \tau_{\mathcal{S}})$.

Note that if $\kappa \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$, as κ is an irreducible representation of the group $\prod_{v\in S_p} \mathrm{GL}_2(k_v)$, $\kappa = \otimes_{v\in S_p}\kappa_v$ where κ_v is a Serre weight for $\mathrm{GL}_2(k_v)$. It follows that $\kappa_v \in \mathrm{JH}(\overline{\sigma}(\lambda_v, \tau_v))$. Conversely, a tensor product of irreducible representations is irreducible as a representation of the product group. Therefore, $\kappa \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$ if and only if $\kappa = \otimes_{v\in S_p}\kappa_v$ where $\kappa_v \in \mathrm{JH}(\overline{\sigma}(\lambda_v, \tau_v))$. We write $\sigma_\kappa = \otimes\sigma_{\kappa_v}$ where σ_{κ_v} is a $\mathcal{O}$-lattice with cosocle κ_v in $\sigma(\lambda_v, \tau_v)$.

Proposition 6.1. Given a Serre weight $\kappa_{\mathcal{S}} \in \sigma(\lambda_{\mathcal{S}}, \tau_{\mathcal{S}})$. Then $M_\infty^{\sigma^{\mathcal{S}}}(\kappa_{\mathcal{S}}) \neq 0$ if and only if $\kappa_v \in W(\bar{r}_v)$ for all $v \in \mathcal{S}$. If that is the case, then $M_\infty(\sigma_{\mathcal{S}})$ is a cyclic $R_\infty(\lambda_{\mathcal{S}}, \tau_{\mathcal{S}})$ module.

Proof. The first part follows from [EGS15, Theorem 9.1.1, Remark 9.1.2]. The second part follows the same argument as in [LLHLM20, Lemma 5.1.2]. In particular, it follows from the method of [Dia97] and our patching functor being minimal. □

Theorem 6.2. *Given a minimal patching functor with unramified coefficients for $\{\overline{\rho}_v\}_{v\in\mathcal{S}}$, where $\overline{\rho}_v$ is $2\ell_v$-generic for some positive integers ℓ_v. Assume $(\lambda_v)_{j,1} - (\lambda_v)_{j,2} \leq \ell_v$. Given any Serre weight $\kappa \in \mathrm{JH}(\overline{\sigma}(\lambda_{\mathcal{S}}, \tau_{\mathcal{S}}))$, $M_\infty(\sigma_\kappa)$ is a cyclic $R_\infty(\lambda_{\mathcal{S}}, \tau_{\mathcal{S}})$ module.*

Proof. If $W(\bar{r}_v, \lambda_v, \tau_v) := W(\bar{r}_v) \cap \mathrm{JH}(\overline{\sigma}(\lambda_v, \tau_v)) = \varnothing$ for some v, then by the exactness of the patching functor and Proposition 6.1, $M_\infty(\kappa) = 0$ for any $\kappa \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$ and hence $M_\infty(\overline{\sigma}_\kappa) = 0$. By Nakayama's lemma $M_\infty(\sigma_\kappa) = 0$. Assume $W(\bar{r}_v, \lambda_v, \tau_v) \neq \varnothing$ for all $v \in \mathcal{S}$, we will prove the statement in the following steps:

Lemma 6.3. *(cf. [BP12, Lemma 12.8]) Assume that $\overline{\rho}$ is $2n$-generic and $\kappa \in \mathrm{Inj}_n \sigma$ for all $\sigma \in W(\overline{\rho})$. We define the distance between $F(\mathfrak{t}_\mu(\omega'))$ and $F(\mathfrak{t}_\mu(\omega))$ to be $\sum_j |\omega'_j - \omega_j|$. There exists a unique α (resp. β) $\in W(\overline{\rho})$ which is the furthest (respectively closest) from κ in the extension graph. Moreover, for any $\sigma \in W(\overline{\rho})$, if $\sigma = \beta$, then $I(\beta, \kappa)$ does not contain any other Serre weights of $W(\overline{\rho})$ as subquotients. If $\sigma \neq \beta$, then β is a subquotient of $I(\sigma, \kappa)$.*

Moreover, assume that $\overline{\rho}$ is $2n$-generic, $(\lambda_v)_{j,1} - (\lambda_v)_{j,2} \leq n$ for all j, v and $W(\overline{\rho}, \lambda, \tau) \neq \varnothing$. There exists a unique $\alpha', \beta' \in W(\overline{\rho}, \lambda, \tau)$ such that $\mathrm{JH}(I(\alpha', \beta')) = W(\overline{\rho}, \lambda, \tau)$, where $I(\alpha', \beta')$ is a Γ-representation.

Proof. As $\overline{\rho}$ is $2n$-generic, by Proposition 3.4, all $\sigma \in W(\overline{\rho})$ is $(2n-1)$-generic and Theorem 2.10 applies. We assume $\kappa = F(\mathfrak{t}_\mu(\omega))$. By Proposition 3.4, if $\sigma \in W(\overline{\rho})$, then $\sigma = F(\mathfrak{t}_\mu(\xi))$ such that $\xi_j = 0$ if $\gamma_{f-1-j} = 0$ and $\xi_j \in \{0, \mathrm{sgn}(s_j)\}$ otherwise. The value of ξ_j for each j is independent. Therefore, we let $\alpha = F(\mathfrak{t}_\mu(\xi'))$ (respectively $\beta = F(\mathfrak{t}_\mu(\xi'')))$, where $|\xi'_j - \omega_j|$ (respectively $|\xi''_j - \omega_j|$) is maximum (respectively minimum) for all j.

By Corollary 2.13, if $\sigma' \in I(\beta, \kappa)$, then σ' is closer to κ than β; therefore $\sigma' \notin W(\overline{\rho})$. Continued with the notation in 1, if $\sigma := F(\mathfrak{t}_\mu(\xi)) \in W(\overline{\rho})$, then for each j, $|\xi''_j - \omega_j| \leq |\xi_j - \omega_j|$ and $|\xi''_j - \xi_j| \leq 1$, by the choice of β and Proposition 3.4. Therefore, we must have for each j, $0 \leq \xi''_j - \omega_j \leq \xi_j - \omega_j$ or $0 \geq \xi''_j - \omega_j \geq \xi_j - \omega_j$, that is $\beta - \kappa \leq \sigma - \kappa$. By Corollary 2.13, this implies that β is a subquotient of $I(\sigma, \kappa)$.

By the proof of Lemma 5.3, $\mathrm{JH}(\overline{\sigma}(\lambda, \tau)) \subseteq \mathrm{JH}(\mathrm{Inj}_n \sigma)$ for all $\sigma \in W(\overline{\rho}, \lambda, \tau)$. Assume $\sigma := F(\mathfrak{t}_\mu(\xi)) \in W(\overline{\rho}, \lambda, \tau)$, then by Proposition 3.1 and Proposition 3.4, it is still the case that the values of ξ_j are independent for each j. By the same argument as in the first paragraph, we can find $\alpha' := F(\mathfrak{t}_\mu(\xi'))$ and $\beta' := F(\mathfrak{t}_\mu(\xi''))$ in $W(\overline{\rho}, \lambda, \tau)$ such that $|\xi''_j - \omega_j|$ is maximal and $|\xi'_j - \omega_j|$ is minimal for all j. Then for any $\sigma = F(\mathfrak{t}_\mu(\xi)) \in W(\overline{\rho}, \lambda, \tau)$, we have $|\xi'_j - \omega_j| \leq |\xi_j - \omega_j| \leq |\xi''_j - \omega_j|$ for all j. Furthermore, by Proposition 3.4, $|\xi'_j - \xi''_j| \leq 1$ for all j. Therefore, for each j, $\xi_j = \xi'_j$ or ξ''_j and hence $\sigma - \alpha' \leq \beta' - \alpha'$. By Corollary 2.13, we deduce that $\sigma \in I(\alpha', \beta')$. The last claim follows from the fact that $|\xi' - \xi''| \leq 1$ and Theorem 2.10. □

By Nakayama's lemma, $M_\infty(\sigma_\kappa)$ is cyclic if and only if $M_\infty(\overline{\sigma}_\kappa)$ is cyclic. For any $v \in \mathcal{S}$, if $\kappa_v \notin W(\overline{r}_v)$, then by Proposition 6.1 and the exactness of the patching functor, $M_\infty(\sigma_v \otimes W^v) = 0$ for any W^v which is a $\prod_{\mathcal{S}\setminus\{v\}} \mathrm{GL}_2(k_v)$-representation over $\mathbb{F}$. Let $W = \otimes_{v \in \mathcal{S}} W_v$. If $\sigma_v \in \mathrm{soc}(W_v)$ and $\sigma_v \notin W(\overline{r}_v)$, by the exactness of the patching functor, $M_\infty(W) = M_\infty(W/(\sigma_v \otimes_{w \in \mathcal{S}\setminus\{v\}} W_w)) = M_\infty((W_v/\sigma_v) \otimes_{w \in \mathcal{S}\setminus\{v\}} W_w)$. Similarly, if $\sigma_v \subseteq \mathrm{cosoc}(W_v)$ and $\sigma_v \notin W(\overline{r}_v)$, we take W'_v to be the pre-image of the quotient map $W_v \twoheadrightarrow \sigma_v$. Then $M_\infty(W) = M_\infty(W'_v \otimes_{w \in \mathcal{S}\setminus\{v\}} W_w)$. Therefore, applying the argument recursively, we reduce it to the subquotient W for which all its socle and cosocle are modular Serre weights. By Proposition 3.4 and Corollary 2.13, we deduce that all the Jordan–Hölder factors of W are modular Serre weights.

By Lemma 6.3, $\overline{\sigma}_{\kappa_v}$, has a subquotient W_v isomorphic to a Γ-representation $I(\alpha_v, \beta_v)$ such that $W(\overline{r}_v, \lambda_v, \tau_v) = \mathrm{JH}(W_v)$. Therefore,

$$M_\infty(\overline{\sigma}_\kappa) \cong M_\infty(\otimes_v W_v) \cong M_\infty(\otimes_v I(\alpha_v, \beta_v)). \tag{6.1}$$

By [EGS15, Proposition 3.5.2], for each $v \in \mathcal{S}$, there exists a tame type τ'_v, such that $\mathrm{JH}(I(\alpha_v, \beta_v)) \subseteq \mathrm{JH}(\overline{\sigma}(\tau'_v))$. We can find a lattice $\sigma(\tau'_v)_{\beta_v} \subseteq \sigma(\tau'_v)$ with cosocle β_v, then $\otimes_{v\in\mathcal{S}} I(\alpha_v, \beta_v)$ is isomorphic to a quotient $\widetilde{W}$ of $\overline{\sigma}(\tau'_{\mathcal{S}})_\beta := \otimes_{v\in\mathcal{S}}\overline{\sigma}(\tau'_v)_{\beta_v}$. We will finish the proof by showing that $M_\infty(\overline{\sigma}(\tau'_{\mathcal{S}})_\beta)$ is cyclic.

By [EGS15, Theorem 7.2.1], the special fibre $\overline{R}_\infty^{\tau'_{\mathcal{S}}}$ (defined in Equation (4.1)) is a power series ring over

$$\widehat{\otimes}_{v\in\mathcal{S}}\mathbb{F}[\![(X'_{j_v}, Y'_{j_v})_{j_v\in\mathcal{K}_v}]\!]/(X'_{j_v}Y'_{j_v})_{j_v\in\mathcal{K}_v}.$$

for some $\mathcal{K}_v \subseteq \{1, \ldots, f_v\}$. Let $\mathcal{K} = \prod_{v\in\mathcal{S}} \mathcal{K}_v$. Using the notation of [EGS15], for each $\prod_{v\in\mathcal{S}} J_v \subseteq \mathcal{K}$, we have $(\sigma_{J_v}) \in W(\overline{r}_v)$. We generalize the proof of [EGS15, Theorem 10.1.1]. (We swapped the notation of $\mathcal{W}$ and $\mathcal{J}$ appearing in [EGS15].) For $\mathcal{W} := \prod_v \mathcal{W}_v \subseteq \mathcal{J}$, we write $J = \prod_v J_v \in \mathcal{W}$ if $J_v \in \mathcal{W}_v$ for all $v \in \mathcal{S}$.

Moreover, given a Serre weight σ_{J_v} for each $v \in \mathcal{S}$, we define $\sigma_J := \otimes_v \sigma_{J_v}$. Then we write $I_{\mathcal{W}}$ for the radical ideal in $\overline{R}^{\tau_{\mathcal{S}}}$, which cuts out the induced reduced structure on the closed subspace $\bigcup_{J'\in\mathcal{W}} X_\infty(\sigma_J)$. The notion of interval (*cf.* [EGS15, Definition 10.1.4]) and capped interval is still well-defined. We define $\mathcal{F}(J_1, J_2)$ and $\mathcal{F}(J_1, J_2)^\times$ analogously to [EGS15, Definition 10.1.5], and generalize [EGS15, Lemmas 10.1.6, 10.1.8] as follows.

Lemma 6.4. *The quotient $I_{\mathcal{F}(J_1,J_2)^\times}/I_{\mathcal{F}(J_1,J_2)}$ is isomorphic to $R_\infty^{\tau_{\mathcal{S}}}/I_{\{J_1\}}$, in particular it is cyclic and is generated by $\prod_{v\in\mathcal{S}}\prod_{j_v\in J_{2v}\setminus J_{1v}} X'_{j_v}$.*

Lemma 6.5. *If $\mathcal{W}_1, \mathcal{W}_2$ are two capped intervals in $\mathcal{J}$ that share a common cap, then $I_{\mathcal{W}_1} + I_{\mathcal{W}_2} = I_{\mathcal{W}_1\cap\mathcal{W}_2}$.*

By the argument in the proof of [EGS15, Theorem 10.1.1], for each interval $\mathcal{W} \subseteq \mathcal{J}$, there is a subquotient $\overline{\sigma}(\tau_{\mathcal{S}})^{\mathcal{W}}$ of $\overline{\sigma}(\tau'_{\mathcal{S}})_\beta$, uniquely characterized by the property that $\mathrm{JH}(\overline{\sigma}(\tau'_{\mathcal{S}})^{\mathcal{W}}) = \{\sigma_{J'}\}_{J'\in\mathcal{W}}$. We finish the proof by proving the following proposition and take $\mathcal{W} = \mathcal{J}$.

Proposition 6.6. For any capped interval $\mathcal{W} \subseteq \mathcal{J}$, $M_\infty(\overline{\sigma}(\tau_{\mathcal{S}})^{\mathcal{W}})$ is cyclic.

Proof. We will prove this by inducting on $|\mathcal{W}|$. If $|\mathcal{W}| = 1$, the ring $R_\infty(\tau'_{\mathcal{S}})$ is regular. As M_∞ is a minimal patching functor, by the method of [Dia97], $M_\infty(\sigma^\circ(\tau'_{\mathcal{S}}))$ is of rank one over $R_\infty(\tau'_{\mathcal{S}})$ for any lattice $\sigma^\circ(\tau'_{\mathcal{S}}) \subseteq \sigma(\tau'_{\mathcal{S}})$. The argument relies on studying R_∞ and is independent of patching using unitary group or quaternion algebra. For the induction step, it follows exactly as in [EGS15, Lemma 10.1.12, 10.1.13] and with Lemmas 10.1.6, 10.1.8 replaced by Lemma 6.4, Lemma 6.5. ☐

□

7 PROPERTIES OF $\pi[\mathfrak{m}_{K_1}^n]$

Proposition 7.1. Given a finite set $\mathcal{D}$ of distinct $(2n-1)$-generic Serre weights. There exists a unique, up to isomorphism, representation $D_0^n(\mathcal{D})$, which is $\mathfrak{m}_{K_1}^n$-torsion such that

(i) $\mathrm{soc}(D_0^n(\mathcal{D})) = \bigoplus_{\sigma\in\mathcal{D}} \sigma$

(ii) $[D_0^n(D)\colon \sigma] = 1$ for all $\sigma \in \mathcal{D}$

(iii) $D_0^n(\mathcal{D})$ is maximal with respect to properties (i) and (ii)

Moreover, there is an isomorphism $D_0^n(\mathcal{D}) = \bigoplus_{\sigma\in D} D_{0,\sigma}^n(\mathcal{D})$ where $D_{0,\sigma}^n(\mathcal{D})$ is the largest subrepresentation of $\mathrm{Inj}_n\,\sigma$ such that $[D_\sigma^n(\mathcal{D})\colon \sigma] = 1$ and $[D_\sigma^n(\mathcal{D})\colon \sigma'] = 0$ for any $\sigma' \in \mathcal{D}$ with $\sigma' \neq \sigma$.

Proof. The first three statements and the isomorphism follow from the same proof in [BP12, Proposition 13.1], replacing Hom_Γ with Hom_{K/Z_1} by [BHH+23, Lemma 2.4.6] and replacing representations of Γ with representations of K/Z_1 which are $\mathfrak{m}_{K_1}^n$-torsion etc. The last statement follows from the same proof in [HW22, Corollary 4.2]. □

Assume that $\overline{\rho}$ is $2n$-generic for some $n \geq 0$. Then by Proposition 3.4, if $\sigma \in W(\overline{\rho})$, σ is $(2n-1)$-generic. In this case, we define $D_0^n(\overline{\rho}) := D_0^n(W(\overline{\rho}))$ and similarly $D_{0,\sigma}^n(\overline{\rho}) := D_{0,\sigma}^n(W(\overline{\rho}))$.

By Corollary 2.13, we have $\dim_{\mathbb{F}}(\mathrm{Hom}(I(\sigma,\tau), I(\sigma,\tau'))) = 1$ if $\tau - \sigma \leq \tau' - \sigma$ and 0 otherwise. If $\mathrm{Hom}(I(\sigma,\tau), I(\sigma,\tau')) \neq 0$, we fix $\iota_\tau\colon \sigma \hookrightarrow I(\sigma,\tau)$, and let $\phi_{\tau,\tau'}\colon I(\sigma,\tau) \hookrightarrow I(\sigma,\tau')$ be the unique embedding such that $\iota_{\tau'} = \phi_{\tau,\tau'} \circ \iota_\tau$.

Lemma 7.2.

1. *We have*
$$D_{0,\sigma}^n(\overline{\rho}) = \varinjlim_{\leq} I(\sigma,\tau),$$
where the inductive limit is taken over $\phi_{\tau,\tau'}$ and such that $I(\sigma,\tau)$ does not contain any other $\sigma' \in W(\overline{\rho})$ if $\sigma' \neq \sigma$.

2. *$D_0^n(\overline{\rho}) = \bigoplus_{\sigma\in W(\overline{\rho})} D_{0,\sigma}^n(\overline{\rho})$ is multiplicity free.*

3. *For any $\sigma \in W(\overline{\rho})$, we have $D_{0,\sigma}(\overline{\rho}) \subseteq D_{0,\sigma}^n(\overline{\rho})$ and $D_{0,\sigma}^n(\overline{\rho})^{K_1} = D_{0,\sigma}(\overline{\rho})$.*

Proof. The proof of 1,2 follows verbatim from [BP12, Proposition 13.4, Corollary 13.5] with Lemma 6.3 in place of [BP12, Lemma 12.8]. The proof of 3 follows from the construction (cf. [HW22, Theorem 4.6]). □

Let F be a totally real number field in which p is unramified. Fix v a place dividing p. Let D be a quaternion algebra with centre F, which splits at exactly one infinite place. Fix a compact open subgroup U^v of $D \otimes_F \mathbb{A}_{F,f}^v$. Given a compact open subgroup U of $(D \otimes_F \mathbb{A}_{F,f})^\times$, we let X_U be the associated smooth projective Shimura curve over F. Letting U_v run over compact open subgroups of $(D \otimes_F F_v)^\times \cong \mathrm{GL}_2(F_v)$, we consider

$$\pi(\overline{\rho}) := \varinjlim_{U_v} \mathrm{Hom}_{G_F}(\overline{r}, H^1_{\text{ét}}(X_{U^v U_v} \times_F \overline{F}, \mathbb{F})),$$

which is an admissible smooth representation of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$. It is expected that π corresponds to $\overline{\rho} := \overline{r}|_{G_{F_v}}$ under the conjectural mod p Langlands Program.

Corollary 7.3. *Assume that $\overline{\rho}$ is $\max\{2n, 12\}$-generic, then*

$$\pi(\overline{\rho})[\mathfrak{m}_{K_1/Z_1}^n] \cong D_0^n(\overline{\rho}),$$

In particular, it follows from Lemma 7.2 that $\pi(\overline{\rho})[\mathfrak{m}_{K_1/Z_1}^n]$ is multiplicity free.

Remark 7.4. The case where $n = 1$ is proven by [LMS22], [HW18] and [Le19]; while the case where $n = 2$ is proven in [BHH+23, Theorem 1.9], [Wan23, Theorem 6.3] (here $r = 1$, as we are considering the case with minimal level), and [HW22, Corollary 8.13].

Proof. To show that $D^n(\overline{\rho}) \subseteq \pi(\overline{\rho})[\mathfrak{m}_{K_1/Z_1}^n]$, we modify the proof of [BHH+23, Theorem 8.4.2] by replacing $\widetilde{D}_{\sigma_v}$ by $D_{0,\sigma}^n$ and $\pi(\overline{\rho})[\mathfrak{m}_{K_1}^2]$ by $\pi(\overline{\rho})[\mathfrak{m}_{K_1}^n]$. We will sketch the proof as follows. In [CEG+18], $\mathbb{M}_\infty$ is constructed so that $\pi(\overline{\rho})^\vee = \mathbb{M}_\infty/\mathfrak{m}_\infty$. Moreover, we have

$$\mathbb{M}_\infty/(p, x_1, \dots, x_{4|S|+q}) \cong \bigoplus_{\sigma \in W(\overline{r}_v^\vee)} (\mathrm{Proj}_{K/Z_1} \sigma^\vee)^{m_\sigma}$$

for some $m_\sigma \geq 1$ and q is an integer greater than or equal to $[F \colon Q]$. Therefore, we can deduce that

$$\begin{aligned}\mathrm{Hom}_{K/Z_1}(D_{0,\sigma}^n, \pi(\overline{\rho})) &= \mathrm{Hom}_{K/Z_1}(D_{0,\sigma}^n, \pi(\overline{\rho})[\mathfrak{m}_{K_1/Z_1}^n]) \\ &\xrightarrow{\sim} \mathrm{Hom}_{K/Z_1}(\sigma, \pi(\overline{\rho})) = \mathrm{Hom}_{K/Z_1}(\sigma, \mathrm{soc}(\pi(\overline{\rho}))).\end{aligned}$$

Since $\mathrm{soc}\, \pi(\overline{\rho}) = \oplus_{\sigma \in W(\overline{r}_v^\vee)} \sigma$, we have indeed $D_0^n(\overline{\rho}) \subseteq \pi(\overline{\rho})[\mathfrak{m}_{K_1/Z_1}^n]$ (*cf.* [Bre14, Lemma 9.2]). If we can show that all $\sigma \in W(\overline{r}_v^\vee)$ appear only once in $\pi(\overline{\rho})[\mathfrak{m}_{K_1/Z_1}^n]$, then by the maximal property of $D^n(\overline{\rho})$, we have the other inclusion.

Since the cases where $n = 1, 2$ are already proven, we assume $n > 2$. By our genericity assumption, all $\sigma \in W(\overline{\rho})$ is $(2n-1)$ generic. Assume for the sake of contradiction that there exists a Serre weight $\sigma \in W(\overline{\rho})$, such that $[\pi(\overline{\rho})[\mathfrak{m}_{K_1}^n] : \sigma] > 1$. Since $\pi(\overline{\rho})[\mathfrak{m}_{K_1}^2]$ is multiplicity free, so is $\mathrm{soc}(\pi(\overline{\rho}))$. Hence, the map

$$f\colon \pi(\overline{\rho})[\mathfrak{m}_{K_1}^n] \hookrightarrow \bigoplus_{\sigma \in W(\overline{\rho})} \mathrm{Inj}_n \sigma$$

is injective as it is injective on the socle. Therefore, $\pi(\overline{\rho})[\mathfrak{m}_{K_1}^n] \cong \oplus_{\sigma \in W(\overline{\rho})} V_\sigma$, where V_σ is the image of $p_n \circ f$ and p_n is the projection map onto $\mathrm{Inj}_n \sigma$. It suffices to show that for $\tau \in W(\overline{\rho})$, $[V_\sigma : \tau] = 1$ if $\tau = \sigma$ and $[V_\sigma : \tau] = 0$ if $\tau \neq \sigma$. Since taking $\mathfrak{m}_{K_1}^2$-torsion is compatible with f, by Lemma 7.2, we deduce that $V_\sigma[\mathfrak{m}_{K_1}^2] \cong D_{0,\sigma}^2(\overline{\rho})$. We finish the proof by the following lemma.

Lemma 7.5. *Let $\sigma, \tau \in W(\overline{\rho})$ be $(2n-1)$-generic and $V \subseteq \mathrm{Inj}_n \sigma$ be a subrepresentation. Assume $V[\mathfrak{m}_{K_1}^2]$ is multiplicity-free, $[V[\mathfrak{m}_{K_1}^2] : \tau] = 1$ if $\tau = \sigma$, and 0 if $\tau \neq \sigma$. Then, $[V : \tau] = 1$ if $\tau = \sigma$ and 0 if $\tau \neq \sigma$.*

Proof. Assume for contradiction that there exists a $(2n-1)$-generic $\tau \in W(\overline{\rho})$ such that $\tau \in \mathrm{JH}(V/V[\mathfrak{m}_{K_1}^2])$. If $\tau \neq \sigma$, by Proposition 3.4 and Theorem 2.10, $I(\sigma, \tau)$, a Γ-representation, is a subrepresentation of $V[\mathfrak{m}_{K_1}^1]$ and hence $I(\sigma, \tau) \subseteq V[\mathfrak{m}_{K_1}^1]$, contradicting our assumption. Now assume $\tau = \sigma$. Considering the image of $\mathrm{Proj}_n \sigma \to V$, without loss of generality, we can replace V with a subrepresentation with cosocle σ. Let $\widetilde{V} := V/\sigma$. By [BHH+23, Lemma 2.4.6], and the fact that $V[\mathfrak{m}_{K_1}^2]$ is multiplicity free, we deduce that $\mathrm{soc}(\widetilde{V}) \subseteq \bigoplus_{\sigma' \in \mathcal{E}(\sigma)} \sigma'$, where $\mathcal{E}(\sigma)$ are the sets of Serre weights adjacent to σ. As $\mathrm{cosoc}(\widetilde{V}) = \sigma$, the image of $\mathrm{Proj}_n \sigma \twoheadrightarrow \widetilde{V}$ lies in $\bigoplus_{\sigma' \in \mathrm{JH}(\mathrm{soc}(\widetilde{V}))} I(\sigma', \sigma)$ which is killed by $\mathfrak{m}_{K_1}$ by Theorem 2.10. Therefore, V is $\mathfrak{m}_{K_1}^2$-torsion but not multiplicity-free, a contradiction. □

□

Using the patching functor M_∞ constructed in [Wan23, § 6], which is based on [BHH+23, § 8], we have the following result.

Corollary 7.6. *Assume that $\overline{\rho}$ is $\max\{2n, 12\}$-generic and $\sigma \in \mathrm{JH}(\pi(\overline{\rho})[\mathfrak{m}_{K_1}^n])$, $M_\infty(\mathrm{Proj}_n \sigma)$ is multiplicity free.*

Proof. By Corollary 7.3, we show that for any $\sigma \in \mathrm{JH}(\pi(\overline{\rho})[\mathfrak{m}_{K_1}^n])$, we have

$$\dim_{\mathbb{F}}(\mathrm{Hom}_{K_1/Z_1}(\mathrm{Proj}_n \sigma, \pi(\overline{\rho}))) = 1.$$

Then, by the proof of [BHH+23, Theorem 8.4.2] we deduce that

$$M_\infty(\mathrm{Proj}_n \sigma)/\mathfrak{m}_\infty \cong \mathrm{Hom}_{K_1/Z_1}(\mathrm{Proj}_n \sigma, \pi(\overline{\rho}))^\vee$$

is cyclic. □

We can extend the result of [BHH+25b, Proposition 5.1] as follows:

Corollary 7.7. *Assume that $\overline{\rho}$ is split reducible and* $\max\{9, 2f+1, 2n+1\}$*-generic.*

1. *Let π' be a subquotient of $\pi(\overline{\rho})$. Then there is a unique subset $\Sigma' \subseteq \{1, \dots, f\}$ such that*

$$\pi'[\mathfrak{m}_{K_1}^n] \cong \bigoplus_{i\in\Sigma'} \bigoplus_{\sigma\in W(\overline{\rho}), |J_\sigma|=i} D_{0,\sigma}^n(\overline{\rho}).$$

2. *Let $\pi_1 \subseteq \pi_2$ be a subrepresentations of $\pi(\overline{\rho})$. Then the induced sequence of $\mathbb{F}[\![K/Z_1]\!]/\mathfrak{m}_{K_1}^n$-modules*

$$0 \to \pi_1[\mathfrak{m}_{K_1}^n] \to \pi_2[\mathfrak{m}_{K_1}^n] \to \pi_1/\pi_2[\mathfrak{m}_{K_1}^n] \to 0$$

is split exact.

Proof. The proof of 1 follows the same argument as in [BHH+25b, Proposition 5.1], with appropriate generalization, such as replacing $\mathfrak{m}_{K_1}^2$ with $\mathfrak{m}_{K_1}^n$, $\widetilde{D}_{0,\sigma}(\overline{\rho})$ with $D_{0,\sigma}^n(\overline{\rho})$, [BHH+25b, Proposition 3.2.8] with Corollary 7.3 and [HW22, Theorem 4.6] with Lemma 7.2.

2. As in the proof of [BHH+25a, Corollary 3.2.5], it suffices to prove the special case $\pi_2 = \pi(\overline{\rho})$. By the argument in [BHH+25b], if 2 does not hold, we have a non-split extension of $\mathbb{F}[\![K/Z_1]\!]/\mathfrak{m}_{K_1}^n$-modules

$$0 \to \bigoplus_{i\in\Sigma'} \bigoplus_{\sigma\in W(\overline{\rho}), |J_\sigma|=i} D_{0,\sigma}^n(\overline{\rho}) \to V \to \tau \to 0$$

where $\tau \in W(\overline{\rho})$. Hence, we have a non-split extension of $\mathbb{F}[\![K/Z_1]\!]/\mathfrak{m}_{K_1}^n$-modules between τ and $D_{0,\sigma}^n(\overline{\rho})$ for some $\sigma \in W(\overline{\rho})$. As this corollary is proved for $n = 2$, it implies that $\tau \notin \mathrm{JH}(V[\mathfrak{m}_{K_1}^2])$, hence $V[\mathfrak{m}_{K_1}^2] \cong D_{0,\sigma}^2(\overline{\rho})$. We can then conclude the proof using Lemma 7.5 . □

BIBLIOGRAPHY

[Alp86] J. L. Alperin, *Local representation theory*, Cambridge Studies in Advanced Mathematics, vol. 11, Cambridge University Press, Cambridge, 1986, Modular representations as an introduction to the local representation theory of finite groups. MR 860771

[BBH⁺24] Rebecca Bellovin, Neelima Borade, Anton Hilado, Kalyani Kansal, Heejong Lee, Brandon Levin, David Savitt, and Hanneke Wiersema, *Irregular loci in the Emerton-Gee stack for* GL_2, J. Reine Angew. Math. **814** (2024), 9–46. MR 4793337

[BDJ10] Kevin Buzzard, Fred Diamond, and Frazer Jarvis, *On Serre's conjecture for mod ℓ Galois representations over totally real fields*, Duke Math. J. **155** (2010), no. 1, 105–161. MR 2730374

[BHH⁺23] Christophe Breuil, Florian Herzig, Yongquan Hu, Stefano Morra, and Benjamin Schraen, *Gelfand-Kirillov dimension and mod p cohomology for* GL_2, Invent. Math. **234** (2023), no. 1, 1–128. MR 4635831

[BHH⁺25a] Christophe Breuil, Florian Herzig, Yongquan Hu, Stefano Morra, and Benjamin Schraen, *Finite length for unramified* GL_2, 2025.

[BHH⁺25b] ———, *On the constituents of the mod p cohomology of shimura curves*, 2025.

[BLGG13] Thomas Barnet-Lamb, Toby Gee, and David Geraghty, *Serre weights for rank two unitary groups*, Math. Ann. **356** (2013), no. 4, 1551–1598. MR 3072811

[BLGHT11] Tom Barnet-Lamb, David Geraghty, Michael Harris, and Richard Taylor, *A family of Calabi-Yau varieties and potential automorphy II*, Publ. Res. Inst. Math. Sci. **47** (2011), no. 1, 29–98. MR 2827723

[BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$, Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart. MR 1944572

[BP12] Christophe Breuil and Vytautas Paškūnas, *Towards a modulo p Langlands correspondence for* GL_2, Mem. Amer. Math. Soc. **216** (2012), no. 1016, vi+114. MR 2931521

[Bre00] Christophe Breuil, *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), no. 2, 489–549. MR 1804530

[Bre03a] ———, *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\boldsymbol{Q}_p)$*. II*, J. Inst. Math. Jussieu **2** (2003), no. 1, 23–58. MR 1955206

[Bre03b] ———, *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\mathbf{Q_p})$*. I*, Compositio Math. **138** (2003), no. 2, 165–188. MR 2018825

[Bre10] ———, *The emerging p-adic Langlands programme*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 203–230. MR 2827792

[Bre14] ———, *Sur un problème de compatibilité local-global modulo p pour* GL_2, J. Reine Angew. Math. **692** (2014), 1–76. MR 3274546

[Cal18] Frank Calegari, *Non-minimal modularity lifting in weight one*, J. Reine Angew. Math. **740** (2018), 41–62. MR 3824782

[Cal23] ———, *Reciprocity in the Langlands program since Fermat's last theorem*, ICM—International Congress of Mathematicians. Vol. 2. Plenary lectures, EMS Press, Berlin, [2023] ©2023, pp. 610–651. MR 4680265

[Car14] Ana Caraiani, *Monodromy and local-global compatibility for* $l = p$, Algebra Number Theory **8** (2014), no. 7, 1597–1646. MR 3272276

[CEG+18] Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paškūnas, and Sug Woo Shin, *Patching and the p-adic Langlands program for* $\mathrm{GL}_2(\mathbb{Q}_p)$, Compos. Math. **154** (2018), no. 3, 503–548. MR 3732208

[CF10] J.W.S. Cassels and A. Fröhlich, *Algebraic number theory*, London Mathematical Society, 2010.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 1–181, With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras. MR 2470687

[CL18] Ana Caraiani and Brandon Levin, *Kisin modules with descent data and parahoric local models*, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no. 1, 181–213. MR 3764041

[Clo91] Laurent Clozel, *Représentations galoisiennes associées aux représentations automorphes autoduales de* $\mathrm{GL}(n)$, Inst. Hautes Études Sci. Publ. Math. (1991), no. 73, 97–145. MR 1114211

[Col10] Pierre Colmez, *Représentations de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *et* (ϕ, Γ)*-modules*, Astérisque (2010), no. 330, 281–509. MR 2642409

[Del71] Pierre Deligne, *Formes modulaires et représentations l-adiques*, Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363, Lecture Notes in Math., vol. 175, Springer, Berlin, 1971, pp. Exp. No. 355, 139–172. MR 3077124

[Dia97] Fred Diamond, *The Taylor-Wiles construction and multiplicity one*, Invent. Math. **128** (1997), no. 2, 379–391. MR 1440309

[DL21] Andrea Dotto and Daniel Le, *Diagrams in the* mod p *cohomology of Shimura curves*, Compos. Math. **157** (2021), no. 8, 1653–1723. MR 4283560

[DS74] Pierre Deligne and Jean-Pierre Serre, *Formes modulaires de poids* 1, Ann. Sci. École Norm. Sup. (4) **7** (1974), 507–530. MR 379379

[EG14] Matthew Emerton and Toby Gee, *A geometric perspective on the Breuil-Mézard conjecture*, J. Inst. Math. Jussieu **13** (2014), no. 1, 183–223. MR 3134019

[EG23] ______, *Moduli stacks of étale* (φ, Γ)*-modules and the existence of crystalline lifts*, Annals of Mathematics Studies, vol. 215, Princeton University Press, Princeton, NJ, [2023] ©2023. MR 4529886

[EGH13] Matthew Emerton, Toby Gee, and Florian Herzig, *Weight cycling and Serre-type conjectures for unitary groups*, Duke Math. J. **162** (2013), no. 9, 1649–1722. MR 3079258

[EGH25] Matthew Emerton, Toby Gee, and Eugen Hellmann, *An introduction to the categorical p-adic langlands program*, 2025.

[EGS15] Matthew Emerton, Toby Gee, and David Savitt, *Lattices in the cohomology of Shimura curves*, Invent. Math. **200** (2015), no. 1, 1–96. MR 3323575

[Elk73] Renée Elkik, *Solutions d'équations à coefficients dans un anneau hensélien*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 553–603. MR 345966

[Eme06] Matthew Emerton, *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms*, Invent. Math. **164** (2006), no. 1, 1–84. MR 2207783

[Eme11] ———, *Local-global compatibility in the p-adic Langlands programme for* $\mathrm{GL}_2/\mathbb{Q}$, preprint **3** (2011), no. 4, 1–119.

[Eme14] ———, *Completed cohomology and the p-adic Langlands program*, Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 319–342. MR 3728617

[FH25] Tony Feng and Bao Le Hung, *Mirror symmetry and the Breuil-Mézard Conjecture*, 2025.

[Fon94] Jean-Marc Fontaine, *Exposé II : Le corps des périodes p-adiques*, Périodes p-adiques - Séminaire de Bures, 1988 (Jean-Marc Fontaine, ed.), Astérisque, no. 223, Société mathématique de France, 1994, talk:2, pp. 59–101 (fr). MR 1293971

[Gee22] Toby Gee, *Modularity lifting theorems*, Essent. Number Theory **1** (2022), no. 1, 73–126. MR 4573253

[GHS18] Toby Gee, Florian Herzig, and David Savitt, *General Serre weight conjectures*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 12, 2859–2949. MR 3871496

[GK14] Toby Gee and Mark Kisin, *The Breuil-Mézard conjecture for potentially Barsotti-Tate representations*, Forum Math. Pi **2** (2014), e1, 56. MR 3292675

[GLS14] Toby Gee, Tong Liu, and David Savitt, *The Buzzard-Diamond-Jarvis conjecture for unitary groups*, J. Amer. Math. Soc. **27** (2014), no. 2, 389–435. MR 3164985

[GV18] S. Galatius and A. Venkatesh, *Derived Galois deformation rings*, Adv. Math. **327** (2018), 470–623. MR 3762000

[Ham75] Eloise Hamann, *On power-invariance*, Pacific J. Math. **61** (1975), no. 1, 153–159. MR 407009

[Hen00] Guy Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique*, Invent. Math. **139** (2000), no. 2, 439–455. MR 1738446

[HP19] Yongquan Hu and Vytautas Paškūnas, *On crystabelline deformation rings of* $\mathrm{G}al(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, Math. Ann. **373** (2019), no. 1-2, 421–487, With an appendix by Jack Shotton. MR 3968877

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR 1876802

[HW18] Yongquan Hu and Haoran Wang, *Multiplicity one for the* mod p *cohomology of Shimura curves: the tame case*, Math. Res. Lett. **25** (2018), no. 3, 843–873. MR 3847337

[HW22] ———, *On the* mod p *cohomology for* GL_2*: the non-semisimple case*, Camb. J. Math. **10** (2022), no. 2, 261–431. MR 4461834

[Kis08] Mark Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), no. 2, 513–546. MR 2373358

[Kis09a] ———, *The Fontaine-Mazur conjecture for* GL_2, J. Amer. Math. Soc. **22** (2009), no. 3, 641–690. MR 2505297

[Kis09b] ———, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180. MR 2600871

[Kot92] Robert E. Kottwitz, *On the λ-adic representations associated to some simple Shimura varieties*, Invent. Math. **108** (1992), no. 3, 653–665. MR 1163241

[KW09a] Chandrashekhar Khare and Jean-Pierre Wintenberger, *Serre's modularity conjecture. I*, Invent. Math. **178** (2009), no. 3, 485–504. MR 2551763

[KW09b] ———, *Serre's modularity conjecture. II*, Invent. Math. **178** (2009), no. 3, 505–586. MR 2551764

[Lab11] J.-P. Labesse, *Changement de base CM et séries discrètes*, On the stabilization of the trace formula, Stab. Trace Formula Shimura Var. Arith. Appl., vol. 1, Int. Press, Somerville, MA, 2011, pp. 429–470. MR 2856380

[Le19] Daniel Le, *Multiplicity one for wildly ramified representations*, Algebra Number Theory **13** (2019), no. 8, 1807–1827. MR 4017535

[LLHL19] Daniel Le, Bao V. Le Hung, and Brandon Levin, *Weight elimination in Serre-type conjectures*, Duke Math. J. **168** (2019), no. 13, 2433–2506. MR 4007598

[LLHLM18] Daniel Le, Bao V. Le Hung, Brandon Levin, and Stefano Morra, *Potentially crystalline deformation rings and Serre weight conjectures: shapes and shadows*, Invent. Math. **212** (2018), no. 1, 1–107. MR 3773788

[LLHLM20] ———, *Serre weights and Breuil's lattice conjecture in dimension three*, Forum Math. Pi **8** (2020), e5, 135. MR 4079756

[LLHLM23] ———, *Local models for Galois deformation rings and applications*, Invent. Math. **231** (2023), no. 3, 1277–1488. MR 4549091

[LLHLM24] Daniel Le, Bao Viet Le Hung, Brandon Levin, and Stefano Morra, *Extremal weights and a tameness criterion for mod p galois representations*, Journal of the European Mathematical Society (2024), published online first.

[LMS22] Daniel Le, Stefano Morra, and Benjamin Schraen, *Multiplicity one at full congruence level*, J. Inst. Math. Jussieu **21** (2022), no. 2, 637–658. MR 4386824

[Ser87] Jean-Pierre Serre, *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Duke Math. J. **54** (1987), no. 1, 179–230. MR 885783

[Sho16] Jack Shotton, *Local deformation rings for* GL_2 *and a Breuil-Mézard conjecture when* $\ell \neq p$, Algebra Number Theory **10** (2016), no. 7, 1437–1475. MR 3554238

[ST68] Jean-Pierre Serre and John Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517. MR 236190

[Tho12] Jack Thorne, *On the automorphy of l-adic Galois representations with small residual image*, J. Inst. Math. Jussieu **11** (2012), no. 4, 855–920, With an appendix by Robert Guralnick, Florian Herzig, Richard Taylor and Thorne. MR 2979825

[Tsu99] Takeshi Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. **137** (1999), no. 2, 233–411. MR 1705837

[TW95] Richard Taylor and Andrew Wiles, *Ring-theoretic properties of certain Hecke algebras*, Ann. of Math. (2) **141** (1995), no. 3, 553–572. MR 1333036

[Wan23] Yitong Wang, *On the* mo*d p cohomology for* GL_2, J. Algebra **636** (2023), 20–41. MR 4637601

[Wil95] Andrew Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2) **141** (1995), no. 3, 443–551. MR 1333035

COLOPHON

This thesis was typeset using the typographical look-and-feel `classicthesis` developed by André Miede and Ivo Pletikosić.

The style was inspired by Robert Bringhurst's seminal book on typography "*The Elements of Typographic Style*".

Figures were created with Geobegra