BROKEN TORIC VARIETIES AND HYPERTORIC HITCHIN SYSTEMS

by

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ABSTRACT

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In the first part of this thesis we study the cohomology of broken toric varieties via the derived push-forward of the constant sheaf to a complex of polytopes, thus proving a Deligne-type decomposition theorem, degeneration of the associated Leray-Serre spectral sequence, and showing that the Leray filtration on their cohomology is equal to the weight filtration (up to re-indexing). We discuss certain maps between broken toric varieties and how they descend to maps of cohomology groups. Furthermore, we give a description of the Betti numbers of some broken toric varieties whose associated complex of polytopes is the n-skeleton of a higher dimensional polytope, encompassing some important examples.

In the second part we investigate hypertoric Hitchin systems, whose cohomology is governed by a subvariety which is broken toric. After reviewing their construction, we give a new proof of a deletion-contraction relationship on these varieties (first proven by Dansco-McBreen-Shende) and refine it to a statement about the cohomology of certain sheaves on the polytope complex. Using these facts and the general results in the first part of the thesis, we develop tools for calculating the cohomology of

some families of hypertoric Hitchin systems inductively, given knowledge of a base case. In particular, this yields explicit formulae for the Poincaré polynomials of hypertoric Hitchin systems associated to graphs with first Betti number 2.

To Kelsey,
without whom this would never have been done,

and Ash,
without whom this would have been done much sooner.

"Tiny plastic polygons
I'm building things that aren't real.
Endless streams of hexagons
and other fun materials."

-Sidney Gish, "Hexagons and other fun materials"

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All the acknowledgements here

PUBLICATIONS

Some of the material in Chapters 2 and 3 (roughly Sections 2.2 - 2.4, 2.6, and 3.1-3.2) has been previously published in [Sun23].

CONTENTS

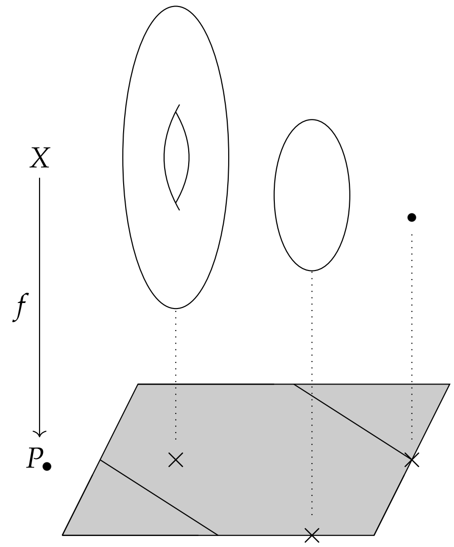
1 INTRODUCTION

1.1 BROKEN TORIC VARIETIES

Broken toric varieties are a class of varieties, essentially combinatorial in nature, which appear in areas such as mirror symmetry, hypertoric geometry, toric topology, and the study of Hitchin systems. To be precise, a *broken toric variety* is a union of (smooth projective) toric varieties glued to each other pairwise along T-invariant toric subvarieties. They are combinatorial in the sense that much of their geometry is captured by a complex of polytopes, which we accordingly call their *polytope complex*, in the same way that the information of a toric variety is studied by looking at its associated polytope or fan.

There is a map from a broken toric variety to its polytope complex which can be viewed alternatively as the quotient map by the compact part of the torus action or as the moment map for that action. By pushing forward the constant sheaf on the broken toric variety along this map, we obtain a complex of sheaves on the polytope complex with nice combinatorial properties (its cohomology objects are *cell-compatible*), which make them amenable to computation.

Example 1.1. Below is pictured a broken toric variety along with its moment map to its complex of polytopes. It is obtained by gluing two copies of $\mathbb{P}^2$ (the toric varieties of the two triangular faces in the arrangement) to a projective toric variety which is $\mathbb{P}^2$ blown up at two points (the toric variety of the hexagon) along copies of $\mathbb{P}^1$. The projective lines along which these varieties are glued to each other correspond to the intersections of the polytopes in the arrangement. Note the important fact that the fibres are tori of dimension equal to the dimension of the face over which they lie.



Broken toric varieties can be thought of as degenerations of toric varieties (a nice overview of this point of view can be found in [Ale15]) when the polytope complex is contractible, and more general examples appear as degenerations of, for example, abelian varieties [Mum72] and of K3 surfaces [FM83, GS10]. After their formal introduction in [Ale02], the structure of the moduli space broken toric varieties is investigated in [AB06] and [AM16]. The main component of the moduli space is compactified in [Ols12] and tropicalized in [MW].

As indicated above, broken toric varieties make appearances in many areas. Fine compactified Jacobians of nodal curves with all rational components are broken toric, as studied in [OS79]. This immediately points to a relationship with the moduli space of Higgs bundles or Hitchin system (see for example [Hit87, Sim92, Ray18, Swo21] and the many references within), of which the fibres above nodal spectral curves are fine compactified Jacobians [BNR89].

In a different direction, broken toric varieties whose polytope complex is a polytopal decomposition of an n-sphere are of interest to the Gross-Siebert program [GW00, GS06, GS10, KX16] in mirror symmetry. Roughly speaking, one wants to study mirror symmetry for families of Calabi-Yau varieties $\mathcal{X} \to S$ by looking at certain maximally singular fibres, so-called large complex structure limits. These are, in general, broken toric varieties (with additional data) and it is conjectured that their polytope complexes are polytopal decompositions of an n-sphere. This is trivially true for elliptic curves and proven for K3 surfaces in [GW00].

One can find broken toric varieties when studying complexity k T-varieties [AIP+12] where the fibres of the natural quotient map, after resolving indeterminacies, are broken toric [Ilt]. They also appear in the field of toric topology [BR08, BP15, BBC20]. For example, the characteristic function of a polytope as defined in [DJ91] describes the stalks of our cell-compatible sheaves (see the proof of Proposition 2.13). In this area one also studies polyhedral products [BBCG09], topological spaces determined by a simplicial complex and a family of pairs of pointed topological spaces. Polyhedral products are a generalization of the moment angle complexes of [DJ91, BP00], and their specializations find applications in many areas (see in particular the table in Chapter 1 of [BBC20]). It turns out that broken toric varieties also fit into this formalism, which we explain in Section 2.6.

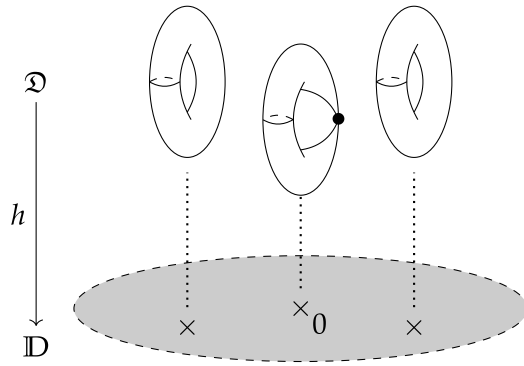
1.2 HYPERTORIC HITCHIN SYSTEMS

In an unpublished note from 2015, Hausel and Proudfoot described a special kind of hypertoric variety associated to a periodic hyperplane arrangement in $\mathbb{R}^n$, which had three incarnations suggestively[1] named the Dolbeault $\mathfrak{D}$, Betti $\mathfrak{B}$, and de Rham $\mathfrak{dR}$ hypertoric varieties. Together they are known as *multiplicative hypertoric varieties* or (due to subsequent authors) *Hausel-Proudfoot varieties*. The motivation for studying such objects comes from several sources. First, the multiplicative hypertoric varieties really do arise naturally as multiplicative versions of the (additive) hypertoric varieties of Bielawski–Dancer [BD00] and Konno [Kon00] (see Section 4.2). Second, they can be thought of as a generalization of the multiplicative quiver constructions of Crawley-Boevey–Shaw [CBS06], objects which are themselves currently attracting some attention, e.g. [Jor14, Fai19].

Our focus is on the Dolbeault multiplicative hypertoric varieties (or *hypertoric Hitchin systems*). They come equipped with a proper morphism to an affine space and indeed with the structure of an integrable system. The generic fibres of this morphism are abelian varieties and there is a deformation retract to the central fibre. One constructs the hypertoric Hitchin systems by starting with the simplest possible example $\mathfrak{D}($○$)$ and then obtaining others as quasi-hyperkähler reductions of products of $\mathfrak{D}($○$)$, controlled by the further ingredient of a connected graph Γ.

Example 1.2. Here is a picture of the Tate curve $\mathfrak{D}($○$)$, the simplest hypertoric Hitchin system.

1 This is in reference to the celebrated non-Abelian Hodge correspondence of Simpson, Hitchin, Donaldson, Corlette, and others which concerns three such named moduli spaces associated to a given Riemann surface. See for example [GRR15] for an overview.



A few uses for this new construction are as follows: the Dolbeault multiplicative hypertoric varieties were constructed to model (at least in terms of cohomology) neighbourhoods of the simplest singular fibres of the Hitchin system, namely those whose spectral curve has only nodal singularities (we detail this relationship in Subsection 4.1.1). Such an approach has the possibility of shedding light on other open problems in the study of the Hitchin system such as the purity conjecture and Hodge-Tate type of the cohomology, as well as on the recently proven Curious Hard Lefschetz [Mel19] and P=W [MS24] conjectures. Closely related is the question of whether the three types of multiplicative hypertoric varieties fit into a version of the non-Abelian Hodge correspondence as suggested by their names.

These threads and others have been picked up by other authors in the following years. Ganev [Gan18] studies quantizations of the hypertoric Betti spaces in terms of certain Azumaya algebras over them. In McBreen–Webster [MW18] and Gammage–McBreen–Webster [GMW19], homological mirror symmetry for multiplicative hypertoric varieties is explored. The spaces $\mathfrak{B}$ and $\mathfrak{D}$ are proven to be (non-canonically) diffeomorphic by Dansco–McBreen–Shende [DMS19], and what's more, they prove the hypertoric version of the P=W conjecture. Groechenig–McBreen [GM20] introduce a formal algebraic analogue of the hypertoric Dolbeault space, of which they compute the Tamagawa numbers and p-adic volumes of the fibres in terms of the Kirchhoff polynomial of certain graphs. In McBreen–Seshmani–Yau [MSY20], multiplicative hypertoric varieties make an appearance when computing the twisted Donaldson-Thomas invariants of a related space, and symplectic duality is discussed.

The relationship between the two broad parts of this thesis can be summarized easily: the central fibre of a hypertoric Hitchin system, which determines the cohomology of the whole space, is a broken toric variety (see Section 4.2.4). This ought not to be too surprising since the construction of a hypertoric Hitchin system is inherently toric. Indeed to construct them one starts with $\mathfrak{D}$(◓), which we construct as a quotient of a space which fibres generically as tori (see Section (4.2.1)), and a short exact sequence of tori induced by a graph Γ.

This can be seen in action in Example (1.2) above. The generic fibres of hypertoric Hitchin map are abelian varieties and the central fibre is a nodal elliptic curve— a broken toric variety whose single component $\mathbb{P}^1$ is glued to itself at a point.

1.3 OVERVIEW

In Chapter 2 we study generic broken toric varieties, starting in Section 2.2 where we state the relevant definitions, give some examples, and prove that the higher derived pushforwards of the constant sheaf on a broken toric variety along its quotient map f to its polytope complex $P_\bullet$ are all cell-compatible subsheaves of a constant sheaf. Our main result appears in Section 2.3, which is that the derived pushforward $Rf_*\underline{\mathbb{Q}}_X$ is a formal complex and so the Leray spectral sequence degenerates.

Theorem 2.21. For X an n-dimensional broken toric variety and f the map from X to its polytope complex $P_\bullet$, there is an isomorphism in $\mathcal{D}^b_c(P_\bullet)$

$$Rf_*\underline{\mathbb{Q}}_X \cong \bigoplus_{i=0}^{2n} R^i f_*\underline{\mathbb{Q}}_X[-i].$$

This implies the degeneration at the E_2 page of the Leray spectral sequence $E_2^{pq} = H^p(P_\bullet, R^q f_*\underline{\mathbb{Q}}_X)$ associated to f.

We note here that this reduces the calculation of the dimension of $H^k(X, \underline{\mathbb{Q}}_X)$ to the calculation of the dimensions of the cohomology groups of the sheaves $R^i f_*\underline{\mathbb{Q}}_X$ on $P_\bullet$. In Section 2.4 we show that the Leray filtration associated to f on the cohomology of a broken toric variety is equal to twice the weight filtration.

Theorem 2.25. For X a broken toric variety,

$$W_{2k}H^i(X, \underline{\mathbb{Q}}_X) = W_{2k+1}H^i(X, \underline{\mathbb{Q}}_X) = L_k H^i(X, \underline{\mathbb{Q}}_X).$$

Section 2.5 is about *balloon animal maps*, which are a kind of quotient map between broken toric varieties whose polytope complexes are related via polytopal subdivisions (see Definition (2.28)). Using such maps we can relate the cohomology groups of certain different broken toric varieties.

Theorem 2.31. Let $b : X \to X^\cup$ be a balloon animal map between broken toric varieties induced by a generic subdivision of the polytope complex of X and $X^\cap$ be the broken toric subvariety of $X^\cup$ associated to $P_\bullet^\cap$. Then there is a long exact sequence

$$\cdots \to H^k(X^\cup, \underline{\mathbb{Q}}_{X^\cup}) \to H^k(X, \underline{\mathbb{Q}}_X) \to H^{k-1}(X^\cap, \underline{\mathbb{Q}}_{X^\cap}) \to \ldots.$$

Next, in Section 2.6 we recall the definition of polyhedral products and show that every broken toric variety can be constructed as a quotient of a polyhedral product.

Chapter 3 is devoted to explicit calculations and finding closed form expressions for the dimensions of cohomology groups of certain types of broken toric varieties. Section 3.1 studies cell-compatible sheaves on this higher dimensional polytope, culminating in a new way to derive the Betti numbers of a smooth projective toric variety (Corollary 3.3). In Section 3.2, we explain how this relates to the Betti numbers of broken toric varieties with skeletal polytope complexes. This includes all polytope complexes which are polytopal decompositions of an n-sphere. Section 3.3 relates cell-compatible sheaves to *cellular sheaves* and discusses a grounded way to calculate their cohomology groups.

In Chapter 4, we study hypertoric Hitchin systems. Section 4.1 contains preliminaries and Section 4.2 explains their construction, states some properties, and gives examples. In Section 4.3, we use the techniques of Section 2.5 to give a new proof of the deletion-contraction relationship of [DMS19] and extend it to a finer result about the higher derived pushforwards:

Theorem 4.12. Let Γ be a graph and e be an edge of Γ which is not a loop. Then there is a long exact sequence

$$\cdots \longrightarrow H^{k-1}(T^{n-1}, R^{i-1}f_*\underline{\mathbb{Q}}_{\Gamma\setminus e}) \longrightarrow H^k(T^n, R^if_*\underline{\mathbb{Q}}_\Gamma) \longrightarrow H^k(T^n, R^if_*\underline{\mathbb{Q}}_{\Gamma/e}) \longrightarrow H^k(T^{n-1}, R^{i-1}f_*\underline{\mathbb{Q}}_{\Gamma\setminus e}) \longrightarrow H^{k+1}(T^n, R^if_*\underline{\mathbb{Q}}_\Gamma) \longrightarrow \cdots$$

Finally, Section 4.4 consists of results concerning the calculation of the Betti numbers of hypertoric Hitchin systems, making use of the tools we have developed along the way.

Theorem 4.14. Let v be a vertex of a connected graph Γ such that $\Gamma \setminus v$ is disconnected with components Γ_i', $i = 1, \ldots, m$. Let $\Gamma_i = \Gamma \setminus \bigsqcup_{j=1, j\neq i}^m \Gamma_i'$ so that $\Gamma = \Gamma_1 \cup_v \Gamma_2 \cup_v \cdots \cup_v \Gamma_m$. Then

$$h_\Gamma^{-1}(0) \cong \prod_{i=1}^m h_{\Gamma_i}^{-1}(0)$$

Theorem 4.18. Let Γ be a graph with $b_1(\Gamma) = n \geq 2$, no bridges or tails, and e an edge of Γ which is incident to a vertex of degree 2. Then

$$H^k(T^n, R^i f_* \underline{\mathbb{Q}}_{\mathfrak{D}(\Gamma)}) \cong H^k(T^n, R^i f_* \underline{\mathbb{Q}}_{\mathfrak{D}(\Gamma/e)}) \oplus H^{k-1}(T^n, R^{i-1} f_* \underline{\mathbb{Q}}_{\mathfrak{D}(\Gamma \backslash e)}).$$

The previous two statements together allow one to calculate the dimensions of the cohomology groups of infinite families of hypertoric Hitchin systems, given knowledge of some base case. This strategy is employed to give a closed form expression for the Betti numbers of hypertoric Hitchin systems associated to graphs of first Betti number 2.

Theorem 4.21. Let Γ be a graph with $b_1(\Gamma) = 2$.

1. If Γ has a disconnecting vertex, then the Poincaré polynomial of $\mathfrak{D}(\Gamma)$ is
$$1 + 2y + (|E(\Gamma)| + 1)y^2 + |E(\Gamma)|y^3 + t(\Gamma)y^4,$$
2. If Γ does not have a disconnecting vertex, then the Poincaré polynomial of $\mathfrak{D}(\Gamma)$ is
$$1 + 2y + |E(\Gamma)|y^2 + (|E(\Gamma)| - 1)y^3 + t(\Gamma)y^4,$$

where $t(\Gamma)$ is the number of spanning trees of Γ.

2

BROKEN TORIC VARIETIES - THEORY

2.1 PRELIMINARIES ON TORIC VARIETIES

We begin with a review of some of the salient features of toric varieties. For more detail, the reader can consult one of the standard references such as [CLS11]. Everything here, and indeed in the whole thesis, takes place over $\mathbb{C}$.

Definition 2.1. An algebraic variety X is *toric* if it contains an algebraic torus $T = (\mathbb{C}^\times)^n$ as a dense open subset such that the action of $(\mathbb{C}^\times)^n$ on itself extends to an action on all of X.

Examples of toric varieties include $(\mathbb{C}^*)^n$, $\mathbb{C}^n$, and less trivially the vanishing locus of $x^3 - y^2$ in $\mathbb{C}^2$, with torus $\{(t^2, t^3) : t \in \mathbb{C}^*\} \cong \mathbb{C}^*$.

In this thesis we focus exclusively on *projective* toric varieties, some examples of which are $\mathbb{P}^n$, $\mathbb{P}^n \times \mathbb{P}^m$, and the blow-up of $\mathbb{P}^2$ at any finite number of points.

Toric varieties are often defined by their associated *fans*, finite collections of polyhedral cones (satisfying certain conditions) in a vector space $N_\mathbb{R} = N \otimes \mathbb{R}$, for N the lattice of one-parameter subgroups of T, $N = \mathrm{Hom}(\mathbb{C}^*, T)$. It is from this combinatorial nature that many of the nice properties of toric varieties descend. A dual point of view to this is to define a (projective) toric variety associated to a polytope in $M_\mathbb{R} = M \otimes \mathbb{R}$, for M the lattice of characters of T, $M = \mathrm{Hom}(T, \mathbb{C}^*)$. This is the vantage most useful for our purposes.

Recall that a polytope in $\mathbb{R}^n$ is the intersection of a finite number of half-planes. We will assume throughout that all of our polytopes are *lattice polytopes*, meaning that their vertices lie on the lattice M. This is not much of a restriction since we mostly care about the topological properties of toric varieties, which do not change under scaling of the underlying polytope. The k-skeleton of a polytope P, denoted by $\mathrm{Sk}_k(P)$, is the union of all polytopes of dimension up to k in P. The *open k-skeleton* of P is $\mathring{\mathrm{Sk}}_k(P) = \mathrm{Sk}_k(P) \setminus \mathrm{Sk}_{k-1}(P)$.

Since this relationship is used extensively in this thesis, let us recall the construction of a projective toric variety from a lattice polytope. Let P be a very ample[1], full-dimensional polytope relative to the n-dimensional lattice M. Enumerate the points of $P \cap M$ as $\{p_1, \ldots, p_s\}$ and recall that a point $m = (m_1, \ldots, m_n)$ in $M \cong \mathbb{Z}^n$ defines a character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ given by $\chi^m(z_1, \ldots, z_n) = z_1^{m_1} \ldots z_n^{m_n}$. So we can use $\{p_1, \ldots, p_s\}$ to define a map $(\mathbb{C}^*)^n \to \mathbb{P}^{s-1}$ by

$$t \mapsto (\chi^{p_1}(t) : \ldots : \chi^{p_n}(t)).$$

The projective toric variety of P is then the Zariski closure of the image of this map.

We can also go in the other direction. Any projective toric variety has a natural symplectic structure given by pulling back the Fubini-Study form from the ambient projective space, and the compact torus $(S^1)^n$ in T provides an effective Hamiltonian group action. A symplectic variety with such an action comes equipped with a symplectic moment map $\mu : X \to M_{\mathbb{R}}$, the image of which is a polytope in $M_{\mathbb{R}}$ by [GS82]. In the smooth case at least, this polytope is the same one defining X. Moreover, this polytope is also the quotient of X by this $(S^1)^n$ action, and the fibre over a point x in the k-skeleton of the polytope is $(S^1)^k$.

Remark 2.2. A smooth projective toric variety has as its image under μ one of the *Delzant polytopes* [Del88]. These are simple polytopes whose edges at each vertex form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.

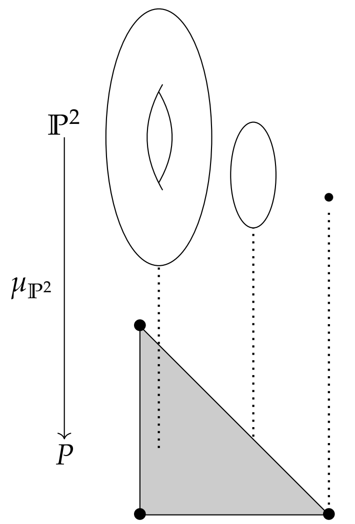
Example 2.3. Let $X = \mathbb{P}^2$, the toric variety of the 2-simplex P in $\mathbb{R}^2$. The moment map $\mu_{\mathbb{P}^2} : \mathbb{P}^2 \to \mathbb{R}^2$ can be written as

$$\mu_{\mathbb{P}^2}((z_0 : z_1 : z_2)) = \left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right),$$

the image of which is the convex hull of the standard basis vectors e_1 and e_2 in $\mathbb{R}^2$; in other words, a 2-simplex.

We can also describe the fibres of $\mu_{\mathbb{P}^2}$. For $x \in \mathring{\mathrm{Sk}}_2(P)$ (the interior of the triangle), the fibre is a real 2-torus. For $x \in \mathring{\mathrm{Sk}}_1(P)$ it is S^1, and for $x \in \mathrm{Sk}_0(P)$, it is simply a point.

1 A lattice polytope $P \subseteq M_{\mathbb{R}}$ is *very ample* if for every vertex $m \in P$, the semigroup S generated by the set $\{m' - m : m' \cap P\}$ has the property that $km \in S$ implies $m \in S$, for all $k \in \mathbb{N} \setminus \{0\}$, $m \in M$. Intuitively, this means that P does not have too few lattice points. Even if P is not very ample, some scaling of it is.



2.2 DEFINITIONS AND CELL-COMPATIBLE SHEAVES

Definition 2.4. A *broken toric variety* is a union of equidimensional smooth projective toric varieties glued to each other pairwise along T-invariant closed toric subvarieties. The *polytope complex* $P_\bullet(X)$ of a broken toric variety X is the union of the polytopes of the components of X glued to each other so that if two components of X meet along a toric variety X', then the polytopes of those components intersect as the polytope of X'.

Remark 2.5. It is perfectly reasonable to define broken toric varieties with components of differing dimensions, or components which are neither smooth nor projective, but we do not investigate this more general definition here.

There is a natural map $f : X \to P_\bullet(X)$ given by gluing together the moment maps of the components of X to their polytopes. In other words, it is the quotient map of X by the T^n action on each of the components.

Remark 2.6. To any complex of polytopes $P_\bullet$ there is not only one, but rather a family, of broken toric varieties which have $P_\bullet$ as their associated polytope complex. Given $P_\bullet$, a broken toric variety X with components X_i is fixed by choosing gluing isomorphisms α_{ij} from a toric subvariety in X_i to an isomorphic subvariety in X_j (satisfying the cocycle condition $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$). This is equivalent to the choice of an element of $H^1(P_\bullet, T)$, that is, to the choice of a T^n-torsor on $P_\bullet$. We further note that there is not necessarily a bijection of $H^1(P_\bullet, T)$ with the space of isomorphism classes of broken toric varieties over $P_\bullet$—that is to say, different gluing isomorphisms may lead to isomorphic broken toric varieties. For example, if $P_\bullet$ is the graph in Example 2.14, $H^1(P_\bullet, T)$ is one-dimensional but all broken toric varieties over it are isomorphic.

Lemma 2.7. *Any n-dimensional broken toric variety X is a quotient of the total space of a T^n-torsor on $P_\bullet$.*

Proof. By Lemma 1.4 of [DJ91], any smooth projective toric variety X with polytope P can be written as a quotient of $P \times T^n$. The proof in fact applies verbatim to broken toric varieties whose polytope complexes $P_\bullet$ are contractible. The idea is to inductively blow up the singular strata of X and obtain a T^n-manifold $\hat{X}$ whose orbit space is identifed with $P_\bullet$, as well as a natural "collapse" map $\hat{X} \to X$. Since $P_\bullet$ contractible, $\hat{X}$ is diffeomorphic to $P_\bullet \times T^n$.

For the general case, let $\mathcal{T}$ be the total space of T^n-torsor defining how the toric components of X are glued together. Locally around any $n-1$ cell of $P_\bullet$, $\mathcal{T}$ is the trivial torsor, so we can define a map from this local neighbourhood to a neighbourhood of $P_\bullet$ by using the above construction. The result is a quotient map from $\mathcal{T}$ to a broken toric variety over $P\bullet$ whose components are glued as prescribed by the T^n-torsor defining X over $P_\bullet$, which is necessarily isomorphic to X. □

Remark 2.8. The proof of Lemma 1.4 of [DJ91] depends on the existence of a smooth T^n action on the toric variety, and as such does not hold for singular toric varieties. This is the main reason for our insistence that the toric components of a broken toric variety be smooth. Our upcoming decomposition result (Theorem 2.21) depends on Lemma 2.7 and so does not hold for the case of singular components. It is even worse, in fact: an explicit example is given in [McC89] of two singular toric varieties with the same polytope but different rational Betti numbers. It would be interesting to investigate a version of the decomposition theorem for broken toric varieties with singular components using intersection cohomology.

Definition 2.9. Let X be a broken toric variety with polytope complex $P_\bullet$. Given a k-cell α of $P_\bullet$ let H_α denote the k-dimensional linear subspace of $(\mathbb{R}^n)^*$ which contains (a shift of) the moment polytope of the k-dimensional smooth projective toric variety defined by α.

One ought to check that this definition is consistent with the way the polytope complex is glued together. This follows since moment polytopes are well-defined up to translation. Consider the set of n-cells of $P_\bullet$ which contain α. The map f restricted to each of the toric components of X corresponding to those cells is a moment map and so we get for each a convex subset of $(\mathbb{R}^n)^*$. By shifting each of these sets to the origin, we see that the image of the toric variety of α under f, as a subset of each, must live in their intersection.

Lemma 2.10. *Let X be a broken toric variety and f be the map from X to its polytope complex $P_\bullet$. The quotient map provided in Lemma 2.7 can be described fibrewise over $x \in \alpha \in \mathring{Sk}_k(P_\bullet)$ as the map which sends $T^n = \mathbb{R}^n/\Gamma$ to $H_\alpha^*/\Gamma \cap H_\alpha^*$ where Γ is the standard lattice in $\mathbb{R}^n$.*

Proof. It is true that when restricted to any toric component of X which contains $f^{-1}(x)$, the fibre of f over x is isomorphic to $H_\alpha^*/\Gamma \cap H_\alpha^*$ where Γ is the standard lattice in $\mathbb{R}^n$. Since the description of H_α is the same for each such component the gluing respects it. So the equivalence relation which describes the quotient map takes $T^n = \mathbb{R}^n/\Gamma$ and identifies all vectors normal to H_α^*. □

Corollary 2.11. *Let X be a broken toric variety, f be the map from X to its polytope complex $P_\bullet$, and $x \in \alpha \in \mathring{Sk}_k(P_\bullet)$. Then $H^i(f^{-1}(x), \underline{Q}) = \bigwedge^i H_\alpha$.*

Definition 2.12. Let X be a topological space with a CW complex structure $\mathcal{E} = \{\mathrm{Sk}_k\}_{k=0}^{\dim X}$ and denote by $\mathring{\mathrm{Sk}}_k$ the complement of Sk_{k-1} in Sk_k. A constructible sheaf $\mathcal{F}$ on X is *cell-compatible* with respect to $\mathcal{E}$ if for all $\alpha \in \mathring{\mathrm{Sk}}_k$, $\mathcal{F}|_\alpha$ is constant on α and if for all $x \in \alpha \in \mathring{\mathrm{Sk}}_k$, the stalk $\mathcal{F}_x$ lies in the intersection $\bigcap_{y \in \beta \in \mathring{\mathrm{Sk}}_{k+1}} \mathcal{F}_y$, where the intersection runs over all $\beta \in \mathring{\mathrm{Sk}}_{k+1}$ for which $\alpha \in \partial\beta$.

For the technically minded reader, the intersection in this definition can be understood by imposing that locally near x, $\mathcal{F}$ is a subsheaf of a locally constant sheaf. For another point of view, see Section 3.3

The following proposition shows how our two previous definitions are related; indeed, our main examples of cell-compatible sheaves come from broken toric varieties.

Proposition 2.13. *Let X be an n-dimensional broken toric variety and $\mathcal{T} \xrightarrow{\pi} P_\bullet$ be the the total space of the T^n-torsor on $P_\bullet$ describing the gluing data of its toric components. The higher derived pushforwards*[2] *$R^i f_* \underline{\mathbb{Q}}_X$ of $\underline{\mathbb{Q}}_X$ along f are subsheaves of $R^i \pi_* \mathcal{T}$ which are cell-compatible with respect to the CW complex structure on $P_\bullet$.*

Proof. For cell-compatibility, take $x \in \alpha \in \mathring{\mathrm{Sk}}_k(P_\bullet)$ and let $B = \{\beta \in \mathring{\mathrm{Sk}}_{k+1}(P_\bullet) : \alpha \in \partial\beta\}$. Recall that α can be thought of as a k-dimensional convex subset in $(\mathbb{R}^n)^*$ and that we denote by H_α the k-dimensional linear subspace which contains that subset shifted to the origin.

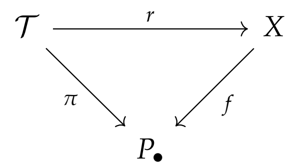
Proper base change (see for example Theorem 2.3.26 of [Dim04]) allows us to describe the stalks of $R^i f_* \underline{\mathbb{Q}}_X$ in terms of the cohomology of the fibres. In particular we have (utilizing Corollary 2.11)

$$(R^i f_* \underline{\mathbb{Q}}_X)_x = H^i(f^{-1}(x), \underline{\mathbb{Q}}_{f^{-1}(x)}) = \bigwedge^i H_\alpha. \tag{2.1}$$

2 Throughout the thesis we make use of a bit of standard machinery regarding derived categories of sheaves. For more background, see a reference such as [KS90] or [Dim04].

This description ensures cell-compatibility of the sheaves $R^i f_* \underline{\mathbb{Q}}_X$, since if $\alpha \subset \bigcap_{\beta \in B} \beta$, then $H_\alpha \subset \bigcap_{\beta \in B} H_\beta$ and thus $\bigcap_{\beta \in B} \bigwedge^i H_\beta$ contains $\bigwedge^i H_\alpha$.

To show that $R^i f_* \underline{\mathbb{Q}}_X$ is a subsheaf of $R^i \pi_* \mathcal{T}$, consider the commutative diagram



where π is the projection onto $P_\bullet$ and r is the quotient map whose existence is assured by Proposition 2.7.

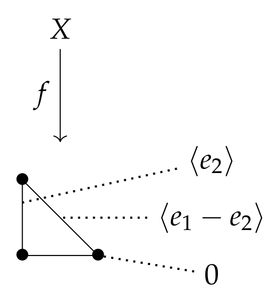
Denote the sheaf complex $Rr_* \underline{\mathbb{Q}}_{\mathcal{T}}$ by $\mathcal{G}^\bullet$. There is a natural map of (complexes of) sheaves on X given by $\alpha : \underline{\mathbb{Q}}_X \to \mathcal{G}^\bullet$, obtained by adjunction from the identity map $r^{-1}\underline{\mathbb{Q}}_X \to \underline{\mathbb{Q}}_{\mathcal{T}}$. This map descends to a map of complexes on $P_\bullet$, $R\alpha : Rf_* \underline{\mathbb{Q}}_X \to Rf_* \mathcal{G}^\bullet$ and further to a map on the cohomology sheaves

$$R^i \alpha : R^i f_* \underline{\mathbb{Q}}_X \to R^i f_* \mathcal{G}^\bullet = R^i \pi_* \mathcal{T},$$

which we will show is injective by checking injectivity on the stalks. For $y \in \mathring{\mathrm{S}}\mathrm{k}_j$, $(R^i \alpha)_y : \bigwedge^i \mathbb{Q}^j \to \bigwedge^i \mathbb{Q}^n$, and we note that $r|_{\pi^{-1}(y)} : T^n \twoheadrightarrow T^j$, which implies that $(R^i \alpha)_y$ is injective. Since this holds for all $0 \le j \le n$, $R^i \alpha$ is injective. □

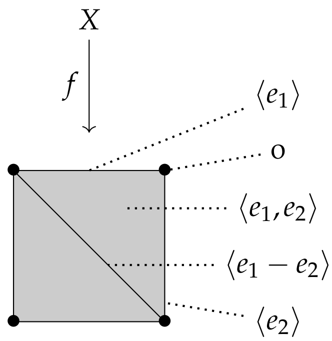
We now have the pleasure of showing some examples.

Example 2.14. First consider the case of a necklace of three copies of the projective line, with $\{0\}$ in one identified with $\{\infty\}$ in the next. The polytope complex of this broken toric variety is a triangulation of S^1 with three 0-cells which we can embed in $\mathbb{R}^2$ with the standard basis $\{e_1, e_2\}$. In the following picture we have labelled some of the stalks of $R^1 f_* \underline{\mathbb{Q}}_X$.



Example 2.15. For a higher-dimensional case, consider two copies of $\mathbb{P}^2$ glued to each other along a copy of $\mathbb{P}^1$. In this case the polytope complex of X is two triangles glued to each other along one edge, which we can

again embed in $\mathbb{R}^2$. As above, we have labelled some of the stalks of $R^1 f_* \underline{\mathbb{Q}}_X$.



For more examples, the reader can look ahead to Subsection 4.2.5 where the varieties $h_\Gamma^{-1}(0)$ which appear are broken toric.

2.3 DECOMPOSITION AND VANISHING RESULTS

In this section we establish some of our key tools, all of which apply to any broken toric variety. This first lemma is not surprising and finds uses throughout the thesis.

Lemma 2.16. *For P a CW complex let us denote by ι_k the inclusion of $\mathring{Sk}_k$ into Sk_k and by κ_k the inclusion of Sk_k into Sk_{k+1}. For $\mathcal{F}$ a cell-compatible sheaf on P and $0 \leq k \leq n$, there is a short exact sequence*

$$0 \to \iota_{k!}\iota_k{}^{-1}\mathcal{F}|_{Sk_k} \to \mathcal{F}|_{Sk_k} \to (\kappa_{k-1})_*(\kappa_{k-1})^{-1}\mathcal{F}|_{Sk_k} \to 0. \tag{2.2}$$

Moreover, when $\mathcal{F} = R^i f_* \underline{\mathbb{Q}}_X$ for a broken toric variety X, one finds $\iota_{k!}\iota_k{}^{-1}\mathcal{F}|_{Sk_k} \cong \iota_{k!} \bigwedge^i \underline{\mathbb{Q}}^k_{\mathring{Sk}_k(P_\bullet)}$.

Proof. Recall that for a sheaf $\mathcal{F}$ on a topological space X with $i : Z \hookrightarrow X$ a closed subspace and $j : U \hookrightarrow X$ its complementary open subspace, there is an exact sequence of sheaves

$$0 \to j_! j^{-1}\mathcal{F} \to \mathcal{F} \to i_* i^{-1}\mathcal{F} \to 0.$$

The sequence in the statement of the lemma is nothing more than this sequence for the complementary open and closed subspaces, in this case to the subspaces Sk_{k-1} and $\mathring{\mathrm{Sk}}_k$ of Sk_k.

Now turning to the situation where $\mathcal{F} = R^i f_* \underline{\mathbb{Q}}_X$ for a broken toric variety X, to show that $\iota_{k!}\iota_k{}^{-1}\mathcal{F}|_{\mathrm{Sk}_k} \cong \iota_{k!} \bigwedge^i \underline{\mathbb{Q}}^k_{\mathring{\mathrm{Sk}}_k(P_\bullet)}$ we note that $\iota_k{}^{-1}\mathcal{F}|_{\mathrm{Sk}_k}$

is a sheaf on $\mathring{\mathrm{Sk}}_k$ (a union of open k-cells) whose stalks are $\bigwedge^i \mathbb{Q}^k$ by proper base change. Hence it is isomorphic to $\bigwedge^i \underline{\mathbb{Q}}^k_{\mathring{\mathrm{Sk}}_k(P_\bullet)}$.

□

The following lemma describes a general vanishing result.

Lemma 2.17. *For X a broken toric variety, $H^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X) = 0$ for all $j < i$.*

Proof. The proof is by induction on k using Lemma 2.16. When $k < i$ the sheaf $R^i f_* \underline{\mathbb{Q}}_X|_{\mathrm{Sk}_k}$ is zero (all the stalks are zero by proper base change), and for $k = i$ we find that $R^i f_* \underline{\mathbb{Q}}_X|_{\mathrm{Sk}_k} \cong \iota_{k!} \underline{\mathbb{Q}}_{\mathring{\mathrm{Sk}}_k(P_\bullet)}$. This is due to ι_k being an open immersion, $(\iota_k)^! = (\iota_k)^*$ and the existence of a natural map

$$\underline{\mathbb{Q}}_{\mathring{\mathrm{Sk}}_k} \to (\iota_k)^!(\kappa_k)_!(\kappa_k)^{-1} R^k f_* \underline{\mathbb{Q}}_X|_{\mathrm{Sk}_k} = (\kappa_k)_!(\kappa_k)^{-1} R^i f_* \underline{\mathbb{Q}}_X|_{\mathring{\mathrm{Sk}}_k} = \underline{\mathbb{Q}}_{\mathring{\mathrm{Sk}}_k}$$

given by, say, the identity. Then, by adjunction, there exists a map $\iota_{k!} \underline{\mathbb{Q}}_{\mathring{\mathrm{Sk}}_k(P_\bullet)} \to R^k f_* \underline{\mathbb{Q}}_X|_{\mathrm{Sk}_k}$ which induces an isomorphism on the stalks and is thus an isomorphism.

Then we can calculate

$$H^j(P_\bullet, R^k f_* \underline{\mathbb{Q}}_X|_{\mathrm{Sk}_k}) = \begin{cases} \mathbb{Q}^{|\mathring{\mathrm{Sk}}_k(P_\bullet)|}, & j = k \\ 0, & \text{otherwise} \end{cases}$$

by applying the localization sequence for compactly supported cohomology, which gives that $H^k(T^n, (\iota_k)_! \underline{\mathbb{Q}}_X) \cong H^k_c(\mathring{\mathrm{Sk}}_k, \underline{\mathbb{Q}}_{\mathring{\mathrm{Sk}}_k})$. Then by Poincaré duality,

$$H^k(T^n, (\iota_k)_! \underline{\mathbb{Q}}_X) \cong H^k(\mathring{\mathrm{Sk}}_k, \underline{\mathbb{Q}}_{\mathring{\mathrm{Sk}}_k})^\vee = \mathbb{Q}^{|\mathring{\mathrm{Sk}}_k(P_\bullet)|}$$

(and similarly, all other cohomology groups are zero).

We now invoke the long exact sequence in cohomology associated to (2.2):

$$\cdots \to H^j(\iota_{k!} \underline{\mathbb{Q}}^{\binom{k}{i}}_{\mathring{\mathrm{Sk}}_k(P_\bullet)}) \to H^j(R^i f_* \underline{\mathbb{Q}}_X|_{\mathrm{Sk}_k}) \to H^j((\kappa_{k-1})_!(\kappa_{k-1})^{-1} R^i f_* \underline{\mathbb{Q}}_X|_{\mathrm{Sk}_k}) \to \cdots.$$

Since the cohomology of $\iota_{k!} \underline{\mathbb{Q}}_{\mathring{\mathrm{Sk}}_k(P_\bullet)}$ vanishes in all degrees except k, and by inductive hypothesis the cohomology of $(\kappa_{k-1})_!(\kappa_{k-1})^{-1} R^i f_* \underline{\mathbb{Q}}_X|_{\mathrm{Sk}_k} \cong (\kappa_{k-1})_! R^i f_* \underline{\mathbb{Q}}_X|_{\mathrm{Sk}_{k-1}}$ vanishes in all degrees less than i, this sequence shows that $H^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X|_{\mathrm{Sk}_k}) = 0$ for $j < i$. The result follows by noticing that $R^i f_* \underline{\mathbb{Q}}_X|_{\mathrm{Sk}_n} \cong R^i f_* \underline{\mathbb{Q}}_X$.

□

As a corollary, we record here a fact about particular cell-compatible sheaves.

Corollary 2.18. *For X an n-dimensional broken toric variety, one has $R^n f_* \underline{\mathbb{Q}}_X = \iota_n \bigwedge^n \underline{\mathbb{Q}}_{\mathring{Sk}_n(P_\bullet)}$. In particular,*

$$H^k(P_\bullet, R^n f_* \underline{\mathbb{Q}}_X) = \begin{cases} \mathbb{Q}^{|\mathring{Sk}_n(P_\bullet)|}, & k = n \\ 0, & \text{otherwise} \end{cases}$$

Before we move on to our decomposition statement in Theorem 2.21 let us go over two lemmas which are needed in the proof.

Lemma 2.19. *Let X be a broken toric variety with polytope complex $P_\bullet$ which is the quotient of the total space of a T^n-torsor on $P_\bullet$ which is torsion. There then exists a positive integer N and an endomorphism $[N]$ of $Rf_* \underline{\mathbb{Q}}_X$ whose induced endomorphism on $Ext^j(\mathcal{F}, R^i f_* \underline{\mathbb{Q}}_X)$ for any $\mathcal{F} \in D^b_c(X)$ acts as multiplication by N^i.*

Proof. If X is a quotient of $T^n \times P_\bullet$ we can define $[N] : T^n \times P_\bullet \to T^n \times P_\bullet$ by $(x, y) \mapsto (x^N, y)$. This action descends to an action on the X which preserves the fibres of $f : X \to P_\bullet$ by definition.

Otherwise, the torsor $\mathcal{T}$ is nontrivial and torsion, meaning that it corresponds to a cocycle in Čech cohomology $\{\alpha_{ij}\} = \alpha \in H^1(P_\bullet, T^n_{\mathbb{C}})$ for which $\alpha^m = 1$ for some integer $m > 1$. We can assume that $\{\alpha_{ij}\}$ actually forms a cocycle in $H^1(P_\bullet, T^n_{\mathbb{C}}[m])$ by looking at the long exact sequence associated to $0 \to T^n_{\mathbb{C}}[m] \to T^n_{\mathbb{C}} \xrightarrow{\times m} T^n_{\mathbb{C}} \to 0$, namely

$$\cdots \to H^1(T^n_{\mathbb{C}}[m]) \to H^1(T^n_{\mathbb{C}}[m]) \xrightarrow{\times m} H^1(T^n_{\mathbb{C}}[m]) \to \dots$$

Since α lies in the kernel of the multiplication by m map, we can pull it back to a a cocycle in $H^1(P_\bullet, T^n_{\mathbb{C}}[m])$ and so $\alpha^m_{ij} = 1$. Now choose a positive integer N which is congruent to 1 modulo m and define $[N]$ acting on $\mathcal{T}$ as follows: Locally on $T^n \times U_i$, let $[N]$ act as $(x, y) \mapsto (x^N, y)$. This action glues to form a global action since if $(x, y) = \alpha_{ij}(x', y')$, then $[N](x, y) = \alpha_{ij}[N](x', y')$ as required. This action then yields an action on X.

With $[N]$ so defined, let us describe how it descends to $Rf_* \underline{\mathbb{Q}}_X$. Adjunction provides a map $\underline{\mathbb{Q}}_X \to R[N]_* R[N]^* \underline{\mathbb{Q}}_X = R[N]_* \underline{\mathbb{Q}}_X$ which we can compose with Rf_* to get a map $Rf_* \underline{\mathbb{Q}}_X \to Rf_* R[N]_* \underline{\mathbb{Q}}_X$. Consider then the diagram

$$\begin{array}{ccc} X & \xrightarrow{[N]} & X \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle f} & \\ & P_\bullet & \end{array}$$

from which we know that $Rf_*\underline{\mathbb{Q}}_X = Rf_*R[N]_*\underline{\mathbb{Q}}_X$. Thus, we have an endomorphism of $Rf_*\underline{\mathbb{Q}}_X$ which we also call $[N]$. Further, this induces endomorphisms $[N]$ of the sheaves $R^if_*\underline{\mathbb{Q}}_X$. By proper base change, we know how $[N]$ acts on the stalks of $R^if_*\underline{\mathbb{Q}}_X$. In particular, recall that $(R^if_*\underline{\mathbb{Q}}_X)_x = H^i(f^{-1}(x), \underline{\mathbb{Q}}_X|_{f^{-1}(x)}) = H^i(T^k)$ for $x \in \mathring{\mathrm{Sk}}_k(P_\bullet)$. Since the action on the fibres of f was induced directly from the map $[N]$ on $T^n \times P_\bullet$, $[N]$ acts as the N-th power map on $f^{-1}(x)$. Hence, the action on $H^i(T^k)$ is given by multiplication by N^i. In particular, $[N]$ acts as multiplication by N^i on $R^if_*\underline{\mathbb{Q}}_X$ and similarly on $\mathrm{Ext}^j(\mathcal{F}, R^if_*\underline{\mathbb{Q}}_X)$.

□

Lemma 2.20. *Let $P_\bullet$ be a polytope complex. If α_0 and α_1 are two T^n-torsors in the same component of $H^1(P_\bullet, T^n_{\mathbb{C}})$, then the constant sheaves on the broken toric varieties over $P_\bullet$ which they define have isomorphic derived pushforward complexes.*

Proof. Let X_0 and X_1 be the broken toric variety over $P_\bullet$ in question and let $\gamma : [0,1]$ be a path between α_0 and α_1 in $H^1(P_\bullet, T^n_{\mathbb{C}})$. This yields a family of broken toric varieties $\mathfrak{X} \xrightarrow{f} P_\bullet \times [0,1]$. Pushing forward the constant sheaf along the family of moment maps yields $\mathcal{F} := Rf_*\underline{\mathbb{Q}}_{\mathfrak{X}}$, a complex of sheaves on $P_\bullet \times [0,1]$. We will show that $\mathcal{F}|_{P_\bullet \times \{t\}} \cong Rf_*\underline{\mathbb{Q}}_{X_t}$ is independent of $t \in [0,1]$.

If g is the contraction map of $[0,1]$ to $\{0\}$, then $g^{-1}g_*\mathcal{F}$ is a sheaf on $P_\bullet$ and adjunction furnishes us with a morphism $g^{-1}g_*\mathcal{F} \to \mathcal{F}$. This is a quasi-isomorphism as can been seen by looking at the stalks of the cohomology sheaves of the complexes: We have $\left(\mathcal{H}^i(g^{-1}g_*\mathcal{F})\right)_{(x,t)} \cong \left(R^if_*\underline{\mathbb{Q}}_{X_0}\right)_x$ and $\left(\mathcal{H}^i(\mathcal{F})\right)_{(x,t)} \cong \left(R^if_*\underline{\mathbb{Q}}_{X_t}\right)_x$. Both of these stalks are described entirely by $P_\bullet$, in particular independently of t, so they are isomorphic. So $\mathcal{F}|_{P_\bullet \times \{t\}} \cong Rf_*\underline{\mathbb{Q}}_{X_t}$ is independent of t and the lemma is proven. □

Theorem 2.21. *For X an n-dimensional broken toric variety and f the map from X to its polytope complex $P_\bullet$, there is an isomorphism in $\mathcal{D}^b_c(P_\bullet)$*

$$Rf_*\underline{\mathbb{Q}}_X \cong \bigoplus_{i=0}^{2n} R^if_*\underline{\mathbb{Q}}_X[-i].$$

This implies the degeneration at the E_2 page of the Leray spectral sequence associated to f

$$E_2^{pq} = H^p(P_\bullet, R^qf_*\underline{\mathbb{Q}}_X).$$

Proof. We first show that the statement holds for a broken toric variety whose gluing data consists of a T^n-torsor which is torsion, using the endomorphism provided by Lemma 2.19. Then we only have to notice that each component of the space $H^1(P_\bullet, T^n_{\mathbb{C}})$ indexing different possible broken toric varieties over $P_\bullet$ contains such a torsor and use that the derived pushforward complex does not vary in components by Lemma (2.20).

The strategy for the first part of the proof will be to show that

$$\operatorname{Hom}(-, Rf_*\underline{\mathbb{Q}}_X) = \bigoplus_{i=0}^{2n} \operatorname{Hom}(-, R^i f_* \underline{\mathbb{Q}}_X[-i]), \tag{2.3}$$

which yields the desired decomposition by the Yoneda Lemma.

Truncating the complex $Rf_*\underline{\mathbb{Q}}_X$ yields distinguished triangles

$$\tau_{\leq i} Rf_*\underline{\mathbb{Q}}_X \to \tau_{\leq i+1} Rf_*\underline{\mathbb{Q}}_X \to \tau_{\geq i+1}\tau_{\leq i+1} Rf_*\underline{\mathbb{Q}}_X \xrightarrow{+1}$$

for $i = 0, \ldots, n-1$. Each of these yields a long exact sequence after applying the functor $\operatorname{Hom}(\mathcal{F}, -)$ for any $\mathcal{F} \in D^b_c(X)$

$$\ldots \to \operatorname{Ext}^j(\mathcal{F}, \tau_{\leq i} Rf_*\underline{\mathbb{Q}}_X) \to \operatorname{Ext}^j(\mathcal{F}, \tau_{\leq i+1} Rf_*\underline{\mathbb{Q}}_X) \to \operatorname{Ext}^j(\mathcal{F}, R^{i+1} f_*\underline{\mathbb{Q}}_X[-(i+1)]) \to \ldots$$

Now we claim that the connecting homomorphisms

$$\operatorname{Ext}^j(\mathcal{F}, R^{i+1} f_*\underline{\mathbb{Q}}_X[-(i+1)]) \xrightarrow{\delta_j} \operatorname{Ext}^{j+1}(\mathcal{F}, \tau_{\leq i} Rf_*\underline{\mathbb{Q}}_X)$$

are zero for all i, j. This will be sufficient to show (2.3) because it implies in particular that

$$\operatorname{Hom}(\mathcal{F}, \tau_{\leq i+1} Rf_*\underline{\mathbb{Q}}_X) \cong \operatorname{Hom}(\mathcal{F}, \tau_{\leq i} Rf_*\underline{\mathbb{Q}}_X) \oplus \operatorname{Hom}(\mathcal{F}, R^{i+1} f_*\underline{\mathbb{Q}}_X[-(i+1)])$$

for all i. The proof of this claim is by induction on i.

First consider the $i = 0$ case. Here we are asking about the map $\operatorname{Ext}^j(\mathcal{F}, R^1 f_*\underline{\mathbb{Q}}_X) \to \operatorname{Ext}^{j+1}(\mathcal{F}, R^0 f_*\underline{\mathbb{Q}}_X)$. By Lemma 2.19, the endomorphism $[N]$ acts on $\operatorname{Ext}^j(\mathcal{F}, R^1 f_*\underline{\mathbb{Q}}_X)$ with eigenvalue N and on $\operatorname{Ext}^{j+1}(\mathcal{F}, R^0 f_*\underline{\mathbb{Q}}_X)$ trivially, and thus the map must be zero.

Now assume that the statement is true for all $k \leq i-1$. This gives us a (non-canonical) isomorphism

$$\operatorname{Ext}^{j+1}(\mathcal{F}, \tau_{\leq k} Rf_*\underline{\mathbb{Q}}_X) \cong \bigoplus_{i=0}^{k} \operatorname{Ext}^{j+1}(\mathcal{F}, R^i f_*\underline{\mathbb{Q}}_X[-i]).$$

The map we are interested in can then be written as

$$\mathrm{Ext}^j(\mathcal{F}, R^{k+1}f_*\underline{\mathbb{Q}}_X[-(k+1)]) \to \bigoplus_{i=0}^{k} \mathrm{Ext}^{j+1}(\mathcal{F}, R^i f_*\underline{\mathbb{Q}}_X[-i]),$$

which then decomposes into maps

$$\mathrm{Ext}^j(\mathcal{F}, R^{k+1}f_*\underline{\mathbb{Q}}_X[-(k+1)]) \to \mathrm{Ext}^{j+1}(\mathcal{F}, R^i f_*\underline{\mathbb{Q}}_X[-i])$$

for $i = 0, \dots, k$, each of which is zero since we have $[N]$ acting as N^{k+1} on the domain and as N^i on the range.

To finish the proof, recall that $H^1(P_\bullet, T^n_{\mathbb{C}})$ are indexed by the torsion elements of $H^1(P_\bullet, \mathbb{Z})$ by the universal coefficient theorem. This description, namely $H^1(P_\bullet, T^n_{\mathbb{C}}) \cong \mathrm{Hom}(H_1(P_\bullet, \mathbb{Z}), T^n_{\mathbb{C}})$, says in particular that there is a torsion element in each component. Then the theorem follows from Lemma 2.20. □

The following corollary is the basis for all of our cohomology calculations.

Corollary 2.22. *For any broken toric variety X there is a canonical decomposition*

$$H^k(X, \underline{\mathbb{Q}}_X) = \bigoplus_{i+j=k} H^j(P_\bullet(X), R^i f_*\underline{\mathbb{Q}}_X).$$

Proof. The existence of some decomposition of the above form follows directly from Theorem 2.21. For the fact that this decomposition is canonical, Lemma 2.19 and Lemma 2.20 together provide us with an action of $[N]$ on $H^k(P_\bullet, Rf_*\underline{\mathbb{Q}}_X)$ for any $N > 1$. The locus in $H^k(P_\bullet, Rf_*\underline{\mathbb{Q}}_X)$ on which $[N]$ acts with trace N^i is identified with $H^j(P_\bullet(X), R^i f_*\underline{\mathbb{Q}}_X)$ in the decomposition.

Lastly we ought to note that this is independent of N. Given two positive integers N and N', applying the two actions in sequence provides some subspace of $H^k(P_\bullet, Rf_*\underline{\mathbb{Q}}_X)$ on which the composition acts as $N^i N'^i$. On the other hand, $[NN']$ defines the same action and the subspace of $H^k(P_\bullet(X), Rf_*\underline{\mathbb{Q}}_X)$ on which it acts is of the same dimension as $H^j(P_\bullet(X), R^i f_*\underline{\mathbb{Q}}_X)$. Hence the decomposition is independent of N. □

One takeaway from this corollary is that the cohomology of a broken toric variety X does not depend on how the components are glued together, rather only on the polytope complex of X. Hence we assume for the rest of the thesis (unless otherwise mentioned) that the torsor defining the gluing is trivial and in particular that $R^i f_*\underline{\mathbb{Q}}_X$ is a subsheaf of $\bigwedge^i \underline{\mathbb{Q}}^n_X$.

2.4 LERAY = WEIGHT

To any continuous map $f : X \to Y$ of topological spaces one can assign the Leray filtration associated with f to the cohomology groups of X. To do this, first apply the truncation functor $\tau_{\leq k}$ to the complex $Rf_*\underline{\mathbb{Q}}_X$

$$\begin{aligned}\tau_{\leq k}\left(\cdots \to (Rf_*\underline{\mathbb{Q}}_X)^{k-1} \xrightarrow{d^{k-1}} (Rf_*\underline{\mathbb{Q}}_X)^k \xrightarrow{d^k} \cdots\right) \\ = \left(\cdots \to (Rf_*\underline{\mathbb{Q}}_X)^{k-1} \xrightarrow{d^{k-1}} \ker d^k \xrightarrow{d^k} 0 \to \cdots\right).\end{aligned}$$

The inclusion map $\tau_{\leq k}Rf_*\underline{\mathbb{Q}}_X \to Rf_*\underline{\mathbb{Q}}_X$ induces a map

$$H^i(Y, \tau_{\leq k}Rf_*\underline{\mathbb{Q}}_X) \to H^i(Y, Rf_*\underline{\mathbb{Q}}_X) \cong H^i(X, \underline{\mathbb{Q}}_X),$$

the image of which is the k-th piece of the *Leray filtration* $L_kH^i(X, \underline{\mathbb{Q}}_X)$.

Theorem 2.21 then gives us an explicit (canonical) formulation for the Leray filtration on the cohomology of a broken toric variety, namely

$$L_kH^i(X, \underline{\mathbb{Q}}_X) = \bigoplus_{\substack{p+q=i \\ q \leq k}} H^p(P_\bullet, R^q f_*\underline{\mathbb{Q}}_X). \tag{2.4}$$

The Leray filtration can equivalently be described as the filtration arising from the degeneration of the Leray spectral sequence. Arising in a similar way is another filtration: the *weight filtration*.

Smooth complex algebraic varieties come equipped with a Hodge structure on their cohomology groups. Deligne [Del71] introduced mixed Hodge structures to extend this theory to more general algebraic varieties. Roughly, a mixed Hodge structure consists of the data of two filtrations on a vector space (in our case, a cohomology group of a complex variety), with one of the filtrations inducing a Hodge structure on the graded pieces of the other.

Broken toric varieties as defined here are singular but projective. Following [GNAPGP88], one defines for such varieties the *weight spectral sequence*

$$E_1^{p,q} = \bigoplus_{|\alpha|=p+1} H^q(X_\alpha, \underline{\mathbb{Q}}) \Rightarrow H^{p+q}(X, \underline{\mathbb{Q}})$$

where $X_\bullet \to X$ is a cubical hyperresolution (whose precise definition would lead a bit far afield and can be found in [GNAPGP88]).

The weight filtration is the filtration on cohomology groups associated to this spectral sequence, namely

$$E_2^{pq} = W_p H^{p+q}(X,\underline{\mathbb{Q}})/W_{p-1}H^{p+q}(X,\underline{\mathbb{Q}}).$$

Definition 2.23. For any broken toric variety X, denote the intersection $W_k H^{i+j}(X,\underline{\mathbb{Q}}_X) \cap H^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X)$ by $W_k H^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X)$.

The main ingredient for proving the equivalence of these two filtrations is the following lemma. Recall that given a polytope P, we denote by $X(P)$ the projective toric variety associated to it.

Lemma 2.24. *Let X be an n-dimensional broken toric variety with polytope complex $P_\bullet$ and set $\tilde{X} := \bigsqcup_{\overline{A}\in Sk_n(P_\bullet)} X(\overline{A})$, the disjoint union of the toric components of X. In addition, let Y be the singular locus of X, i.e. if we set*

$$S = Sk_{n-1}(P_\bullet) \setminus \{\alpha \in \mathring{S}k_{n-1}(P_\bullet) : \alpha \in \partial A \text{ for only one } \overline{A} \in Sk_n(P_\bullet)\},$$

then $Y = X|_{f^{-1}(S)}$ and let $\tilde{Y}$ be the preimage of Y in $\tilde{X}$.

Then

1. *There is a long exact sequence*

$$\cdots \to H^j(X) \to H^j(Y) \oplus H^j(\tilde{X}) \to H^j(\tilde{Y}) \to \cdots$$

which respects both the weight and Leray filtrations.

2. *There is a long exact sequence*

$$\cdots \to W_k H^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X) \to W_k H^j((P_\bullet)_Y, R^i f_* \underline{\mathbb{Q}}_Y) \oplus W_k H^j((P_\bullet)_{\tilde{X}}, R^i f_* \underline{\mathbb{Q}}_{\tilde{X}})$$
$$\to W_k H^j((P_\bullet)_{\tilde{Y}}, R^i f_* \underline{\mathbb{Q}}_{\tilde{Y}}) \to W_k H^{j+1}(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X) \to \cdots.$$

Proof. The first sequence is the Mayer-Vietoris sequence associated to the Cartesian square

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

and it respects the weight filtration by construction (see [GNAPGP88]). Take this square and look at the higher derived pushforwards to their polytope complexes, from which we can get Mayer-Vietoris sequences

$$\cdots \to H^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X) \to H^j((P_\bullet)_Y, R^i f_* \underline{\mathbb{Q}}_Y) \oplus H^j((P_\bullet)_{\tilde{X}}, R^i f_* \underline{\mathbb{Q}}_{\tilde{X}}) \quad (2.5)$$
$$\to H^j((P_\bullet)_{\tilde{Y}}, R^i f_* \underline{\mathbb{Q}}_{\tilde{Y}}) \to H^{j+1}(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X) \to \cdots$$

for each i. Taking direct sums of these sequences (in view of the description (2.4) of the Leray filtration) shows that the sequence in part (1) of the lemma respects the Leray filtration.

The sequence in (1) preserving the weight filtration means that there is a long exact sequence

$$\cdots \to W_k H^j(X) \to W_k H^j(Y) \oplus W_k H^j(\tilde{X}) \to W_k H^j(\tilde{Y}) \to \cdots .$$

The sequence in part (2) is constructed by taking the intersection of this sequence with the one in Equation (2.5).

□

Theorem 2.25. *For X a broken toric variety,*

$$W_{2k} H^i(X, \underline{\mathbb{Q}}_X) = W_{2k+1} H^i(X, \underline{\mathbb{Q}}_X) = L_k H^i(X, \underline{\mathbb{Q}}_X).$$

Proof. For smooth projective toric varieties it is well-known that their odd degree cohomology vanishes and that they have pure cohomology. In this context this means that $W_k H^{2i}(X, \underline{\mathbb{Q}}_X) = 0$ for all $k \leq 2i - 1$ and that $W_{2i} H^{2i}(X, \underline{\mathbb{Q}}_X) = H^{2i}(X, \underline{\mathbb{Q}}_X)$. Regarding the Leray filtration, Lemma 2.17 and Corollary 3.2 together show that $H^p(P_\bullet, R^q f_* \underline{\mathbb{Q}}_X)$ is only nonzero for $p = q$, so that $L_k H^{2i}(X, \underline{\mathbb{Q}}_X) = 0$ for $k \leq i - 1$ and that $L_i H^{2i}(X, \underline{\mathbb{Q}}_X) = H^{2i}(X, \underline{\mathbb{Q}}_X)$.

To extend this result to all broken toric varieties, we first notice that the statement of the theorem is equivalent to asking that

$$W_{2k} H^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X) = W_{2k+1} H^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X) = H^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X) \tag{2.6}$$

for all $k \geq i$. This is because, again in view of Theorem 2.21, we have

$$W_{2k} H^p(X, \underline{\mathbb{Q}}_X) = W_{2k} \bigoplus_{i+j=p} H^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X) = \bigoplus_{i+j=p} W_{2k} H^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X)$$

which is equal to

$$\bigoplus_{\substack{i+j=p \\ i \leq k}} H^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X) = L_k H^p(X, \underline{\mathbb{Q}}_X)$$

if and only if Equation (2.6) is true (and similary for W_{2k+1}).

The result now follows by induction. Starting with $n = 1$, we see that the result is trivially true for Y and $\tilde{Y}$ and true for $\tilde{X}$ since it is a disjoint union of toric varieties. To show that (2.6) is true for X, consider the inclusion map of complexes (which exist by Lemma 2.24)

$$\begin{array}{ccccccc}
\cdots \to W_{2k}H^j(R^if_*\underline{\mathbb{Q}}_X) & \to & W_{2k}H^j(R^if_*\underline{\mathbb{Q}}_Y) \oplus W_{2k}H^j(R^if_*\underline{\mathbb{Q}}_{\tilde{X}}) & \to & W_{2k}H^j(R^if_*\underline{\mathbb{Q}}_{\tilde{Y}}) \to \cdots \\
\downarrow & & \downarrow \sim & & \downarrow \sim \\
\cdots \longrightarrow H^j(R^if_*\underline{\mathbb{Q}}_X) & \longrightarrow & H^j(R^if_*\underline{\mathbb{Q}}_Y) \oplus H^j(R^if_*\underline{\mathbb{Q}}_{\tilde{X}}) & \longrightarrow & H^j(R^if_*\underline{\mathbb{Q}}_{\tilde{Y}}) \longrightarrow \cdots
\end{array}$$

We see that the inclusion $W_{2k}H^j(R^if_*\underline{\mathbb{Q}}_X) \hookrightarrow H^j(R^if_*\underline{\mathbb{Q}}_X)$ is in fact an isomorphism by the 5-lemma and so the proof is complete.

□

2.5 BALLOON ANIMAL MAPS

In this section we will construct long exact sequences comparing the cohomology of broken toric varieties related to each other via what we call balloon animal maps.

Definition 2.26. A subdivision $(P_\bullet, \mathcal{P})$ of an n-dimensional polytope complex $P_\bullet$ is a finite collection $\mathcal{P} = \{P_1, \ldots, P_m\}$ of subsets of $P_\bullet$ such that

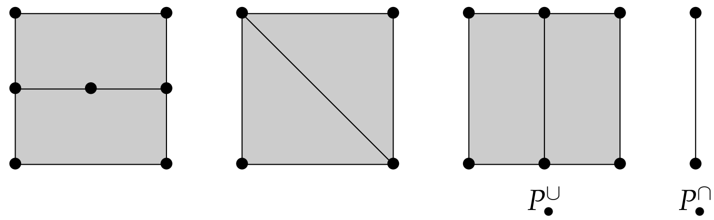
1. For each i, P_i is an n-dimensional polytope.
2. $P_\bullet = P_1 \cup \ldots \cup P_m$.
3. If $i \neq j$, then $P_i \cap P_j$ is a common (possibly empty) proper face of both P_i and P_j.
4. No polytope of $P_\bullet$ is properly contained within one of the P_i.

Such a subdivision is called *generic* if for every i, every k-face α of P_i which does not lie in the k-skeleton of $P_\bullet$ has the property that $\mathring{\mathrm{S}}\mathrm{k}_{k-1}(\alpha) \subset \mathring{\mathrm{S}}\mathrm{k}_k(P_\bullet)$.

This definition is the obvious generalization to polytope complexes of the definition of a subdivision of a polytope (see for example Chapter 16 of [GOT18]).

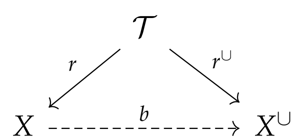
Note that a generic dissection $(P_\bullet, \mathcal{P})$ of an n-dimensional polyope complex $P_\bullet$ gives rise two new polytope complexes: first $P_\bullet^\cup$, defined by taking the set of all P_j and remembering the skeleton structure, and secondly $P_\bullet^\cap$, which is the $(n-1)$-dimensional polytope complex consisting of all faces among the $P_i \in \mathcal{P}$ which are of dimension $k \leq n-1$ and which do not lie in a k-face of $P_\bullet$.

Example 2.27. Pictured below are three attempts at subdivisions of a square $P_\bullet$. The first is not a subdivision (it fails item (3) of the definition) while the second is a non-generic subdivision. The third is a generic subdivision of $P_\bullet$, pictured as $P_\bullet^\cup$, for which $P_\bullet^\cap$ is also pictured.



Now let X be a broken toric variety over $P_\bullet$ with the gluing of its toric components determined by the T^n-torsor $\alpha \in H^1(P_\bullet, T^n_{\mathbb{C}})$. Since the polytope complex $P_\bullet^\cup$ is of the same topological type as $P_\bullet$, α also defines a broken toric variety $X^\cup$ over $P_\bullet^\cup$.

With this setup we can define a map between X and $X^\cup$. The geometric realizations of $P_\bullet$ and $P_\bullet^\cup$ are the same and that X and $X^\cup$ are determined by the same T^n torsor whose total space we write as $\mathcal{T}$. So both X and $X^\cap$ are quotients of $\mathcal{T}$ and the proof of Lemma 2.10 describes the equivalence relation defining the quotients. Each equivalence relation depends only on the polytope complex: more specifically it is fibrewise determined at x by the dimension k for which x lies in the open k-skeleton of the polytope complex. Since $P_\bullet$ can be viewed as a refinement of $P_\bullet$, the equivalence relation defining $X^\cup$ is a refinement of the one defining X. This induces a map from X to $X^\cup$.



Definition 2.28. Given a generic subdivision $(P_\bullet, \mathcal{P})$, the *balloon animal map* is the continous map $b : X \to X^\cup$ induced by the refinement of the equivalence relations defining X and X'.

Example 2.29. One example to have in mind is that of the polytope complexes $P_\bullet =$  and $P_\bullet^\cup =$  . In this case we have a map of broken toric varieties where X is the torus with one pinched point and $X^\cup$ is the torus with two pinched points (so we are "twisting" the "balloon animal").

Lemma 2.30. *Let $b : X \to X^\cup$ be a balloon animal map between broken toric varieties induced by a generic subdivision of the polytope complex of X and $X^\cap$ be the broken toric subvariety of $X^\cup$ associated to $P_\bullet^\cap$. Then*

1. $R^1 b_* \underline{\mathbb{Q}}_X \cong \underline{\mathbb{Q}}_{X^\cap}$.

2. *There is a distinguished triangle in the derived category of sheaves on* $X^{\cup}$

$$\underline{\mathbb{Q}}_{X^{\cup}} \to Rb_*\underline{\mathbb{Q}}_X \to \underline{\mathbb{Q}}_{X^{\cap}}[-1] \xrightarrow{+1}$$

Proof. For part (2), consider the distinguished triangle from truncation of the pushforward of the constant sheaf on X along the balloon animal map:

$$\tau_{\leq 0} Rb_*\underline{\mathbb{Q}}_X \to Rb_*\underline{\mathbb{Q}}_X \to \tau_{\geq 1} Rb_*\underline{\mathbb{Q}}_X \xrightarrow{+1} . \tag{2.7}$$

By definition of the truncation functor, $\tau_{\leq 0} Rb_*\underline{\mathbb{Q}}_X$ is $\underline{\mathbb{Q}}_{X^{\cup}}$, and for dimension reasons $R^i b_*\underline{\mathbb{Q}}_X$ is zero for all $i > 1$, so $\tau_{\geq 1} Rb_*\underline{\mathbb{Q}}_X = R^1 b_*\underline{\mathbb{Q}}_X[-1]$. The statement now follows directly from (1), the proof of which is easiest to state using the technology of cellular sheaf cohomology and so is postponed to Section 3.3. □

Theorem 2.31. *Let* $b : X \to X^{\cup}$ *be a balloon animal map between broken toric varieties induced by a generic subdivision of the polytope complex of* X *and* $X^{\cap}$ *be the broken toric subvariety of* $X^{\cup}$ *associated to* $P_{\bullet}^{\cap}$*. Then there is a long exact sequence*

$$\cdots \to H^k(X^{\cup}, \underline{\mathbb{Q}}_{X^{\cup}}) \to H^k(X, \underline{\mathbb{Q}}_X) \to H^{k-1}(X^{\cap}, \underline{\mathbb{Q}}_{X^{\cap}}) \to \ldots .$$

Proof. This follows by applying the hypercohomology functor to the distinguished triangle of Lemma 2.30 (2) and the fact that $\mathbb{H}^{\bullet}(X^{\cup}, Rb_*\underline{\mathbb{Q}}_X) \cong H^{\bullet}(X, \underline{\mathbb{Q}}_X)$.

□

It turns out that the above statement can be refined further after decomposing via Corollary 2.22.

Lemma 2.32. *Let* $b : X \to X^{\cup}$ *be a balloon animal map between broken toric varieties induced by a generic subdivision of the polytope complex of* X *and* $X^{\cap}$ *be the broken toric subvariety of* $X^{\cup}$ *associated to* $P_{\bullet}^{\cap}$*. For* $i \geq 0$ *there exist short exact sequences*

$$0 \to R^i f_*\underline{\mathbb{Q}}_{X^{\cup}} \to R^i f_*\underline{\mathbb{Q}}_X \to R^{i-1} f_*\underline{\mathbb{Q}}_{X^{\cap}} \to 0.$$

Proof. Taking the derived pushforward to $P_{\bullet}^{\cup}$ of the distinguished triangle of Lemma 2.30 (2), we obtain the distinguished triangle

$$Rf_*\underline{\mathbb{Q}}_{X^{\cup}} \to Rf_* Rb_*\underline{\mathbb{Q}}_{X^{\cup}} \cong Rf_*\underline{\mathbb{Q}}_X \to Rf_*\underline{\mathbb{Q}}_{X^{\cap}}[-1] \xrightarrow{+1},$$

of which we can take the cohomology objects to obtain the following long exact sequence of sheaves on $P_{\bullet}$,

$$\cdots \to R^i f_* \underline{\mathbb{Q}}_{X^\cup} \to R^i f_* \underline{\mathbb{Q}}_X \to R^{i-1} f_* \underline{\mathbb{Q}}_{X^\cap} \to \dots \tag{2.8}$$

We want to show that the boundary maps are zero or equivalently that the maps $b_i : R^i f_* \underline{\mathbb{Q}}_{X^\cup} \to R^i f_* \underline{\mathbb{Q}}_X$ are injective. This follows directly from the construction as we will see by observing what is happening on the stalks.

Let $x \in \alpha \in \mathring{\mathrm{Sk}}_k(P_\bullet)$. The map b_i descends from a balloon animal map and so is the identity away from the 'new' cells in the decomposition. That is to say, if x lies within the relative interior of a k-face in both $P_\bullet$ and $P_\bullet^\cup$, then $(b_i)_x : \bigwedge^i H_\alpha \to \bigwedge^i H_\alpha$ (cf. Definition 2.9) is the identity.

If α does not lie within a k-face of $P_\bullet$, then we are looking at $(b_i)_x : \bigwedge^i H_\alpha \to \bigwedge^i H_\beta$ where β is the $k+1$-face of $P_\bullet$ in which α lies. This map is really a map of cohomology groups (recall Corollary 2.11) induced by the map of tori $H^*_\beta / \Gamma \cap H^*_\beta \to H^* \alpha / \Gamma \cap H^*_\alpha$. This map forms part of a split short exact sequence since H_α and H_β are defined so that $H^*_\beta / \Gamma \cap H^*_\beta \cong (H^*_\alpha / \Gamma \cap H^*_\alpha) \times S^1$. Thus $(b_i)_x$ is injective and the boundary maps of the sequence in Equation (2.8) are zero.

□

Theorem 2.33. *Let $b : X \to X^\cup$ be a balloon animal map between broken toric varieties induced by a generic subdivision of the polytope complex of X and $X^\cap$ be the broken toric subvariety of $X^\cup$ associated to $P_\bullet^\cap$. For $i \geq 0$ there exists a long exact sequence*

$$\cdots \longrightarrow H^{k-1}(P_\bullet^\cap, R^{i-1} f_* \underline{\mathbb{Q}}_{X^\cap}) \longrightarrow H^k(P_\bullet^\cup, R^i f_* \underline{\mathbb{Q}}_{X^\cup}) \longrightarrow H^k(P_\bullet, R^i f_* \underline{\mathbb{Q}}_X) \longrightarrow H^k(P_\bullet^\cap, R^{i-1} f_* \underline{\mathbb{Q}}_{X^\cap}) \longrightarrow H^{k+1}(P_\bullet^\cup, R^i f_* \underline{\mathbb{Q}}_{X^\cup}) \longrightarrow \cdots$$

Proof. This is the long exact sequence in cohomology associated to the short exact sequence in Lemma 2.32. □

2.6 broken toric varieties as quotients of polyhedral products

In this section we will describe how broken toric varieties appear in the study of polyhedral products, further cementing them as objects with widespread appeal.

Given a family of based CW pairs $(\underline{X}, \underline{A}) := (X_i, A_i)_{i=1}^m$ and a simplicial complex K on m vertices, the polyhedral product is a tool developed in the field of toric topology to produce a subspace $\mathcal{Z}(K, (\underline{X}, \underline{A}))$ of the Cartesian product $X_1 \times \cdots \times X_m$. To be precise, for **SCpx** the category of simplicial complexes and $\mathcal{C}[m]$ the category of m-tuples of based CW pairs, it is a functor

$$\mathcal{Z}(-,-) : \mathbf{SCpx} \times \mathcal{C}[m] \to \mathbf{Top}$$

satisfying (following [BBC20])

$$\mathcal{Z}(K, (\underline{X}, \underline{A})) \subseteq X_1 \times \cdots \times X_m$$

and which is the colimit of a diagram D in the category CW_* of pointed CW pairs, defined for $\sigma \in K$ by

$$W_i = \begin{cases} X_i, & i \in \sigma \\ A_i, & i \notin \sigma \end{cases}$$

and

$$D(\sigma) = W_1 \times \cdots \times W_m.$$

If we fix $(\underline{X}, \underline{A})$ with A_i being the basepoint of X_i for all i, one can think of the polyhedral product for different K as interpolating between $X_1 \vee \cdots \vee X_m$ (when K is m discrete points) and $X_1 \times \cdots \times X_m$ (when K is the full $(m-1)$-simplex). Another relevant example is the case where $(\underline{X}, \underline{A}) = ((\mathbb{P}^2, \mathbb{P}^1), (\mathbb{P}^2, \mathbb{P}^1))$ and K is the simplicial complex consisting of two distinct points. Here we find that

$$\mathcal{Z}(\{\{1\}, \{2\}\}, ((\mathbb{P}^2, \mathbb{P}^1), (\mathbb{P}^2, \mathbb{P}^1))) \cong \mathbb{P}^2 \times \mathbb{P}^1 \cup_{\mathbb{P}^1 \times \mathbb{P}^1} \mathbb{P}^1 \times \mathbb{P}^2.$$

The diagonal torus $\Delta = \mathbb{C}^*$ in $\mathbb{P}^1 \times \mathbb{P}^1$ acts in a natural way with quotient $\mathbb{P}^1$ and further on $\mathbb{P}^2 \times \mathbb{P}^1$ (and $\mathbb{P}^1 \times \mathbb{P}^2$) with quotient $\mathbb{P}^2$. Thus, taking the quotient of $\mathcal{Z}(\{\{1\}, \{2\}\}, ((\mathbb{P}^2, \mathbb{P}^1), (\mathbb{P}^2, \mathbb{P}^1)))$ by this diagonal subgroup yields the broken toric variety of Example 2.15.

The above example generalizes. For any two (potentially broken) toric varieties X and X' which we wish to glue along a common (potentially broken) toric subvariety $Y \subset X, X'$, take the polyhedral product

$$\mathcal{Z} := \mathcal{Z}(\{\{1\}, \{2\}\}, ((X, Y), (X', Y))) = X \times Y \cup_{Y \times Y} Y \times X$$

and quotient out by the diagonal torus $\Delta \subset Y \times Y$. This quotient is best viewed from the point of view of the corresponding fans[3] (*cf.* [KSZ91]). The quotient of a toric variety X with fan $\mathcal{F}_X \subset T^{\vee}_{\mathbb{R}}$ by a subgroup H of its torus can be described as the toric variety associated to the fan one gets

3 We have not emphasized the fact that a toric variety is completely determined by its corresponding fan. Many of the results in this thesis could be reformulated from the fan point of view.

by projecting $\mathcal{F}_X$ onto the rational subspace of $T^\vee_\mathbb{R}$ defined by H. So, the quotient of $Y \times Y$ by Δ is the toric variety associated to the fan $\mathcal{F}_Y \times \mathcal{F}_Y$ projected to the diagonal hyperplane, which is just $\mathcal{F}_Y$ itself. In a similar way, the quotient of $X \times Y$ by Δ is the toric variety associated to the fan $\mathcal{F}_X \times \mathcal{F}_Y$ projected to the diagonal hyperplane, which is $\mathcal{F}_X$. All together, the quotient of $\mathcal{Z}$ by Δ is the desired broken toric variety.

Iterating the above construction shows that any broken toric variety is the quotient of a polyhedral product.

3

BROKEN TORIC VARIETIES - COMPUTATIONS

3.1 CELL-COMPATIBLE SHEAVES ON POLYTOPES

Here we will define a sequence of cell-compatible sheaves which are then used to reproduce the well-known formula for the Betti numbers of a toric variety and discuss some related facts about certain cell-compatible sheaves on non-simple polytopes.

Take any polytope P of dimension n and let us define a particular sequence of subsets A_m of $\mathring{\mathrm{Sk}}_{n-1}(P)$ by letting $\alpha_1, \ldots, \alpha_{|\mathring{\mathrm{Sk}}_{n-1}(P)|}$ be an enumeration of the n-cells of $\mathring{\mathrm{Sk}}_{n-1}(P)$ such that $A_m := \bigcup_{i=1}^{m} \overline{\alpha_i}$ is contractible. Define a subsheaf $\mathcal{S}^i_P(A_m)$ of $\bigwedge^i \underline{\mathbb{Q}}^n_P$ by the following restriction of the stalks:

$$\left(\mathcal{S}^i_P(A_m)\right)_x = \bigwedge^i \left(\bigcap_{\substack{\overline{\alpha}_j \ni x \\ 0 \leq j \leq m}} H_{\alpha_j} \right),$$

where $H_{\alpha_j} \subset (\mathbb{R}^n)^*$ is the $n-1$ dimensional subspace associated to the $n-1$ cell α_j.

The following proposition is a bit technical and its use is to prove Corollary 3.2.

Proposition 3.1. *Let P be a polytope of dimension n and $\{A_m\}$ a sequence of subsets of $\mathring{Sk}_{n-1}(P)$ as defined above. Then:*

1. *$\mathcal{S}^i_P(\varnothing) \cong \bigwedge^i \mathbb{Q}^n_P$ and if P is the polytope of a smooth projective toric variety X then $\mathcal{S}^i_P(\mathring{Sk}_{n-1}) \cong R^i f_* \underline{\mathbb{Q}}_X$.*

2. *If P is simple there is a short exact sequence of sheaves*

$$0 \to \mathcal{S}^i_P(A_m) \to \mathcal{S}^i_P(A_{m-1}) \to \mathcal{S}^{i-1}_{a_m}(A_{m-1} \cap \{\alpha_m\}) \to 0.$$

3. *$H^j(\mathcal{S}^i_P(A_m)) = 0$ for all $i < j$. If P is simple, then in addition $H^j(\mathcal{S}^i_P(A_m)) = 0$ for all $j > i$.*

Proof. The first part of (1) follows directly from the definition, as does the second since if P is the polytope of a smooth projective toric variety X then we know that $R^i f_* \underline{\mathbb{Q}}_X$ is the subsheaf of $\bigwedge^i \underline{\mathbb{Q}}_P^n$ with stalks

$$(R^i f_* \underline{\mathbb{Q}}_X)_x = \bigwedge^i \left(\bigcap_{\substack{\alpha \in \mathring{\mathrm{Sk}}_{n-1}(P) \\ \overline{\alpha} \ni x}} H_\alpha \right).$$

This is simply another way of writing the description of the stalks that appears in the proof of Proposition 2.13[1].

For (2), consider the surjective morphism of sheaves

$$q : \mathcal{S}_P^i(A_{m-1}) \to \mathcal{S}_{\alpha_m}^{i-1}(A_{m-1} \cap \{\alpha_m\})$$

defined on stalks in the following way: First, if $x \notin \overline{\alpha_m}$, then q_x is the zero map and if $x \in \overline{\alpha_m}$, then

$$q_x : \bigwedge^i \left(\bigcap_{\overline{\alpha_k} \ni x} H_{\alpha_k} \right) \to \bigwedge^i \left(\bigcap_{\overline{\alpha_k} \ni x} H_{\alpha_k} \cap H_{\alpha_m} \right)$$

where $\bigcap_{\overline{\alpha_k} \ni x} H_{\alpha_k}$ is an l-dimensional vector space with a chosen basis $\{b_1, \ldots, b_l\}$ and so $\bigcap_{\overline{\alpha_k} \ni x} H_{\alpha_k} \cap H_{\alpha_m}$ is $(l-1)$-dimensional with basis $\{b_1, \ldots, b_{l-1}\}$. Then q_x is defined as taking $b_{r_1} \wedge \cdots \wedge b_{r_i}$ to 0 if b_l is not among the b_r and to $b_{r_1} \wedge \cdots \wedge \hat{b}_l \wedge \cdots \wedge b_{r_i}$ otherwise.

The kernel of q is a subsheaf of $\bigwedge^i \underline{\mathbb{Q}}_P^n$ (since it is a subsheaf of $\mathcal{S}_P^i(A_{m-1})$) whose stalks match those of $\mathcal{S}_P^i(A_m)$, hence it is $\mathcal{S}_P^i(A_m)$ itself.

Part(3) is reminiscent of the proof of Lemma 2.17, proceeding by induction on the dimension n of P. For the 1-dimensional simple polytope P, we can see

$$\mathcal{S}_P^0(A) \cong R^0 f_* \underline{\mathbb{Q}}_X \text{ for all } A \subseteq \mathring{\mathrm{Sk}}_0(P)$$

and

$$\mathcal{S}_P^1(\varnothing) \cong \mathbb{Q}_P,$$

$$\mathcal{S}_P^1(\{\alpha_i\}) \cong j_! \mathbb{Q}_{P \setminus \{\alpha_i\}},$$

$$\mathcal{S}_P^1(\{\alpha_1, \alpha_2\}) \cong j_! \mathbb{Q}_{P \setminus \{\alpha_1, \alpha_2\}} \cong R^1 f_* \mathbb{Q}_X$$

These are fairly innocent cell-compatible sheaves with the property that $H^j(\mathcal{S}_P^i(A)) = 0$ for all $j > i$. Assuming that this holds for a dimension n polytope, consider the long exact sequence in cohomology arising from the short exact sequence of (2) with $\alpha_m = P$ and $P_\bullet$ equal to some $n+1$-

1 Although there is a small abuse of notation where we write $\alpha \in \mathring{\mathrm{Sk}}_{n-1}(P)$ to mean that α is an $(n-1)$-cell of $\mathrm{Sk}_{n-1}(P)$.

dimensional polytope with P as a face. Note that it is here where we use simplicity of the polytope; the statement of (2) does not hold for P non-simple. This gives the desired result for $n+1$ as long as we recall that for any n, $H^j(P, \mathcal{S}_P^i(\varnothing)) = H^j(P, \bigwedge^i \underline{\mathbb{Q}}_P^n) = 0$ for all $j > i$.

□

Corollary 3.2. *Let X be a smooth projective toric variety with polytope P. Then $H^j(P, R^i f_* \underline{\mathbb{Q}}_X) = 0$ for all $j > i$.*

Proof. Combine (1) and (3) of Proposition 3.1. □

This can now be used to give a new proof of the following result of Danilov.

Corollary 3.3 ([Dan78])**.** *Let X be a smooth projective toric variety of dimension n with polytope P. Then the odd-dimensional cohomology of X vanishes and*

$$h^{2i}(X, \underline{\mathbb{Q}}_X) = h^i(P, R^i f_* \underline{\mathbb{Q}}_X) = \sum_{j=i}^{n} (-1)^{i-j} \binom{j}{i} |\mathring{S}k_j(P)|.$$

Proof. Assume that $i > 0$, since the $i = 0$ case is trivial. Denote $\mathcal{S}_P^i(\mathring{\mathrm{Sk}}_{n-1}(P))$ by $\mathcal{F}$ and consider the long exact sequence associated to 2.2, which tells us that

$$H^l(P, (\kappa_k)_!(\kappa_k)^{-1}\mathcal{F}) = H^l(P, (\kappa_{k-1})_!(\kappa_{k-1})^{-1}\mathcal{F}) \tag{3.9}$$

for all $l \neq k, k-1$, since $H^l(P, (\iota_k)_! \mathcal{F}|_{\mathring{\mathrm{Sk}}_k(P)}) = H^l(P, (\iota_{l+1})_! \bigwedge^i \underline{\mathbb{Q}}^n_{\mathring{\mathrm{Sk}}_k(P)})$ is only nonzero for $l = k$. Since $H^l(P, (\kappa_0)_!(\kappa_0)^{-1}\mathcal{F}) = 0$ for all l, we find

$$H^l(P, (\kappa_k)_!(\kappa_k)^{-1}\mathcal{F}) = 0$$

for all $l > k$. On the other hand, setting $k = n$ in (3.9) yields

$$H^l(P, \mathcal{F}) = H^l(P, (\kappa_{n-1})_!(\kappa_{n-1})^{-1}\mathcal{F})$$

for all $l \neq n, n-1$, and so in particular by Corollary 3.2 we have

$$H^l(P, (\kappa_{n-1})_!(\kappa_{n-1})^{-1}\mathcal{F}) = 0$$

for all $l \neq n, n-1, i$. We can again use (3.9) to say that

$$H^l(P, (\kappa_k)_!(\bar{\iota}_k)^{-1}\mathcal{F}) = 0$$

for all $i \neq l < k$ (as well as for all $k < i$, as in the proof of Lemma 2.16). These vanishings together imply that

$$h^l(P,(\kappa_l)_!(\kappa_l)^{-1}\mathcal{F}) = h^l(P,(\iota_l)_!\bigwedge^i \underline{\mathbb{Q}}^n_{\mathring{\mathrm{S}}\mathrm{k}_k(P)}) - h^{l-1}(P,(\kappa_{l-1})_!(\kappa_{l-1})^{-1}\mathcal{F}) \tag{3.10}$$

for $l \leq i$, and

$$h^l(P,(\kappa_l)_!(\kappa_l)^{-1}\mathcal{F}) = h^{l+1}(P,(\iota_{l+1})_!\bigwedge^i \underline{\mathbb{Q}}^n_{\mathring{\mathrm{S}}\mathrm{k}_k(P)}) - h^{l+1}(P,(\kappa_{l+1})_!(\kappa_{l+1})^{-1}\mathcal{F}) \tag{3.11}$$

for $l > i$.

Of course, what we are interested in is $H^i(P,\mathcal{F})$, which is isomorphic to $H^i(P,(\kappa_{i+1})_!(\kappa_{i+1})^{-1}\mathcal{F})$ by (3.9). The long exact sequence associated to (2.2) for $k = i+1$ tells us that

$$\begin{aligned} h^i(P,\mathcal{F}) = h^i(P,(\kappa_i)_!(\kappa_i)^{-1}\mathcal{F}) - h^{i+1}(P,(\iota_{i+1})_!\bigwedge^i \underline{\mathbb{Q}}^{i+1}_{\mathring{\mathrm{S}}\mathrm{k}_{i+1}(P_\bullet)})+ \\ + h^{i+1}(P,(\kappa_{i+1})_!(\kappa_{i+1})^{-1}\mathcal{F}). \end{aligned} \tag{3.12}$$

Repeatedly applying Equations (3.10) and (3.11) calculates

$$\begin{aligned} h^i(P,(\kappa_i)_!(\kappa_i)^{-1}\mathcal{F}) &= \sum_{j=0}^{i}(-1)^{i-j}h^j(P,(\iota_j)_!\bigwedge^i \underline{\mathbb{Q}}^j_{\mathring{\mathrm{S}}\mathrm{k}_j(P)}) \\ &= \sum_{j=0}^{i}(-1)^{i-j}\binom{j}{i}|\mathring{\mathrm{S}}\mathrm{k}_j(P)| \\ &= |\mathring{\mathrm{S}}\mathrm{k}_i(P)| \end{aligned}$$

and

$$\begin{aligned} h^{i+1}(P,(\kappa_{i+1})_!(\kappa_{i+1})^{-1}\mathcal{F}) &= \sum_{j=i+2}^{n}(-1)^{i-j}h^j(P,(\iota_j)_!\bigwedge^i \underline{\mathbb{Q}}^j_{\mathring{\mathrm{S}}\mathrm{k}_j(P)}) \\ &= \sum_{j=i+2}^{n}(-1)^{i-j}\binom{j}{i}|\mathring{\mathrm{S}}\mathrm{k}_j(P)| \end{aligned}$$

so that (3.12) simplifies to

$$\begin{aligned} h^i(P,\mathcal{F}) = |\mathring{\mathrm{S}}\mathrm{k}_i(P)| - \binom{i+1}{i}|\mathring{\mathrm{S}}\mathrm{k}_{i+1}(P)|+ \\ + \sum_{j=i+2}^{n}(-1)^{i-j}\binom{j}{i}|\mathring{\mathrm{S}}\mathrm{k}_j(P)| \\ = \sum_{j=i}^{n}(-1)^{i-j}\binom{j}{i}|\mathring{\mathrm{S}}\mathrm{k}_j(P)| \end{aligned}$$

This is an expression for the $2i$-th Betti number of X thanks to Theorem 2.21 and the vanishings of Lemma 2.17 and Corollary 3.2. □

3.2 SKELETAL POLYTOPE COMPLEXES

In this section we describe the cohomology of a broad class of broken toric varieties, namely those X whose polytope complexes comprise the n-skeleton of a higher dimensional polytope P'. The idea is that the sheaves $R^i f_* \underline{\mathbb{Q}}_X$ which come into play here are subsheaves of the cell-compatible sheaves $\mathcal{S}^i_{P'}(\mathring{\mathrm{Sk}}_{\dim(P')-1}(P'))$ (which we understand by Section 3.1) and we can use this relationship to extract information about the former.

Definition 3.4. We say that a polytope complex $P_\bullet$ of dimension n is *skeletal* if there exists a polytope P' such that $P_\bullet = \mathrm{Sk}_n(P')$.

To get a feel for this definition, note that the polytope complex in Example 2.14 is skeletal for P' being the 2-simplex, while the polytope complex in Example 2.15 is not skeletal.

Proposition 3.5. *If X is a broken toric variety of dimension n with skeletal polytope complex $P_\bullet$ and P' is a polytope of dimension n' such that $P_\bullet = Sk_n(P')$, then*

$$h^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_{P_\bullet}) = \begin{cases} h^j(P', R^i f_* \underline{\mathbb{Q}}_{X(P')}), & i \le j < n \\ \sum_{l=n}^{n'} (-1)^{n+l} h^l(P', R^i f_* \underline{\mathbb{Q}}_{X(P')}) \\ \qquad + \sum_{l=n+1}^{n'} (-1)^{n+l+1} \binom{l}{i} |\mathring{Sk}_l(P')|, & i \le j = n \\ 0, & \textit{otherwise} \end{cases}$$

In particular, if P' is simple, then

$$h^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_{P_\bullet}) = \begin{cases} h^i(P', R^i f_* \underline{\mathbb{Q}}_{X(P')}), & i = j < n \\ \sum_{l=n+1}^{n'} (-1)^{n+l+1} \binom{l}{i} |\mathring{Sk}_l(P')|, & i < j = n \\ h^n(P', R^n f_* \underline{\mathbb{Q}}_{X(P')}) + \sum_{l=n+1}^{n'} (-1)^{n+l+1} \binom{l}{i} |\mathring{Sk}_l(P')|, & i = j = n \\ 0, & \textit{otherwise} \end{cases}$$

Proof. This result follows from inductively applying Lemma 2.16 first to $\mathcal{F} = R^i f_* \underline{\mathbb{Q}}_{P'}$ for $k = \dim(P')$, then to $\mathcal{F} = R^i f_* \underline{\mathbb{Q}}_{\mathrm{Sk}_{\dim(P')-1}(P')}$ for $k = \dim(P') - 1$, etc. In the first step, we find the desired formula by using the vanishings given by Lemma 2.17. For P' simple, we apply Corollary 3.2 to get the second part of the proposition. □

The second part of this proposition allows for an interested party to write out an explicit formula for the Betti numbers of any broken toric variety whose polytope complex is the n-skeleton of a higher dimensional simple polytope. We will content ourselves by writing this out for the case $\dim(P') = n+1$.

Corollary 3.6. *If X is a broken toric variety of dimension n with skeletal polytope complex $P_\bullet = Sk_n(P')$ for P' a simple polytope of dimension $n+1$, then*

$$h^j(P_\bullet, R^i f_* \underline{\mathbb{Q}}_{P_\bullet}) = \begin{cases} h^j(P', R^i f_* \underline{\mathbb{Q}}_{X(P')}), & i = j < n \\ \binom{n+1}{i}, & i < j = n \\ |\mathring{Sk}_n(P_\bullet)|, & i = j = n \\ 0, & \text{otherwise} \end{cases}$$

and

$$h^j(X, \underline{\mathbb{Q}}_X) = \begin{cases} (-1)^{n+1-2i}\binom{n+1}{i} + \sum_{k=i}^{n}(-1)^{i-k}\binom{k}{i}|\mathring{Sk}_k(P)|, & j = 2i < n \\ 0, & j = 2i+1 < n \\ \binom{n+1}{2i+1-n}, & n \le j = 2i+1 \le 2n \\ \binom{n+1}{2i-n} + (-1)^{n+1-2i}\binom{n+1}{i} + \sum_{k=i}^{n}(-1)^{i-k}\binom{k}{i}|\mathring{Sk}_k(P)|, & n \le j = 2i \le 2n \end{cases}$$

3.3 CELLULAR SHEAVES

One final way to approach the problem of calculating the cohomology of cell-compatible sheaves is to view them as cellular sheaves. Let us take a brief look the definition and their cohomology, following [Ghr14, Cur14]. First, just a bit of notation: for a CW complex P and $\sigma \in \mathring{\mathrm{Sk}}_k(P)$, $\tau \in \mathring{\mathrm{Sk}}_{k+l}(P)$, let us write $\sigma \le_l \tau$ if $\sigma \in \mathring{\mathrm{Sk}}_k(\tau)$.

Definition 3.7. A *cellular sheaf* (of vector spaces) $\mathcal{F}$ on a CW complex P is

1. the assignment of a vector space $\mathcal{F}(\sigma)$ to each cell σ of P
2. a linear map $\rho_{\sigma,\tau} : \mathcal{F}(\sigma) \to \mathcal{F}(\tau)$ to every pair of cells $\sigma \le_l \tau$. These *restriction* maps are required to satisfy
   - $\rho_{\sigma,\sigma} = \mathrm{id}_{\mathcal{F}(\sigma)}$
   - If $\sigma \le_1 \tau \le_1 \kappa$, then $\rho_{\sigma,\kappa} = \rho_{\sigma,\tau} \circ \rho_{\tau,\kappa}$.

Now we can define the cohomology of a cellular sheaf. For a cellular sheaf $\mathcal{F}$ on P, define the k-th cochain space of $\mathcal{F}$ to be

$$C^k(P, \mathcal{F}) = \bigoplus_{\sigma \in \mathring{\mathrm{Sk}}_k(P)} \mathcal{F}(\sigma).$$

To define coboundary maps, we first need to choose an orientation for each cell in our complex. Then for every pair of cells $\sigma \le_1 \tau$, we get a number

$[\sigma : \tau] = \pm 1$, which is equal to 1 if the orientations of σ as a subcomplex of τ matches the orientation of σ and -1 if not. The k-th coboundary map $\delta^k : C^k(P, \mathcal{F}) \to C^{k+1}(P, \mathcal{F})$ is then defined as the sum over $\sigma \in \mathring{\mathrm{S}}\mathrm{k}_k(P)$ of the maps $\delta^k_\sigma : \mathcal{F}(\sigma) \subseteq C^k(P, \mathcal{F}) \to C^{k+1}(P, \mathcal{F})$ defined by

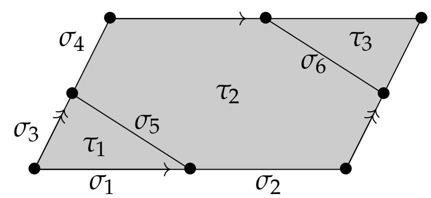
$$\delta^k_\sigma(v) = \sum_{\sigma \leq_1 \tau} [\sigma : \tau] \rho_{\sigma,\tau}(v).$$

Then the k-th cellular cohomology group of $\mathcal{F}$ on P is

$$H^k(P, \mathcal{F}) = \frac{\ker \delta^k}{\operatorname{im} \delta^{k-1}}.$$

This down-to-earth method is useful for calculation when the polytope complexes involved are simple enough.

Example 3.8. Let $\mathcal{F}$ be the cellular sheaf on T^2 with cell complex structure as pictured here:



Letting (e_1, e_2) be the standard basis vectors, say that the 0-cells are all assigned the zero vector space, σ_1 and σ_2 are assigned $\langle e_1 \rangle$, σ_3 and σ_4 are assigned $\langle e_2 \rangle$, σ_5 and σ_6 are assigned $\langle e_1 - e_2 \rangle$, τ_i are all assigned $\langle e_1, e_2 \rangle$, and the restriction maps are all inclusions. Finally, let us orient σ_1 and σ_2 towards the right, σ_3 and σ_4 upwards, σ_5 and σ_6 pointing up and to the left, and each 2-cell with the counterclockwise orientation. The cochain complex of this cellular (and cell-compatible) sheaf is

$$0 \to \langle e_1 \rangle^{\oplus 2} \oplus \langle e_2 \rangle^{\oplus 2} \oplus \langle e_2 - e_1 \rangle^{\oplus 2} \to \langle e_1, e_2 \rangle^{\oplus 3} \to 0.$$

So the only nontrivial coboundary morphism is δ^1. We can calculate, for example, that

$$\begin{aligned} \delta^k(1,0,0,0,0,0) &= \delta^k_{\sigma_1}(1) \\ &= \sum_{\sigma_1 \leq_1 \tau} [\sigma_1 : \tau] \rho_{\sigma_1,\tau}(1) \\ &= [\sigma_1, \tau_1] \rho_{\sigma_1,\tau_1}(1) + [\sigma_1, \tau_2] \rho_{\sigma_1,\tau_2}(1) \\ &= (1, 0, -1, 0, 0, 0) \in \langle e_1, e_2 \rangle^{\oplus 3} \end{aligned}$$

Altogether δ^k can be represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

which has rank 4. Accordingly the cellular cohomology groups of $\mathcal{F}$ on T^2 are

$$H^k(T^2, \mathcal{F}) = \begin{cases} \mathbb{Q}^{\oplus 2}, & k = 1,2 \\ 0, & \text{otherwise} \end{cases}$$

One can also think of cellular sheaves on P as usual sheaves on P equipped with the *Alexandrov topology* (see, for example, [Cur14]). In this context, cellular sheaf cohomology is nothing but the Čech cohomology of the sheaf.

Cell-compatible sheaves have nice interpretations from this point of view—they are in one-to-one correspondence with cellular sheaves where all the restriction maps are inclusions[2]. Moreover, the sheaf cohomology of the cell-compatible sheaf is the same as the sheaf cohomology of its associated sheaf in the Alexandrov topology. The sheaf cohomology of a cell-compatible sheaf then matches with the cellular cohomology of its associated cellular sheaf by Proposition A.15 of [Rus22].

Using these ideas, we can now provide the postponed proof to part (1) of Lemma 2.30.

Proof of Lemma 2.30 (1). First note that $R^1 b_* \underline{\mathbb{Q}}_X$ is supported exactly on $Y = f^{-1}(P^{\cap}_{\bullet})$ with stalks $(R^1 b_* \underline{\mathbb{Q}}_X)_y = (R^1 f_* \underline{\mathbb{Q}}_X)_{f(y)} / (R^1 f_* \underline{\mathbb{Q}}_{X^{\cup}})_{f(y)}$. In particular all the stalks are 1-dimensional.

Denote by $\mathcal{F}$ and $\mathcal{G}$ the associated cellular sheaves of $R^1 b_* \underline{\mathbb{Q}}_X$ and $\underline{\mathbb{Q}}_{X^{\cap}}$ respectively. It is easier to check that we can define an isomorphism of sheaves here than in the classical case. The restriction maps of $\mathcal{G}$ are the identity by definition and those of $\mathcal{F}$ take the unique generator of $\mathcal{F}(\sigma)$ to that of $\mathcal{F}(\tau)$. This follows from the definition of $R^1 b_* \underline{\mathbb{Q}}_X$ as the sheafification of the sheaf which takes an open set U to $H^1(b^{-1}(U), \underline{\mathbb{Q}}_X)$. The map $\phi : \mathcal{F}(\sigma) \to \mathcal{G}(\sigma)$ which sends generator to generator is then seen to define a cellular sheaf morphism since it commutes with restriction. This morphism is an isomorphism of cellular sheaves since it is on each cell.

2 Some cellular sheaves which are not necessarily cell-compatible have also made an appearance in this thesis, namely the higher derived pushforwards of the constant sheaf on a broken toric variety along a balloon animal map as seen in Section 2.5.

Since $\mathcal{F}$ and $\mathcal{G}$ are isomorphic, so are their associated sheaves $R^1 b_* \underline{\mathbb{Q}}_X$ and $\underline{\mathbb{Q}}_{X^\cap}$.

□

4

HYPERTORIC HITCHIN SYSTEMS

4.1 PRELIMINARIES

4.1.1 *Higgs Bundles and Their Relationship to Hypertoric Hitchin Systems*

Before recalling the definition of the hypertoric Hitchin system $\mathfrak{D}(\Gamma)$ associated to a graph Γ in the next section, let us start here with some motivation. We will review some of the geometry of the moduli space of Higgs bundles and explain the relevance to it of the hypertoric Hitchin systems.

Definition 4.1. Let X be a compact Riemann surface of genus $g \geq 2$. A *Higgs bundle* on X consists of a holomorphic vector bundle E on X along with a twisted endomorphism $\phi : E \to E \otimes K_X$. A Higgs bundle is *stable* if $\frac{\deg F}{\mathrm{rk} F} < \frac{\deg E}{\mathrm{rk} E}$ for all sub-bundles $F \subset E$ for which $\phi(F) \subseteq F \otimes K_X$.

The moduli space of rank r, degree d stable Higgs bundles on X, $\mathcal{M}_X(r,d)$ is called a *Hitchin system*. These moduli spaces have more fascinating structures than one can shake a stick at: they are hyperkähler completely integrable systems which come equipped with a proper morphism, the *Hitchin map* h, to an affine space (the space of spectral curves). A generic fibre of this morphism is an abelian variety. Through the non-Abelian Hodge theorem there is a one-to-one correspondence between stable Higgs bundles on X, flat connections on X, and GL_r-representations of the fundamental group of X. These three moduli spaces are sometimes called the *Dolbeault*, *de Rham*, and *Betti* moduli spaces, respectively.

Importantly for our purposes, the BNR Correspondence [BNR89] states that for a smooth spectral curve Σ, the fibre $h^{-1}(\Sigma)$ is the Jacobian variety of Σ. This also works for nodal spectral curves, where the fibre is a fine compactified Jacobian[1]. Further, it follows from a proposition of Ngô (Proposition (7.5.1) in [Ng0]) that

$$H^\bullet(\overline{\mathrm{Jac}(\Sigma)}) \cong H^\bullet(\mathrm{Jac}(\tilde{\Sigma})) \otimes D(\Sigma),$$

1 *a priori* there is more than one way to compactify a Jacobian, depending on a stability condition, but the cohomology of the compactification is independent of the chosen (generic) stability condition by [MSV21].

where $\tilde{\Sigma}$ is the normalization of X and $D(\Gamma_\Sigma)$ is some vector space depending only on the dual graph[2] Γ_Σ of Σ.

In the case that all of the components of Σ are rational, $H^\bullet(\overline{\mathrm{Jac}(X)}) \cong D(\Sigma)$. We can alternatively describe the compactified Jacobian as a broken toric variety. The Jacobian $\mathrm{Jac}(X) \cong (\mathbb{C}^*)^d$ (where d is the number of components of X) acts on $\overline{\mathrm{Jac}(X)}$ by tensoring and as a subvariety it is dense by construction. Taking the quotient by the compact part $U(1)^d$ of $(\mathbb{C}^*)^d$ yields a periodic hyperplane arrangement as described in [OS79]. Further, this is the same hyperplane arrangement governing the geometric structure of the central fibre of $\mathfrak{D}(\Gamma_\Sigma)$, as we will describe in Subsection 4.2.4. In particular, this shows that in the case of all rational components, $H^\bullet(\overline{\mathrm{Jac}(X)}) \cong H^\bullet(\mathfrak{D}(\Gamma_\Sigma))$. It is true for any nodal spectral curve that $D(\Gamma_\Sigma) \cong H^\bullet(\mathfrak{D}(\Gamma_\Sigma))$. This is what we mean when we say that the hypertoric Hitchin systems are "combinatorial toy models" for the cohomology around singular fibres of the Hitchin system.

4.1.2 *Additive Hypertoric Varieties*

In [BD00], Bielawski and Dancer introduce a hyperkähler analogue of toric varieties which are now called *additive hypertoric varieties* to distinguish from the multiplicative version introduced later. Here is a quick overview of their construction which can serve as a blueprint for the multiplicative case.

Start with the most basic possible hyperkähler manifold $T^*\mathbb{C}$ and let the real torus $T^n = \{(\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n : \|\zeta_i\| = 1\}$ act on $(T^*\mathbb{C})^n \cong T^*\mathbb{C}^n$ by $\zeta(z, w) = (\zeta z, \zeta^{-1} w)$. Now choose a connected subtorus K of T^n and denote by D the quotient torus T^n/K. Then we have short exact sequences of the corresponding Lie algebras and their duals

$$0 \to \mathfrak{k} \to \mathfrak{t} \to \mathfrak{d} \to 0$$

and

$$0 \to \mathfrak{d}^* \to \mathfrak{t}^* \to \mathfrak{k}^* \to 0.$$

The action of K on $T^*\mathbb{C}^n$ induced by the above action of T^n has a hyperkähler moment map

$$\mu = \mu_\mathbb{R} + \mu_\mathbb{C} : T^*\mathbb{C}^n \to \mathfrak{k}^* \times \mathfrak{k}^*_\mathbb{C}$$

2 The *dual graph* of a nodal curve Σ consists of a vertex for every component of Σ, and an edge between two vertices for every node between their corresponding components.

which allows us to take a hyperkähler quotient and thus define the *additive hypertoric variety* associated to $K \hookrightarrow T^n$ and $(\alpha, \beta) \in \mathfrak{k}^* \times \mathfrak{k}^*_{\mathbb{C}}$ to be

$$\mathfrak{M}_{(\alpha,\beta)}(K \hookrightarrow T^n) = T^*\mathbb{C}^n \;/\!/\!/_{(\alpha,\beta)}\; K = \mu^{-1}(\alpha, \beta)/K.$$

One can read about the geometry of such spaces in, for example, [Kon00].

4.2 CONSTRUCTION AND EXAMPLES

We will mimic the above additive construction but with the basic space $T^*\mathbb{C}$ replaced with something else, leading to a "multiplicative" theory. This section and the next attempt to present a somewhat contained presentation of these ideas, following [PH06, MW18, DMS19].

4.2.1 *The Base Space*

Let us begin by defining the space which we will be using as our basic building block. For all $m \in \mathbb{Z}$, let $X_m = \{(x_m, y_m) : x_m, y_m \in \mathbb{C}\} \cong \mathbb{C}^2$ and consider the isomorphism

$$f_m : X_m \setminus \{x_m = 0\} \to X_{m+1} \setminus \{y_{m+1} = 0\}$$

given by

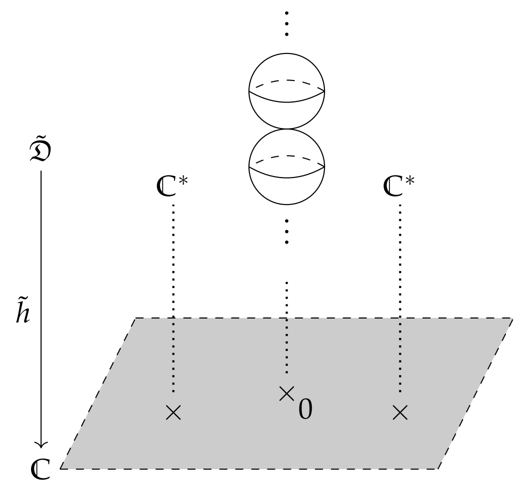
$$(x_m, y_m) \mapsto (x_m^2 y_m, x_m^{-1}).$$

With these maps, we can define

$$\tilde{\mathfrak{D}} = \left(\bigsqcup_{m \in \mathbb{Z}} X_m \right) / \sim,$$

where the copies of $\mathbb{C}^2$ are glued to each other by the maps f_m. Put differently, this gluing identifies (x_m, y_m) and (x_{m+1}, y_{m+1}) if $x_m = y_{m+1}^{-1}$ and $x_m y_m = x_{m+1} y_{m+1}$. More generally,$(x_m, y_m)$ and (x_{m+k}, y_{m+k}) are equivalent if $x_{m+k} = (x_m y_m)^k x_m$ and $y_{m+k} = (x_m y_m)^{-k} y_m$. The resulting space is a complex surface which inherits a symplectic form by gluing together the pullbacks $f^*\omega_{m+1} = \omega_m$ of the standard forms $\omega_m = dx_m \wedge dy_m$ on X_m.

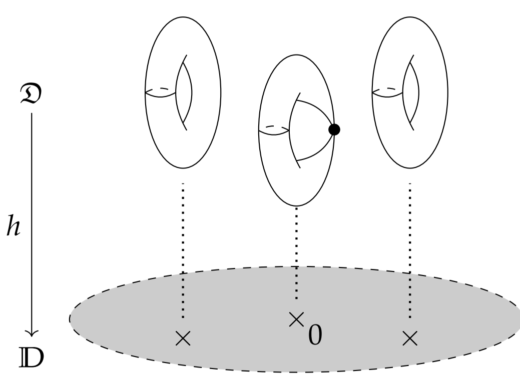
Importantly, there is also a proper morphism $\tilde{h} : \tilde{\mathfrak{D}} \to \mathbb{C}$ given by $\overline{(x_m, y_m)} \mapsto x_m y_m$, which is well-defined by our second description of the gluing. The generic fibre of h is $\{\overline{(x_m, y_m)} \in \tilde{\mathfrak{D}} : x_m y_m = c \neq 0\} \cong \mathbb{C}^*$ while the fibre over 0 is an infinite chain of copies of $\mathbb{P}^1$. This can be seen by first considering the preimage of 0 in $\bigsqcup_{m \in \mathbb{Z}} X_m$: The preimage of 0 in each copy of X_m is the union of $\mathbb{C} = \{(x_m, 0)\}$ and $\mathbb{C} = \{(0, y_m)\}$ intersecting at the origin. Our equivalence then glues $\{(x_m, 0)\}$ to $\{(0, y_{m+1})\}$ via $x_m = y_{m+1}^{-1}$ for all $m \in \mathbb{Z}$, which yields an infinite number of copies of $\mathbb{P}^1$, each one glued to the next at a single point.



There is a also a natural action of $\mathbb{Z}$ on $\tilde{\mathfrak{D}}$ generated by the shift $X_m \to X_{m+1}$, $1 : \overline{(x_m, y_m)} \mapsto \overline{(x_m, y_m)}$. From this description, a point of $\tilde{\mathfrak{D}}$ is fixed by k if $\overline{(x_m, y_m)} = \overline{((x_m y_m)^k x_m, (x_m y_m)^{-k} y_m)}$, i.e. if it lies over a k-th root of unity. In particular, this tells us that the action of $\mathbb{Z}$ on $\tilde{\mathfrak{D}}$ is free as long as we restrict to points which are mapped by $\tilde{h}$ to the unit disc $\mathbb{D} \subset \mathbb{C}$. Accordingly, we define

$$\mathfrak{D} = \tilde{h}^{-1}(\mathbb{D})/\mathbb{Z}.$$

It is from this space (the *Tate curve*) that all other hypertoric Hitchin systems are defined. Clearly $\tilde{h}$ descends to a proper map $h : \mathfrak{D} \to \mathbb{D}$. The generic fibre of h is $\mathbb{C}^*/\mathbb{Z}$, an elliptic curve, while the fibre over 0 is the nodal elliptic curve, as illustrated by how $\mathbb{Z}$ acts on the chain of $\mathbb{P}^1$'s: it maps one to the next, and so in the quotient they are all identified.



4.2.2 *A Quasi-Hyperkähler Reduction*

As in the additive case, there is an action of $U(1)$ on $\tilde{\mathfrak{D}}$ given by $\zeta\overline{(x,y)} = \overline{(\zeta x, \zeta^{-1}y)}$ with moment map $\tilde{\mu} : \tilde{\mathfrak{D}} \to \mathbb{R}$. This action descends to an action of $U(1)$ on $\mathfrak{D}$ with a group-valued moment map $\mu^{\mathfrak{D}}_{U(1)} : \mathfrak{D} \to \mathbb{R}/\mathbb{Z}$.

So, given a short exact sequence of tori

$$1 \to K \to T^n \to D \to 1,$$

the action of T^n on $\mathfrak{D}^n$ induces an action of K with a quasi-hyperkähler moment map

$$\mu : \mathfrak{D}^n \to K^* \times \mathfrak{k}^*_{\mathbb{C}}.$$

That is, the action is Hamiltonian for two of the Kähler forms but quasi-hamiltonian with respect to the third.

There is a general theory of group-valued moment maps (see for example [AMM98]), but it is enough for us to simply define a quasi-hyperkähler reduction in effectively the same way as the hyperkähler case. That is, the *multiplicative hypertoric Dolbeault space* or *hypertoric Hitchin system* associated to $K \hookrightarrow T^n$ and $\alpha \in K^*$ is defined to be

$$\mathfrak{D}_\alpha(K \hookrightarrow T^n) = \mathfrak{D}^n \,/\!/\!/_{(\alpha,0)}\, K = \mu^{-1}(\alpha, 0)/K.$$

4.2.3 *Cographical Hypertoric Hitchin Systems*

The hypertoric Hitchin systems we are interested in correspond to embeddings $K \hookrightarrow T^n$ arising from graphs. Such hypertoric Hitchin systems are called *cographical*, and they include many interesting examples. In particular, they are exactly the spaces that model neighbourhoods of certain fibres of the moduli space of Higgs bundles.

Let Γ be a graph with no bridges or tails[3], and choose an orientation. This is required for the construction of the (co)homology groups of the graph, but all resulting spaces are independent of the orientation.

From such an oriented graph and a group G, let $V(\Gamma)$ and $E(\Gamma)$ denote the vertex and edge sets, respectively. Viewing these sets as the 0- and 1-simplices of the CW complex Γ, we obtain chain complexes

$$0 \to C_1(\Gamma, G) = G^{E(\Gamma)} \xrightarrow{d_\Gamma} C_0(\Gamma, G) = G^{V(\Gamma)} \to 0$$

3 Bridges or tails in a graph Γ do not contribute any information involved in the construction of $\mathfrak{D}(\Gamma)$ and so can be safely ignored. To be precise, if e is a bridge or tail of Γ, then $\mathfrak{D}(\Gamma) = \mathfrak{D}(\Gamma/e)$.

and

$$0 \to C^0(\Gamma, G) = G^{V(\Gamma)} \xrightarrow{d_\Gamma^*} C^1(\Gamma, G) = G^{E(\Gamma)} \to 0.$$

We sometimes denote the dimension of the first homology group of Γ as $b_1(\Gamma)$, standing for the *first Betti number* of Γ.

Taking (co)homology with $U(1)$ coefficients, we find dual short exact sequences of tori

$$1 \to H_1(\Gamma, U(1)) \to C_1(\Gamma, U(1)) \to \mathrm{im} d_\Gamma \to 1$$

and

$$1 \to \mathrm{im} d_\Gamma^* \to C^1(\Gamma, U(1)) \to H^1(\Gamma, U(1)) \to 1.$$

Using this second sequence (setting $K = \mathrm{im} d_\Gamma^*$, etc.), we can define the hypertoric Hitchin system associated to the graph Γ and $\alpha \in \mathrm{im} d_\Gamma$ to be

$$\mathfrak{D}(\Gamma) = \mathfrak{D}^{E(\Gamma)} /\!/\!/_{(\alpha,0)} \mathrm{im} d_\Gamma^*.$$

Given Γ and $A \in U(1)^{E(\Gamma)}$, set $S(A)$ to be the set of edges for which A_e is zero. We say that α is *generic* if for all $A \in d_\Gamma^{-1}(\alpha)$, the graph $\Gamma \setminus S(A)$ is connected. For two stability parameters generic in this sense, the spaces are diffeomorphic, which is why it has been dropped from the notation. From here on, "hypertoric Hitchin system" will be taken to mean a cographical hypertoric Hitchin system with α chosen generically.

The moment map $\mu : \mathfrak{D}^{E(\Gamma)} \to K^* \times \mathfrak{k}_{\mathbb{C}}^* \subset U(1)^{V(\Gamma)} \times \mathbb{C}^{V(\Gamma)}$ can be described as a composition of the moment map on $\mathfrak{D}$ with d_Γ:

$$\mathfrak{D}^{E(\Gamma)} \xrightarrow{(\mu^{\mathfrak{D}}_{U(1)} \times h)^{E(\Gamma)}} (U(1) \times \mathbb{C})^{E(\Gamma)} = C_1(\Gamma, U(1) \times \mathbb{C}) \xrightarrow{d_\Gamma} C_0(\Gamma, U(1) \times \mathbb{C}) = (U(1) \times \mathbb{C})^{V(\Gamma)}.$$

From this, one can describe the inverse image of $(\alpha, 0)$: a point

$$(\overline{(x_m^1, y_m^1)}, \ldots, \overline{(x_m^{|E(\Gamma)|}, y_m^{|E(\Gamma)|})}) \in \mathfrak{D}^{E(\Gamma)}$$

lies in $\mu^{-1}(\alpha, 0)$ if

$$\sum_{\text{edges } e \text{ entering } v} x_m^e y_m^e - \sum_{\text{edges } e \text{ exiting } v} x_m^e y_m^e = 0$$

and

$$\prod_{\text{edges } e \text{ entering } v} \mu^{\mathfrak{D}}_{U(1)}(x_m^e, y_m^e) \prod_{\text{edges } e \text{ exiting } v} \mu^{\mathfrak{D}}_{U(1)}(x_m^e, y_m^e)^{-1} = \eta_v$$

for all vertices $v \in V(\Gamma)$.

4.2.4 *Properties*

$\mathfrak{D}(\Gamma)$ is a smooth manifold of complex dimension $2b_1(\Gamma)$. It comes equipped with a proper morphism

$$h_\Gamma : \mathfrak{D}(\Gamma) \to \mathbb{D}^{b_1(\Gamma)},$$

the hypertoric analogue of the Hitchin map, which endows it with the structure of an integrable system. As with the usual Hitchin system, there is a deformation retract of $\mathfrak{D}(\Gamma)$ to its central fibre. In particular this implies that the inclusion $h_\Gamma^{-1}(0) \hookrightarrow \mathfrak{D}(\Gamma)$ induces an isomorphism on cohomology. We denote the constant sheaf with rational coefficients on $\mathfrak{D}(\Gamma)$ by $\underline{\mathbb{Q}}_{\mathfrak{D}(\Gamma)}$ and on $h^{-1}(0)_\Gamma$ by $\underline{\mathbb{Q}}_\Gamma$.

As a seeming aside, let us describe how a graph Γ gives rise to a periodic arrangement of hyperplanes in a vector space. There is a natural pairing

$$H^1(\Gamma, \mathbb{R}) \times H_1(\Gamma, \mathbb{R}) \to \mathbb{R}$$

so the class $[e] \in H^1(\Gamma, \mathbb{R})$ of an edge $e \in E(\Gamma)$ defines a linear form

$$[e] : H_1(\Gamma, \mathbb{R}) \to \mathbb{R}.$$

The kernel of this map is a hyperplane H_e in $H_1(\Gamma, \mathbb{R})$ so we can find an arrangement of hyperplanes by considering all $e \in E(\Gamma)$. We can pass to a periodic arrangement of hyperplanes in $H_1(\Gamma, \mathbb{R})$ by adding in the hyperplanes associated to $\langle e, \eta \rangle = n$ for $n \in \mathbb{Z}$, then to an arrangement of hyperplanes in $H_1(\Gamma, \mathbb{R})/H_1(\Gamma, \mathbb{Z})$ by simply taking the quotient.

We would like this arrangement to have the property that any k hyperplanes intersect in dimension k. This can be ensured by defining the hyperplanes not by $\langle e, \eta \rangle = 0$ but by a "shifted" condition depending on a generic $\alpha \in \mathrm{im} d_\Gamma$. Let $\alpha^* \in K \subset C^0(\Gamma, U(1))$ be the dual of α and let $\tilde{\alpha} = (\tilde{\alpha}_{e_1}, \ldots, \tilde{\alpha}_{e_{E(\Gamma)}})$ be the image of α^* under d_Γ^*. Finally, we can define the hyperplane h_e in $H_1(\Gamma, \mathbb{R})$ associated to e by $\langle e, \eta \rangle = \tilde{\alpha}_e$. In other words, we have associated to each edge of Γ an element of $U(1)$ and then taken each central hyperplane H_e, shifted it by that element, and called the resulting hyperplane h_e. See the next section for some examples.

The relevance of this construction is evident from the following theorem of McBreen–Webster.

Theorem 4.2. *([MW18]) The central fibre $h_\Gamma^{-1}(0)$ of $\mathfrak{D}(\Gamma)$ is a broken toric variety whose polytope complex has the topological type of real torus of dimension $b_1(\Gamma)$. Further, the polytopal decomposition on the torus which gives the polytope complex of $h_\Gamma^{-1}(0)$ is described by the periodic hyperplane arrangement of* Γ.

The idea of the proof is to start with the observation that $\tilde{h}_\Gamma^{-1}(0)$ is a symplectic reduction of $\tilde{h}^{-1}(0)^{|E(\Gamma)|}$, which is an infinite grid of copies of $\prod_{e\in E(\Gamma)} \mathbb{P}^1$. The reduction is then a quotient of each component of $\tilde{h}^{-1}(0)$ which lies in the fibre of the moment map over α. The quotient of a toric variety by a torus action is toric, so the result is a periodic, non-compact broken toric variety whose polytope complex is also periodic, with the topological type of $\mathbb{R}^{b_1(\Gamma)}$. Taking the quotient by $\mathbb{Z}^{b_1(\Gamma)}$ then recovers $h_\Gamma^{-1}(0)$.

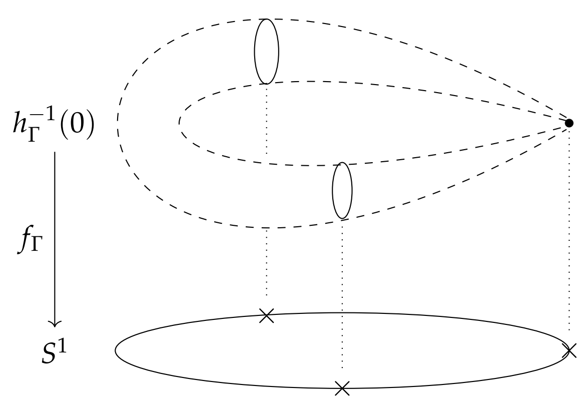
4.2.5 *Examples*

Example 4.3. Let $\Gamma =$ ◓. According to the construction of the previous section, we define

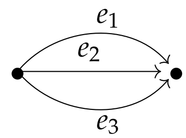
$$\mathfrak{D}(\Gamma) = \mathfrak{D} /\!/\!/_{(\alpha,0)} \operatorname{im} d_\Gamma^*.$$

In this case, d_Γ^* is the zero map and every $(x,y) \in \mathfrak{D}$ lies in $\mu^{-1}(\alpha,0)$. Thus $\mathfrak{D}(\Gamma) = \mathfrak{D}$.

We can also verify that the central fibre of $\mathfrak{D}$ is what we expect it to be. Recall that by construction, $h_\Gamma^{-1}(0)$ should be a pinched torus. On the other hand, the periodic hyperplane arrangement associated to $\Gamma =$ ◓ is $\mathbb{Z} \subset \mathbb{R}$, the quotient of which by $H^1(\Gamma,\mathbb{Z}) = \mathbb{Z}$ gives the hyperplane arrangement of a single point on S^1. The fibre of the map $f : h_\Gamma^{-1}(0) \to S^1$ is a copy of S^1 away from this hyperplane and a point over the hyperplane.



Example 4.4. Let $\Gamma =$  . For this example, let us go carefully through the construction of the periodic hyperplane arrangement of Γ. We need to choose an orientation for the graph:



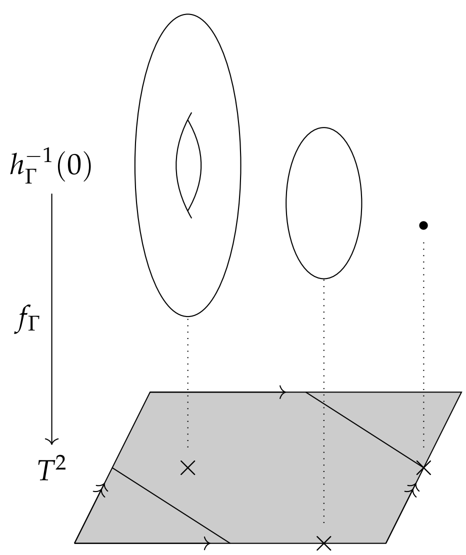
Then we can write $H_1(\Gamma, \mathbb{R}) = \langle [e_1 - e_2], [e_2 - e_3] \rangle$ and we have linear forms $[e_i] : H_1(\Gamma, \mathbb{R}) \to \mathbb{R}$. Given $a, b \in \mathbb{R}$, these forms look like

$$[e_1](a[e_1 - e_2], b[e_2 - e_3]) = a$$

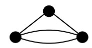
$$[e_2](a[e_1 - e_2], b[e_2 - e_3]) = -a + b$$

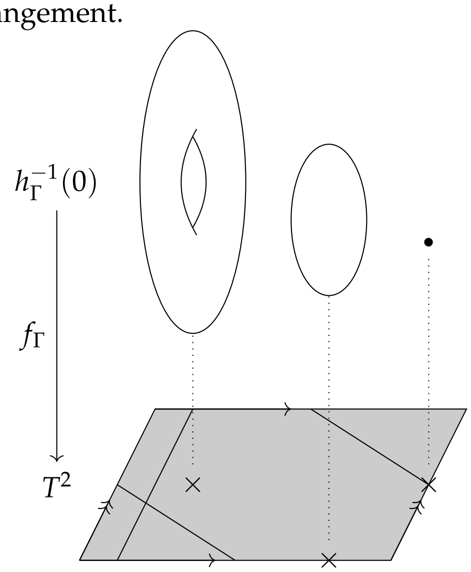
$$[e_3](a[e_1 - e_2], b[e_2 - e_3]) = b$$

and thus the arrangement of central hyperplanes they define in $H_1(\Gamma, \mathbb{R})$ is given by the x and y axes along with the line $-x + y = 0$. However, this arrangement is not stable (there are 3 hyperplanes meeting at 0) so we shift the hyperplane H_{e_2} by some $\tilde{\alpha}_{e_2} \in S^1$. We then pass to a periodic arrangement and then to an arrangement on $T^2 = H_1(\Gamma, \mathbb{R}) / H_1(\Gamma, \mathbb{Z})$. This is the base in the following picture, which also visualizes some of the fibres of the map $f : h_\Gamma^{-1}(0) \to T^2$.

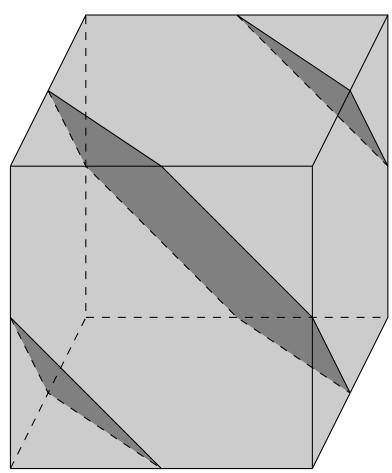


The irreducible components of the broken toric variety $h_\Gamma^{-1}(0)$ are two copies of $\mathbb{P}^2$ (the toric varieties of the two triangular cells in the arrangement) along with a copy of $\mathbb{P}^2$ which is blown up at two points (the toric variety of the hexagon).

Example 4.5. Let $\Gamma =$  . This example is nearly the same as the previous, but with the edge e_1 divided into two edges (e_1' and e_1''). They both define the same central hyperplane, so one must be shifted in order to get a stable arrangement.

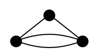


Example 4.6. Let $\Gamma =$  . This gives a higher-dimensional example since $b_1(\Gamma) = 3$, the central hyperplanes of which are given by the planes defined by $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 0$. Pictured below is a cube representing T^3 with the faces of the cube and the interior diagonal plane representing the shifted periodic hyperplane arrangement. Analogously to the previous examples, the fibre of f_Γ over a point in $\mathring{\mathrm{Sk}}_k$ is T^k. Further, it appears that the broken toric variety $h_\Gamma^{-1}(0)$ has four irreducible components (see Lemma 4.17).



4.3 DELETION-CONTRACTION

Deletion-contraction relations are often useful when graphs are concerned, as evidenced for example by the number of spanning trees of a graph or its chromatic polynomial (both of which are really specializations of the Tutte polynomial of a graph, which is universal among multiplicative graph invariants that exhibit deletion-contraction relationships). Hypertoric Hitchin systems are no exception to this rule. Recall that for a graph Γ and a non-loop edge e, the *deletion* $\Gamma \setminus e$ is the graph which has the same vertices of Γ and all the same edges, sans e. The *contraction* Γ/e is the graph obtained by replacing the two vertices $\{v_1, v_2\}$ to which e is incident with a single vertex, with any edge that was incident to v_1 or v_2 now incident to the new vertex (and so an edge incident to both v_1 and v_2 becomes a loop in the contraction graph).

Example 4.7. Let $\Gamma =$  and e be one of the edges adjacent to the unique vertex of degree two. Then $\Gamma/e =$  and $\Gamma \setminus e =$  .

Example 4.8. Let Γ be a connected graph, e a non-loop edge of Γ, and $t(\Gamma)$ be the number of spanning trees of Γ. Then

$$t(\Gamma) = t(\Gamma \setminus e) + t(\Gamma/e).$$

This is a prototypical example of a deletion-contraction relationship.

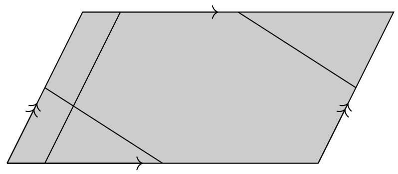
Now we can express some of our earlier machinery involving broken toric varieties in terms of deletion-contraction.

Proposition 4.9. *Let Γ be a graph and e be a non-loop edge of Γ. Associated to the contraction $\Gamma \to \Gamma/e$ is a balloon animal map* (cf. *Section 2.5*) *of broken toric varieties $h^{-1}_{\mathfrak{D}(\Gamma/e)}(0) \to h^{-1}_{\mathfrak{D}(\Gamma)}(0)$, for which the broken toric variety $X^{\cap}$ associated to the intersection of the polytopes in the dissection is $h^{-1}_{\mathfrak{D}(\Gamma\setminus e)}(0)$.*

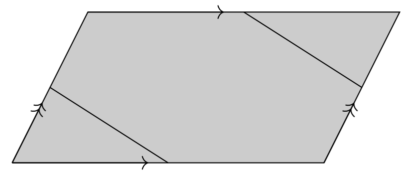
Proof. The passage from Γ/e to Γ at the level of the periodic hyperplane arrangements describing the varieties amounts to the addition of a single hyperplane to the arrangement. Since the hyperplane arrangement is stable, the subdivision of polytope complexes that it induces is generic, and so there exists a balloon animal map between the two broken toric varieties.

The hyperplane arrangement on the hyperplane corresponding to e is equivalently either the polytope complex $P^{\cap}_{\bullet}$ associated to the intersection of the polytopes in the subdivision or the hyperplane arrangement induced by the inclusion of $H_1(\Gamma \setminus e, \mathbb{R}) \subset H_1(\Gamma, \mathbb{R})$. The latter is precisely the hyperplane arrangement associated to $\Gamma \setminus e$ by definition: For any non-bridge, non-tail edge $e' \in \Gamma \setminus e$, the kernel of $[e'] : H_1(\Gamma \setminus e, \mathbb{R}) \to \mathbb{R}$ is the intersection of the kernels of $[e] : H_1(\Gamma, \mathbb{R}) \to \mathbb{R}$ and $[e'] : H_1(\Gamma, \mathbb{R}) \to \mathbb{R}$. □

Example 4.10. Let Γ and e be as in Example 4.7. Pictured below are the polytope complexes of the central fibres of the hypertoric Hitchin systems $\mathfrak{D}(\Gamma)$ and $\mathfrak{D}(\Gamma/e)$. The polytope complex of the central fibre of $\mathfrak{D}(\Gamma \setminus e)$ is the circle with two marked points.



$P_\bullet(h_\Gamma^{-1}(0))$



$P_\bullet(h_{\Gamma/e}^{-1}(0))$

The next theorem is originally due to Dansco–McBreen–Shende. Our investigation of balloon animal maps provides an alternative, more geometric proof.

Theorem 4.11. *([DMS19]) Let Γ be a graph and e be an edge of Γ which is not a loop. Then there is a long exact sequence relating the cohomologies of hypertoric Hitchin systems:*

$$\cdots \to H^k(\mathfrak{D}(\Gamma), \underline{\mathbb{Q}}_{\mathfrak{D}(\Gamma)}) \to H^k(\mathfrak{D}(\Gamma/e), \underline{\mathbb{Q}}_{\mathfrak{D}(\Gamma/e)}) \to H^{k-1}(\mathfrak{D}(\Gamma \setminus e), \underline{\mathbb{Q}}_{\mathfrak{D}(\Gamma \setminus e)}) \to \ldots$$

Proof. Theorem 2.31 and Proposition 4.9 combine to yield the result at the level of central fibres $h^{-1}(0)$, and the inclusion of these fibres into their respective ambient hypertoric Hitchin systems induces an isomorphism on cohomology. □

Theorem 4.12. *Let Γ be a graph and e be an edge of Γ which is not a loop. Then there is a long exact sequence*

$$\cdots \longrightarrow H^{k-1}(T^{n-1}, R^{i-1} f_* \underline{\mathbb{Q}}_{\Gamma \setminus e})$$
$$\longrightarrow H^k(T^n, R^i f_* \underline{\mathbb{Q}}_\Gamma) \longrightarrow H^k(T^n, R^i f_* \underline{\mathbb{Q}}_{\Gamma/e}) \longrightarrow H^k(T^{n-1}, R^{i-1} f_* \underline{\mathbb{Q}}_{\Gamma \setminus e})$$
$$\longrightarrow H^{k+1}(T^n, R^i f_* \underline{\mathbb{Q}}_\Gamma) \longrightarrow \cdots$$

Proof. This is a special case of Theorem 2.33 in the case that the balloon animal maps are between special fibres of hypertoric Hitchin systems and arising from a deletion-contraction relation. □

4.4 COHOMOLOGY

In this section we describe a potpourri of results regarding actual calculations of the cohomology of hypertoric Hitchin systems. We start with a statement that allows us to 'break up' the central fibre $\mathfrak{D}(\Gamma)$ for graphs which exhibit a certain kind of decomposability.

Definition 4.13. A *disconnecting vertex* of a connected graph Γ is a vertex v with the property that Γ can be written as the union of a finite number of connected graphs all joined together at a single common vertex v.

Theorem 4.14. *Let v be a disconnecting vertex of a connected graph Γ so that Γ can be written as the union of a finite number of connected graphs $\Gamma_1, \dots, \Gamma_m$ all joined together at v. Then*

$$h_\Gamma^{-1}(0) \cong \prod_{i=1}^{m} h_{\Gamma_i}^{-1}(0).$$

Proof. This statement follows from the description of the hyperplane arrangements of Γ_i, which describe the polytope complexes of $h_{\Gamma_i}^{-1}(0)$. Each Γ_i defines an arrangement A_i on $\mathbb{R}^{b_1(\Gamma_i)}$, and we claim that the arrangement $\prod_{i=1}^m A_i$ on $\prod_{i=1}^m \mathbb{R}^{b_1(\Gamma_i)}$ is the same as the arrangement A on $\mathbb{R}^{b_1(\Gamma)}$ defined by Γ.

This is true, since given an edge e in Γ_i with associated central hyperplane H'_e, the morphism $[e] : H(\Gamma_j, \mathbb{R}) \to \mathbb{R}$ is zero for any $j \neq i$. That is to say, the central hyperplane H_e in $\mathbb{R}^{b_1(\Gamma)}$ can be written as $H'_e \times \prod_{k=1, k\neq i}^m \mathbb{R}^{b_1(\Gamma_i)}$. □

As a corollary, we can decompose the cohomology of $\mathfrak{D}(\Gamma)$ in such a way as well.

Corollary 4.15. *Let v, Γ, and Γ_i be as above. Then the Poincaré polynomial of $\mathfrak{D}(\Gamma)$ is the product of the Poincaré polynomials of $\mathfrak{D}(\Gamma_i)$, or equivalently*

$$H^k(\mathfrak{D}(\Gamma), \underline{\mathbb{Q}}_{\mathfrak{D}(\Gamma)}) \cong \bigoplus_{k_1+\dots+k_m=k} \left(\bigotimes_{i=1}^{m} H^{k_i}(\mathfrak{D}(\Gamma_i), \underline{\mathbb{Q}}_{\mathfrak{D}(\Gamma_i)}) \right)$$

Next up we calculate the top-degree cohomology of $\mathfrak{D}(\Gamma)$ for any graph. Along the way we will use the following definition.

Definition 4.16. Let Θ_k denote the graph with two vertices and k edges between them.

Lemma 4.17. *For Γ a graph with $b_1(\Gamma) = n$ we have*

$$h^{2n}(\mathfrak{D}(\Gamma), \underline{\mathbb{Q}}_{\mathfrak{D}(\Gamma)}) = h^n(T^n, R^n f_* \underline{\mathbb{Q}}_\Gamma) = |\mathring{S}k_n(P_\bullet(h_\Gamma^{-1}(0)))| = t(\Gamma),$$

where $t(\Gamma)$ is the number of spanning trees of Γ.

Proof. The first equality follows from Theorem 2.21 along with the vanishing of Lemma 2.17 and the second equality from Corollary 2.18. The nontrivial part of this statement is that the number of n-cells in $P_\bullet(h_\Gamma^{-1}(0))$ is equal to the number of spanning trees of Γ.

We will ignore the case of graphs with a disconnecting vertex since the central fibres of the associated hypertoric Hitchin systems can be decomposed by Theorem 4.14. The proof proceeds by induction since any graph Γ with no disconnecting vertices can be reduced to $\Theta_{b_1(\Gamma)}$ via contraction. The statement is true for Θ_k, whose associated (unstable) periodic hyperplane arrangements consist of the sides of the unit cube $[0,1]^{k-1}$ and the hyperplane given by $x_1 + \ldots + x_{k-1} = 0$, the periodic version of which passes through $[0,1]^{k-1}$ $k-2$ times. To get a stable arrangement we shift the final hyperplane slightly, yielding k-many $(k-1)$-cells, which agrees with $t(\Theta_k)$.

The inductive step is as follows. Let Γ be a graph with $b_1(\Gamma) = n$ and choose $e \in E(\Gamma)$. The hyperplane h_e in T^n corresponding to e has on it the arrangement corresponding to $\Gamma \setminus e$. On the other hand, removing this hyperplane leaves us with the arrangement on T^n corresponding to Γ/e. Going back to the arrangement corresponding to Γ by adding back in h_e, we note that each $n-1$ cell on h_e intersects an n-cell of the arrangement corresponding to Γ/e, dividing it into two n-cells. The upshot is that $|\mathring{\mathrm{Sk}}_n(P_\bullet(h_\Gamma^{-1}(0)))| = |\mathring{\mathrm{Sk}}_n(P_\bullet(h_{\Gamma/e}^{-1}(0)))| + |\mathring{\mathrm{Sk}}_n(P_\bullet(h_{\Gamma\setminus e}^{-1}(0)))| = t(\Gamma/e) + t(\Gamma \setminus e) = t(\Gamma)$, where the last equality comes about via the deletion-contraction relationship on the number of spanning trees. □

Another useful tool for computing cohomologies of hypertoric Hitchin systems is the following, which can be thought of as a further refinement of Proposition 4.12.

Theorem 4.18. *Let Γ be a graph with $b_1(\Gamma) = n \geq 2$ and e an edge of Γ which is incident to a vertex of degree 2. Then*

$$H^k(T^n, R^i f_* \underline{\mathbb{Q}}_\Gamma) \cong H^k(T^n, R^i f_* \underline{\mathbb{Q}}_{\Gamma/e}) \oplus H^{k-1}(T^n, R^{i-1} f_* \underline{\mathbb{Q}}_{\Gamma\setminus e}).$$

Proof. Consider the long exact sequence of Proposition 4.12:

$$\cdots \longrightarrow H^{k-1}(T^{n-1}, R^{i-1} f_* \underline{\mathbb{Q}}_{\Gamma\setminus e}) \longrightarrow$$
$$H^k(T^n, R^i f_* \underline{\mathbb{Q}}_\Gamma) \xrightarrow{b_k} H^k(T^n, R^i f_* \underline{\mathbb{Q}}_{\Gamma/e}) \xrightarrow{\eta_k} H^k(T^{n-1}, R^{i-1} f_* \underline{\mathbb{Q}}_{\Gamma\setminus e}) \longrightarrow$$
$$H^{k+1}(T^n, R^i f_* \underline{\mathbb{Q}}_\Gamma) \longrightarrow \cdots$$

The statement of the theorem is that $\eta_k = 0$ for all k which is in turn equivalent to the statement that the map on cohomology b_k induced by the balloon animal map has a section.

To construct such a section, consider the periodic hyperplane arrangements A_Γ and $A_{\Gamma/e}$ associated to Γ and Γ/e respectively, which describe the cell-compatible sheaves $R^i f_* \underline{\mathbb{Q}}_\Gamma$ and $R^i f_* \underline{\mathbb{Q}}_{\Gamma/e}$. The only difference between these arrangements is the presence in A_Γ of the shifted hyperplane h_e which is parallel to $h_{e'}$ where e' is the other edge incident to v. These two hyperplanes are parallel since e and e' lie on all the same cycles in Γ so they define the same central hyperplane. They are shifted by different amounts $\tilde{\alpha}_e, \tilde{\alpha}_{e'}$ to ensure genericity of the arrangement. Taking the difference of their shifts $\tilde{\alpha}_e - \tilde{\alpha}_{e'}$ to zero, one essentially forgets the hyperplane h_e and so defines a map of cell-compatible sheaves $s : R^i f_* \underline{\mathbb{Q}}_\Gamma \to R^i f_* \underline{\mathbb{Q}}_{\Gamma/e}$. Cell-compatibility is preserved by this operation of changing the shifts since the stalks over h_e and $h_{e'}$ are the same and the stalks over all the n-cells surrounding them also all match. The map s induces a map in cohomology $H^k(R^i f_* \underline{\mathbb{Q}}_{\Gamma/e}) \to H^k(R^i f_* \underline{\mathbb{Q}}_\Gamma)$ which is a section of b_k. □

Let Γ be a graph with no disconnecting vertices. Denote by $\hat{\Gamma}$ its *base graph*, the graph we get by repeatedly contracting an edge in Γ which is incident to a vertex of degree 2. For a fixed first Betti number, there are a finite number of graphs with no degree 2 vertices. Theorem 4.18 allows one to calculate the cohomology of any hypertoric Hitchin systems from knowledge of these 'base cases', formulated in the following corollary.

Corollary 4.19. *Let Γ be a graph with no disconnecting vertices, $\hat{\Gamma}$ its base graph, and $e_1, \ldots, e_m$ an enumeration of the edges of $\hat{\Gamma}$. Then*

$$H^k(T^n, R^i f_* \underline{\mathbb{Q}}_\Gamma) = H^k(T^n, R^i f_* \underline{\mathbb{Q}}_{\hat{\Gamma}}) \oplus \bigoplus_{1 \le a_1 < \cdots < a_p \le m} H^{k-p}(T^n, R^{i-p} f_* \underline{\mathbb{Q}}_{\hat{\Gamma} \setminus e_1, \ldots, e_p})^{\oplus \sum_{j=1}^{p} |S_{a_j}|}$$

where $\bigsqcup S_i$ is a partition of the set of degree 2 vertices in Γ, with $v \in S_i$ if there is a path from v to $\tilde{v}$ which goes through only degree two vertices and $\tilde{v}$ is adjacent to e_i.

Proof. This statement follows from repeated applications of Theorem 4.18 and reorganization of the terms.

Let us see some of the details. We write $S_i = \{e^i_1, \ldots, e^i_{|S_{a_i}|}\}$ and suppress "T^n" in each term. By repeatedly applying Theorem 4.18 to the k-th degree cohomology (and noting that $H^k(R^i f_* \underline{\mathbb{Q}}_{\Gamma \setminus e'}) = H^k(R^i f_* \underline{\mathbb{Q}}_{\Gamma \setminus e''})$ for two degree two edges in the same S_i), we find

$$
\begin{aligned}
H^k(R^i f_* \underline{\mathbb{Q}}_\Gamma) &= H^k(R^i f_* \underline{\mathbb{Q}}_{\Gamma/e_1^1}) \oplus H^{k-1}(R^{i-1} f_* \underline{\mathbb{Q}}_{\Gamma\backslash e_1^1}) \\
&= H^k(R^i f_* \underline{\mathbb{Q}}_{\Gamma/e_1^1,e_2^1}) \oplus H^{k-1}(R^{i-1} f_* \underline{\mathbb{Q}}_{\Gamma\backslash e_1^1,e_2^1})^{\oplus 2} \\
&\vdots \\
&= H^k(R^i f_* \underline{\mathbb{Q}}_{\Gamma/S_1}) \oplus H^{k-1}(R^{i-1} f_* \underline{\mathbb{Q}}_{\Gamma\backslash S_1})^{\oplus |S_1|} \\
&= H^k(R^i f_* \underline{\mathbb{Q}}_{(\Gamma/S_1)/e_1^2}) \oplus H^{k-1}(R^{i-1} f_* \underline{\mathbb{Q}}_{(\Gamma/S_1)\backslash e_1^2}) \\
&\quad \oplus H^{k-1}(R^{i-1} f_* \underline{\mathbb{Q}}_{\Gamma\backslash S_1})^{\oplus |S_{a_1}|} \\
&\vdots \\
&= H^k(R^i f_* \underline{\mathbb{Q}}_{\hat{\Gamma}}) \oplus \bigoplus_{a_1=1}^{m} H^{k-1}(R^{i-1} f_* \underline{\mathbb{Q}}_{(\Gamma/S_1,\ldots,S_{a_1-1})\backslash S_{a_1}})^{\oplus |S_{a_1}|}
\end{aligned}
\tag{4.13}
$$

Repeating this process on the degree $k-1$ cohomology terms (and so on) yields the following, which only a mother could love:

$$
\begin{aligned}
H^k(R^i f_* \underline{\mathbb{Q}}_\Gamma) = H^k(R^i f_* \underline{\mathbb{Q}}_{\hat{\Gamma}}) \oplus \bigoplus_{a_1=1}^{m} \Bigg(& H^{k-1}(R^{i-1} f_* \underline{\mathbb{Q}}_{\hat{\Gamma}\backslash e_{a_1}}) \\
\oplus \bigoplus_{a_2=a_1+1}^{m} \Bigg(& H^{k-2}(R^{i-2} f_* \underline{\mathbb{Q}}_{\hat{\Gamma}\backslash e_{a_1},e_{a_2}}) \\
\oplus \bigoplus_{a_3=a_2+1}^{m} \Bigg(& H^{k-3}(R^{i-3} f_* \underline{\mathbb{Q}}_{\hat{\Gamma}\backslash e_{a_1},e_{a_2},e_{a_3}}) \\
\oplus \bigoplus_{a_4=a_3+1}^{m} \Bigg(& \ldots \Bigg)^{\oplus |S_{a_3}|} \Bigg)^{\oplus |S_{a_2}|} \Bigg)^{\oplus |S_{a_1}|}
\end{aligned}
$$

To rearrange this into something manageable, observe that collecting the $k-p$ degree cohomology groups above yields exactly

$$
\bigoplus_{1 \le a_1 < \cdots < a_p \le m} H^{k-p}(T^n, R^{i-p} f_* \underline{\mathbb{Q}}_{\hat{\Gamma}\backslash e_1,\ldots,e_p})^{\oplus \sum_{j=1}^{p} |S_{a_j}|}.
$$

□

The next couple of results cover the low-dimensional examples.

Proposition 4.20. *If Γ is a graph with $b_1(\Gamma) = 1$, then the Poincaré polynomial of $\mathfrak{D}(\Gamma)$ is*

$$
1 + y + |E(\Gamma)| y^2.
$$

Proof. This is nearly trivial, since a graph with $b_1(\Gamma) = 1$ (and no bridges or tails) is just a loop of $|E(\Gamma)|$ edges. Thus, $h_\Gamma^{-1}(0)$ is a necklace of $|E(\Gamma)|$ copies of $\mathbb{P}^1$, which has the stated Poincaré polynomial. □

Theorem 4.21. *Let Γ be a graph with $b_1(\Gamma) = 2$.*

1. *If Γ has a disconnecting vertex, then the Poincaré polynomial of $\mathfrak{D}(\Gamma)$ is*

$$1 + 2y + (|E(\Gamma)| + 1)y^2 + |E(\Gamma)|y^3 + t(\Gamma)y^4.$$

2. *If Γ does not have a disconnecting vertex, then the Poincaré polynomial of $\mathfrak{D}(\Gamma)$ is*

$$1 + 2y + |E(\Gamma)|y^2 + (|E(\Gamma)| - 1)y^3 + t(\Gamma)y^4,$$

where $t(\Gamma)$ is the number of spanning trees of Γ.

Proof. For part (1), note that Corollary 4.15 implies that the Poincaré polynomial of $\mathfrak{D}(\Gamma)$ is $\mathcal{P}_y(\mathfrak{D}(\Gamma)) = \mathcal{P}_y(\mathfrak{D}(\Gamma_1))\mathcal{P}_y(\mathfrak{D}(\Gamma_2))$ and Proposition 4.20 simplifies this to

$$\begin{aligned}\mathcal{P}_y(\mathfrak{D}(\Gamma)) &= \left(1 + y + |E(\Gamma_1)|y^2\right)\left(1 + y + |E(\Gamma_2)|y^2\right) \\ &= 1 + 2y + (|E(\Gamma_1)| + |E(\Gamma_2)| + 1)\,y^2 \\ &\qquad + (|E(\Gamma_1)| + |E(\Gamma_2)|)\,y^3 + |E(\Gamma_1)||E(\Gamma_2)|y^4 \\ &= 1 + 2y + (|E(\Gamma)| + 1)y^2 + |E(\Gamma)|y^3 + t(\Gamma)y^4\end{aligned}$$

Part (2) requires a bit more work. The idea is to first find the cohomology groups of $\mathfrak{D}(\Theta_3)$ and then apply Theorem 4.18, since any graph with first Betti number 2 and no disconnecting vertices can be reduced to $\Theta =$  via contractions of edges which are incident to vertices of degree 2.

To calculate $H^\bullet(\mathfrak{D}(\Theta_3), \underline{\mathbb{Q}}_{\mathfrak{D}(\Theta_3)})$, we recall that thanks to Corollary 2.22 it suffices to find the cohomology groups of the cell-compatible sheaves $R^i f_* \underline{\mathbb{Q}}_{\Theta_3}$ on $P_\bullet(h_{\Theta_3}^{-1}(0))$. In this case, $P_\bullet(h_{\Theta_3}^{-1}(0))$ has the topological type of a 2-torus. So we find

$$H^k(T^2, R^0 f_* \underline{\mathbb{Q}}_{\Theta_3}) = \begin{cases} \mathbb{Q}^1, & k = 0, 2 \\ \mathbb{Q}^2, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

trivially, and

$$H^k(T^2, R^2 f_* \underline{\mathbb{Q}}_{\Theta_3}) = \begin{cases} \mathbb{Q}^3, & k = 2 \\ 0, & \text{otherwise} \end{cases}$$

by Corollary 2.18. Lastly, the calculation of the cohomology groups of $R^1 f_* \underline{\mathbb{Q}}_{\Theta_3}$ was cunningly placed earlier in the thesis as Example 3.8, where we found that

$$H^k(T^2, R^1 f_* \underline{\mathbb{Q}}_{\Theta_3}) = \begin{cases} \mathbb{Q}^{\oplus 2}, & k = 1,2 \\ 0, & \text{otherwise} \end{cases} \tag{4.14}$$

Now let us prove by induction that

$$H^k(T^2, R^1 f_* \underline{\mathbb{Q}}_{\Omega}) = \begin{cases} \mathbb{Q}^{|E(\Omega)|-1}, & k = 1,2 \\ 0, & \text{otherwise} \end{cases}$$

for Ω a graph with $b_1(\Omega) = 2$ and no disconnecting vertices. Equation (4.14) serves as our base case. Assuming that the statement holds for a graph with k edges, let Ω be a graph with $|E(\Omega)| = k + 1$ and e an edge of Ω which is adjacent to a vertex of degree 2. Then Theorem 4.18 yields

$$\begin{aligned} H^1(T^2, R^1 f_* \underline{\mathbb{Q}}_{\Omega}) &\cong H^1(T^2, R^1 f_* \underline{\mathbb{Q}}_{\Omega/e}) \oplus H^0(T^2, R^0 f_* \underline{\mathbb{Q}}_{\Omega \setminus e}) \\ &\cong \mathbb{Q}^{|E(\Omega/e)|-1} \oplus \mathbb{Q} \\ &\cong \mathbb{Q}^{|E(\Omega)|-2} \oplus \mathbb{Q} \\ &\cong \mathbb{Q}^{|E(\Omega)|-1} \end{aligned}$$

and similarly for $H^2(T^2, R^1 f_* \underline{\mathbb{Q}}_{\Omega})$, so we are done, in view of the fact that Corollary 2.18 again yields

$$H^k(T^2, R^2 f_* \underline{\mathbb{Q}}_{\Omega}) = \begin{cases} \mathbb{Q}^{|\mathring{\mathrm{Sk}}_2|}, & k = 2 \\ 0, & \text{otherwise} \end{cases}$$

and $|\mathring{\mathrm{Sk}}_2| = t(\Gamma)$ by Lemma 4.17. □

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