# Scattering Constants and Cuspidal Divisor Class Groups of Some Noncongruence Subgroups

by

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# Abstract

This thesis presents computations involving certain noncongruence subgroups of the modular group. Using the Kronecker limit formula, we calculate the scattering constants for three families of nonelliptic noncongruence subgroups. For these same groups, we also determine their cuspidal divisor class groups and show that these groups are finite.

These findings relate to the Manin–Drinfeld theorem, which states that cuspidal divisor class groups are finite for congruence subgroups. Since no general finiteness result is known for noncongruence subgroups, these examples provide additional insight into their arithmetic properties.

Furthermore, we compute cuspidal divisor class groups for several congruence and noncongruence subgroups associated with elliptic curves. In some of the noncongruence cases, the cuspidal divisor class groups are finite, indicating that finiteness can occur outside the congruence setting.

Overall, these computations offer explicit data in a field where such examples are relatively rare and may contribute to further understanding of noncongruence modular curves.

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# Contents

# Chapter 1

# Introduction

Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{Z})$ be a finite index subgroup. To each cusp $S_j$ of $\Gamma$, we can associate a non-holomorphic Eisenstein series $E_j^\Gamma(z,s)$. For two cusps $S_j$ and $S_k$, the *scattering constant* $C_{jk}^\Gamma$ is defined via the constant term of a Dirichlet series $\phi_{jk}^\Gamma(s)$ arising from the Fourier expansion of $E_j^\Gamma(z,s)$ at the cusp $S_k$. Collecting these functions into a matrix $(\phi_{jk}^\Gamma(s))$ gives the *scattering matrix*, whose determinant is called the *scattering determinant*.

The scattering determinant plays an important role in the analytic theory of Eisenstein series: its pole corresponds to the pole of the Eisenstein series itself, and its logarithmic derivative accounts for the contribution of the continuous spectrum to the Selberg trace formula. For congruence subgroups, its computation is quite involved and has been studied extensively by Hejhal [14] and Huxley [17]; see also [11].

In contrast, for *noncongruence subgroups*, very little is known. One notable example is due to Posingies [24], who computed the scattering constants of noncongruence subgroups associated to Fermat curves using a method based on the *Kronecker limit formula*. The classical Kronecker limit formula for $\Gamma(1) = \mathrm{PSL}_2(\mathbb{Z})$ is

$$4\pi \lim_{s\to 1}\left(E^{\Gamma(1)}(z,s) - \frac{3/\pi}{s-1}\right) = -\log\|\Delta(z)\|^2 + 24\left(\frac{\zeta'(-1)}{\zeta(-1)} - \log(4\pi) + 1\right), \tag{1.1}$$

where $\Delta(z)$ is the Ramanujan $\Delta$-function, relating the behavior of Eisenstein series at $s = 1$ to modular forms.

The Kronecker limit formula approach hinges on the existence of modular forms that vanish at a single cusp and are nonvanishing at all others, allowing one to isolate the contribution of individual cusps to the scattering constants. A natural way to study the existence of such modular forms is through the arithmetic of divisors supported on the cusps.

To make this precise, consider the group $\mathscr{D}^\infty$ of degree-zero divisors supported on the cusps of $\Gamma$, and let $\mathscr{F}^\infty$ be the subgroup of principal divisors of modular functions whose zeros and poles lie only at the cusps. The quotient

$$\mathrm{CL}(\Gamma) = \mathscr{D}^\infty/\mathscr{F}^\infty$$

is called the *cuspidal divisor class group* of $\Gamma$ and measures the obstruction to constructing modular functions with prescribed vanishing behavior at the cusps.

If $\mathrm{CL}(\Gamma)$ is finite, we are able to construct modular forms that vanish at a given cusp and nowhere else — precisely the kind needed for applying the Kronecker limit formula. We prove this sufficiency result in Proposition 3.1, and it forms the foundation of our method for computing scattering constants.

For congruence subgroups, the finiteness of $\mathrm{CL}(\Gamma)$ is guaranteed by the Manin–Drinfeld theorem [22, Theorem 3.5]. For noncongruence subgroups, however, the structure of $\mathrm{CL}(\Gamma)$ is more subtle and generally unknown, making its determination a key part of this thesis.

Rohrlich [27] computed $\mathrm{CL}(\Gamma_N)$ for noncongruence subgroups $\Gamma_N$ associated to Fermat curves $x^N + y^N = 1$ $(N > 2)$, showing that

$$\mathrm{CL}(\Gamma_N) \simeq \begin{cases} (\mathbb{Z}/N\mathbb{Z})^{3N-7}, & \text{if } N \text{ odd,} \\ (\mathbb{Z}/N\mathbb{Z})^{3N-7} \times \mathbb{Z}/2\mathbb{Z}, & \text{if } N \text{ even.} \end{cases}$$

Scholl [28] later related these groups to the Fourier coefficients of certain weight-2 modular forms constructed from derivatives of Eisenstein series, showing in particular that $\mathrm{CL}(\Gamma)$ is finite if and only if all such coefficients are algebraic.

In this thesis, we compute scattering constants for several families of noncongruence subgroups using the Kronecker limit formula, supported by explicit constructions of modular forms made possible by proving that the corresponding cuspidal divisor class groups are finite.

The first family we consider is the group $\Gamma_{6N}$, introduced by Sansone [23], which forms an infinite family of genus zero noncongruence subgroups. Since the associated modular curves have genus zero, the cuspidal divisor class group of $\Gamma_{6N}$ is trivial, and we compute its associated scattering constants (see Theorem 2.8).

The second and third families are the groups $Y_2(N)$ and $Y_3(N)$, studied respectively by Yang–Yin [32] and Ye [33]. Building on their results, we determine the cuspidal divisor class groups and compute the corresponding scattering constants, including for several variants of these groups (see Theorem 2.10, Theorem 2.11, Theorem 2.12, and Theorem 2.13).

Finally, we study cuspidal divisor class groups associated with certain elliptic curves. By Belyi's theorem, any elliptic curve arises from a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, which in many cases is noncongruence. For several such examples we compute $\mathrm{CL}(\Gamma_E)$ explicitly, while illustrating the additional difficulties that arise when no explicit Belyi map is known (see Section 3.2.1, Section 3.2.3, and Section 3.2.2).

In summary, the finiteness of the cuspidal divisor class group governs the existence of modular forms required for the Kronecker limit method and serves as a unifying thread linking the analytic, algebraic, and arithmetic aspects of the noncongruence subgroups studied in this thesis.

# Chapter 2

# Scattering Constants

The material in this chapter corresponds to the article [16].

## 2.1 Preliminaries

Let $\Gamma$ be a finite index subgroup of $\Gamma(1) := \mathrm{PSL}_2(\mathbb{Z})$. For a cusp $S_j \in \mathbb{P}^1(\mathbb{Q})$ of $\Gamma$, we denote by $\gamma_j$ the matrix in $\Gamma(1)$ with $\gamma_j(\infty) = S_j$, which always exists. Furthermore, we normalize $\gamma_j$ to

$$\sigma_j := \gamma_j \cdot \begin{pmatrix} \sqrt{b_j} & 0 \\ 0 & 1/\sqrt{b_j} \end{pmatrix}$$

with $b_j \in \mathbb{N}$ such that $\sigma_j^{-1}\Gamma_j\sigma_j = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$, where $\Gamma_j := \mathrm{Stab}_\Gamma(S_j)$. We call $b_j$ the width of the cusp $S_j$. Then $\Gamma_j = \langle \gamma_j \left(\begin{smallmatrix} 1 & b_j \\ 0 & 1 \end{smallmatrix}\right) \gamma_j^{-1} \rangle$.

We introduce the notion of a non-holomorphic Eisenstein series attached to the finite index subgroup $\Gamma$:

**Definition 2.1** ([24, Definition 2.2])**.** *Let $\Gamma \subseteq \Gamma(1)$ be a finite index subgroup. For each cusp $S_j$ there is a non-holomorphic Eisenstein series $E_j^\Gamma(z,s)$, which for $z \in \mathbb{H}$, $s \in \mathbb{C}$ and $\Re(s) > 1$ is defined by the convergent series*

$$E_j^\Gamma(z,s) = \sum_{\sigma_j \in \Gamma_j \backslash \Gamma} \Im(\sigma_j^{-1}\sigma_j(z))^s$$

An important property of the Eisenstein series is that it admits a Fourier expansion.

**Proposition 2.2** ([24, Proposition 2.5])**.** *The Fourier expansion of $E_j^\Gamma(z,s)$ at the cusp $S_k$ is given by*

$$\begin{aligned} E_j^\Gamma(\gamma_k(z),s) = \delta_{jk}\frac{y^s}{b_j^s} &+ \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{1}{b_j^s b_k}\phi_{jk,0}^\Gamma(s)y^{1-s} \\ &+ \sum_{m\neq 0}\frac{1}{b_j^s b_k}\phi_{jk,m}^\Gamma(s)2\pi^s\left|\frac{m}{b_k}\right|^{s-1/2}\Gamma(s)^{-1}y^{1/2}K_{s-1/2}(2\pi|m|y/b_k)e^{2\pi imx/b_k}, \end{aligned}$$

*where $z = x+iy$ and $K_*(\cdot)$ the modified K-Bessel function, defined by $K_s(y) = \frac{1}{2}\int_0^\infty e^{-y\frac{t+t^{-1}}{2}}t^s\,\frac{dt}{t}$,* *and*

$$\phi^{\Gamma}_{jk,m}(s) \coloneqq \sum_{c>0} \frac{1}{c^{2s}} \sum_{d\ mod\ b_k c} e^{2\pi i m \frac{d}{b_k c}}$$

*where $d$ runs through $d \pmod{b_k c}$ such that there exists* $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1}\Gamma\gamma_k$.

This allows us to define the notion of scattering constants:

**Definition 2.3** ([24, Definition 2.7 and Remark 2.8])**.** *For $\Gamma \subseteq \Gamma(1)$ be a subgroup of finite index we define the scattering matrix to be*

$$\Phi_\Gamma(s) \coloneqq \left(\pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)} \cdot \frac{1}{b_j^s b_k}\phi^{\Gamma}_{jk,0}\right)_{j,k},$$

*where $j$ and $k$ run over all cusps of $\Gamma$.*
*For all pairs $j, k$ we define the scattering constant $C^{\Gamma}_{jk}$ to be the constant term at $s = 1$ of Dirichlet series $(\Phi_\Gamma)_{jk}(s)$:*

$$C^{\Gamma}_{jk} \coloneqq \lim_{s\to 1}\left(\Phi_\Gamma(s)_{j,k} - \frac{1}{\mathrm{vol}(\Gamma)(s-1)}\right) - \frac{\log(b_k)}{\mathrm{vol}(\Gamma)}.$$

**Notations.** *For convenience, we will specify the set of cusps in order such as $j, k \in \{s_1, s_2, \cdots, s_n\}$ so that the entry in the $r$-th row and $c$-th column of the matrix $(C^{\Gamma}_{jk})$ is the scattering constant $C^{\Gamma}_{s_r s_c}$. Similarly, we record the results of the transformations $\gamma_k$ acting on the modular forms $G^{\Gamma}_j$ (with divisor $\operatorname{div} G^{\Gamma}_j = m \cdot j$ for some $m \in \mathbb{N}$) by a matrix $\left(G^{\Gamma}_j|\gamma_k\right)$, where the entry in the $r$-th row and $c$-th column records $G^{\Gamma}_{s_r}|\gamma_{s_c}$.*

*Furthermore, we denote $I_n$ to be the $n \times n$ identity matrix, and $J_{m\times n}$ be the matrix with 1 in every entry. Also, $\mathbf{0}_{m\times n}$ represents the $m \times n$ zero matrix. When we write $a + A$, where $a \in \mathbb{R}$, and $A = (a_{ij})$ is an $m \times n$ matrix, it is understood as an $m \times n$ matrix $(a + a_{ij})$, namely, $aJ_{m\times n} + A$.*

The scattering constants can be computed directly from this definition only in very simple cases, and even then, the calculation is already quite involved.

*Example.* We compute the scattering constants of the principal congruence subgroup $\Gamma(3)$ by first considering the degenerate Kloosterman sum

$$r^{\Gamma(3)}_{jk} \coloneqq \phi^{\Gamma(3)}_{jk,0} = \#\left\{d \pmod{3c} \,\middle|\, \exists\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma\gamma_k\right\}$$

in Definition 2.3 for $\Gamma(3)$. Recall that $\Gamma(3)$ is a subgroup of $\Gamma(1) = \mathrm{PSL}_2(\mathbb{Z})$, so

$$\begin{aligned}
j = k = \infty\colon \quad & \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1}\Gamma(3)\gamma_k \iff (c, d) \equiv \pm(0, 1) \pmod 3 \\
j = 0, k = \infty\colon \quad & \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1}\Gamma(3)\gamma_k \iff (c, d) \equiv \pm(1, 0) \pmod 3 \\
j = 1, k = \infty\colon \quad & \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1}\Gamma(3)\gamma_k \iff (c, d) \equiv \pm(1, 1) \pmod 3 \\
j = 2, k = \infty\colon \quad & \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1}\Gamma(3)\gamma_k \iff (c, d) \equiv \pm(1, 2) \pmod 3
\end{aligned}$$

by an easy calculation. It follows that

$$\text{if } j = k, \text{ then } r_{jk}^{\Gamma(3)} = \begin{cases} 3\phi(c), & \text{if } 3 \mid c \\ 0, & \text{if } 3 \nmid c \end{cases}$$

$$\text{if } j \neq k, \text{ then } r_{jk}^{\Gamma(3)} = \begin{cases} 0, & \text{if } 3 \mid c \\ \phi(c), & \text{if } 3 \nmid c \end{cases}$$

because $\Gamma(3)$ is a normal subgroup of $\Gamma(1)$ ([4, Proposition 6.2.4 (3)]).

We then make the observation that

$$\begin{aligned} L(s) &\coloneqq \sum_{3\mid c} \frac{3\phi(c)}{c^{2s}} = \sum_{c>0} \frac{3\phi(3c)}{(3c)^{2s}} = \sum_{3\mid c} \frac{3\phi(3c)}{(3c)^{2s}} + \sum_{3\nmid c} \frac{3\phi(3c)}{(3c)^{2s}} = \sum_{3\mid c} \frac{9\phi(c)}{(3c)^{2s}} + \sum_{3\nmid c} \frac{6\phi(c)}{(3c)^{2s}} \\ &= \sum_{3\mid c} \frac{3\phi(c)}{(3c)^{2s}} + \sum_{c>0} \frac{6\phi(c)}{(3c)^{2s}} \overset{(*)}{=} 3^{-2s} L(s) + (6 \cdot 3^{-2s}) \frac{\zeta(2s-1)}{\zeta(2s)} \end{aligned}$$

where $(*)$ uses the identity $\displaystyle\sum_{c>0} \frac{\phi(c)}{c^s} = \frac{\zeta(s-1)}{\zeta(s)}$ ([12, Theorem 288]). This gives

$$\sum_{c>0} \frac{1}{c^{2s}} r_{jk}^{\Gamma(3)}(c) = L(s) = \frac{6 \cdot 3^{-2s}}{1-3^{-2s}} \cdot \frac{\zeta(2s-1)}{\zeta(2s)} = \frac{6}{3^{2s}-1} \cdot \frac{\zeta(2s-1)}{\zeta(2s)} \qquad \text{for } j = k$$

$$\sum_{c>0} \frac{1}{c^{2s}} r_{jk}^{\Gamma(3)}(c) = \sum_{3\nmid c} \frac{\phi(c)}{c^{2s}} = \sum_{c>0} \frac{\phi(c)}{c^{2s}} - \frac{1}{3} \cdot L(s) = \left(1 - \frac{2}{3^{2s}-1}\right) \cdot \frac{\zeta(2s-1)}{\zeta(2s)} \qquad \text{for } j \neq k.$$

We can finally compute the scattering constants for $\Gamma(3)$. Note that we have expansions

$$\frac{1}{9^s} = \frac{1}{9} - \frac{2\log(3)}{9}(s-1) + O((s-1)^2) \qquad \text{and} \qquad \frac{1}{3^{2s}-1} = \frac{1}{8} - \frac{9\log(3)}{32}(s-1) + O((s-1)^2);$$

and by the definition of scattering constants for $\Gamma(1)$, we also have

$$\sqrt{\pi} \cdot \frac{\Gamma(s-1/2)}{\Gamma(s)} \cdot \frac{\zeta(2s-1)}{\zeta(2s)} = \frac{3}{\pi(s-1)} + C^{\Gamma(1)} + O(s-1).$$

It thus follows that

$$\frac{\sqrt{\pi}}{9^s} \cdot \frac{\Gamma(s-1/2)}{\Gamma(s)} \cdot \left(\frac{6}{3^{2s}-1} \cdot \frac{\zeta(2s-1)}{\zeta(2s)}\right) = \frac{1}{4\pi(s-1)} + \left(\frac{1}{12} C^{\Gamma(1)} - \frac{17\log(3)}{16\pi}\right) + O(s-1)$$

$$\frac{\sqrt{\pi}}{9^s} \cdot \frac{\Gamma(s-1/2)}{\Gamma(s)} \cdot \left(\left(1 - \frac{2}{3^{2s}-1}\right) \cdot \frac{\zeta(2s-1)}{\zeta(2s)}\right) = \frac{1}{4\pi(s-1)} + \left(\frac{1}{12} C^{\Gamma(1)} - \frac{5\log(3)}{16\pi}\right) + O(s-1)$$

and therefore

$$C_{jk}^{\Gamma(3)} = \lim_{s\to 1} \left( \frac{\sqrt{\pi}}{9^s} \cdot \frac{\Gamma(s-1/2)}{\Gamma(s)} \cdot \sum_{c>0} \frac{1}{c^{2s}} r_{jk}^{\Gamma(3)}(c) - \frac{1}{4\pi(s-1)} \right) = \frac{1}{12} C^{\Gamma(1)} - \left(\frac{5}{4} + 3\delta_{jk}\right) \frac{\log(3)}{4\pi}. \tag{2.1}$$

□

In limited cases where the cusps of $\Gamma \subseteq \Gamma(1)$ are totally ramified from cusps of a larger group $\Gamma'$, the following linear relations of scattering constants can be helpful:

**Proposition 2.4** ([3, Proposition 1.3])**.** *Let $\Gamma \subseteq \Gamma'$ be finite index subgroups of $\Gamma(1)$, $S_{j'}^{\Gamma'}$ and $S_{k'}^{\Gamma'}$ two cusps of $\Gamma'$. Then*

$$\sum_{S_j^\Gamma \subseteq S_{j'}^\Gamma} \frac{b_j}{b'_j} C_{jk}^{\Gamma} = C_{j'k'}^{\Gamma'} - \frac{1}{\mathrm{vol}(\Gamma)} \sum_{S_j^\Gamma \subseteq S_{j'}^\Gamma} \frac{b_j}{b'_j} \log\left(\frac{b_j b_k}{b'_j b'_k}\right)$$

*where we sum over a system of representatives $\left\{S_j^\Gamma\right\}$ of cusps of $\Gamma$ such that $S_j^\Gamma \sim_{\Gamma'} S_{j'}^{\Gamma'}$ and $S_k^\Gamma$ is any cusp of $\Gamma$ with $S_k^\Gamma \sim_{\Gamma'} S_{k'}^{\Gamma'}$. The $b_j, b_k$ and $b'_j, b'_k$ respectively denote the width of the cusps of $\Gamma$ and $\Gamma'$.*

*Example.* Let $p$ and $q$ be distinct prime numbers such that $q \geq 5$ and $q$ does not divide $p-1$. We define

$$\Delta(z)^{1/q} := e^{2\pi i z/q} \prod_{k=1}^{\infty} (1 - e^{2\pi i k z})^{24/q}, \qquad \text{for } z \in \mathbb{H}$$

by taking the branch so that $(1 - e^{2\pi i k z})^{24/q}$ is positive when $z$ is purely imaginary. Put

$$f(z) := \Delta(pz)^{1/q} / \Delta(z)^{1/q}.$$

Then $f(z)^q$ is an automorphic function with respect to $\Gamma_0(p)$ because $\Delta(pz) \in S_{12}(\Gamma_0(p))$. For $\gamma \in \Gamma_0(p)$, put

$$\chi(\gamma) = f(\gamma z)/f(z).$$

Since $\chi(\gamma)^q = 1$, we see that $\chi(\gamma)$ does not depend on $z$ and so $\chi$ is a character on $\Gamma_0(p)$. Let $\Gamma_\chi$ be the kernel of $\chi$. It is a noncongruence subgroup ([34, Theorem 3.1]) such that $[\Gamma_0(p) \colon \Gamma_\chi] = q$ with two cusps $\infty$ and 0 ([34, §4]).

The computation of the scattering constants of $\Gamma_\chi$ can be reduced to that of $\Gamma_0(p)$ using Proposition 2.4:

$$q \cdot C_{jk}^{\Gamma_\chi} = C_{jk}^{\Gamma_0(p)} - \frac{1}{\mathrm{vol}(\Gamma_\chi)} \cdot q \log(q^2) \tag{2.2}$$

because the cusps $\infty, 0$ of $\Gamma_0(p)$ completely ramifies at the level of $\Gamma_\chi$. Meanwhile, Proposition 2.4 alone does not help reduce that scattering constants of $\Gamma_0(p)$ to that of $\Gamma(1)$ since the cusp $\infty$ of $\Gamma(1)$ splits into cusps $\infty, 0$ at the level of $\Gamma_0(p)$. Luckily, enough entries in the scattering matrix of $\Gamma_0(p)$ are known

$$C_{jk}^{\Gamma_0(p)} = \frac{1}{[\Gamma(1) \colon \Gamma_0(p)]} \cdot C_{\infty\infty}^{\Gamma(1)} - \frac{1}{\mathrm{vol}(\Gamma_0(p))} \cdot \begin{pmatrix} 2p^2 & p^2 - 2p - 1 \\ p^2 - 2p - 1 & ??? \end{pmatrix} \cdot \frac{\log(p)}{p^2 - 1} \tag{2.3}$$

([11, Example 2]) where $[\Gamma(1) \colon \Gamma_0(p)] = p + 1$ ([4, Corollary 6.2.13]). Using Proposition 2.4 then gives

$$1 \cdot C_{\infty 0}^{\Gamma_0(p)} + p \cdot C_{00}^{\Gamma_0(p)} = C_{\infty\infty}^{\Gamma(1)} - \frac{1}{\mathrm{vol}(\Gamma_0(p))} \cdot \left(1 \cdot \log(p) + p \cdot \log(p^2)\right)$$

and hence

$$\begin{aligned} C_{00}^{\Gamma_0(p)} &= \frac{1}{p}\left(\frac{p}{p+1}\cdot C_{\infty\infty}^{\Gamma(1)} - \frac{1}{\mathrm{vol}(\Gamma_0(p))}\cdot\left(1+2p-\frac{p^2-2p-1}{p^2-1}\right)\cdot\log(p)\right) \\ &= \frac{1}{[\Gamma(1)\colon\Gamma_0(p)]}\cdot C_{\infty\infty}^{\Gamma(1)} - \frac{1}{\mathrm{vol}(\Gamma_0(p))}\cdot\frac{2p^2}{p^2-1}\cdot\log(p). \end{aligned} \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), we obtain the scattering constants for $\Gamma_\chi$

$$\begin{aligned} C_{jk}^{\Gamma_\chi} &= \frac{1}{[\Gamma_0(p)\colon\Gamma_\chi]}\cdot C_{jk}^{\Gamma_0(p)} - \frac{1}{\mathrm{vol}(\Gamma_\chi)}\cdot 2\log(q) \\ &= \frac{1}{[\Gamma(1)\colon\Gamma_\chi]}C_{\infty\infty}^{\Gamma(1)} - \frac{1}{\mathrm{vol}(\Gamma_\chi)}\begin{pmatrix} 2p^2 & p^2-2p-1 \\ p^2-2p-1 & 2p^2 \end{pmatrix}\frac{\log(p)}{p^2-1} - \frac{1}{\mathrm{vol}(\Gamma_\chi)}\cdot 2\log(q). \end{aligned}$$

□

However, in most cases, we cannot compute the scattering constants from Proposition 2.4. So we survey Posingies' framework briefly, and apply it to three families of noncongruence subgroups mentioned in the introduction.

### 2.1.1 Posingies' Framework

To compute the scattering constants of what are known to be the Fermat groups $\Gamma_N$ associated to Fermat curves $F_N$ ([24, §5]), Posingies made use of Kronecker Limit formulas:

**Proposition 2.5** ([24, Proposition 3.4])**.** *Let $\Gamma$ be a finite index subgroup of $\Gamma(1)$, $S_j$ be a cusp of $\Gamma$. Suppose there is a modular form $G_j^\Gamma \in \mathrm{M}_k(\Gamma)$ ($k \in \mathbb{N}$), that only vanishes at the cusp $S_j$. Then there is a constant $A_j^\Gamma \in \mathbb{R}$ (depending on $G_j^\Gamma$) such that*

$$4\pi \lim_{s\to 1}\left(E_j^\Gamma(z,s) - \frac{1}{\mathrm{vol}(\Gamma)(s-1)}\right) = \frac{-1}{[\Gamma(1)\colon\Gamma]\cdot\frac{k}{12}}\log\left\|G_j^\Gamma\right\|^2 + A_j^\Gamma.$$

This observation is useful because we have the following relations for the scattering constants:

**Proposition 2.6** ([24, Corollary 2.9])**.** *Let $\Gamma$ be a finite index subgroup of $\Gamma(1)$, $S_j$ and $S_k$ cusps of $\Gamma$. We have the following Fourier expansion*

$$\lim_{s\to 1}\left(E_j^\Gamma(\gamma_k z,s) - \frac{1}{\mathrm{vol}(\Gamma)(s-1)}\right) = \delta_{jk}\frac{y}{b_j} + C_{jk}^\Gamma + \frac{\log(b_k)}{\mathrm{vol}(\Gamma)} - \frac{\log(y)}{\mathrm{vol}(\Gamma)} + \text{higher order terms} \tag{2.5}$$

In addition, the constant $A_j^\Gamma$ in Proposition 2.5 can be determined by considering the Eisenstein series of some ambient group:

**Proposition 2.7** ([24, Corollary 2.12])**.** *Let $\Gamma \subseteq \Gamma'$ be finite index subgroups of $\Gamma(1)$, and $S_j$ be a cusp of cusp width $b_j$ in $\Gamma$ and $w_j$ in $\Gamma'$, respectively. Then we have*

$$\begin{aligned} &\sum_{\gamma\in\Gamma\backslash\Gamma'} \lim_{s\to 1}\left(E_j^\Gamma(\gamma\gamma_k z,s) - \frac{1}{\mathrm{vol}(\Gamma)(s-1)}\right) \\ &\qquad = \lim_{s\to 1}\left(E_j^{\Gamma'}(\gamma_k z,s) - \frac{1}{\mathrm{vol}(\Gamma')(s-1)}\right) - \frac{1}{\mathrm{vol}(\Gamma')}\cdot\log\left(\frac{b_j}{w_j}\right). \end{aligned}$$

In other words, Posingies' framework reduces the computation of scattering constants to finding modular forms $G_j^\Gamma$ with divisors $\operatorname{div} G_j^\Gamma = m \cdot j$ for some $m \in \mathbb{N}$. A simple calculation for the principal congruence subgroup $\Gamma(3)$ using Posingies' framework is in Section 2.4.3.

## 2.2 Sansone's Group $\Gamma_{6N}$

We begin with a variant of the Fermat groups, which we call Sansone's group $\Gamma_{6N}$ ([23]). Let

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad U := ST^{-1}S^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{2.6}$$

be in $\Gamma(1)$, and recall that $\Gamma(2)$ is a free group with generators $\{T^2, U^2\}$ ([26, Appendix]). Then $\Gamma_{6N}$ is defined as the subgroup of $\Gamma(2)$ with words in $T^2, U^2$ for which the sum of the exponents of $T^2$ is divisible by $N$ ([23, Theorem 1]).[1] It has index $[\Gamma(1) \colon \Gamma_{6N}] = 6N$ ([23, §2]) with coset representatives $\Gamma_{6N} \setminus \Gamma(2) = \{T^{2n} \mid 0 \le n < N\}$ ([26, §1]). There are $N+2$ cusps. A system of cusp representatives are $S_0 = \{0, 2, \cdots, 2N-2\}$ with cusp width 2, and $S_1 = \{1\}$, $S_\infty = \{\infty\}$ both with cusp width $2N$. Moreover, $\Gamma_{6N}$ is a noncongruence subgroup if and only if $N \nmid 8$ ([23, Theorem 2]).

Recall the following modular forms of weight 2 ([24]):

$$G_0^{\Gamma(2)}(z) := \frac{\lambda(z)}{(1-\lambda)(z)}\theta^2(z), \quad G_1^{\Gamma(2)}(z) := \theta^2(z), \quad G_\infty^{\Gamma(2)}(z) := \frac{1}{(1-\lambda)(z)}\theta^2(z),$$

where

$$\lambda(z) = \frac{-1}{16} q^{-1/2} \prod_{n \ge 1} \left( \frac{1 - q^{n-1/2}}{1+q^n} \right), \quad (1-\lambda)(z) = \frac{1}{16} q^{-1/2} \prod_{n \ge 1} \left( \frac{1 + q^{n-1/2}}{1+q^n} \right), \tag{2.7}$$

and $\theta^2(z) = \prod_{n \ge 1}(1-q^n)^4(1+q^{n-1/2})^8$ are modular functions. Then $\operatorname{div} G_j^{\Gamma(2)} = 1 \cdot j$ on $\Gamma(2)$ at all cusps $j \in \{0, 1, \infty\}$ ([24, §4]).

We can show as in [26, §1] that the $N$-th root $y := (G_1^{\Gamma(2)}/G_\infty^{\Gamma(2)})^{1/N}$ is a modular function on $\Gamma_{6N}$, and we have

$$\operatorname{div} y = 1 - \infty, \quad \operatorname{div}(y - \zeta^n) = a_j - \infty \quad \text{and} \quad \operatorname{div} \theta^2 = N \cdot 1,$$

where $\zeta = e^{\frac{2\pi i}{N}}$. Thus, taking for $n = 0, 1, ..., N-1$,

$$G_{2n}^{\Gamma_{6N}} := \left( \frac{y - \zeta^n}{y} \right)^N \cdot \theta^2, \quad G_1^{\Gamma_{6N}} := \theta^2 \quad \text{and} \quad G_\infty^{\Gamma_{6N}} := \left( \frac{1}{y} \right)^N \cdot \theta^2$$

gives $\operatorname{div} G_j^{\Gamma_{6N}} = N \cdot j$ at all cusps $j \in S_0 \cup S_1 \cup S_\infty$. These are the modular forms we need to apply Posingies' framework to $\Gamma_{6N}$.

Using the transformation rules ([24, Lemma 6.4])

$$y \circ T^2 = \zeta^{-1} y \quad \text{and} \quad y \circ U^2 = y,$$

[1] This is a variant of the Fermat groups in the sense that Fermat group $\Gamma_N \subseteq \Gamma(2)$ consists of words in $T^2, U^2$ for which the sum of the exponents of both $T^2$ and $U^2$ are divisible by $N$ ([24, Lemma 5.3]).

we get the products

$$\begin{aligned}\prod_{\gamma\in\Gamma_{6N}\backslash\Gamma(2)} G_{2n}^{\Gamma_{6N}}\big|_2\gamma &= \left(-G_0^{\Gamma(2)}\right)^N\\ \prod_{\gamma\in\Gamma_{6N}\backslash\Gamma(2)} G_{1}^{\Gamma_{6N}}\big|_2\gamma &= \left(G_1^{\Gamma(2)}\right)^N\\ \prod_{\gamma\in\Gamma_{6N}\backslash\Gamma(2)} G_{\infty}^{\Gamma_{6N}}\big|_2\gamma &= \left(G_\infty^{\Gamma(2)}\right)^N.\end{aligned} \tag{2.8}$$

This allows us to obtain the Kronecker limit formula for $\Gamma_{6N}$: note that

$$\begin{aligned}&\sum_{\gamma\in\Gamma_{6N}\backslash\Gamma(2)} 4\pi\lim_{s\to1}\left(E_j^{\Gamma_{6N}}(\gamma z,s)-\frac{1}{\mathrm{vol}(\Gamma_{6N})(s-1)}\right)\\ &\quad= 4\pi\lim_{s\to1}\left(E_j^{\Gamma(2)}(z,s)-\frac{1}{\mathrm{vol}(\Gamma(2))(s-1)}-\frac{\log(1)}{\mathrm{vol}(\Gamma(2))}\right) && \text{(by Proposition 2.7)}\\ &\quad= -\log\left\|G_\infty^{\Gamma(2)}(z)\right\|^2+A_\infty^{\Gamma(2)} && \text{(by Proposition 2.5)}\\ &\quad= -\frac{1}{N}\sum_{\gamma\in\Gamma_{6N}\backslash\Gamma(2)}\log\left\|G_j^{\Gamma_{6N}}|_2\gamma(z)\right\|^2+A_\infty^{\Gamma(2)} && \text{(by eq. (2.8))}\end{aligned}$$

at all cusps $j\in S_0$, and likewise

$$\begin{aligned}&\sum_{\gamma\in\Gamma_{6N}\backslash\Gamma(2)} 4\pi\cdot\lim_{s\to1}\left(E_j^{\Gamma_{6N}}(\gamma z,s)-\frac{1}{\mathrm{vol}(\Gamma_{6N})(s-1)}\right)\\ &\quad=\frac{-1}{N}\sum_{\gamma\in\Gamma_{6N}\backslash\Gamma(2)}\log\left\|G_j^{\Gamma_{6N}}\big|_2\gamma(z)\right\|^2+A_j^{\Gamma(2)}-2\log(N)\end{aligned}$$

at all cusps $j\in\{1,\infty\}$, where

$$A_j^{\Gamma(2)}=4\left(\frac{6}{\pi}C^{\Gamma(1)}+\frac{1}{6}\log(2)\right)$$

for all $j\in\{0,1,\infty\}$ ([24, Proposition 4.1]); so the fact that the constant $A_j^{\Gamma_{6N}}$ is independent of the cusp in which the Fourier expansion is taken ([24, Remark 3.5 and §7]) implies that

$$\begin{aligned}&4\pi\lim_{s\to1}\left(E_j^{\Gamma_{6N}}(z,s)-\frac{1}{\mathrm{vol}(\Gamma_{6N})(s-1)}\right)\\ &\quad=-\frac{1}{N}\log\left\|G_j^{\Gamma_{6N}}\right\|^2+\frac{1}{N}\left(\frac{24}{\pi}C^{\Gamma(1)}+\frac{2}{3}\log(2)-2\log(N)\delta_j\right)\end{aligned} \tag{2.9}$$

where $\delta_j=1$ if $j\in\{1,\infty\}$ and $\delta_j=0$ otherwise. Then we have

**Theorem 2.8.** *Let $\Gamma_{6N}$ be Sansone's group defined above. With respect to its cusps $S_0\cup S_1\cup S_\infty$ in order, its scattering constants are given by:*

$$\left(C_{jk}^{\Gamma_{6N}}\right)=\frac{1}{6N}\left(C^{\Gamma(1)}-\frac{2}{\pi}\log(2)-\frac{2}{\pi}\begin{pmatrix}3\mathbf{C}_N & 3\log(N)\cdot J_{N\times2}\\ 3\log(N)\cdot J_{2\times N} & 12\log(2)\cdot I_2+6\log(N)\cdot J_{2\times2}\end{pmatrix}\right).$$

*Proof.* We compare the constant terms on both sides of (2.9) with

$$\gamma_{2n} = \begin{pmatrix} 2n & -1 \\ 1 & 0 \end{pmatrix} = T^{2n}S \quad \text{for all } n \in \{0, 1, \cdots, N-1\},$$
$$\gamma_1 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = TS \quad \text{and} \quad \gamma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

acting on $z$ as in [24, Proposition 4.1]. On the left hand side, we use (2.5) to get

$$\begin{aligned} &4\pi \begin{pmatrix} \frac{y}{2} \cdot I_N & \mathbf{0}_{N\times 2} \\ \mathbf{0}_{2\times N} & \frac{y}{2N} \cdot I_2 \end{pmatrix} + \left(C_{jk}^{\Gamma_{6N}}\right) \\ &\quad + \frac{1}{\operatorname{vol}(\Gamma_{6N})} \left( \begin{pmatrix} \log(2) \cdot J_{(N+2)\times N} & \log(2N) \cdot J_{(N+2)\times 2} \end{pmatrix} - \frac{\log(y)}{\operatorname{vol}(\Gamma_{6N})} \cdot I_{N+2} \right). \end{aligned} \tag{2.10}$$

On the right hand side, we have transformation rules ([26, Proposition 1.3] and [4, Corollary 2.3.11])

$$\begin{aligned} x \circ T &= y, & y \circ T &= \zeta^{-1}x, & \theta^2 \circ T &= \left(\frac{-x^N}{y^N}\right)\theta^2 \\ x \circ S &= \zeta(1/x), & y \circ S &= \epsilon\left(\frac{y}{x}\right), & \theta^2 \circ S &= -\theta^2, \end{aligned}$$

where $\epsilon = e^{\frac{\pi i}{N}}$, and $x := \left(G_0^{\Gamma(2)}/G_\infty^{\Gamma(2)}\right)^{1/N}$ is the $N$-th root. Then we have the actions $G_j^{\Gamma_{6N}}\big|_2\gamma_k$ tabulated in Table 2.1; hence, the constant term on the right hand side of (2.9) is

$$\begin{aligned} &\frac{-1}{N}\left(2 \cdot \begin{pmatrix} \mathbf{C}_N + N\log\left|q^{1/2}\right| \cdot I_N & \mathbf{0}_{N\times 2} \\ \mathbf{0}_{2\times N} & \log\left|16q^{1/2}\right| \cdot I_2 \end{pmatrix} + \log(y)\right) \\ &\qquad + \frac{1}{N}\left(A^{\Gamma(2)} - 2 \cdot \begin{pmatrix} \mathbf{0}_{N\times(N+2)} \\ \log(N) \cdot J_{2\times(N+2)} \end{pmatrix}\right), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} \mathbf{C}_N &= N\left(\log\left|\frac{16}{N}\right| \cdot I_N + \left(\log\left|1 - \zeta^{\overline{n-m}}\right|\right)_{m,n}\right) \\ &= N\left((4\log(2) - \log(N)) \cdot I_N + \left(\log\left|1 - \zeta^{\overline{n-m}}\right|\right)_{m,n}\right). \end{aligned}$$

($\overline{n-m}$ denotes the image of the integer $n-m$ in $\mathbb{Z}/N\mathbb{Z}$.) Note that $\dfrac{4\pi}{\operatorname{vol}(\Gamma_{6N})} = \dfrac{2}{N}$ and $\log|q| = \log\left|e^{2\pi i(x+iy)}\right| = -2\pi y$. Hence

$$\begin{aligned} \left(C_{jk}^{\Gamma_{6N}}\right) &= \frac{1}{N}\left(\frac{1}{6}C^{\Gamma(1)} + \frac{1}{\pi}\left(\frac{1}{6}\log(2) - \frac{1}{2}\begin{pmatrix} \mathbf{0}_{N\times(N+2)} \\ \log(N) \cdot J_{2\times(N+2)} \end{pmatrix}\right)\right) \\ &\quad + \begin{pmatrix} \mathbf{C}_N & \mathbf{0}_{N\times 2} \\ \mathbf{0}_{2\times N} & \log(16) \cdot I_2 \end{pmatrix} + \begin{pmatrix} \log(2) \cdot J_{(N+2)\times N} & \log(2N) \cdot J_{(N+2)\times N} \end{pmatrix} \end{aligned}$$

by equating the constant-term matrices (2.10) and (2.11). Rearranging the terms gives rise to the scattering constants for $\Gamma_{6N}$.

∎

| $G_j^{\Gamma_{6N}}\big\|_2\gamma_k$ | $\gamma_{2n}$ | $\gamma_1$ | $\gamma_\infty$ |
|---|---|---|---|
| $G_{2n}^{\Gamma_{6N}}$ | $(-1)^N\left(\frac{x-e^{\pi i/N}\zeta^{N-n+1}y}{y}\right)^N\theta^2$ | $\left(\frac{x-\zeta^{N-n+1}}{y}\right)^N\theta^2$ | $G_{2n}^{\Gamma_{6N}}$ |
| $G_1^{\Gamma_{6N}}$ | $-G_1^{\Gamma_{6N}}$ | $-G_\infty^{\Gamma_{6N}}$ | $G_1^{\Gamma_{6N}}$ |
| $G_\infty^{\Gamma_{6N}}$ | $G_0^{\Gamma_{6N}}$ | $-G_0^{\Gamma_{6N}}$ | $G_\infty^{\Gamma_{6N}}$ |

Table 2.1: The action of the coset representatives on $G_j^{\Gamma_{6N}}$

## 2.3 Yang and Yin's Group $Y_2(N)$

Let

$$B_2 := ST^{-2}S^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix},$$

and for a fixed positive integer $N$, we define Yang and Yin's group $Y_2(N)$ to be the subgroup of $\Gamma_0(2)$, generated by its commutator subgroup $\Gamma_0(2)'$, $T^N$ and $B_2^N$. It is a noncongruence group if and only if $N \nmid 24$ ([32, Theorem 2.7]). Since the group $Y_2(N)$ and hence the modular curve $X_{Y_2(N)}$ have significant differences in their properties and ramification behaviors for the cases $N$ is odd and $N$ is even, we split our computation into two cases, based on the parity of $N$.

### 2.3.1 The Case $2 \nmid N$

When $N$ is odd, the modular curve $X_{Y_2(N)}$ has only two cusps 0 and $\infty$ ([32, Theorem 6.3]), so the cover $X_{Y_2(N)} \to X_{\Gamma_0(2)}$ is totally ramified at both cusps. By Proposition 2.4 and the fact that $[\Gamma_0(2) \colon Y_2(N)] = N$ ([32, Corollary 2.5]), we see that

$$N \cdot C_{jk}^{Y_2(N)} = C_{jk}^{\Gamma_0(2)} - \frac{2}{\pi} \cdot \log(N) \tag{2.12}$$

for $j, k \in \{0, \infty\}$ ([4, Proposition 6.3.8(b)]). In other words, in order to compute the scattering constants $C_{jk}^{Y_2(N)}$ when $N$ is odd, it suffices to compute $C_{jk}^{\Gamma_0(2)}$.

The first step is to find the constant $A_j^{\Gamma_0(2)}$ in the Kronecker limit formula (Proposition 2.5) for $\Gamma_0(2)$. Recall that $\Gamma_0(2)$ has two cusps with stabilizer group and cusp width tabulated as follows:

| $S_j$ | $0$ | $\infty$ |
|---|---|---|
| $\mathrm{Stab}_{\Gamma_0(2)}(S_j)$ | $\left\langle\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\right\rangle$ | $\left\langle\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\rangle$ |
| $b_j$ | 2 | 1 |

and $[\Gamma(1)\colon \Gamma_0(2)] = 3$ with coset representatives

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}. \tag{2.13}$$

Moreover, we have a modular form of weight 8

$$f(z) \coloneqq \eta^8(z)\eta^8(2z) = q \prod_{n=1}^{\infty} (1-q^n)^8 (1-q^{2n})^8$$

and a modular function

$$\omega_2(z) \coloneqq 2^{12} \cdot \frac{\Delta(2z)}{\Delta(z)} = 2^{12} q \prod_{n>0} (1+q^n)^{24} \tag{2.14}$$

on $\Gamma_0(2)$ ([32, eq (1.2)]), where

$$\eta(z) = q^{1/24} \prod_{n \geq 1} (1-q^n) \quad \text{and} \quad \Delta(z) = \eta^{24}(z) = q \prod_{n \geq 1} (1-q^n)^{24}.$$

Note that $\operatorname{div} f = 1 \cdot 0 + 1 \cdot \infty$ and $\operatorname{div} \omega_2 = -1 \cdot 0 + 1 \cdot \infty$ due to the following proposition.[2]

**Proposition 2.9** ([4, Proposition 5.9.3(1)])**.** *Let $a/c \in \mathbb{P}^1(\mathbb{Q})$ be a cusp with* $\gcd(a,c) = 1$ *and* $c > 0$*. The order of $\eta(mz)$ at $a/c$ is given by*

$$\nu_{a/c}(\eta(mz)) = \frac{1}{24} \frac{\gcd(m,c)^2}{m}.$$

Therefore, we can construct modular forms

$$\begin{aligned} G_0^{\Gamma_0(2)} &\coloneqq \frac{f}{\omega_2} = \frac{1}{2^{12}} \prod_{n \geq 1} \left( \frac{1-q^n}{1+q^n} \right)^{16} = \frac{1}{2^{12}} \cdot \frac{\eta^{32}(z)}{\eta^{16}(2z)} \\ G_\infty^{\Gamma_0(2)} &\coloneqq f \cdot \omega_2 = 2^{12} q^2 \prod_{n \geq 1} (1+q^n)^{16} (1-q^{2n})^{16} = 2^{12} \cdot \frac{\eta^{32}(2z)}{\eta^{16}(z)} \end{aligned} \tag{2.15}$$

on $\Gamma_0(2)$ with divisors $\operatorname{div} G_0^{\Gamma_0(2)} = 1 \cdot 0$ and $\operatorname{div} G_\infty^{\Gamma_0(2)} = 2 \cdot \infty$, satisfying the conditions of Proposition 2.5.

To determine the constants $A_j^{\Gamma_0(2)}$ in Proposition 2.5, we compute the products

$$\prod_{\gamma \in \Gamma_0(2) \backslash \Gamma(1)} G_j^{\Gamma_0(2)}|_8 \gamma(z), \quad \text{where } j \in \{0, \infty\}$$

as in [24, Lemma 6.4]. This involves finding the actions of the coset representatives (2.13) on the modular forms (2.15) using ([4, Theorem 5.8.1(a)])

$$\eta^4(z+1) = e^{\pi i/3} \eta^4(z) \quad \text{and} \quad \eta^4\left(\frac{-1}{z}\right) = -z^2 \cdot \eta^4(z). \tag{2.16}$$

[2] Note that the cusp width at 0 is 2.

| $\gamma \in \Gamma_0(2)\backslash\Gamma(1)$ | $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ | $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ | $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ |
|---|---|---|---|
| $G_0^{\Gamma_0(2)}\vert_8\gamma(z)$ | $\frac{1}{2^{12}}\cdot\frac{\eta^{32}(z)}{\eta^{16}(2z)}$ | $\frac{1}{2^{4}}\cdot\frac{\eta^{32}(z)}{\eta^{16}(z/2)}$ | $\frac{1}{2^{4}}\cdot\frac{\eta^{32}(z+1)}{\eta^{16}((z+1)/2)}$ |
| $G_\infty^{\Gamma_0(2)}\vert_8\gamma(z)$ | $2^{12}\cdot\frac{\eta^{32}(2z)}{\eta^{16}(z)}$ | $\frac{1}{2^{4}}\cdot\frac{\eta^{32}(z/2)}{\eta^{16}(z)}$ | $\frac{1}{2^{4}}\cdot\frac{\eta^{32}((z+1)/2)}{\eta^{16}(z+1)}$ |

Table 2.2: The action of the coset representatives on $G_j^{\Gamma_0(2)}$

For demonstration purposes, we compute

$$\begin{aligned} G_0^{\Gamma_0(2)}\Big|_8\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}(z) &= \frac{1}{2^{12}}\cdot\frac{1}{(z+1)^8}\cdot\frac{\eta^{32}(z/(z+1))}{\eta^{16}(2z/(z+1))} \\ &= \frac{1}{2^{12}}\cdot\frac{e^{12\pi i/3}}{(z+1)^8}\cdot\frac{\eta^{32}(-1/(z+1))}{\eta^{16}(-2/(z+1))} \\ &= \frac{1}{2^{12}}\cdot\frac{1}{(z+1)^8}\cdot\frac{(z+1)^{16}}{((z+1)/2)^8}\cdot\frac{\eta^{32}(z+1)}{\eta^{16}((z+1)/2)} \\ &= \frac{1}{2^{4}}\cdot\frac{\eta^{32}(z+1)}{\eta^{16}((z+1)/2)} \end{aligned}$$

and the remaining results are tabulated in Table 2.2. Recall that $\dfrac{\eta^2((z+1)/2}{\eta(z+1)} = \dfrac{\eta^5(z)}{\eta^2(z/2)\eta^2(2z)}$ ([4, Proposition 2.3.13]), so

$$\prod_{\gamma\in\Gamma_0(2)\backslash\Gamma(1)} G_0^{\Gamma_0(2)}|_8\gamma(z) = \frac{1}{2^{20}}\cdot\Delta^2(z) \quad\text{and}\quad \prod_{\gamma\in\Gamma_0(2)\backslash\Gamma(1)} G_\infty^{\Gamma_0(2)}|_8\gamma(z) = 2^4\cdot\Delta^2(z). \tag{2.17}$$

We are now ready to compute the constants $A_j^{\Gamma_0(2)}$ in Proposition 2.5, which is possible thanks to Proposition 2.7. In particular, we have

$$\begin{aligned} &\sum_{\gamma\in\Gamma_0(2)\backslash\Gamma(1)} 4\pi\lim_{s\to1}\left(E_0^{\Gamma_0(2)}(\gamma z,s)-\frac{1}{\mathrm{vol}(\Gamma_0(2))(s-1)}\right) \\ &= 4\pi\lim_{s\to1}\left(E^{\Gamma(1)}(z,s)-\frac{1}{\mathrm{vol}(\Gamma(1))(s-1)}\right)-\frac{4\pi}{\mathrm{vol}(\Gamma(1))}\cdot\log 2 \quad\text{(by Proposition 2.7)} \\ &= -\log\|\Delta(z)\|^2 + A^{\Gamma(1)} \quad\text{(by eq (1.1))} \\ &= -\log\|\Delta(z)\|^2 + 4\pi C^{\Gamma(1)} \quad\text{(by [24, eq (1.0.1)]) and [3, eq (0.1)])} \\ &= -\frac{1}{2}\log\left\|\frac{1}{2^{20}}\cdot\Delta^2(z)\right\|^2 + 4\pi C^{\Gamma(1)} - 32\log(2) \\ &= -\frac{1}{2}\log\left\|\prod_{\gamma\in\Gamma_0(2)\backslash\Gamma(1)} G_0^{\Gamma_0(2)}|_8\gamma(z)\right\|^2 + 4\pi C^{\Gamma(1)} - 32\log(2) \quad\text{(by eq (2.17))} \\ &= -\frac{1}{2}\sum_{\gamma\in\Gamma_0(2)\backslash\Gamma(1)}\log\left\|G_0^{\Gamma_0(2)}|_8\gamma(z)\right\|^2 + 4\pi C^{\Gamma(1)} - 32\log(2) \end{aligned}$$

and likewise

$$\sum_{\gamma\in\Gamma_0(2)\backslash\Gamma(1)} 4\pi \lim_{s\to 1}\left(E_\infty^{\Gamma_0(2)}(\gamma z,s)-\frac{1}{\mathrm{vol}(\Gamma_0(2))(s-1)}\right)$$
$$=-\frac{1}{2}\sum_{\gamma\in\Gamma_0(2)\backslash\Gamma(1)}\log\left\|G_\infty^{\Gamma_0(2)}|_8\gamma(z)\right\|^2+4\pi C^{\Gamma(1)}+4\log(2).$$

Since the constant $A_j^{\Gamma_0(2)}$ is independent of the cusp in which the Fourier expansion is taken ([24, Remark 3.5 and §7]) for all $j\in\{0,\infty\}$, it follows that

$$A_j^{\Gamma_0(2)}=\frac{4\pi}{3}C^{\Gamma(1)}+\begin{cases}-\frac{32}{3}\log(2), & \text{if } j=0\\ \frac{4}{3}\log(2), & \text{if } j=\infty.\end{cases}$$

From the Kronecker limit formula

$$4\pi\lim_{s\to 1}\left(E_j^{\Gamma_0(2)}(z,s)-\frac{1}{\mathrm{vol}(\Gamma_0(2))(s-1)}\right)=\frac{-1}{3\cdot\frac{8}{12}}\log\left\|G_j^{\Gamma_0(2)}\right\|^2+A_j^{\Gamma_0(2)},$$

we then have

**Theorem 2.10.** *Let $Y_2(N)$ be Yang and Yin's group defined above with $2\nmid N$. With respect to the cusps $0,\infty$ in order, its scattering constants are:*

$$C_{jk}^{Y_2(N)}=\frac{1}{3N}C^{\Gamma(1)}+\left(\frac{1}{3}-3\delta_{jk}\right)\frac{\log(2)}{N\pi}-\frac{2}{N\pi}\log(N)$$
$$=\frac{1}{3N}C^{\Gamma(1)}+\frac{1}{N\pi}\left(\left(\frac{1}{3}-3\delta_{jk}\right)\log(2)-2\log(N)\right).$$

*Proof.* We compare the constant term matrices on both sides with

$$\gamma_0=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \gamma_\infty=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

acting on $z$ as in [24, Proposition 4.1]. On the left hand side, we use Proposition 2.6 to get

$$4\pi\left(\begin{pmatrix}y/2&0\\0&y\end{pmatrix}+\left(C_{jk}^{\Gamma_0(2)}\right)+\frac{\log(2)}{\mathrm{vol}(\Gamma_0(2))}I_2-\frac{\log(y)}{\mathrm{vol}(\Gamma_0(2))}.\right).$$

On the right hand side, we look up our computations in Table 2.2 and find

$$-\frac{1}{2}\left(2\cdot\begin{pmatrix}\log|q/2^4| & \log|1/2^{12}|\\ \log|1/2^4| & \log|2^{12}q^2|\end{pmatrix}+\log(y^8)\right)+A_j^{\Gamma_0(2)}.$$

Note that $\dfrac{4\pi}{\mathrm{vol}(\Gamma_0(2))}=4$, so equating the constant term matrices gives

$$\left(C_{jk}^{\Gamma_0(2)}\right)=-\frac{\log(2)}{\pi}I_2+\frac{1}{4\pi}\left(\begin{pmatrix}4&12\\4&-12\end{pmatrix}\log(2)+A_j^{\Gamma_0(2)}\right)$$

and so

$$C_{jk}^{\Gamma_0(2)} = \frac{1}{3}C^{\Gamma(1)} + \left(\frac{1}{3} - 3\delta_{jk}\right)\frac{\log(2)}{\pi}.$$

This agrees with Proposition 2.4 for $\Gamma_0(2) \subseteq \Gamma(1)$:

$$\begin{aligned} 2C_{00}^{\Gamma_0(2)} + C_{\infty 0}^{\Gamma_0(2)} &= C^{\Gamma(1)} - \frac{5\log(2)}{\pi} \\ 2C_{0\infty}^{\Gamma_0(2)} + C_{\infty\infty}^{\Gamma_0(2)} &= C^{\Gamma(1)} - \frac{2\log(2)}{\pi}. \end{aligned}$$

Therefore, we plug $C_{jk}^{\Gamma_0(2)}$ back into (2.12) and obtain our result. □

### 2.3.2 The Case $2 \mid N$

When $N$ is even, $Y_2(N)$ has four cusps with the stabilizer group and cusp width tabulated as follows:

| $S_j$ | $0$ | $1$ | $\infty$ | $1/2$ |
|---|---|---|---|---|
| $\mathrm{Stab}_{Y_2(N)}(S_j)$ | $\langle B_2^N \rangle$ | $\langle (T^2B_2)^N \rangle$ | $\langle T^N \rangle$ | $\langle (TB_2^2)^N \rangle$ |
| $b_j$ | 2N | 2N | N | N |

where

$$T^2B_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad TB_2^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

It is also known that $[\Gamma(2)\colon Y_2(N)] = N$ ([32, Lemma 4.1]) with coset representatives

$$\Gamma(2)/Y_2(N) = \left\{\mathrm{id}, T^2, \cdots, T^{N-2}, B_2, T^2B_2, \cdots, T^{N-2}B_2\right\}.$$

Moreover, consider the modular forms of weight 2 from [32, eq (6.1)]):

$$G_0^{Y_2(N)} := \frac{\lambda}{1-\lambda}\theta^2, \quad G_1^{Y_2(N)} := \theta^2, \quad G_\infty^{Y_2(N)} := h_1^2\theta^2 \qquad \text{and} \quad G_{1/2}^{Y_2(N)} := h_3^2\theta^2,$$

where $h_1 := \lambda\omega_2^{1/2} + 4$ and $h_3 := -\lambda\omega_2^{1/2} + 4$. Since the cover $X_{Y_2(N)} \to X_{\Gamma(2)}$ is totally ramified at the cusps $0, 1$ with ramification index $N$ ([32, §5]), we have $\operatorname{div} G_j^{Y_2(N)} = N \cdot j$ at all cusps $j \in \{0, 1, \infty, 1/2\}$. For future reference, we record the relation ([32, eq (4.1)]):

$$\omega_2 = \frac{16}{\lambda(\lambda-1)}. \tag{2.18}$$

To compute the products $\prod_{\gamma \in Y_2(N)\backslash\Gamma(2)} G_j^{Y_2(N)}|_2\gamma(z)$, we use the transformation laws

$$\begin{aligned} T^2 \circ h_1 = h_1 \quad &\text{and} \quad T^2 \circ h_3 = h_3 \\ B_2 \circ h_1 = -h_3 \quad &\text{and} \quad B_2 \circ h_3 = h_1 \end{aligned} \tag{2.19}$$

([32, Lemma 6.1]) for the coset representatives $\Gamma(2)/Y_2(N)$ to act on $G_\infty^{Y_2(N)}$ and $G_{1/2}^{Y_2(N)}$, which gives

$$
\begin{aligned}
&\prod_{\gamma\in Y_2(N)\backslash\Gamma(2)} G_\infty^{Y_2(N)}|_2\gamma(z)\\
&= \left(h_1^2 G_1^{Y_2(N)}\right)\left(h_1^2 G_1^{Y_2(N)}\right)\cdots\left(h_1^2 G_1^{Y_2(N)}\right)\left(h_3^2 G_1^{Y_2(N)}\right)\left(h_3^2 G_1^{Y_2(N)}\right)\cdots\left(h_3^2 G_1^{Y_2(N)}\right)\\
&= (h_1h_3)^N\left(G_1^{Y_2(N)}\right)^N = (-\lambda^2\omega_2+16)^N\left(G_1^{Y_2(N)}\right)^N \\
&\overset{(*)}{=} 16^N\left(-\lambda^2\frac{1}{(\lambda-1)\lambda}+1\right)^N\left(G_1^{Y_2(N)}\right)^N = 16^N\left(\frac{1}{1-\lambda}\right)^N\left(G_1^{Y_2(N)}\right)^N\\
&= 16^N\left(G_\infty^{Y_2(N)}\right)^N
\end{aligned}
\tag{2.20}
$$

(where the equation $(*)$ follows from (2.18)) and likewise

$$
\prod_{\gamma\in Y_2(N)\backslash\Gamma(2)} G_{1/2}^{Y_2(N)}|_2\gamma(z) = 16^N\left(G_\infty^{Y_2(N)}\right)^N. \tag{2.21}
$$

This allows us to obtain the Kronecker limit formula for $Y_2(N)$: Let $j\in\{\infty,1/2\}$ and note that

$$
\begin{aligned}
&\sum_{\gamma\in Y_2(N)\backslash\Gamma(2)} 4\pi\lim_{s\to1}\left(E_j^{Y_2(N)}(\gamma z,s)-\frac{1}{\mathrm{vol}(Y_2(N))(s-1)}\right)\\
&= 4\pi\lim_{s\to1}\left(E_j^{\Gamma(2)}(z,s)-\frac{1}{\mathrm{vol}(\Gamma(2))(s-1)}-\frac{\log(N/2)}{\mathrm{vol}(\Gamma(2))}\right) && \text{(by Proposition 2.7)}\\
&= -\log\left\|G_\infty^{\Gamma(2)}(z)\right\|^2 + A_\infty^{\Gamma(2)} - 2\log\left(\frac{N}{2}\right) && \text{(by Proposition 2.5)}\\
&= -\frac{1}{N}\sum_{\gamma\in Y_2(N)\backslash\Gamma(2)}\log\left\|G_j^{Y_2(N)}|_2\gamma(z)\right\|^2 + A_\infty^{\Gamma(2)} - 2\log(N)+10\log(2) && \text{(by eq (2.20) and eq (2.21))},
\end{aligned}
$$

where

$$
A_\infty^{\Gamma(2)} = 4\left(\frac{\zeta'(-1)}{\zeta(-1)}-\log(4\pi)+1+\frac{1}{6}\log(2)\right),
$$

([24, Proposition 4.1]); so the fact that the constant $A_j^{Y_2(N)}$ is independent of the cusp in which the Fourier expansion is taken ([24, Remark 3.5 and §7]) implies that

$$
4\pi\lim_{s\to1}\left(E_j^{Y_2(N)}(z,s)-\frac{1}{\mathrm{vol}(Y_2(N))(s-1)}\right) = -\frac{1}{N}\log\left\|G_j^{Y_2(N)}\right\|^2 + A_j^{Y_2(N)}, \tag{2.22}
$$

where

$$
A_j^{Y_2(N)} = \frac{1}{N}\left(A_\infty^{\Gamma(2)}-2\log(N)+10\log(2)\right) = \frac{4}{N}\left(\frac{\zeta'(-1)}{\zeta(-1)}-\log(4\pi)+1+\frac{8}{3}\log(2)-\frac{1}{2}\log(N)\right).
$$

Then we have

**Theorem 2.11.** *Let $Y_2(N)$ be Yang and Yin's group defined above with $2\mid N$. With respect to the* *cusps* $0, 1, \infty, 1/2$ *in order, its scattering constants are:*

$$\left(C_{jk}^{Y_2(N)}\right) = \frac{1}{6N}C^{\Gamma(1)} - \frac{1}{N\pi}\left(\frac{\log(2)}{6}\begin{pmatrix} 14 & 2 & -1 & -1 \\ 2 & 14 & -1 & -1 \\ -1 & -1 & 14 & 2 \\ -1 & -1 & 2 & 14 \end{pmatrix} + \log(N)\right).$$

*Proof.* To compute the scattering constants $C_{jk}^{Y_2(N)}$ for $j, k \in \{\infty, 1/2\}$, we compare the constant terms on both sides of (2.22) with

$$\gamma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_{1/2} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = B_2^{-1}$$

acting on $z$ as in [24, Proposition 4.1]. On the left hand side, we use (2.5) to get

$$4\pi\left(\frac{y}{N}I_2 + \left(C_{jk}^{Y_2(N)}\right) + \frac{\log(N)}{\mathrm{vol}(Y_2(N))} - \frac{\log(y)}{\mathrm{vol}(Y_2(N))}\right).$$

On the right hand side, we use the transformation laws (2.19) to get

$$h_1|_0\gamma_{1/2}(z) = h_1 \quad \text{and} \quad h_3|_0\gamma_{1/2}(z) = h_3,$$

and the transformation laws ([4, Corollary 2.3.11])

$$G_1^{Y_2(N)}(z+2) = G_1^{Y_2(N)}(z) \quad \text{and} \quad G_1^{Y_2(N)}\left(\frac{-1}{z}\right) = -z^2 G_1^{Y_2(N)}(z)$$

to get

$$\begin{aligned} G_1^{Y_2(N)}|_2\gamma_{1/2}(z) &= \frac{1}{(2z+1)^2}G_1^{Y_2(N)}\left(\frac{z}{2z+1}\right) = \frac{-1}{z^2}G_1^{Y_2(N)}\left(\frac{2z+1}{-z}\right) \\ &= \frac{-1}{z^2}G_1^{Y_2(N)}\left(\frac{-1}{z}\right) = G_1^{Y_2(N)}(z); \end{aligned}$$

so the constant term on the right hand side of (2.22) is

$$\frac{-4}{N}\begin{pmatrix} \log|32q^{1/2}| & \log(8) \\ \log(8) & \log|32q^{1/2}| \end{pmatrix} - \frac{\log(y^2)}{N} + A_j^{Y_2(N)}.$$

Note that $\dfrac{4\pi}{\mathrm{vol}(Y_2(N))} = \dfrac{2}{N}$; so equating the constant term matrices gives

$$\begin{aligned} \left(C_{jk}^{Y_2(N)}\right) &= \frac{-1}{\pi N}\left(\frac{1}{2}\log(N) + \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\log(2) - \left(\frac{\zeta'(-1)}{\zeta(-1)} - \log(4\pi) + 1 + \frac{8}{3}\log(2) - \frac{1}{2}\log(N)\right)\right) \\ &= \frac{1}{\pi N}\left(\frac{\zeta'(-1)}{\zeta(-1)} - \log(4\pi) + 1 - \frac{\log(2)}{3}\begin{pmatrix} 7 & 1 \\ 1 & 7 \end{pmatrix} - \log(N)\right) \end{aligned}$$

for $j,k \in \{\infty, 1/2\}$, which agrees with Proposition 2.4 for $Y_2(N) \subseteq \Gamma(2)$:

$$\frac{N}{2}\left(C^{Y_2(N)}_{\infty\infty} + C^{Y_2(N)}_{\frac{1}{2}\infty}\right) = \frac{1}{\pi}\left(\frac{\zeta'(-1)}{\zeta(-1)} - \log(4\pi) + 1 - \frac{4\log(2)}{3} - \log(N)\right) = C^{\Gamma(2)}_{\infty\infty} - \frac{\log(N/2)}{\pi}.$$

In fact, using Proposition 2.4 further gives

$$\begin{aligned} NC^{Y_2(N)}_{jk} &= C^{\Gamma(2)}_{jk} - \frac{1}{2\pi}\log\left(N^2\right) \quad \text{for } j,k \in \{0,1\} \\ NC^{Y_2(N)}_{jk} &= C^{\Gamma(2)}_{jk} - \frac{1}{2\pi}\log\left(\frac{N^2}{2}\right) \quad \text{for } j \in \{0,1\} \text{ and } k \in \{\infty, 1/2\} \end{aligned}$$

and so we deduce from the identity

$$C^{\Gamma(2)}_{jk} = \frac{1}{6}C^{\Gamma(1)} - \left(2\cdot\delta_{jk} + \frac{1}{3}\right)\frac{\log(2)}{\pi}$$

for $j,k \in \{0,1,\infty\}$ ([3, eq (1.4) and eq (1.5)]) that

$$\left(C^{Y_2(N)}_{jk}\right) = \frac{1}{6N}C^{\Gamma(1)} - \frac{1}{N\pi}\left(\frac{\log(2)}{6}\begin{pmatrix} 14 & 2 & -1 & -1 \\ 2 & 14 & -1 & -1 \\ * & * & 14 & 2 \\ * & * & 2 & 14 \end{pmatrix} + \log(N)\right)$$

for $j,k \in \{0,1,\infty,1/2\}$. Since the scattering matrix is symmetric ([20, Theorem 2.2.1]), we have our result. $\square$

*Remark.* Define $Y_{2,s}(N) = \langle Y_2(N), TB_2^{s+1}\rangle$, where the relation $Y_2(N) \subseteq Y_{2,s}(N) \subseteq \Gamma_0(2)$ holds. It is a noncongruence group if and only if $N \nmid 24$ ([32, Theorem 2.10]). We are interested in the case where $Y_{2,s}(N)$ has more than two cusps, where it has three cusps $\{0,\infty,1/2\}$, when ([32, Lemma 3.3])

1. $N \mid 2s, N \nmid s$ and $N \equiv 2 \pmod 4$

   In this case, $[\Gamma_0(2)\colon Y_{2,\frac{N}{2}}(N)] = N$ ([32, Lemma 2.9]), so $[Y_{2,\frac{N}{2}}(N)\colon Y_2(N)] = 2$, and the cusp widths at $0,\infty,1/2$ are $2N, N/2, N/2$, resp. Using Proposition 2.4, we obtain from the above theorem that

$$\left(C^{Y_{2,\frac{N}{2}}(N)}_{jk}\right) = \frac{1}{3N}C^{\Gamma(1)} - \frac{1}{N\pi}\left(\frac{\log(2)}{3}\begin{pmatrix} 8 & -4 & -4 \\ -4 & 8 & -4 \\ -4 & -4 & 8 \end{pmatrix} + 2\log(N)\right).$$

2. $N \mid 4s, N \nmid 2s$ and $N \equiv 4 \pmod 8$

   In this, $[\Gamma_0(2)\colon Y_{2,\pm\frac{N}{4}}(N)] = \frac{N}{2}$ ([32, Lemma 2.9]), so $[Y_{2,\pm\frac{N}{4}}(N)\colon Y_2(N)] = 4$, and the cusp widths at $0,\infty,1/2$ are $N, N/4, N/4$, resp. Using Proposition 2.4, we obtain from the above theorem that

$$\left(C^{Y_{2,\pm\frac{N}{4}}(N)}_{jk}\right) = \frac{2}{3N}C^{\Gamma(1)} - \frac{1}{N\pi}\left(\frac{\log(2)}{3}\begin{pmatrix} 4 & -20 & -20 \\ -20 & 4 & -20 \\ -20 & -20 & 4 \end{pmatrix} + 4\log(N)\right).$$

## 2.4 Ye's Group $Y_3(N)$

Let

$$B_3 \coloneqq ST^3S^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix},$$

and for a fixed positive integer $N$, we define Ye's group $Y_3(N)$ to be the subgroup of $\Gamma_0(3)$ generated by its commutator subgroup $\Gamma_0(3)'$, $T^N$ and $B_3^N$. It is a noncongruence group if and only if $N \nmid 12$ ([33, Theorem 2.6]). Since the group $Y_3(N)$ and hence the modular curve $X_{Y_3(N)}$ have significant differences in their properties and ramification behaviors for the cases $3 \nmid N$ and $3 \mid N$, we split our computation into two cases, based on the divisibility of $N$ by 3.

### 2.4.1 The Case $3 \nmid N$

When $3 \nmid N$, the modular curve $X_{Y_3(N)}$ has only two cusps 0 and $\infty$ ([33, Theorem 3.1]), so the cover $X_{Y_3(N)} \to X_{\Gamma_0(3)}$ is totally ramified at both cusps. By Proposition 2.4 and the fact that $[\Gamma_0(3)\colon Y_3(N)] = N$ ([33, Theorem 4.2]), we see that

$$N \cdot C_{jk}^{Y_3(N)} = C_{jk}^{\Gamma_0(3)} - \frac{3}{2\pi} \cdot \log(N) \tag{2.23}$$

for $j, k \in \{0, \infty\}$ ([4, Proposition 6.3.8(b)]). In other words, in order to compute the scattering constants $C_{jk}^{\Phi_0(N)}$ when $3 \nmid N$, it suffices to compute $C_{jk}^{\Gamma_0(3)}$.

The first step is to find the constant $A_j^{\Gamma_0(3)}$ in the Kronecker limit formula (Proposition 2.5) for $\Gamma_0(3)$. Recall that $\Gamma_0(3)$ has two cusps with stabilizer group and cusp width tabulated as follows:

| $S_j$ | $0$ | $\infty$ |
|---|---|---|
| $\mathrm{Stab}_{\Gamma_0(2)}(S_j)$ | $\langle B_3^{-1} \rangle$ | $\langle T \rangle$ |
| $b_j$ | $3$ | $1$ |

and $[\Gamma(1)\colon \Gamma_0(3)] = 4$ with coset representatives

$$\Gamma_0(3)\backslash\Gamma(1) = \left\{\mathrm{id}, S, ST, ST^2\right\} \tag{2.24}$$

([33, Lemma 2.3]). Moreover, we have a modular form $f(z) \coloneqq \eta^{15}(z)\eta^3(3z)$ and a modular function

$$g_3(z) \coloneqq 3^6 \cdot \left(\frac{\eta(3z)}{\eta(z)}\right)^{12}$$

on $\Gamma_0(3)$ ([33, §2]). Note that $\operatorname{div} f = 2 \cdot 0 + 1 \cdot \infty$ and $\operatorname{div} g_3 = -1 \cdot 0 + 1 \cdot \infty$ by Proposition 2.9[3], so we can construct modular forms

$$\begin{aligned} G_0(z) &\coloneqq \frac{f}{g_3} = \frac{1}{3^6} \cdot \frac{\eta^{27}(z)}{\eta^9(3z)} \\ G_\infty(z) &\coloneqq f \cdot g_3^2 = 3^{12} \cdot \frac{\eta^{27}(3z)}{\eta^9(z)} \end{aligned} \tag{2.25}$$

[3] Note that the cusp width at 0 is 3.

| $G_j^{\Gamma_0(3)}\big\vert_9\gamma_k$ | id | $S$ | $ST$ | $ST^2$ |
|---|---|---|---|---|
| $G_0^{\Gamma_0(3)}$ | $\frac{1}{3^6}\cdot\frac{\eta^{27}(z)}{\eta^9(3z)}$ | $\frac{1}{3^{3/2}}\cdot\frac{\eta^{27}(z)}{\eta^9(z/3)}$ | $\frac{1}{3^{3/2}}\cdot\frac{\eta^{27}(z+1)}{\eta^9((z+1)/3)}$ | $\frac{1}{3^{3/2}}\cdot\frac{\eta^{27}(z+2)}{\eta^9((z+2)/3)}$ |
| $G_\infty^{\Gamma_0(3)}$ | $3^{12}\cdot\frac{\eta^{27}(3z)}{\eta^9(z)}$ | $\frac{1}{3^{3/2}}\cdot\frac{\eta^{27}(z/3)}{\eta^9(z)}$ | $\frac{1}{3^{3/2}}\cdot\frac{\eta^{27}((z+1)/3)}{\eta^9(z+1)}$ | $\frac{1}{3^{3/2}}\cdot\frac{\eta^{27}((z+2)/3)}{\eta^9(z+2)}$ |

Table 2.3: The action of the coset representatives on $G_j^{\Gamma_0(3)}$ (up to some constant with norm 1)

on $\Gamma_0(3)$ with divisors $\operatorname{div} G_0 = 1\cdot 0$ and $\operatorname{div} G_\infty = 3\cdot\infty$, satisfying the conditions of Proposition 2.5.

To determine the constants $A_j^{\Gamma_0(3)}$ in Proposition 2.5, we compute the products

$$\prod_{\gamma\in\Gamma_0(3)\backslash\Gamma(1)} G_j^{\Gamma_0(3)}\big\vert_9\gamma(z), \quad \text{where } j\in\{0,\infty\}$$

as in [24, Lemma 6.4], which amounts to finding the actions of the coset representatives (2.24) on the modular forms (2.25) using the transformation rules (2.16). For demonstration purposes, we compute

$$\begin{aligned} G_0^{\Gamma_0(3)}\big\vert_9\begin{pmatrix}1&0\\1&1\end{pmatrix}(z) &= \frac{1}{3^6}\cdot\frac{1}{(z+1)^9}\cdot\frac{\eta^{27}(z/(z+1))}{\eta^9(3z/(z+1))} = \frac{1}{3^6}\cdot\frac{1}{(z+1)^9}\cdot\frac{\eta^{27}(-1/(z+1))}{\eta^9(-3/(z+1))}\\ &= \frac{1}{3^6}\cdot\frac{1}{(z+1)^9}\cdot\frac{(z+1)^{27/2}}{((z+1)/3)^{9/2}}\cdot\frac{\eta^{27}(z+1)}{\eta^9((z+1)/3)} = \frac{1}{3^{3/2}}\cdot\frac{\eta^{27}(z+1)}{\eta^9((z+1)/3)}\end{aligned}$$

and the remaining results are tabulated in Table 2.3. Recall that $\displaystyle\prod_{i=0}^{2}\frac{\eta((z+i)/3)}{\eta(z+i)} = \frac{\eta(z)}{\eta(3z)}$ ([33, §1]), so

$$\prod_{\gamma\in\Gamma_0(3)\backslash\Gamma(1)} G_0^{\Gamma_0(3)}\big\vert_9\gamma = \frac{1}{3^{21/2}}\cdot\Delta^3 \quad\text{and}\quad \prod_{\gamma\in\Gamma_0(3)\backslash\Gamma(1)} G_\infty^{\Gamma_0(3)}\big\vert_9\gamma = 3^{15/2}\cdot\Delta^3 \tag{2.26}$$

We are now ready to compute the constants $A_j^{\Gamma_0(3)}$ in Proposition 2.5, which is possible thanks to Proposition 2.7. In particular, we have

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_0(3)\backslash\Gamma(1)} 4\pi \lim_{s\to 1}\left(E_0^{\Gamma_0(3)}(\gamma z,s)-\frac{1}{\mathrm{vol}(\Gamma_0(3))(s-1)}\right)\\
&\quad = 4\pi\lim_{s\to 1}\left(E^{\Gamma(1)}(z,s)-\frac{1}{\mathrm{vol}(\Gamma(1))(s-1)}\right)-\frac{4\pi}{\mathrm{vol}(\Gamma(1))}\cdot\log(3) \quad \text{(by Proposition 2.7)}\\
&\quad = -\log\|\Delta(z)\|^2 + A^{\Gamma(1)} - 12\log(3) \quad \text{(by eq (1.1))}\\
&\quad = -\log\|\Delta(z)\|^2 + 4\pi C^{\Gamma(1)} - 12\log(3) \quad \text{(by [24, eq (1.0.1)] and [3, eq (0.1)])}\\
&\quad = -\frac{1}{3}\log\left\|\frac{1}{3^{21/2}}\cdot\Delta^3(z)\right\|^2 + 4\pi C^{\Gamma(1)} - 19\log(3)\\
&\quad = -\frac{1}{3}\log\left\|\prod_{\gamma\in\Gamma_0(3)\backslash\Gamma(1)} G_0^{\Gamma_0(3)}\big|_9\gamma(z)\right\|^2 + 4\pi C^{\Gamma(1)} - 19\log(3) \quad \text{(by eq (2.26))}\\
&\quad = \frac{-1}{3}\sum_{\gamma\in\Gamma_0(3)\backslash\Gamma(1)}\log\left\|G_0^{\Gamma_0(3)}\big|_9\gamma(z)\right\|^2 + 4\pi C^{\Gamma(1)} - 19\log(3),
\end{aligned}$$

and likewise

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_0(3)\backslash\Gamma(1)} 4\pi \lim_{s\to 1}\left(E_\infty^{\Gamma_0(3)}(\gamma z,s)-\frac{1}{\mathrm{vol}(\Gamma_0(3))(s-1)}\right)\\
&\quad = \frac{-1}{3}\sum_{\gamma\in\Gamma_0(3)\backslash\Gamma(1)}\log\left\|G_\infty^{\Gamma_0(3)}\big|_9\gamma(z)\right\|^2 + 4\pi C^{\Gamma(1)} + 5\log(3).
\end{aligned}$$

Since the constant $A_j^{\Gamma_0(3)}$ is independent of the cusp in which the Fourier expansion is taken ([24, Remark 3.5 and §7]) for all $j\in\{0,\infty\}$, it follows that

$$A_j^{\Gamma_0(3)} = \pi\cdot C^{\Gamma(1)} + \begin{cases} -\frac{19}{4}\log(3), & \text{if } j=0\\ \frac{5}{4}\log(3), & \text{if } j=\infty.\end{cases}$$

From the Kronecker limit formula

$$4\pi\lim_{s\to 1}\left(E_j^{\Gamma_0(3)}(z,s)-\frac{1}{\mathrm{vol}(\Gamma_0(3))(s-1)}\right) = \frac{-1}{4\cdot\frac{9}{12}}\log\left\|G_j^{\Gamma_0(3)}\right\|^2 + A_j^{\Gamma_0(3)},$$

we then have

**Theorem 2.12.** *Let $Y_3(N)$ be Ye's group defined above with $3\nmid N$. With respect to the cusps $0,\infty$ in order, its scattering constants are:*

$$\begin{aligned}
C_{jk}^{Y_3(N)} &= \frac{1}{3N}C^{\Gamma(1)} - \left(\frac{3}{16}-\frac{3}{2}\delta_{jk}\right)\frac{\log(3)}{\pi N} - \frac{3}{2\pi N}\log(N)\\
&= \frac{1}{\pi N}\left(2\left(\frac{\zeta'(-1)}{\zeta(-1)}-\log(4\pi)+1\right) - \left(\frac{3}{16}-\frac{3}{2}\delta_{jk}\right)\log(3) - \frac{3}{2}\log(N)\right).
\end{aligned}$$

*Proof.* We compare the constant term matrices on both sides with $\gamma_0$ and $\gamma_\infty$ acting on $z$ (as in [24, Proposition 4.1]). On the left hand side, we use Proposition 2.6 to get

$$4\pi\left(\begin{pmatrix} y/3 & 0 \\ 0 & y \end{pmatrix} + \left(C_{jk}^{\Gamma_0(3)}\right) + \frac{\log(3)}{\mathrm{vol}(\Gamma_0(3))} I_2 - \frac{\log(y)}{\mathrm{vol}(\Gamma_0(3))}\right).$$

On the right hand side, we look up our computations in Table 2.3 and find

$$-\frac{1}{3}\left(2\cdot\begin{pmatrix} \log\left|q/3^{3/2}\right| & \log\left|1/3^6\right| \\ \log\left|1/3^{3/2}\right| & \log\left|3^{12}q^3\right| \end{pmatrix} + \log(y^9)\right) + A_j^{\Gamma_0(3)}.$$

Note that $\frac{4\pi}{\mathrm{vol}(\Gamma_0(3))} = 3$, so equating the constant term matrices gives

$$\left(C_{jk}^{\Gamma_0(3)}\right) = \frac{-3\log(3)}{4\pi} I_2 + \frac{1}{4\pi}\left(\begin{pmatrix} 1 & 4 \\ 1 & -8 \end{pmatrix}\log(3) + A_j^{\Gamma_0(3)}\right).$$

Hence

$$C_{jk}^{\Gamma_0(3)} = \frac{1}{4}C^{\Gamma(1)} - \left(\frac{3}{16} - \frac{3}{2}\delta_{jk}\right)\frac{\log(3)}{\pi}.$$

This agrees with Proposition 2.4 for $\Gamma_0(3) \subseteq \Gamma(1)$:

$$3C_{00}^{\Gamma_0(3)} + C_{\infty 0}^{\Gamma_0(3)} = C^{\Gamma(1)} - \frac{21\log(3)}{4\pi}$$
$$3C_{0\infty}^{\Gamma_0(3)} + C_{\infty\infty}^{\Gamma_0(3)} = C^{\Gamma(1)} - \frac{9\log(3)}{4\pi}.$$

Therefore, we plug $C_{jk}^{\Gamma_0(3)}$ back into (2.23) and obtain our result. □

### 2.4.2 The Case $3 \mid N$

When $3 \mid N$, $Y_3(N)$ has six cusps with stabilizer group and cusp width tabulated as follows:

| $S_j$ | $0$ | $1$ | $2$ | $\infty$ | $1/3$ | $1/6$ |
|---|---|---|---|---|---|---|
| $\mathrm{Stab}_{Y_3(N)}(S_j)$ | $\langle B_3^N\rangle$ | $\langle TB_3^NT^{-1}\rangle$ | $\langle T^2B_3^NT^{-2}\rangle$ | $\langle T^N\rangle$ | $\langle B_3^{-1}T^NB_3\rangle$ | $\langle B_3^{-2}T^NB_3^2\rangle$ |
| $b_j$ | $3N$ | $3N$ | $3N$ | $N$ | $N$ | $N$ |

Here

$$TB_3^NT^{-1} = \begin{pmatrix} 1-3N & 3N \\ -3N & 1+3N \end{pmatrix}, \quad T^2B_3^NT^{-2} = \begin{pmatrix} 1-6N & 12N \\ -3N & 1+6N \end{pmatrix},$$
$$B_3^{-1}T^NB_3 = \begin{pmatrix} 1-3N & N \\ -9N & 1+3N \end{pmatrix}, \quad B_3^{-2}T^NB_3^2 = \begin{pmatrix} 1-6N & N \\ -36N & 1+6N \end{pmatrix}.$$

It is also known that $[\Gamma(3)\colon Y_3(N)] = N$ ([33, Lemma 4.1]) with coset representatives

$$\Gamma(3)/Y_3(N) = \left\{\mathrm{id}, T^3, \cdots, T^{N-3}, B_3, T^3B_3, \cdots, T^{N-3}B_3, B_3^2, T^3B_3^2, \cdots, T^{N-3}B_3^2\right\}.$$

Moreover, let

$$F(z) = \frac{\eta(z/3)^3}{\eta(z)}, \tag{2.27}$$

which is a modular form of weight 1 on $\Gamma(3)$ with $\operatorname{div} F = 1 \cdot 0$ ([33, §5 (p.304)]). Consider also the so-called modular $\mu$-function ([33, §4 (p.301)]), which is a modular function on $\Gamma(3)$ with

$$\mu(\infty) = \infty, \quad \mu(0) = 1, \quad \mu(1) = \zeta_3^2 \quad \text{and} \quad \mu(2) = \zeta_3 \tag{2.28}$$

(here, $\zeta_3 = e^{2\pi i/3}$). For future reference, we also note the relation ([33, Theorem 4.3]):

$$\mu^3(z) - 1 = 27 g_3^{-1}, \quad \mu(z) = 1 + \frac{1}{3}\frac{\eta(\frac{z}{3})^3}{\eta(3z)^3} = \frac{1}{3}q^{-\frac{1}{3}} + 1 + \cdots \tag{2.29}$$

This can be proved by observing that $F|_1 S = -i\sqrt{3}\frac{F}{\mu-1}$.[4] Now, consider the modular forms of weight 1:

$$G_\infty^{Y_3(N)} := F \cdot \frac{\mu/3 - g_3^{-1/3}}{\mu - 1}, \quad G_{1/3}^{Y_3(N)} := F \cdot \frac{\mu/3 - \zeta_3 g_3^{-1/3}}{\mu - 1} \quad \text{and} \quad G_{1/6}^{Y_3(N)} := F \cdot \frac{\mu/3 - \zeta_3^2 g_3^{-1/3}}{\mu - 1},$$

$$G_0^{Y_3(N)} := F, \quad G_1^{Y_3(N)} := F \cdot \frac{\mu - \zeta_3^2}{\mu - 1} \quad \text{and} \quad G_2^{Y_3(N)} := F \cdot \frac{\mu - \zeta_3}{\mu - 1}.$$

We note that

$$\begin{aligned}
\operatorname{div}(\mu - 3g_3^{-\frac{1}{3}}) &= \frac{2N}{3} \cdot \infty - \frac{N}{3} \cdot \frac{1}{3} - \frac{N}{3} \cdot \frac{1}{6} \\
\operatorname{div}(\mu - 3\zeta_3 g_3^{-\frac{1}{3}}) &= \frac{2N}{3} \cdot \frac{1}{3} - \frac{N}{3} \cdot \infty - \frac{N}{3} \cdot \frac{1}{6} \\
\operatorname{div}(\mu - 3\zeta_3^2 g_3^{-\frac{1}{3}}) &= \frac{2N}{3} \cdot \frac{1}{6} - \frac{N}{3} \cdot \frac{1}{3} - \frac{N}{3} \cdot \infty.
\end{aligned} \tag{2.30}$$

Since the cover $X_{Y_3(N)} \to X_{\Gamma(3)}$ ramifies at the cusp $\infty$ as $\infty, 1/3, 1/6$ each of index $N/3$ ([33, §5]), and totally ramifies at the cusps $0, 1, 2$ each of index $N$ (cf. (2.36)), we have $\operatorname{div} G_j^{Y_3(N)} = N \cdot j$ at each cusp.

To compute the products $\prod_{\gamma \in \Gamma \backslash \Gamma(3)} G_j^{Y_3(N)}\big|_1 \gamma$, we use the transformation laws

$$\begin{aligned}
\mu \circ T = \zeta_3^{-1}\mu \quad &\text{and} \quad g_3^{-1/3} \circ T = \zeta_3^{-1} g_3^{-1/3} \\
\mu \circ B_3 = \mu \quad &\text{and} \quad g_3^{-1/3} \circ B_3 = \zeta_3 g_3^{-1/3}
\end{aligned} \tag{2.31}$$

([33, Corollary 5.4]) for the coset representatives $\Gamma(3)/Y_3(N)$ to act on $G_\infty^{Y_3(N)}$, $G_{1/3}^{Y_3(N)}$ and $G_{1/6}^{Y_3(N)}$, which gives

$$\prod_{\gamma \in \Gamma \backslash \Gamma(3)} G_\infty^{Y_3(N)}\big|_1 \gamma = (G_\infty^{Y_3(N)} \cdot G_{1/3}^{Y_3(N)} \cdot G_{1/6}^{Y_3(N)})^{N/3} \stackrel{(*)}{=} (G_\infty^{\Gamma(3)}/9)^N \tag{2.32}$$

(where the equation $(*)$ follows from (2.29)) and likewise

$$\prod_{\gamma \in \Gamma \backslash \Gamma(3)} G_{1/3}^{Y_3(N)}\big|_1 \gamma = \left(G_\infty^{\Gamma(3)}/9\right)^N = \prod_{\gamma \in \Gamma \backslash \Gamma(3)} G_{1/6}^{Y_3(N)}. \tag{2.33}$$

[4] It follows from the fact that $\dim S_1(\Gamma(3)) = 0$ [4, p. 262], and by comparing Fourier coefficients.

This allows us to obtain the Kronecker limit formula for $Y_3(N)$: let $j \in \{\infty, 1/3, 1/6\}$, and note that

$$\begin{aligned}
&\sum_{\gamma \in Y_3(N)\backslash\Gamma(3)} 4\pi \lim_{s\to 1}\left(E_j^{Y_3(N)}(\gamma z, s) - \frac{1}{\mathrm{vol}(Y_3(N))(s-1)}\right) \\
&= 4\pi \lim_{s\to 1}\left(E_j^{\Gamma(3)}(z,s) - \frac{1}{\mathrm{vol}(\Gamma(3))(s-1)} - \frac{\log(N/3)}{\mathrm{vol}(\Gamma(3))}\right) && \text{(by Proposition 2.7)} \\
&= -\log\left\|G_\infty^{\Gamma(3)}(z)\right\|^2 + A_\infty^{\Gamma(3)} - \log\left(\frac{N}{3}\right) && \text{(by Proposition 2.5)} \\
&= -\frac{1}{N}\sum_{\gamma \in Y_3(N)\backslash\Gamma(3)} \log\left\|G_j^{Y_3(N)}\big|_2 \gamma(z)\right\|^2 + A_\infty^{\Gamma(3)} - \log(N) - 2\log(3). && \text{(by eq (2.32) and eq (2.33))}
\end{aligned}$$

where $A_\infty^{\Gamma(3)} = \frac{1}{12}(4\pi C^{\Gamma(1)} - 3\log(3))$ from (2.39); so the fact that the constant $A_j^{Y_3(N)}$ is independent of the cusp in which the Fourier expansion is taken ([24, Remark 3.5 and §7]) implies that

$$4\pi \lim_{s\to 1}\left(E_j^{Y_3(N)}(z,s) - \frac{1}{\mathrm{vol}(Y_3(N))(s-1)}\right) = -\frac{1}{N}\log\left\|G_j^{Y_3(N)}\right\|^2 + A_j^{Y_3(N)} \tag{2.34}$$

where

$$A_j^{Y_3(N)} = \frac{1}{N}\left(A_\infty^{\Gamma(3)} - \log(N) - 2\log(3)\right) = \frac{1}{N}\left(\frac{1}{3}\pi C^{\Gamma(1)} - \frac{9}{4}\log(3) - \log(N)\right).$$

Then we have

**Theorem 2.13.** *Let $Y_3(N)$ be Ye's group defined above with $3 \mid N$. With respect to the cusps $0, 1, 2, \infty, 1/3, 1/6$ in order, its scattering constants are:*

$$\left(C_{jk}^{Y_3(N)}\right) = \frac{1}{12N}C^{\Gamma(1)} - \frac{1}{\pi N}\left(\frac{\log(3)}{16}\begin{pmatrix} 17 & 5 & 5 & 1 & 1 & 1 \\ 5 & 17 & 5 & 1 & 1 & 1 \\ 5 & 5 & 17 & 1 & 1 & 1 \\ 1 & 1 & 1 & 17 & 5 & 5 \\ 1 & 1 & 1 & 5 & 17 & 5 \\ 1 & 1 & 1 & 5 & 5 & 17 \end{pmatrix} + \frac{\log(N)}{2}\right).$$

*Proof.* To compute the scattering constants $C_{jk}^{Y_3(N)}$ for $j, k \in \{\infty, 1/3, 1/6\}$, we compare the constant terms on both sides of (2.34) with

$$\gamma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_{1/3} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = B_3^{-1} \quad \text{and} \quad \gamma_{1/6} = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix} = B_3^{-2}$$

acting on $z$ as in [24, Proposition 4.1]. On the left hand side, we use (2.5) to get

$$4\pi\left(y/NI_3 + \left(C_{jk}^{Y_3(N)}\right) + \frac{\log(N)}{\mathrm{vol}(Y_3(N))} - \frac{\log(y)}{\mathrm{vol}(Y_3(N))}\right).$$

On the right hand side, we use the transformation laws (2.31) to get the constant term on the right hand side of (2.34):

$$-\frac{2}{N}\begin{pmatrix} \log|3q| & \log|(1-\zeta_3)/3| & \log\left|(1-\zeta_3^2)/3\right| \\ \log\left|(1-\zeta_3^2)/3\right| & \log|3q| & \log|(1-\zeta_3)/3| \\ \log|(1-\zeta_3)/3| & \log\left|(1-\zeta_3^2)/3\right| & \log|3q| \end{pmatrix} - \frac{\log(y)}{N} + A_j^{Y_3(N)}.$$

Note that $\dfrac{4\pi}{\mathrm{vol}(Y_3(N))} = \dfrac{1}{N}$; so equating the constant term matrices gives

$$\begin{aligned}\left(C_{jk}^{Y_3(N)}\right) &= \frac{-1}{\pi N}\left(\frac{\log(N)}{4} + \frac{\log(3)}{2}\begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} - \frac{1}{4}\left(\frac{1}{3}\pi C^{\Gamma(1)} - \frac{9}{4}\log(3) - \log(N)\right)\right) \\ &= \frac{1}{12N}C^{\Gamma(1)} - \frac{1}{\pi N}\left(\frac{\log(3)}{16}\begin{pmatrix} 17 & 5 & 5 \\ 5 & 17 & 5 \\ 5 & 5 & 17 \end{pmatrix} + \frac{1}{2}\log(N)\right)\end{aligned}$$

for $j,k \in \{\infty, 1/3, 1/6\}$, which agrees with Proposition 2.4 for $Y_3(N) \subseteq \Gamma(3)$:

$$\frac{N}{3}\left(C_{\infty\infty}^{Y_3(N)} + C_{\frac{1}{3}\infty}^{Y_3(N)} + C_{\frac{1}{6}\infty}^{Y_3(N)}\right) = \frac{1}{12}C^{\Gamma(1)} - \frac{9\log(3)}{16\pi} - \frac{\log(N)}{2\pi} = C_{\infty\infty}^{\Gamma(3)} - \frac{\log(N/3)}{2\pi}.$$

In fact, using Proposition 2.4 further gives

$$\begin{aligned} NC_{jk}^{Y_3(N)} &= C_{jk}^{\Gamma(3)} - \frac{1}{4\pi}\log\left(N^2\right) && \text{for } j,k \in \{0,1,2\} \\ NC_{jk}^{Y_3(N)} &= C_{jk}^{\Gamma(3)} - \frac{1}{4\pi}\log\left(\frac{N^2}{3}\right) && \text{for } j \in \{0,1,2\} \text{ and } k \in \{\infty, 1/3, 1/6\} \end{aligned}$$

and so we deduce from identity (2.42)

$$\left(C_{jk}^{\Gamma(3)}\right) = \frac{1}{12}C^{\Gamma(1)} - \left(3\cdot\delta_{jk} + \frac{5}{4}\right)\frac{\log(3)}{4\pi}$$

for $j,k \in \{0,1,2,\infty\}$ that

$$\left(C_{jk}^{Y_3(N)}\right) = \frac{1}{12N}C^{\Gamma(1)} - \frac{1}{\pi N}\left(\frac{\log(3)}{16}\begin{pmatrix} 17 & 5 & 5 & 1 & 1 & 1 \\ 5 & 17 & 5 & 1 & 1 & 1 \\ 5 & 5 & 17 & 1 & 1 & 1 \\ * & * & * & 17 & 5 & 5 \\ * & * & * & 5 & 17 & 5 \\ * & * & * & 5 & 5 & 17 \end{pmatrix} + \frac{\log(N)}{2}\right)$$

for $j,k \in \{0,1,2,\infty,1/3,1/6\}$. Since the scattering matrix is symmetric ([20, Theorem 2.2.1]), we obtain the result. □

*Remark.* Define $Y_{3,s}(N) = \langle Y_3(N), TB_3^{s+1}\rangle$, where the relation $Y_3(N) \subseteq Y_{3,s}(N) \subseteq \Gamma_0(3)$ holds. It is a noncongruence group if and only if $N \nmid 12$ ([33, Theorem 2.9]). We are interested in the case where $Y_{3,s}(N)$ has more than two cusps, where it has four cusps $\{0, \infty, 1/3, 2/3\}$ when $N \mid 3s$, $N \nmid s$ and $\pm\frac{N}{3} \equiv -1 \pmod 3$ ([33, Lemma 3.2]). In this case, $[\Gamma_0(3)\colon Y_{3,\pm\frac{N}{3}}(N)] = N$ ([33, Lemma 2.9]), so $[Y_{3,\pm\frac{N}{3}}(N)\colon Y_3(N)] = 3$. Using Proposition 2.4, we obtain from the above theorem that

$$\left(C_{jk}^{Y_{3,\pm\frac{N}{3}}(N)}\right) = \frac{1}{4N}C^{\Gamma(1)} - \frac{1}{N\pi}\left(\frac{\log(3)}{16}\begin{pmatrix} 27 & -9 & -9 & -9 \\ -9 & 27 & -9 & -9 \\ -9 & -9 & 27 & -9 \\ -9 & -9 & -9 & 27 \end{pmatrix} + \frac{3}{2}\log(N)\right).$$

### 2.4.3 The Group $\Gamma(3)$

We include another computation of the scattering constants for the principal congruence subgroup $\Gamma(3)$, whose Kronecker limit formula is needed for computing $C_{jk}^{Y_3(N)}$. The coset representatives of $\Gamma(3)$ in $\Gamma(1)$ are

$$\left\{T^a, ST^a, TST^a, T^2ST^a\right\} \qquad \text{for each } a \in \{0,1,2\}. \tag{2.35}$$

where $S$ and $T$ are defined in (2.6), and the recall auxiliary modular functions $F$ in (2.27) and $\mu$ in (2.28). They together give modular forms of weight 1 on $\Gamma(3)$ with $\operatorname{div} G_j^{\Gamma(3)} = 1 \cdot j$ for all $j \in \{\infty, 0, 1, 2\}$:

$$\begin{aligned} G_\infty^{\Gamma(3)}(z) &:= F(z)\cdot\frac{1/3}{\mu(z)-1}, \quad G_0^{\Gamma(3)}(z) := F(z) \\ G_1^{\Gamma(3)}(z) &:= F(z)\cdot\frac{\mu(z)-\zeta_3^2}{\mu(z)-1}, \quad G_2^{\Gamma(3)}(z) := F(z)\cdot\frac{\mu(z)-\zeta_3}{\mu(z)-1}. \end{aligned} \tag{2.36}$$

To determine $A_j^{\Gamma(3)}$ in Proposition 2.5, we compute the products $\prod_{\gamma\in\Gamma(3)\backslash\Gamma(1)} G_j^{\Gamma(3)}|_1\gamma(z)$, or equivalently finding the actions of the coset representatives (2.35) on the modular forms (2.36). The actions can be determined by the location of their (only) simple zero, up to leading coefficients, because there are no cusp forms on $\Gamma(3)$ ([6, Theorem 3.6.1]). We tabulate the result of all actions in Table 2.4 (up to a constant with absolute value 1), where $\overline{n}$ denotes the image of the integer $n$ in $\mathbb{Z}/3\mathbb{Z}$. Note that

$$G_\infty^{\Gamma(3)}(z) = \frac{G_0^{\Gamma(3)}|_1S(z)}{3\sqrt{3}} = \frac{z^{-1}}{3\sqrt{3}}\cdot\left(\frac{\eta^3(-1/3z)}{\eta(-1/z)}\right) \overset{(*)}{=} \frac{\eta^3(3z)}{\eta(z)} \tag{2.37}$$

where $(*)$ uses the transformation rules (2.16).

This gives the products

$$\begin{aligned} \prod_{\gamma\in\Gamma(3)\backslash\Gamma(1)} G_j^{\Gamma(3)}|_1\gamma(z) &= \left(\sqrt{3}(1-\zeta_3)(1-\zeta_3^2)\cdot G_\infty^{\Gamma(3)}(z)\cdot G_0^{\Gamma(3)}(z)\cdot G_1^{\Gamma(3)}(z)\cdot G_2^{\Gamma(3)}(z)\right)^3 \\ &= \left(3\sqrt{3}\left(F(z)\cdot\frac{1/3}{\mu(z)-1}\right)^4(\mu^3(z)-1)\right)^3 \\ &\overset{(*)}{=} \left(3\sqrt{3}\cdot G_\infty^4(z)\left(\frac{\eta^3(3z)}{27\eta(z)}\right)\right)^3 \\ &\overset{(**)}{=} \left(3\sqrt{3}\cdot\left(\frac{\eta^3(3z)}{\eta(z)}\right)^4\left(\frac{\eta^3(3z)}{27\eta(z)}\right)\right)^3 = 3^{9/2}\Delta(z) \end{aligned} \tag{2.38}$$

for all $j \in \{\infty, 0, 1, 2\}$, where $(*)$ and $(**)$ respectively use (2.29) and (2.37). We then find the sum of the residues of the Eisenstein series for $\Gamma(3)$:

$$\begin{aligned}
&\sum_{\gamma\in\Gamma(3)\backslash\Gamma(1)} 4\pi \lim_{s\to 1}\left(E_j^{\Gamma(3)}(\gamma z,s)-\frac{1}{\mathrm{vol}(\Gamma(3))(s-1)}\right)\\
&\quad= 4\pi\lim_{s\to 1}\left(E_\infty^{\Gamma(1)}(z,s)-\frac{1}{\mathrm{vol}(\Gamma(1))(s-1)}\right)-\frac{4\pi}{\mathrm{vol}(\Gamma(1))}\cdot\log(3) && \text{(Proposition 2.7)}\\
&\quad= -\log\|\Delta(z)\|^2 + A^{\Gamma(1)} - \frac{4\pi}{\pi/3}\log(3) && \text{(Proposition 2.5)}\\
&\quad= -\log\left\|3^{9/2}\Delta(z)\right\|^2 + A^{\Gamma(1)} - 3\log(3)\\
&\quad= -\log\left\|\prod_{\gamma\in\Gamma(3)\backslash\Gamma(1)} G_j|_2^{\Gamma(3)}\gamma(z)\right\|^2 + A^{\Gamma(1)} - 3\log(3) && (2.38)\\
&\quad= -\sum_{\gamma\in\Gamma(3)\backslash\Gamma(1)}\log\left\|G_j|_2^{\Gamma(3)}\gamma(z)\right\|^2 + 4\pi C^{\Gamma(1)} - 3\log(3) && \text{(see footnote[5])}
\end{aligned}$$

for all $j\in\{\infty,0,1,2\}$. The fact that the constant $A_j^{\Gamma(3)}$ is independent of the cusp in which the Fourier expansion is taken ([24, Remark 3.5 and §7]) then gives the Kronecker limit formula for $\Gamma(3)$:

$$4\pi\lim_{s\to 1}\left(E_j^{\Gamma(3)}(z,s)-\frac{1}{\mathrm{vol}(\Gamma(3))(s-1)}\right) = -\log\left\|G_j^{\Gamma(3)}(z)\right\|^2 + A_j^{\Gamma(3)} \tag{2.39}$$

where $A_j^{\Gamma(3)} = \frac{1}{12}(4\pi C^{\Gamma(1)} - 3\log(3))$ for all $j\in\{\infty,0,1,2\}$.

| $\gamma\in\Gamma(3)\backslash\Gamma(1)$ | $T^a$ | $ST^a$ | $TST^a$ | $T^2ST^a$ |
|---|---|---|---|---|
| $G_\infty^{\Gamma(3)}\|_1\gamma(z)$ | $G_\infty^{\Gamma(3)}(z)$ | $\sqrt{3}\cdot G_{0-a}^{\Gamma(3)}(z)$ | $(1-\zeta_3)\cdot G_{2-a}^{\Gamma(3)}(z)$ | $(1-\zeta_3^2)\cdot G_{1-a}^{\Gamma(3)}(z)$ |
| $G_0^{\Gamma(3)}\|_1\gamma(z)$ | $G_{0-a}^{\Gamma(3)}(z)$ | $3\sqrt{3}\cdot G_\infty^{\Gamma(3)}(z)$ | $\frac{1-\zeta_3}{\sqrt{3}}\cdot G_{1-a}^{\Gamma(3)}(z)$ | $\frac{1-\zeta_3^2}{\sqrt{3}}\cdot G_{2-a}^{\Gamma(3)}(z)$ |
| $G_1^{\Gamma(3)}\|_1\gamma(z)$ | $G_{1-a}^{\Gamma(3)}(z)$ | $\frac{1-\zeta_3^2}{\sqrt{3}}\cdot G_{2-a}^{\Gamma(3)}(z)$ | $3\sqrt{3}\cdot G_\infty^{\Gamma(3)}(z)$ | $\frac{1-\zeta_3}{\sqrt{3}}\cdot G_{0-a}^{\Gamma(3)}(z)$ |
| $G_2^{\Gamma(3)}\|_1\gamma(z)$ | $G_{2-a}^{\Gamma(3)}(z)$ | $\frac{1-\zeta_3}{\sqrt{3}}\cdot G_{1-a}^{\Gamma(3)}(z)$ | $\frac{1-\zeta_3^2}{\sqrt{3}}\cdot G_{2-a}^{\Gamma(3)}(z)$ | $3\sqrt{3}\cdot G_\infty^{\Gamma(3)}(z)$ |

Table 2.4: The actions of the coset representatives on $G_j^{\Gamma(3)}$ (up to a constant with absolute value 1)

Finally, to compute the scattering constants, we compare the constant term matrices on both sides of (2.39) with

$$\gamma_\infty = \begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \gamma_0=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \gamma_1=\begin{pmatrix}1&-1\\1&0\end{pmatrix}\quad\text{and}\quad \gamma_2=\begin{pmatrix}2&-1\\1&0\end{pmatrix}$$

[5]By comparing the constant terms on both sides of Kronecker limit formula (see [24, Proposition 3.4]) for $\Gamma(1)$, it is easy to see that $A^{\Gamma(1)} = 4\pi C^{\Gamma(1)}$.

acting on $z$ as in [24, Proposition 4.1]. On the left hand side, we use (2.5) to get

$$4\pi\left(\frac{y}{3}\delta_{jk}+C_{jk}^{\Gamma(3)}+\frac{\log(3)}{\mathrm{vol}(\Gamma(3))}-\frac{\log(y)}{\mathrm{vol}(\Gamma(3))}\right). \tag{2.40}$$

On the right hand side, we look up our computations in Table 2.4 and find

$$-\left(2\log\left|3\sqrt{3}q^{1/3}\right|\cdot\delta_{jk}+\log(y)\right)+A_j^{\Gamma(3)}. \tag{2.41}$$

Note that $\dfrac{4\pi}{\mathrm{vol}(\Gamma(3))}=1$, so equating the constant term matrices (2.40) and (2.41) gives

$$C_{jk}^{\Gamma(3)}=\frac{1}{4\pi}\left((-3\log(3))\cdot\delta_{jk}-\log(3)+A_j^{\Gamma(3)}\right)=\frac{1}{12}C^{\Gamma(1)}-\left(\frac{5}{4}+3\delta_{jk}\right)\frac{\log(3)}{4\pi}. \tag{2.42}$$

This agrees with our earlier result (2.1).

# Chapter 3

# Cuspidal Divisor Class Groups

The material in this chapter corresponds to the article [15].

## 3.1 Examples from Chapter 2

Recall that Posingies' framework reduces the computation of scattering constants to finding modular forms $G_j^\Gamma$ with divisors $\operatorname{div} G_j^\Gamma = m \cdot j$ for some $m \in \mathbb{N}$. A sufficient condition for which this is possible is given in the following proposition

**Proposition 3.1.** *If the cuspidal divisor class group* $\mathrm{CL}(\Gamma) = \mathscr{D}^\infty / \mathscr{F}^\infty$ *is finite, then there exists modular forms* $G_j^\Gamma$ *with divisors* $\operatorname{div} G_j^\Gamma = m \cdot j$ *for some* $m \in \mathbb{N}$ *at each cusp* $S_j$ *of* $\Gamma$.

*Here,* $\mathscr{D}^\infty$ *denotes the free abelian group of degree 0 divisors supported on the cusps, and* $\mathscr{F}^\infty$ *denotes its subgroup of divisor* $\operatorname{div}(f)$ *for some modular function* $f$ *whose divisor is supported on the cusps.*

*Proof.* Let $\infty = S_1, S_2, ..., S_n$ be cusps of $\Gamma$ over $\infty$. Suppose $\operatorname{div} \Delta = a_1 S_1 + \cdots + a_n S_n$, where $a_i \in \mathbb{Z}$, and $a_i > 0$ for all $i$. Then for any cusp $S$ of $\Gamma$, consider the divisor $bS - (a_1 S_1 + \cdots + a_n S_n) \in \mathscr{D}^\infty$, where $b = a_1 + \cdots + a_n$. Since $CL(\Gamma)$ is finite, there exists $m \in \mathbb{Z}$ such that $m(bS - (a_1 S_1 + \cdots + a_n S_n)) \in \mathscr{F}^\infty$. Hence there exists a modular function $f$ such that $\operatorname{div}(f) = m(bS - (a_1 S_1 + \cdots + a_n S_n))$. Then $\operatorname{div} f \Delta^m = mbS$. $\square$

Note that $\mathrm{CL}(\Gamma_{6N})$ is trivial since $\Gamma_{6N}$ has genus zero ([23, §5]). Moreover, Yang and Yin have computed the cuspidal divisor class group of $Y_2(N)$ to be

$$\mathrm{CL}(Y_2(N)) = \begin{cases} 0 & \text{if } N \text{ is odd,} \\ (\mathbb{Z}/\frac{N}{2}\mathbb{Z})^2 & \text{if } N \text{ is even,} \end{cases}$$

as stated in [32, Theorem 6.3]. In this section, we compute the cuspidal divisor class groups for the additional groups considered in Chapter 2, further illustrating and confirming the validity of Proposition 3.1.

### 3.1.1 Yang and Yin's Variant $Y_{2,s}(N)$

Recall that $Y_{2,s}(N) = \langle Y_2(N), TB_2^{s+1}\rangle$, where the relation $Y_2(N) \subseteq Y_{2,s}(N) \subseteq \Gamma_0(2)$ holds, and it is noncongruence if and only if $N \nmid 24$. The cases we computed scattering constants for were

1. $N \mid 2s, N \nmid s$ and $N \equiv 2 \pmod 4$

   In this case, recall that $[Y_{2,\frac{N}{2}}(N) \colon Y_2(N)] = 2$, and the group $Y_{2,\frac{N}{2}}(N)$ has three cusps $0, \infty, 1/2$ with cusp widths $2N, \frac{N}{2}, \frac{N}{2}$, respectively. Moreover, the function field of $X_{Y_{2,\frac{N}{2}}(N)}$ is given by

   $$\mathbb{C}(Y_{2,\frac{N}{2}}(N)) = \mathbb{C}\left(\omega_2^{\frac{2}{N}}, (1-2\lambda)\omega_2^{\frac{1}{N}}\right)$$

   ([32, Theorem 4.4]), where $\omega_2$ and $\lambda$ are respectively defined in (2.14) and (2.7). Using the fact that $N \equiv 2 \pmod 4$, we can choose our generators $\omega_2^{\frac{2}{N}}$ and $(1-2\lambda)\omega_2^{\frac{1}{2}} - 8$ to be from $\mathscr{F}^\infty$ instead. To be precise, we have

   $$\operatorname{div}\omega_2^{\frac{2}{N}} = -2\cdot 0 + 1\cdot\infty + 1\cdot\frac{1}{2} \qquad \text{and} \qquad \operatorname{div}[(1-2\lambda)\omega_2^{\frac{1}{2}} - 8] = -\frac{N}{2}\cdot 0 + \frac{N}{2}\cdot\infty$$

   ([32, Table 1 and 2]), noting the fact that the cups 0 of $Y_{2,\frac{N}{2}}(N)$ splits into 0 and 1 under the branched cover $X_{Y_2(N)} \to X_{Y_{2,\frac{N}{2}}(N)}$. It follows that $\mathrm{CL}(Y_{2,\frac{N}{2}}(N)) \cong \mathbb{Z}/\frac{N}{2}\mathbb{Z}$.

2. $N \mid 4s, N \nmid 2s$ and $N \equiv 4 \pmod 8$

   In this case, recall that $[Y_{2,\pm\frac{N}{4}}(N) \colon Y_2(N)] = 4$, and the group $Y_{2,\pm\frac{N}{4}}(N)$ has three cusps $0, \infty, 1/2$ with cusp widths $N, \frac{N}{4}, \frac{N}{4}$, respectively. Moreover, the function field of $X_{Y_{2,\pm\frac{N}{4}}(N)}$ is given by

   $$\mathbb{C}(Y_{2,\pm\frac{N}{4}}(N)) = \mathbb{C}\left(\omega_2^{\frac{4}{N}}, (1-2\lambda)\omega_2^{\frac{2}{N}}\right)$$

   ([32, Theorem 4.4]), where $\omega_2$ and $\lambda$ are respectively defined in (2.14) and (2.7). Using the fact that $N \equiv 4 \pmod 8$, we can choose our generators $\omega_2^{\frac{4}{N}}$ and $(1-2\lambda)\omega_2^{\frac{1}{2}} - 8$ to be from $\mathscr{F}^\infty$ instead. To be precise, we have

   $$\operatorname{div}\omega_4^{\frac{4}{N}} = -2\cdot 0 + 1\cdot\infty + 1\cdot\frac{1}{2} \qquad \text{and} \qquad \operatorname{div}[(1-2\lambda)\omega_2^{\frac{1}{2}} - 8] = -\frac{N}{4}\cdot 0 + \frac{N}{4}\cdot\infty$$

   ([32, Table 1 and 2]), noting the fact that the cups 0 of $Y_{2,\pm\frac{N}{4}}(N)$ splits into 0 and 1 under the branched cover $X_{Y_2(N)} \to X_{Y_{2,\pm\frac{N}{4}}(N)}$. It follows that $\mathrm{CL}(Y_{2,\pm\frac{N}{4}}(N)) \cong \mathbb{Z}/\frac{N}{4}\mathbb{Z}$.

### 3.1.2 Ye's Group $Y_3(N)$

Recall that $Y_3(N) = \langle \Gamma_0(3)', T^N, B_3^N\rangle$ is a subgroup of $\Gamma_0(3)$, and it is noncongruence if and only if $N \nmid 12$. If $3 \nmid N$, it is known that $Y_3(N)$ has genus zero ([33, Theorem 3.1]), so $\mathrm{CL}(Y_3(N))$ is automatically trivial. Suppose now $3 \mid N$. It has three cusps $\infty, 1/3, 2/3$ with cusp width $N$ and three other cusps $0, 1, 2$ with cups width $3N$. We claim that

$$f_1 \coloneqq g_3^{\frac{1}{N}}, \quad f_2 \colon \mu - 1, \quad f_3 \coloneqq \mu - \zeta_3, \quad f_4 = \mu - 3g_3^{-\frac{1}{3}}, \quad f_5 \coloneqq \mu - 3\zeta_3 g_3^{-\frac{1}{3}}.$$

are generators of $\mathscr{F}^\infty$.

Indeed, we apply the transformation rules

$$\mu \circ T = \zeta_3^{-1}\mu, \qquad \mu \circ B = \mu, \qquad g_3^{-\frac{1}{3}} \circ T = \zeta_3^{-1} g_3^{-\frac{1}{3}} \qquad \text{and} \qquad g_3^{-\frac{1}{3}} \circ B = \zeta_3 g_3^{-\frac{1}{3}} \tag{3.1}$$

([33, Corollay 5.4]) and the case $G = \Gamma_0(3)$, $\gamma = S$ and $m = 3$ of the following proposition

**Proposition 3.2** ([26, Proposition 1.1])**.** *Let $G$ be a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$ and for some fixed $\gamma$ in $\mathrm{SL}_2(\mathbb{Z})$, let $m$ be the smallest positive integer such that $\gamma T^m \gamma^{-1}$ is in $G$. Suppose $f$ is modular with respect to $G$ and holomorphic and never zero on the upper half plane. Then*

$$(\log f) \circ \gamma T^m \gamma^{-1} - (\log f) = 2\pi i \cdot \mathrm{ord}^G_{\gamma\cdot\infty} f.$$

to the generators $f_2, f_3, f_4, f_5$ as well as

$$f_{23} \coloneqq \mu - \zeta_3^2 \qquad \text{and} \qquad f_{45} \coloneqq \mu - 3\zeta_3^2 g_3^{-\frac{1}{3}},$$

which gives [1]

$$\begin{gathered} f_2^{\frac{1}{c}} \circ B = \zeta_c^{-1} f_2^{\frac{1}{c}}, \quad f_3^{\frac{1}{c}} \circ B = f_3^{\frac{1}{c}}, \quad f_{23}^{\frac{1}{c}} \circ B = f_3^{\frac{1}{c}}, \\ f_4^{\frac{1}{c}} \circ B = f_5^{\frac{1}{c}}, \quad f_5^{\frac{1}{c}} \circ B = f_{45}^{\frac{1}{c}}, \quad f_{45}^{\frac{1}{c}} \circ B = f_4^{\frac{1}{c}} \end{gathered} \tag{3.2}$$

where $c$ is a positive integer and $\zeta_c = e^{2\pi i/c}$. In addition, by comparing the constant term of the expansions on both sides (see for example, [26, Proposition 1.3]), we find that

$$f_2^{\frac{1}{c}} \circ T = \zeta_{3c}^{-1} f_3^{\frac{1}{c}}, \quad f_3^{\frac{1}{c}} \circ T = \zeta_{3c}^{-1} f_{23}^{\frac{1}{c}}, \quad f_{23}^{\frac{1}{c}} \circ T = \zeta_{3c}^{-1} f_2^{\frac{1}{c}}. \tag{3.3}$$

Also, since $\log f_4 = \log 3 + \frac{4\pi i}{3} z + \log(1 + (*)q + \cdots)$ and $\log f_5 = \log \frac{1-\zeta_3}{3} - \frac{2\pi i}{3} z + \log(1 + (*)q + \cdots)$, we have

$$f_4^{\frac{1}{c}} \circ T = \zeta_{3c}^2 f_4^{\frac{1}{c}}, \qquad f_5^{\frac{1}{c}} \circ T = \zeta_{3c}^{-1} f_5^{\frac{1}{c}} \qquad \text{and} \qquad f_6^{\frac{1}{c}} \circ T = \zeta_{3c}^{-1} f_6^{\frac{1}{c}}. \tag{3.4}$$

Using these transformation rules (3.2), (3.3) and (3.4), we tabulate the actions of some relevant commuatators on the proposed generators in 3.1. It can then be deduced that if $f = (f_1^{a_1} f_2^{a_2} f_3^{a_3} f_4^{a_4} f_5^{a_5})^{1/c}$ is modular with respect to $Y_3(N)$, then

$$-a_2 - a_4 + a_5 \equiv a_2 - a_4 \equiv -a_2 + a_3 \equiv a_2 - a_3 - a_4 + a_5 \equiv 0 \pmod{c}$$

It follows that $a_2 \equiv a_3 \equiv a_4 \equiv a_5 \equiv 0 \pmod{c}$ and hence we must also have $a_1 \equiv 0 \pmod{c}$. Now the function field of $X(Y_3(N))$ is given by $\mathbb{C}(X(Y_3(N))) = \mathbb{C}(g_3^{\frac{1}{N}}, \mu)$. It is clear that

$$\mathbb{C}(g_3^{\frac{1}{N}}, \mu) = \mathbb{C}(f_1, ..., f_5),$$

i.e. $\mathscr{F}^\infty$ is generated by $f_1, ..., f_5$.

By identifying $\mathscr{D}^\infty$ with the subgroup $\left\{(x_i) \in \mathbb{Z}^6 \,\middle|\, \sum x_i = 0\right\}$ via the mapping

$$x_1 \cdot \infty + x_2 \cdot \frac{1}{3} + x_3 \cdot \frac{2}{3} + x_4 \cdot 0 + x_5 \cdot 1 + x_6 \cdot \frac{1}{2} \longmapsto (x_1, \ldots, x_6).$$

[1] This uses the fact that if $f$ has no zeros and no poles in the upper half plane, then we have $f^{\frac{1}{c}} = e^{\frac{1}{c}\log f}$

| $f_j^{1/c} \circ \gamma$ | $f_2^{1/c}$ | $f_3^{1/c}$ | $f_{23}^{1/c}$ | $f_4^{1/c}$ | $f_5^{1/c}$ | $f_{45}^{1/c}$ |
|---|---|---|---|---|---|---|
| $T^{-1}BTB^{-1}$ | $\zeta_c^{-1} f_2^{1/c}$ | $f_3^{1/c}$ | $\zeta_c f_{23}^{1/c}$ | $\zeta_c^{-1} f_4^{1/c}$ | $\zeta_c f_5^{1/c}$ | $f_{45}^{1/c}$ |
| $T^{-1}B^{-1}TB$ | $\zeta_c f_2^{1/c}$ | $f_3^{1/c}$ | $\zeta_c^{-1} f_{23}^{1/c}$ | $\zeta_c^{-1} f_4^{1/c}$ | $f_5^{1/c}$ | $\zeta_c f_{45}^{1/c}$ |
| $B^{-1}TBT^{-1}$ | $\zeta_c^{-1} f_2^{1/c}$ | $\zeta_c f_3^{1/c}$ | $f_{23}^{1/c}$ | $f_4^{1/c}$ | $f_5^{1/c}$ | $f_{45}^{1/c}$ |
| $BTB^{-1}T^{-1}$ | $\zeta_c f_2^{1/c}$ | $\zeta_c^{-1} f_3^{1/c}$ | $f_{23}^{1/c}$ | $\zeta_c^{-1} f_4^{1/c}$ | $\zeta_c f_5^{1/c}$ | $f_{45}^{1/c}$ |

Table 3.1: The action of commutators on the generators of $\mathrm{CL}(\Phi^0(N))$

so that

$$\alpha \coloneqq (1,0,0,0,0,-1), \qquad \beta \coloneqq (0,1,0,0,0,-1), \qquad \gamma \coloneqq (0,0,1,0,0,-1),$$
$$\delta \coloneqq (0,0,0,1,0,-1) \qquad \text{and} \qquad \epsilon \coloneqq (0,0,0,0,1,-1)$$

form a basis for $\mathscr{D}^\infty$, we may write

$$\begin{aligned}
\operatorname{div} f_1 &= (1,1,1,-1,-1,-1) = \alpha+\beta+\gamma-\delta-\epsilon,\\
\operatorname{div} f_2 &= (-\frac{N}{3},-\frac{N}{3},-\frac{N}{3},N,0,0) = -\frac{N}{3}(\alpha+\beta+\gamma-3\delta),\\
\operatorname{div} f_3 &= (-\frac{N}{3},-\frac{N}{3},-\frac{N}{3},0,0,N) = \frac{N}{3}(-\alpha-\beta-\gamma),\\
\operatorname{div} f_4 &= (\frac{2N}{3},-\frac{N}{3},-\frac{N}{3},0,0,0) = \frac{2N}{3}\alpha-\frac{N}{3}\beta-\frac{N}{3}\gamma,\\
\operatorname{div} f_5 &= (-\frac{N}{3},\frac{2N}{3},-\frac{N}{3},0,0,0) = -\frac{N}{3}\alpha+\frac{2N}{3}\beta-\frac{N}{3}\gamma
\end{aligned}$$

([33, Table 1] and (2.30)). It is then straightforward to compute (as in [19, Theorem IV.4.56]) that

$$\alpha+\beta+\gamma-\delta-\epsilon, \quad -\frac{1}{3}(\alpha+\beta+\gamma-3\delta), \quad -\alpha-\beta-\gamma, \quad \frac{1}{3}(2\alpha-\beta-\gamma), \quad \frac{1}{3}(-\alpha+2\beta-\gamma)$$

form a $\mathbb{Z}$-basis for $\mathscr{D}^\infty$, and therefore

$$\mathrm{CL}(Y_3(N)) \cong (\mathbb{Z}/N\mathbb{Z})^3 \times \mathbb{Z}/\tfrac{N}{3}\mathbb{Z}. \tag{3.5}$$

### 3.1.3 Ye's Variant $Y_{3,\pm\frac{N}{3}}(N)$

Recall that $Y_{3,s}(N) = \langle Y_3(N), TB_3^{s+1}\rangle$, where the relation $Y_3(N) \subseteq Y_{3,s}(N) \subseteq \Gamma_0(3)$ holds, and it is noncongruence if and only if $N \nmid 12$. The case we computed scattering constants for was $N \mid 3s$, $N \nmid s$ and $\pm\frac{N}{3} \equiv -1 \pmod 3$ ([33, Lemma 3.2]). In this case, recall that $[Y_{3,s}(N)\colon Y_3(N)] = 3$ and the group $Y_{3,s}(N)$ has four cusps $0, \infty, 1/3, 2/3$ with cusp widths $3N, \frac{N}{3}, \frac{N}{3}, \frac{N}{3}$, respectively.

Using the transformation rules (3.1), we get $g_3^{\frac{1}{N}} \circ TB^{\pm\frac{N}{3}+1} = \zeta_3^{\mp 1} g_3^{\frac{1}{N}}$ as well as

$$(\mu - 3g_3^{-\frac{1}{3}}) \circ TB^{\pm\frac{N}{3}+1} = \zeta_3^{-1}(\mu - 3\zeta_3^{\pm\frac{N}{3}+1} g_3^{-\frac{1}{3}}) \quad \text{and} \quad (\mu - 3\zeta_3 g_3^{-\frac{1}{3}}) \circ TB^{\pm\frac{N}{3}+1} = \zeta_3^{-1}(\mu - 3\zeta_3^{\pm\frac{N}{3}+2} g_3^{-\frac{1}{3}}).$$

It follows that $(\mu - 3g_3^{-\frac{1}{3}})g_3^{\mp\frac{1}{N}}$ and $(\mu - 3\zeta_3 g_3^{-\frac{1}{3}})g_3^{\mp\frac{1}{N}}$ are modular functions with respect to $\Phi^0_{\pm\frac{N}{3}}(N)$ and thus are in $\mathscr{F}^\infty$, when $\frac{N}{3} \equiv \mp 1 \pmod 3$. In fact, by writing $N = \mp 3 + 9k$, we find

$$\begin{aligned}
\operatorname{div} g_3^{\frac{3}{N}} &= 1 \cdot \infty + 1 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} - 3 \cdot 0 \\
\operatorname{div}(\mu - 3g_3^{-\frac{1}{3}})g_3^{\mp\frac{1}{N}} &= (2k \mp 1) \cdot \infty - k \cdot \frac{1}{3} - k \cdot \frac{2}{3} \pm 1 \cdot 0 \\
\operatorname{div}(\mu - 3\zeta_3 g_3^{-\frac{1}{3}})g_3^{\mp\frac{1}{N}} &= -k \cdot \infty + (2k \mp 1) \cdot \frac{1}{3} - k \cdot \frac{2}{3} \pm 1 \cdot 0.
\end{aligned} \tag{3.6}$$

It is easy to see that, under the condition $\frac{N}{3} \equiv \mp 1 \pmod 3$, we have

$$\mathbb{C}(X(Y_{3,\pm\frac{N}{3}}(N))) = \mathbb{C}(g_3^{\frac{3}{N}}, g_3^{\frac{1}{N}}\mu^{\mp 1}) = \mathbb{C}(g_3^{\frac{3}{N}}, (\mu - 3g_3^{-\frac{1}{3}})g_3^{\mp\frac{1}{N}}, (\mu - 3\zeta_3 g_3^{-\frac{1}{3}})g_3^{\mp\frac{1}{N}}).$$

In other words, $\mathscr{F}^\infty$ is generated by $g_3^{\frac{3}{N}}, (\mu - 3g_3^{-\frac{1}{3}})g_3^{\mp\frac{1}{N}}, (\mu - 3\zeta_3 g_3^{-\frac{1}{3}})g_3^{\mp\frac{1}{N}}$.

Again, by identifying $\mathscr{D}^\infty$ with the subgroup $\{x = (x_i) \in \mathbb{Z}^4 \mid \sum x_i = 0\}$ via the mapping

$$x_1 \cdot \infty + x_2 \cdot \frac{1}{3} + x_3 \cdot \frac{2}{3} + x_4 \cdot 0 \longmapsto (x_1, x_2, x_3, x_4).$$

so that

$$\alpha \coloneqq (1, 0, 0, -1) \qquad \beta \coloneqq (0, 1, 0, -1) \qquad \text{and} \qquad \gamma \coloneqq (0, 0, 1, -1).$$

forms a basis for $\mathscr{D}^\infty$, the divisors in (3.6) can then be rewritten as

$$\begin{aligned}
\operatorname{div} g_3^{\frac{3}{N}} &= \alpha + \beta + \gamma \\
\operatorname{div}(\mu - 3g_3^{-\frac{1}{3}})g_3^{-\frac{1}{N}} &= (2k \mp 1)\alpha - k\beta - k\gamma = (3k \mp 1)\alpha - k(\alpha + \beta + \gamma) \\
\operatorname{div}(\mu - 3\zeta_3 g_3^{-\frac{1}{3}})g_3^{-\frac{1}{N}} &= -k\alpha + (2k \mp 1)\beta - k\gamma = (3k \mp 1)\beta - k(\alpha + \beta + \gamma).
\end{aligned}$$

Therefore, a straightforward computation (as in [19, Theorem IV.4.56]) gives

$$\mathrm{CL}(Y_{3,\pm\frac{N}{3}}(N)) = (\mathbb{Z}/(3k \mp 1)\mathbb{Z})^2 = (\mathbb{Z}/\tfrac{N}{3}\mathbb{Z})^2.$$

## 3.2 Examples from Elliptic Curves

Let $E$ be an elliptic curve. Let $\operatorname{Pic}^0(E) = \operatorname{Div}^0(E)/\operatorname{Div}^l(E)$ be the Picard group, where $\operatorname{Div}^0(E)$ is the group of divisors of degree 0 and $\operatorname{Div}^l(E)$ is the subgroup of principal divisors, i.e.

$$\operatorname{Div}^l(E) = \{\delta \in \operatorname{Div}^0(X) \mid \delta = \operatorname{div}(f) \text{ for some } f \in \mathbb{C}(X)\}$$

Then we have the isomorphism ([29, Proposition 3.4])

$$E(\mathbb{C}) \cong \operatorname{Pic}^0(E), \tag{3.7}$$

given by $P \mapsto [P - \mathcal{O}]$, where $\mathcal{O}$ is the identity element in $E(\mathbb{C})$ and $[P - \mathcal{O}]$ is the class of the divisor $P - \mathcal{O}$.

Recall that the Belyi map is a finite morphism $\beta : E \longrightarrow \mathbb{P}^1$ with at most three critical values $0, 1, \infty$. It is equivalent to the fact that there exists $\Gamma_E$, a finite index subgroup of $\Gamma(2)$ such that $\overline{\Gamma_E \backslash \mathbb{H}} \cong E(\mathbb{C})$ ([10, Theorem 4.31]). In many cases, $\Gamma_E$ is a noncongruence subgroup. Moreover, since $\mathscr{D}^\infty$ can be regarded as a subgroup of $\mathrm{Div}^0(E)$, we can consider $\mathrm{CL}(\Gamma_E)$ as a subgroup of $E(\mathbb{C})$ under the above isomorphism (3.7).

### 3.2.1 Elkies' Examples

*Example 1.* Let $E$ be the elliptic curve $y^2 = x^3 + 5x + 10$ studied in [3].[2] It has a degree 5 Belyi map $\beta(x, y) = \frac{(x-5)y+16}{32}$ ([3, Proposition 3.1]), and the cusps correspond to points in $E(\mathbb{C})$ as follows:

$$S_0 = \mathcal{O},\ S_1 = (1, 4),\ S_2 = -4S_1 = (6, -16),\ S_3 = -S_1 = (1, -4),\ S_4 = 4S_1 = (6, 16)$$

([3, eq (3.3)]). Here under the map $E \longrightarrow \mathbb{P}^1$, $S_0$ lies above $\infty$; $S_1$ and $S_2$ lie above 0; $S_3$ and $S_4$ lie above 1. Therefore, we can identify $\mathrm{CL}(\Gamma_E)$ with the subgroup

$$\left\{ \sum_{i=0}^{4} x_i S_i \;\middle|\; \sum_{i=0}^{4} x_i = 0 \right\}.$$

in $E(\mathbb{C})$, where

$$\sum_{i=0}^{4} x_i S_i = x_1 S_1 - 4x_2 S_1 - x_3 S_1 + 4x_4 S_1 = (x_1 - 4x_2 - x_3 + 4x_4) S_1.$$

Note that the Mordell-Weil group $E(\mathbb{Q}) \cong \mathbb{Z}$ is generated by $S_1 = (1, 4)$ ([3, Proposition 3.2]). Therefore,

$$\mathrm{CL}(\Gamma_E) \cong \mathbb{Z}.$$

As a consequence, it follows from Manin-Drinfeld theorem that $\Gamma_E$ must be a noncongruence subgroup. $\square$

*Example 2.* Let $E$ be the elliptic curve $y^2 = x^3 - 120x + 740$ in the list [7] given by Elkies. Its Mordell-Weil group is isomorphic to $\mathbb{Z}$, and is generated by $S_0 = (16, 54)$. The integral points are

$$(-11, \pm 27), \quad (-4, \pm 34), \quad (4, \pm 18) \quad \text{and} \quad (16, \pm 54)$$

([21, 900.e2]). It has a degree 5 Belyi map $\phi(x, y) = \frac{(x+5)y+162}{324}$, where

- Over $\infty$, there is only one cusp $\mathcal{O}$.
- Over 0, there are two cusps $3S_0 = (-11, 27)$ and $-2S_0 = (4, -18)$. Here $3S_0$ has multiplicity 3, and $-2S_0$ has multiplicity 2.
- Over 1, there are two cusps $2S_0 = (4, 18)$ and $-3S_0 = (-11, -27)$. Here $2S_0$ has multiplicity 3, and $-3S_0$ has multiplicity 2.

[2] The authors in this article attributed the example to Elkies' paper [8].

So again $\mathrm{CL}(\Gamma_E) \cong \mathbb{Z}$. As a consequence, it follows from Manin-Drinfeld theorem that $\Gamma_E$ must be a noncongruence subgroup. □

### 3.2.2 Berger's Examples

We reproduce the setup in Berger's paper [2]. Recall that the congruence subgroup $\Gamma_0(16)$ has six inequivalent cusps $0, 1/2, \pm 1/4, 1/8, \infty$ ([6, p.103]) and no elliptic points ([6, Corollary 3.7.2]). Moreover, the covering map $X_0(16) \to X(1)$ is of degree

$$\frac{16^3 \cdot \left(1 - \frac{1}{2^2}\right)}{16 \cdot \phi(16)} = 24$$

([6, p.107]), so the modular curve $X_0(16)$ has genus $1 + \frac{24}{12} - \frac{6}{2} = 0$ ([6, Theorem 3.1.1]). By the uniformization theorem, the open modular curve $Y_0(16)$ is then topologically a Riemann sphere with six punctures. This means that its fundamental group is a free group with five generators ([13, Example 1.21]), or equivalently

$$\pi_1(Y_0(16)) \cong \left\langle \alpha_1, \cdots, \alpha_6 \;\middle|\; \prod_{i=1}^{6} \alpha_i = 1 \right\rangle.$$

Since the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ is properly discontinuous ([6, Proposition 2.1.1]) and $\Gamma_0(16)$ has no elliptic points, we have the isomorphism $\Gamma_0(16) \cong \pi_1(\Gamma_0(16)\backslash\mathbb{H})$ ([13, Proposition 1.40(c)]). In particular, the generator of the stabilizer of each cusp can be identified with the loop around the corresponding puncture on the Riemann sphere ([1, Example 6.3.5]). Therefore, we have the group presentation

$$\Gamma_0(16) \cong \left\langle \gamma_0, \gamma_{1/2}, \gamma_{1/4}, \gamma_{-1/4}, \gamma_{1/8}, \gamma_\infty \;\middle|\; \prod_{t\colon \text{cusp of } \Gamma_0(16)} \gamma_t = 1 \right\rangle,$$

where $\gamma_t$ is a generator of the stabilizer $\Gamma_0(16)$ of the cusp $t$.

This allows us to define homomorphisms

$$\psi\colon \Gamma_0(16) \to \{\pm 1\}$$

by sending two of the generators to 1 and the remaining four to $-1$ so that their kernels $\Delta$ have index 2 in $\Gamma_0(16)$. These homomorphisms induce canonical maps $X_\Delta \to X_0(16)$ of degree 2 ([6, p.66]) ramified over the four points corresponding to the generators sent to $-1$ ([6, p.67]). In particular, we define four homomorphisms

$$\psi_1\colon \gamma_{1/8}, \gamma_\infty \mapsto 1, \qquad \psi_2\colon \gamma_{1/4}, \gamma_{-1/4} \mapsto 1, \qquad \psi_3\colon \gamma_0, \gamma_\infty \mapsto 1 \quad \text{and} \quad \psi_4\colon \gamma_0, \gamma_{1/8} \mapsto 1$$

via the generators

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 16 & 1 \end{pmatrix}, \qquad \gamma_{1/2} = \begin{pmatrix} -7 & 4 \\ -16 & 9 \end{pmatrix}, \qquad \gamma_{1/4} = \begin{pmatrix} -3 & 1 \\ -16 & 5 \end{pmatrix}$$
$$\gamma_{-1/4} = \begin{pmatrix} -3 & -1 \\ 16 & 5 \end{pmatrix}, \qquad \gamma_{1/8} = \begin{pmatrix} -7 & 1 \\ -64 & 9 \end{pmatrix} \quad \text{and} \quad \gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with respective kernels $\Delta_1, \Delta_2, \Delta_3, \Delta_4$.

Now, since $X_0(16)$ has genus 0, we may specify a Hauptmodul $x$ satisfying

$$x(0) = 0, \qquad x(1/2) = \infty \quad \text{and} \quad x(\infty) = -1$$

so that $\mathbb{C}(X_0(16)) = \mathbb{C}(x)$. By our construction of $\Delta_i$, it follows that $\mathbb{C}(X_{\Delta_i}) = \mathbb{C}(x, y)$ where $y^2 = f_i(x)$ is a polynomial in $x$. In fact, the Riemann-Hurwitz formula ([29, Theorem II.5.9]) suggests that the modular curves $X_{\Delta_i}$ have genus 1, so the polynomials $f_i$ must be cubic polynomials in $x$ ([29, Proposition III.3.1(a)]). We thus have a commutative diagram

$$\begin{tikzcd} X_{\Delta_i} \arrow[r, "x,y"] \arrow[d] & E_i \colon y^2 = f_i(x) \arrow[d] \\ X_0(16) \arrow[r, "x"] & \mathbb{P}^1 \end{tikzcd} \tag{3.8}$$

where the maps $E_i \to \mathbb{P}^1$ are not Belyi because the map of modular curves $X_{\Delta_i} \to X_0(16)$ ramifies over four points.

With some algebra, one can also show that the Hauptmodul $x$ has Fourier series

$$x(z) = -\prod_{n=1}^{\infty}(1 + q^{2n})^2(1 - q^{2n-1})^4 = -1 + 4q - 8q^2 + \cdots .$$

with additional special values $x\left(\{\pm\frac{1}{4}\}\right) = \{\pm i\}$ and $x(\frac{1}{8}) = 1$ ([2, Theorem 1]).

*Example 1.* By construction and diagram (3.8), the map $E_1 \to \mathbb{P}^1$ ramifies over the four points $x(\{0, 1/2, \pm 1/4\}) = \{0, \infty, \pm i\}$. For convenience, we work with[3] the elliptic curve $E_1 \colon y^2 = x^3 + x$, so that the cusps of $\Delta_1$ correspond to points in $E_1(\mathbb{C})$ as follows:

$$S_0 = \mathcal{O}, \quad S_1 = (0,0), \quad S_2 = (i,0), \quad S_3 = -S_2 = (-i,0)$$
$$S_4 = (-1, \sqrt{2}i), \quad S_5 = -S_4 = (-1, -\sqrt{2}i), \quad S_6 = (1, \sqrt{2}), \quad S_7 = -S_6 = (1, -\sqrt{2}).$$

Using computational tools like SageMath (example code provided in Appendix A.1), we have the identities

$$2S_1 = 2S_2 = 2S_3 = \mathcal{O}, \qquad S_1 + S_2 = S_3$$
$$2S_4 = S_1, \qquad 2S_6 = S_1, \qquad S_4 + S_6 = S_2.$$

As a result, $\mathrm{CL}(\Delta_1)$ can be identified with the subgroup

$$\left\{ \sum_{i=0}^{7} x_i S_i \,\middle|\, \sum_{i=0}^{7} x_i = 0 \right\}$$

[3] Our choice differs from that made in [2] because we are not concerned with the rationality of the Fourier coefficients of $y$.

where

$$\begin{aligned}\sum_{i=0}^{7} x_i S_i &= 2x_1 S_4 + x_2 S_2 + x_3(2S_4 + S_2) + x_4 S_4 - x_5 S_4 + x_6(S_2 - S_4) - x_7(S_2 - S_4)\\ &= (x_2 + x_3 + x_6 - x_7)S_2 + (2x_1 + 2x_3 + x_4 - x_5 + x_7)S_4.\end{aligned}$$

Therefore, we have

$$\mathrm{CL}(\Delta_1) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$

□

*Example 2.* By construction and diagram (3.8), the map $E_2 \to \mathbb{P}^1$ ramifies over the four points $x(\{0, 1/2, 1/8, \infty\}) = \{0, \infty, 1, -1\}$. For convenience, we work with[4] the elliptic curve $E_2\colon y^2 = x^3 - x$, so that the cusps of $\Delta_2$ correspond to points in $E_2(\mathbb{C})$ as follows:

$$S_0 = \mathcal{O}, \quad S_1 = (0,0), \quad S_2 = (1,0), \quad S_3 = -S_2 = (-1,0)$$
$$S_4 = (i, \sqrt{2}(1-i)), \quad S_5 = -S_4 = (i, -\sqrt{2}(1-i)), \quad S_6 = (-i, \sqrt{2}(1+i)), \quad S_7 = -S_6 = (-i, -\sqrt{2}(1+i)).$$

Using computational tools like SageMath (example code provided in Appendix A.1), we have the identities

$$\begin{gathered}2S_1 = 2S_2 = 2S_3 = \mathcal{O}, \qquad S_1 + S_2 = S_3\\ 2S_4 = S_1, \qquad 2S_6 = S_1, \qquad S_4 + S_6 = S_3.\end{gathered}$$

As a result, $\mathrm{CL}(\Delta_2)$ can be identified with the subgroup

$$\left\{ \sum_{i=0}^{7} x_i S_i \;\middle|\; \sum_{i=0}^{7} x_i = 0 \right\}$$

where

$$\begin{aligned}\sum_{i=0}^{7} x_i S_i &= 2x_1 S_4 + x_2(2S_4 - S_3) + x_3 S_3 + x_4 S_4 - x_5 S_4 + x_6(S_3 - S_4) - x_7(S_3 - S_4)\\ &= (-x_2 + x_3 + x_6 - x_7)S_3 + (2x_1 + 2x_2 + x_4 - x_5 - x_6 + x_7)S_4.\end{aligned}$$

Therefore, we have

$$\mathrm{CL}(\Delta_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$

□

*Example 3.* By construction and diagram (3.8), the map $E_3 \to \mathbb{P}^1$ ramifies over the four points $x(\{1/2, \pm 1/4, 1/8\}) = \{\infty, \pm i, 1\}$. For convenience, we work with[5] the elliptic curve $E_3\colon y^2 =$

[4] see Footnote 3
[5] see Footnote 3

$x^3 - x^2 + x - 1$, so that the cusps of $\Delta_3$ correspond to points in $E_3(\mathbb{C})$ as follows:

$$S_0 = \mathcal{O}, \quad S_1 = (1,0), \quad S_2 = (i,0), \quad S_3 = (-i,0)$$
$$S_4 = (0,i), \quad S_5 = -S_4 = (0,-i), \quad S_6 = (-1,2i), \quad S_7 = -S_6 = (-1,-2i).$$

Using computational tools like SageMath (example code provided in Appendix A.1), we find that $S_4$ is of infinite order and

$$2S_1 = 2S_2 = 2S_3 = \mathcal{O}, \qquad S_2 + S_3 = S_1, \qquad S_4 + S_6 = S_1.$$

As a result, $\mathrm{CL}(\Delta_3)$ can be identified with the subgroup

$$\left\{ \sum_{i=0}^{7} x_i S_i \,\middle|\, \sum_{i=0}^{7} x_i = 0 \right\}$$

where

$$\begin{aligned}\sum_{i=0}^{7} x_i S_i &= x_1 S_1 + x_2 S_2 + x_3(S_1 - S_2) + x_4 S_4 - x_5 S_4 + x_6(S_1 - S_4) + x_7(S_4 - S_1)\\ &= (x_1 + x_3 + x_6 - x_7)S_1 + (x_2 - x_3)S_2 + (x_4 - x_5 - x_6 + x_7)S_4.\end{aligned}$$

Therefore, we have

$$\mathrm{CL}(\Delta_3) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}.$$

□

*Example 4.* By construction and diagram (3.8), the map $E_4 \to \mathbb{P}^1$ ramifies over the four points $x(\{1/2, \pm 1/4, \infty\}) = \{\infty, \pm i, -1\}$. For convenience, we work with[6] the elliptic curve $E_4\colon y^2 = x^3 + x^2 + x + 1$, so that the cusps of $\Delta_4$ correspond to points in $E_4(\mathbb{C})$ as follows:

$$S_0 = \mathcal{O}, \quad S_1 = (-1,0), \quad S_2 = (i,0), \quad S_3 = (-i,0)$$
$$S_4 = (0,1), \quad S_5 = -S_4 = (0,-1), \quad S_6 = (1,2), \quad S_7 = -S_6 = (1,-2).$$

Using computational tools like SageMath (example code provided in Appendix A.1), we find that $S_4$ is of infinite order[7] and

$$2S_1 = 2S_2 = 2S_3 = \mathcal{O},$$
$$S_1 + S_2 = S_3, \qquad S_1 + S_4 = S_7, \qquad S_4 + S_6 = S_1.$$

As a result, $\mathrm{CL}(\Delta_4)$ can be identified with the subgroup

$$\left\{ \sum_{i=0}^{7} x_i S_i \,\middle|\, \sum_{i=0}^{7} x_i = 0 \right\}$$

[6] see Footnote 3

[7] Alternatively, one can check that $2^5 S_4 \neq 0$. By [5], the order of the 2-torsion subgroup of $E(K)$ is at most $2^5$, where $K = \mathbb{Q}(i, \sqrt{2})$. Hence $S_4$ has infinite order.

where

$$\sum_{i=0}^{7} x_i S_i = x_1 S_1 + x_2 S_2 + x_3(S_1 - S_2) + x_4 S_4 - x_5 S_4 + x_6(S_1 - S_4) + x_7(S_4 - S_1)$$
$$= (x_1 + x_3 + x_6 + x_7)S_1 + (x_2 - x_3)S_2 + (x_4 - x_5 - x_6 + x_7)S_4.$$

Therefore, we have

$$\mathrm{CL}(\Delta_4) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}.$$

□

### 3.2.3 Special Elliptic Curves

In this section, we consider special classes of elliptic curves and their cuspidal divisor class groups. For more discussions on their Belyi maps, see [18].

*Example 1.* Consider the elliptic curve $E_3 : y^2 = x^3 + D^2$, where $D$ is a positive integer. It has zero $j$-invariant and a degree 3 Belyi map $\beta_3\colon E_3 \to \mathbb{P}^1$ given by $\beta_3(x, y) = \frac{y+D}{2D}$. Note that $S_0 = \mathcal{O}$ lies over $\infty$; $S_1 = (0, -D)$ lies over 0; $S_2 = -S_1 = (0, D)$ lies over 1. Hence the cusps corresponding to $\Gamma_{E_3}$ are $\{S_0, S_1, S_2\}$. Therefore, $\mathrm{CL}(\Gamma_{E_3})$ corresponds to the subgroup $\left\{\sum_{i=0}^{2} x_i S_i \,\middle|\, \sum_{i=0}^{2} x_i = 0\right\}$ where

$$\sum_{i=0}^{2} x_i S_i = x_1 S_1 - x_2 S_1 = (x_1 - x_2)S_1.$$

Since $3S_1 = \mathcal{O}$, we conclude that

$$\mathrm{CL}(\Gamma_{E_3}) \cong \mathbb{Z}/3\mathbb{Z}.$$

□

*Example 2.* Consider the elliptic curve $E_4 : y^2 = x^3 + Ax$. It has $j$-invariant 1728 and degree 4 Belyi map $\beta_4\colon E \to \mathbb{P}^1$ given by $\beta_4(x, y) = -\frac{x^2}{A}$. Note that $S_0 = \mathcal{O}$ lies over $\infty$; $S_1 = (0, 0)$ lies over 0; $S_2 = (\sqrt{-A}, 0)$ and $S_3 = (-\sqrt{-A}, 0)$ lie over 1. Hence the cusps corresponding to $\Gamma_{E_4}$ are $\{S_0, S_1, S_2, S_3\}$. It can be computed that $S_2 + S_3 = S_1$, so $\mathrm{CL}(\Gamma_{E_4})$ corresponds to $\left\{\sum_{i=0}^{3} x_i S_i : \sum_{i=0}^{3} x_i = 0\right\}$ where

$$\sum_{i=0}^{3} x_i S_i = x_1(S_2 + S_3) + x_2 S_2 + x_3 S_3 = (x_1 + x_2)S_2 + (x_1 + x_3)S_3.$$

Since $2S_1 = \mathcal{O}$, we conclude that

$$\mathrm{CL}(\Gamma_{E_4}) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

□

*Example 3.* Consider the elliptic curve $E_6 : y^2 = x^3 + B$. It has a degree 6 Belyi map $\beta_6 : E_6 \to \mathbb{P}^1$ given by $\beta_6(x, y) = -\frac{x^3}{B}$. Note that $S_0 = \mathcal{O}$ lies over $\infty$; $S_1 = (0, \sqrt{B})$ and $S_2 = -S_1 = (0, -\sqrt{B})$ lie over 0;

$$S_3 = (x_1, 0), \qquad S_4 = (x_2, 0), \qquad S_5 = (x_3, 0)$$

lie over 1, where $x^3 + B = (x - x_1)(x - x_2)(x - x_3)$. Hence the cusps corresponding to $\Gamma_{E_6}$ are $\{S_0, S_1, ..., S_5\}$. It can be computed that

$$2S_1 = S_2 \quad \text{and} \quad S_3 + S_4 = S_5$$

so $\mathrm{CL}(\Gamma_{E_6})$ corresponds to the subgroup $\left\{\sum_{i=0}^{5} x_i S_i \,\middle|\, \sum_{i=0}^{5} x_i = 0\right\}$ where

$$\begin{aligned}\sum_{i=0}^{5} x_i S_i &= x_1 S_1 + x_2(2S_1) + x_3 S_3 + x_4 S_4 + x_5(S_3 + S_4)\\ &= (x_1 + 2x_2)S_1 + (x_3 + x_5)S_3 + (x_4 + x_5)S_4.\end{aligned}$$

Since $3S_1 = 2S_3 = 2S_4 = \mathcal{O}$, we conclude that

$$\mathrm{CL}(\Gamma_{E_6}) \cong \mathbb{Z}/3\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2.$$

□

For the remaining of the subsection, we prove the following:

**Theorem 3.3.** *The modular groups $\Gamma_{E_3}$ and $\Gamma_{E_4}$ are congruence, while $\Gamma_{E_6}$ is noncongruence.*

We first explain the correspondence

**Proposition 3.4.** *The following are in one to one correspondences:*

1. *Belyi maps of degree $d$;*
2. *permutation triples $(\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$, $\sigma_0\sigma_1\sigma_\infty = 1$, which give rise to the monodromy map $M : \Gamma(2) \longrightarrow S_d$ such that $\gamma_i \longmapsto \sigma_i$ for $i = 0, 1$;*
3. *finite index subgroup of $\Gamma(2)$ of index $d$ given by $M^{-1}(\mathrm{Stab}(1))$.*

where

$$\gamma_0 = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} \qquad \text{and} \qquad \gamma_1 = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

are the generators of $\Gamma(2)$. Recall the definition of a dessin:

**Definition 3.5** ([10, Definition 4.1])**.** *A* dessin d'enfant*, or simply a* dessin*, is a pair $(X, \mathcal{D})$ where $X$ is an oriented compact Riemann surface, and $\mathcal{D} \subseteq X$ is a finite graph such that:*

*(i) $\mathcal{D}$ is connected*

*(ii) $\mathcal{D}$ is bi-coloured (i.e. the vertices have been given either* white *or* black *color and vertices connected by an edge have different colors)*

*(iii) $X \setminus \mathcal{D}$ is the union of finitely many topological disks, which we call faces of $\mathcal{D}$*

Let $(E, \beta)$ be the Belyi pair given by one of those special elliptic curves, and let $(\mathcal{D}, X)$ be its corresponding dessin (given by [10, Proposition 4.25]). Suppose $\mathcal{D}$ has $d$ edges labelled from 1 to $d$. We can define its *permutation pair* $(\sigma_0, \sigma_1)$ as follows. Draw small (topological) circles around each of the white (resp. black) vertices and set $\sigma_0(i) = j$ (resp. $\sigma_1(i) = j$) if $j$ is the edge that follows $i$ under a positive orientation (c.f. [10, §1.2.1]). The permutation pair is also called the *monodromy group of the dessin* (or *monodromy group of the Belyi pair*). This is because it is precisely the inverse of the pair $M_\beta(\gamma_0, \gamma_1)$ ([10, Proposition 4.29]), where $M_\beta$ is the monodromy map from $\Gamma(2)$ to the symmetric group $S_d$ determined by regarding the restriction $\beta_0 \colon E \setminus \beta^{-1}(\{0, 1, \infty\}) \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ as a map between covering spaces (see [10, p.229]). The finite index subgroup $\Gamma_E$ for which $\overline{\Gamma_E \setminus H} \simeq E(\mathbb{C})$ can then be recovered as $M_\beta^{-1}(\mathrm{Stab}(1))$, the preimage of the stabilizer of 1 under the monodromy map $M_\beta$.

In particular, the cycle types of the permutation triples are determined by the splitting behavior of the cusps $0, 1, \infty$. For example, for $\Gamma_{E_3}$, there is only one cusp lying over each of the cusps $0, 1, \infty$. Hence $\sigma_0, \sigma_1, \sigma_\infty$ should all be 3-cycles. So the only choice would be $\sigma_0 = \sigma_1 = (123)$. Let $\gamma_\infty = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ so that $\gamma_0\gamma_1\gamma_\infty = 1$. The permutation triples of our special elliptic curves are given by

$$\begin{aligned} M_{\beta_3} \colon \Gamma(2) &\to S_3 & M_{\beta_4} \colon \Gamma(2) &\to S_4 & M_{\beta_6} \colon \Gamma(2) &\to S_6 \\ \gamma_0 &\mapsto (123) & \gamma_0 &\mapsto (1234) & \gamma_0 &\mapsto (154)(263) \\ \gamma_1 &\mapsto (123) & \gamma_1 &\mapsto (13)(24), & \gamma_1 &\mapsto (12)(34)(56) \\ \gamma_\infty &\mapsto (123) & \gamma_\infty &\mapsto (1234). & \gamma_\infty &\mapsto (135246) \end{aligned} \tag{3.9}$$

(See [9, Section 8] for $\Gamma_{E_4}$ and [10, Example 4.14] for $\Gamma_{E_6}$). Note that the image $\mathrm{image}(M_{\beta_i})$ is all cyclic, i.e.

$$\mathrm{image}(M_{\beta_3}) = \langle (123) \rangle, \qquad \mathrm{image}(M_{\beta_4}) = \langle (1234) \rangle, \quad \text{and} \quad \mathrm{image}(M_{\beta_6}) = \langle (135246) \rangle.$$

Hence the intersection $\mathrm{Stab}(1) \cap \mathrm{image}(M_{\beta_i}) = \{1\}$ is trivial and so $\Gamma_{E_i} = \ker(M_{\beta_i})$.

Now, we use the result of Wohlfahrt [31]

**Proposition 3.6.** *Suppose $\Gamma$ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $N$ be the level, namely, the least common multiple of the cusp widths of $\Gamma$. If $\Gamma$ is a congruence subgroup, then $\Gamma$ contains $\Gamma(N)$.*

to determine whether $\Gamma_E$ is a congruence subgroup. The level of $\Gamma_{E_i}$ is 6,8,12, respectively. So we should check that

$$\Gamma(6) \subseteq \Gamma_{E_3}, \qquad \Gamma(8) \subseteq \Gamma_{E_4} \quad \text{and} \quad \Gamma(12) \not\subseteq \Gamma_{E_6}.$$

This can be checked programatically, for example, using SageMath (example code provided in Appendix A.2) by checking if the generators of $\Gamma(N)$ gets mapped to $\{1\}$ under the map $M_\beta$.

- For example, one can check that for each $\gamma$ of 13 generators of $\Gamma(6)$, if we write $\gamma = \prod_{i=1}^n \gamma_i^{r_i}$ with $\gamma_i \in \{\gamma_0, \gamma_1\}$, then $\sum_{i=1}^n r_i \equiv 0 \pmod 3$.
- Similarly, for each $\gamma$ of 33 generators of $\Gamma(8)$, if we write $\gamma = \prod_{i=1}^n \gamma_i^{r_i}$ with $\gamma_i \in \{\gamma_0, \gamma_1\}$, then $\sum_{i=1}^n a_i r_i \equiv 0 \pmod 4$ where $a_i = \begin{cases} 1, & \text{if } \gamma_i = \gamma_0 \\ 2, & \text{if } \gamma_i = \gamma_1 \end{cases}$.
- For $\Gamma_{E_6}$, note that $\sigma_0 = \sigma_\infty^2$ and $\sigma_1 = \sigma_\infty^3$. Consider the generator

$$\gamma = \begin{pmatrix} 157 & -36 \\ 48 & -11 \end{pmatrix}$$

of $\Gamma(12)$, which we write $\gamma = \prod_{i=1}^{n} \gamma_i^{r_i}$ with $\gamma_i \in \{\gamma_0, \gamma_1\}$. It can be shown that $\sum_{i=1}^{n} a_i r_i \not\equiv 0 \pmod 6$, where $a_i = \begin{cases} 2, & \text{if } \gamma_i = \gamma_0 \\ 3, & \text{if } \gamma_i = \gamma_1 \end{cases}$.

# Chapter 4

# Future Direction(s)

As discussed in Chapter 1, Posingies computed the scattering constants, and Rohrlich computed the cuspidal divisor class groups, for the Fermat groups $\Phi(N)$ associated with the Fermat curves $F(N)$. A central theme of this thesis has been to extend these ideas to other families of noncongruence subgroups for which such arithmetic and analytic invariants can be made explicit.

A natural direction for future work is to further generalize the class of Fermat-type groups and to investigate their associated scattering constants and cuspidal divisor class groups, which remain largely unexplored. In particular, one may ask whether the guiding role played by cuspidal divisor class groups in the Fermat case persists more broadly.

Recall that the Fermat groups are defined (see [26, Proposition 1.2]) by

$$\Phi(N) \coloneqq \{\sigma \in \Gamma(2) \mid \alpha(\sigma) \equiv \beta(\sigma) \equiv 0 \pmod N\},$$

where $\alpha, \beta : \Gamma(2) \to \mathbb{Z}$ are homomorphisms determined by

$$\begin{cases} \alpha(T^2) = 1, \\ \alpha(B_2) = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} \beta(T^2) = 0, \\ \beta(B_2) = 1. \end{cases}$$

Equivalently, $\Phi(N)$ is generated by the commutator subgroup $\Gamma(2)'$ together with $T^{2N}$ and $B_2^N$. Since $\Gamma(2)$ itself is generated by $T^2$ and $B_2$ (see [26, Appendix]), this characterization admits a more conceptual reformulation:

$$\Phi(N) = \langle \Gamma(2)', \, \gamma^N \mid \gamma \in \Gamma(2) \rangle.$$

That is, $\Phi(N)$ is the subgroup of $\Gamma(2)$ generated by its commutator subgroup together with all $N$-th powers of elements of $\Gamma(2)$.

This description suggests a natural generalization to higher levels. Let $p$ be a prime. We define $\Omega(p, N)$ to be the subgroup of $\Gamma(p)$ generated by the commutator subgroup $\Gamma(p)'$ together with the set

$$\{\gamma^N \mid \gamma \in \Gamma(p)\}.$$

With this notation, the classical Fermat groups $\Phi(N)$ are precisely the groups $\Omega(2, N)$.

The groups $\Omega(p, N)$ form a rich family of noncongruence subgroups. Indeed, it was shown in [25] that $\Omega(p, N)$ is noncongruence whenever $N > 1$ and $(N, p) = 1$.[1] Despite their natural definition and close relationship to the classical Fermat groups, neither the scattering constants nor the cuspidal divisor class groups of $\Omega(p, N)$ have been computed.

From the perspective developed in this thesis, a first step toward understanding these invariants is to determine whether $\mathrm{CL}(\Omega(p, N))$ is finite and, if so, to describe its structure explicitly. Such information is essential for constructing modular forms with prescribed vanishing behavior at the cusps, which in turn is the key input for applying Posingies' Kronecker limit formula framework (see Section 2.1.1).

To illustrate the nature of the problem, consider the case $p = 3$. It is well known that the matrices

$$\gamma_0 \coloneqq \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \qquad \gamma_1 \coloneqq \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}, \quad \text{and} \quad \gamma_\infty \coloneqq \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

generate the principal congruence subgroup $\Gamma(3)$ in $\mathrm{PSL}_2(\mathbb{Z})$.[2] In this setting, the subgroup $\Omega(3, N)$ admits a particularly concrete description.

*Fact.* We have $\Omega(3, N) \triangleleft \Gamma(1)$, with index $[\Gamma(1) : \Omega(3, N)] = 12N^3$. The group has $4N^2$ cusps, all of the same width $3N$. A system of representatives for the cosets $\Omega(3, N)\backslash\Gamma(3)$ is given by

$$\left\{\gamma_0^i \gamma_1^j \gamma_\infty^k\right\}, \qquad i, j, k \in \{0, 1, \ldots, N-1\}.$$

A system of representatives for the cusps is $S = S_0 \cup S_1 \cup S_2 \cup S_\infty$, where

$$S_0 = \{\gamma_1^i \gamma_\infty^j 0\}, \qquad S_1 = \{\gamma_0^i \gamma_\infty^j 1\}, \qquad S_2 = \{\gamma_0^i \gamma_\infty^j 2\}, \quad \text{and} \quad S_\infty = \left\{\gamma_0^i \gamma_1^j \infty\right\},$$

with $i, j \in \{0, 1, \ldots, N-1\}$. The cusps in $S_i$ are $\Gamma(3)$-equivalent to $i$ for each $i \in \{0, 1, 2, \infty\}$.

The natural goal is to construct modular forms

$$G_{0_j}^{\Omega(3,N)}, \quad G_{1_j}^{\Omega(3,N)}, \quad G_{2_j}^{\Omega(3,N)}, \quad G_{\infty,j}^{\Omega(3,N)}$$

for $\Omega(3, N)$ whose divisors satisfy

$$\operatorname{div} G_{i_j}^{\Omega(3,N)} = m \cdot i_j$$

for some integer $m$, where $i_j$ denotes a cusp in the above system of representatives. This construction would be directly analogous to [24, Lemma 6.3]. Moreover, for the purpose of computing scattering constants, these modular forms should arise from suitable factorizations of the corresponding forms $G_i^{\Gamma(3)}$, as in [24, Lemma 6.4].

Progress along these lines would extend the explicit computations known for Fermat groups to a broader and more systematic family of noncongruence subgroups. More broadly, it would provide further evidence that cuspidal divisor class groups play a fundamental role in the analytic study of Eisenstein series for noncongruence subgroups, reinforcing the central theme of this thesis.

[1] This condition is not necessary: for example, the Fermat groups $\Phi(N)$ are noncongruence if and only if $N \nmid 8$; see [26, Appendix].

[2] Note that the matrix $\gamma_1 \gamma_0 \gamma_\infty^{-1}$ fixes the cusp 2 of $\Gamma(3)$.

# Appendix A

# SageMath Code

## A.1 Berger's Examples

For reader's convenience, we include the SageMath setup to perform various group law computations in Section 3.2.2.

```
# SageMath code for Example 1.
x = polygen(ZZ, 'x')
K.<i> = NumberField(x^2 + 1)        # i
L.<sqrt2> = K.extension(x^2 - 2)    # sqrt(2)
E_1 = EllipticCurve(L, [0,0,0,1,0])
S_1 = E_1(0, 0)
S_2 = E_1(i, 0)
S_3 = E_1(-i, 0)
S_4 = E_1(-1, sqrt2 * i)
S_5 = E_1(-1, -sqrt2 * i)
S_6 = E_1(1, sqrt2)
S_7 = E_1(1, -sqrt2)
```

```
# SageMath code for Example 2.
x = polygen(ZZ, 'x')
K.<i> = NumberField(x^2 + 1)        # i
E_2 = EllipticCurve(K, [0,0,0,-1,0])
S_1 = E_2(0, 0)
S_2 = E_2(1, 0)
S_3 = E_2(-1, 0)
S_4 = E_2(i, 1 - i)
S_5 = E_2(i, -(1 - i))
S_6 = E_2(-i, 1 + i)
S_7 = E_2(-i, -(1 + i))
```

```
# SageMath code for Example 3.
x = polygen(ZZ, 'x')
K.<i> = NumberField(x^2 + 1)        # i
E_3 = EllipticCurve(K, [0,-1,0,1,-1])
S_1 = E_3(1, 0)
S_2 = E_3(i, 0)
S_3 = E_3(-i, 0)
S_4 = E_3(0, i)
S_5 = E_3(0, -i)
S_6 = E_3(-1, 2 * i)
S_7 = E_3(-1, -2 * i)
```

```
# SageMath code for Example 4.
x = polygen(ZZ, 'x')
K.<i> = NumberField(x^2 + 1)        # i
E_4 = EllipticCurve(K, [0,1,0,1,1])
S_1 = E_4(-1, 0)
S_2 = E_4(i, 0)
S_3 = E_4(-i, 0)
S_4 = E_4(0, 1)
S_5 = E_4(0, -1)
S_6 = E_4(1, 2)
S_7 = E_4(1, -2)
```

## A.2 Special Elliptic Curves

For reader's convenience, we include the SageMath code to verify Theorem 3.3. The common setup for all three families of special elliptic curves is the matrices $g_0, g_1, g_{00}$, which are respective stabilizers of the cusps $\{0, 1, \infty\}$ of $\Gamma(2)$. We use the notation $g_{00}$ rather than $g_\infty$ to simplify the code.

```
# g0, g1, g00 resp fixes 0, 1 and 00
# {go, g1} is an indep set of generators for \Gamma(2)
# 'h0, h1 and h00 are the resp inverse of g0, g1 and g00'
g0 = Matrix([[-1,0],[2,-1]])
g1 = Matrix([[-1,2],[-2,3]])
h0 = g0.inverse()
h1 = g1.inverse()
g00 = h1 * h0
h00 = g00.inverse()

# output command in interactive python session:
# https://ask.sagemath.org/question/9837/how-are-list-of-matrices-printed-by-sage/
# the `None` on the last line of the output shouldn't be there
import sys
print('[[\ng0, g1, g00\n],\n[\nh0, h1, h00\n]] = ')
print(sys.displayhook([[g0, g1, g00], [h0, h1, h00]]))
```

### A.2.1 The Elliptic Curve $E_3$

We call the built-in generators of $\Gamma(6)$, factorize the generators, and check that we have the correct factorization.

```
# SageMath built-in generators of Gamma(6)
G6 = FareySymbol(Gamma(6)).generators()
G6
```

```
# factorize each of the generators in G(6)
F6 = [Matrix([[1,0],[0,1]])] * len(G6)

F6[0] = g00^3
F6[1] = -g00^3 * g0^3
F6[2] = -g00 * h1 * h00 * g0
F6[3] = -g00 * h1 * g00 * h1
F6[4] = h1^3
F6[5] = -g00^2 * g1 * h00 * g1
F6[6] = -g00^2 * h0 * g00 * g1
F6[7] = -g00 * h0^2 * g1
F6[8] = -g00^2 * h1 * h00 * g0 * h00
F6[9] = -g00^2 * h1 * g00 * h1 * h00
F6[10] = g00 * h1^3 * h00
```

```
F6[11] = -g00^2 * h0^3 * h00^2
F6[12] = g00^2 * h1^3 * h00^2

F6
```

```
# check entries in F6 are indeed factorizations of elements in G6
b = true

for i in range(len(G6)):
    if (F6[i] != G6[i]):
        b = false
        print('----------')
        print(f'[\nF6[{i}], G6[{i}]\n] = ')
        print(sys.displayhook([F6[i], G6[i]]))

print('==========')
print(f"F6 {'=' if b else '!'}= G6")
```

We now check that entries in F6 are mapped trivially under the monodromy map $M_{\beta_3}$ (see (3.9)) associated to the Belyi pair $E_3 : y^2 = x^3 + D^2$ and $\beta_3(x, y) = \frac{y+D}{2D}$.

```
# data manipulation: convert the expressions in F6 to strings
F6_str = """
F6[0] = g00^3
F6[1] = -g00^3 * g0^3
F6[2] = -g00 * h1 * h00 * g0
F6[3] = -g00 * h1 * g00 * h1
F6[4] = h1^3
F6[5] = -g00^2 * g1 * h00 * g1
F6[6] = -g00^2 * h0 * g00 * g1
F6[7] = -g00 * h0^2 * g1
F6[8] = -g00^2 * h1 * h00 * g0 * h00
F6[9] = -g00^2 * h1 * g00 * h1 * h00
F6[10] = g00 * h1^3 * h00
F6[11] = -g00^2 * h0^3 * h00^2
F6[12] = g00^2 * h1^3 * h00^2
"""
FAC6 = [line.split('=')[1].strip() for line in F6_str.split('\n') if line.strip()]
```

```
# check entries in FAC6 are contained in Gamma_{E_3}
# by checking the images of FAC6 under M_{\beta_3} are trivial in S_3
b = true
exp = 0
for i in range(len(FAC6)):
```

```
    fac = FAC6[i]
    print('-------------------------------------------------------------------------------')

    # get rid of minus signs in front
    if (fac[0] == '-'):
        fac = fac[1:]

    # sum the effective exponents until running out of '*'
    ind_star = 0
    while (ind_star != -1):
        ind_star = fac.find(' * ')
        term = fac if ind_star == -1 else fac[:ind_star]

        # *** main logic
        ind_power = term.find('^')
        pow = 1 if ind_power == -1 else int(term[-1])
        exp += pow if term[0] == 'g' else -pow
        # ***

        fac = fac[ind_star + 3:]

    print(f'F6[{i}] = {FAC6[i]} has effective exponent {exp}')
    if (exp % 3 != 0):
        b = false
        perm = '(123)' if exp % 3 == 1 else '(132)'
        print(f'so its image under phi_{{E_1}} is {perm} in S_3')
    exp = 0


print('===============================================================================')
print(f"Gamma(6) is {'' if b else 'NOT '}contained in Gamma_{{E_3}}")
```

### A.2.2 The Elliptic Curve $E_4$

We call the built-in generators of $\Gamma(8)$, factorize the generators, and check that we have the correct factorization.

```
# SageMath built-in generators of Gamma(8)
G8 = FareySymbol(Gamma(8)).generators()
G8
```

```
# factorize each of the generators in G(8)
F8 = [Matrix([[1,0],[0,1]])] * len(G8)

F8[0] = g00^4
F8[1] = g00^3 * h1 * g0^3
F8[2] = g00^2 * h1 * g0 * h1 * g0
```

```
F8[3] = h1 * g00 * g1^2 * g0
F8[4] = g00 * h1 * h00 * h0 * g1 * g0
F8[5] = g00^2 * g1 * h00 * g0
F8[6] = g00^2 * h0 * g00 * h1
F8[7] = h1 * h00 * g1 * h0 * h1
F8[8] = g00 * h1 * g00 * h0^2 * h1
F8[9] = g00 * h1^2 * g0 * h1
F8[10] = h1^4
F8[11] = g00^3 * g1 * h0 * g1^2
F8[12] = g00 * h0 * h1 * g0 * h00 * g1
F8[13] = g00^2 * h1 * h00^2 * g1
F8[14] = g00^2 * h1 * g00^2 * g1
F8[15] = g00 * g1 * g0^2 * g00 * g1
F8[16] = g00^2 * h0 * g1 * h0 * g1
F8[17] = g00 * h0^3 * g1
F8[18] = g00^3 * h1 * g0 * h1 * g0 * h00
F8[19] = g00 * h1 * g00 * g1^2 * g0 * h00
F8[20] = g00^3 * g1 * h00 * g0 * h00
F8[21] = g00^3 * h0 * g00 * h1 * h00
F8[22] = g00 * h1 * h00 * g1^2 * g00 * h1 * h00
F8[23] = g00^2 * h1^2 * g0 * h1 * h00
F8[24] = g00 * h1^4 * h00
F8[25] = g00^3 * h1 * h00^2 * g1 * h00
F8[26] = g00^3 * h1 * g00^2 * g1 * h00
F8[27] = g00^3 * h0 * g1 * h0 * g1 * h00
F8[28] = g00^2 * h0^3 * g1 * h00
F8[29] = g00^3 * h1^2 * g0 * h1 * h00^2
F8[30] = g00^2 * h1^4 * h00^2
F8[31] = g00^3 * h0^4 * h00^3
F8[32] = g00^3 * h1^4 * h00^3

F8
```

```
# check the entries in F8 are indeed factorizations of elements in G8
b = true

for i in range(len(G8)):
    if (F8[i] != G8[i]):
        b = false
        print('----------')
        print(f'[\nF8[{i}], G8[{i}]\n] = ')
        print(sys.displayhook([F8[i], G8[i]]))

print('==========')
print(f"F8 {'=' if b else '!'}= G8")
```

We now check that entries in F8 are mapped trivially under the monodromy map $M_{\beta_4}$ (see (3.9)) associated to the Belyi pair $E_4 : y^2 = x^3 + Ax$ and $\beta_4(x, y) = \frac{-x^2}{A}$.

```
# data manipulation: convert the expressions in F8 to strings
F8_str = """
F8[0] = g00^4
F8[1] = g00^3 * h1 * g0^3
F8[2] = g00^2 * h1 * g0 * h1 * g0
F8[3] = h1 * g00 * g1^2 * g0
F8[4] = g00 * h1 * h00 * h0 * g1 * g0
F8[5] = g00^2 * g1 * h00 * g0
F8[6] = g00^2 * h0 * g00 * h1
F8[7] = h1 * h00 * g1 * h0 * h1
F8[8] = g00 * h1 * g00 * h0^2 * h1
F8[9] = g00 * h1^2 * g0 * h1
F8[10] = h1^4
F8[11] = g00^3 * g1 * h0 * g1^2
F8[12] = g00 * h0 * h1 * g0 * h00 * g1
F8[13] = g00^2 * h1 * h00^2 * g1
F8[14] = g00^2 * h1 * g00^2 * g1
F8[15] = g00 * g1 * g0^2 * g00 * g1
F8[16] = g00^2 * h0 * g1 * h0 * g1
F8[17] = g00 * h0^3 * g1
F8[18] = g00^3 * h1 * g0 * h1 * g0 * h00
F8[19] = g00 * h1 * g00 * g1^2 * g0 * h00
F8[20] = g00^3 * g1 * h00 * g0 * h00
F8[21] = g00^3 * h0 * g00 * h1 * h00
F8[22] = g00 * h1 * h00 * g1^2 * g00 * h1 * h00
F8[23] = g00^2 * h1^2 * g0 * h1 * h00
F8[24] = g00 * h1^4 * h00
F8[25] = g00^3 * h1 * h00^2 * g1 * h00
F8[26] = g00^3 * h1 * g00^2 * g1 * h00
F8[27] = g00^3 * h0 * g1 * h0 * g1 * h00
F8[28] = g00^2 * h0^3 * g1 * h00
F8[29] = g00^3 * h1^2 * g0 * h1 * h00^2
F8[30] = g00^2 * h1^4 * h00^2
F8[31] = g00^3 * h0^4 * h00^3
F8[32] = g00^3 * h1^4 * h00^3
"""
FAC8 = [line.split('=')[1].strip() for line in F8_str.split('\n') if line.strip()]
```

```
# check entries in FAC8 are contained in Gamma_{E_4}
# by checking the images of FAC8 under M_{\beta_4} are trivial in S_4
b = true
exp = 0
for i in range(len(FAC8)):
    fac = FAC8[i]
    print('------------------------------------------------------------')

    # get rid of minus signs in front
    if (fac[0] == '-'):
        fac = fac[1:]
```

```
    # sum the effective exponents until running out of '*'
    ind_star = 0
    while (ind_star != -1):
        ind_star = fac.find(' * ')
        term = fac if ind_star == -1 else fac[:ind_star]

        # *** main logic
        ind_power = term.find('^')
        pow = 1 if ind_power == -1 else int(term[-1])

        cusp = term[1:] if ind_power == -1 else term[1:-2]
        if (cusp == '1'):
            pow *= 2

        exp += pow if term[0] == 'g' else -pow
        # ***

        fac = fac[ind_star + 3:]

    print(f'F8[{i}] = {FAC8[i]} has effective exponent {exp}')
    if (exp % 4 != 0):
        b = false
        perm = '(1234)' if exp % 4 == 1 else '(13)(24)' if exp % 4 == 2 else '(1432)'
        print(f'so its image under phi_{{E_3}} is {perm} in S_4')
    exp = 0


print('============================================================')
print(f"Gamma(8) is {'' if b else 'NOT '}contained in Gamma_{{E_4}}")
```

### A.2.3 The Elliptic Curve $E_6$

We call the built-in generators of $\Gamma(12)$, factorize the generators, and check that we have the correct factorization.

```
# SageMath built-in generators of Gamma(12)
G12 = FareySymbol(Gamma(12)).generators()
G12
```

```
# factorize each of the generators in G(12)
F12 = [matrix.identity(2)] * len(G12)
FAC12 = [''] * len(G12)

for i in range(len(G12)):
    mat = G12[i]

    while (mat != matrix.identity(2)):
```

```
        a = mat[0][0]
        b = mat[0][1]

        if (a * b > 0):
            if (abs(a) >= abs(b)):
                mat *= g0
                F12[i] = h0 * F12[i]
                FAC12[i] = ' * h0' + FAC12[i]
            else:
                mat *= h00
                F12[i] = g00 * F12[i]
                FAC12[i] = ' * g00' + FAC12[i]
        else:
            if (abs(a) >= abs(b)):
                mat *= h0
                F12[i] = g0 * F12[i]
                FAC12[i] = ' * g0' + FAC12[i]
            else:
                mat *= g00
                F12[i] = h00 * F12[i]
                FAC12[i] = ' * h00' + FAC12[i]

    FAC12[i] = FAC12[i][3:]

F12
```

```
# check the entries in F12 are indeed factorizations of elements in G12
b = true

for i in range(len(G12)):
    if (F12[i] != G12[i]):
        b = false
        print('----------')
        print(f'[\nF12[{i}], G12[{i}]\n] = ')
        print(sys.displayhook([F12[i], G12[i]]))

print('==========')
print(f"F12 {'=' if b else '!'}= G12")
```

We now check that entries in FAC12 are mapped trivially under the monodromy map $M_{\beta_6}$ (see (3.9)) associated to the Belyi pair $E_6 : y^2 = x^3 + B$ and $\beta_6(x, y) = \frac{-x^3}{B}$.

```
# check entries in FAC12 are contained in Gamma_{E_6}
# by checking the images of FAC12 under phi_{E_2} are trivial in S_6
b = true
exp = 0
```

```
for i in range(len(FAC12)):
    fac = FAC12[i]
    print('---------------------------------------------------------')
    print('---------------------------------------------------------')

    # get rid of minus signs in front
    if (fac[0] == '-'):
        fac = fac[1:]

    # sum the effective exponents until running out of '*'
    ind = 0
    while (ind != -1):
        ind = fac.find(' * ')
        term = fac if ind == -1 else fac[:ind]

        # *** main logic
        cusp = term[1:]
        pow = 2 if (cusp == '0') else 1

        exp += pow if term[0] == 'g' else -pow
        # ***

        fac = fac[ind + 3:]

    print(f'F12[{i}] = {FAC12[i]} has effective exponent {exp}')
    if (exp % 6 != 0):
        b = false
        perm = '(135246)' if exp % 6 == 1 else '(154)(263)' if exp % 6 == 2 else '(12)(34)(56)' if exp % 6 == 3 else '(145)(236)' if exp % 6 == 4 else '(164253)'
        print(f'so its image under phi_{{E_2}} is {perm} in S_6')
    exp = 0

print('=============================================================')
print('=============================================================')
print(f"Gamma(12) is {'' if b else 'NOT '}contained in Gamma_{{E_6}}")
```

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