PROPER FORCING, THE P-IDEAL DICHOTOMY, AND THE S-SPACE PROBLEM

by

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ABSTRACT

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In this thesis, we will explore the relationship between the S-space problem and the Proper Forcing Axiom (PFA). In particular, we will show that many consequences of PFA are compatible with the existence of various S-spaces.

We will start by studying the notion of solid graph, first introduced by Soukup, and show this can be used to encode many objects whose existence follows from $\diamondsuit$ or CH. We will show that Neeman iterations preserve solid graphs, providing a general tool for preserving such objects while forcing consequences of PFA.

Next, we will then introduce two new types of graph, namely (m, n)-solid graphs and HF graphs. These new notions can encode many different S-spaces, including strong HFD spaces, strong HFD_w spaces, and first countable strong O-spaces. We will show that much of the theory of solid graphs extends to these new notions, including the preservation by Neeman iterations.

From here, we will be able to construct models with many consequences of PFA plus the aforementioned S-spaces. Using (m, n)-solid graphs, we

will construct models with $\mathfrak{p} = \aleph_2$, the Mapping Reflection Principle, the Open Graph Axiom, Baumgartner's Axiom, and all Aronszajn trees are club isomorphic. Using HF graphs, we will construct models with $\mathfrak{s} = \aleph_1$, $\mathrm{add}(\mathcal{M}) = \aleph_2$, the Mapping Reflection Principle, and the *P*-Ideal Dichotomy. Most notably, this provides a partial negative answer to a question of Todorčević about whether the *P*-Ideal Dichotomy and $\mathfrak{b} > \aleph_1$ implies there are no *S*-spaces.

To Anna,
without whom none of this would be possible.
I look forward to growing old with you.

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PUBLICATIONS

Parts of Chapter 3 and Chapter 4 are based on [7] (accepted for publication by the Annals of Pure and Applied Logic).

CONTENTS

1

INTRODUCTION

1.1 S-spaces and L-spaces

A topological space is *separable* if it contains a countable dense subset, and is *Lindelöf* if every open cover has a countable subcover. Similarly, we say that a space is *hereditarily separable* (*hereditarily Lindelöf*) if all of its subspaces are separable (resp. Lindelöf).

Definition 1.1. Let X be a regular topological space.

- X is an *S-space* if it is hereditarily separable but not hereditarily Lindelöf.
- X is an *L-space* if it is hereditarily Lindelöf but not hereditarily separable.
- X is a *strong S-space* (*strong L-space*) if X^k is an S-space (resp. L-space) for all $k < \omega$.

Research into the existence of S and L-spaces mainly started in the 70s. Many S-spaces and L-spaces were discovered to exist under CH, and Roitman showed adding a Cohen real adds strong S- and L-spaces [25]. Similarly, Szentmiklóssy [32] and Abraham and Todorčević constructed models [2] of MA containing S- and L-spaces. Conversely, various forcing axioms restrict their existence:

Theorem 1.2 (Kunen, 1977 [13])**.** MA *implies there are no strong S- or L-spaces.*

Theorem 1.3 (Szentmiklóssy, 1980 [31])**.** MA *implies there are no compact S- or L-spaces.*

Definition 1.4. We write PFA for the following statement: For every proper forcing $\mathbb{P}$ and sequence $\{D_\alpha : \alpha < \omega_1\}$ of dense-open subsets of $\mathbb{P}$, there is a filter $\mathcal{F} \subseteq \mathbb{P}$ such that $\mathcal{F} \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

Theorem 1.5 (Todorčević, 1981)**.** *The Proper Forcing Axiom (*PFA*) implies there are no S-spaces.*

All of the above naturally leads to the following two questions:

- **S-space problem:** When do S-spaces exist?

- **L-space problem:** When do L-spaces exist?

Due to the symmetry between S- and L-spaces, it was believed that the these two problems might be equivalent. However, Moore showed this is not the case by solving the L-space problem:

Theorem 1.6 (Moore, 2006 [21])**.** ZFC *implies there is an L-space.*

An interested reader should see [27] and [36] for a more in-depth introduction into S- and L-spaces.

1.2 THE P-IDEAL DICHOTOMY

An ideal $\mathcal{I} \subseteq [\nu]^{\aleph_0}$ is called a *P-ideal* if every countable collection of elements in $\mathcal{I}$ has a pseudo-intersection, i.e. given $\{I_n : n < \omega\} \subseteq \mathcal{I}$, there is some $I \in \mathcal{O}$ such that $I_n \subseteq^* I$ for all $n < \omega$.

Definition 1.7. The *P-Ideal Dichotomy* (PID) is the following statement: For every P-ideal of countable sets $\mathcal{I} \subseteq [\nu]^{\aleph_0}$, one of the following holds:

1. There is an uncountable $T \subseteq \nu$ such that $[T]^{\aleph_0} \subseteq \mathcal{I}$,
2. ν can be partitioned into countably many sets $\{S_i : i < \omega\}$ such that $S_i \in \mathcal{I}^\perp$ for all $i < \omega$.

The P-Ideal Dichotomy was introduced by Todorčević in [37] as a powerful consequence of the Proper Forcing Axiom (PFA) that is not implied by Martin's Axiom and is also compatible with CH. In particular, many other consequences of PFA become equivalent to a statement about cardinal characteristics in the presence of PID. For example:

Theorem 1.8. *Under* PID*, the following statements are equivalent:*

1. $\mathfrak{b} > \aleph_1$*;*
2. $\mathfrak{b} = \aleph_2$ *[39, §21];*
3. $\omega_1 \to (\omega_1, \omega + 2)$ *[23];*
4. *Every ω_1-tower is Hausdorff [5].*

Theorem 1.9 ([23])**.** *Under* PID*, the following statements are equivalent:*

1. $\min\{\mathfrak{b}, \mathrm{cof}(\mathcal{F}_\sigma)\} > \aleph_1$*;*
2. *There are only five Tukey types of size at most $\aleph_1$: 1, ω, ω_1, $\omega \times \omega_1$, and $[\omega_1]^{<\aleph_0}$.*

The reason that $\mathfrak{b} > \aleph_1$ is such a strong statement under PID is that it gives rise to many P-ideals:

Lemma 1.10. *Let $\mathcal{F} \subseteq [\nu]^{\aleph_0}$ be such that $|\mathcal{F}| < \mathfrak{b}$. Then $\mathcal{F}^\perp$ is a P-ideal.*

Proof. It is clear that $\mathcal{F}^\perp$ is an ideal, so it suffices to show that given $\{X_n : n < \omega\} \subseteq \mathcal{F}^\perp$ we can find a pseudo-intersection. Let $\kappa = |\mathcal{F}|$, enumerate $\mathcal{F}$ as $\{Y_\alpha : \alpha < \kappa\}$, and pick some bijection $\psi\colon \omega \to \bigcup_{n<\omega} X_n$. We can now define maps $f_\alpha\colon \omega \to \omega$ for all $\alpha < \kappa$ as follows:

$$f_\alpha(n) = \min\left(\omega \setminus (\psi[X_n \cap Y_\alpha])\right).$$

Since $\kappa < \mathfrak{b}$, there is some function $f\colon \omega \to \omega$ such that $f \geq^* f_\alpha$ for all $\alpha < \kappa$. Let

$$X = \bigcup_{n<\omega} X_n \setminus \psi[f(n)].$$

We claim X is our required pseudo-intersection. Since $X_n \setminus X$ is finite for all $n < \omega$, it remains to prove $X \in \mathcal{F}^\perp$. Let $Y_\alpha \in \mathcal{F}$. Since $f \geq^* f_\alpha$, we can find some $N < \omega$ such that $f(n) \geq f_\alpha(n)$ for all $n \geq N$. Thus for all $n \geq N$, $Y_\alpha \cap (X_n \setminus \psi[f(n)]) = \emptyset$. Therefore

$$\begin{aligned} Y_\alpha \cap X &= \bigcup_{n<\omega} Y_\alpha \cap (X_n \setminus \psi[f(n)]) \\ &= \bigcup_{n<N} Y_\alpha \cap (X_n \setminus \psi[f(n)]) \\ &\subseteq Y_\alpha \cap (\bigcup_{n<N} X_n) \end{aligned}$$

and since $\bigcup_{n<N} X_n \in \mathcal{F}^\perp$ we have that $Y_\alpha \cap X$ is finite as required. □

However, PID first arose from attempting to understand the aforementioned S-space problem. For example:

- PID + $\mathfrak{p} > \aleph_1$ implies there are no S-spaces [39].
- $\mathfrak{b} = \aleph_1$ implies the existence of a first-countable S-space [36, Theorem 0.6].
- PID + $\mathfrak{b} > \aleph_1$ implies there are no sequential compact S-spaces [38].
- $\mathfrak{b} = \aleph_1$ implies the existence of a Fréchet-Urysohn compact S-space [36, Theorem 2.4].

These results lead naturally to the following question:

Question 1.11 ([39, Question 23.8])**.** Are any of the following equivalent in the presence of PID?

1. There are no S-spaces.
2. There are no first-countable S-spaces.
3. There are no compact S-spaces.
4. $\mathfrak{b} > \aleph_1$.
5. $\mathfrak{p} > \aleph_1$.

Remark 1.12. Yorioka has proven that $\mathsf{PID} + \mathrm{cov}(\mathcal{N}) = \aleph_1$ is compatible with no S-spaces [45].

A reader interested in learning more about PID should see [37], [39, Part III], and [41].

1.3 RELATIVISING PFA

A fruitful area of recent research has been separating properties of PFA by considering 'relativised' axioms, where we only have PFA for proper forcings that preserve the existence of some object otherwise forbidden by PFA. The most notable example is Todorčević's $\mathsf{PFA}(\mathbb{S})$:

Definition 1.13 ([38])**.** Let $\mathbb{S}$ be a coherent Souslin tree. We write $\mathsf{PFA}(\mathbb{S})$ for the following statement: For every proper forcing $\mathbb{P}$ that preserves that $\mathbb{S}$ is a Souslin tree and sequence $\{D_\alpha : \alpha < \omega_1\}$ of dense-open subsets of $\mathbb{P}$, there is a filter $\mathcal{F} \subseteq \mathbb{P}$ such that $\mathcal{F} \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

We write $\mathsf{PFA}(\mathbb{S})[\mathbb{S}]$ for models obtained by forcing with $\mathbb{S}$ over a model of $\mathsf{PFA}(\mathbb{S})$. If we say '$\mathsf{PFA}(\mathbb{S})[\mathbb{S}]$ implies P', we mean P holds in all such models.

The consistency of $\mathsf{PFA}(\mathbb{S})$ (relative to a supercompact cardinal) was originally shown using a countable-support iteration, but Venturi showed that models of $\mathsf{PFA}([S])$ could also be obtained using Neeman iterations [42]. $\mathsf{PFA}(\mathbb{S})$ implies several consequences of PFA:

Theorem 1.14. $\mathsf{PFA}(\mathbb{S})$ *implies the following:*

- $\mathfrak{p} = \mathrm{add}(\mathcal{N}) = \mathfrak{c} = \aleph_2$ *[38], [24].*
- *The Open Graph Axiom* (OGA) *[38].*
- *Baumgartner's Axiom* (BA)*.*
- $\neg\square(\theta)$ *for ordinals with* $\mathrm{cof}(\theta) > \aleph_1$ *[38].*
- *The Mapping Reflection Principle* (MRP) *[19].*

However, the true strength of this axiom lies with $\mathsf{PFA}(\mathbb{S})[\mathbb{S}]$:

Theorem 1.15. $\mathsf{PFA}(\mathbb{S})[\mathbb{S}]$ *implies the following:*

- $\mathfrak{p} = \aleph_1$ *and* $\mathfrak{c} = \aleph_2$.
- $\mathrm{add}(\mathcal{N}) = \mathfrak{h} = \aleph_2$ *[24].*
- OGA *[8].*
- ¬BA *[8].*
- PID *[38].*
- $\mathcal{K}_2(rec)$ *[15].*
- *Compact countably-tight spaces are sequential [38].*

Since $\mathsf{PFA}(\mathbb{S})[\mathbb{S}]$ is a model of PID with $\mathfrak{p} = \aleph_1$ but almost all of the other interesting cardinal characteristics equal to $\aleph_2$, it begets the following question:

Question 1.16.

- Do all models of the form $\mathsf{PFA}(\mathbb{S})[\mathbb{S}]$ have no *S*-spaces?
- Do any models of the form $\mathsf{PFA}(\mathbb{S})[\mathbb{S}]$ have no *S*-spaces?

While this question remains open, some partial progress has been made:

Theorem 1.17 ([44])**.** *Under* $\mathsf{PFA}(\mathbb{S})$, $\mathbb{S}$ *forces that every topology on* ω_1 *generated by a basis in the ground model is not an S-topology.*

Later, Guzmán-González and Todorčević were able to relativise PFA to a 2-entangled set of reals:

Theorem 1.18 ([10])**.** *Assuming a supercompact cardinal,* PFA *relativised to a* 2*-entangled set of reals is consistent, and implies* PID + MA. *In particular, this shows that* PID + MA *does not imply* OGA *or* BA.

This model was again constructed using a Neeman iteration. Building on these results, the author was able to prove the following:

Theorem 1.19 ([7])**.** *Assuming a supercompact cardinal,* PFA *relativised to a weak* HFD *space is consistent, and implies* $\mathsf{OGA} + \mathfrak{p} > \aleph_1$.

1.4 CHAPTER OVERVIEW

This thesis will be structured as follows. Chapter 2 will provide a brief summary of proper forcing and strong proper forcing, with an emphasis on the tools we will use later in the thesis. In particular, we will discuss Neeman iterations and cite some of the fundamental results about such forcings. Chapter 3 will introduce the basic objects of investigation, solid graphs and (m, n)-solid graphs, and explore both their combinatorial properties and preserving them with proper forcings. Most notably, we will show that we can preserve solid graphs with Neeman iterations. This will allow us to construct models of PFA relativised to the existence of some solid graph. We will then in Chapter 4 explore the consequences of said relativised PFA-like axioms, showing that the standard forcings for the Open Graph Axiom, Baumgartner's Axiom, and forcing that all Aronszajn trees are club-isomorphic all preserve all (m, n)-solid graphs. In Chapter 5 we will introduce HF graphs, a strengthening of the notion of (m, n)-solid graph. We will similarly show that HF graphs are preserved by Neeman iterations, but also that they are preserved by the standard forcings for $\mathfrak{b} = \aleph_2$ and the P-Ideal Dichotomy. Chapter 6 will provide several examples of both (m, n)-solid graphs and HF graphs that encode the existence of various S- and L-spaces and strong colourings. Finally, we will summarise our findings and discuss avenues of further research in Chapter 7.

2 PROPER FORCING & SIDE CONDITIONS

2.1 PROPER FORCING

Given a sufficiently large regular cardinal λ, we will often implicitly fix a well-order $<_w$ on $H(\lambda)$. Similarly, whenever we talk about a countable elementary submodel $M \prec H(\lambda)$, we mean that

$$(M, \in, <_w) \prec (H(\lambda), \in, <_w).$$

Furthermore, we will write δ_M for $\omega_1 \cap M$.

Definition 2.1. Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$, λ a sufficiently large cardinal, and M an elementary submodel of $H(\lambda)$ containing $\mathbb{P}$. We say that p is $(M, \mathbb{P})$-*generic* if for all dense-open subsets $\mathcal{D} \subseteq \mathbb{P}$ with $\mathcal{D} \in M$,

$$p \Vdash \text{`}\dot{G} \cap M \cap \mathcal{D} \neq \emptyset\text{'}$$

where $\dot{G}$ is a $\mathbb{P}$-name for a generic filter of $\mathbb{P}$.

Remark 2.2.

- Equivalently, p is $(M, \mathbb{P})$-generic if it forces that $\dot{G} \cap M$ is an M-generic filter.
- We can replace dense-open with dense or pre-dense in this definition.
- The set of generic conditions is open in $\mathbb{P}$: if p is $(M, \mathbb{P})$-generic and $q \leq p$, then q is $(M, \mathbb{P})$-generic too.

Lemma 2.3 ([29, Chapter III, Corollary 2.13]). *Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$, λ a sufficiently large cardinal, and M an elementary submodel of $H(\lambda)$ containing $\mathbb{P}$. The following are equivalent:*

1. *p is $(M, \mathbb{P})$-generic;*
2. *$p \Vdash$ '$M[\dot{G}] \cap Ord = M \cap Ord$';*
3. *$p \Vdash$ '$M[\dot{G}] \cap V = M$'.*

The following combinatorial characterisation of proper forcing will be incredibly useful and lead to combinatorial characterisations for future properties:

Lemma 2.4. *Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$, λ a sufficiently large cardinal, and M an elementary submodel of $H(\lambda)$ containing $\mathbb{P}$. The following are equivalent:*

1. *p is $(M, \mathbb{P})$-generic;*
2. *For all $q \leq p$ and dense-open subsets $\mathcal{D} \subseteq \mathbb{P}$ with $\mathcal{D} \in M$, there is $q' \in \mathcal{D} \cap M$ that is compatible with q.*

As with the definition of generic condition, we can replace dense-open with dense or pre-dense.

Proof. Suppose p is $(M, \mathbb{P})$-generic. Let $q \leq p$ and $\mathcal{D} \subseteq \mathbb{P}$ be dense-open such that $\mathcal{D} \in M$. q is also $(M, \mathbb{P})$-generic, so $q \Vdash$ '$\dot{G} \cap M \cap \mathcal{D} \neq \emptyset$'. Take some $q' \in \mathbb{P}$ such that $q \Vdash$ '$q' \in \dot{G} \cap M \cap \mathcal{D}$'. Then since $q \Vdash$ '$q' \in \dot{G}$', q and q' are compatible.

Conversely, suppose p is not $(M, \mathbb{P})$-generic. Then there is some dense-open $\mathcal{D} \subseteq \mathbb{P}$ with $\mathcal{D} \in M$ such that $p \nVdash$ '$\dot{G} \cap M \cap \mathcal{D} \neq \emptyset$'. Take $q \leq p$ such that $q \Vdash$ '$\dot{G} \cap M \cap \mathcal{D} = \emptyset$'. Then q is not compatible with any $q' \in \mathcal{D} \cap M$, since any such q' forces that $q' \in \dot{G} \cap M \cap \mathcal{D}$. □

Definition 2.5. Let $\mathbb{P}$ be a forcing notion. We say that $\mathbb{P}$ is *proper* if for all sufficiently large cardinals λ, there is a club (in $[H(\lambda)]^{\aleph_0}$) of countable elementary submodels $M \prec H(\lambda)$ containing $\mathbb{P}$ such that every condition in $\mathbb{P} \cap M$ can be extended to an $(M, \mathbb{P})$-generic condition.

We give some natural examples of proper forcings.

Proposition 2.6. *Let $\mathbb{P}$ be a ccc forcing notion, λ a sufficiently large cardinal, and M an elementary submodel of $H(\lambda)$ containing $\mathbb{P}$. Then $\mathbb{1}_{\mathbb{P}}$ is $(M, \mathbb{P})$-generic. (In particular, $\mathbb{P}$ is proper.)*

Proof. Let λ be a sufficiently large cardinal, and M an elementary submodel of $H(\lambda)$ containing $\mathbb{P}$. By Lemma 2.4, it is sufficient to show that for all $p \in \mathbb{P}$ and dense-open $\mathcal{D} \subseteq \mathbb{P}$ with $\mathcal{D} \in M$, there is some $q \in \mathcal{D} \cap M$ compatible with p. Let $X \subseteq \mathcal{D}$ be a maximal antichain. By elementarity, we can assume that $X \in M$. Since $\mathbb{P}$ is ccc, X is countable and thus $X \subseteq M$. Take some $p' \in \mathcal{D}$ extending p. By maximality of X, there is some $q \in X$ be such that q and p' are compatible. But then q and p are compatible as required. □

Proposition 2.7. *Let $\mathbb{P}$ be a σ-closed forcing notion (i.e. every countable decreasing sequence has a lower bound). Then $\mathbb{P}$ is proper.*

Proof. Let λ be a sufficiently large cardinal and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$. Let $p_0 \in \mathbb{P}$, and enumerate the dense-open subsets of $\mathbb{P}$ in M as $\langle \mathcal{D}_n \rangle_{n<\omega}$.

We now inductively pick a decreasing sequence $\langle p_n \rangle_{n<\omega} \subseteq \mathbb{P} \cap M$ as follows: given $\langle p_n \rangle_{n<m}$, pick $p_m \in \mathcal{D}_{m-1} \cap M$ such that $p_m \leq p_{m-1}$. Let q be a lower bound for $\langle p_n \rangle_{n<m}$. Then for all n,

$$q \Vdash `p_{n+1} \in \dot{G} \cap M \cap \mathcal{D}_n'$$

and thus q is $(M, \mathbb{P})$-generic as required. □

Why are proper forcings interesting?

Lemma 2.8. *Let $\mathbb{P}$ be a proper forcing and $\dot{\tau}$ be a $\mathbb{P}$-name for a countable set of ordinals. Then there is a countable set of ordinals $X \in V$ such that $\mathbb{P} \Vdash `\tau' \subseteq X$.*

In particular, this implies that proper forcings preserves the property $\mathrm{cof}(\alpha) > \aleph_0$ for all such ordinals α. Thus they also preserve $\aleph_1$.

Proof. Let λ be a sufficiently large cardinal and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$ and $\dot{\tau}$. For each $n < \omega$, we can define the $\mathbb{P}$-name $\dot{\sigma}_n$ for the nth element of $\dot{\tau}$ in increasing order. By elementarity, we can assume that $\dot{\sigma}_n \in M$ for all $n < \omega$.

Let $q \in \mathbb{P}$ be $(M, \mathbb{P})$-generic. Then by Lemma 2.3, we have that $q \Vdash$ '$\dot{\sigma}_n \in M \cap \mathrm{Ord}$' for all $n < \omega$. Thus

$$q \Vdash `\dot{\tau} \subseteq M \cap \mathrm{Ord}'$$

as required. □

Theorem 2.9 ([29, Chapter III, Theorem 2.8])**.** *Let $\mathbb{P}$ be a forcing notion. The following are equivalent:*

1. *$\mathbb{P}$ is proper;*
2. *$\mathbb{P}$ preserves all stationary sets in $[\lambda]^{\aleph_0}$ for all uncountable cardinals λ.*

As with ccc forcings, we can iterate proper forcings:

Proposition 2.10. *Let $\mathbb{P}$ be a forcing notion, $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a forcing notion, λ a sufficiently large regular cardinal, and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$ and $\dot{\mathbb{Q}}$. Let $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$. The following are equivalent:*

1. *$(p, \dot{q})$ is $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic,*
2. *p is $(M, \mathbb{P})$-generic and p forces that $\dot{q}$ is $(M[\dot{G}_{\mathbb{P}}], \dot{\mathbb{Q}})$-generic (where $\dot{G}_{\mathbb{P}}$ is the canonical name for the $\mathbb{P}$-generic filter).*

Proof. Follows from Lemma 2.3. □

Corollary 2.11. *Let $\mathbb{P}$ be a proper forcing notion and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a proper forcing notion. Then $\mathbb{P} * \dot{\mathbb{Q}}$ is proper.*

Theorem 2.12 ([29, III §3])**.** *Countable-support iterations of proper forcings are proper.*

For a more in-depth introduction to proper forcing, see [29, Chapter III] and [3].

2.2 STRONG PROPER FORCING

Definition 2.13. Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$, and X a set. We say that p is $(X, \mathbb{P})$-*strongly generic* if for all dense-open subsets $\mathcal{D} \subseteq \mathbb{P}$ with $\mathcal{D} \subseteq X$,

$$p \Vdash \text{`}\dot{G} \cap X \cap \mathcal{D} \neq \emptyset\text{'}$$

where $\dot{G}$ is a $\mathbb{P}$-name for a generic filter of $\mathbb{P}$.

The notion of strong generic conditions was first introduced by Mitchell in [18]. We have the following useful combinatorial condition for strong genericity:

Proposition 2.14 ([18, Proposition 2.15])**.** *Let $\mathbb{P}$ be a forcing notion and X a set. A condition $p \in \mathbb{P}$ is $(X, \mathbb{P})$-strongly generic iff for every $q \leq p$, there is some condition $q \restriction X \in X \cap \mathbb{P}$ such that every condition $r \leq q \restriction X$ in X is compatible with q.*

Definition 2.15. Let $\mathbb{P}$ be a forcing notion. We say that $\mathbb{P}$ is *strongly proper* if for all sufficiently large cardinals λ, there is a club (in $[H(\lambda)]^{\aleph_0}$) of countable elementary submodels $M \prec H(\lambda)$ containing $\mathbb{P}$ such that every condition in $\mathbb{P} \cap M$ can be extended to an $(M, \mathbb{P})$-strongly generic condition.

The most well-known examples of strong proper forcing are the $\in$-collapse (see next chapter) or Cohen forcing. Strong proper forcings have a lot of nice properties:

Lemma 2.16. *Strongly proper forcings don't add new cofinal branches to ω_1-trees.*

Proof. Let $\mathbb{P}$ be strongly proper, T an ω_1-tree, $\dot{b}$ a $\mathbb{P}$-name for a branch of T, and $p_0 \in \mathbb{P}$. Let λ be a sufficiently large cardinal and let M be a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$, T, $\dot{b}$, and p_0. Finally, pick $p \leq p_0$ and $t \in T \setminus M$ such that p is $(M, \mathbb{P})$-strongly generic and

$p \Vdash `t \in \dot{b}'$. (Such a t exists since $\dot{b}$ is forced to be uncountable.) Consider the following subset of T:

$$S = \{a \in T : (\exists r \leq p{\restriction}M)\ r \Vdash `a \in \dot{b}'\}.$$

Note that S is a downwards-closed subtree of T. By elementarity $S \in M$, and thus since $t \in S$ we have that S is uncountable.

We claim that S is a branch. (If so, then $p \Vdash `\dot{b} = S'$ and we're done.) Suppose not, then by elementarity there must be $q_0, q_1 \in \mathbb{P} \cap M$ and $a_0, a_1 \in T \cap M$ such that:

- $q_i \leq p{\restriction}M$ for $i < 2$,
- $q_i \Vdash `a_i \in T'$ for $i < 2$,
- $q_0 \perp q_1$.

But since $\mathrm{ht}(a_0), \mathrm{ht}(a_1) < \delta_M \leq \mathrm{ht}(t)$, at least one of a_0 and a_1 must be incompatible with t. Assume $a_0 \perp t$. Then since $q_0 \leq p{\restriction}M$, there is some condition p' extending q_0 and p. But then $p' \Vdash `a_0, t \in \dot{b}'$, contradicting that $\dot{b}$ is a branch. □

Lemma 2.17 ([17])**.** *The only reals that strongly proper forcings can add are Cohen reals.*

Definition 2.18 ([19])**.** Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$, λ a sufficiently large cardinal, and M an elementary submodel of $H(\lambda)$ containing $[\mathbb{P}]^{\aleph_0}$. We say that p is $(M, \mathbb{P})$*-almost strongly generic* if for all $q \leq p$ and sets $U \in M$ that are unbounded in $[\mathbb{P}]^{\aleph_0}$, there is some $X \in U \cap M$ and $q{\restriction}X \in X$ such that every condition $r \leq q{\restriction}X$ in X is compatible with q. We say $\mathbb{P}$ is *almost strongly proper* if for all sufficiently large cardinals λ, there is a club of countable elementary submodels $M \prec H(\lambda)$ containing $[\mathbb{P}]^{\aleph_0}$ such that every condition in $\mathbb{P} \cap M$ can be extended to an $(M, \mathbb{P})$-almost strongly generic condition.

Almost strongly proper forcings share many properties with strongly proper forcings (see [19] for examples).

Question 2.19. Can almost strongly proper forcings add non-Cohen reals?

2.3 THE SIDE CONDITION HULL

Definition 2.20 ($\in$-collapse)**.** Let θ be a sufficiently large regular cardinal. Then the $\in$*-collapse of* $H(\theta)$ is the poset of all finite $\in$-chains of countable elementary submodels of $H(\theta)$ ordered by reverse-inclusion.

More information about the $\in$-collapse can be found in [40, §7.1].

Lemma 2.21. *The $\in$-collapse poset is strongly proper.*

Proof. Let $\mathbb{P}$ the $\in$-collapse of $H(\theta)$ for some regular uncountable cardinal θ and $p_0 \in \mathbb{P}$. Pick $\lambda \gg \theta$ a sufficiently large cardinal and $\overline{M}$ a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$ and p_0, and denote $M = \overline{M} \cap H(\theta)$. Note that $p = p_0 \cup \{M\} \in \mathbb{P}$ and $p \leq p_0$. We claim that p is $(\overline{M}, \mathbb{P})$-strongly proper. Let $q \leq p$, and note $q \cap M \in \mathbb{P} \cap \overline{M}$. If $r \in \mathbb{P} \cap \overline{M}$ is such that $r \leq q \cap M$, then $r \cup q = r \cup (q \setminus M)$ is an $\in$-chain extending q. Thus r and q are compatible, so letting $q \restriction M = q \cap M$ we are done by Proposition 2.14. $\square$

Definition 2.22. Let $\mathbb{P}$ be a forcing notion and $\mathcal{S}$ a family of countable sets. We say that $\mathbb{P}$ is *$\mathcal{S}$-proper* (*$\mathcal{S}$-strongly proper*) if for every λ a sufficiently large regular cardinal and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$, if $M \cap \bigcup \mathcal{S} \in \mathcal{S}$ then every condition in $\mathbb{P} \cap M$ can be extended to an $(M, \mathbb{P})$-generic (resp. $(M, \mathbb{P})$-strongly generic) condition.

Definition 2.23. Let $\mathbb{P}$ be a forcing notion, θ a sufficiently large regular cardinal, and $\mathcal{S} \subseteq [H(\theta)]^{\aleph_0}$ be a stationary set. The *side condition hull of* $\mathbb{P}$ *with respect to* $\mathcal{S}$, denoted $\mathbb{S}_\in(\mathbb{P}, \mathcal{S})$, is the set of all pairs (p, a) such that:

1. $p = \{M_0, \ldots, M_{n-1}\} \subseteq \mathcal{S}$ is an $\in$-chain of countable elementary submodels of $H(\theta)$ all containing $\mathbb{P}$,
2. $a \in \mathbb{P}$ is a $(M_i, \mathbb{P})$-generic condition for all $i < n$.

We order $\mathbb{S}_\in(\mathbb{P}, \mathcal{S})$ as follows: $(q, b) \leq (p, a)$ if $p \subseteq q$ and $a \leq_{\mathbb{P}} b$.

Proposition 2.24 ([10, Proposition 50])**.** *Let $\mathbb{P}$ be a forcing notion, θ a sufficiently large regular cardinal, and $\mathcal{S} \subseteq [H(\theta)]^{\aleph_0}$ be a stationary set such that $\mathbb{P}$ is $\mathcal{S}$-proper. Then $\mathbb{S}_\in(\mathbb{P}, \mathcal{S})$ is $\mathcal{S}$-proper. (In particular, if $\mathcal{S}$ is a club, then $\mathbb{S}_\in(\mathbb{P}, \mathcal{S})$ is proper.)*

We can define the function $\pi \colon \mathbb{S}_\in(\mathbb{P}, \mathcal{S}) \to \mathbb{P}$ given by $\pi(p, a) = a$. π is a projection [10, Lemma 52], so given some generic filter $G_{\mathbb{P}} \subseteq \mathbb{P}$ we can consider the quotient forcing $\mathbb{S}_\in(\mathbb{P}, \mathcal{S}) / G_{\mathbb{P}}$.

Proposition 2.25. *Suppose that $\mathbb{P}$ is a forcing, θ a sufficiently large cardinal, $\mathcal{S} \subseteq [H(\theta)]^{\aleph_0}$ a stationary set such that $\mathbb{P}$ is $\mathcal{S}$-proper. Let $G_{\mathbb{P}} \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter. Then $\mathbb{S}_\in(\mathbb{P}, \mathcal{S}) / G_{\mathbb{P}}$ is $\mathcal{S}$-strongly proper. (In particular, if $\mathcal{S}$ is a club, then $\mathbb{S}_\in(\mathbb{P}, \mathcal{S}) / G_{\mathbb{P}}$ is strongly proper.)*

The proof of this result is very similar to Lemma 2.21, but we include it both for completeness and because this result will save us some work in Section 3.4 and Section 5.2.

Proof. Let $(p_0, a_0) \in \mathbb{S}_\in(\mathbb{P}, \mathcal{S})/G_\mathbb{P}$. Take λ a sufficiently large cardinal. Since $\mathcal{S}$ is stationary, we can find $\overline{M}$ a countable elementary submodel of $H(\lambda)$ such that $(p_0, a_0) \in \overline{M}$ and $M = \overline{M} \cap H(\theta) \in \mathcal{S}$. Since $\mathbb{P}$ is $\mathcal{S}$-proper, we can find $(p, a) \in \mathbb{S}_\in(\mathbb{P}, \mathcal{S})/G_\mathbb{P}$ extending (p_0, a_0) such that $M \in p$. Our goal is to show that (p, a) is $(\overline{M}, \mathbb{S}_\in(\mathbb{P}, \mathcal{S})/G_\mathbb{P})$-strongly generic.

Let $\mathcal{D} \subseteq \mathbb{S}_\in(\mathbb{P}, \mathcal{S})/G_\mathbb{P} \cap \overline{M}$ be dense, and take $(\overline{p}, \overline{a}) \leq (p, a)$. By elementarity, we can find $\overline{a}_M \in \mathbb{P} \cap M$ such that $(\overline{p} \cap M, \overline{a}_M) \in \mathbb{S}_\in(\mathbb{P}, \mathcal{S})/G_\mathbb{P}$. Now take some $(q, b) \in \mathcal{D}$ extending $(\overline{p} \cap M, \overline{a}_M)$. But since $a, b \in G_\mathbb{P}$ and $q \in M$, we have that $(\overline{p}, \overline{a})$ and (q, b) are compatible as required. □

2.4 TWO-TYPE SIDE CONDITIONS

The classical theory of proper forcings relies heavily on countable-support iterations of proper forcings. In 2014, Neeman introduced a new approach: iterating proper forcings by using finite sequences of models of two types as side conditions. Such forcings are colloquially referred to as *Neeman iterations*. This approach allows us to iterate proper forcings while taking advantage of the flexibility of finite-support iterations.

Neeman initially used two-type side conditions to give an alternative proof of the consistency of PFA [22]. Since then, Neeman iterations have been used to prove the consistency of various relativised versions of PFA (see [42], [14], and [10]).

An interested reader should see [22] and [10] for a more complete exposition on this topic. In this section, we briefly review the definitions and results we will need for the rest of the thesis. We will use the notation from [10].

For the definitions in this section, we fix a sufficiently large inaccessible cardinal θ, well-order $<_w$ of $H(\theta)$, and function $J\colon \theta \to H(\theta)$. If $A \in H(\theta)$, we write $A \prec H(\theta)$ if $(A, \in, <_w, J\restriction(A \cap \theta))$ is an elementary submodel of $(H(\theta), \in, <_w, J)$. We also fix

$$\mathcal{S} \subseteq \{M \in [H(\theta)]^{\aleph_0} : M \prec H(\theta)\}$$

$$\mathcal{T} = \{H(\lambda) : (H(\lambda) \leq H(\theta)) \wedge (\mathrm{cof}(\lambda) > \omega)\}$$

such that $\mathcal{S}$ is stationary in $[H(\theta)]^{\aleph_0}$ and $\mathcal{S} \cup \mathcal{T}$ is closed under intersection. Given an element $Z \in \mathcal{T}$, we write Z^+ to denote the successor of Z in $\mathcal{T}$ and write $J(Z) = J(\lambda)$, where λ is the cardinal such that $Z = H(\lambda)$.

Definition 2.26. We define $\mathbb{P}_\in^{\mathcal{S},\mathcal{T}}$ to be the collection of all finite sets $p \subseteq \mathcal{S} \cup \mathcal{T}$ such that:

- $p = \{A_0, \ldots, A_{n-1}\}$ is an $\in$-path, i.e. $A_i \in A_{i+1}$ for all $i < n - 1$,

- p is closed under intersection.

We order $\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$ by reverse inclusion.

Theorem 2.27.

- *$\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$ is $\mathcal{S}\cup\mathcal{T}$-strongly proper [22, Claim 4.1]. In particular, if $\mathcal{S}$ is a club, then $\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$ is strongly proper.*
- *$\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$ has the θ-chain condition [10, Proposition 68].*
- *$\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}} \Vdash$ '$\check{\theta} = \aleph_2$'.*

Definition 2.28 (Neeman iterations)**.** We recursively define $\mathbb{P}(J)$ as the set of all pairs (p, f_p) with the following properties:

1. $p \in \mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$,
2. Given $Z \in \mathcal{T}$ and $G_Z \subseteq \mathbb{P}(J) \cap Z$ a generic filter, we define the following sets:

$$\mathcal{S}_Z[G_Z] = \{M[G_Z] : (M \in \mathcal{S}) \wedge (Z \in M) \wedge ((\{M \cap Z\}, \emptyset) \in G_Z)\}$$

$$\mathcal{S}_{(Z,Z^+)}[G_Z] = \{M[G_Z] : (M \in \mathcal{S}) \wedge (Z \in M \in Z^+) \\ \wedge ((\{M \cap Z\}, \emptyset) \in G_Z)\}$$

   and say that $Z \in \mathcal{T}$ is *non-trivial* if

$$\mathbb{1}_{\mathbb{P}(J)\cap Z} \Vdash \text{`}J(Z) \text{ is a } \mathcal{S}_{(Z,Z^+)}[G_Z]\text{-proper forcing'},$$

3. f_p is a function such that

$$\mathrm{dom}(f_p) \subseteq \{Z \in \mathcal{T} \cap p : Z \text{ is non-trivial}\},$$

4. If $Z \in \mathrm{dom}(f_p)$, then $(p \cap Z, f_p \restriction Z) \Vdash$ '$f_p(Z) \in J(Z)$',
5. If $Z \in \mathrm{dom}(f_p)$, $M \in \mathcal{S} \cap p$, and $Z \in M$, then

$$(p \cap Z, f_p \restriction Z) \Vdash \text{`}f_p(Z) \text{ is an } (M[\dot{G}_Z], J(Z)[\dot{G}_Z])\text{-generic condition'}.$$

We order $\mathbb{P}(J)$ as follows: $(q, f_q) \leq (p, f_p)$ if:

1. $q \supseteq p$,
2. $\mathrm{dom}(f_q) \supseteq \mathrm{dom}(f_p)$,

3. For all $Z \in \mathrm{dom}(f_p)$ we have that $(q \cap Z, f_q \restriction Z) \Vdash$ '$f_q(Z) \leq f_p(Z)$'.

We will make use of the following results for working with Neeman iterations:

Lemma 2.29 ([10, Lemma 77]). *Let $(p, f) \in \mathbb{P}(J)$ and $X \in \mathcal{T}$ be non-trivial. Then there is $(q, g) \in \mathbb{P}(J)$ such that:*

1. *$X \in q$,*
2. *$(q, g) \leq (p, f)$,*
3. $\mathrm{dom}(g) = \mathrm{dom}(f) \cup \{X\}$,
4. *For all $A \in q \setminus p$, either*
   a) *A is transitive, or*
   b) *There is $N \in \mathcal{S} \cap p$ and $W \in \mathcal{T} \cap q$ such that $A = N \cap W$.*

Proposition 2.30 ([10, Proposition 78]). *Let $(p, f) \in \mathbb{P}(J)$ and $M \in \mathcal{S}$. Then there is $(q, g) \in \mathbb{P}(J)$ such that:*

1. *$M \in q$,*
2. *$(q, g) \leq (p, f)$,*
3. $\mathrm{dom}(g) = \mathrm{dom}(f)$.

Proposition 2.31 ([10, Proposition 83]). *Let $X \in \mathcal{T}$ and let $\dot{G}_X$ be the canonical name for the $\mathbb{P}(J) \cap X$ generic filter.*

1. *If X is non-trivial, $\mathbb{P}(J) \cap X^+$ and $(\mathbb{P}(J) \cap X) * \mathbb{S}_\in(J(X), \mathcal{S}_{X,X^+}[\dot{G}_X])$ are forcing equivalent.*
2. *If X is trivial, $\mathbb{P}(J) \cap X^+$ and $(\mathbb{P}(J) \cap X) * \mathbb{S}_\in(1, \mathcal{S}_{X,X^+}[\dot{G}_X])$ are forcing equivalent (where 1 is the trivial forcing.)*

Lemma 2.32 ([10, Lemma 85]). *Let $X, Y \in \mathcal{T}$ and $M \in \mathcal{S}$ such that $X \in Y$ and $X, Y \in M$. Let $(p, f), (q, g) \in \mathbb{P}(J) \cap Y$ be such that:*

1. *$X, M \cap Y \in p$,*
2. $\mathrm{dom}(f) \cap M \subseteq X$,
3. *$(q, g) \in M$,*
4. *$p \cap M \sqsubseteq q$,*
5. *$(p \cap X, f \restriction X)$ and $(q \cap X, g \restriction X)$ are compatible in $\mathbb{P}(J) \cap X$.*

Then (p, f) and (q, g) are compatible in $\mathbb{P}(J) \cap Y$.

Theorem 2.33 ([10, Theorem 89]). *If $Z \in \mathcal{T}$, then $\mathbb{P}(J) \cap Z$ is $\mathcal{S}$-proper. In particular, if $\kappa \gg \theta$ is a sufficiently large regular cardinal and $\overline{M} \prec H(\kappa)$ such that $\mathbb{P}(J), Z \in \overline{M}$ and $M = \overline{M} \cap H(\theta) \in \mathcal{S}$, then given any $(p, f) \in \mathbb{P}(J) \cap Z$ with $M \cap Z \in p$, it follows that (p, f) is $(\overline{M}, \mathbb{P}(J) \cap Z)$-generic.*

Lemma 2.34 ([22, Lemma 6.7]). *$\mathbb{P}(J)$ is $\mathcal{T}$-strongly proper. In particular, if $X \in \mathcal{T}$ is such that $(p, f) \in X$, then $(p \cup \{X\}, f)$ extends (p, f) and is $(X, \mathbb{P}(J))$-strongly generic.*

Finally, we present some of the results about Neeman iterations:

Proposition 2.35 ([10, Theorem 89, Proposition 90]).

- *$\mathbb{P}(J)$ preserves $\aleph_1$ and collapses all cardinals $\aleph_1 < \kappa < \theta$.*
- *$\mathbb{P}(J)$ has the θ-chain condition.*
- *$\mathbb{P}(J) \Vdash$ '$\check{\theta} = \aleph_2$'.*
- *$\mathbb{P}(J)$ is $\mathcal{S}$-proper. In particular, if $\mathcal{S}$ is a club, then $\mathbb{P}(J)$ is proper.*

Definition 2.36. A function $J \colon \theta \to H(\theta)$ is a *Laver function* if for every set X and sufficiently large cardinal λ, there is an elementary embedding $j \colon V \to M$ such that:

1. $\mathrm{crit}(j) = \theta$,
2. $j(\theta) > \lambda$,
3. $[M]^\lambda \subseteq M$,
4. $j(J)(\theta) = X$.

Theorem 2.37 ([16]). *If θ is supercompact, then there is a Laver function $J \colon \theta \to H(\theta)$.*

Theorem 2.38 ([22, Lemma 6.14]). *Let $J \colon \theta \to H(\theta)$ be a Laver function. Then $\mathbb{P}(J)$ forces* PFA.

We repeat the proof of this result, since we will use minor modifications of this argument later.

Proof. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}(J)$-name for a proper forcing, $\langle \dot{D}_\alpha \rangle_{\alpha < \omega_1}$ be a sequence $\mathbb{P}$-names for dense-open subsets of $\dot{\mathbb{Q}}$, and $(p_0, f) \in \mathbb{P}(J)$. Let κ be sufficiently large that $\dot{\mathbb{Q}}$ and $\langle \dot{D}_\alpha \rangle_{\alpha < \omega_1}$ belong to $H(\kappa)$. Since J is a Laver function, we can find $\lambda \gg \theta, \kappa$ and an elementary embedding $j \colon V \to M$ such that:

1. $\mathrm{crit}(j) = \theta$,
2. $j(\theta) > \lambda$,
3. $[M]^\lambda \subseteq M$,
4. $j(J)(\theta) = \dot{\mathbb{Q}}$.

Since M is closed under taking subsets of size $\leq \lambda$, $j \restriction H(\kappa) \in M$. Thus M models that there exists $\theta', \kappa' < \lambda$, $\dot{\mathbb{A}}$, $\langle \dot{E}_\alpha \rangle_{\alpha<\omega_1} \in H(\kappa')$, and elementary embedding

$$j' \colon (H(\kappa'), J \restriction \theta', \dot{\mathbb{A}}, \langle \dot{E}_\alpha \rangle_{\alpha<\omega_1}) \to j(H(\kappa), J, \dot{\mathbb{Q}}, \langle \dot{D}_\alpha \rangle_{\alpha<\omega_1})$$

with critical point θ' such that $j(J(\theta')) = \dot{\mathbb{Q}} = j(\dot{\mathbb{A}})$ (since this is witnessed by $j' = j \restriction H(\kappa)$). Thus if we pull back via j, this means that V models that there exists $\theta', \kappa' < \theta$, $\dot{\mathbb{A}}$, $\langle \dot{E}_\alpha \rangle_{\alpha<\omega_1} \in H(\kappa')$ and elementary embedding

$$j' \colon (H(\kappa'), J \restriction \theta', \dot{\mathbb{A}}, \langle \dot{E}_\alpha \rangle_{\alpha<\omega_1}) \to (H(\kappa), J, \dot{\mathbb{Q}}, \langle \dot{D}_\alpha \rangle_{\alpha<\omega_1})$$

with critical point θ' such that $J(\theta') = \dot{\mathbb{A}}$. Moreover, we can pick θ' to be sufficiently large such that $(p_0, f) \in H(\theta')$. Thus via Lemma 2.34, letting $p = p_0 \cup \{H(\theta')\}$ we have that (p, f) is $(H(\theta'), \mathbb{P}(J))$-strongly generic.

Let $G_{(p,f)} \subseteq \mathbb{P}(J)$ be a generic filter containing (p, f). By strong properness, $G_{(p,f)} \cap H(\theta')$ is a V-generic filter for $\mathbb{P}(J) \cap H(\theta')$, and thus we can extend j' into an elementary embedding $j'' \colon H(\kappa')[G_{(p,f)} \cap H(\kappa')] \to H(\kappa)[G_{(p,f)}]$.

Consider the following set:

$$F = \{f(\theta')[G_{(p,f)} \cap H(\kappa')] \colon ((p, f) \in G_{(p,f)}) \wedge (\theta' \in p)\}.$$

By definition of $\mathbb{P}(J)$, F is a generic pre-filter for $\dot{\mathbb{A}}[G_{(p,f)} \cap H(\kappa')]$ over $H(\kappa')[G_{(p,f)} \cap H(\kappa')]$. Thus F meets all of the $\dot{E}_\alpha$ for $\alpha < \omega_1$. So taking $j''(F)$, we have a pre-filter for $\dot{\mathbb{Q}}$ in $V[G_{(p,f)}]$ that meets all the $\dot{D}_\alpha$ as required. □

3 SOLID GRAPHS

3.1 BLOCK-SEQUENCES AND DOM-BLOCK-SEQUENCES

We start by fixing some notation:

- Let a and b be sets of ordinals. We say that $a < b$ if $\alpha < \beta$ for all $\alpha \in a$ and $\beta \in b$.
- Given a set K and $m < \omega$, let $\mathrm{Fn}_m(\omega_1, K)$ be the family of partial functions from ω_1 to K whose domain are size m. We also write $\mathrm{Fn}(\omega_1, K) = \bigcup_{m<\omega} \mathrm{Fn}_m(\omega_1, K)$.
- Given two disjoint sets a, b we write

$$[a, b] = \{\{x, y\} : (x \in a) \wedge (y \in b)\}.$$

Given $a \in [\omega_1]^{<\aleph_0}$, we will implicitly use the enumeration $a = \{a(i) : i < |a|\}$ such that $a(i) < a(j)$ whenever $i < j$. Similarly, given $b \in \mathrm{Fn}_m(\omega_1, K)$, we will implicitly use the enumeration $b = \{b(i) \in \omega_1 \times K : i < m\}$ such that $\pi_0(b(i)) < \pi_0(b(j))$ whenever $i < j$.

Definition 3.1. We say that a sequence $\mathcal{C} \subseteq [\omega_1]^{<\aleph_0}$ is a *block-sequence* if $<$ is a total order on $\mathcal{C}$.

We will be mostly concerned with uncountable block-sequences $\mathcal{C} \subseteq [\omega_1]^m$ for a specific $m < \omega$.

Definition 3.2. Given a sequence $\mathcal{B} \subseteq \mathrm{Fn}(\omega_1, K)$, we say that $\mathcal{B}$ is a *dom-block-sequence* if:

1. For distinct $a, a' \in \mathcal{B}$, $\mathrm{dom}(a) \neq \mathrm{dom}(a')$;
2. $\{\mathrm{dom}(a) : a \in \mathcal{B}\}$ is a block-sequence.

We will frequently use the following basic properties of dom-block-sequences:

Proposition 3.3. *Let $\mathcal{B} \subseteq \mathrm{Fn}(\omega_1, K)$, λ a sufficiently large regular cardinal, and M a countable elementary submodel of $H(\lambda)$ containing $\mathcal{B}$. If there is a $b \in \mathcal{B}$ with $\mathrm{dom}(b) \cap M = \emptyset$, then $\mathcal{B}$ contains an uncountable dom-block-sequence.*

Proof. We will inductively generate a dom-block-sequence $\langle a_\xi \rangle \subseteq \mathcal{B}$ as follows:

- Let a_0 be the $<_w$-least element of $\mathcal{B}$;
- Given $\langle a_\xi : \xi < \alpha \rangle$ for $\alpha < \omega_1$, we let a_α be the $<_w$-least element of $\mathcal{B}$ such that $\operatorname{dom}(a_\xi) < \operatorname{dom}(a_\alpha)$ for all $\xi < \alpha$.

If this process continues for all $\alpha < \omega_1$, we are done. Suppose instead that this process stops at some $\alpha < \omega_1$, i.e. for every $c \in \mathcal{B}$ there is some $\xi < \alpha$ such that $\operatorname{dom}(a_\xi) \not< \operatorname{dom}(c)$. By elementarity, $\langle a_\xi : \xi < \alpha \rangle \in M$ and thus $\langle a_\xi : \xi < \alpha \rangle \subseteq M$. But then we have that $\operatorname{dom}(a_\xi) < \delta_M \leq \operatorname{dom}(b)$ for all $\xi < \alpha$, a contradiction. $\square$

Proposition 3.4. *Let $\mathcal{B}$ be an uncountable dom-block-sequence, λ a sufficiently large regular cardinal, and M a countable elementary submodel of $H(\lambda)$ containing $\mathcal{B}$. If $b \in \mathcal{B}$ is such that* $\operatorname{dom} b \cap M \neq \emptyset$, *then $b \in M$.*

Proof. Note that if $\mathcal{B}$ is a dom-block-sequence, then the elements of $\mathcal{B}$ must have pairwise-disjoint domains. Define $f\colon \bigcup\{\operatorname{dom}(b) : b \in \mathcal{B}\} \to \mathcal{B}$ which maps $\alpha \in \omega_1$ to the unique $b \in \mathcal{B}$ with $\alpha \in b$. By elementarity, f is in M. Thus for any $\alpha \in \operatorname{dom}(b) \cap M$ we have that $f(\alpha) = b \in M$. $\square$

3.2 SOLID GRAPHS

Definition 3.5 ([30, Definition 2.1])**.** Let K be a set and $m < \omega$. We say that a graph $\mathcal{G}$ on $\omega_1 \times K$ is *m-solid* if given any uncountable dom-block-sequence $\mathcal{B} = \{a_\alpha : \alpha < \omega_1\} \subseteq \operatorname{Fn}_m(\omega_1, K)$ there are $\alpha < \beta < \omega_1$ such that such that $[a_\alpha, a_\beta] \subseteq \mathcal{G}$.

We say that $\mathcal{G}$ is *strongly solid* if $\mathcal{G}$ is m-solid for every $m < \omega$.

Many interesting objects can be encoded as solid graphs:

Example 3.6. Suppose that $\mathbb{P}$ is a forcing notion and define $\mathcal{G}_\mathbb{P} \subseteq [\omega_1 \times \mathbb{P}]^2$ as follows:

$$\mathcal{G}_\mathbb{P} = \{\{\langle \nu_0, p_0 \rangle, \langle \nu_1, p_1 \rangle\} : p_0 \not\perp_\mathbb{P} p_1\}.$$

Then $\mathbb{P}$ is ccc iff $\mathcal{G}_\mathbb{P}$ is 1-solid.

Example 3.7 ([30])**.** Given $X = \{x_\alpha : \alpha < \omega_1\} \subseteq 2^{\omega_1}$, one can define two graphs $\mathcal{G}_X^<$ and $\mathcal{G}_X^>$ such that X is HFD_w^m iff $\mathcal{G}_X^<$ is m-solid and X is HFC_w^m iff $\mathcal{G}_X^>$ is m-solid.

Example 3.8. Suppose that $E = \{e_\alpha : \alpha < \omega_1\} \subseteq \mathbb{R}$. Define $\mathcal{G}_E \subseteq [\omega_1 \times \omega_1^m \times \{>, <\}^m]^2$ to be the set of pairs $\{\langle \nu_0, a_0, R_0 \rangle, \langle \nu_1, a_1, R_1 \rangle\}$ such that $\nu_0 < \nu_1$ and either:

- $a_0 \not< a_1$, or
- $R_0 \neq R_1$, or
- for all $i < m$, $e_{a_0(i)} \ R_1(i) \ e_{a_1(i)}$.

Then E is m-entangled iff $\mathcal{G}_E$ is 1-solid.

Proposition 3.9. *Let $\mathcal{G}$ be a graph on $\omega_1 \times K$. The following are equivalent:*

1. *$\mathcal{G}$ is m-solid;*
2. *For every λ a sufficiently large regular cardinal and M a countable elementary submodel of $H(\lambda)$ containing $\mathcal{G}$, if $\mathcal{B} \in M$ is an uncountable subset of $\mathrm{Fn}_m(\omega_1, K)$, for every $b \in \mathcal{B}$ with $\mathrm{dom}(b) \cap M = \emptyset$ there is some $a \in \mathcal{B} \cap M$ such that $[a, b] \subseteq \mathcal{G}$;*
3. *For every uncountable dom-block-sequence $\mathcal{B} \subseteq \mathrm{Fn}_m(\omega_1, K)$, there is a countable $\mathcal{B}' \subseteq \mathcal{B}$ and $\delta < \omega_1$ such that for all $b \in \mathcal{B}$ with $\mathrm{dom}(b) > \{\delta\}$, there is $a \in \mathcal{B}'$ such that $[a, b] \subseteq \mathcal{G}$;*
4. *For every λ a sufficiently large regular cardinal and M a countable elementary submodel of $H(\lambda)$ containing $\mathcal{G}$, if $\mathcal{B} \in M$ is an uncountable dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$, there is $b \in \mathcal{B} \setminus M$ and $a \in \mathcal{B} \cap M$ such that $[a, b] \subseteq \mathcal{G}$.*

Proof. Note that (2) $\implies$ (3) $\implies$ (4) $\implies$ (1) is immediate, so it suffices to prove that (1) $\implies$ (2). To this end, let $\mathcal{G}$ be an m-solid graph, λ a sufficiently large regular cardinal, $M \prec H(\lambda)$ a countable elementary submodel containing $\mathcal{G}$, $\mathcal{B} \in M$ be an uncountable subset of $\mathrm{Fn}_m(\omega_1, K)$, and $b \in \mathcal{B}$ such that $\mathrm{dom}(b) \cap M = \emptyset$. Inductively generate a sequence $\langle a_\xi \rangle \subseteq \mathcal{B}$ as follows: a_0 is the $<_w$-least element of $\mathcal{B}$, and given $\langle a_\xi \rangle$ for all $\xi < \alpha$, let a_α be the $<_w$ least element of $\mathcal{B}$ such that for all $\xi < \alpha$:

- $\mathrm{dom}(a_\xi) < \mathrm{dom}(a_\alpha)$,
- $[a_\xi, a_\alpha] \not\subseteq \mathcal{G}$.

If no such element exists, we stop this process. In fact, since $\mathcal{G}$ is m-solid this process must stop at some countable $\alpha < \omega_1$. By elementarity, it thus holds that $\{a_\xi : \xi < \alpha\} \subseteq M$. Since $b \notin M$, b is not part of our sequence $\{a_\xi : \xi < \alpha\}$. But since $\mathrm{dom}(a_\xi) < \delta_M \leq \mathrm{dom}(b)$ for all $\xi < \alpha$, it must thus follow that there is some $\xi < \alpha$ such that $[a_\xi, b] \subseteq \mathcal{G}$. □

We can think of the above proposition as an extension of the following well-known characterisation of ccc forcings:

Corollary 3.10. *Let $\mathbb{P}$ be a forcing notion. The following are equivalent:*

1. *$\mathbb{P}$ is ccc;*

2. *Let λ be a sufficiently large regular cardinal, A an uncountable subset of $\mathbb{P}$, and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$ and A. If $p \in A$, there is some $q \in A \cap M$ that is compatible with p.*

Definition 3.11. Let $\mathcal{G}$ be an m-solid graph on $\omega_1 \times K$ and $\mathbb{P}$ be a forcing notion.

- $\mathbb{P}$ *preserves* $\mathcal{G}$ if $\mathbb{P}$ forces that $\mathcal{G}$ is an m-solid graph.
- $\mathbb{P}$ *destroys* $\mathcal{G}$ if $\mathbb{P}$ forces that $\mathcal{G}$ is not an m-solid graph.

We have the following result that reduces preserving a m-solid graph to a combinatorial problem:

Proposition 3.12. *Let $\mathcal{G}$ be an m-solid graph on $\omega_1 \times K$ and $\mathbb{P}$ be a proper forcing notion. The following are equivalent:*

1. *$\mathbb{P}$ preserves $\mathcal{G}$;*

2. *Let λ be a sufficiently large regular cardinal, $\dot{\mathcal{B}}$ a $\mathbb{P}$-name for an uncountable subset of $\mathrm{Fn}_m(\omega_1, K)$, and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$, $\mathcal{G}$, and $\dot{\mathcal{B}}$. If $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic, and $b \in \mathrm{Fn}_m(\omega_1, K)$ such that $\mathrm{dom}(b) \cap M = \emptyset$ and $p \Vdash$ '$b \in \dot{\mathcal{B}}$', then there is $q \in \mathbb{P} \cap M$ and $a \in \mathrm{Fn}_m(\omega_1, K) \cap M$ such that $q \Vdash$ '$a \in \dot{\mathcal{B}}$', q is compatible with p, and $[a, b] \subseteq \mathcal{G}$;*

3. *Let λ be a sufficiently large regular cardinal, $\dot{\mathcal{B}}$ a $\mathbb{P}$-name for an uncountable dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$, and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$, $\mathcal{G}$, and $\dot{\mathcal{B}}$. If $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic, there is $p' \leq p$ and $b \in \mathrm{Fn}_m(\omega_1, K) \setminus M$ such that $p' \Vdash$ '$b \in \dot{\mathcal{B}}$', and there is $q \in \mathbb{P} \cap M$ and $a \in \mathrm{Fn}_m(\omega_1, K) \cap M$ such that $q \Vdash$ '$a \in \dot{\mathcal{B}}$', q is compatible with p', and $[a, b] \subseteq \mathcal{G}$.*

Proof.

- **(1) implies (2).** Suppose $\mathbb{P}$ preserves $\mathcal{G}$ and take λ, $\dot{\mathcal{B}}$, M, p, and b as in the statement of the proposition. Let $G_p \subseteq \mathbb{P}$ be a generic filter containing p. We work in $V[G_p]$. Since $\mathbb{P}$ is proper and $\mathbb{P}$ preserves $\mathcal{G}$, by Proposition 3.9 we can find $a \in \dot{\mathcal{B}}[G_p] \cap M[G_p]$ such that $[a, b] \subseteq \check{\mathcal{G}}$. Then pick $q \in M \cap G_p$ such that $q \Vdash$ '$a \in \dot{\mathcal{B}}$'. As G_p is a filter, p and q are compatible as required.
- **(2) implies (3).** Immediate.

- **(3) implies (1).** Let $r \in \mathbb{P}$ and $\dot{\mathcal{B}}$ be a $\mathbb{P}$-name for a dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$, and take λ a sufficiently large regular cardinal and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$, $\mathcal{G}$, $\dot{\mathcal{B}}$, and r. Pick some $p \le r$ and $b \in \mathrm{Fn}_m(\omega_1, K)$ such that p is $(M, \mathbb{P})$-generic. Then by our hypothesis, we know that there is $p' \le p$ and $b \in \mathrm{Fn}_m(\omega_1, K) \setminus M$ such that $p' \Vdash$ '$b \in \dot{\mathcal{B}}$', and $q \in \mathbb{P} \cap M$ compatible with p' and $a \in \mathrm{Fn}_m(\omega_1, K) \cap M$ such that $q \Vdash$ '$a \in \dot{\mathcal{B}}$' and $[a, b] \subseteq \check{G}$. Thus via Proposition 3.9, any condition extending p' and q is sufficient. $\square$

The following result formalises the notion that preserving strongly solid graphs and m-solid graphs is more difficult than preserving 1-solid graphs.

Proposition 3.13. *Let $\mathcal{G} \subseteq [\omega_1 \times K]^2$. For each of the below properties, we can find some set K' and graph $\mathcal{G}' \subseteq [\omega_1 \times K']^2$ such that $|K'| \le \max\{\aleph_0, |K|\}$ and $\mathcal{G}$ has the given property iff $\mathcal{G}'$ is 1-solid.*

- *$\mathcal{G}$ is m-solid (for some $m < \omega$).*
- *$\mathcal{G}$ is strongly solid.*

Proof. We will focus on the case of m-solid in full, since the proof for strongly solid is analogous. Pick some bijection $f : \omega_1 \to [\omega_1]^m$. We then define the graph $\mathcal{G}' \subseteq [\omega_1 \times K^m]$ to be the set of all pairs $\{\langle \nu_0, \vec{k}_0 \rangle, \langle \nu_1, \vec{k}_1 \rangle\}$ such that $\nu_0 < \nu_1$ and either:

- $f(\nu_0) \not< f(\nu_1)$, or
- for all $i, j < m$, $\{\langle f(\nu_0)(i), \vec{k}_0(i) \rangle, \langle f(\nu_1)(j), \vec{k}_1(j) \rangle\} \in \mathcal{G}$.

Suppose that $\mathcal{G}$ is m-solid and let $\mathcal{A} = \{\langle a_\alpha, \vec{k}_\alpha \rangle : \alpha < \omega_1\}$ be an uncountable dom-block-sequence contained in $\mathrm{Fn}_1(\omega_1, K^m)$. If there is $\alpha < \beta < \omega_1$ such that $f(a_\alpha) \cap f(a_\beta) \not< \emptyset$ then we are done, so we can assume that for all $\alpha < \beta < \omega_1$, $f(a_\alpha) < f(a_\beta)$. Define $\mathcal{A}'$ as follows:

$$\mathcal{A}' = \{\{\langle f(a_\alpha)(i), \vec{k}_\alpha(i) \rangle : i < m\} : \alpha < \omega_1\}.$$

Then $\mathcal{A}'$ is a dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$, so since $\mathcal{G}$ is m-solid we can find some $\alpha < \beta$ such that for all $i, j < m$,

$$\{\langle f(a_\alpha)(i), \vec{k}_\alpha(i) \rangle, \langle f(a_\beta)(j), \vec{k}_\beta(j) \rangle\} \in \mathcal{G}.$$

But then $\{\langle a_\alpha, \vec{k}_\alpha \rangle, \langle a_\beta, \vec{k}_\beta \rangle\} \in \mathcal{G}'$ as required, so $\mathcal{G}'$ is 1-solid.

Now suppose that $\mathcal{G}'$ is 1-solid and $\mathcal{B} = \{b_\alpha : \alpha < \omega_1\}$ is a dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$. Define $\mathcal{B}'$ as follows:

$$\mathcal{B}' = \{\langle f^{-1}(\mathrm{dom}(b_\alpha)), \langle \pi_1(b_\alpha(i)) \rangle_{i<m} \rangle\}.$$

Then $\mathcal{B}'$ is a dom-block-sequence contained in $\mathrm{Fn}_1(\omega_1, K^m)$, so we can find some $\alpha < \beta$ such that for all $i, j < m$, $\{b_\alpha(i), b_\beta(j)\} \in \mathcal{G}$. In other words, we have that $[b_\alpha, b_\beta] \subseteq \mathcal{G}$ and thus $\mathcal{G}$ is m-solid. □

Question 3.14. In general, for $m > n > 1$ can we encode m-solid graphs as n-solid graphs without changing the cardinality of K? In other words: if for some infinite cardinal κ a proper forcing $\mathbb{P}$ preserves all n-solid graphs on $\omega_1 \times \kappa$, must it preserve all m-solid graphs on $\omega_1 \times \kappa$?

3.3 (m,n)-SOLID GRAPHS

In this section, we'll define a new, slight modification of the notion of m-solid graph. We will discover that this modified notion will be preserved by a larger collection of interesting proper forcings.

Definition 3.15. Let K be a set and $m, n < \omega$. We say that a graph $\mathcal{G}$ on $\omega_1 \times K$ is *(m,n)-solid* if given any uncountable dom-block-sequences $\mathcal{A} = \langle a_\xi \rangle_{\xi<\omega_1} \subseteq \mathrm{Fn}_m(\omega_1, K)$ and $\mathcal{B} = \langle b_\xi \rangle_{\xi<\omega_1} \subseteq \mathrm{Fn}_n(\omega_1, K)$, there are $\alpha < \beta < \omega_1$ such that $[a_\alpha, b_\beta] \subseteq \mathcal{G}$.

We say that $\mathcal{G}$ is *$(m,{<}\omega)$-solid* (*$({<}\omega,n)$-solid*) if $\mathcal{G}$ is (m,n)-solid for every $n < \omega$ (resp. for every $m < \omega$).

We can think of the notion of (m,n)-solid graphs being a 'rectangularisation' of the notion of m-solid graph. This notion can be connected to Soukup's notion of solid graph as follows:

Proposition 3.16. *Let $\mathcal{G} \subseteq [\omega_1, K]^2$ and $m, n < \omega$.*

1. *If $\mathcal{G}$ is $m+n$-solid, it is (m,n)-solid.*
2. *If $\mathcal{G}$ is (m,n)-solid, it is $\min\{m,n\}$-solid.*

In particular, this implies that a graph $\mathcal{G}$ is strongly solid iff it is (m,n)-solid for all $m, n < \omega$.

Proof. Suppose $\mathcal{G}$ is $m+n$-solid. Let $\mathcal{A} = \langle a_\xi \rangle_{\xi<\omega_1} \subseteq \mathrm{Fn}_m(\omega_1, K)$ and $\mathcal{B} = \langle b_\xi \rangle_{\xi<\omega_1} \subseteq \mathrm{Fn}_n(\omega_1, K)$ be uncountable dom-block-sequences. We will inductively construct an increasing sequence of ordinals $\{\eta_\xi \in \omega_1 : \xi < \omega_1\}$ and a third dom-block-sequence $\mathcal{C} = \langle c_\xi \rangle_{\xi<\omega_1}$ as follows: given $\langle \eta_\xi \rangle_{\xi<\delta}$ and $\langle c_\xi \rangle_{\xi<\delta}$ for some $\delta < \omega_1$, let γ_δ be the minimal ordinal such that $\{\gamma_\delta\} > \mathrm{dom}(c_\xi)$ for all $\xi < \delta$. Then let η_δ be the least ordinal such that for all $\xi < \delta$:

- $\eta_\delta > \eta_\xi$,
- $\mathrm{dom}(a_{\eta_\delta}) > \mathrm{dom}(a_{\eta_\xi})$,

- $\text{dom}(b_{\eta_\delta}) > \text{dom}(b_{\eta_\xi})$.

Then let $c_\delta = a_{\eta_\delta} \cup b_{\eta_\delta}$. Since $\mathcal{G}$ is $m+n$-solid we can find $\alpha < \beta < \omega_1$ such that $[c_\alpha, c_\beta] \subseteq \mathcal{G}$. But then $[a_{\eta_\alpha}, b_{\eta_\beta}] \subseteq \mathcal{G}$ as required.

Conversely, let $\mathcal{G}$ be (m,n) solid. We can assume that $m > n$ (the proof of the other case is identical). Let $\mathcal{B} = \langle b_\xi \rangle_{\xi<\omega_1} \subseteq \text{Fn}_n(\omega_1, K)$ be an uncountable dom-block-sequence. We will again inductively construct a second dom-block-sequence $\mathcal{A} = \langle a_\xi \rangle_{\xi<\omega_1}$ as follows: given $\langle a_\xi \rangle_{\xi<\delta}$ for some $\delta < \omega_1$, let γ_δ be the minimal ordinal such that $\text{dom}(b_{\gamma_\delta}) > \text{dom}(a_\xi)$ for all $\xi < \delta$. Then pick some $a_\delta \in \text{Fn}_m(\omega_1, K)$ such that $b_{\gamma_\delta} \sqsubseteq a_\delta$. Since $\mathcal{G}$ is (m,n)-solid, we can find some $\alpha < \beta < \omega_1$ such that $[a_\alpha, a_\beta \restriction n] \subseteq \mathcal{G}$. But this implies that there are some $\alpha' < \beta' < \omega_1$ such that $[b_{\alpha'}, b_{\beta'}] \subseteq \mathcal{G}$ as required. $\square$

Just as with the notion of m-solid, there is a nice combinatorial characterisation of a proper forcing notion preserving an (m,n)-solid graph:

Proposition 3.17. *Let $\mathcal{G} \subseteq [\omega_1, K]^2$. The following are equivalent:*

1. *$\mathcal{G}$ is (m,n)-solid;*
2. *For all uncountable dom-block-sequences $\mathcal{A} \subseteq \text{Fn}_m(\omega_1, K)$ and $\mathcal{B} \subseteq \text{Fn}_n(\omega_1, K)$, there is a countable subsequence $\mathcal{A}' \subseteq \mathcal{A}$ and $\delta < \omega_1$ such that for all $b \in \mathcal{B}$ with $\text{dom}(b) > \{\delta\}$, there is $a \in \mathcal{A}'$ with $[a,b] \subseteq \mathcal{G}$;*
3. *For every uncountable dom-block-sequence $\mathcal{A} \subseteq \text{Fn}_m(\omega_1, K)$, there is a countable $\mathcal{A}' \subseteq \mathcal{A}$ and $\delta < \omega_1$ such that for all $b \in \text{Fn}_n(\omega_1, K)$ with $\text{dom}(b) > \{\delta\}$, there is $a \in \mathcal{A}'$ such that $[a,b] \subseteq \mathcal{G}$;*
4. *For every λ a sufficiently large regular cardinal and M a countable elementary submodel of $H(\lambda)$ containing $\mathcal{G}$, if $\mathcal{A} \in M$ is an uncountable dom-block-sequence contained in $\text{Fn}_m(\omega_1, K)$, for every $b \in \text{Fn}_n(\omega_1, K)$ with $\text{dom}(b) \cap M = \emptyset$ there is some $a \in \mathcal{A} \cap M$ such that $[a,b] \subseteq \mathcal{G}$.*

Proof. Again, (3) $\iff$ (4) $\implies$ (2) $\implies$ (1) is immediate, so we focus on $\neg$(3) $\implies$ $\neg$(1). Suppose that $\mathcal{A} = \langle a_\xi \rangle_{\xi<\omega_1} \subseteq \text{Fn}_m(\omega_1, K)$ witnesses that $\mathcal{G}$ does not satisfy (3). We will iteratively construct a dom-block-sequence $\mathcal{B} = \langle b_\xi \rangle_{\xi<\omega_1} \subseteq \text{Fn}_n(\omega_1, K)$ as follows: given $\langle b_\xi \rangle_{\xi<\delta}$ for some $\delta < \omega_1$, let γ_δ be the least ordinal such that $\text{dom}(b_\xi) < \{\gamma_\delta\}$ for all $\xi < \delta$. Then since $\langle a_\xi \rangle_{\xi<\delta}$ is countable, by $\neg$(3) we can find some $b_\delta \in \text{Fn}_n(\omega_1 \setminus \gamma_\delta, K)$ such that $[a_\xi, b_\delta] \not\subseteq \mathcal{G}$ for all $\xi < \delta$. Then $\mathcal{A}$, $\mathcal{B}$ witness that $\mathcal{G}$ is not (m,n)-solid. $\square$

Proposition 3.18. *Let $\mathcal{G} \subseteq [\omega_1 \times K]^2$ be an (m,n)-solid graph and $\mathbb{P}$ be a proper forcing notion. The following are equivalent:*

1. *$\mathbb{P}$ preserves that $\mathcal{G}$ is (m,n)-solid;*

2. *Let λ be a sufficiently large regular cardinal, $\dot{\mathcal{A}}$ a $\mathbb{P}$-name for an uncountable dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$, and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$, $\mathcal{G}$, and $\dot{\mathcal{A}}$. If $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic and $b \in \mathrm{Fn}_m(\omega_1, K)$ such that $\mathrm{dom}(b) \cap M = \emptyset$, then there is $q \in \mathbb{P} \cap M$ and $a \in \mathrm{Fn}_m(\omega_1, K) \cap M$ such that $q \Vdash$ '$a \in \dot{\mathcal{A}}$', q is compatible with p, and $[a, b] \subseteq \mathcal{G}$.*

Proof. Similar to Proposition 3.12, but using Proposition 3.17. □

Remark 3.19. Comparing Proposition 3.12 and Proposition 3.18 allows us to see why preserving (m,n)-solid graphs is 'easier' than preserving m-solid graphs. Furthermore, all of the arguments in following sections that a given forcing notion preserves an m-solid graph will be able to be easily modified to preserve (m,n)-solid graphs. The converse is not true, however: we will give examples of proper forcing notions that preserve (m,n)-solid graphs but destroy certain 1-solid graphs.

The following results provides a partial formalisation of the above remark.

Proposition 3.20. *Let $\mathcal{G} \subseteq [\omega_1 \times K]^2$ and $0 < m, n < \omega$. There is a set K' and graph $\mathcal{G}' \subseteq [\omega_1 \times K']^2$ such that $|K'| \leq \max\{\aleph_0, |K|\}$ and $\mathcal{G}$ is (m,n)-solid iff $\mathcal{G}'$ is 1-solid. (An analogous result is true for $(m, {<}\omega)$-solid and $({<}\omega, n)$-solid.)*

Proof. We will focus on the case of $(1,1)$-solid, as the argument for (m,n)-solid is very similar (with some addition of ideas from Proposition 3.13). Pick some bijection $f : \omega_1 \to \omega_1^2$, and let $f_0, f_1 : \omega_1 \to \omega_1$ be such that $f(\alpha) = \langle f_0(\alpha), f_1(\alpha)\rangle$. We then define the graph $\mathcal{G}' \subseteq [\omega_1 \times K^2]$ to be the set of all pairs $\{\langle \nu_0, \vec{k}_0\rangle, \langle \nu_1, \vec{k}_1\rangle\}$ such that $\nu_0 < \nu_1$ and either:

- $f_0(\nu_0) = f_0(\nu_1)$, or
- $f_1(\nu_0) = f_1(\nu_1)$, or
- $\{\langle f_0(\nu_0), \vec{k}_0(0)\rangle, \langle f_1(\nu_1), \vec{k}_1(1)\rangle\} \in \mathcal{G}$.

Suppose that $\mathcal{G}$ is $(1,1)$-solid and let $\mathcal{A} = \{\langle a_\xi, \vec{k}_\xi\rangle : \xi < \omega_1\}$ be an uncountable dom-block-sequence contained in $\mathrm{Fn}_1(\omega_1, K^2)$. If there is α, β distinct with either $f_0(a_\alpha) = f_0(a_\beta)$ or $f_1(a_\alpha) = f_1(a_\beta)$ we are done, so we can assume (thinning $\mathcal{A}$ if necessary) that for all $\alpha < \beta < \omega_1$ and $i < 2$, $f_i(a_\alpha) < f_i(a_\beta)$. For $i < 2$, define $\mathcal{A}^{(i)}$ as follows:

$$\mathcal{A}^{(i)} = \{\langle f_i(a_\xi), \vec{k}_\xi(i)\rangle : \xi < \omega_1\}.$$

Then $\mathcal{A}^{(i)}$ is an uncountable dom-block-sequence contained in $\mathrm{Fn}_1(\omega_1, K)$ for $i < 2$. Thus as $\mathcal{G}$ is $(1,1)$-solid we can find some $\alpha < \beta$ such that

$$\{\langle f_0(a_\alpha), \vec{k}_\alpha(0)\rangle, \langle f_1(a_\beta), \vec{k}_\beta(1)\rangle\} \in \mathcal{G}$$

and thus $\{\langle a_\alpha, \vec{k}_\alpha\rangle, \langle a_\beta, \vec{k}_\beta\rangle\} \in \mathcal{G}'$. Hence $\mathcal{G}'$ is 1-solid as required.

Now suppose that $\mathcal{G}'$ is 1-solid and for $i < 2$, $\mathcal{B}^{(i)} = \{\langle b^{(i)}_\alpha, k^{(i)}_\alpha\rangle : \alpha < \omega_1\}$ is an uncountable dom-block-sequence contained in $\mathrm{Fn}_1(\omega_1, K)$. For $\xi < \omega_1$, define $c_\xi \in \mathrm{Fn}_1(\omega_1, K^2)$ as follows:

$$c_\xi = \langle f^{-1}(\langle b^{(0)}_\xi, b^{(1)}_\xi\rangle), \langle k^{(0)}_\xi, k^{(1)}_\xi\rangle\rangle.$$

Thus $\mathcal{C} = \langle c_\xi\rangle_{\xi<\omega_1}$ is an uncountable dom-block-sequence contained in $\mathrm{Fn}_1(\omega_1, K^2)$, so we can find some $\alpha < \beta < \omega_1$ such that $[c_\alpha, c_\beta] \subseteq \mathcal{G}'$. This implies that

$$\{\langle b^{(0)}_\alpha, k^{(0)}_\alpha\rangle, \langle b^{(1)}_\beta, k^{(1)}_\beta\rangle\} \in \mathcal{G}$$

and thus $\mathcal{G}$ is $(1,1)$-solid. □

Proposition 3.21. *Let $\mathcal{G} \subseteq [\omega_1 \times K]^2$ and $0 < m, n < \omega$. There is a set K' and graph $\mathcal{G}' \subseteq [\omega_1 \times K']^2$ such that $|K'| \leq \max\{\aleph_0, |K|\}$ and $\mathcal{G}$ is (m,n)-solid iff $\mathcal{G}'$ is $(m,1)$-solid. (An analogous result is true for $({<}\omega, n)$-solid.)*

Proof. Pick some bijection $f\colon \omega_1 \to [\omega_1]^n$. We then define the graph $\mathcal{G}' \subseteq [\omega_1 \times K^n]$ to be the set of all pairs $\{\langle \nu_0, \vec{k}_0\rangle, \langle \nu_1, \vec{k}_1\rangle\}$ such that $\nu_0 < \nu_1$ and either:

- $\{\nu_0\} \not< f(\nu_1)$, or
- for all $j < n$, $\{\langle \nu_0, \vec{k}_0(0)\rangle, \langle f(\nu_1)(j), \vec{k}_1(j)\rangle\} \in \mathcal{G}$.

Suppose that $\mathcal{G}$ is (m,n)-solid and let $\mathcal{A}$ be an uncountable dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K^n)$. Let λ be a sufficiently large regular cardinal and M a countable elementary submodel of $H(\lambda)$ containing all relevant objects. By Proposition 3.17, we need to show that for every $\langle \beta, \vec{k}\rangle \in \mathrm{Fn}_1(\omega_1, K^n)$ with $\beta > \delta_M = \emptyset$ there is some $a \in \mathcal{A} \cap M$ such that $[a, \{\langle \beta, \vec{k}\rangle\}] \subseteq \mathcal{G}'$. Note that if $f(\beta) \cap M \neq \emptyset$ then we can just pick $a \in \mathcal{A} \cap M$ such that $\mathrm{dom}(a) \not< f(\beta)$, so we can assume that $f(\beta) \cap M = \emptyset$. For each $a \in \mathcal{A}$, define $a^{(0)} = \{\langle \mathrm{dom}(a)(i), a(i)(0)\rangle : i < m\}$. Then $\mathcal{A}^{(0)} = \{a^{(0)} : a \in \mathcal{A}\}$ is an uncountable dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K) \cap M$. Let $b = \{\langle f(\beta)(j), \vec{k}(j) : j < n\}$. Then since $\mathcal{G}$ is (m,n)-solid there is some $a \in \mathcal{A} \cap M$ such that $[a^{(0)}, b] \subseteq \mathcal{G}$. But this implies that $[a, \{\langle \beta, \vec{k}\rangle\}] \subseteq \mathcal{G}'$ as required.

Conversely, suppose that $\mathcal{G}'$ is $(m,1)$-solid and let $\mathcal{C}$ be an uncountable dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$. Again, let λ be a sufficiently large regular cardinal and M a countable elementary submodel of

$H(\lambda)$ containing all relevant objects. For all $c \in \mathcal{C}$, let $\bar{c} = \langle \mathrm{dom}(c)(i), \langle < c(i) \rangle_{j<n} : i < m \rangle$. Then $\overline{\mathcal{C}} = \{\bar{c} : c \in \mathcal{C}\}$ is an uncountable dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K^n) \cap M$. Suppose $b \in \mathrm{Fn}_n(\omega_1, K)$ such that $\mathrm{dom}(b) \cap M = \emptyset$, and let $b' = \langle f^{-1}(\mathrm{dom}(b)), \langle b(j) \rangle_{j<n} \rangle \in \mathrm{Fn}_1(\omega_1, K^n)$. Then since $f^{-1}(\mathrm{dom}(b)) > \delta_M$ and $\mathcal{G}'$ is $(m, 1)$ solid, we can find some $c \in \mathcal{C} \cap M$ such that $[\bar{c}, b'] \subseteq \mathcal{G}'$. But this implies that $[c, b] \subseteq \mathcal{G}$ as required. □

Question 3.22. In general, for $m, n > 1$ is it possible to encode (m, n)-solid graphs via $(1, 1)$-solid graphs or $\min\{m, n\}$-solid graphs without changing the cardinality of K?

3.4 TWO-TYPE FORCINGS AND SOLID GRAPHS

We now work to show that Neeman iterations of proper forcings preserving an m-solid or (m, n)-solid graph $\mathcal{G}$ also preserves $\mathcal{G}$. All the results in this chapter will be proved for m-solid graphs, since by Proposition 3.20 this also implies the results hold for (m, n)-solid graphs.

Proposition 3.23. *Let $\mathbb{P}$ be strongly proper and $\mathcal{G}$ be an m-solid or (m, n)-solid graph. Then $\mathbb{P}$ preserves $\mathcal{G}$.*

Proof. We will prove this result (and all other results in the rest of this chapter) via Proposition 3.12. Suppose $\mathcal{G} \subseteq [\omega_1 \times K]^2$ is m-solid. Let λ be a sufficiently large regular cardinal, $\dot{\mathcal{B}}$ a $\mathbb{P}$-name for an uncountable dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$, and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$, $\dot{\mathcal{B}}$, and $\mathcal{G}$. Let $p \in \mathbb{P}$ be $(M, \mathbb{P})$-generic and $b \in \mathrm{Fn}_m(\omega_1, K)$ such that $\mathrm{dom}(b) \cap M = \emptyset$ and $p \Vdash$ '$b \in \dot{\mathcal{B}}$'. Pick $p' \leq p$ such that p' is strongly $(M, \mathbb{P})$-generic. Then define the set W of pairs $(q, c) \in \mathbb{P} \cap \mathrm{Fn}_m(\omega_1, K)$ such that:

1. $q \leq p' \restriction M$,
2. $q \Vdash$ '$c \in \dot{\mathcal{B}}$'.

Note that $(p', b) \in W$ and by elementarity $(W \in M)$. Thus by Proposition 3.9, we can find some $(q, a) \in W \cap M$ such that $[a, b] \subseteq \mathcal{G}$. Since $q \leq p' \restriction M$, q and p' (and thus q and p) are compatible as required. □

Proposition 3.24. *Let $\mathcal{G}$ be an m-solid or (m, n)-solid graph on $\omega_1 \times K$, $\mathbb{P}$ be a proper forcing preserving $\mathcal{G}$, and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a proper forcing that preserves $\mathcal{G}$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ preserves $\mathcal{G}$.*

Proof. Suppose $\mathcal{G} \subseteq [\omega_1 \times K]^2$ is m-solid. Let λ be a sufficiently large regular cardinal, $\dot{\mathcal{B}}$ a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a dom-block-sequence contained in

$\mathrm{Fn}_m(\omega_1, K)$, and M a countable elementary submodel of $H(\lambda)$ containing all relevant objects. Let $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ be $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic and $b \in \mathrm{Fn}_m(\omega_1, K)$ such that $\mathrm{dom}(b) \cap M = \emptyset$ and $(p, \dot{q}) \Vdash$ '$b \in \dot{\mathcal{B}}$'. Let $G_{\mathbb{P}} \subseteq \mathbb{P}$ be a generic filter containing p.

Working in $V[G_{\mathbb{P}}]$, we have that $\dot{\mathcal{B}}[G_{\mathbb{P}}]$ is a $\dot{\mathbb{Q}}[G_{\mathbb{P}}]$-name for a dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$, $\dot{q}$ is $(M[G_{\mathbb{P}}], \dot{\mathbb{Q}}[G_{\mathbb{P}}])$-generic, $\mathrm{dom}(b) \cap M[G_{\mathbb{P}}] = \emptyset$, and $\dot{q} \Vdash$ '$b \in \dot{\mathcal{B}}$'. Thus since $\dot{\mathbb{Q}}[G_{\mathbb{P}}])$ preserves that $\mathcal{G}$ is m-solid, we can find $\dot{q}' \in \dot{\mathbb{Q}}[G_{\mathbb{P}}] \cap M[G_{\mathbb{P}}]$ and $a \in \mathrm{Fn}_m(\omega_1, K) \cap M[G_{\mathbb{P}}]$ such that $\dot{q} \Vdash$ '$a \in \dot{\mathcal{B}}$', q' is compatible with q, and $[a, b] \subseteq \mathcal{G}$. Since p is $(M, \mathbb{P})$-generic and $\mathrm{Fn}_m(\omega_1, K) \in M$, we have that $a \in M$. Pick $p' \in G_{\mathbb{P}} \cap M$ such that $(p', \dot{q}') \in \dot{\mathbb{Q}} \cap M$ and $(p', \dot{q}') \Vdash$ '$a \in \dot{\mathcal{B}}$'. Then $(p, \dot{q})$ and $(p', \dot{q}')$ are compatible as required. □

Theorem 3.25. *Let θ be an inaccessible cardinal, and J, $\mathcal{S}$, $\mathcal{T}$, and $\mathbb{P}(J)$ as in Section 2.4. Suppose that $\mathcal{G}$ is an m-solid or (m, n)-solid graph. If for every $Y \in \mathcal{T}$ either Y is trivial or $\mathbb{P}(J) \cap Y$ forces that $J(Y)$ preserves $\mathcal{G}$, then $\mathbb{P}(J)$ preserves $\mathcal{G}$.*

Proof. Suppose $\mathcal{G} \subseteq [\omega_1 \times K]^2$ is m-solid. Since $\mathbb{P}(J)$ has the θ-chain condition and preserves $\aleph_1$, it is sufficient to show that if $Z \in \mathcal{T}$ then $\mathbb{P}(J) \cap Z$ preserves $\mathcal{G}$. We prove this via induction. Note that:

- If Z is the smallest element of $\mathcal{T}$, then $\mathbb{P}(J) \cap Z$ is strongly proper, so this holds via Proposition 3.23.
- If Z is a successor, then (using Proposition 2.31 and Proposition 2.25) we can rewrite $\mathbb{P}(J) \cap Z$ as a finite iteration of proper forcings preserving $\mathcal{G}$, so the result holds via Proposition 3.24.

Thus we can assume that Z is a limit element of $\mathcal{T}$. Let $(p_0, f_0) \in \mathbb{P}(J) \cap Z$, $\dot{\mathcal{B}}$ be a $\mathbb{P}(J) \cap Z$-name for an uncountable dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$, and $\kappa \gg \theta$ be a sufficiently large regular cardinal. Pick a countable elementary submodel $\overline{M} \prec H(\kappa)$ containing $\mathbb{P}(J)$, Z, $\mathcal{G}$, $\dot{\mathcal{B}}$, and (p_0, f_0). Let $M = \overline{M} \cap H(\theta) \in \mathcal{S}$. By Proposition 2.30, we can find $(p_1, f) \leq (p_0, f_0)$ such that $M \cap Z \in p_1$. We can assume (by extending (p_1, f) if necessary) that there is some $b \in \mathrm{Fn}_m(\omega_1, K) \setminus M$ such that $(p_1, f) \Vdash$ '$b \in \dot{\mathcal{B}}$'.

Since Z is a limit model contained in M, we can find $X \in M \cap Z \cap \mathcal{T}$ such that $p_1 \cap M \subseteq X$. Thus via Lemma 2.29, there is some $p \in \mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$ such that $p_1 \cup \{X\} \subseteq p$ and $(p, f) \in \mathbb{P}(J)$. By Theorem 2.33, $(p \cap X, f \restriction X)$ is an $(\overline{M}, \mathbb{P}(J) \cap X)$-generic condition. Now define the following set:

$$\begin{aligned}\dot{W} = \{(c, (q \cap X, g \restriction X)) :& (c \in \mathrm{Fn}_m(\omega_1, K)) \\ & \wedge (p \cap M \sqsubseteq q) \\ & \wedge ((q, g) \Vdash \text{`} c \in \dot{\mathcal{B}}\text{'})\}.\end{aligned}$$

We have that $\dot{W} \in \overline{M}$ by elementarity, $(b, (p \cap X, f \restriction X)) \in \dot{W}$, and hence that $\dot{W}$ is a $\mathbb{P}(J) \cap X$-name for an uncountable subset of $\mathrm{Fn}_m(\omega_1, K)$. Thus by our inductive hypothesis and Proposition 3.12 we can find $(q, g) \in \mathbb{P}(J) \cap Z$ and $a \in \mathrm{Fn}_m(\omega_1, K)$ such that:

1. $(q, g), a \in M$,
2. $p \cap M \sqsubseteq q$,
3. $(q \cap X, g \restriction X)$ and $(p \cap X, f \restriction X)$ are compatible in $\mathbb{P}(J) \cap X$,
4. $[a, b] \subseteq \mathcal{G}$,
5. $(q, g) \Vdash \text{`} a \in \dot{\mathcal{B}}\text{'}$.

Thus Lemma 2.32 implies (q, g) and (p, f) are compatible as required. □

Remark 3.26. Note that the above proof suggests that properties preserved by strongly proper forcings and two-step iterations are very close to being preserved by Neeman iterations. This observation was first made in [10]. We were able to make this connection more apparent by showing that such properties are also preserved by side condition hulls. Further investigation is required to see if the last remaining part of the above proof (being preserved at limit steps) also follows from some basic preservation conditions.

Definition 3.27. If $\mathcal{G}$ is an m-solid graph, we write $\mathsf{PFA}(\mathcal{G}, m)$ for the following statement: For every proper forcing $\mathbb{P}$ that preserves that $\mathcal{G}$ is m-solid and sequence $\{D_\alpha : \alpha < \omega_1\}$ of dense-open subsets of $\mathbb{P}$, there is a filter $\mathcal{F} \subseteq \mathbb{P}$ such that $\mathcal{F} \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

We similarly define $\mathsf{PFA}(\mathcal{G}, {<}\omega)$, $\mathsf{PFA}(\mathcal{G}, (m, n))$, $\mathsf{PFA}(\mathcal{G}, (m, {<}\omega))$ for strongly solid, (m, n)-solid, and $(m, < \omega)$-solid graphs respectively.

In cases where the type of solid graph is clear or irrelevant, we may write $\mathsf{PFA}(\mathcal{G})$ for convenience.

Corollary 3.28. *Assuming the existence of a supercompact cardinal, if $\mathcal{G}$ is an m-solid graph, there is a proper forcing $\mathbb{P}$ that preserves $\mathcal{G}$ such that $V^{\mathbb{P}} \models$ $\mathsf{PFA}(\mathcal{G}, m)$. A similar result holds for strongly solid, (m, n)-solid, and $(m, {<}\omega)$-solid graphs.*

Proof. Let θ be a supercompact cardinal and $J\colon \theta \to H(\theta)$ a Laver function. Define $\mathcal{S}$ and $\mathcal{T}$ as in Section 2.4. Recursively define a new function $J'\colon \theta \to H(\theta)$ and Neeman iteration $\mathbb{P}(J')$ as follows. Given $\alpha < \theta$,

- If $H(\alpha) \notin \mathcal{T}$, then $J'(\alpha) = \emptyset$.
- If $H(\alpha) \in \mathcal{T}$:
  - If $J(\alpha)$ is a $\mathbb{P}(J') \cap H(\alpha)$-name for a proper forcing that preserves $\mathcal{G}$, then $J'(\alpha) = J(\alpha)$.
  - Otherwise, let $J'(\alpha)$ be the $\mathbb{P}(J') \cap H(\alpha)$-name for the trivial forcing.

By Theorem 3.25, $\mathbb{P}(J')$ preserves $\mathcal{G}$. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}(J')$-name for a proper forcing that preserves $\mathcal{G}$ and $\langle \dot{D}_\alpha \rangle_{\alpha<\omega_1}$ a sequence of $\mathbb{P}$-names for dense-open subsets of $\mathbb{Q}$. Let κ be sufficiently large that $\dot{\mathbb{Q}}$ and $\langle \dot{D}_\alpha \rangle_{\alpha<\omega_1}$ belong to $H(\kappa)$. Since J is a Laver function, there is an elementary embedding $j\colon V \to M$ such that:

1. $\mathrm{crit}(j) = \theta$,
2. $j(\theta) > \lambda$,
3. $[M]^\lambda \subseteq M$,
4. $j(J')(\theta) = j(J)(\theta) = \dot{\mathbb{Q}}$.

Let $G \subseteq \mathbb{P}(J')$ be a generic filter. Via the argument of Theorem 2.38, $V[G]$ contains a filter for $\dot{\mathbb{Q}}$ that meets all the $\dot{D}_\alpha$ as required. □

3.5 PRESERVING SOLID GRAPHS

In this section, we show that $\mathsf{PFA}(\mathcal{G})$ implies both $\mathfrak{p} > \aleph_1$ and the Mapping Reflection Principle (defined below). As in the previous section, we focus on m-solid graphs since we can use Proposition 3.20 to generalise these results to (m,n)-solid graphs.

Lemma 3.29. *Let $\mathbb{P}$ be σ-centred and $\mathcal{G}$ an m-solid graph or (m,n)-solid graph. Then $\mathbb{P}$ preserves $\mathcal{G}$.*

Proof. Suppose $\mathcal{G} \subseteq [\omega_1 \times K]^2$ is m-solid. Let $\dot{\mathcal{B}}$ be a $\mathbb{P}$-name for an uncountable dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$, and $p_0 \in \mathbb{P}$. Let λ be a sufficiently large regular cardinal and M a countable elementary submodel of $H(\lambda)$ containing p_0, $\mathbb{P}$, $\mathcal{G}$, and $\dot{\mathcal{B}}$. Then we can find some $p \leq p_0$ such that p is $(M, \mathbb{P})$-generic. We can also assume, by extending p if necessary, that there is some $b \in \mathrm{Fn}_m(\omega_1, K)$ such that $\mathrm{dom}(b) \cap M = \emptyset$ and $p \Vdash$ '$b \in \dot{\mathcal{B}}$'.

Since $\mathbb{P}$ is σ-centred, we can write $\mathbb{P} = \bigcup_{i<\omega} \mathbb{P}_i$ such that each $\mathbb{P}_i$ is centred. Pick n such that $p \in \mathbb{P}_n$. Then define the set W of pairs $(q, c) \in \mathbb{P} \times \mathrm{Fn}_m(\omega_1, K)$ such that:

1. $q \in \mathbb{P}_n$,
2. $q \Vdash$ '$c \in \dot{\mathcal{B}}$'.

Note that $(p, b) \in W$ and by elementarity $W \in M$. Thus by Proposition 3.9, we can find some $(q, a) \in W \cap M$ such that $[a, b] \subseteq \mathcal{G}$. Since $q, p \in \mathbb{P}_n$, q and p are compatible as required. □

Proposition 3.30. *Let $\mathbb{P}$ be almost strongly proper and $\mathcal{G}$ an m-solid or (m, n)-solid graph. Then $\mathbb{P}$ preserves $\mathcal{G}$.*

Proof. Suppose $\mathcal{G} \subseteq [\omega_1 \times K]^2$ is m-solid. Let $\lambda < \lambda'$ be sufficiently large regular cardinals, $\dot{\mathcal{B}}$ a $\mathbb{P}$-name for an uncountable dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$, and M a countable elementary submodel of $H(\lambda')$ containing λ, $\mathbb{P}$, $\dot{\mathcal{B}}$, and $\mathcal{G}$. Let $p \in \mathbb{P}$ be $(M, \mathbb{P})$-almost strongly generic. We can also assume, by extending p if necessary, that there is some $b \in \mathrm{Fn}_m(\omega_1, K)$ such that $\mathrm{dom}(b) \cap M = \emptyset$ and $p \Vdash$ '$b \in \dot{\mathcal{B}}$'. Consider the following set:

$$\mathcal{U} = \{N \cap \mathbb{P} : (\mathbb{P}, \dot{\mathcal{B}}, \mathcal{G} \in N) \wedge (N \text{ is a countable elementary submodel of } H(\lambda))\}.$$

$\mathcal{U}$ is unbounded in $[\mathbb{P}]^{\aleph_0}$ and by elementarity is contained in M. Thus we can find some countable elementary submodel $N \prec H(\lambda)$ such that $\mathbb{P}, \dot{\mathcal{B}}, \mathcal{G} \in N \in M$ and conditions $p' \leq p$ and $p' \restriction N \in N$ such that every condition $r \leq p' \restriction N$ in N is compatible with p'. Define the following set:

$$\mathcal{C} = \{a \in \mathrm{Fn}_m(\omega_1, K) : (\exists r \leq p' \restriction N)\ (r \Vdash \text{'}a \in \dot{\mathcal{B}}\text{'})\}.$$

By elementarity, $\mathcal{C} \in N$. Since p' and $p' \restriction M$ are compatible, we also have that $b \in \mathcal{C}$, so $\mathcal{C}$ contains an uncountable block sequence. Thus by Proposition 3.9 there must be some $a \in \mathcal{C} \cap N$ such that $[a, b] \subseteq \mathcal{G}$. Pick a condition $q \in \mathbb{P} \cap N$ witnessing that $a \in \mathcal{C}$. Then q and p' are compatible, $q \Vdash$ '$a \in \dot{\mathcal{B}}$', and $[a, b] \subseteq \mathcal{G}$ as required. □

Definition 3.31. Let X be an uncountable set and M be a countable elementary submodel of $H(\theta)$ for some regular θ such that $[X]^{\aleph_0} \in M$. A subset Σ of $[X]^{\aleph_0}$ is *M-stationary* if whenever $E \subseteq [X]^{\aleph_0}$ is a club in M, there is $E \cap \Sigma \cap M \neq \emptyset$.

Definition 3.32. A set mapping Σ is *open stationary* if there is some uncountable set X and regular cardinal θ with $X \in H(\theta)$ such that:

- The domain of Σ consists of elementary submodels of $H(\theta)$ that contain M,
- for all $M \in \mathrm{dom}(\Sigma)$, the set $\Sigma(M) \subseteq [X]^{\aleph_0}$ is open (with respect to the Ellentuck topology on $[X]^{\aleph_0}$) and M-stationary.

We denote the parameters of Σ as X_Σ and θ_Σ.

Definition 3.33 ([20])**.** The *Mapping Reflection Principle* (MRP) is the following statement: If Σ is an open stationary set mapping whose domain is a club, then there is a continuous $\in$-chain $\langle N_\nu : \nu < \omega_1 \rangle$ in the domain of Σ such that for all limit ordinals $\nu \in (0, \omega_1)$, there is some $\nu_0 < \nu$ such that for all $\xi < \nu$ with $\nu_0 \in N_\xi$, $N_\xi \cap X_\Sigma \in \Sigma(N_\nu)$. (Such a chain is called a *reflecting sequence*.)

Theorem 3.34 ([19, Theorem 2.7])**.** *Suppose that Σ is an open stationary mapping. Then there is an almost strongly proper forcing $\mathbb{P}_\Sigma$ which adds a reflecting sequence for Σ.*

MRP implies many well-known consequences of PFA:

- $\mathfrak{c} = \aleph_2$ [20].
- The failure of $\square(\kappa)$ for all regular $\kappa > \aleph_1$ [20].
- The Singular Cardinal Hypothesis [43].

Corollary 3.35. PFA($\mathcal{G}$) *implies* MRP *and* $p = \aleph_2$.

4

PRESERVING (m,n)-SOLID GRAPHS

In this chapter, we will show that many of the consequences of PFA are also consequences of PFA relativised to an (m,n)-solid graph. We will also show that that the notion of (m,n)-solid graph is necessary, since many of these consequences do not automatically hold in PFA relativised to particular 1-solid graphs.

For the rest of this chapter, fix $m,n < \omega$, K a countable set, and $\mathcal{G}$ an (m,n)-solid graph on $\omega_1 \times K$. (The requirement that K is countable is to make sure that K is a subset of all relevant countable transitive submodels.)

4.1 THE OPEN GRAPH AXIOM

Definition 4.1. Let (V,E) be an open graph such that (V,E) is not countably chromatic, $\mathcal{I}$ be the σ-ideal of countably chromatic subsets of V, and θ be a sufficiently large regular cardinal. Define the poset $\mathbb{P}(V,E)$ to be the set of all finite functions $p\colon \mathcal{N}_p \to V$ such that:

1. $\mathcal{N}_p$ is an $\in$-chain of countable elementary submodels of $H(\theta)$ containing all above objects,
2. For all $N \in \mathcal{N}_p$, $p(N) \notin \bigcup(\mathcal{I} \cap N)$,
3. For all $M,N \in \mathcal{N}_p$ with $M \in N$: $p(M) \in N$ and $\{p(M),p(N)\} \in E$.

We order $\mathbb{P}(V,E)$ by reverse inclusion.

By selecting the $<_w$-least countable open basis of V, we can (and will) assume that all aforementioned models agree on the basis of V.

Theorem 4.2 ([40, §7.2])**.** $\mathbb{P}(V,E)$ *is proper and adds an uncountable complete subgraph to* (V,E)*.*

Proposition 4.3. *Let M be a countable elementary submodel of $H(\theta)$ containing all relevant objects, and $v \in V$ be such that $v \notin \bigcup(\mathcal{I} \cap M)$. Then if there is some set $Y \subseteq V$ with $v \in Y \in M$, there is some $w \in Y$ with $\{v,w\} \in E$.*

Proof. Consider the set $Y' = \{u \in Y : (\forall u' \in Y)\ \{u,u'\} \notin E\}$. Since $Y' \in M$ and Y' is discrete, it follows that $v \notin Y'$ as required. □

Theorem 4.4. *$\mathbb{P}(V,E)$ preserves that $\mathcal{G}$ is (m,n)-solid.*

Proof. We prove this via Proposition 3.18. Let $\dot{A}$ be a $\mathbb{P}(V,E)$-name for a dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1,K)$, and $p_0 \in \mathbb{P}(V,E)$. Choose $\kappa \gg \theta$ a sufficiently large regular cardinal, pick a countable elementary submodel $\overline{M} \prec H(\kappa)$ containing all relevant objects, and denote $M = \overline{M} \cap H(\theta)$. By the proof that $\mathbb{P}(V,E)$ is proper, we can pick $p \le p_0$ such that p is $(\overline{M},\mathbb{P}(V,E))$-generic and $M \in \mathcal{N}_p$. We can also assume, by extending p if necessary, that there is some $b \in \mathrm{Fn}_m(\omega_1,K)$ such that $\mathrm{dom}(b) \cap M = \emptyset$ and $p \Vdash$ '$b \in \dot{A}$'. Finally, fix some $d \in \mathrm{Fn}_n(\omega_1,K)$ such that $\mathrm{dom}(d) \cap M = \emptyset$.

Enumerate $\mathcal{N}_p \setminus M$ in $\in$-increasing order as $M_0, M_1, \dots, M_{k-1}$ (where $M_0 = M$.) Denote $v_i = p(M_i)$ for $i < k$ and let $\vec{v} = \langle v_i \rangle_{i<k}$. Since (V,E) is an open graph, we can find some $\vec{U} = \langle U_i \rangle_{i<k}$ such that:

- U_i is a basic open set for all $i < k$,
- $\vec{v} \in \vec{U}$,
- For every $\vec{w} = \langle w_i \rangle_{i<k} \in \vec{U}$, $\{v_i, w_j\} \in E$ for all $i \neq j$.

Let $\mathcal{H}_n$ be the set of pairs $\langle c, \vec{w} \rangle \in \mathrm{Fn}_m(\omega_1,K) \times V^k$ such that there is some $r \in \mathbb{P}(V,E)$ with the following properties:

1. $p \cap M \sqsubseteq r$,
2. $|r \setminus (p \cap M)| = k$ (enumerated as $N_0 \in N_1 \in \dots \in N_{n-1}$),
3. $\vec{w} = \langle r(N_i) \rangle_{i<k}$,
4. $\vec{w} \in \vec{U}$,
5. $r \Vdash$ '$c \in \dot{A}$'.

Note that $(b, \vec{v}) \in \mathcal{H}_n$ and by elementarity $\mathcal{H}_n \in M$. Denote $b_n = b$.

Claim 4.4.1. *We can find sequences $\langle b_i \rangle_{i<k}$, $\langle U'_i \rangle_{i<k}$ and $\langle \mathcal{H}_i \rangle_{i<k}$ such that for all $j < k$:*

1. *$b_j \in \mathrm{Fn}_m(\omega_1,K)$,*
2. *$b_j \cap b \sqsubseteq b_j, b$,*
3. *$\mathrm{dom}(b_j \setminus b) \cap M_j = \emptyset$,*
4. *$U'_j \subseteq U_j$ is a basic open set,*
5. *$\{v_j\} \times U'_j \subseteq E$,*

6. $\mathcal{H}_j = \{(c, \vec{w}) \in \mathrm{Fn}_m(\omega_1, K) \times V^j : (\exists w' \in U'_j)\ (c, \vec{w} ^\frown \langle w' \rangle) \in \mathcal{H}_{j+1}\}$,
7. $\mathcal{H}_j \in M$ *(note this follows from the previous point),*
8. $(b_j, \langle v_i \rangle_{i<j}) \in \mathcal{H}_j$.

Proof of Claim. We prove this by reverse induction. We'll focus on the inductive step, since the claim for $j = k - 1$ follows from the same argument. Suppose we've defined our above sequences down to $j + 1$. We can assume that $b_{j+1} \setminus M_j \neq \emptyset$, since this makes our life more difficult. (Note that since $K \in M_j$ and K is countable, $\mathrm{dom}(b_{j+1}) \setminus M_j \neq \emptyset$ iff $b_{j+1} \setminus M_j \neq \emptyset$.) Let $\tilde{b}_j = b_{j+1} \cap M_j$, $m_j = m - |\tilde{b}_j|$, and let $\beta_j = \max(\mathrm{dom}(\tilde{b}_j))$. Now for all $\alpha \in (\beta_j, \omega_1)$, consider the following set:

$$Z_j^{(\alpha)} = \{(c, w) \in \mathrm{Fn}_{m_j}(\omega_1 \setminus \alpha, K) \times V : (\tilde{b}_j \cup c, v_0, \dots, v_{j-1}, w) \in \mathcal{H}_{j+1}\}.$$

Note that for all $\alpha \in (\beta_j, \delta_{M_j})$, $(b_{j+1} \setminus M_j, v_j) \in Z_j^{(\alpha)}$. Thus by Proposition 4.3 M_j models that there must be two disjoint basic open sets $W_{\alpha,0}, W_{\alpha,1} \subseteq U_j$ such that $W_{\alpha,0} \times W_{\alpha,1} \subseteq E$ and $W_{\alpha,l} \cap \pi_1(Z_j^{(\alpha)}) \neq \emptyset$ for $l = 0, 1$. (Note also that we could choose such $W_{\alpha,0}$, $W_{\alpha,1}$ such that $v_j \in W_{\alpha,1}$, but M_j obviously cannot model this.) Thus by elementarity this must be true for all $\alpha \in (\beta_j, \omega_1)$. Now let $\mathcal{W}_j$ be the collection of all basic open sets $W \subseteq U_j$ such that for all $\alpha \in (\beta_j, \omega_1)$, there is a basic open set $W' \subseteq U_j$ with $W \times W' \subseteq E$, $W \cap \pi_1(Z_j^{(\alpha)}) \neq \emptyset$, and $W' \cap \pi_1(Z_j^{(\alpha)}) \neq \emptyset$. By elementarity, $\mathcal{W}_j \in M_j$. Since for $\alpha \in (\beta_j, \delta_{M_j})$ we could always pick our $W_{\alpha,1}$ such that $v_j \in W_{\alpha,1}$ and there are only countably many basic sets containing v_j, by elementarity there must be some basic open set $\tilde{U}_j \subseteq U_j$ with $v_j \in \tilde{U}_j$ and $\tilde{U}_j \in \mathcal{W}_j$. Now take $\alpha > \delta_{M_j}$ and some basic open set U'_j with $\tilde{U}_j \times U'_j \subseteq E$ and $U'_j \cap \pi_1(Z_j^{(\alpha)}) \neq \emptyset$. Finally, take some $(b'_j, w_j) \in Z_j^{(\alpha)}$ such that $w_j \in U'_j$, and denote $b_j = \tilde{b}_j \cup b'_j$. Then b_j and U'_j are as required. □

From the claim, we have that $\mathcal{H}_0 \in M$, $b_0 \in \mathcal{H}_0$, and $\mathrm{dom}(b_0) \cap M_0 = \emptyset$. $\mathcal{H}_0$ is an uncountable dom-block-sequence and thus by Proposition 3.17 there is some $a \in \mathcal{H}_0 \cap M$ such that $[a, d] \subseteq \mathcal{G}$. Pick some sequence $\langle w_i \rangle_{i<k} \in V^k \cap M$ such that $w_i \in U'_i$ and $(a, w_0, \dots, w_{i-1}) \in \mathcal{H}_i$ for all $i \leq k$. Let $q \in \mathbb{P}(V, E) \cap M$ witness that $(a, w_0, \dots, w_{k-1}) \in \mathcal{H}_k$. Then $p \cup q \in \mathbb{P}(V, E)$, $q \Vdash$ '$a \in \dot{A}$', and $[a, d] \subseteq \mathcal{G}$ as required. □

Remark 4.5. Theorem 4.4 implies that 'can be encoded by an (m, n)-solid graph' and 'can be encoded by an m-solid graph' are very different conditions, since 2-entangled sets can be encoded by a 1-solid graph but OGA implies that 2-entangled sets of reals cannot exist [10, Proposition 14]. Moreover, it implies that 2-entangled sets cannot be encoded by (m,n)-solid graphs. However, the encoding given by Example 3.8 uses an uncountable set K, so one might wonder whether the cardinality of K is a deciding factor. To show this is not the case, let $m < \omega$, $E = \{e_\alpha : \alpha < \omega_1\} \subseteq \mathbb{R}$ and $f\colon \omega_1 \to [\omega_1]^m$ be a bijection. Let $K = \{>,<\}^m$. Then define $\mathcal{G}_{E,f} \subseteq [\omega_1 \times K]^2$ to be the set of pairs $\{\langle \nu_0, R_0\rangle, \langle \nu_1, R_1\rangle\}$ such that $\nu_0 < \nu_1$ and either:

- $f(\nu_0) \not< f(\nu_1)$, or
- For all $i < m$, $e_{f(\nu_0)(i)}\ R_1(i)\ e_{f(\nu_1)(i)}$.

Then using similar arguments to Proposition 3.13, $\mathcal{G}_{E,f}$ is 1-solid iff E is m-entangled.

4.2 BAUMGARTNER'S AXIOM

Definition 4.6. Let K, L be linear orders.

- We write $K \leq L$ if there is a strictly increasing function $f\colon K \to L$. (In other words, L has an isomorphic copy of K.)
- We say K and L are *equivalent*, denoted $K \equiv L$, if $K \leq L$ and $L \leq K$.

Definition 4.7. Let κ be an infinite cardinal and L a linear order.

- L is κ-dense if for any $x, y \in L$ with $x \lneq y$, $|(x,y)_L| = \kappa$.
- L is κ-scattered if it does not contain any κ-dense suborder (other than singletons).
- L is *separable* if it contains a countable, $\aleph_0$-dense subset $D \subseteq L$ such that for any $x, y \in L$ with $x \lneq y$, $(x,y)_L \cap D \neq \emptyset$.

Note that every separable linear order embeds into the real line.

Definition 4.8 (Baumgartner's Axiom)**.** Let κ be an infinite cardinal.

- We write $\mathsf{BA}(\kappa)$ for the statement: Every two κ-dense sets of reals are isomorphic.
- We write $\mathsf{BA}_{\equiv}(\kappa)$ for the statement: Every two sets of reals of size κ are equivalent.

Lemma 4.9 ([36, Corollary 8.3])**.** $\mathsf{BA}_{\equiv}(\aleph_1) + \mathfrak{p} > \aleph_1$ *implies* $\mathsf{BA}(\aleph_1)$.

Definition 4.10. Let N be a countable elementary submodel of $H(\mathfrak{c}^+)$. We say that N is a *strong model* if there is $\overline{N}$ a countable elementary submodel $H((2^{\mathfrak{c}})^+)$ such that $N = \overline{N} \cap H(\mathfrak{c}^+)$.

Definition 4.11. Let X, Y be sets of reals of size $\aleph_1$. Define the poset $\mathbb{P}(X, Y)$ to be set of all pairs $p = (f_p, \mathcal{N}_p)$ such that:

1. f_p is a finite partial increasing map from X to Y,
2. $\mathcal{N}_p$ is a finite $\in$-chain of countable elementary submodels of $H(\mathfrak{c}^+)$ containing X and Y,
3. The range of f_p is separated by $\mathcal{N}_p$,
4. For all $x \in \mathrm{dom}(f_p)$, there is some $N \in \mathcal{N}_p$ such that $x \in N$ but $f_p(x) \notin N$,
5. For all $x \in \mathrm{dom}(f_p)$, if $N \in \mathcal{N}_p$ is a strong model, then N does not separate x and $f_p(x)$.

We order $\mathbb{P}(X, Y)$ by reverse-inclusion.

Theorem 4.12 ([40, §15.2])**.** $\mathbb{P}(X, Y)$ *is proper and adds a strictly increasing function* $f : X \to Y$.

Theorem 4.13. $\mathbb{P}(X, Y)$ *preserves that* $\mathcal{G}$ *is* (m, n)*-solid.*

Proof. Let $\dot{\mathcal{A}}$ be a $\mathbb{P}(X, Y)$-name for a dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$ and $p_0 \in \mathbb{P}(X, Y)$. Choose κ a sufficiently large regular cardinal, pick a countable elementary submodel $\overline{M} \prec H(\kappa)$ containing all relevant objects, and denote $M = \overline{M} \cap H(\mathfrak{c}^+)$. By the proof that $\mathbb{P}(X, Y)$ is proper, we can pick $p \leq p_0$ such that p is $(\overline{M}, \mathbb{P}(X, Y))$-generic and $M \in \mathcal{N}_p$. We can also assume, by extending p if necessary, that there is some $b \in \mathrm{Fn}_m(\omega_1, K)$ such that $\mathrm{dom}(b) \cap M = \emptyset$ and $p \Vdash$ '$b \in \dot{\mathcal{A}}$'. Finally, fix some $d \in \mathrm{Fn}_n(\omega_1, K)$ such that $\mathrm{dom}(d) \cap M = \emptyset$.

Let $k = |\,\mathrm{dom}(f_p) \setminus M|$, and enumerate $\mathrm{dom}(f_p) \setminus M$ as $x_0, x_1, \ldots, x_{k-1}$. Pick rational open intervals $\{U_i : i < k\}$ and $\{U'_i : i < k\}$ such that:

- The $\{U_i : i < k\}$ are pairwise-disjoint,
- The $\{U'_i : i < k\}$ are pairwise-disjoint,
- $x_i \in U_i$ and $f_p(x_i) \in U'_i$ for all $i < k$,
- $(\mathrm{dom}(f_p) \setminus M) \cap U'_i = \emptyset$ for all $i < k$,
- $(\mathrm{range}(f_p) \setminus M) \cap U_i = \emptyset$ for all $i < k$,

Now let $\mathcal{F}$ be the set of triples $\langle c, \{V_i : i < k\}, \{V'_i : i < k\}\rangle$ such that there is some $r \in \mathbb{P}(X, Y)$ with the following properties:

1. $c \in \mathrm{Fn}_m(\omega_1, K)$,

2. $\mathcal{N}_p \cap M \sqsubseteq \mathcal{N}_r$,
3. $|\operatorname{dom}(f_r) \setminus M| = k$ (enumerated as $z_0 < z_1 < \cdots < z_{n-1}$),
4. For all $i < k$,
   a) V_i and V_i' are rational open intervals,
   b) $V_i \subseteq U_i$ and $V_i' \subseteq U_i'$,
   c) $z_i \in V_i$ and $f_r(z_i) \in V_i'$.
5. $r \Vdash$ '$c \in \dot{\mathcal{A}}$'.

For all $\alpha < \omega_1$, let $\mathcal{H}_\alpha$ be the set of pairs $\langle \{V_i : i < k\}, \{V_i' : i < k\} \rangle$ such that there is some $c \in \mathrm{Fn}_m(\omega_1 \setminus \alpha, K)$ such that $\langle c, \{V_i : i < k\}, \{V_i' : i < k\} \rangle \in \mathcal{F}$. Finally, let $\mathcal{H}_{\omega_1} = \bigcap_{\alpha<\omega_1} \mathcal{H}_\alpha$. Note that:

- $\mathcal{F}, \mathcal{H}_{\omega_1} \in M$,
- For all $\alpha < \delta_M$, $\mathcal{H}_\alpha \in M$,
- If $\langle \{V_i : i < k\}, \{V_i' : i < k\} \rangle$ is such that $x_i \in V_i \subseteq U_i$ and $f_p(x_i) \in V_i' \subseteq U_i'$ for all $i < k$, then since $\operatorname{dom}(b) \cap M = \emptyset$ we have that for all $\alpha < \delta_M$, $\langle \{V_i : i < k\}, \{V_i' : i < k\} \rangle \in \mathcal{H}_\alpha$ and thus by elementarity
$$\langle \{V_i : i < k\}, \{V_i' : i < k\} \rangle \in \mathcal{H}_{\omega_1}.$$

Thus by following the usual proof of Theorem 4.12, we get that for all $\alpha < \delta_M$, there are rational open intervals $\{W_i : i < k\}$ and $\{W_i' : i < k\}$ such that

- $\langle \{W_i : i < k\}, \{W_i' : i < k\} \rangle \in \mathcal{H}_\alpha$,
- For all $i < k$,
  – $x_i \notin W_i$, $f_p(x_i) \notin W_i'$,
  – $x_i \leq \inf(W_i)$ iff $f_p(x_i) \leq \inf(W_i')$.

Thus since there are only countably many $2n$-tuples of rational open intervals, we can find rational open intervals $\{W_i : i < k\}$ and $\{W_i' : i < k\}$ with these properties and also that $\langle \{W_i : i < k\}, \{W_i' : i < k\} \rangle \in \mathcal{H}_{\omega_1}$. Now consider the set
$$\mathcal{A} = \{c \in \mathrm{Fn}_m(\omega_1, K) : \langle c, \{W_i : i < k\}, \{W_i' : i < k\} \rangle \in \mathcal{F}\}.$$

We have shown that $\mathcal{A}$ contains an uncountable block-sequence, so by Proposition 3.17 we can find some we can find some $a \in \mathcal{A} \cap M$ with $[a, d] \subseteq \mathcal{G}$. Pick some $q \in \mathbb{P} \cap M$ witnessing that $\langle a, \{W_i : i < k\}, \{W_i' : i < k\} \rangle \in \mathcal{F}$. Then $q \Vdash$ '$a \in \dot{\mathcal{A}}$', $[a, d] \subseteq \mathcal{G}$, and p and q are compatible as required. □

Remark 4.14. As with OGA, BA($\aleph_1$) implies that no m-entangled sets of reals can exist [10, Proposition 14]. Thus no forcing notion for BA($\aleph_1$) can preserve all 1-solid graphs.

4.3 CLUB-ISOMORPHISM OF ARONSZAJN TREES

Definition 4.15.

- An ω_1-tree is a tree of height ω_1 with no uncountable levels.
- An *Aronszajn tree* is an ω_1-tree with no uncountable branches.
- An ω_1-tree is *special* if it can be covered by countably many antichains.
- Two ω_1-trees T, S are *club-isomorphic* if there is some club $C \subseteq \omega_1$ such that $T \restriction C$ is isomorphic to $S \restriction C$.

Lemma 4.16. *Let T be an ω_1-tree and $C \subseteq \omega_1$ a club such that $T \restriction X$ is special. Then T is special. In particular, if all Aronszajn trees are club-isomorphic, they are all special.*

Note that in [1], it is claimed that this result was known since 1974. The below proof is based on [34, §2].

Proof. We will assume for simplicity that $\emptyset \in C$. Define a map $f\colon T \restriction (\omega_1 \setminus C) \to T$ by $f(t) = t \restriction \max(C \cap \mathrm{ht}(t))$. Note that since C is closed, f is well-defined. Moreover, f is regressive.

We claim that for all $t \in T$, $f^{-1}(t)$ is countable. Indeed, given $t \in T$, let $\delta \in C$ be such that $\delta > \mathrm{ht}(t)$. Then $f^{-1}(t) \subseteq T \restriction \delta$ and thus is countable. For each $t \in T$, let $g_t\colon f^{-1}(t) \to \omega$ be an injection.

Let $h'\colon T \restriction C \to \omega$ specialise $T \restriction C$. We now define a map $h\colon T \to \omega^{<\omega}$ as follows:

$$h(t) = \begin{cases} h'(t) & t \in T \restriction C, \\ h(f(t)) ^\frown \langle g_{f(t)}(t) \rangle & \text{otherwise.} \end{cases}$$

Now we show that h specialises T. Suppose that $t, t' \in T$ with $h(t) = h(t') = \langle n_0, n_1, \ldots, n_k \rangle$. We prove our result by induction on k. The $k = 0$ case follows from the fact that h' specialises $T \restriction C$. Now suppose that this result holds for all $i < k$. Then since $h(f(t)) = h(f(t')) = \langle n_0, n_1, \ldots, n_{k-1} \rangle$, either $f(t) \perp f(t')$ or $f(t) = f(t')$. In the first case, then $t \perp t'$ since $f(t) \leq t$ and $f(t') \leq t'$. In the second case, then since $g_{f(t)}(t) = g_{f(t)}(t')$ and $g_{f(t)}$ is an injection, $t = t'$. □

Remark 4.17. Note that every special tree is Aronszajn, and every Souslin tree is non-special. Thus since a Souslin tree can be encoded by a 1-solid graph, no forcing notion for forcing that all Aronszajn trees are club-isomorphic can preserve all 1-solid graphs. Similarly, non-special Aronszajn trees cannot be encoded by (m, n)-solid graphs.

Lemma 4.18 ([33, Lemma 5.9])**.** *Let $k < \omega$ and $\{T^i : i < k\}$ be a collection of Aronszajn trees. Let $\{t^i_\alpha \in T^i : \alpha < \omega_1, i < k\}$ be such that for all $\beta < \alpha < \omega_1$ and $i, i' < k$:*

- $\mathrm{ht}(t^i_\alpha) = \mathrm{ht}(t^{i'}_\alpha)$,
- $\mathrm{ht}(t^i_\beta) < \mathrm{ht}(t^i_\alpha)$.

*Then for any $l < \omega$, there is $\delta < \omega_1$ such that $\{t^i_\alpha \in T^i : \alpha < \omega_1, i < l\}$ is (l, δ)-*distributed*, i.e. for all collections $\{s^i_j \in T^i_\delta : j < l, i < k\}$ there are uncountably many α such that for all $j < l$ and $i < k$, $s^i_j \perp_{T^i} t^i_\alpha$.*

Definition 4.19. Let T and S be normal Aronszajn trees. Define $\mathbb{P}(T, S)$ to be the poset of pairs $p = (A^p, f^p)$ such that:

1. A^p is a finite subset of ω_1,
2. f^p is a partial bijection $T_{\max(A^p)} \to S_{\max(A^p)}$,
3. For all $\alpha \in A^p$ and $t_0, t_1 \in \mathrm{dom}(f^p)$,

$$(\exists x \in T_\alpha)\ x \leq_T t_0, t_1 \iff (\exists y \in S_\alpha)\ y \leq_S f^p(t_0), f^p(t_1).$$

The following extra notation will be very useful:

- Given a condition p, we write α^p for the maximum element of A^p.
- Given $\alpha < \omega_1$ and $f \in \mathrm{Fn}(T_\alpha, S_\alpha)$, we write $\mathrm{ht}(f) = \alpha$.
- Given $f, g \in \mathrm{Fn}(T, S)$, we write $f \perp g$ if for all $t \in \mathrm{dom}(f)$ and $t' \in \mathrm{dom}(g)$, $t \perp_T t'$ and $f(t) \perp_S g(t')$.
- Let $\alpha < \omega_1$ and $f \in \mathrm{Fn}(T_\alpha, S_\alpha)$. Given $\beta < \alpha$, we write $\pi_\beta(f)$ for the element of $\mathrm{Fn}(T_\beta, S_\beta)$ given by $\pi_\beta(f)(x \restriction \beta) = f(x) \restriction \beta$ for $x \in \mathrm{dom}(f)$ (and undefined otherwise.)
- Let $p = (A^p, f^p) \in \mathbb{P}(T, S)$. For $\beta < \alpha^p$ we write $\pi_\beta(p)$ for the pair $(A^p \cap \beta \cup \{\beta\}, \pi_\beta(f^p))$ if this is a condition (and undefined otherwise.) Note that if $\beta \in A^p$, then $\pi_\beta(p)$ is a condition.

We order $\mathbb{P}(T, S)$ as follows: $q \leq p$ if:

1. $A^p \subseteq A^q$,
2. $\pi_{\alpha^p}(f^q) \restriction A^p = f^p$.

Theorem 4.20 ([1, §5]). *$\mathbb{P}(T,S)$ is a proper forcing that forces that T and S are club-isomorphic.*

The following elementary propositions will be useful:

Proposition 4.21. *Let $p \in \mathbb{P}(T,S)$ and $\beta < \alpha^p$ such that*

$$|\operatorname{dom}(f^p)| = |\operatorname{dom}(\pi_\beta(f^p))| = |\operatorname{range}(\pi_\beta(f^p))|.$$

Then $(A^p \cup \{\beta\}, f^p) \in \mathbb{P}(T,S)$.

Proposition 4.22. *Let $p, q \in \mathbb{P}(T,S)$ and $\beta < \gamma < \omega_1$ be such that*

1. *$\gamma < \alpha^p, \alpha^q$,*
2. *$\pi_\beta(f^p) = \pi_\beta(f^q)$,*
3. *$|\operatorname{dom}(f^p)| = |\operatorname{dom}(f^q)| = |\operatorname{dom}(\pi_\beta(f^p))| = |\operatorname{range}(\pi_\beta(f^p))|$,*
4. *$\pi_\gamma(f^p) \perp \pi_\gamma(f^q)$,*
5. *$(A^p \cup A^q) \cap (\beta, \gamma) = \emptyset$.*

Then p, q are compatible elements.

Proof. By extending our elements if necessary, we can assume that $\alpha^p = \alpha^q$. Moreover, by the previous proposition we can also assume that $\{\beta, \gamma\} \subseteq A^p, A^q$. It will suffice to show that $r = (A^p \cup A^q, f^p \cup f^q)$ is a condition. Since $f^p \perp f^q$, we have that f^r is a bijection, so we only need to check the third criterion for being in $\mathbb{P}(T,S)$. Pick $\alpha \in A^r$ and consider the following cases:

- If $\alpha \leq \beta$, since $\pi_\beta(f^p) = \pi_\beta(f^q)$, for any $t_0, t_1 \in \operatorname{dom}(f^r)$:

$$\begin{aligned} & (\exists x \in T_\alpha)\ x \leq_T t_0, t_1 \\ \iff & (\exists x \in T_\alpha)\ x \leq_T t_0 \restriction \beta, t_1 \restriction \beta \\ \iff & (\exists y \in S_\alpha)\ y \leq_S \pi_\beta(f^r)(t_0 \restriction \beta), \pi_\beta(f^r)(t_1 \restriction \beta) \\ \iff & (\exists y \in S_\alpha)\ y \leq_S f^r(t_0), f^r(t_1). \end{aligned}$$

- If $\alpha \geq \gamma$, since $\pi_\gamma(f^p) \perp \pi_\gamma(f^q)$ and no pairs of elements of $\operatorname{dom}(f^r)$ or $\operatorname{range}(f^r)$ meet above β, it follows that for any $t_0, t_1 \in \operatorname{dom}(f^r)$, $t_0 \restriction \alpha = t_1 \restriction \alpha \iff t_0 = t_1$ and $f^r(t_0) \restriction \alpha = f^r(t_1) \restriction \alpha \iff f^r(t_0) = f^r(t_1)$.

Thus since $A^r \cap (\beta, \gamma) = \emptyset$, $r \in \mathbb{P}(T, S)$. $\square$

Theorem 4.23. *$\mathbb{P}(T, S)$ preserves that $\mathcal{G}$ is (m, n)-solid.*

Proof. Let $p_0 \in \mathbb{P}(T, S)$, and $\dot{\mathcal{A}}$ be a $\mathbb{P}(T, S)$-name for a dom-block-sequence contained in $\mathrm{Fn}_m(\omega_1, K)$. Choose κ a sufficiently large regular cardinal and pick a countable elementary submodel $M \prec H(\kappa)$ containing all relevant objects. By the usual proof of Theorem 4.20, we can pick $p \leq p_0$ such that p is $(M, \mathbb{P}(T, S))$-generic and $\delta_M \in A^p$. We can also assume, by extending p if necessary, that there is some $b \in \mathrm{Fn}_m(\omega_1, K)$ such that $\mathrm{dom}(b) \cap M = \emptyset$ and $p \Vdash$ '$b \in \dot{\mathcal{A}}$'. Finally, pick some $d \in \mathrm{Fn}_n(\omega_1, K)$ such that $\mathrm{dom}(d) \cap M = \emptyset$.

Let $k = |\,\mathrm{dom}(f^p)|$ and pick some $\tilde{\alpha} < \delta_M$ such that

$$k = |\,\mathrm{dom}(\pi_{\tilde{\alpha}}(f^p))| = |\,\mathrm{range}(\pi_{\tilde{\alpha}}(f^p))|$$

and let $p_M = \pi_{\tilde{\alpha}}(p)$. (Note that this is possible since S and T are normal trees and δ_M is a limit ordinal.) Then $p_M \in \mathbb{P}(T, S)$ and since $T_{\tilde{\alpha}}, S_{\tilde{\alpha}} \subseteq M$, we have that $p_M \in M$ and $p \leq p_M$.

Now let $\mathcal{F}$ be the set of all pairs $(c, g) \in \mathrm{Fn}_m(\omega_1, K) \times \mathrm{Fn}_k(T, S)$ such that there is some $r \in \mathbb{P}(T, S)$ and $\beta < \omega_1$ with the following properties:

1. $g = \pi_\beta(f^r)$,
2. $r \leq p_M$,
3. $A_r \cap (\alpha^{p_M}, \beta) = \emptyset$,
4. $r \Vdash$ '$c \in \dot{\mathcal{A}}$'.

Then $(b, \pi_{\delta_M}(f^p)) \in \mathcal{F}$ and by elementarity $\mathcal{F} \in M$ and $\mathcal{F}$ is uncountable. By the same argument as Proposition 3.3, we can find a subset $\mathcal{F}' \subseteq \mathcal{F}$ such that for all $(c, g), (c', g')$, we have that $\mathrm{dom}(c) \cup \{\mathrm{ht}(g)\} < \mathrm{dom}(c') \cup \{\mathrm{ht}(g')\}$ (or vice-versa). Again by elementarity $\mathcal{F}' \in M$. By Proposition 3.17 we can find $\gamma < \omega_1$ such that $\{g : (\exists c \in \mathrm{Fn}_m(\omega_1, K))\ (c, g) \in \mathcal{F}'\}$ is $(2k, \gamma)$-distributed. Note that by elementarity we can pick γ such that $\alpha^{p_M} < \gamma < \delta_M$.

Finally, define $\mathcal{A}'$ be the collection of elements $c \in \mathrm{Fn}_m(\omega_1, K)$ such that there is $g \in \mathrm{Fn}_k(T, S)$ with the following properties:

1. $(c, g) \in \mathcal{F}'$,
2. $\mathrm{ht}(g) \geq \gamma$,
3. $g \perp \pi_\gamma(f^p)$.

By elementarity $\mathcal{A}' \in M$. Moreover, by choice of $\mathcal{F}'$ and use of the aforementioned lemma, $\mathcal{A}'$ is an uncountable dom-block-sequence. Thus by Proposition 3.17 we can pick some $a \in \mathcal{A}' \cap M$ such that $[a, d] \subseteq \mathcal{G}$. Pick some $g \in \mathrm{Fn}_k(T, S) \cap M$ that witnesses $a \in \mathcal{A}'$, and $q \in \mathbb{P}(T, S) \cap M$ witnessing $(a, g) \in \mathcal{F}$. Then by Proposition 4.22 we have that $q \Vdash$ '$a \in \dot{\mathcal{A}}'$', $[a, d] \subseteq \mathcal{G}$, and p and q are compatible as required. $\square$

5 HF GRAPHS

5.1 INTRODUCTION

Definition 5.1. Let $\mathcal{A} \subseteq \mathrm{Fn}(\omega_1, K)$. We say that $\mathcal{A}$ is *K-aligned* if:

- $\mathcal{A}$ is a dom-block-sequence,
- There is $m < \omega$ such that $\mathcal{A} \subseteq \mathrm{Fn}_m(\omega_1, K)$,
- There is some function $f : m \to K$ such that for all $a \in \mathcal{A}$,

$$a = \{\langle \mathrm{dom}(a)(i), f(i)\rangle : i < m\}.$$

Note that if K is countable, every uncountable dom-block-sequence in $\mathrm{Fn}(\omega_1, K)$ contains an uncountable K-aligned subset.

Definition 5.2. Let $\mathcal{G} \subseteq [\omega_1 \times K]^2$. For $b \in \mathrm{Fn}(\omega_1, K)$ and $\epsilon \in 2^{|b|}$, define $U^{\mathcal{G}}_{b,\epsilon} \subseteq \mathrm{Fn}(\omega_1, K)$ as follows:

$$\begin{aligned} U^{\mathcal{G}}_{b,\epsilon} = \{a \in \mathrm{Fn}(\omega_1, K) : (a < b) \\ \wedge ((\forall i < |a|)(\forall j < |b|)\ \{a(i), b(j)\} \in \mathcal{G} \iff \epsilon(j) = 1)\}.\end{aligned}$$

Definition 5.3. Let K be a countable set, $\mathcal{G} \subseteq [\omega_1 \times K]^2$, and $m, n < \omega$.

- $\mathcal{G}$ is an $\mathrm{HF}^{m,n}$ *graph* if for every K-aligned countably infinite subset $\mathcal{A} \subseteq \mathrm{Fn}_m(\omega_1, K)$, there is a $\delta < \omega_1$ such that for all $b \in \mathrm{Fn}_n(\omega_1 \setminus \delta, K)$ and $\epsilon \in 2^{|b|}$, $U^{\mathcal{G}}_{b,\epsilon} \cap \mathcal{A}$ is infinite.
- $\mathcal{G}$ is an HFD^m *graph* if it is an $\mathrm{HF}^{m,n}$ graph for all $n < \omega$.
- $\mathcal{G}$ is an HFC^n *graph* if it is an $\mathrm{HF}^{m,n}$ graph for all $m < \omega$.
- $\mathcal{G}$ is a *strong* HF *graph* if it is an $\mathrm{HF}^{m,n}$ graph for all $m, n < \omega$.

We will say 'HF graph' (resp. 'HFD graph', 'HFC graph') to refer to any graph that is $\mathrm{HF}^{m,n}$ (resp. HFD^m, HFC^n) for some $m, n < \omega$.

Note that every $\mathrm{HF}^{m,n}$ graph is (m,n)-solid. Just as with solid graphs, we can characterise HF graphs in terms of countable elementary submodels:

Proposition 5.4. *Let $\mathcal{G}$ be a graph on $\omega_1 \times K$, where K is countable. The following are equivalent:*

1. *$\mathcal{G}$ is $\mathrm{HF}^{m,n}$;*
2. *For every K-aligned countably infinite subset $\mathcal{A} \subseteq \mathrm{Fn}_m(\omega_1, K)$, uncountable dom-block-sequence $\mathcal{B} \subseteq \mathrm{Fn}_n(\omega_1, K)$, and $\epsilon \in 2^n$ there is $\delta < \omega_1$ such that for all $b \in \mathcal{B}$ with $\mathrm{dom}(b) > \{\delta\}$, $U^{\mathcal{G}}_{b,\epsilon} \cap \mathcal{A}$ is infinite;*
3. *For every λ a sufficiently large regular cardinal and M a countable elementary submodel of $H(\lambda)$ containing $\mathcal{G}$, if $\mathcal{A} \in M$ is a K-aligned countably infinite subset of $\mathrm{Fn}_m(\omega_1, K)$, for every $b \in \mathrm{Fn}_n(\omega_1, K) \setminus M$ and $\epsilon \in 2^{|b|}$, $U^{\mathcal{G}}_{b,\epsilon} \cap \mathcal{A}$ is infinite.*

Proof. Since (3) $\iff$ (1) $\implies$ (2) is immediate we focus on $\neg(1) \implies \neg(2)$. Take a K-aligned countably infinite subset $\mathcal{A} \subseteq \mathrm{Fn}_m(\omega_1, K)$ that witnesses $\neg(1)$. We will iteratively construct an uncountable dom-block-sequence $\mathcal{B} = \langle b_\xi \rangle_{\xi<\omega_1} \subseteq \mathrm{Fn}_n(\omega_1, K)$ and sequence $\langle \epsilon_\xi \rangle_{\xi<\omega_1}$ of elements of 2^n as follows: given $\langle b_\xi \rangle_{\xi<\delta}$ and $\langle \epsilon_\xi \rangle_{\xi<\delta}$ for some $\delta < \omega_1$, let γ_δ be the least ordinal such that $\mathrm{dom}(b_\xi) < \{y_\delta\}$ for all $\xi < \delta$. Then since $\mathcal{A}$ witnesses $\neg(2)$, we can find some $b_\delta \in \mathrm{Fn}_n(\omega_1 \setminus \gamma_\delta, K)$ and $\epsilon_\delta \in 2^n$ such that $U^{\mathcal{G}}_{b_\delta,\epsilon_\delta} \cap \mathcal{A}$ is finite. By thinning our sequence if necessary, we can assume that there is some $\epsilon \in 2^n$ such that $\epsilon_\xi = \epsilon$ for all $\xi < \omega_1$. Then $\mathcal{A}$, $\mathcal{B}$, and ϵ witness that $\neg(2)$ holds. ☐

Given two sets X, Y, we say that X *splits* Y if $X \cap Y$ and $Y \setminus X$ are both infinite. $S \subseteq [\omega]^{\aleph_0}$ is called a *splitting family* if for all $Y \subseteq \omega$, there is some $X \in S$ that splits Y. The cardinal characteristic $\mathfrak{s}$ is defined to be the size of the smallest splitting family.

Definition 5.5 ([4]). Let $\mathcal{X} = \langle x_\alpha : \alpha < \eta \rangle$ be a sequence of elements of $[\omega]^{\aleph_0}$ of length η, where $\mathrm{cof}(\eta) > \aleph_0$.

- We say that $\mathcal{X}$ is *eventually narrow* if for all $y \in [\omega]^{\aleph_0}$, there is some $\delta < \eta$ such that for all $\alpha > \delta$, $y \setminus x_\alpha$ is infinite.
- We say that $\mathcal{X}$ is *eventually splitting* if for all $y \in [\omega]^{\aleph_0}$, there is some $\delta < \eta$ such that for all $\alpha > \delta$, x_α splits y.

Just like the existence of a countable HFD space implies the existence of an uncountable HFD space ([11, p. 4.23]) or that the existence a Luzin set of reals implies the existence of a Luzin subset of 2^{ω_1} ([36, Theorem 6.2]), we have a similar result for HF graphs:

Proposition 5.6. *There is an $\mathrm{HF}^{1,1}$ graph iff there is an eventually splitting sequence of length ω_1.*

Proof. We prove the converse direction. As with the aforementioned proofs, fix a sequence of functions $\{e_\alpha : \alpha < \omega_1\}$ such that $e_\alpha : \alpha \to \omega$ is an injection for all α, and given $\alpha < \beta < \omega_1$, $e_\alpha =^* e_\beta \restriction \alpha$. Let $\mathcal{X} = \langle x_\alpha : \alpha < \omega_1 \rangle$ be an eventually splitting sequence in $[\omega]^{\aleph_0}$. Define the graph $\mathcal{G}_\mathcal{X} \subseteq [\omega_1 \times 1]^2$ as follows:

$$\mathcal{G}_\mathcal{X} = \{\{\langle \nu_0, 0\rangle, \langle \nu_1, 0\rangle\} : (\nu_0 < \nu_1) \wedge (e_{\nu_1}(\nu_0) \in x_{\nu_1})\}.$$

(For ease of notation, we will identify $\omega_1 \times 1$ with 1.) Let $A \in [\omega_1]^{\aleph_0}$. We need to show that there is some $\delta < \omega_1$ such that for all $\delta < \beta < \omega_1$, there is $\alpha, \alpha' \in A$ with $\{\alpha, \beta\} \in \mathcal{G}_X$ and $\{\alpha', \beta\} \notin \mathcal{G}_X$.

Let $\gamma = \bigcup A$. Since $\mathcal{X}$ is eventually splitting, we can find some $\delta < \omega_1$ such that $\delta \geq \gamma$ and for all $\beta > \delta$, x_β splits $e_\gamma[A]$. Let $\beta > \delta$. Since $\{\alpha < \gamma : e_\gamma(\alpha) \neq e_\beta(\alpha)\}$ is finite, x_β also splits $e_\beta[A]$. Thus we can find $\alpha, \alpha' \in A$ such that $e_\beta(\alpha) \in x_\beta$ and $e_\beta(\alpha') \notin x_\beta$ as required. ☐

The existence of an eventually splitting sequence of length ω_1 implies that $\mathfrak{s} = \aleph_1$, but it turns out even more is true:

Proposition 5.7 ([9]). *There is an eventually splitting family of length ω_1 iff*

$$\binom{\omega_1}{\omega} \not\to \binom{\omega_1}{\omega}^{1,1}_2.$$

Theorem 5.8 ([6, Theorem 40]). *It is consistent that $\mathfrak{s} = \aleph_1$ but there are no eventually splitting sequences.*

5.2 PRESERVING HF GRAPHS

Proposition 5.9. *Let $\mathcal{G}$ be an $\mathrm{HF}^{m,n}$ graph on $\omega_1 \times K$ and $\mathbb{P}$ a proper forcing. The following are equivalent:*

1. *$\mathbb{P}$ preserves that $\mathcal{G}$ is $\mathrm{HF}^{m,n}$;*
2. *Let λ be a sufficiently large regular cardinal, $\dot{\mathcal{A}}$ a $\mathbb{P}$-name for a countable K-aligned subset of $\mathrm{Fn}_m(\omega_1, K)$, and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$, $\mathcal{G}$, and $\dot{\mathcal{A}}$. If $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic, $b \in \mathrm{Fn}_n(\omega_1, K) \setminus M$, and $\epsilon \in 2^n$, there is some $q \in \mathbb{P} \cap M$ and $a \in \mathrm{Fn}_m(\omega_1, K) \cap M$ such that $q \Vdash$ '$a \in \dot{\mathcal{A}}$', q is compatible with p, and $a \in U^{\mathcal{G}}_{b,\epsilon}$.*

Proof.

- **(1) implies (2).** Suppose $\mathbb{P}$ preserves that $\mathcal{G}$ is $\mathrm{HF}^{m,n}$ and take λ, $\dot{\mathcal{A}}$, M, p, b, and ϵ as in the statement of the proposition. Let $G_p \subseteq \mathbb{P}$ be a generic filter containing p. We work in $V[G_p]$. Since $\mathbb{P}$ is proper and $\mathbb{P}$ preserves $\mathcal{G}$, we can find $a \in \dot{A}[G_p]$ such that $a \in U^{\mathcal{G}}_{b,\epsilon}$. Then pick $q \in M \cap G_p$ such that $q \Vdash$ '$a \in \dot{A}$'. As G_p is a filter, p and q are compatible as required.

- **(2) implies (1).** Follows from Proposition 5.4. □

The following proposition will allow us to show that many useful proper forcing notions preserve HF graphs:

Proposition 5.10. *Let $\mathbb{P}$ be a proper forcing notion, and consider the following properties:*

1. *Let $p_0 \in \mathbb{P}$, λ be a sufficiently large regular cardinal and M be a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$ and p_0. If there is $b \in [\omega]^{\aleph_0}$ such that for all $a \in M \cap [\omega]^{\aleph_0}$, $|a \cap b| = \aleph_0$, then there is $p \leq p_0$ that is $(M, \mathbb{P})$-generic such that*

$$p \Vdash \text{`}(\forall \dot{a} \in M[G_{\mathbb{P}}] \cap [\omega]^{\aleph_0})\ |\dot{a} \cap b| = \aleph_0\text{'}.$$

2. *$\mathbb{P}$ preserves all eventually narrow sequences of length ω_1 (and hence all such eventually splitting sequences).*

3. *$\mathbb{P}$ preserves all $\mathrm{HF}^{1,1}$ graphs.*

Then $(1) \implies (2) \implies (3)$.

Proof.

- **(1) implies (2).** Suppose $\mathcal{X} = \{x_\alpha : \alpha < \omega_1\}$ is an eventually narrow sequence. Let $p_0 \in \mathbb{P}$, λ a sufficiently large regular cardinal, and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$, p_0, and $\mathcal{X}$. Since $\mathcal{X}$ is eventually narrow and M is countable, there is some $\delta < \omega_1 \setminus \delta_M$ such that for all $\alpha \in \omega_1 \setminus \delta$ and $a \in M \cap [\omega]^{\aleph_0}$, $a \setminus x_\alpha$ is infinite. Thus by our hypothesis, for each $\alpha \in \omega_1 \setminus \delta$ we can find $p \leq p_0$ that is $(M, \mathbb{P})$-generic such that

$$p \Vdash \text{`}(\forall \dot{a} \in M[G_{\mathbb{P}}] \cap [\omega]^{\aleph_0})\ |\dot{a} \cap (\omega \setminus x_\alpha)| = \aleph_0\text{'}.$$

  Thus $\mathbb{P}$ preserves that $\mathcal{X}$ is eventually narrow as required.

- **(2) implies (3).** Let $\mathcal{G} \subseteq [\omega_1 \times K]^2$ for some countable set K. Pick a sequence of functions $\{e_\alpha : \omega \leq \alpha < \omega_1\}$ such that $e_\alpha \colon \alpha \to \omega$ is a bijection. For every $k, k' \in K$ and $\omega \leq \alpha < \beta < \omega_1$, define $x^{k,k'}_{\alpha,\beta} \in [\omega]^{\aleph_0}$ as follows:

$$x^{k,k'}_{\alpha,\beta} = \{n \in \omega : \{\langle e_\alpha^{-1}(n), k\rangle, \langle \beta, k'\rangle\} \in \mathcal{G}\}.$$

For each $\alpha \in \omega_1 \setminus \omega$ and $k \in K$, let $\mathcal{X}^k_\alpha = \{x^{k,k'}_{\alpha,\beta} : \beta \in \omega_1 \setminus \alpha, k' \in K\}$. Note that $\mathcal{G}$ is $\mathrm{HF}^{1,1}$ iff $\mathcal{X}^k_\alpha$ is eventually splitting for all $\alpha \in \omega_1 \setminus \omega$ and $k \in K$. Thus any proper forcing notion preserving all eventually splitting sequences of length ω_1 preserves all $\mathrm{HF}^{1,1}$ graphs. $\square$

To extend this proposition to all HF graphs, we'll need the following definition. We write $\mathcal{D}^m_{\aleph_0}(\omega)$ for the collection of all sets $\mathcal{A} \subseteq [\omega]^m$ of cardinality $\aleph_0$ such that the elements of $\mathcal{A}$ are pairwise disjoint.

Definition 5.11. Let $\mathbb{P}$ be a proper forcing notion. For $m < \omega$, we write $(*_m)$ for the following property: Let $p_0 \in \mathbb{P}$, λ be a sufficiently large regular cardinal and M be a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$ and p_0. If there is $\langle y_i \rangle_{i<m} \in ([\omega]^{\aleph_0})^m$ such that for every $\mathcal{A} \in M \cap \mathcal{D}^m_{\aleph_0}(\omega)$ we have that

$$\{a \in \mathcal{A} : (\forall i < m)\ a(i) \in y_i\}$$

is infinite, then there is $p \leq p_0$ that is $(M, \mathbb{P})$-generic such that

$$p \Vdash `(\forall \dot{\mathcal{A}} \in M[G_{\mathbb{P}}] \cap \mathcal{D}^m_{\aleph_0}(\omega))\ |\{a \in \dot{\mathcal{A}} : (\forall i < m)\ a(i) \in y_i\}| = \aleph_0'.$$

Proposition 5.12. *Let $\mathbb{P}$ be a proper forcing notion with property $(*_m)$. Then $\mathbb{P}$ preserves all $\mathrm{HF}^{m,n}$ graphs for all $n < \omega$.*

Proof. Let $\mathcal{G} \subseteq [\omega_1 \times K]^2$ be an $\mathrm{HF}^{m,n}$ graph, λ a sufficiently large regular cardinal, $\dot{\mathcal{A}}$ a $\mathbb{P}$-name for a countable K-aligned subset of $\mathrm{Fn}_m(\omega_1, K)$, $p_0 \in \mathbb{P}$, and M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$, $\mathcal{G}$, and $\dot{\mathcal{A}}$. We can assume (by strengthening p_0 if necessary) that there is some $\vec{k} \in K$ such that

$$p_0 \Vdash `(\forall a \in \dot{\mathcal{A}})(\forall i < m)\ a(i) = \vec{k}(i)'$$

and $\gamma < \delta_M$ such that $p_0 \Vdash `(\forall a \in \dot{\mathcal{A}})\ \mathrm{dom}(a) < \{\gamma\}'$. Pick some bijection $e\colon \gamma \to \omega$. Let $b \in \mathrm{Fn}_n(\omega_1, K) \setminus M$ and $\epsilon \in 2^n$. Similarly, by extending p_0 we can find a bijection $\sigma\colon m \to m$ such that p_0 forces that for infinitely many $a \in \dot{\mathcal{A}}$, $e(\mathrm{dom}(a)(\sigma(i))) = e[\mathrm{dom}(a)](i)$. For each $i < m$, define $y_i \subseteq \omega$ as follows:

$$y_i = \{e(\alpha) : (\alpha < \gamma) \vee ((\forall j < n)\ \{\langle \alpha, \vec{k}(\sigma(i)) \rangle, b(j)\} \in \mathcal{G} \iff \epsilon(j) = 1)\}.$$

Using Proposition 5.4, we have that for every $\mathcal{C} \in \mathcal{D}^m_{\aleph_0}(\omega) \cap M$ the set $\{c \in \mathcal{C} : (\forall i < m)\ c(i) \in y_i\}$ is infinite. Since $\mathbb{P}$ satisfies $(*_m)$, we can find some $p \leq p_0$ that is $(M, \mathbb{P})$-generic such that

$$p \Vdash `(\forall \dot{\mathcal{C}} \in M[G_{\mathbb{P}}] \cap \mathcal{D}^m_{\aleph_0}(\omega))\ |\{c \in \dot{\mathcal{C}} : (\forall i < m)\ c(i) \in y_i\}| = \aleph_0'.$$

Let $\dot{C} = \{e[\mathrm{dom}(a)] : a \in \dot{\mathcal{A}}\}$. Then we can find some $q \leq p$ and $c \in [\omega]^m$ such that $q \Vdash$ '$c \in \dot{C}$' and for all $i < m$, $c(i) \in y_i$. But then if $a = \{\langle e^{-1}[c](i), \vec{k}(i)\rangle : i < m\}$, we have that $q \Vdash$ '$a \in \dot{A}$' and $a \in U^{\mathcal{G}}_{b,\epsilon}$ as required. □

Corollary 5.13. *Suppose $\mathbb{P}$ is a proper forcing that doesn't add any reals. Then $\mathbb{P}$ preserves all* HF *graphs.*

We will now prove that adding Hechler reals preserves all HF graphs. This will be a necessary step to showing that both that Neeman iterations preserve HF graphs, and that we can also force $\mathrm{add}(\mathcal{M}) > \aleph_1$.

Theorem 5.14. *Adding any number of Hechler reals preserves all* HF *graphs.*

This theorem will be an extension of the following result, and our proof will follow their original argument closely.

Theorem 5.15 ([4, Theorem 3.1])**.** *Adding any number of Hechler reals preserves all eventually narrow sequences.*

Definition 5.16 (Hechler forcing)**.** We write $\omega^{\uparrow<\omega}$ and $\omega^{\uparrow\omega}$ for the sets of all strictly increasing functions into ω of finite length and length ω respectively. We now define the forcing $\mathbb{H}$ as the collection of all $(s, f) \in (\omega^{\uparrow<\omega}, \omega^{\uparrow\omega})$ with the ordering that $(t, g) \leq (s, f)$ if:

1. $t \sqsupseteq s$,
2. $g \geq f$,
3. For all $i \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$, $t(i) \geq f(i)$.

Given an dense-open set $D \subseteq \mathbb{H}$, we define the sequence $\langle D_\alpha\rangle_{\alpha<\omega_1}$ of *derivatives* of D as follows:

- $D_0 = \{s \in \omega^{\uparrow<\omega} : (\exists f \in \omega^{\uparrow\omega})\ (s, f) \in D\}$.
- Given D_α, we define $D_{\alpha+1}$ as

$$D_{\alpha+1} = D_\alpha \cup \{s \in \omega^{\uparrow<\omega} : (\exists n > |s|)(\forall k < \omega)(\exists t \in D_\alpha) \\ (s \sqsubseteq t) \wedge (|t| = n) \wedge (t(|s|) > k)\}.$$

- For limit $\beta < \omega_1$, we let $D_\beta = \bigcup_{\alpha<\lambda} D_\alpha$.

Theorem 5.17 ([4, Theorem 2.1])**.** *For every dense-open set $D \subseteq \mathbb{H}$, there is $\gamma < \omega_1$ such that $D_\gamma = \omega^{\uparrow<\omega}$.*

Lemma 5.18. *For all $m < \omega$, $\mathbb{H}$ has property $(*_m)$.*

Proof. Suppose not. We will assume that $m > 0$ to ignore the trivial case. Then there is some $\dot{\mathcal{A}}$ a $\mathbb{H}$-name for an element of $\mathcal{D}^m_{\aleph_0}(\omega)$, λ a sufficiently large regular cardinal, M a countable elementary submodel of $H(\lambda)$ containing $\mathbb{H}$ and $\dot{\mathcal{A}}$, $\langle y_i \rangle_{i<m} \in ([\omega]^{\aleph_0})^m$ such that for every $\mathcal{B} \in M \cap \mathcal{D}^m_{\aleph_0}(\omega)$ the set $\{b \in \mathcal{B} : (\forall i < m)\ b(i) \in y_i\}$ is infinite, and $(s, f) \in \mathbb{H}$ that is $(M, \mathbb{H})$-generic such that

$$(s, f) \Vdash `|\{a \in \dot{\mathcal{A}} : (\forall i < m)\ a(i) \in y_i\}| < \aleph_0'.$$

Extending (s, f) if necessary, we can find some $n < \omega$ such that

$$(s, f) \Vdash `(\forall a \in [\omega \setminus n]^m)\ a \in \dot{\mathcal{A}} \implies (\exists i < m)\ a(i) \notin y_i'. \qquad (\dagger)$$

Let $\dot{h}$ be the $\mathbb{H}$-name for the enumeration of $\dot{\mathcal{A}}$ in $<$-increasing order. By elementarity, $\dot{h} \in M$. For each $t \in \omega^{\uparrow<\omega}$ with $(t, f) \leq (s, f)$ and each $i \geq n$, define

$$Z_t(i) = \{a \in [\omega]^m : (\forall g \in \omega^{\uparrow\omega})(\exists (t', g') \leq (t, g))\ (t', g') \Vdash `\dot{h}(i) = a'\}.$$

Claim 5.18.1. *For all $t \in \omega^{\uparrow<\omega}$ with $(t, f) \leq (s, f)$ and all $i \in \omega \setminus n$, we have $Z_t(i) \neq \emptyset$.*

Proof of Claim. Let t, i be as in the statement of the claim and define the set

$$D = \{p \in \mathbb{H} : (\exists a \in [\omega]^m)\ p \Vdash `\dot{h}(i) = a'\}.$$

Note D is dense-open and $D \in M$. We will inductively prove that for all $\alpha < \omega_1$, if $t \in D_\alpha$ then the claim holds. This is sufficient by Theorem 5.17. Since the claim holds for $t \in D_0$, and the inductive step for limit ordinals follows immediately, suppose that the inductive hypothesis is true at $\alpha < \omega_1$ and $t \in D_{\alpha+1} \setminus D_\alpha$. Thus there is a sequence $\langle t_k \rangle_{k<\omega}$ of elements of D_α and $l < \omega$ such that for all $k < \omega$ we have $|t_k| = l$, $t_k \sqsupseteq t$, and $t_k(|t|) > k$. By elementarity, assume $\langle t_k \rangle_{k<\omega} \in M$.

If there is some $a \in [\omega]^m$ such that a belongs to infinitely many $Z_{t_k}(i)$, then $a \in Z_t(i)$ and we are done. Suppose instead this is not the case. For all $k < \omega$ let a_k be the $<_w$-least element of $Z_{t_k}(i)$ and let $\mathcal{B} = \{a_k : k < \omega\}$. By our assumption $\mathcal{B}$ is infinite, but also by elementarity $\mathcal{B} \in M$. Thus $\{a \in \mathcal{B} : (\forall i < m)\ a(i) \in y_i\}$ is infinite. So by picking k sufficiently large such that $\min(a_k) \geq n$, $k \geq f(l-1)$, and $(\forall i < m)\ a_k(i) \in y_i$, we have that $(t_k, f) \leq (t, f)$ and there is some $(u, g) \leq (t_k, f)$ with $(u, g) \Vdash `\dot{h}(i) = a_k'$, contradicting ($\dagger$). □

For every $i < \omega$ pick the $<_w$-least $a_i \in Z_s(i)$ and let $\mathcal{B} = \{a_i : i < \omega\}$. Again, we have that $\mathcal{B} \in M$ and is infinite, so $\{a \in \mathcal{B} : a \cap b = \emptyset\}$ is infinite. Thus again we can pick some i sufficiently large such that $\min(a_i) \geq n$, $a_i \cap b = \emptyset$, and there is some $(s', f') \leq (s, f)$ such that $(s', f') \Vdash$ '$\dot{h}(i) = a_i$', again contradicting (†). □

Theorem 5.19. *Let $\mathcal{G} \subseteq [\omega_1 \times K]^2$ be an $\mathrm{HF}^{m,n}$ graph and $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi \leq \nu \rangle$ be a finite-support iteration such that for all $\xi < \nu$, $\mathbb{P}_\xi$ forces that $\dot{\mathbb{Q}}_\xi$ is ccc and preserves that $\mathcal{G}$ is $\mathrm{HF}^{m,n}$. Then $\mathbb{P}_\nu$ preserves that $\mathcal{X}$ is $\mathrm{HF}^{m,n}$.*

Proof. Let $\mathcal{G} \subseteq [\omega_1 \times K]^2$ and $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi \leq \nu \rangle$ be as in the statement of the theorem. We will prove this via induction on ξ. The successor case follows by our hypothesis, and limit cases of uncountable cofinality don't add any reals, so we can assume that $\mathrm{cof}(\xi) = \aleph_0$. Let $\langle \xi_k \rangle_{k<\omega}$ be a cofinal sequence in ξ.

Suppose the theorem is false at ξ, i.e. there is some $\mathbb{P}_\xi$-name $\dot{\mathcal{A}}$ for a countably infinite K-aligned subset of $\mathrm{Fn}_m(\omega_1, K)$ and $p \in \mathbb{P}_\xi$ such that

$$p \Vdash \text{`}(\forall \alpha < \omega_1)(\exists b \in \mathrm{Fn}_n(\omega_1 \setminus \alpha, K))(\exists \epsilon \in 2^n)\ |\dot{\mathcal{A}} \cap U^{\mathcal{G}}_{b,\epsilon}| < \aleph_0\text{'}.$$

We can also presume (by extending p if necessary) that there is some $\gamma < \omega_1$ such that $p \Vdash$ '$\gamma = \sup\{\bigcup \mathrm{dom}(a) : a \in \dot{\mathcal{A}}\}$'. Let $G_\xi \subseteq \mathbb{P}_\xi$ be a generic filter containing p. Then we can find some uncountable subset $\mathcal{B} \subseteq \mathrm{Fn}_n(\omega_1, K)$ such that for all $b \in \mathcal{B}$, there is some $p_b \in G_\xi$ and $\eta_b < \gamma$ such that

$$p_b \Vdash \text{`}(\forall a \in \mathrm{Fn}_m(\gamma \setminus \eta_b, K))\ a \in \dot{\mathcal{A}} \implies a \notin U^{\mathcal{G}}_{b,\epsilon}\text{'}.$$

For all $\zeta < \xi$, let $G_\zeta = G_\xi \cap \mathbb{P}_\zeta$. Since $\mathbb{P}_\xi$ is a direct limit of $\langle \mathbb{P}_{\xi_k} \rangle_{k<\omega}$, we can find some $k < \omega$, $\eta < \gamma$, and uncountable $\mathcal{C} \subseteq \mathcal{B}$ such that for all $b \in \mathcal{C}$, $p_b \in G_{\xi_k}$ and $\eta_b = \eta$. Note that $\mathcal{C} \in V[G_{\xi_k}]$. Working in $V[G_{\xi_k}]$, define the set $\mathcal{A}'$ as follows:

$$\mathcal{A}' = \bigcap_{b \in \mathcal{C}} \{a \in \mathrm{Fn}_m(\gamma \setminus \eta) : a \notin U^{\mathcal{G}}_{b,\epsilon}\}.$$

For all $b \in \mathcal{C}$, we have that $\mathcal{A}' \cap U^{\mathcal{G}}_{b,\epsilon}$. But also, we have that

$$G_{\xi_k} \Vdash \text{`}\dot{\mathcal{A}} \cap \mathrm{Fn}_m(\gamma \setminus \eta) \subseteq \mathcal{A}'\text{'}$$

which implies that $\mathcal{A}'$ contains an infinite K-aligned set. Thus $\mathbb{P}_{\xi_k}$ does not preserve that $\mathcal{G}$ is $\mathrm{HF}^{m,n}$, contradicting our hypothesis. □

Proof of Theorem 5.14. Follows from Lemma 5.18 and Theorem 5.19. □

Corollary 5.20.

1. *Adding any number of Cohen reals preserves* HF *graphs.*
2. *Strongly proper forcings preserve* HF *graphs.*

Proof.

1. If $g \in \omega^\omega$ is a Hechler real, then $g \pmod 2 \in 2^\omega$ is a Cohen real. Thus since adding any number of Hechler reals preserves HF graphs, so does adding any number of Cohen reals.
2. By Lemma 2.17, strongly proper forcings only add Cohen reals. Since the only way to destroy a graph being HF is to add new reals (Corollary 5.13), the result follows. □

Theorem 5.21. *Let $\mathcal{G} \subseteq [\omega_1 \times K]^2$ be an* $\mathrm{HF}^{m,n}$ *graph, θ be an inaccessible cardinal, and J, $\mathcal{S}$, $\mathcal{T}$, and $\mathbb{P}(J)$ as expected. If every $Y \in \mathcal{T}$ either Y is trivial or $\mathbb{P}(J) \cap Y$ forces that $J(Y)$ preserves that $\mathcal{G}$ is* $\mathrm{HF}^{m,n}$*, then $\mathbb{P})(J)$ preserves that $\mathcal{G}$ is* $\mathrm{HF}^{m,n}$.

Proof. As with the proof of Theorem 3.25, since strongly proper forcings preserve HF graphs, it is sufficient to show that if $Z \in \mathcal{T}$ is a limit element, and $\mathbb{P}(J) \cap Y$ preserves that $\mathcal{G}$ is $\mathrm{HF}^{m,n}$ for all $Y \in \mathcal{T} \cap Z$, then $\mathbb{P}(J) \cap Z$ preserves that $\mathcal{G}$ is $\mathrm{HF}^{m,n}$, which can be shown via Proposition 5.9.

Let $(p_0, f_0) \in \mathbb{P}(J) \cap Z$, $\dot{\mathcal{A}}$ be a $\mathbb{P}(J) \cap Z$ for a countable K-aligned subset of $\mathrm{Fn}_m(\omega_1, K)$, and $\kappa \gg \theta$ be a sufficiently large regular cardinal. Pick a countable elementary submodel $\overline{M} \prec H(\kappa)$ containing all relevant objects and let $M = \overline{M} \cap H(\theta) \in \mathcal{S}$. By Proposition 2.30, we can find $(p_1, f) \leq (p_0, f_0)$ such that $M \cap Z \in p_1$. Since Z is a limit model contained in M, we can find some $Z \in M \cap Z \cap \mathcal{T}$ such that $p_1 \cap M \subseteq X$. Thus via Lemma 2.29 there is some $p \in \mathbb{P}^{\mathcal{S},\mathcal{T}}_{\in}$ such that $p_1 \cup \{X\} \subseteq p$ and $(p, f) \in \mathbb{P}$. By Theorem 2.33, $(p \cap X, f\restriction X)$ is an $(\overline{M}, \mathbb{P}(J) \cap X)$-generic condition. Let $b \in \mathrm{Fn}_n(\omega_1, K) \setminus \overline{M}$ and $\epsilon \in 2^n$. Now define the following set:

$$\dot{W} = \{(a, (q \cap X, g\restriction X)) : (c \in \mathrm{Fn}_m(\omega_1, K)) \wedge (p \cap M \sqsubseteq q) \\ \wedge ((q, g) \Vdash \text{`}a \in \dot{\mathcal{A}}\text{'})\}.$$

Note that $\dot{W} \in \overline{M}$ by elementarity and that $\dot{W}$ is a $\mathbb{P}(J) \cap X$-name for an infinite subset of $\mathrm{Fn}_m(\omega_1, K)$. Thus by our inductive hypothesis and Proposition 5.9 we can find $(q, g) \in \mathbb{P} \cap Z$ and $a \in \mathrm{Fn}_m(\omega_1, K)$ such that:

1. $(q, g), a \in M$,
2. $p \cap M \sqsubseteq q$,

3. $(q \cap X, g \restriction X)$ and $(p \cap X, f \restriction X)$ are compatible in $\mathbb{P}(J) \cap X$,
4. $a \in U^{\mathcal{G}}_{b,\epsilon}$,
5. $(q, g) \Vdash$ '$a \in \dot{\mathcal{A}}$'.

Thus by Lemma 2.32, (q, g) and (p, f) are compatible as required. □

Definition 5.22. If $\mathcal{G}$ is an $\mathrm{HF}^{m,n}$ graph, we write $\mathsf{PFA}(\mathcal{G}, \mathrm{HF}^{m,n})$ for the following statement, For every proper forcing $\mathbb{P}$ that preserves that $\mathcal{G}$ is $\mathrm{HF}^{m,n}$ and sequence $\{D_\alpha : \alpha < \omega_1\}$ of dense-open subsets of $\mathbb{P}$, there is a filter $\mathcal{F} \subseteq \mathbb{P}$ such that $\mathcal{F} \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

We define analogous forcing axioms for HFD^m, HFC^n, and strong HF graphs. In the case where the type of HF graph is clear or irrelevant, we may write $\mathsf{PFA}(\mathcal{G}, \mathrm{HF})$ for convenience.

As with solid graphs, we have the following:

Corollary 5.23. *Assuming the existence of a supercompact cardinal, if $\mathcal{G}$ is an $\mathrm{HF}^{m,n}$ graph, there is a proper forcing $\mathbb{P}$ that preserves $\mathcal{G}$ such that $V^{\mathbb{P}} \models$ $\mathsf{PFA}(\mathcal{G}, \mathrm{HF}^{m,n})$. The analogous result holds for HFD^m, HFC^n, and strong HF graphs.*

Proof. Analogous to the proof of Corollary 3.28. □

Remark 5.24. Since adding Hechler reals preserve HF graphs, $\mathsf{PFA}(\mathcal{G}, \mathrm{HF})$ implies $\mathrm{add}(\mathcal{M}) > \aleph_1$.

Question 5.25. Does adding random reals preserve HF graphs?

Finally, we show that almost strongly proper forcings preserve HF graphs (and thus $\mathsf{PFA}(\mathcal{G}, \mathrm{HF})$ implies the Mapping Reflection Principle):

Proposition 5.26. *Let $\mathbb{P}$ be an almost strongly proper forcing. Then $\mathbb{P}$ has property $(*_m)$ for all $m < \omega$.*

Proof. Let $p_0 \in \mathbb{P}$, $\lambda < \lambda'$ be sufficiently large regular cardinals, M a countable elementary submodel of $H(\lambda')$ containing $\mathbb{P}$ and p_0, and $\langle y_i \rangle_{i<m} \in ([\omega]^{\aleph_0})^m$ such that for every $\mathcal{A} \in M \cap \mathcal{D}^m_{\aleph_0}(\omega)$ we have that $\{a \in \mathcal{A} : (\forall i < m)\ a(i) \in y_i\}$ is infinite. Let $\dot{\mathcal{A}} \in M$ be a $\mathbb{P}$-name for an element of $\mathcal{D}^m_{\aleph_0}(\omega)$, and take $p \leq p_0$ that is $(M, \mathbb{P})$-almost strongly generic. Consider the following set:

$$\begin{aligned} \mathcal{U} = \{N \cap \mathbb{P} : (&\mathbb{P}, \dot{\mathcal{A}} \in N) \\ &\wedge (N \text{ is a countable elementary submodel of } H(\lambda))\}. \end{aligned}$$

$\mathcal{U}$ is unbounded in $[\mathbb{P}]^{\aleph_0}$ and by elementarity is contained in M. Thus we can find some countable elementary submodel $N \prec H(\lambda)$ such that $\mathbb{P}, \dot{\mathcal{A}} \in N \in M$ and conditions $p' \leq p$ and $p' \restriction N \in N$ such that every condition $r \leq p' \restriction N \in N$ is compatible with p'. Define the following set:

$$\mathcal{B} = \{a \in [\omega]^m : (\exists r \leq p' \restriction N)\ (r \Vdash `a \in \dot{\mathcal{A}}')\}.$$

By elementarity, $\mathcal{B} \in \mathcal{N}$. Moreover, by definition of $\dot{\mathcal{A}}$, for all $n < \omega$ there is some $a \in [\omega \setminus n]^m \cap \mathcal{B}$. Thus $\mathcal{B}$ contains a subset $\mathcal{B}' \in \mathcal{D}^m_{\aleph_0}(\omega)$, so $\{a \in \mathcal{B} : (\forall i < m)\ a(i) \in y_i\}$ is infinite. This implies that

$$p' \Vdash `|\{a \in \dot{\mathcal{A}} : (\forall i < m)\ a(i) \in y_i\}| = \aleph_0'$$

as required. □

Corollary 5.27. *Let $\mathcal{G}$ be an* HF *graph. Then* $\mathsf{PFA}(\mathcal{G}, \mathrm{HF})$ *implies* $\mathfrak{p} = \mathfrak{s} = \aleph_1$ *and* $\mathrm{add}(\mathcal{M}) = \mathfrak{c} = \aleph_2$.

5.3 the P-ideal dichotomy and HF graphs

In this section, we will show that the standard forcing notion with side conditions for forcing PID preserves HF graphs. Note that there is a proper forcing notion that forces PID without adding reals ([37, §5]), but this proof provides some insight into which forcing notions with side conditions might preserve HF graphs.

Let $\mathcal{I}$ be a P-ideal on some ordinal ν that does not satisfy the second alternative of the P-ideal dichotomy. (This implies that $\mathcal{I}^{\perp}$ generates a proper σ-ideal on ν.) Let $\mathcal{J}$ be the σ-ideal generated by $\mathcal{I}^{\perp}$. Furthermore, fix a sufficiently large cardinal θ such that $H(\theta)$ contains all the above objects and fix a well-order $<_w$ on $H(\theta)$. Finally, given a countable elementary submodel $N \prec (H(\theta), \in, <_w)$ containing $\mathcal{I}$, let ξ_N be the $<_w$-least element of ν such that $\xi_N \notin \bigcup(\mathcal{J} \cap N)$.

Definition 5.28. Define the poset $\mathbb{P}(\mathcal{I})$ to be the set of finite functions $p \colon \mathcal{N}_p \to \mathcal{I}$ such that:

1. $\mathcal{N}_p$ is an $\in$-chain of countable elementary submodels of $H(\theta)$ containing all above objects,
2. For all $N \in \mathcal{N}_p$, $p(N)$ is a pseudounion of $N \cap \mathcal{I}$,
3. For all $M, N \in \mathcal{N}_p$ with $M \in N$, $p(M) \in N$.

We order $\mathbb{P}(\mathcal{I})$ by saying that $q \leq p$ if:

4. $\mathcal{N}_q \supseteq \mathcal{N}_p$,

5. $q \restriction \mathcal{N}_p = p$,

6. For all $N \in \mathcal{N}_p$ and all $M \in (\mathcal{N}_q \setminus \mathcal{N}_p) \cap N$, $\xi_M \in p(N)$.

Theorem 5.29 ([40, §8.3])**.** *$\mathbb{P}(\mathcal{I})$ is a proper forcing that adds an uncountable subset of ν witnessing the first alternative of the P-ideal dichotomy for $\mathcal{I}$.*

Theorem 5.30. *$\mathbb{P}(\mathcal{I})$ has property $(*_m)$ for all $m < \omega$ (and hence preserves all* HF *graphs).*

Proof. Let $p_0 \in \mathbb{P}(\mathcal{I})$ and $\dot{\mathcal{B}}$ be a $\mathbb{P}(\mathcal{I})$-name for an element of $\mathcal{D}^m_{\aleph_0}(\omega)$. Choose κ a sufficiently large regular cardinal, pick a countable elementary submodel $\overline{M} \prec H(\kappa)$ containing all relevant objects, and denote $M = \overline{M} \cap H(\theta)$. By the usual proof that $\mathbb{P}(\mathcal{I})$ is proper, we can pick $p \leq p_0$ such that p is $(\overline{M}, \mathbb{P}(\mathcal{I}))$-generic and $M \in \mathcal{N}_p$. Let $\langle y_i \rangle_{i<m} \in [\omega]^{\aleph_0}$ be such that for all $\mathcal{A} \in \overline{M} \cap \mathcal{D}^m_{\aleph_0}(\omega)$, $|\{a \in \mathcal{A} : (\forall i < m)\ a(i) \in y_i\}| = \aleph_0$.

Claim 5.30.1. *Let $X \in [p(M)]^{<\aleph_0}$ and let $n < \omega$. Then there is $a \in [\omega \setminus n]^m$ and $q \in \mathbb{P} \cap M$ such that $q \Vdash$ '$a \in \dot{\mathcal{A}}$' and for all $N \in \mathcal{N}_q \setminus \mathcal{N}_p$, $\xi_N \in p(M) \setminus X$.*

Proof of Claim. Let M' be a countable elementary submodel of $H(\theta)$ containing p, and $I' \in \mathcal{I}$ be such that I' is a pseudounion of $M' \cap \mathcal{I}$ and $I' \cap X = \emptyset$. Then $p' = p \cup \{\langle M', I' \rangle\} \in \mathbb{P}$ and extends p. Since $\dot{\mathcal{A}}$ is forced to be infinite, we can find some $a \in [\omega \setminus n]^m$ and $p'' \leq p'$ such that $p'' \Vdash$ '$a \in \dot{\mathcal{A}}$'. Since $\mathbb{P}$ is proper, we can find some $q \in \mathbb{P} \cap M$ such that q is compatible with p' and $q \Vdash$ '$a \in \dot{\mathcal{A}}$'. Then since $q \cup p' \leq p'$ and $\mathcal{N}_q \in M, M'$, it follows that for all $N \in \mathcal{N}_q \setminus \mathcal{N}_p$, $\xi_N \in p(M) \cap I' \subseteq p(M) \setminus X$. □

Given a set $I \in \mathcal{I}$, let $\mathcal{C}_I$ be the set of $a \in [\omega]^m$ such that there is some $q \in \mathbb{P}$ such that:

1. $q \leq p \cap M$,

2. $q \Vdash$ '$a \in \dot{\mathcal{A}}$',

3. For all $N \in \mathcal{N}_q \setminus (\mathcal{N}_p \cap M)$, $\xi_N \in I$.

By elementarity and our claim, there is some $I \in M$ such that for all $X \in [I]^{<\aleph_0}$, the set $\mathcal{C}_{I \setminus X}$ contains a subset in $\mathcal{D}^m_{\aleph_0}(\omega)$. Pick X such that $I \setminus X \subseteq I_N$ for all $N \in p \setminus M$. Then since $\mathcal{C}_{I \setminus X} \in M$, we can find some $a \in \mathcal{C}_{I \setminus X} \cap [\omega \setminus n]^m$ such that $(\forall i < m)\ a(i) \in y_i$, and thus some $q \in \mathbb{P} \cap M$ such that $q \Vdash$ '$a \in \dot{\mathcal{A}}$' and for all $N \in \mathcal{N}_q \setminus (\mathcal{N}_p \cap M)$, $\xi_N \in I \setminus X$. But then q is compatible with p as required. □

Corollary 5.31. *$\mathsf{PID} + \mathrm{add}(\mathcal{M}) > \aleph_1$ is compatible with an $\mathcal{S}$-space.*

6

EXAMPLES OF SOLID AND HF GRAPHS

6.1 HFD and HFC type spaces

Definition 6.1. Let $X = \{x_\xi : \xi < \omega_1\} \subseteq 2^{\omega_1}$.

- X is an HFD^m *space* if for all $n < \omega$, all block sequences $\mathcal{A} = \{a_\alpha : \alpha < \omega\} \subseteq [\omega_1]^m$ and $\mathcal{B} = \{b_\beta : \beta < \omega_1\} \subseteq [\omega_1]^n$, and all $H\colon m \times n \to 2$, there is $\alpha < \omega$ and $\beta < \omega_1$ such that
$$(\forall i < m)(\forall j < n)\ x_{a_\alpha(i)}(b_\beta(j)) = H(i,j).$$
- X is an HFC^n *space* if for all $m < \omega$, all block sequences $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\} \subseteq [\omega_1]^m$ and $\mathcal{B} = \{b_\beta : \beta < \omega\} \subseteq [\omega_1]^n$, and all $H\colon m \times n \to 2$, there is $\alpha < \omega$ and $\beta < \omega_1$ such that
$$(\forall i < m)(\forall j < n)\ x_{a_\alpha(i)}(b_\beta(j)) = H(i,j).$$
- X is an HFD^m_w *space* if for all $n < \omega$, all block sequences $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\} \subseteq [\omega_1]^m$ and $\mathcal{B} = \{b_\beta : \beta < \omega_1\} \subseteq [\omega_1]^n$, and all $H\colon m \times n \to 2$, there is $\alpha < \beta < \omega_1$ such that
$$(\forall i < m)(\forall j < n)\ x_{a_\alpha(i)}(b_\beta(j)) = H(i,j).$$
- X is an HFC^n_w *space* if for all $m < \omega$, all block sequences $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\} \subseteq [\omega_1]^m$ and $\mathcal{B} = \{b_\beta : \beta < \omega_1\} \subseteq [\omega_1]^n$, and all $H\colon m \times n \to 2$, there is $\beta < \alpha < \omega_1$ such that
$$(\forall i < m)(\forall j < n)\ x_{a_\alpha(i)}(b_\beta(j)) = H(i,j).$$

We say that X is a *strong* HFD *space* (*strong* HFD_w *space*) if X is HFD^m (resp. HFD^m_w) for all $m < \omega$. Similarly, we say that X is a *strong* HFC *space* (*strong* HFC_w *space*) if X is HFC^n (resp. HFC^n_w) for all $n < \omega$.

More information about HFD and HFC type spaces can be found in [11]. In particular, note that a strong HFD space exists iff a strong HFC space exists.

Lemma 6.2. *Let* $X = \{x_\xi : \xi < \omega_1\} \subseteq 2^{\omega_1}$ *and* $K = \mathrm{Fn}(\omega, 2) \times \omega$. *Then define the graph* $\mathcal{G}_X^< \subseteq [\omega_1 \times K]^2$ *to be the set of pairs* $\{\langle \nu_0, \langle \epsilon_0, k_0 \rangle\rangle, \langle \nu_1, \langle \epsilon_1, k_1 \rangle\rangle\}$ *such that* $\nu_0 < \nu_1$ *and either:*

- $k_1 \notin \mathrm{dom}(\epsilon_0)$, *or*
- $x_{\nu_0}(\nu_1) = \epsilon_0(k_1)$.

Then X *is an* HFD_w^m *space iff* $\mathcal{G}_X^<$ *is* $(m, {<}\omega)$*-solid. Moreover,* X *is an* HFD^m *space iff* $\mathcal{G}_X^<$ *is an* HFD^m *graph.*

Proof. We focus on the HFD_w^m equivalence, since the second equivalence uses almost identical (but slightly simpler) arguments.

Let X be an HFD_w^m space, and suppose that $\mathcal{G}_X^<$ is not $(m, {<}\omega)$-solid. Via Proposition 3.20, there is an uncountable block sequence $\mathcal{A} = \langle a_\alpha \rangle_{\alpha<\omega_1}$ contained in $\mathrm{Fn}_m(\omega_1, K)$ such that for all $\delta < \omega_1$ there is some $b \in \mathrm{Fn}(\omega_1, K)$ with $\mathrm{dom}(b) > \{\delta\}$ and for all $\alpha < \delta$ we have $[a_\alpha, b] \not\subseteq \mathcal{G}_X^<$. Let $\mathcal{B} \subseteq \mathrm{Fn}(\omega_1, K)$ be the collection of all such b. By shrinking $\mathcal{A}$ and $\mathcal{B}$, we can assume that:

- $\mathcal{B}$ is an uncountable dom-block-sequence (enumerated $\langle b_\beta \rangle_{\beta<\omega_1}$),
- For all $\beta < \omega_1$ and all $\alpha < \beta$ we have $[a_\alpha, b_\beta] \not\subseteq \mathcal{G}$,
- There is $n < \omega$ with $\mathcal{B} \subseteq \mathrm{Fn}_n(\omega_1, K)$,
- $\mathcal{A}$, $\mathcal{B}$ are K-aligned (witnessed by $f\colon m \to K$, $g\colon n \to K$ respectively).

Denote $\epsilon_i = \pi_0(f(i))$ and $k_j = \pi_1(g(j))$ for $i < m$ and $j < n$. We can assume that for all $i < m$ and $j < n$, $k_j \in \mathrm{dom}(\epsilon_i)$ since this makes our life more difficult. Let $H\colon m \times n \to 2$ be given by $H(i, j) = \epsilon_i(k_j)$. Then since X is HFD_w^m, there is $\alpha < \beta$ such that

$$(\forall i < m)(\forall j < n)\ x_{\mathrm{dom}(a_\alpha)(i)}(\mathrm{dom}(b_\beta)(j)) = H(i, j) = \epsilon_i(k_j).$$

But this implies that $[a_\alpha, b_\beta] \subseteq \mathcal{G}_X^<$, contradicting the definition of $\mathcal{B}$ and thus that $\mathcal{G}_X^<$ is not $(m, {<}\omega)$-solid.

Conversely, suppose that $\mathcal{G}_X^<$ is $(m, {<}\omega)$-solid. Let $n < \omega$, let $\mathcal{A} = \langle a_\alpha \rangle_{\alpha<\omega_1} \subseteq [\omega_1]^m$ and $\mathcal{B} = \langle b_\beta \rangle_{\beta<\omega_1} \subseteq [\omega_1]^n$ be block sequences, and $H\colon m \times n \to 2$. For $i < m$, let $\epsilon_i\colon n \to 2$ be given by $\epsilon_i(j) = H(i, j)$. For $\alpha < \omega_1$, define $\tilde{a}_\alpha = \{\langle a_\alpha(i), \langle \epsilon_i, i \rangle\rangle : i < m\}$ and $\tilde{b}_\alpha = \{\langle b_\alpha(j), \langle \emptyset, j \rangle\rangle : j < n\}$. Let $\tilde{\mathcal{A}} = \langle \tilde{a}_\alpha \rangle_{\alpha<\omega_1}$ and $\tilde{\mathcal{B}} = \langle \tilde{b}_\beta \rangle_{\beta<\omega_1}$. Then since $\mathcal{G}_X^<$ is $(m, {<}\omega)$-solid, we can find some $\alpha < \beta$ such that $[\tilde{a}_\alpha, \tilde{b}_\beta] \subseteq \mathcal{G}_X^<$, which implies that

$$(\forall i < m)(\forall j < n)\ x_{a_\alpha(i)}(b_\beta(j)) = H(i, j)$$

as required. □

The same argument also gives us the analogous lemma for HFC type spaces:

Lemma 6.3. *Let $X = \{x_\xi : \xi < \omega_1\} \subseteq 2^{\omega_1}$ and $K = \mathrm{Fn}(\omega, 2) \times \omega$. Then define the graph $\mathcal{G}_X^{\gtrless} \subseteq [\omega_1 \times K]^2$ to be the set of pairs $\{\langle \nu_0, \langle \epsilon_0, k_0\rangle\rangle, \langle \nu_1, \langle \epsilon_1, k_1\rangle\rangle\}$ such that $\nu_0 < \nu_1$ and either:*

- *$k_0 \notin \mathrm{dom}(\epsilon_1)$, or*
- *$x_{\nu_1}(\nu_0) = \epsilon_1(k_0)$.*

Then X is an HFC^n_w space iff $\mathcal{G}_X^{\gtrless}$ is $(<\omega, n)$-solid. Moreover, X is an HFC^n space iff $\mathcal{G}_X^{\gtrless}$ is an HFC^n graph.

Just as HFD and HFC spaces lead to HFD and HFC graphs, the converse is also true:

Proposition 6.4. *Let $\mathcal{G} \subseteq [\omega_1 \times K]^2$ and pick $k \in K$. For each $\alpha < \omega_1$, define $f_\alpha : \omega_1 \to 2$ as follows:*

$$f_\alpha(\beta) = \begin{cases} 0 & \{\langle \alpha, k\rangle, \langle \beta, k\rangle\} \notin \mathcal{G} \\ 1 & \{\langle \alpha, k\rangle, \langle \beta, k\rangle\} \in \mathcal{G} \end{cases}$$

(The value of $f_\alpha(\alpha)$ is unimportant.) Let $X = \{f_\alpha : \alpha < \omega_1\} \subseteq 2^{\omega_1}$. Then if $\mathcal{G}$ is an HFD^m (HFC^n) graph, X is an HFD^m (resp. HFC^n) space.

Proof. We just prove the result for $\mathcal{G}$ an HFD^m graph. Let $\mathcal{A} \subseteq [\omega_1]^m$ be a countably infinite block sequence. For each $a \in A$, define the function $\overline{a} \in \mathrm{Fn}_m(\omega_1, K)$ as $\overline{a}(\beta) = k$ if $\beta \in \mathrm{dom}(a)$ and undefined otherwise. Then $\overline{\mathcal{A}} = \{\overline{a} : a \in \mathcal{A}\}$ is a countable K-aligned subset, so there is some $\delta < \omega_1$ such that for all $b \in \mathrm{Fn}_n(\omega_1 \setminus \delta, K)$ and $\epsilon \in 2^{|b|}$, the set $U^{\mathcal{G}}_{b,\epsilon} \cap \overline{\mathcal{A}}$ is infinite.

Now suppose that we have some finite function $\sigma\colon \omega_1 \setminus \delta_M \to 2$. Let $\epsilon \in 2^{|\sigma|}$ be given by $\epsilon(j) = \sigma(\mathrm{dom}(\sigma)(j))$ and let $b \in \mathrm{Fn}(\omega_1, K)$ be given by $b(\beta) = k$ if $\beta \in \mathrm{dom}(\sigma)$ and undefined otherwise. Let $\mathcal{A}' = \{a \in \mathcal{A} : \overline{a} \in U^{\mathcal{G}}_{b,\epsilon}\}$. Then $\mathcal{A}'$ is infinite and contained in $[\sigma]$ as required. □

Remark 6.5.

- It is well known that CH implies the existence of a strong HFD space, and $\mathfrak{p} > \aleph_1$ implies that HFD and HFC spaces cannot exist (see e.g. [27]). These bounds can be improved: $\mathrm{non}(\mathcal{M}) = \aleph_1$ is sufficient for the existence of a strong HFD space ([12]), and our discussion of HF graphs shows that $\mathfrak{s} > \aleph_1$ implies no HFD or HFC spaces exist.
- For HFD_w and HFC_w spaces, the situation is different. While $\mathsf{MA}_{\aleph_1}$ implies that no strong S-spaces exist, it is compatible with $\mathsf{MA}_{\aleph_1}$ that there are HFD^m_w and HFC^n_ω spaces for all $m, n < \omega$ ([30, Theorem 3.5]).

- HFD and HFC type spaces can be used to construct various S-groups and L-groups ([26]). Thus we can use our models of PFA relativised to a (strong) HFD or HFC type space to show that these fragments of PFA are compatible with (strong) S-groups or L-groups.

6.2 STRONG COLOURINGS

To simplify notation in this section, given a function $c\colon [\lambda]^2 \to \kappa$ and $\alpha < \beta < \lambda$, we will write $c(\alpha, \beta)$ to mean $c(\{\alpha, \beta\})$.

Definition 6.6 ([28]). Let $\lambda, \mu, \kappa, \theta$ be cardinals with μ infinite, $\lambda \geq \mu \geq \kappa$, and $\theta \leq \aleph_0$. Then $\mathsf{Pr}_0(\lambda, \mu, \kappa, \theta)$ is the following statement: there exists a colouring $c : [\lambda]^2 \to \kappa$ such that whenever $m < \theta$, $\mathcal{A} \subseteq [\lambda]^m$ is a block-sequence with $|\mathcal{A}| = \mu$, and $H\colon m^2 \to \kappa$ there is $a, b \in \mathcal{A}$ with $a < b$ such that for all $i, j < m$,

$$c(a(i), b(j)) = H(i, j).$$

Definition 6.7. Let $\lambda, \mu, \mu', \kappa, \theta$ be cardinals with μ, μ' infinite, $\lambda \geq \mu, \mu' \geq \kappa$, and $\theta \leq \aleph_0$. Then $\mathsf{Pr}_0(\lambda, \mu \circledast \mu', \kappa, \theta)$ is the following statement: there exists a colouring $c : [\lambda]^2 \to \kappa$ such that whenever $m < \theta$, $\mathcal{A}, \mathcal{B} \subseteq [\lambda]^m$ are block-sequences with $|\mathcal{A}| = \mu$ and $|\mathcal{B}| = \mu'$, and $H\colon m^2 \to \kappa$ there is $a \in \mathcal{A}$ and $b \in \mathcal{B}$ with $a < b$ such that for all $i, j < m$,

$$c(a(i), b(j)) = H(i, j).$$

The following will be very useful for us:

Proposition 6.8.

- *There is a colouring witnessing* $\mathsf{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)$ *iff there is a colouring witnessing* $\mathsf{Pr}_0(\aleph_1, \aleph_1, \aleph_0, \aleph_0)$.
- *There is a colouring witnessing* $\mathsf{Pr}_0(\aleph_1, \aleph_0 \circledast \aleph_1, \aleph_1, \aleph_0)$ *iff there is a colouring witnessing* $\mathsf{Pr}_0(\aleph_1, \aleph_0 \circledast \aleph_1, \aleph_0, \aleph_0)$.

Proof. Note that the first half of this proposition is a well-known result (see e.g. [35, §4]). Thus we just show the non-trivial direction of the second half. Suppose that $c\colon [\aleph_1]^2 \to \aleph_0$ witnesses $\mathsf{Pr}_0(\aleph_1, \aleph_0 \circledast \aleph_1, \aleph_0, \aleph_0)$. For all $\beta < \aleph_1$, fix a bijection $e_\beta\colon \beta \to \aleph_0$. Then define $d\colon [\aleph_1]^2 \to \aleph_0$ as follows:

$$d(\alpha, \beta) = e_\beta^{-1}(c(\alpha, \beta)).$$

Now let $m < \aleph_0$, $\mathcal{A}, \mathcal{B} \subseteq [\aleph_1]^m$ be block sequences of cardinality $\aleph_0$ and $\aleph_1$ respectively, and let $H\colon i, j \to \aleph_1$. Pick $\gamma < \aleph_1$ that is larger than any ordinal in the image of H. By repeatedly applying the pressing-down lemma, we can find an uncountable subsequence $\mathcal{B}' \subseteq \mathcal{B}$ and function $h\colon m^2 \to \aleph_0$ such that for all $b \in \mathcal{B}'$ we have that $b > \{\gamma\}$, and for all $i, j < m$,

$$e_{b(j)}(H(i,j)) = h(i,j).$$

Since c witnesses $\mathsf{Pr}_0(\aleph_1, \aleph_0 \circledast \aleph_1, \aleph_0, \aleph_0)$, we can find some $a \in \mathcal{A}$ and $b \in \mathcal{B}'$ with $b > a$ such that for all $i, j < m$, $c(a(i), b(j)) = h(i,j)$. But then for all $i, j < m$, we have that

$$c'(a(i), b(j)) = e^{-1}_{b(j)}(c(a(i), b(j))) = e^{-1}_{b(j)}(h(i,j)) = H(i,j)$$

as required. □

Lemma 6.9. *Let $c\colon [\aleph_1]^2 \to \aleph_0$ and let $K = \mathrm{Fn}(\aleph_0, \aleph_0) \times \aleph_0$. Then define the graph $\mathcal{G}_c \subseteq [\omega_1 \times K]^2$ to be the set of pairs $\{\langle \nu_0, \langle \epsilon_0, n_0 \rangle\rangle, \langle \nu_1, \langle \epsilon_1, n_1 \rangle\rangle\}$ such that $\nu_0 < \nu_1$ and either:*

- *$n_1 \notin \mathrm{dom}(\epsilon_0)$, or*
- *$c(\nu_0, \nu_1) = \epsilon_0(k_1)$.*

Then c witnesses $\mathsf{Pr}_0(\aleph_1, \aleph_1, \aleph_0, \aleph_0)$ iff $\mathcal{G}_c$ is strongly solid. Moreover, c witnesses $\mathsf{Pr}_0(\aleph_1, \aleph_0 \circledast \aleph_1, \aleph_0, \aleph_0)$ iff c is a strong HF graph.

Note the proof of this is almost identical to the proof of Lemma 6.2. In fact, the existence of a strong HFD_w space is equivalent to $\mathsf{Pr}_0(\aleph_1, \aleph_1, 2, \aleph_0)$.

Proof. As with Lemma 6.2, we will only prove the first result. Assume that c witnesses $\mathsf{Pr}_0(\aleph_1, \aleph_1, \aleph_0, \aleph_0)$ and let $m < \omega$ and $\mathcal{C} \subseteq \mathrm{Fn}_m(\omega_1, K)$ be an uncountable dom-block-sequence. By shrinking $\mathcal{C}$, we can assume that $\mathcal{C}$ is K-aligned. Let $f = \{\langle i, \langle \epsilon_i, n_i \rangle\rangle : i < m\}$ be the function that witnesses that $\mathcal{C}$ is K-aligned. We can also assume that $n_j \in \mathrm{dom}(\epsilon_i)$ for all $i, j < m$, since this makes our life more difficult. Define a function $H\colon m^2 \to \aleph_0$ by $H(i,j) = \epsilon_i(n_j)$. Since c witnesses $\mathsf{Pr}_0(\aleph_1, \aleph_1, \aleph_0, \aleph_0)$, we can find some $a, b \in \mathcal{A}$ with $\mathrm{dom}(a) < \mathrm{dom}(b)$ and $c(a(i), b(j)) = H(i,j) = \epsilon_i(n_j)$ for all $i, j < m$. Thus $\mathcal{G}_c$ is strongly solid.

For the converse, assume that $\mathcal{G}_c$ is strongly solid, let $m < \omega$, $\mathcal{A} \subseteq [\omega_1]^m$ be an uncountable block-sequence, and let $H\colon m^2 \to \aleph_0$. For each $i < m$, let $\epsilon_i\colon m \to \aleph_0$ be given by $\epsilon_i(j) = H(i,j)$. Define the following dom-block-sequence:

$$\mathcal{A}' = \{\{\langle a(i), \langle \epsilon_i, i \rangle\rangle : i < m\} : a \in \mathcal{A}\}.$$

Then since $\mathcal{G}_c$ is strongly solid, we can find $a, b \in \mathcal{A}'$ with $a < b$ and $[a, b] \subseteq \mathcal{G}_c$. In other words, for all $i, j < m$

$$c(a(i), b(j)) = \epsilon_i(j) = H(i,j)$$

as required. □

Theorem 6.10 ([12]). *There is a non-meagre set of reals of size* $\aleph_1$ *iff there is a colouring* $c\colon [\aleph_1]^2 \to \aleph_1$ *such that for all* $m, n < \omega$*, for every infinite block-sequence* $\mathcal{A} \subseteq [\aleph_1]^m$ *and uncountable block-sequence* $\mathcal{B} \subseteq [\aleph_1]^n$*, there is* $a \in \mathcal{A}$ *such that for every function* $H\colon m \times n \to \aleph_1$*, there is* $b \in \mathcal{B}$ *with* $a < b$ *such that* $c(a(i), b(j)) = H(i, j)$ *for all* $i < m$*,* $j < n$*.*

Note that such a colouring witnesses $\mathsf{Pr}_0(\aleph_1, \aleph_0 \circledast \aleph_1, \aleph_1, \aleph_0)$, and thus $\mathrm{non}(\mathcal{M}) = \aleph_1$ implies the existence of such a colouring.

Remark 6.11. Since a colouring witnessing $\mathsf{Pr}_0(\aleph_1, \aleph_0 \circledast \aleph_1, \aleph_1, \aleph_0)$ is encodable with a strong HF graph and $\mathsf{PFA}(\mathcal{G}, \mathrm{HF})$ implies that $\mathrm{add}(\mathcal{M}) = \aleph_2$, then the existence of such a colouring is strictly weaker than the existence of a non-meagre set of reals of size $\aleph_1$.

6.3 PRESERVING A FIRST-COUNTABLE STRONG S-SPACE

Definition 6.12. An uncountable T_3 topological space is called an *O-space* if all of its open subsets are countable or co-countable.

Note that all O-spaces are S-spaces [11, §2]. In this section we will show that we can construct a particular first-countable O-space that can be encoded by a strongly solid graph. This space was first constructed in [30].

Definition 6.13. Let $\mathbb{Q}$ be the poset of triples $r = \langle I, n, u\rangle$ such that $I \in [\omega_1]^{<\aleph_0}$, $n \in \omega$, and $u\colon I \times n \to \mathcal{P}(I)$ is such that for all $k < n$,

$$\alpha \in u(\alpha, k) \subseteq u(\alpha, 0) = I \cap (\alpha + 1).$$

We order $\mathbb{Q}$ as follows: for $q = \langle I, n, u\rangle, q' = \langle I', n', u'\rangle \in \mathbb{Q}$, $q \leq q'$ if:

1. $I' \subseteq I$,
2. $n' \leq n$,
3. For all $\alpha \in I'$ and $1 \leq k < n'$, $u'(\alpha, k) = u(\alpha, k) \cap I'$,
4. For all $\alpha, \beta \in I'$ and $1 \leq i, j < n'$:
   a) If $u'(\alpha, i) \cap u'(\beta, j) = \emptyset$, then $u(\alpha, i) \cap u(\beta, j) = \emptyset$,
   b) If $u'(\alpha, i) \subseteq u'(\beta, j)$, then $u(\alpha, i) \subseteq u(\beta, j)$.

For $q \in \mathbb{Q}$, write $q = \langle I^q, n^q, u^q\rangle$.

Lemma 6.14. *[30] Let $G \subseteq \mathbb{Q}$ be a generic filter and work in $V[G]$. Then for all $\alpha < \omega_1$ and $k < \omega$ let*

$$U(\alpha, k) = \bigcup \{u^q(\alpha, k) : q \in G, \alpha \in I^q, k < n^q\}.$$

Let $\mathcal{B}_+ = \{U(\alpha, k) : \alpha < \omega_1, 1 \leq k < \omega\}$. Then $\mathcal{B}_+$ is a clopen base for a T_2 topological space $X = (\omega_1, \tau)$.

Proposition 6.15. *For every countable pairwise-disjoint set $\{c_i \in [\omega_1]^{<\aleph_0} : i < \omega\}$, the set*

$$\mathcal{D} = \{q \in \mathbb{Q} : (\exists i < \omega, \alpha \in I^q)\ c_i \subseteq u^q(\alpha, n^q - 1)\}$$

is dense in $\mathbb{Q}$.

Proof. Let $q = \langle I^q, n^q, u^q \rangle$. Since I^q is finite, we can find some $i < \omega$ with $c_i \cap I^q = \emptyset$. Let $\beta = \max(I^q)$ and define $u' \colon (I^q \cup c_i) \times n^q \to 2$ as follows:

$$u'(\alpha, k) = \begin{cases} u^q(\alpha, k) & \alpha \in I^q \setminus \{\beta\}, \\ u^q(\beta, k) \cup c_i & \alpha = \beta, \\ \alpha + 1 & \alpha \in c_i. \end{cases}$$

Then $\langle I^q \cup c_i, u', n^q \rangle \in \mathbb{Q}$ and extends q. □

In particular, note that ω is dense in X.

Proposition 6.16. *Let $q_0 = \langle I^0, n^0, u^0 \rangle, q_1 = \langle I^1, n^1, u^1 \rangle \in \mathbb{Q}$ be conditions such that:*

- $I^0 \cap I^1 < I^0 \setminus I^1 < I^1 \setminus I^0$,
- $n^0 \geq n^1$,
- $u^0 \restriction (I^0 \cap I^1) \times n_1 = u^1 \restriction (I^0 \cap I^1) \times n_1$.

Then define $u \colon (I_0 \cup I_1) \times n_0 \to (I_0 \cup I_1)$ as follows:

$$u(\nu, i) = \begin{cases} u_0(\nu, i) & \nu \in I^0, \\ u_1(\nu, i) & (\nu \in I^1 \setminus I^0) \wedge (i < n_1), \\ u(\nu, n_1 - 1) & (\nu \in I^1 \setminus I^0) \wedge (i \geq n_1). \end{cases}$$

Then defining $q = \langle u, I^0 \cup I^1, n_0 \rangle \in \mathbb{Q}$, we have that $q \leq q_0, q_1$.

Proof. Follows immediately from the fact that $u \restriction I_0 \times n_0 = u_0$ and $u \restriction I_1 \times n_1 = u_1$. □

Lemma 6.17. *$\mathbb{Q}$ is strongly proper.*

Proof. Let $q_0 \in \mathbb{Q}$, λ be a sufficiently large cardinal, and $M \prec H(\lambda)$ a countable elementary submodel such that $q_0, \mathbb{Q} \in M$. We show that q_0 is strongly $(M, \mathbb{Q})$-generic. Take some $q \leq q_0$ and some dense-open set $\mathcal{D} \subseteq \mathbb{Q} \cap M$. Now define the condition $q_M = \langle I^q \cap \delta_M, n^q, u^q \restriction (I^q \cap \delta_M) \times n^q \rangle$. Note that $q_M \in M$, so we can find some condition $q' \in \mathbb{Q} \cap M$ such that $q' \in \mathcal{D}$ and $q' \leq q_M$. Then by Proposition 6.16, it follows that q' and q are compatible. □

Definition 6.18. Let $K = \omega \times \omega$. Then define the graph $\mathcal{G} \subseteq [\omega_1 \times K]^2$ as the set of all pairs $\{\langle \alpha_0, \langle k_0, d_0 \rangle \rangle, \langle \alpha_1, \langle k_1, d_1 \rangle \rangle\}$ such that $\alpha_0 < \alpha_1$ and either

- $d_i \notin U(a_i, k_i)$ for some $i < 2$, or
- $d_0 \neq d_1$, or
- $a_0 \in U(a_1, k_1)$.

Lemma 6.19 ([30, Lemma 3.8])**.** $\mathbb{Q}$ *forces that $\mathcal{G}_X$ is strongly solid.*

In fact, $\mathbb{Q}$ forces something even stronger:

Proposition 6.20. *Let $q_0 = \langle I^0, n^0, u^0 \rangle, q_1 = \langle I^1, n^1, u^1 \rangle \in \mathbb{Q}$ be conditions such that:*

- $I^0 \cap I^1 < I^0 \setminus I^1 < I^1 \setminus I^0$,
- $n^0 \geq n^1$,
- $u^0 \restriction (I^0 \cap I^1) \times n_1 = u^1 \restriction (I^0 \cap I^1) \times n_1$.

Pick some $\gamma \in I^0 \cap I^1$. Then define $u \colon (I_0 \cup I_1) \times n_0 \to (I_0 \cup I_1)$ as follows:

$$u(\nu, i) = \begin{cases} u_0(\nu, i) & \nu \in I^0, \\ u_1(\nu, i) & (\nu \in I^1 \setminus I^0) \wedge (i < n_1) \wedge (\gamma \notin u_1(\nu, i)), \\ u_1(\nu, i) \cup (I^0 \setminus I^1) & (\nu \in I^1 \setminus I^0) \wedge (i < n_1) \wedge (\gamma \in u_1(\nu, i)), \\ u(\nu, n_1 - 1) & (\nu \in I^1 \setminus I^0) \wedge (i \geq n_1). \end{cases}$$

Then defining $q = \langle u, I^0 \cup I^1, n_0 \rangle \in \mathbb{Q}$, we have that $q \leq q_0, q_1$.

Proof. Note that $q \leq q_0$ is immediate, since $u \restriction I_0 \times n_0 = u_0$ by definition. To show that $q \leq q_1$, it is sufficient to consider $\nu, \nu' \in I^1 \setminus I^0$ with $\nu < \nu'$ and $i, j < n_1$. We now check all possible cases:

- Suppose that $u_1(\nu, i) \cap u_1(\nu', j) = \emptyset$. In particular, this implies that γ is in at most one of these two sets. Thus $(I^0 \setminus I^1)$ is contained in at most one of $u(\nu, i)$ and $u(\nu', j)$, so $u(\nu, i) \cap u(\nu, j) = \emptyset$.

- Suppose that $u_1(\nu, i) \subseteq u_1(\nu', j)$ and $\gamma \notin u_1(\nu, i)$. Then

$$u(\nu, i) = u_1(\nu, i) \subseteq u_1(\nu', j) \subseteq u(\nu', j)$$

as required.

- Suppose that $u_1(\nu, i) \subseteq u_1(\nu', j)$ and $\gamma \in u_1(\nu, i)$. Then

$$u(\nu, i) = u_1(\nu, i) \cup (I^0 \setminus I^1) \subseteq u_1(\nu', j) \cup (I^0 \setminus I^1) = u(\nu', j)$$

as required. □

Lemma 6.21. *$\mathbb{Q}$ forces that for every countable pairwise-disjoint set $A \subseteq [\omega_1]^{<\aleph_0}$, there is some $\delta < \omega_1$ such that given $\{(\beta_i, k_i) \in (\omega_1 \setminus \delta) \times \omega : i < m\}$ such that there is some $d \in \omega$ with $d \in U(\beta_i, k_i)$ for all $i < m$, there is some $a \in A$ such that $a \subseteq U(\beta_i, k_i)$ for all $i < m$. (Thus $\mathbb{Q}$ forces that $\mathcal{G}_X$ is a strongly solid graph.)*

Proof. Let $\dot{A}$ be a $\mathbb{Q}$-name for a countable pairwise-disjoint subset of $[\omega_1]^{<\aleph_0}$. Let λ be a sufficiently large regular cardinal, and let M be a countable elementary submodel of $H(\lambda)$ containing $\mathbb{Q}$ and $\dot{A}$. Finally, let $q \in \mathbb{Q}$, $d < \omega$, and $\{(\beta_i, k_i) \in (\omega_1 \setminus \delta_M) \times \omega : i < m\}$ be such that $q \Vdash$ '$d \in U(\beta_i, k_i)$' for all $i < m$. (Note this implies that $\{d\} \cup \{\beta_i : i < m\} \in I^q$ and $d \in u^q(\beta_i, k_i)$ for all i.) Now define the condition $q_M = \langle I^q \cap \delta_M, n^q, u^q \upharpoonright (I^q \cap \delta_M) \times n^q \rangle$. Note that $q_M \in M$ by elementarity. By definition of $\dot{A}$, we can find some $a \in [\omega_1]^{<\aleph_0} \cap M$ and $p \in \mathbb{Q} \cap M$ such that $p \leq q_M$, $p \Vdash$ '$a \in \dot{A}$', and $a \cap I^{q_M} = \emptyset$. Moreover, we can assume that $a \subseteq I^p$. Finally, by Proposition 6.20, we can find some $r \leq p, q$ such that $a \subseteq u^r(\beta_i, k_i)$ for all $i < m$. □

Soukup showed that $\mathcal{G}_X$ encodes that X is an O-space:

Lemma 6.22 ([30, Lemma 3.9])**.** *If $\mathcal{G}_X$ is 2-solid, then every open set in X is either countable or co-countable.*

However, we can extend the results even further.

Definition 6.23. An uncountable T_3 topological space X is called a *strong O-space* if for all $m < \omega$, given an open subset $U \subseteq X^m$ such that $\pi_j(U)$ is uncountable for all $j < m$, then U is co-countable.

Proposition 6.24. *Strong O-spaces are strong S-spaces.*

The proof of this will extend the proof of [11, p. 2.25].

Proof. Let X be a strong O-space. As in [11, p. 2.25], we have that X is not Lindelöf and we can assume that every point $p \in X$ has a countable open subset. Note that 'every open subset U such that $\pi_j(U)$ is uncountable for all $j < m$ is co-countable' is inherited by all subspaces of X^m.

First, we will show that every subspace $Y \subseteq X^m$ is ccc for all $m < \omega$ by induction. The case $m = 1$ is given by the original proof. Suppose that the result holds for $m - 1$ and let $\mathcal{U}$ be an uncountable collection of open subsets of Y. If there is any $j < m$ such that $\bigcup\{\pi_j(U) : U \in \mathcal{U}\}$ is countable, by thinning $\mathcal{U}$ we can assume there is some $p \in X$ such that $p \in \pi_j(U)$ for all $U \in \mathcal{U}$ and then apply our inductive hypothesis. If not, then since X is a strong O-space, we can partition $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$ such that $\bigcup \mathcal{U}_0$ and $\bigcup \mathcal{U}_1$ are co-countable open sets. Thus $\bigcup \mathcal{U}_0 \cap \bigcup \mathcal{U}_1 \neq \emptyset$, so the elements of $\mathcal{U}$ are not pairwise-disjoint.

To see that X^m is hereditarily separable, let $Y \subseteq X^m$ and let $\mathcal{U}$ be a maximal pairwise-disjoint family of countable open subsets of Y. Since Y is ccc, $\mathcal{U}$ is countable and thus $\bigcup \mathcal{U}$ is a countable dense subset of Y. This shows that Y is separable, and thus X^m is hereditarily separable as required. □

Proposition 6.25. *If $\mathcal{G}_X$ is strongly solid, then X is a strong O-space.*

The proof of this will be very similar to [30, Lemma 3.9].

Proof. Let $V \subseteq X^k$ be an open set such that $\pi_j(V)$ is uncountable for all $j < m$, and $Y \in [X^m]^{\aleph_1}$. Inductively construct sequences $\langle b_\alpha \rangle_{\alpha<\omega_1} \subseteq \omega_1^m$, $\langle k_\alpha \rangle_{\alpha<\omega_1} \subseteq \omega$, $\langle \vec{d}_\alpha \rangle_{\alpha<\omega_1} \subseteq \omega^m$, and $\langle a_\alpha \rangle_{\alpha<\omega_1} \subseteq \omega_1^m$ such that:

1. $\prod_{j<m} U(b_\alpha(j), \vec{d}_\alpha(j)) \subseteq V$,
2. $\vec{d}_\alpha \in \prod_{j<m} U(b_\alpha(j), \vec{d}_\alpha(j))$,
3. $a_\alpha \in Y$,
4. $\omega < a_\alpha < b_\alpha < a_\beta < b_\beta$ for $\alpha < \beta < \omega_1$.

By thinning out our sequence, we can pick some $\vec{d} \in \omega^m$ such that $\vec{d}_\alpha = \vec{d}$ for all $\alpha < \omega_1$. Then for $\alpha < \omega_1$, define s_α as follows:

$$s_\alpha = \{\langle a_\alpha(j), \langle 0, \vec{d}(j)\rangle\rangle : j < m\} \cup \{\langle b_\alpha(j), \langle k_\alpha, \vec{d}(j)\rangle\rangle : j < m\}.$$

Since $\mathcal{G}$ is strongly solid, there is $\alpha < \beta < \omega_1$ such that $[s_\alpha, s_\beta] \subseteq \mathcal{G}$. In particular,

$$\{\langle a_\alpha(j), \langle 0, \vec{d}(j)\rangle\rangle, \langle b_\beta(j), \langle k_\alpha, \vec{d}(j)\rangle\rangle\} \in \mathcal{G}$$

for all $j < m$. But since $\vec{d}(j) \in \omega \subseteq U(a_\alpha(j), 0)$ and $b_\beta(j) \notin U(a_\alpha(j), 0)$, it follows that $a_\alpha(j) \in U(b_\beta(j), \vec{d}(j))$ for all $j < m$ and thus $a_\alpha \in V$ as required. □

Based on Lemma 6.21, one might ask the following: does $\mathbb{Q}$ force that $\mathcal{G}_X$ is strongly HF? This is not the case: if $(\beta_0, k_0), (\beta_1, k_1)$ and $d < \omega$ is such that $d \in U(\beta_0, k_0) \subseteq U(\beta_1, k_1)$, then there is no $a \in \mathrm{Fn}(\omega_1, \omega \times \omega)$ such that $[a, \{\langle \beta_0, \langle k_0, d\rangle\rangle\}] \subseteq \mathcal{G}_X$ but $[a, \{\langle \beta_1, \langle k_1, d\rangle\rangle\}] \cap \mathcal{G}_X = \emptyset$. However, it is 'almost' true in the following sense:

Corollary 6.26. $\mathbb{Q}$ *forces that for every K-aligned countably infinite subset $\mathcal{A} \subseteq \mathrm{Fn}(\omega_1, K)$, there is a $\delta < \omega_1$ such that for all $b \in \mathrm{Fn}(\omega_1 \setminus \delta, K)$, there exists $a, a' \in \mathcal{A}$ such that $[a, b] \subseteq \mathcal{G}$ and $[a', b] \cap \mathcal{G} = \emptyset$.*

Proof. The existence of such an a is given by Lemma 6.21. The existence of a' can be given by the same proof, replacing the usage of Proposition 6.20 with Proposition 6.16. □

This property implies that $\mathcal{G}_X$ is both HFD[1] and strongly solid. Moreover, the proofs in Section 5.2 can easily be modified to show that this property is preserved by

- Proper forcing satisfying $(*_m)$ for all $m < \omega$,
- Finite support ccc iterations,
- Neeman iterations.

Thus while $\mathcal{G}_X$ is not strongly HF, we still end up with the desired result:

Theorem 6.27. *Relative to a supercompact cardinal, there is a model with a first-countable strong O-space satisfying the following:*

- PID;
- MRP;
- $\mathrm{add}(\mathcal{M}) = \aleph_2$;
- $\binom{\omega_1}{\omega} \not\to \binom{\omega_1}{\omega}^{1,1}_2$.

7 SUMMARY AND OPEN QUESTIONS

SUMMARY OF RESULTS

Given a graph $\mathcal{G} \subseteq [\omega_1 \times K]^2$ for some set K, relative to a supercompact cardinal, we can use Neeman iterations to construct models of $\mathsf{PFA}(\mathcal{G})$. Moreover, depending on the properties of $\mathcal{G}$, $\mathsf{PFA}(\mathcal{G})$ implies the following:

	m-solid	(m,n)-solid	$\mathrm{HF}^{m,n}$
$\mathfrak{p}, \mathfrak{s}$	$\aleph_2$	$\aleph_2$	$\aleph_1$
$\mathrm{add}(\mathcal{M}), \mathfrak{c}$	$\aleph_2$	$\aleph_2$	$\aleph_2$
$\binom{\omega_1}{\omega} \to \binom{\omega_1}{\omega}^{1,1}_2$	✓	✓	✗
MRP	✓	✓	✓
OGA	!	✓	?
$\mathsf{BA}(\aleph_1)$	!	✓	?
All A-trees are club-isomorphic	!	✓	?
PID	✗	✗	✓

(In this table ! means the consequence holds for some but not all such graphs with the given property, and ? means that it is unknown.)

The following objects can be encoded by a graph $\mathcal{G} \subseteq [\omega_1 \times K]^2$ for some countable set K:

Strongly solid graph	Strong HF graph
Strong HFD_w space	Strong HFD space
Strong HFC_w space	Strong HFC space
$\mathsf{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)$ colouring	$\mathsf{Pr}_0(\aleph_1, \aleph_0 \circledast \aleph_1, \aleph_1, \aleph_0)$ colouring
First-countable strong O-space	

Moreover, we showed that (relative to a supercompact cardinal) we can also construct a model of with a first-countable O-space with all the properties given in our table for $\mathsf{PFA}(\mathcal{G}, \mathrm{HF})$.

PID, S-SPACES, AND CARDINAL CHARACTERISTICS

In particular, we have shown that PID + $\mathfrak{b} > \aleph_0$ is compatible with first-countable S-spaces, providing a partial answer to Question 1.11. In light of the fact that $\mathfrak{s}$ plays an important role, we can ask the following:

Question 7.1. Under PID, do any of the following properties imply there are no S-spaces? (Note each property implies the one underneath it.)

- $\min\{\mathfrak{b}, \mathfrak{s}\} > \aleph_1$.
- $\mathfrak{h} > \aleph_1$.

Question 7.2. Do any of the following properties imply the existence of an S-space? (Note each property implies the one underneath it.)

- $\binom{\omega_1}{\omega} \not\to \binom{\omega_1}{\omega}^{1,1}_2$.
- $\mathfrak{s} = \aleph_1$.
- $\mathfrak{p} = \aleph_1$.

Question 7.3.

- Is there a model of PID + an S-space + no first-countable S-spaces?
- Is there a model of PID + an S-space + no strong S-spaces?

Since $\binom{\omega_1}{\omega} \not\to \binom{\omega_1}{\omega}^{1,1}_2$ implies the existence of an $\mathrm{HF}^{1,1}$ graph, and an HFD^1 graph implies the existence of an S-space, we can also ask the following:

Question 7.4. Does the existence of an $\mathrm{HF}^{1,1}$ graph imply the existence of an HFD^1 graph?

We've only been able to show that PID + $\mathfrak{b} > \aleph_1$ is compatible with 'HFD-like' S-spaces and not 'HFD_w-like' S-spaces. This leads to the following question:

Question 7.5. Is there a model of PID + an HFD_w-space + no HFD spaces?

Note that since $\mathfrak{h} = \mathfrak{s} = \aleph_2$ in $\mathsf{PFA}(\mathbb{S})[\mathbb{S}]$, our results do not elucidate whether S-spaces do or do not exist here.

$\mathsf{PFA}(\mathcal{G})$ and $\mathsf{PFA}(\mathcal{G}, \mathrm{HF})$

If $\mathcal{G}$ is (m, n)-solid, then $\mathsf{PFA}(\mathcal{G})$ implies the Mapping Reflection Principle, Baumgartner's Axiom, and that all Aronszajn trees are club-isomorphic. In light of this, we might ask the following:

Question 7.6. If $\mathcal{G}$ is (m, n) solid, does $\mathsf{PFA}(\mathcal{G})$ imply that the class of uncountable linear orderings has a five-element basis?

Outside of PID and some cardinal characteristics, we still do not know much about how much of PFA is compatible with HF graphs. In particular:

Question 7.7.

- Does $\mathsf{PFA}(\mathcal{G}, \mathrm{HF})$ imply OGA?
- Is OGA compatible with HF (HFD, HFC) graphs?

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colophon

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