Curvatures in generalized Kähler geometry

by

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Contents

Chapter 1

Introduction

1.1 History and Motivation

In generalized Kähler geometry, there are always two main viewpoints: a viewpoint which comes from Hitchin's generalized geometry program, and a bi-Hermitian viewpoint which comes from the physics literature. Many aspects of the geometry have been clarified using one or both of these viewpoints. In this thesis we study the appearance of curvature in generalized Kähler geometry, relating curvatures that appear in the bi-Hermitian viewpoint to generalized geometric curvatures.

Generalized Kähler geometry (or Bi-Hermitian geometry as it was originally known) originated in the 1980s in the work of string theorists, who were studying nonlinear σ-models. These models describe the embedding of a string into spacetime. Mathematically, such embeddings are modeled as maps ϕ from a two dimensional manifold Σ to a n-dimensional manifold M. The two dimensional manifold is known as the worldsheet while the n-dimensional manifold represents spacetime. In order to describe the dynamics of a propagating spring, the worldsheet and spacetime are endowed with metrics h and g. This data allows us to write down the *Polyakov action functional* S:

$$S(\phi) = \int_\Sigma \|d\phi\|^2 \operatorname{vol}_h = \int_\Sigma g(d\phi \wedge \star d\phi)$$

where we think of $d\phi$ as a 1-form on Σ valued in ϕ^*TM. The Hodge star $\star$ comes from the worldsheet metric h, while g is used to contract target indices. Given a two form b one can modify the Polyakov action functional by adding ϕ^*b to the integrand. This extra term is known as a Wess-Zumino-Witten (WZW) term.

Much can be learned by studying the symmetries of this action. Importantly, because the worldsheet is two dimensional, this action is invariant under Weyl transformations of h (that is, conformal rescalings of h) and thus it is typical to assume that the worldsheet is in fact a conformal manifold (or equivalently, a Riemann surface).

However, in the process of quantizing these models this conformal invariance is lost. Without giving details, the idea is as follows. In order to quantize a σ-model a renormalization scheme is used which involves a parameter λ representing the scale or energy at which the theory is valid. The introduction of this *ultra-violet cutoff* breaks the conformal invariance of the action resulting in what as known as the *Weyl anomaly.* The dependence of the theory on λ is controlled by the *renormalization group flow.* Various tricks are used to argue that the renormalization group flow only need be computed up to a certain order in λ. An important result of Friedan [14] says that in the case of the 2-dimensional nonlinear σ-model, the 1st order part of the renormalization group flow is given precisely by the Ricci tensor. This result was later generalized to the case with nonzero Wess-Zumino-Witten term, giving the *generalized Ricci flow* [7]:

$$\begin{aligned}\dot{g} &= -\operatorname{Rc} + \frac{1}{4}H^2 \\ \dot{b} &= -\frac{1}{2}d^* H\end{aligned}$$

where $H = H_0 + db$ for H_0 a closed 3-form and H^2 is a contraction of H using the metric g.

In addition to the Weyl transformations, physicists often search for supersymmetries of the action. A key connection to complex geometry was made by Zumino [49] who found that for the unmodified Polyakov action functional the quantized σ-model has $N = (2, 2)$ supersymmetry precisely when the spacetime (M, g) is Kähler. This was later extended to the full action by Gates, Hull and Roček in their 1984 paper [16], which showed that the most general form of $N = (2, 2)$ supersymmetry is obtained when the spacetime is bi-Hermitian, that is, it is equipped with a pair of complex structures $I_\pm$ which are compatible with g and whose Hermitian forms $\omega_\pm = gI_\pm$ satisfy the integrability conditions

$$d^c_+\omega_+ + d^c_-\omega- = 0, \qquad dd^c_\pm\omega_\pm = 0 \tag{1.1}$$

where $d^c_\pm = i\left(\overline{\partial}_\pm - \partial_\pm\right)$.

Nearly 30 years later, Streets and Tian [46] found a way of extending the generalized Ricci flow to bi-Hermitian structures, by deforming both complex structures simultaneously, obtaining the generalized Kähler-Ricci flow:

$$\begin{aligned}\dot{g} &= -\operatorname{Rc} + \frac{1}{4}H^2 \\ \dot{b} &= -\frac{1}{2}d^* H \\ \dot{I_\pm} &= \frac{1}{2}\mathcal{L}_{\theta^\sharp_\pm} I_\pm\end{aligned}$$

where $\theta_\pm = -d^*\omega_\pm \circ I_\pm$ is the Lee form.

In between these two developments, bi-Hermitian geometry was rediscovered by Gualtieri [21] in the context of Hitchin's generalized geometry program [26]. This discovery led to many connections between generalized Kähler geometry and Poisson geometry [24, 25, 6, 29, 27] and greatly clarified the structure of generalized Kähler geometry. In particular, to any generalized Kähler manifold is associated a pair of real Poisson tensors:

$$\pi_A = -\frac{1}{2}(I_+ - I_-)g^{-1} \qquad \pi_B = -\frac{1}{2}(I_+ + I_-)g^{-1}$$

and a pair of Poisson structures $\sigma_\pm$, holomorphic with respect to $I_\pm$ who share an imaginary part $Q = \frac{1}{2}[I_+, I_-]g^{-1}$.

In this thesis we make a contribution to the study of generalized Kähler-Ricci flow from the point of view of the generalized geometry program. In particular, we identify a plethora of generalized curvatures associated to any generalized Kähler manifold and describe these in terms of the bi-Hermitian geometry. In doing so we are able to describe the generalized Kähler-Ricci flow as the action of a certain curvature on a triple of data which includes both of the real Poisson structures and either one of the holomorphic Poisson structures.

The main tool that we use comes from graded symplectic geometry. Namely, Roytenberg established a connection between graded symplectic manifolds and Courant algebroids which turns out to be crucial in understanding curvatures of generalized connections.

1.2 Outline of the thesis

This thesis is organized as follows. Chapter 2 gives an introduction to Hermitian geometry, culminating in some curvature identites which will be used later on. This material is well known and mostly found in standard textbooks on differential geometry.

Chapter 3 explains a construction of Roytenberg which associates a Poisson algebra to any pseudo-Euclidean vector bundle. This material is also standard, though the exposition is distinct from that of Roytenberg.

In chapter 4 we give a review of spin geometry, starting with the basic linear algebraic background on Clifford algebras before focusing on the case of interest in this thesis: the differential forms as a spinor bundle for a split exact Courant algebroid. In section 4.3 we give a relationship between this spinor bundle and the Roytenberg algebra. This section is the least well known of the chapter, though the material is not novel. Section 4.4 gives review of the calculus of Dirac structures which focuses on their pure spinors.

Chapter 5 explains the main conceptual understanding of Courant connections that we will use, which ties Courant connections to the Roytenberg algebra. Section 5.1 and section 5.2 give expositions of material found in [12] and [20] respectively. In section 5.3 we give a novel notion of torsion for Courant connections and its accompanying algebraic Bianchi identities. This notion is not used elsewhere in the thesis but we believe it may be of interest.

Chapter 6 reviews the notion of generalized complex geometry with a focus on the interplay between generalized complex structures and the Roytenberg algebra. Section 6.2 contains an expanded characterization of the integrability condition for a generalized complex structure. In Section 6.5 we discuss generalized Chern connections, giving a generalization to an earlier version of the Poincaré-Lelong formula and a relationship between the generalized Chern connection of the canonical bundle and modular vector fields.

Chapter 7 and chapter 8 contain most of the novel results of this thesis. They explain generalized Chern connections of two types that are present on generalized Kähler manifold. The first focuses on those generalized Chern connections which arise from line bundles holomorphic with respect to the generalized complex structures constituting the generalized Kähler structure. The second describes induced generalized Chern connections, which come from holomorphic Poisson modules. In particular, in section 8.4 we give a novel characterization of the generalized Kähler-Ricci flow which results from these generalized Chern curvatures.

Chapter 2

Hermitian geometry

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In this section we give an introduction to complex and Hermitian geometry. A more comprehensive introduction to this topic may be found in any of [40, 4, 30]. The material in section **??** can be found in many textbooks on differential geometry (e.g. [32]) while the material in **??** is explained in more detail in [15]. The thesis of Barbaro is also a useful source for some of the material [5].

2.1 Derivations of the algebra of differential forms

{sec:derivofdiffe

Since we will use this often, we begin with a brief explanation of the derivations of $\Omega^\bullet(M)$, the algebra of differential forms on a manifold M. The differential forms on M form a graded commutative algebra. We denote by $\mathrm{Der}^k(\Omega^\bullet(M))$ the space of derivations of degree k of $\Omega^\bullet(M)$, that is, $\mathbb{R}$-linear maps $D : \Omega^\bullet(M) \to \Omega^{\bullet+k}(M)$ satisfying the following Leibniz rule:

$$(D(\alpha \wedge \beta) = D\alpha \wedge \beta + (-1)^{kl}\alpha \wedge D\beta$$

for $\alpha \in \Omega^l(M)$. For D_1 and D_2 a pair of derivations of degrees k and l we denote by $[D_1, D_2] = D_1D_2 - (-1)^{kl}D_2D_1$ the graded commutator of D_1 and D_2 which gives $\mathrm{Der}^\bullet(\Omega^\bullet(M))$ the structure of a graded Lie algebra.

A derivation D of degree k is called *algebraic* when its restriction to $\Omega^0(M)$ vanishes. In this case D is tensorial since the Leibniz rule gives $D(f\alpha) = fD\alpha$. Thus D is determined by its action on 1-forms which can be described as a vector bundle map $T^*M \to \wedge^{k+1}T^*M$ or equivalently a TM-valued $(k+1)$-form. For $K \in \Omega^{k+1}(M, TM)$ we label the resulting degree k derivation by ι_K. In particular for α a differential form and X a vector field, $\iota_{\alpha\otimes X} = \alpha \wedge \iota_X$.

2.2 Complex geometry

{sec:cxgeom}

An *almost complex structure* on a manifold M is an endomorphism $I : TM \to TM$ whose square is minus the identity, $I^2 = -\mathrm{id}$. In the presence of an almost complex structure, the complexified tangent bundle while decompose into $\pm i$-eigenbundles for I. We call these $T^{1,0}M$ and $T^{0,1}M$ respectively and denote by $\pi_{1,0}$ and $\pi_{0,1}$ the projections onto these bundles. These projections are given by

$$\pi_{1,0} = \frac{1 - iI}{2} \qquad\qquad \pi_{0,1} = \frac{1 + iI}{2}.$$

It is useful to note that while the bundles $T^{1,0}M$ and $T^{0,1}M$ are complex vector bundles, their underlying real vector bundles are isomorphic to TM via projection onto the real part. These isomorphisms identify the complex structure i with $\pm I$ respectively. The inverses are given by $\pi_{1,0}$ and $\pi_{0,1}$.

As a result of the decomposition of TM, the differential forms have their own decomposition

$$\Omega^\bullet(M, \mathbb{C}) = \bigoplus_{p,q \in \mathbb{N}} \Omega^{p,q}(M)$$

where $\Omega^{p,q}$ is the space of sections of $\wedge^p (T^{1,0})^* \otimes \wedge^q (T^{0,1})^*$. These spaces turn out to also be eigenbundles for an action of I, namely, the action which extends $I^* : T^* \to T^*$ as a degree 0 derivation. This action may also be described as the action of ι_I. Since the bundles $(T^{1,0})^*$ and $(T^{0,1})^*$ have eigenvalues $\pm i$ respectively, the space $\Omega^{p,q}$ has eigenvalue $i(p-q)$ for ι_I.

It is natural to ask how the exterior derivative behaves with respect to this decomposition. Applying degree considerations to the Cartan formula

$$[[d, \iota_X], \iota_Y] = \iota_{[X,Y]}$$

shows that d can only have components of type $(1,0)$, $(0,1)$, $(2,-1)$ and $(-1,2)$. The components of type $(1,0)$ and $(0,1)$, called the Dolbeault operators, are denoted by ∂ and $\overline{\partial}$. Since d is a real operator ∂ and $\overline{\partial}$ are complex conjugates, justifying the choice of notation. Similarly, the type $(2,-1)$ and $(-1,2)$ parts are complex conjugates. It is straightforward to check that they are algebraic derivations and are thus given by $-\iota_\eta$ and $-\iota_{\overline{\eta}}$ where $\eta(X,Y) = \pi_{0,1}[\pi_{1,0}X, \pi_{1,0}Y]$ defines a section of $\wedge^2 (T^{1,0})^* \otimes T^{0,1}$. The equation $\frac{1}{2}[d,d] = d^2 = 0$ then translates to

$$\begin{aligned}
\iota_\eta^2 &= 0 \\
\iota_\eta \partial + \partial \iota_\eta &= 0 \\
\partial^2 - \overline{\partial}\iota_\eta - \iota_\eta \overline{\partial} &= 0 \\
\partial\overline{\partial} + \overline{\partial}\partial + \iota_\eta \iota_{\overline{\eta}} + \iota_{\overline{\eta}} \iota_\eta &= 0
\end{aligned}$$

The 2-form η may be identified with a real 2-form valued in TM by taking its real part. The result is $\frac{1}{8}N$ where N is the *Nijenhuis tensor*:

$$N(X,Y) = [X,Y] - [IX, IY] + I[IX, Y] + I[X, IY].$$

A straightforward corollary of the above discussion is the following.

`{cxinteg}`

Proposition 1. The following are equivalent:

- $N = 0$
- $d = \partial + \overline{\partial}$

- $[[d, \iota_I], \iota_I] = -d$

A result of Newlander and Nirenberg shows that these conditions are equivalent to M being a complex manifold:

Definition 1. An almost complex structure I is called *integrable* when there exist an atlas consisting of local coordinates $(x^1, y^1, \ldots, x^n, y^n)$ such that

$$I\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}$$
$$I\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}$$

In this case we say that these coordinates are *local holomorphic coordinates.*

Any integrable almost complex structure gives rise to a unique complex structure on M, that is, an equivalence class of atlases of holomorphic charts.

Theorem 1. [41] An almost complex structure is integrable if and only if the equivalent conditions in Proposition 1 are satisfied.

The exterior derivative may be twisted by a complex structure I resulting in a new operator

$$d^c := [d, \iota_I] = i(\overline{\partial} - \partial).$$

2.3 Hermitian structures

{sec:hermitianstr

A Hermitian manifold is a complex manifold (M, I) equipped with a metric g such that I is orthogonal with respect to g:

$$g(I\cdot, I\cdot) = g$$

As a result there is an induced 2-form $\omega(\cdot, \cdot) = g(I\cdot, \cdot)$ which is of type (1,1) with respect to I. This $(1,1)$-form is positive in the sense that

$$-i\omega(Z, \overline{Z}) > 0$$

for all holomorphic tangent vectors $Z \in T^{1,0}$. In fact any positive $(1,1)$-form will induce a Hermitian metric $g = -\omega I$ on M. If $d\omega = 0$ we say that M is a Kähler manifold. If $dd^c\omega = 0$ the manifold is called *pluriclosed* (also called *strong Kähler with torsion (SKT)* in some literature).

2.4 Hodge star

{sec:hodgestar}

The metric g on an n-dimensional manifold M induces a fiberwise innerproduct on $\wedge^\bullet T^*M$ which we characterise by setting $\{e^{i_1} \wedge \cdots \wedge e^{i_k} | 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$ to be an orthonormal basis where $e_1, \ldots e_n$ is an orthonormal basis for TM and $e^1, \ldots, e^n$ its dual basis. For this inner product $\langle \cdot, \cdot \rangle$, the interior product and exterior product are adjoints in the following sense:

$$\langle \iota_X \alpha, \beta \rangle = \langle \alpha, g(X) \wedge \beta \rangle \tag{2.1}$$

{{extint}}

where X is a vector field and α and β are real-valued differential forms.

The Hodge star operator associated to g is then defined by

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \operatorname{vol}_g$$

where vol_g is the volume form induced by g. The Hodge star satisfies the following properties:

$$\begin{aligned}
\star^2 &= (-1)^{k(n-k)} \text{ on } \wedge^k T^*M \qquad (2.2)\\
\star 1 &= \operatorname{vol}_g \qquad (2.3)\\
g(X) \wedge \star\alpha &= -(-1)^k \star \iota_X \alpha \qquad (2.4)\\
\iota_X \star \alpha &= (-1)^k \star gX \wedge \alpha \qquad (2.5)
\end{aligned}$$

{eq:hodgeduality

Note that when extending the Hodge star to the complex differential forms one must choose whether to extend complex linearly or anti-linearly. The choice to extend anti-linearly will result in a relation similar to the one above, replacing $\langle \cdot, \cdot \rangle$ with the Hermitian metric induced on $\Omega^\bullet(M, \mathbb{C})$ while for the linear extension, $\langle \cdot, \cdot \rangle$ must be replaced with its complexification. We will choose for $\star$ to be complex linear.

As a result we get a inner product on $\Omega^\bullet(M, \mathbb{C})$

$$(\alpha, \beta) = \int_M \alpha \wedge \star\overline{\beta}$$

With these conventions, interior product with a vector field X and wedge product with $g(\overline{X})$ are formal adjoints. The adjoints of d, ∂ and $\overline{\partial}$ with respect to this inner product are given by

$$d^* = -(-1)^{n(k+1)} \star d\star \qquad \partial^* = -\star\partial\star \qquad \overline{\partial}^* = -\star\overline{\partial}\star$$

on k-forms. On a Hermitian manifold, there is another set of important operators:

$$L(\alpha) = \omega \wedge \alpha, \qquad \Lambda(\alpha) = -\iota_{\omega^{-1}}\alpha$$

Using 2.1 we can see that these operators are fiberwise adjoints.

The *Lee form* θ associated to (g, I) is given

$$\theta = -\iota_I d^*\omega$$

2.5 Hermitian connections

{sec:hermitiancon

Unlike in the case of Kähler manifolds, for Hermitian manifolds it is not generally the case that the Levi-Cevita connection preserves the complex structure, that is, that $\nabla^{LC} I = 0$. In fact, this condition turns out to be equivalent to the metric being Kähler (see e.g. [40, Theorem 5.5]). Thus the relevant connections on a Hermitian manifold are generally not equal to the Levi-Cevita connection.

We say that a connection is Hermitian if it preserves both g and I: $\nabla g = 0 = \nabla I$. The isomorphism $(TM, I) := (T^{1,0}, i)$ induces an equivalence between Hermitian connections on TM and connections on the complex vector bundle $T^{1,0}$.

The fact that $T^{1,0}$ is a holomorphic vector bundle, equipped with a Hermitian metric, means that there is a natural choice for connection as we now explain.

Any holomorphic vector bundle V has a Dolbeault operator

$$\overline{\partial} : \Gamma(V) \to \Gamma((T^{0,1})^* \otimes V)$$

which satisfies $\overline{\partial}(fs) = (\overline{\partial}f)s + f\overline{\partial}s$. Here both the Dolbeault operator for V and the Dolbeault operator on functions appear. On the other hand a Hermitian metric h identifies V with $\overline{V}^*$, an antiholomorphic vector bundle. This means that V may also be equipped with an antiholomorphic Dolbeault operator $\partial : \Gamma(V) \to \Gamma((T^{1,0})^* \otimes V)$ which satisfies $\partial(fs) = (\partial f)s + f\partial s$. The combination of these gives the *Chern connection*:

$$\nabla = \partial + \overline{\partial}.$$

This connection is uniquely specified by the following properties:

- the component of ∇ in the $T^{0,1}$ direction is $\overline{\partial}$,
- ∇ is a unitary connection: $X \cdot h(u,v) = h(\nabla_X u, v) + h(u, \nabla_X v)$.

In particular, on a Hermitian manifold (M, g, I), the tangent bundle $T_{1,0}M$ is holomorphic and Hermitian, with Dolbeault operator

$$\overline{\partial}_X Y = [X, Y]_{1,0}$$

for X and Y sections of $T_{0,1}M$ and $T_{1,0}M$ respectively.

As a connection on the real tangent bundle TM the Chern connection ∇^{Ch} is defined by

$$\begin{aligned} g(\nabla^{Ch}_X Y, Z) &= g(\nabla^{LC}_X Y, Z) + \frac{1}{2} d\omega(IX, Y, Z) \qquad (2.6) \\ &= g(\nabla^{LC}_X Y, Z) + \frac{1}{2} d^c\omega(X, IY, IZ) \qquad (2.7) \end{aligned}$$

{{Chern_connectio

where we have used the fact that $d^c\omega(X,Y,Z) = -d\omega(IX, IY, IZ)$ for ω a (1,1)-form. The Chern connection has torsion T^C given by

$$g(T^C(X,Y), Z) = -\frac{1}{2}\left(d\omega(IX, Y, Z) + d\omega(X, IY, Z)\right)$$

which is a 2-form with values in TM. From this expression one can check that the $(1,1)$ part of T^C vanishes, a property which may also be used to characterize ∇^C uniquely.

To any pair (g, H) where g is a Riemannian metric and H a 3-form, we may associate a connection ∇^B known as the Bismut connection:

$$g(\nabla^{Bis}_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} H(X, Y, Z). \tag{2.8}$$

{{Bismut_connecti

The Bismut connection is the unique metric preserving connection with torsion H. On a Hermitian manifold (M, g, I) there is a unique choice of H for which the Bismut connection preserves I, namely $H = -d^c\omega$.

The curvature $R^{Ch/Bis}$ of $\nabla^{Ch/Bis}$ give rise to Ricci tensors:

$$\mathrm{Rc}^{Ch/Bis}(X,Y) = \sum_{i=1}^{2n} R^{Ch/Bis}(e_i, X, Y, e_i)$$

where $e_1, \ldots, e_{2n}$ is an orthonormal frame for TM. The Bismut Ricci tensor has the following characterization.

Proposition 2. [28] Let (M,g) be a Riemannian manifold and H a closed 3-form. The Ricci tensor of the Bismut connection is given by

$$\mathrm{Rc}^{Bis} = \mathrm{Rc} - \frac{1}{4}H^2 - \frac{1}{2}d^*H \tag{2.9}$$

where Rc is the Ricci tensor for the Levi-Cevita connection and $H^2(X,Y) = \langle \iota_X H, \iota_Y H\rangle$.

The Bismut and Chern connections, being compatible with I, also give rise to connections on the holomorphic tangent bundle $T_{1,0}M$ and on the anticanonical bundle $K^* = \wedge^\bullet T_{1,0}M$ which we also refer to as the Bismut and Chern connections. The curvature of K^* with respect to either of these is a two form $i\rho_{Ch/Bis}$, where $\rho_{Ch/Bis}$ is a real 2-form known as the Chern/Bismut Ricci form.

Equivalently,

$$\rho_{Ch/Bis}(X,Y) = \frac{1}{2} g(R^{Ch/Bis}(X,Y)Ie_i, e_i)$$

Indeed, this is a direct consequence of the following linear algebraic fact.

Lemma 1. If $A^{\mathbb{C}}$ is a complex, hermitian endomorphism of the holomorphic tangent bundle, and $A^{\mathbb{R}}$ is the corresponding real endomorphism of the tangent bundle then

$$\mathrm{tr}^{\mathbb{C}}(A^{\mathbb{C}}) = \frac{1}{2i}\,\mathrm{tr}^{\mathbb{R}}(A^{\mathbb{R}} \circ I)$$

Proof. This is a pointwise relationship so we compute for a matrix $A+iB$. Such a matrix is Hermitian precisely when A is skew-symmetric and B is symmetric. Under the isomorphism $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ the complex structure i and Hermitian matrix $A^{\mathbb{C}}$ take the forms

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad A^{\mathbb{R}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

Since A is skew-symmetric, the trace of $A+iB$ is given by $i\,\mathrm{tr}\,B$. On the other hand the trace of $A^{\mathbb{R}} \circ I$ is $-2\,\mathrm{tr}\,B$. □

2.6 Curvature identities on Hermitian manifolds

{sec:curvatureide

The Bismut and Chern connections determine connections on the canonical bundle $K = \Omega^{n,0}(M)$ which we label by $\overline{\nabla}^B$ and $\overline{\nabla}^C$. The curvatures of these are $-i\rho_C$ and $-i\rho_B$ where ρ_C and ρ_B are real closed 2-forms known as the Ricci forms. The difference between the connections on K is

$$\begin{aligned}\overline{\nabla}^C - \overline{\nabla}^B &= -\operatorname{tr}^{\mathbb{C}}(\nabla^C - \nabla^B)^{\mathbb{C}} \\ &= -\frac{1}{2i}\operatorname{tr}^{\mathbb{R}}((\nabla^C - \nabla^B)^{\mathbb{R}} \circ I) \\ &= -\frac{1}{2i} g((\nabla^C - \nabla^B)^{\mathbb{R}} I e_i, e_i) \\ &= -\frac{1}{2i} d^c\omega(\cdot, I e_i, e_i)\end{aligned}$$

where $e_1, \ldots, e_n$ is an orthonormal frame for TM. Using the expression $d^* = -\sum_i \iota_{e_i} \nabla^{LC}_{e_i}$ and the fact that the Bismut connection preserves ω as well as 2.8, we find that $d^*\omega = \frac{1}{2} d^c\omega(\cdot, Ie_i, e_i)$ proving the following:

$$\overline{\nabla}^C - \overline{\nabla}^B = i d^*\omega \tag{2.10}$$ {{ChernMinusBismu

This also gives us the relationship between the two Ricci forms:

$$\rho_B - \rho_C = dd^*\omega.$$

2.7 Pluriclosed flow

A manifold is called pluriclosed if its Hermitian form satisfies $dd^c\omega = 0$. The *pluriclosed flow* of a pluriclosed Hermitian structure is given by

$$\begin{aligned}\dot{g} &= -\operatorname{Rc} + \frac{1}{4}H^2 - \frac{1}{2}\mathcal{L}_{\theta^\sharp} g \\ \dot{I} &= 0\end{aligned} \tag{2.11}$$ {{eq:pcf}}

where $H = -d^c\omega$ and $\theta = -d^*\omega \circ I$ the Lee form of (g, I).

The reader may be used to the identification in Kähler geometry between the Ricci form and tensor $\rho = \operatorname{Rc} I$. We advise that this relationship does not hold for the Bismut connection on an arbitrary Hermitian manifold.

Proposition 3. [28] Let (M, b, I) be a pluriclosed Hermitian manifold. The Bismut-Ricci form is given by

$$\rho_{Bis}(X, Y) = -\operatorname{Rc}^{Bis}(X, IY) - (\nabla_X \theta)(IY)$$

where $\theta = -d^*\omega \circ I$ is the Lee form.

Corollary 1. (c.f. [15]) If (M, g, I) is a pluriclosed Hermitian manifold, the Bismut-Ricci form ρ_{Bis} satisfies

$$\begin{aligned}I^* \circ \rho^{1,1}_{Bis} &= -\operatorname{Rc} + \frac{1}{4}H^2 - \frac{1}{2}\mathcal{L}_{\theta^\sharp} g \\ I^* \circ \rho^{2,0+0,2}_{Bis} &= -\frac{1}{2}d^*H + \frac{1}{2}d\theta - \frac{1}{2}\iota_{\theta^\sharp} H\end{aligned}$$

where $H^2(X, Y) = \langle \iota_X H, \iota_Y H \rangle$ and $\theta = -d^*\omega \circ I$ is the Lee form, with corresponding vector field $\theta^\sharp = g^{-1}\theta$.

Chapter 3

The Roytenberg algebra

{chpt:roytenberga

In order to arrive at a generalized geometric formulation of bi-Hermitian geometry, it is necessary to discuss Courant algebroids. We take a slightly ahistorical approach here, making use of the correspondence of Roytenberg [44, 43] between Courant algebroids and symplectic NQ-manifolds of degree 2.

3.1 The Roytenberg algebra

{sec:roytenbergal

Our starting point is the formulation of Courant algebroids in terms of symplectic supermanifolds [42, 43]. This an enhancement of an equivalence, proved by Roytenberg, between pseudo-Euclidean vector bundles and symplectic N-manifolds of degree 2. We will only describe one direction of this correspondence.

Throughout, we let $E \to M$ be a vector bundle equipped with a nondegenerate bilinear pairing $\langle \cdot, \cdot \rangle$ which we will refer to as a metric. Associated to any such vector bundle is its *Atiyah algebroid* $\mathrm{At}(E)$, sections of the Atiyah algebroid may be thought of as vector fields on the total space of E which preserve the bundle structure and the metric. Equivalently, these are differential operators D on E for which there exists a vector field σ_D such that

$$D(fu) = (\sigma_D \cdot f)u + fDu \tag{3.1}$$

$$\sigma_D \langle u, v \rangle = \langle Du, v \rangle + \langle u, Dv \rangle \tag{3.2}$$

for any function f and sections u and v of E. Sections of the Atiyah algebroid are also known as derivations. The Atiyah algebroid is an extension of TM by $\mathfrak{so}(E)$, the bundle whose fibers are the orthogonal Lie algebras for the fibers of E:

$$0 \mapsto \mathfrak{so}(E) \to \mathrm{At}(E) \xrightarrow{\sigma} TM \to 0 \tag{3.3}$$

and splittings of this sequence are given by metric connections on E. We will frequently make use of the fact that $\wedge^2 E = \mathfrak{so}(E)$ with the equivalence given by $u \wedge v \mapsto \phi_{u \wedge v}$ where $\phi_{u \wedge v}(w) = \langle u, w \rangle v - \langle v, w \rangle u$.

Roughly speaking, the Roytenberg algebra of E is the bundle of graded commutative algebras generated by $\mathrm{At}(E)$ in degree 2 and E in degree 1 which satisfies the relation $u \wedge v = \phi_{u \wedge v}$ for

u and v in E. To spell this out in detail we consider the bundle of graded algebras given by $\mathrm{Sym}\,(\mathrm{At}(E)[2] \oplus E[1])$. Here the bracket notation indicates the degree of the vector bundle and Sym denotes the symmetric algebra for graded vector bundles so that as vector bundles, $\mathrm{Sym}(\mathrm{At}(E)[2] \oplus E[1]) = \wedge^{\bullet} E \otimes \mathrm{Sym}(\mathrm{At}(E))$. The Roytenberg algebra of E is then defined to be the space of sections of the bundle

$$C^{\bullet}(E) \coloneqq \Gamma(\mathrm{Sym}(\mathrm{At}(E)[2] \oplus E[1])/I) \tag{3.4}$$

where I is the ideal generated by elements of the form $\omega \otimes 1 - 1 \otimes \phi_{\omega}$ for $\omega \in \wedge^2 E$. By construction this is a bundle of graded algebras, with product denoted by $\wedge$. In fact, the sections of this bundle are also equipped with a Poisson bracket $[\cdot,\cdot]$ of degree -2. By the Leibniz rule, it suffices to specify the Poisson bracket on the generators of the algebra. For $f, g \in C^{\infty}(M)$, $u, v \in \Gamma(E)$ and $D \in \Gamma(\mathrm{At}(E))$ it is given by

$$[f, g] = 0 = [f, u] \tag{3.5}$$

$$[u, v] = \langle u, v\rangle \tag{3.6}$$

$$[f, D] = \sigma_D \cdot f \tag{3.7}$$

$$[u, D] = Du \tag{3.8}$$

$$[D_1, D_2] = D_2 D_1 - D_1 D_2 \tag{3.9}$$

By degree considerations, there are only 4 combinations for which the Jacobi identity needs to be checked:

$$\begin{aligned}
[D_1, [D_2, D_3]] &= [[D_1, D_2], D_3] + [D_2, [D_1, D_3]] \\
[u, [D_1, D_2]] &= [[u, D_1], D_2] + [D_1, [u, D_2]] \\
[f, [D_1, D_2]] &= [[f, D_1], D_2] + [D_1, [f, D_2]] \\
[D, [u, v]] &= [[D, u], v] + [u, [D, v]].
\end{aligned}$$

These are easily verified from the properties of the Atiyah algebroid. Given an element ω in $C^k(E)$, we obtain a map

$$\begin{aligned}
\widetilde{\omega} : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{k} &\to C^{\infty}(M) \\
(u_1, \ldots, u_k) &\mapsto [u_k, [u_{k-1}, \ldots, [u_1, \omega] \ldots]]
\end{aligned} \tag{3.10}$$

`{{eq:poissoniso}}`

which satisfies:

$$\widetilde{\omega}(u_1, \ldots, u_{k-1}, f u_k) = f \widetilde{\omega}(u_1, \ldots, u_k) \tag{3.11}$$

`{{eq:tensoriality`

for any function f and sections u_i of E. Moreover, for $k \geq 2$, the map $\widetilde{\omega}$ has the property that there exists a *symbol map*,

$$\sigma_{\widetilde{\omega}} : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{k-2} \to \mathcal{X}(M)$$

such that

$$\widetilde{\omega}(\dots,u,v,\dots)+\widetilde{\omega}(\dots,v,u,\dots)=\sigma_{\widetilde{\omega}}(\dots)\cdot\langle u,v\rangle \tag{3.12}$$ {eq:KWsymbol}

Indeed, since $D \coloneqq [u_{k-2},[u_{k-3},\dots[u_1,\omega]\dots]]$ is a section of $\mathrm{At}(E)$, by setting $\sigma_{\widetilde{\omega}}(u_1,\dots,u_{k-2}) \coloneqq \sigma_D$ and by judicious application of the Jacobi identity we arrive at eq. (3.12). In particular, for a section u of E, the corresponding map is given by $\tilde{u}=\langle u,\cdot\rangle$, and for a derivation D the map is given by $\tilde{D}=\langle D\cdot,\cdot\rangle$.

The Keller-Waldmann algebra [31] is the graded commutative algebra $\widetilde{C}^\bullet(E)$ whose elements of degree k are functions

$$\omega:\underbrace{\Gamma(E)\times\cdots\times\Gamma(E)}_{k}\to C^\infty(M)$$

which satisfy eq. (3.11) and which have a symbol map σ_ω satisfying eq. (3.12). It has graded commutative product given by

$$(\alpha\wedge\beta)(u_1,\dots,u_{k+l})=\sum_\tau \mathrm{sgn}(\tau)\alpha(u_{\tau(1)},\dots,u_{\tau(k)})\beta(u_{\tau(k+1)},\dots,u_{\tau(k+l)})$$

where the sum is taken over (k,l)-shuffle permutations τ. It is a straightforward verification, using tensoriality of the last argument, that the Keller-Waldmann algebra coincides with the Roytenberg algebra in degrees 0, 1, and 2.

In fact, $\widetilde{C}^\bullet(E)$ has a Poisson bracket of degree -2 [31] and by a theorem of Cueca and Mehta [12], the map $\omega\mapsto\widetilde{\omega}$ as defined by eq. (3.10) gives an isomorphism of Poisson algebras. For this reason, we will not distinguish between a section ω of $C^\bullet(E)$ and its corresponding map $\widetilde{\omega}\in\widetilde{C}^\bullet(E)$, referring to them as ω. We will often use the notation $\iota_u\omega=[u,\omega]$

Remark 1. The Roytenberg algebra is the algebra of functions on a degree 2 symplectic N-manifold, that is, a non-negatively graded manifold with symplectic form of degree 2 [43]. In fact, the Roytenberg algebra gives a correspondence between symplectic N-manifolds of degree 2 and pseudo-Euclidean vector bundles. Since we will not use graded geometry in this thesis, we do not describe this correspondence in detail. We remark, however, that the perspective of graded geometry often gives a useful conceptual framework, when working with the Roytenberg algebra.

We conclude this section with a useful formula involving the Poisson bracket.

{symbolcalc}

Proposition 4. For any $\omega\in C^k(E)$, $f\in C^\infty(M)$ we have $[f,\omega]=\sigma_\omega\cdot f$.

Proof. By definition

$$\begin{aligned}\sigma_\omega(u_1,\dots u_{k-2})\cdot f &= \sigma_{[u_{k-2},[u_{k-3},\dots[u_1,\omega]\dots]]}\cdot f\\ &= [f,[u_{k-2},[u_{k-1},\dots[u_1,\omega]\dots]]]\\ &= [u_{k-2},[u_{k-1},\dots[u_1,[f,\omega]]\dots]]\\ &= [f,\omega](u_1,\dots,u_{k-2})\end{aligned}$$

where the last line follows by the Jacobi identity. □

3.2 The adjoint representation

{sec:adjointrep}

In this section we describe an action of the bundle of orthogonal Lie groups $\mathrm{O}(E)$ on $C^\bullet(E)$. This will be used in chapter 6 to give a decomposition of $C^\bullet(E)$ corresponding to a generalized complex structure. It is intuitively clear that such an action exists, since $\mathrm{O}(E)$ acts naturally on E and $\mathrm{At}(E)$, the generators for $C^\bullet(E)$, and since the extension of this action preserves the ideal I.

Explicitly, the adjoint representation of $\mathrm{O}(E)$ on $C^\bullet(E)$ is given by

$$\begin{aligned}\Gamma(\mathrm{O}(E)) &\to \mathrm{Aut}(C^\bullet(E))\\ g &\mapsto \mathrm{Ad}_g\end{aligned}$$

where $(\mathrm{Ad}_g\,\omega)(u_1,\ldots,u_k)=\omega(gu_1,\ldots,gu_k)$. In particular, for $D\in C^2(E)$ a derivation, we have $\mathrm{Ad}_g D=g^{-1}Dg$. Importantly, Ad_g preserves the graded commutative product and the Poisson bracket.

A simple calculation shows that the corresponding Lie algebra action of sections of the adjoint bundle is given by

$$\begin{aligned}\Gamma(\mathfrak{so}(E)) &\to \mathrm{Der}(C^\bullet(E))\\ \phi &\mapsto \mathrm{ad}_\phi\end{aligned} \tag{3.13}$$

{{adjoint}}

is given by $(\mathrm{ad}_\phi\,\omega)(u_1,\ldots,u_k)=\sum_{i=1}^k\omega(u_1,\ldots,\phi(u_i),\ldots u_k)$.

Lemma 2. If ϕ is a section of $\mathfrak{so}(E)$ and $\omega\in C^\bullet(E)$ then $\mathrm{ad}_\phi\,\omega=[\phi,\omega]$.

Proof. Note that

$$\begin{aligned}\iota_u[\phi,\omega] &= [u,[\phi,\omega]]\\ &= [[u,\phi],\omega]+[\phi,[u,\omega]]\\ &= \iota_{\phi(u)}\omega+[\phi,\iota_u\omega].\end{aligned}$$

The argument then follows by induction on the degree of ω. □

{ex:roytenbergdec

Example 1. If E decomposes as a sum of complementary maximal isotropics $E=L\oplus L'$ with projection onto L denoted by π_L then the element

$$2\pi_L-\mathrm{id}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

is a semisimple section of $\mathfrak{so}(E)$. Thus, its action on $C^\bullet(E)$ yields an eigendecomposition

$$C^k(E)=\bigoplus_{s=-k}^{k}C^k_s(E)$$

where $C^k_s(E)$ is the s-eigenbundle for the action of $2\pi_L-\mathrm{id}$. In particular, the exterior algebras of L^* and $(L')^*$ embed as components of this decomposition:

$$\wedge^k L^*=C^k_k(E)\qquad\qquad \wedge^k(L')^*=C^k_{-k}(E)$$

By the Jacobi identity, the Poisson bracket is degree 0 with respect to this secondary grading. That is, if $\alpha \in C^k_a(E)$ and $\beta \in C^k_b(E)$ then $[\alpha, \beta] \in C^k_{a+b}(E)$. As a result, the Poisson bracket gives a derivation of the algebra Ω_L of degree $k-2$ for any element $\omega \in C^k_{k-2}$.

{rmk:polyderivati

Remark 2. This last example has a convenient explanation in the language of graded symplectic geometry [43, 42]. Namely, the vector bundle $E = L \oplus L^*$ with its natural inner product corresponds to the graded symplectic manifold $T^*[2]L[1]$ whose algebra of functions consists of polyvectors on $L[1]$:

$$C^\bullet(E) = \mathrm{Sym}(\mathrm{Der}(\Omega_L)[-2])$$

where $\Omega_L = \Gamma(\wedge^\bullet L^*)$. This algebra has a total degree and also a polynomial weight coming from the symmetric powers. In particular,

$$C^k_{k-2}(E) = \mathrm{Der}^{k-2}(\Omega_L)[-1]$$

and under this identification the Poisson bracket corresponds to the graded commutator, while the Poisson bracket of a derivation D with $\omega \in \Omega_L$ is given by $[D, \omega] = D\omega$.

3.3 Courant brackets as Hamiltonian functions

{sec:courantbrack

Any element $\Theta \in C^3(E)$ has symbol $\sigma_\Theta \in C^1(E; TM) = \Gamma(E^* \otimes TM)$ known as the *anchor map* $\mathsf{a} : E \to TM$. Using the inner product we may identify such elements $\Theta \in C^3(E)$ with brackets $[\![\cdot, \cdot]\!] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying the properties

- $[\![u, v]\!] + [\![v, u]\!] = \mathsf{a}^t d_{dR} \langle u, v\rangle$
- $\langle [\![u, v]\!], w\rangle + \langle v, [\![u, w]\!]\rangle = \mathsf{a}(u)\langle v, w\rangle$

where a^t is the composition a^* with the inverse of the metric $\langle \cdot, \cdot\rangle$ thought of as a map from E to E^*. Explicitly, Θ and $[\![\cdot, \cdot]\!]$ are related by the formula $\Theta = \langle [\![\cdot, \cdot]\!], \cdot\rangle$. Roughly speaking, the first property says that the failure of the bracket to be skew-symmetric is measured by the inner product while the second property can be thought of as a derivation property for the inner product.

The element Θ also produces a degree 1 derivation $d_E = [\Theta, \cdot]$ on $(C^\bullet(E), \wedge)$ and the following are equivalent:

- $[\Theta, \Theta] = 0$,
- $[\![\cdot, \cdot]\!]$ satisfies the Jacobi identity: $[\![u, [\![v, w]\!]]\!] = [\![[\![u, v]\!], w]\!] + [\![v, [\![u, w]\!]]\!]$,
- $d_E^2 = 0$.

Indeed we find that the Jacabiator $J(u, v, w) = [\![[\![u, v]\!], w]\!] + [\![v, [\![u, w]\!]]\!] - [\![u, [\![v, w]\!]]\!]$ satsifies

$$[\Theta, \Theta](u, v, w, x) = 2\langle J(u, v, w), x\rangle$$

If any of these equivalent conditions hold then we say that $[\![\cdot, \cdot]\!]$ is a *Courant bracket* and that the data $(E, \langle \cdot, \cdot\rangle, [\![\cdot, \cdot]\!])$ constitutes a *Courant algebroid.*

From these axioms we may derive a few important properties of Courant algebroids (see [47] for example):

- $\mathsf{a}[\![u,v]\!] = [\![\mathsf{a}(u),\mathsf{a}(v)]\!]$
- $[\![u,fv]\!] = (\mathsf{a}(u)f)v + f[\![u,v]\!]$
- $\mathsf{a}\circ\mathsf{a}^t = 0$

The last of these suggests we consider the sequence

$$0 \to T^* \xrightarrow{\mathsf{a}^t} E \xrightarrow{\mathsf{a}} T \to 0$$

In the case that this sequence is exact, we call E an exact Courant algebroid.

In [12] Cueca and Mehta develop a Cartan calculus for $C^\bullet(E)$ so that, in particular, d_E has an invariant formula

$$d\omega(u_0,\dots,u_k) = \sum_i (-1)^i u_i \cdot \omega(u_0,\dots,\widehat{u_i},\dots,u_k) - \sum_{i<j}(-1)^i\omega(u_0,\dots,\widehat{u_i},\dots,[\![u_i,u_j]\!],\dots,u_k) \tag{3.14}$$

`{{invariant_formu`

where the action of u_i on functions is understood to be given by the anchor map.

In low degrees we have the following identifications:

- On $C^0(E) = C^\infty(M)$, the differential is a^*d.
- On $C^1(E) = \Gamma(E)$, the differential is $u \mapsto -\operatorname{ad}_u$.
- On $C^2(E)$ the differential is $D \mapsto -D[\![\cdot,\cdot]\!] + [\![D\cdot,\cdot]\!] + [\![\cdot,D\cdot]\!]$ so 2-cocycles are precisely derivations of E.

In particular, the derivations of E fit into an exact sequence

$$0 \to Z^1(E) \to \Gamma(E) \xrightarrow{\operatorname{ad}} \operatorname{Der}(E) \to H^2(E) \to 0.$$

Proposition 5. Let $(E,\langle\cdot,\cdot\rangle,[\![\cdot,\cdot]\!])$ be a Courant algebroid. An element $\varepsilon \in C^3(E)$ defines a deformation of the Courant bracket if and only if ε satisfies the Maurer-Cartan equation:

$$d\varepsilon + \frac{1}{2}[\varepsilon,\varepsilon] = 0$$

Proof. Let $\Theta = \langle[\![\cdot,\cdot]\!],\cdot\rangle$. Then the Jacobi identity is equivalent to the master equation so we compute

$$\begin{aligned}[\Theta+\varepsilon,\Theta+\varepsilon] &= [\Theta,\Theta] + 2[\Theta,\varepsilon] + [\varepsilon,\varepsilon]\\ &= 2d\varepsilon + [\varepsilon,\varepsilon]\end{aligned}$$

□

As a result, the nontrivial infinitesimal symmetries of a Courant algebroid E are measured by $H^2(E)$ and the nontrivial infinitesimal deformations are given by $H^3(E)$.

Example 2. [43] Consider the vector bundle $\mathbb{T}M \coloneqq TM \oplus T^*M$ with its natural pairing

$$\langle X+\xi, Y+\eta\rangle = \xi(Y) + \eta(X)$$

for vectors X, Y and 1-forms ξ, η. The sequence

$$0 \to \mathfrak{so}(\mathbb{T}M) \to \mathrm{At}(\mathbb{T}M) \xrightarrow{\sigma} TM \to 0$$

has no canonical splitting (a splitting is equivalent to a metric connection on $\mathbb{T}$, of which there are many). Yet there is a natural description of sections of the Atiyah algebroid using the Lie derivative. Indeed, if D is a section of $\mathrm{At}(\mathbb{T})$ with symbol X then $D - \mathcal{L}_X$ is a derivation with symbol 0, that is a section of $\mathfrak{so}(\mathbb{T})$. Thus derivations of $(\mathbb{T}, \langle \cdot, \cdot \rangle)$ consist of pairs $(X, \phi) \in \Gamma(T) \oplus \Gamma(\mathfrak{so}(\mathbb{T}))$ which act via

$$(X, \phi) \cdot (Y + \eta) = \mathcal{L}_X(Y + \eta) + \phi(Y + \eta).$$

By Remark 2 the de Rham derivative on M, d_{dR} may be thought of as an element of $C^{1,2}$ which satisfies the classical master equation $[d_{dR}, d_{dR}] = 2d_{dR}^2 = 0$. Its symbol is the projection map onto T. Thus there is a natural exact Courant bracket on $\mathbb{T}M$ given by

$$\begin{aligned} [\![X + \xi, Y + \eta]\!] &= [[d_{dR}, X + \xi], Y + \eta] \\ &= [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi \end{aligned}$$

In addition to the Hamiltonian vector field $d_E = [d_{dR}, \cdot]$ which has degree (0,1), there is also a degree $(0, -1)$ vector field ι defined on generators $\omega \in \Omega_L = C^{0,\bullet}$ and $D \in \mathrm{Der}(\Omega_L) = C^{1,\bullet}$ by

$$\iota\omega = 0 \qquad\qquad \iota D = \iota_K$$

where $K \in \Omega(M; TM)$ is the vector-valued differential form given by the restriction of D to $C^\infty(M)$. By construction, ι gives a null homotopy of the polynomial degree operator:

$$d\iota + \iota d = \varepsilon_1.$$

As a result, the complexes $(C^{k,\bullet}, d_0)$ are acyclic for $k > 0$ and nontrivial infinitesimal deformations of the standard Courant bracket are given by $H^3(M)$.

Chapter 4

The spinor bundle of an exact Courant algebroid

{chpt:spinorbdle}

In this section we give an exposition of the linear algebra of the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$ of a manifold M. with nondegenerate symmetric bilinear pairings of *split signature*, meaning that they have signature (n, n) (and in particular are even dimensional). For such vector spaces it is particularly straightforward to construct the spin representation. The resulting linear algebra forms the basis of generalized geometry. Detailed references for this material may be found in [35, 39, 38].

4.1 Clifford algebras and associated groups

{sec:cliffordalg}

In this section we describe the Clifford algebra associated to a vector space with nondegenerate symmetric pairing. Though we will ultimately apply this construction to the generalized tangent bundle $\mathbb{T}M \coloneqq TM \oplus T^*M$, we keep this section quite general.

Let E be a real vector space equipped with a nondegenerate, symmetric bilinear form $\langle \cdot, \cdot \rangle$. The *Clifford algebra* $Cl(E, \langle \cdot, \cdot \rangle)$ associated to $(E, \langle \cdot, \cdot \rangle)$ is the quotient

$$Cl(E, \langle \cdot, \cdot \rangle) = T(E)/I$$

of the tensor algebra $T(E)$ by the two-sided ideal I generated by elements of the form $u \otimes v + v \otimes u - \langle u, v \rangle$ for $u, v \in E$ [1]. When the bilinear form is clear from the context, we will denote the Clifford algebra of $(E, \langle \cdot, \cdot \rangle)$ simply by $Cl(E)$. We will use $[\cdot, \cdot]$ to denote the graded commutator on $Cl(E)$ so that the defining equation for the Clifford algebra takes the form

$$[u, v] = \langle u, v \rangle.$$

The tensor algebra is a $\mathbb{Z}$-graded algebra and while this grading is lost when we descend to the Clifford algebra, two important pieces of structure remain. Namely, if we think of $T(E)$ as a filtered superalgebra, inheriting both the filtration and $\mathbb{Z}_2$-grading from the $\mathbb{Z}$-grading, then the Clifford

---

[1] This convention differs from the standard one by a factor of 2

algebra inherits this structure of a filtered superalgebra. Simply stated, the reason for this is that the generators of I are even and of filtration degree 2. The $\mathbb{Z}_2$ grading is realized by the *parity automorphism* $\Pi : Cl(E) \to Cl(E)$ which acts by $(-1)^k$ on $Cl^k(E)$. We denote by $Cl_i(E)$ for $i \in \mathbb{Z}_2$ the components of the $\mathbb{Z}_2$-grading and by $\mathcal{F}^i Cl(E)$ the filtration components. We will also make use of the *transposition antiautomorphism* $(u_1 \dots u_n)^T = u_n \dots u_1$ for $u_i \in E$.

The primary utility of the Clifford algebra is in constructing the Spin groups and Spin representations. This construction proceeds as follows. Given a vector $u \in E$ with nonzero norm, there is an associated orthogonal transformation, namely, the reflection through the hyperplane perpendicular to u:

$$v \mapsto v - 2\frac{\langle u, v\rangle}{\langle u, u\rangle}u.$$

In terms of the Clifford algebra, this reflection has the expression

$$v \mapsto -uvu^{-1}$$

motivating the following definitions. The *twisted adjoint* actions of $a \in Cl(E)^\times$ and $b \in Cl(E)$ (which is the Lie algebra of $Cl(E)^\times$) on $v \in E$ are given by

$$\begin{aligned} \mathrm{Ad}_a(v) &= \Pi(a)va^{-1} \\ \mathrm{ad}_b(v) &= [b, v] \end{aligned}$$

where $[\cdot, \cdot]$ is the graded commutator. In general such an operation will not preserve E. The *Clifford group* of E is the subgroup $\mathrm{CG}(E)$ of $Cl(E)^\times$ consisting of $a \in Cl(E)^\times$ such that

$$\mathrm{Ad}_a E \subseteq E.$$

By definition $\mathrm{CG}(E)$ comes equipped with a representation:

$$\mathrm{Ad} : \mathrm{CG}(E) \to \mathrm{GL}(E).$$

A straightforward calculation shows that the image of Ad lies on $\mathrm{O}(E)$. In fact, any reflection may be expressed as Ad_u for some nonzero $u \in E$, demonstrating that Ad surjects onto $\mathrm{O}(E)$ (this follows from the Cartan-Dieudonné theorem). This also implies that any element in $\mathrm{CG}(E)$ is a product of non-isotropic vectors and, in particular, is homogeneous with respect to the $\mathbb{Z}_2$ grading.

The kernel of Ad is given by elements $a \in Cl(E)^\times$ such that $[a, v] = 0$ for all $v \in V$, that is, elements elements in $Cl(E)^\times$ which lie in the (super)center. Since the center of $Cl(E)$ is $\mathbb{R}$ (see [39, Lemma 2.1] for example), we have a short exact sequence:

$$0 \to \mathbb{R}^\times \to \mathrm{CG}(E) \xrightarrow{\mathrm{Ad}} \mathrm{O}(E) \to 0 \tag{4.1}$$

If we let $\mathrm{SCG}(E) := \mathrm{CG}(E) \cap Cl^0(E)$ denote the *special Clifford group*, then the above exact sequence restricts to

$$0 \to \mathbb{R}^\times \to \mathrm{SCG}(E) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(E) \to 0 \tag{4.2}$$

We may then define the *Pin group* $\mathrm{Pin}(E)$ to be the preimage of $\{1, -1\}$ under the norm homomorphism

$$\begin{aligned} N : \mathrm{CG}(E) &\to \mathbb{R}^\times \\ a &\mapsto a^T a \end{aligned}$$

whose image lies in $\mathbb{R}^\times$ since a is a product of non-isotropic vectors. The *Spin group* $\mathrm{Spin}(E)$ is the intersection of the Pin group with $\mathrm{SCG}(E)$, and the above short exact sequences restrict to

$$\begin{aligned} 0 \to \{1,-1\} \to &\,\mathrm{Pin}(E) \xrightarrow{\mathrm{Ad}} \mathrm{O}(E) \to 0 \\ 0 \to \{1,-1\} \to &\,\mathrm{Spin}(E) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(E) \to 0. \end{aligned}$$

It is natural also to consider the preimage of $-1 \in \mathrm{O}(E)$. This is a $\{1,-1\}$-torsor and a choice of generator is equivalent to an orientation of E. Indeed, given an oriented orthonormal basis $a_1, \cdots, a_m$ of E the *chirality* or *volume* element $\Gamma = a_1 \ldots a_m$ gives such a generator.

From the above short exact sequences, it follows that the Lie algebra of $\mathrm{Spin}(E)$ and $\mathrm{Pin}(E)$, which we denote by $\mathfrak{spin}(E)$, is isomorphic to $\mathfrak{so}(E)$ via ad. Since $\mathrm{Spin}(E)$ is a subgroup of $Cl^0(E)^\times$, which has Lie algebra $Cl^0(E)$ with Lie bracket being the commutator, it is useful (when constructing the spin representation for example) to represent this Lie algebra as a Lie subalgebra of $Cl^0(E)$. Explicitly, it is the subspace

$$\mathfrak{spin}(E) = \{b \in Cl^0(E) \mid \mathrm{ad}_b(E) \subseteq E, b^T + b = 0\}.$$

The result is the following commutative diagram:

$$\begin{tikzcd} \mathrm{Spin}(E) \arrow[r, hook] & Cl^0(E)^\times \\ \mathfrak{spin}(E) \arrow[r, hook] \arrow[u, "\exp"] & Cl^0(E) \arrow[u, "\exp"] \end{tikzcd}$$

which says simply that the exponential map of the Spin group is given by the usual formula for the exponential map, once the element of $\mathfrak{so}(E) = \mathfrak{spin}(E)$ is represented as an element of the Clifford algebra.

4.2 The spin representation

{sec:spinrep}

A *spinor module* or *spin representation* is an irreducible $Cl(E)$-module. If we assume that E is of dimension $2m$ and the bilinear form $\langle \cdot, \cdot \rangle$ is of split signature then E may be decomposed as a sum of complementary maximal isotropic subspaces $E = L \oplus L'$. Such a choice identifies E with $L \oplus L^*$ where the inner product is given by

$$\langle a + \alpha, b + \beta \rangle = \alpha(b) + \beta(a)$$

for $a, b \in L$ and $\alpha, \beta \in L^*$. A spin representation may be realized by $\wedge^\bullet L^*$ with action given by

$$(a + \alpha) \cdot \phi = \iota_a \phi + \alpha \wedge \phi.$$

One checks that this indeed an irreducible Clifford module.

In particular, using the identification of $\mathfrak{so}(E)$ with $\mathfrak{spin}(E)$ we realize the spin representation. Here there is a choice to be made, and we take the convention that ensures that an element $B \in \wedge^2 L^*$ thought of as a map $L \mapsto L^*$ acts on $\bigwedge L^*$ by wedge product and an element $\beta \in \wedge^2 L$ acts by $-\iota_\beta$.

The spin module comes equipped with the *Chevalley pairing* which, in terms of the above model, may be described as:

$$\begin{aligned} \wedge^\bullet L^* \otimes \wedge^\bullet L^* &\to \det L^* \\ (\alpha, \beta) &\mapsto [\alpha \wedge \beta^T]_m \end{aligned}$$

where $[\cdot]_m$ represents projection onto the forms of degree m. Provided that m is even (as we shall assume for much of this thesis), the Chevalley pairing satisfies an invariance property with respect to the action of the Clifford algebra:

$$(a \cdot \alpha, \beta) = (\alpha, a^T \cdot \beta) \tag{4.3}$$

{{eq:chevalleyinv

from which it follows that for elements a in the Clifford group $\Gamma(E)$,

$$(a \cdot \alpha, a \cdot \beta) = N(a)(\alpha, \beta). \tag{4.4}$$

One checks that the Chevalley pairing is nondegenerate and either symmetric or antisymmetric, depending on the dimension of M:

$$(\alpha, \beta) = (-1)^{\frac{m(m-1)}{2}} (\beta, \alpha) \tag{4.5}$$

{{eq:chevalleysym

4.3 The Dirac generating operator

{sec:diracgenop}

In this section we apply the previous constructions to the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$ resulting in a bundle of Clifford algebra and a natural bundle of spinors. Here we depart from linear algebra, and in order to relate the bundle of spinors to the Roytenberg algebra, it is fruitful to study the differential operators on the spinors.

Applying the construction of the previous section to $\mathbb{T}M$ we find that $Cl(\mathbb{T}M)$ acts naturally on $S = \Omega^\bullet(M)$, the differential forms on M. The differential forms inherit a $\mathbb{Z}_2$ grading, $S = S^0 \oplus S^1$ from $Cl(\mathbb{T}M)$ which is nothing but their decomposition into even and odd forms. We let $\mathcal{D}(S)$ denote the differential operators on S. Then $\mathcal{D}(S)$ is naturally $\mathbb{Z}_2$-graded with

$$\mathcal{D}_0(S) = \{P \in \mathcal{D}(S) \mid P : S^{0/1} \to S^{0/1}\} \qquad \mathcal{D}_1(S) = \{P \in \mathcal{D}(S) \mid P : S^{0/1} \to S^{1/0}\}$$

so that $\mathcal{D}(S)$ becomes a graded Lie algebra with the graded commutator. It is typical to consider the increasing filtration $\mathcal{F}^\bullet$ where the k-th piece is given by differential operators of order at most k,

$$\mathcal{F}^k\mathcal{D}(S) = \{P \in \mathcal{D}(S) \mid [f_k, [f_{k-1}, \dots [f_0, P] \dots]] = 0 \text{ for functions } f_0, \dots, f_k\}.$$

The associated graded algebra is then given by $\mathrm{Sym}^\bullet(TM) \otimes \mathrm{End}(S)$. Instead, we consider the increasing filtration with k-th piece given by

$$\widetilde{\mathcal{F}}^k\mathcal{D}(S) = \{P \in \mathcal{D}(S) \mid [u_k, [u_{k-1}, \dots [u_0, P] \dots]] = 0 \text{ for } u_i \in \Gamma(\mathbb{T}M)\}$$

In particular, note that any differential operator $P \in \widetilde{\mathcal{F}}^1\mathcal{D}(S)$ is a tensor since any function may expressed in the form $\langle u, v\rangle$ for some sections u and v of $\mathbb{T}M$ and $[\langle u, v\rangle, P] = [[u, v], P] = [u, [v, P]] + [v, [u, P]] = 0$. In particular, we find $\widetilde{\mathcal{F}}^0\mathcal{D}(S) = C^\infty(M)$, since the supercenter of a Clifford algebra is the scalars. Also, we find $\widetilde{\mathcal{F}}^1\mathcal{D}(S) = \Gamma(\mathcal{F}^1\, Cl(\mathbb{T}M))$.

If $P \in \widetilde{\mathcal{F}}^2\mathcal{D}(S)$ then the map $f \mapsto [f, P]$ defines a derivation of the ring of functions and thus a vector field.

As a result there is a natural map $\widetilde{\mathcal{F}}^k\mathcal{D}(S) \to C^k(\mathbb{T}M)$ which sends a differential operator P to the function $\omega_P : (u_1, \dots, u_k) \mapsto [u_k, [u_{k-1}, \dots [u_0, P] \dots]]$. Indeed, the Leibniz rule and Jacobi identity imply that eq. (3.11) and eq. (3.12) hold for ω_P with $\sigma_{\omega_P}(u_1, \dots, u_{k-2}) \cdot f := [f, [u_{k-2}, [u_{k-3}, \dots [u_1, P] \dots]]]$, demonstrating that ω_P is an element of $C^k(\mathbb{T}M)$. It is clear that the kernel of this map is $\widetilde{\mathcal{F}}^{k-1}\mathcal{D}(S)$ and an important theorem of Grutzmann, Michel and Xu shows that the resulting map from the quotient is surjective:

Theorem 2. [19] The natural map

$$\begin{aligned} \operatorname{Gr}^\bullet \mathcal{D}(S) &\to C^\bullet(\mathbb{T}M) \\ [P] &\mapsto \omega_P \end{aligned} \tag{4.6}$$

where $\omega_P(u_1, \dots, u_k) = [u_k, [u_{k-1}, \dots [u_1, P] \dots]]$ and $\sigma_{\omega_P}(u_1, \dots, u_{k-2}) = [u_{k-2}, [u_{k-3}, \dots [u_1, P]]$ is an isomorphism of graded Poisson algebras.

In particular, the operator $d_H = d + H\wedge$ on $\Omega^\bullet(M)$ satisfies

$$[Y + \eta, [X + \xi, d_H]] = -[X, Y] - \mathcal{L}_X\eta + \iota_Y d\xi - \iota_Y\iota_X H =: -[\![X + \xi, Y + \eta]\!]_H \tag{4.7}$$

demonstrating that $d_H \in \widetilde{\mathcal{F}}^3\mathcal{D}(S)$. Moreover, $[d_H, d_H] = 2d_H^2 = 0$, demonstrating

4.4 The Dirac calculus

{sec:diraccalc}

In this section we focus on exact Courant algebroids $\mathbb{T}M = TM \oplus T^*M$ with Courant bracket twisted by a 3-form H. An almost Dirac structure is a maximal isotropic subbundle $L \subseteq \mathbb{T}M$.

For any almost Dirac structure, the exterior algebra $\wedge^\bullet L$ is a subalgebra of $Cl(\mathbb{T}M)$ and as a result, there is an increasing filtration of $S = \Omega^\bullet(M)$, given by

$$\mathcal{F}_L^k S = \{\phi \in \Omega^\bullet(M) \mid \wedge^{k+1} L \cdot \phi = 0\}.$$

The line bundle $K_L := \mathcal{F}^0 S$ is known as the pure spinor line bundle of the almost Dirac structure L, and the filtration can be equivalently expressed as $\mathcal{F}_L^k S = \mathcal{F}^k\, Cl(E) \cdot K_L$.

Given a complementary Dirac structure L' to a Dirac structure L, the triple $(\mathbb{T}M, L, L')$ is called a Manin triple. In this case the filtration $\mathcal{F}^\bullet S$ is upgraded to a grading

$$S = U_{-n} \oplus \cdots \oplus U_n$$

where $U_k = \wedge^{n-k} L \cdot K_L$. Equivalently, U_k is the k-eigenbundle for the spin action of the operator

$$T := \begin{pmatrix} \mathrm{id} & \\ & -\mathrm{id} \end{pmatrix}$$

on $L \oplus L'$.

The Dirac calculus provides a pair of (partially defined) operations on the space of Dirac structures [25]. Namely, given a pair of almost Dirac structures L and L' which are transverse in the sense that $\mathsf{a}(L) + \mathsf{a}(L') = TM$, their Dirac sum is

$$L + L' = \{X + \xi + \eta \mid X + \xi \in L,\ X + \eta \in L'\}$$

If K and K' are the pure spinor lines for L and L' then $K \wedge K'$ is the pure spinor line for $L + L'$. Furthermore, a Dirac structure L may be rescaled by a nonzero $\lambda \in \mathbb{R}$:

$$\lambda L = \{X + \lambda\xi \mid X + \xi \in L\}$$

Example 3.

1. The cotangent bundle T^*M is a maximal isotropic and is involutive with respect to any 3-form H. It has pure spinor line $K_{T^*M} = \det T^*M$
2. The tangent bundle TM is also maximal isotropic but is involutive only with respect to $H = 0$. It has pure spinor line $K_{TM} = \Omega^0_M$.
3. If π is a bivector then the graph Γ_π is maximal isotropic. It is integrable with respect to $H = 0$ precisely when π is Poisson. The pure spinor line of π is $K_\pi = e^\pi \Omega^n_M$. Indeed, by **??**

   $$(\pi\xi + \xi)e^{\iota_\pi}\Omega^n = e^{\iota_\pi}e^{-\iota_\pi}(\pi\xi + \xi)e^{\iota_\pi}\Omega^n = e^{\iota_\pi}\xi \cdot \Omega^n = 0$$

4. Similarly, if B is a 2-form, then its graph $\Gamma_B := \{X + \iota_X B \mid X \in TM\}$ is maximal isotropic. The graph of B is integrable with respect to $H = -dB$. The canonical bundle of Γ_B is given by

   $$K_B = \langle e^{-B} \rangle \tag{4.8}$$

   respectively. Indeed, $\iota_X e^{-B} = -(\iota_X B) \wedge e^{-B}$ so that $(X + B(X)) \cdot e^{-B} = 0$.

Associated to any Poisson structure π and volume form μ is its modular vector field X_π defined by

$$X_\pi f := \frac{\mathcal{L}_{\pi(df)}\mu}{\mu} \tag{4.9}$$

Proposition 6. The modular vector field X_π satisfies:

$$de^{\iota_\pi}\mu = \iota_{X_\pi} e^{\iota_\pi}\mu$$

Proof. First note that this relation is equivalent to

$$\partial^\pi \mu = \iota_{X_\pi}\mu$$

where $\partial^\pi = [d, \iota_\pi]$ is the Koszul operator. Indeed,...

Now observe that for any function f,

$$\begin{aligned}0 &= \iota_{X_\pi}(df \wedge \mu)\\ &= (X_\pi f)\mu - df \wedge \iota_{X_\pi}\mu\\ &= \mathcal{L}_{\pi(df)}\mu - df \wedge \iota_{X_\pi}\mu\\ &= d\iota_{\pi(df)}\mu - df \wedge \iota_{X_\pi}\mu\\ &= d(df \wedge \iota_\pi \mu) - df \wedge \iota_{X_\pi}\mu\\ &= df \wedge d\iota_\pi\mu - df \wedge \iota_{X_\pi}\mu\end{aligned}$$

Since this is true for any function f, the result holds in any local neighbourhood, and thus is valid globally. □

Proposition 7. [10] Let K be the canonical line bundle of a Dirac structure L. Then there is an isomorphism of L-modules, given by

$$K_L \otimes K_L \mapsto Q_L$$

where $Q_L = \det L \otimes \det T^*$ is the Lu-Evens-Weinstein module.

Proof. □

Proposition 8. If ϕ is a (local) pure spinor generating the canonical bundle of a Manin triple $(\mathbb{T}M, L, L')$ and $e \in \Gamma_{\overline{L}}$ is the unique section such that $d^H\phi = e \cdot \phi$ then $\langle e, \cdot\rangle \in \Omega^1_L$ is the modular cocycle of the Lie algebroid L.

Though much of what follows is true for Courant algebroids in generality, in this thesis we shall focus on split exact Courant algebroids. On of the main tools that we use in this thesis is the *Dirac calculus*, from [22, 25], which we now explain.

Suppose that L_1 and L_2 are Dirac structures, integrable with respect to three forms H_1 and H_2 respectively. If L_1 and L_2 are transverse in the sense that $\mathsf{a}(L_1) + \mathsf{a}(L_2) = TM$ then their Dirac sum,

$$L_1 + L_2 \coloneqq \{X + \xi_1 + \xi_2 \mid X + \xi_i \in L_i\} \tag{4.10}$$

is a smooth, maximal isotropic subbundle of $\mathbb{T}M$ and is a Dirac structure with respect $H_1 + H_2$. Also for any Dirac structure L integrable with respect to H and any scalar λ, we define

$$\lambda L \coloneqq \{X + \lambda\xi \mid X + \xi \in L\} \tag{4.11}$$

which is a Dirac structure with respect to λH.

One useful feature of the Dirac calculus is that it encodes gauge transformations. Given any 2-form B, the bundle $\Gamma_B \coloneqq \{X + \iota_X B | X \in TM\}$ is a Dirac structure for $-dB$. Then observe that for any Dirac structure L, integrable with respect to H,

$$e^B(L) = \Gamma_B + L$$

which is a Dirac structure with respect to $H - db$.

An important observation of Mackenzie and Xu [37, 1] is that if L_1 and L_2 make up a Manin triple $\mathbb{T}M = L_1 \oplus L_2$, then their Dirac difference $L_2 - L_1 := L_2 + (-1)L_1$ is a Poisson structure.

Associated to any Manin triple $(\mathbb{T}, A, B)$, is a Poisson structure, realized through the Dirac calculus as $\Gamma_\pi = A - B$. As a result there is a diagram of Lie algebroids

$$\begin{tikzcd} \Gamma_\pi \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & TM \end{tikzcd}$$

and any A-module (resp. B-module) inherits a Poisson module structure for π. In particular, the tensor product $K_A \otimes K_B$ is a Poisson module which is isomorphic to the pure spinor line bundle for π:

`{lemma:poissoniso`

Lemma 3. For $(\mathbb{T}, A, B)$ a Manin triple with spinor line bundles K_A and K_B respectively, and $\Gamma_\pi = A - B$, the map

$$K_A \otimes K_B \to \det T^*M, \qquad \phi \otimes \psi \mapsto (\phi, \psi) \tag{4.12}$$

is an isomorphism of π-modules.

Proof.

$$\begin{aligned} \xi \wedge d\iota_\pi(\phi,\psi) &= d\xi \wedge \iota_\pi(\phi,\psi) - d(\xi \wedge \iota_\pi(\phi,\psi)) \\ &= d\xi \wedge \iota_\pi(\phi,\psi) - d\iota_{\pi(\xi)}(\phi,\psi) \\ &= \end{aligned}$$

□

Chapter 5

Courant connections and curvature

{chpt:gconnection

In this chapter we review Courant connections and curvature. In our exposition, we make use of the framework developed in [12]. They key is the following analogy: when one wants to replace the notion of a connection with a Courant connection, the algebra of differential forms must be replaced with the Roytenberg algebra.

5.1 Courant connections and the Roytenberg algebra

{sec:connectionsa

Given a Courant algebroid $E \to M$ and a vector bundle $V \to M$, an E-connection on V is a map

$$\begin{aligned} \mathbb{D} : \Gamma(V) &\to \Gamma(E^* \otimes V) \\ \phi &\mapsto \mathbb{D}\phi \end{aligned}$$

satisfying the Leibniz rule:

$$\mathbb{D}(f\phi) = \mathsf{a}^* df \otimes \phi + f\mathbb{D}\phi$$

for any $f \in C^\infty(M)$. We will find it convenient to define $C^\bullet(E;V) = C^\bullet(E) \otimes_{C^\infty(M)} \Gamma(V)$. An E-connection may then be thought of as a map $\mathbb{D} : C^0(E;V) \to C^1(E;V)$, which then extends uniquely to a map

$$d^{\mathbb{D}} : C^k(E;V) \to C^{k+1}(E;V)$$

satisfying the Leibniz rule:

$$d^{\mathbb{D}}(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d^{\mathbb{D}}\tau$$

where $\omega \in C^k(E)$ and $\tau \in C^\bullet(E;V)$. There is also an invariant formula for this *exterior covariant derivative*:

$$(d^{\mathbb{D}}\tau)(u_0, \ldots, u_k) = \sum_{j=0}^{k} (-1)^j \mathbb{D}_{u_j} \tau(u_0, \ldots, \hat{u}_j, \ldots, u_k) - \sum_{i<j} (-1)^i \tau(u_0, \ldots, \hat{u}_i, \ldots, [\![u_i, u_j]\!], \ldots, u_k)$$

The curvature of an E-connection $\mathbb{D}$ is a co-chain $R \in C^2(E; \mathrm{End}(V))$ defined by

$$R(u,v) = [\mathbb{D}_u, \mathbb{D}_v] - \mathbb{D}_{[u,v]}.$$

or equivalently by $R(u,v)s = (d^{\mathbb{D}})^2(s)(u,v)$ for u and v sections of E and s a section of V.

Any E-connection $\mathbb{D}$ on V induces an E-connection $\tilde{\mathbb{D}}$ on $\mathrm{End}(V)$ by the formula

$$\tilde{\mathbb{D}}\phi = [\mathbb{D}, \phi].$$

There is a differential Bianchi identity for R which simply says that R is covariant constant:

Proposition 9. [12] If R is the curvature of an E-connection $\mathbb{D}$ on V then $d^{\tilde{\mathbb{D}}}R = 0$.

Proof. The computation proceeds as follows.

$$\begin{aligned}
(d^{\tilde{\mathbb{D}}}R)(u,v,w) &= \tilde{\mathbb{D}}_u R(v,w) - \tilde{\mathbb{D}}_v R(u,w) + \tilde{\mathbb{D}}_w R(u,v) - R([\![u,v]\!],w) + R(u,[\![v,w]\!]) - R(v,[\![u,w]\!]) \\
&= [\mathbb{D}_u, R(v,w)] - [\mathbb{D}_v, R(u,w)] + [\mathbb{D}_w, R(u,v)] - R([\![u,v]\!],w) + R(u,[\![v,w]\!]) - R(v,[\![u,w]\!]) \\
&= [\mathbb{D}_u, [\mathbb{D}_v,\mathbb{D}_w] - \mathbb{D}_{[\![v,w]\!]}] - [\mathbb{D}_v, [\mathbb{D}_u,\mathbb{D}_w] - \mathbb{D}_{[\![u,w]\!]}] + [\mathbb{D}_w, [\mathbb{D}_u,\mathbb{D}_v] - \mathbb{D}_{[\![u,v]\!]}] \\
&- [\mathbb{D}_{[\![u,v]\!]}, \mathbb{D}_w] + \mathbb{D}_{[\![[\![u,v]\!],w]\!]} + [\mathbb{D}_u, \mathbb{D}_{[\![v,w]\!]}] - \mathbb{D}_{[\![u,[\![v,w]\!]]\!]} - [\mathbb{D}_v, \mathbb{D}_{[\![u,w]\!]}] + \mathbb{D}_{[\![v,[\![u,w]\!]]\!]} \\
&= 0
\end{aligned}$$

where the last line uses the Jacobi identity of both the Courant bracket and the commutator. $\square$

We end this section by remarking that any E-connection $\mathbb{D}$ gives rise to a vector field $\mathcal{X} \in \Gamma(T \otimes \mathrm{End}(V))$, by composing with $\mathsf{a}^t : T^* \to E$. Indeed, since $\mathsf{a} \circ \mathsf{a}^t = 0$, the Leibniz rule gives

$$\mathbb{D}_{\mathsf{a}^t\xi}(fs) = f\mathbb{D}_{\mathsf{a}^t\xi}s$$

demonstrating that $\mathcal{X} = -\mathbb{D}_{\mathsf{a}^t(\cdot)}$ is tensorial. In fact, this vector field may also be recovered as the symbol of the curvature of $\mathbb{D}$, since $R(u,v) + R(v,u) = \mathcal{X} \cdot \langle u, v\rangle$.

5.2 Courant connections on exact Courant algebroids

{sec:ecas}

Given a split exact Courant algebroid $E = T \oplus T^*$, an E-connection $\mathbb{D}$ decomposes

$$\mathbb{D} = \nabla - \mathcal{X}$$

where ∇ is a genuine connection on E. The curvature R of $\mathbb{D}$ is then a symmetry of $\mathbb{T}M$ with the H-twisted Courant bracket (c.f. **??**) which is related to the curvature F of ∇ via the formula

$$R = (\mathcal{X}, F + \iota_{\mathcal{X}} H). \tag{5.1}$$

{{eq:gcurvature}}

Indeed, $R(X,Y) = [\nabla_X, \nabla_Y] - \mathbb{D}_{[X,Y]+\iota_Y\iota_X H} = (F + \iota_{\mathcal{X}}H)(X,Y)$.

In the case that V is a line bundle the following expressions will also be useful:

$$\begin{aligned}
R(\xi,\eta) &= 0 \\
R(X,\eta) &= -\iota_{[X,\mathcal{X}]}\eta \\
R(\xi,Y) &= \iota_Y \mathcal{L}_{\mathcal{X}}\xi
\end{aligned}$$

for X and Y vector fields and ξ and η 1-forms. This implies, in particular, that the derivation corresponding to R is $(\chi, F - \iota_\chi H)$.

If, in addition, E has generalized metric $C_\pm$, we may decompose $\mathbb{D}$ into operators $\Gamma(V) \to \Gamma(C_\pm^* \otimes V)$. Using the splittings $s_\pm$ to identify $C_\pm$ with T we find that the members of this decomposition $\mathbb{D} = \nabla^+ + \nabla^-$ are nothing but connections. The relation of this decomposition to the one arising from the splitting $E = T \oplus T^*$ is as follows:

$$\nabla = \frac{1}{2}(\nabla^+ + \nabla^-) - \iota_\chi b \tag{5.2}$$

$$-g(\chi) = \frac{1}{2}(\nabla^+ - \nabla^-) \tag{5.3}$$

{{gconnection-sym

Proposition 10. If $\mathbb{D} = \nabla - \chi$ is an E-connection on a line bundle and F is the curvature of ∇ then under the isomorphism $C_+ \otimes C_- = T \otimes T$ we have

$$R|_{C_+ \otimes C_-} = F - \iota_\chi H + \mathcal{L}_\chi(b + g)$$

Proof. Compute

$$\begin{aligned} R(X + (b+g)(X), Y + (b-g)(Y)) &= (F + \iota_\chi H)(X,Y) + \iota_{[X,\chi]}(b-g)(Y) - \iota_Y \mathcal{L}_\chi(b+g)(X) \\ &= (F + \iota_\chi H - \mathcal{L}_\chi(b+g))(X,Y). \end{aligned}$$

Similarly,

$$\begin{aligned} R(X + (b-g)(X), Y + (b+g)(X)) &= -R(Y + (b+g)(Y), X + (b-g)(X)) \\ &= (F + \iota_\chi H - \mathcal{L}_\chi(b+g))(Y,X) \\ &= -(F + \iota_\chi H - \mathcal{L}_\chi(b-g))(X,Y) \end{aligned}$$

□

5.3 Torsion and the algebraic Bianchi identity

{sec:torsion}

Though we will not make use of it in this thesis, we remark here that there is a definition of torsion for E-connections on E or TM which results in a particularly pleasing algebraic Bianchi identity. This is distinct from the definition of torsion of an E-connection which is tensorial found elsewhere in the literature [20, 15].

Given an E-connection $\mathbb{D}$ on E there is a natural element id in $C^1(E;E)$ and we can define the torsion τ as $\tau = d^{\mathbb{D}}\,\mathrm{id}$. More explicitly the torsion is

$$\tau(u,v) = \mathbb{D}_u v - \mathbb{D}_v u - [\![u,v]\!].$$

From its definition it is clear that this torsion is not tensorial in the sense of the torsion of [20], but an element of $C^2(E;E)$. By applying $d^{\mathbb{D}}$ again to the defining equation for τ we get the *algebraic Bianchi identity*:

$$(d^{\mathbb{D}}\tau)(u,v,w) = R(u,v)w - R(u,w)v + R(v,w)u \tag{5.4}$$

{{algBianchi1}}

There is a similar notion of torsion for an E-connection $\mathbb{D}$ on TM. In this context we may use the anchor map a to define a torsion $\tau = d^{\mathbb{D}}\mathsf{a}$. Here the anchor map is thought of as an element of $C^1(E;T)$. In this case the torsion has the explicit expression

$$\tau(u,v) = \mathbb{D}_u\mathsf{a}(v) - \mathbb{D}_v\mathsf{a}(u) - \mathsf{a}([\![u,v]\!])$$

and the algebraic Bianchi identity says

$$(d^{\mathbb{D}}\tau)(u,v,w) = R(u,v)\mathsf{a}(w) - R(u,w)\mathsf{a}(v) + R(v,w)\mathsf{a}(u). \tag{5.5}$$ {algBianchi2}

These algebraic Bianchi identities are most useful in cases when the torsion is zero, placing strong conditions on the curvature R.

Chapter 6

Generalized complex geometry and generalized Chern connections

{chpt:GCgeometry}

In this chapter we review generalized complex geometry. For a comprehensive overview we refer the reader to any of [21, 22, 8]. We will focus on the interplay between the Roytenberg algebra and a generalized complex structure.

6.1 Generalized complex geometry

{sec:gcgeom}

A *generalized almost complex structure* on an exact Courant algebroid E is an orthogonal endomorphism $\mathbb{J}$ of E which satisfies $\mathbb{J}^2 = -1$. This gives a decomposition of $E \otimes \mathbb{C}$ into $\pm i$-eigenspaces for $\mathbb{J}$, L and $\overline{L}$. A generalized almost complex structure $\mathbb{J}$ is called *integrable* if L is involutive with respect to the Courant bracket, in which case we say that J is a *generalized complex structure*.

The canonical bundle of a generalized complex structure $\mathbb{J}$ is the line bundle K consisting of those spinors $\phi \in S$ such that $L \cdot \phi = 0$.

Example 4. If I is an almost complex structure on M then

$$\mathbb{J}_I = \begin{pmatrix} -I & \\ & I^* \end{pmatrix} \tag{6.1}$$

is a generalized almost complex structure on $\mathbb{T}M$. The $+i$-eigenbundle for $\mathbb{J}_I$ is $L = T_{0,1} \oplus T^*_{1,0}$ so the integrability condition for $\mathbb{J}$ with respect to the untwisted ($H = 0$) Courant bracket is precisely the integrability condition for I. The canonical bundle of $\mathbb{J}$ is $K = \Omega^{n,0}$, in agreement with the canonical bundle for the complex structure I.

Example 5. If ω is a nondegenerate 2-form on M then

$$\mathbb{J}_\omega = \begin{pmatrix} & \omega^{-1} \\ -\omega & \end{pmatrix} \tag{6.2}$$

is an almost generalized complex structure. The i-eigenspace for $\mathbb{J}$ is $L = \Gamma_{i\omega}$ and one checks that the integrability condition for $\mathbb{J}$ is precisely the condition $d\omega = 0$. The canonical bundle of $\mathbb{J}_\omega$ has a global section $e^{-i\omega}$.

Example 6. If a generalized almost complex structure on $\mathbb{T}M$ preserves TM, then it must be of the form

$$\mathbb{J}_\sigma = \begin{pmatrix} -I & Q \\ & I^* \end{pmatrix}$$

where I is an almost complex structure and Q is a bivector of type $(2,0)+(0,2)$. The $+i$-eigenbundle for $\mathbb{J}_\sigma$ is

$$L_\sigma = \Gamma_\sigma \oplus T_{0,1}M$$

where $\sigma = -\frac{1}{4}(IQ + iQ)$. A routine verification shows that L_σ is involutive precisely when I is integrable and σ is holomorphic Poisson.

A generalized complex structure is, in particular, a Manin triple $(E_{\mathbb{C}}, L, \overline{L})$ and thus comes with a Poisson structure π whose graph is realized as

$$\Gamma_\pi = \frac{1}{2i}(L - \overline{L}).$$

With this scaling factor, the Poisson structure π turns out to be equivalent to $\pi = \mathsf{a}\mathbb{J}\mathsf{a}^t$. Indeed, if α and β and ξ are 1-forms satisfying

$$\begin{gathered} \alpha - \beta = 2i\xi \\ \pi\xi + \alpha \in L \\ \pi\xi + \beta \in \overline{L} \end{gathered}$$

then $2i\mathsf{a}\mathbb{J}\mathsf{a}^t\xi = \mathsf{a}\mathbb{J}(\pi\xi + \alpha - \pi\xi - \beta) = 2i\pi\xi$.

6.2 The Roytenberg algebra and integrability

{sec:roytenbergan

In this section we explain the interaction between generalized complex structures and the Roytenberg $C^\bullet(E)$. has a secondary grading in the presence of an almost generalized complex structure. This bigrading is due to Roytenberg [43] but the presentation we give here makes use of the Keller-Waldmann description of the Roytenberg-Ševera algebra described in section 3.1.

Following example 1 we note that the adjoint representation (c.f. section 3.2) gives, in particular, an action of any almost generalized complex structure on $C^\bullet(E_{\mathbb{C}})$. Since $\mathbb{J}$ is a semi-simple element, its action has an eigendecomposition

$$C^k(E_{\mathbb{C}}) = \oplus_s C^k_s(E_{\mathbb{C}})$$

where $[\mathbb{J}, \omega] = is\omega$ for any $\omega \in C^k_s(E_{\mathbb{C}})$. For any element $\omega \in C^\bullet(E)$ we denote by ω^k_s the projection of ω onto $C^k_s(E)$. Observe that the interior product by sections of L and $\overline{L}$ act respectively as lowering and raising operators for the action of $\mathbb{J}$. Since the functions $C^\infty(M)$ are of degree 0 with respect to $\mathbb{J}$, this means that for $\omega \in C^k_s(E)$

$$\omega(u_1, \cdots, u_k)$$

is zero if more than $\frac{k+s}{2}$ of $u_1, \dots, u_k$ have no component in $\overline{L}$ or if more than $\frac{k-s}{2}$ of $u_1, \dots, u_k$ have no component in L. Thus, elements of $C^k_s(E)$ may be thought of as those elements of $C^\bullet(E)$ which take as inputs $\frac{k+s}{2}$ sections of L and $\frac{k-s}{2}$ sections of $\overline{L}$, though of course, they can also take mixed sections as inputs.

It may be helpful for the reader to compare with the decomposition of differential forms arising from a complex structure I on a manifold M. In this case I gives rise to a derivation ι_I of the algebra of differential forms and forms of type (p, q) have eigenvalue $i(p-q)$ with respect to this derivation.

Continuing with this analogy, we define Dolbeault operators $d_L : C^k_s(E) \to C^{k+1}_{s+1}(E)$ and $\overline{d}_L : C^k_s(E) \to C^{k+1}_{s-1}(E)$ by composing d_E with the projections onto $C^{k+1}_{s+1}(E)$ and $C^{k+1}_{s-1}(E)$ respectively. We have the following characterizations of integrability of the generalized complex structure.

{thm:integrabilit

Theorem 3. If $\Theta = \langle [\![\cdot,\cdot]\!], \cdot\rangle$ and $\mathbb{J}$ is a generalized almost complex structure on E then the following are equivalent:

1. $\mathbb{J}$ is integrable with respect to $[\![\cdot,\cdot]\!]$
2. $[\mathbb{J}, [\mathbb{J}, \Theta]] = -\Theta$ {grabowski}
3. $d_E = d_L + \overline{d}_L$
4. $\Theta^3_3 = 0$

The component $4\left(\Theta^3_3 + \Theta^3_{-3}\right) = 8\,\mathrm{Re}(\Theta^3_3) \in \Omega^3_E$ is known as the Nijenhuis tensor of $\mathbb{J}$.

Proof. Decomposing $\Theta = \Theta^3_3 + \Theta^3_1 + \Theta^3_{-1} + \Theta^3_{-3}$ we note that for u, v, and w sections of E, $\Theta^3_3(u, v, w) = \langle [\![\pi_L(u), \pi_L(v)]\!], \pi_L(w)\rangle$ where π_L is the projection onto L . Thus L is involutive if and only if $\Theta^3_3 = 0$. We also make note of the fact that $\Theta^3_3 = \overline{\Theta^3_{-3}}$. The equivalence of 2 and 4 then follows from the computation

$$[\mathbb{J}, [\mathbb{J}, \Theta]] = -9\left(\Theta^3_3 + \Theta^3_{-3}\right) - \Theta^3_1 - \Theta^3_{-1} = -\Theta - 8\left(\Theta^3_3 + \Theta^3_{-3}\right).$$

On the other hand, since the Poisson bracket is degree 0 with respect to action of $\mathbb{J}$, it follows that

$$\overline{d}_L = [\Theta^2_{-1}, \cdot] \qquad\qquad d_L = [\Theta^2_1, \cdot] \tag{6.3}$$

so that $d_E = d_L + \overline{d}_L$ if and only if $[\Theta^3_3, \cdot] = 0$ identically. Since the Poisson structure is nondegenerate, it is clear that this happens only when $\Theta^3_3 = 0$, demonstrating the equivalence of 4 and 3. □

Remark 3. While the above theorem does not appear in the literature as stated, similar results have been observed elsewhere. The equivalence between 1 and 2 was proved by Grabowski [18] using the language of graded symplectic geometry. Meanwhile the equivalence between 1, 3, and 4 follows from a result of Roytenberg [42].

When $\mathbb{J}$ is integrable, it is clear that d_L and $\overline{d}_L$ satisfy the relations

$$d_L^2 = 0 = \overline{d}_L^2, \qquad\qquad d_L\overline{d}_L + \overline{d}_L d_L = 0.$$

In particular, if we restrict our attention to the loci $C^k_k(E)$ and $C^k_{-k}(E)$, we obtain chain complexes (Ω_L, d_L) and $(\Omega_{\overline{L}}, \overline{d}_L)$ which are the de Rham complexes for the Lie algebroids L and $\overline{L}$. We will also make use of the real operator $d^c_{\mathbb{J}} = [[\Theta, \mathbb{J}], \cdot] = i(\bar{d}_L - d_L)$.

Corollary 2. If Θ is a Courant bracket on E and $\mathbb{J}$ is an integrable generalized complex structure with respect to Θ then $[\mathbb{J}, \Theta]$ gives a new Courant bracket $[\![\cdot,\cdot]\!]_{\mathbb{J}}$ on E by the formula

$$[\![u, v]\!]_{\mathbb{J}} = \mathbb{J}[\![u, v]\!] - [\![\mathbb{J}u, v]\!] - [\![u, \mathbb{J}v]\!].$$

Corollary 3 ([11])**.** If $\mathbb{J}$ is an almost generalized complex structure and $\mathbb{D}$ is an E-connection on E that preserves $\mathbb{J}$ then $(T_{\mathbb{D}})_3 = -\Theta^{3,0}$. In particular, $\mathbb{J}$ is integrable if and only if $T_{\mathbb{D}}$ is of type $(2,1)+(1,2)$.

6.3 Spinors and integrability

{sec:spinorsandin

There is a deep relationship between the bundle of spinors of a Courant algebroid and the Roytenberg algebra, allowing much of the content of the previous section to carry over to differential forms $\Omega^\bullet(M)$. While we do not explain this relationship in detail, we will spell out the analogous results for $\Omega^\bullet(M)$. These are largely taken from [22].

For any generalized complex structure $\mathbb{J}$ on $\mathbb{T}M$ with its H-twisted Courant bracket, the exterior algebra $\wedge^\bullet\overline{L}$ of the $-i$-eigenbundle $\overline{L}$ of $\mathbb{J}$ embeds as a subalgebra inside the Clifford algebra $Cl(\mathbb{T}M)$. As a result there is a canonical isomorphism of Clifford modules given by the Clifford action:

$$\wedge^\bullet \overline{L} \otimes K \to \Omega^\bullet(M) \tag{6.4}$$

Indeed this is a nonzero morphism of irreducible Clifford modules and thus an isomorphism. The decomposition of $\wedge^\bullet\overline{L}$ into exterior powers of $\overline{L}$ gives a decomposition of $\Omega^\bullet(M)$ into subbundles. As explained by Gualtieri, this decomposition turns out to be the eigendecomposition for the spin action of $\mathbb{J}$.

Proposition 11. [22] For $k \in \{-n, -n+1, \ldots, n-1, n\}$ the subspace $U_k = \wedge^{n-k}\overline{L} \cdot K \subseteq S$ is the ik-eigenspace for the action of $\mathbb{J}$ on S. {decomposition}

Proof. Let $e_1, \ldots, e_n$ be a basis for L and $f_1, \ldots, f_n$ be a basis for $\overline{L}$, chosen so that $\langle e_i, f_j\rangle = \delta_{ij}$. With respect to this combined basis $e_1, \ldots, e_n, f_1, \ldots f_n$, we have

$$\mathbb{J} = \begin{pmatrix} i & \\ & -i \end{pmatrix}$$

and using **??** we find that the corresponding element in $\mathfrak{spin}(\mathbb{T}) \subset Cl(\mathbb{T})$ is $\mathbb{J}_{spin} := i\sum_{k=1}^n e_k f_k = in - i\sum_{k=1}^n f_k e_k$. In particular, since $e_k \cdot \phi = 0$ we find that K is in the in eigenspace for $\mathbb{J}_{spin}$. To compute the eigenvalues for the rest of the decomposition we use the relation $[\mathbb{J}_{spin}, u] = \mathbb{J}(u) = -iu$ for $u \in \overline{L}$, to show that u acts as a lowering operator for the $\mathbb{J}_{spin}$. □

It will be useful to us to understand how the Chevalley pairing interacts with the decomposition of spinors.

Proposition 12. [9] With respect to the Chevalley paring, the subspace U_k is orthogonal to U_l unless $k = -l$, in which case it provides an isomorphism $U_k \to U^*_{-k}$. {prop:chevalleyde

We also find it useful to denote the projection onto U_k by π_k. By applying a degree argument to the derived bracket formula

$$[[d_H, u], v] = [\![u, v]\!]$$

we find that d_H necessarily decomposes as $d_H = \eta_L + \delta_L + \overline{\delta}_L + \overline{\eta}_L$ where

$$\begin{aligned}
\eta_L &= \pi_{k+3} \circ d_H : \Gamma(U_k) \to \Gamma(U_{k+3}) \\
\partial_L &= \pi_{k+1} \circ d_H : \Gamma(U_k) \to \Gamma(U_{k+1}) \\
\overline{\partial}_L &= \pi_{k-1} \circ d_H : \Gamma(U_k) \to \Gamma(U_{k-1}) \\
\overline{\eta}_L &= \pi_{k-3} \circ d_H : \Gamma(U_k) \to \Gamma(U_{k-3})
\end{aligned}$$

Moreover, η_L is tensorial and is given by Clifford action of the section $\langle [\![\cdot, \cdot]\!], \cdot \rangle$ of $\wedge^3 L$

Theorem 4. [22] If H is a closed 3-form and $\mathbb{J}$ is a generalized almost complex structure on $\mathbb{T}M$ then the following are equivalent: {thm:integrabilit

1. $\mathbb{J}$ is integrable with respect to the H-twisted Courant bracket
2. $[\mathbb{J}, [\mathbb{J}, d_H]] = -d_H$
3. $d_H = \partial_L + \overline{\partial}_L$
4. $\eta_L = 0$

Remark 4. Though we don't expand upon this here, we emphasize that the similarity between theorem 3 and theorem 4 is due to the relationship between spinors and the Roytenberg algebra explained in section 4.3.

6.4 Formal deformations of generalized complex structures

{sec:formaldeform

In this section we describe the tangent space to the space of generalized complex structures. The deformation theory of generalized complex structures was studied by Gualtieri [22]. Here we will focus only on formal aspects of the deformation theory.

Let E be an exact Courant algebroid over a manifold M. We denote by $\mathcal{GC}(E)$ the space of generalized complex structures on E.

In order to describe the tangent to these spaces at $\mathbb{J}$, notice that a deformation of an almost generalized complex structure is given by a section ϕ of $\mathfrak{so}(E) = \wedge^2 E^*$:

$$\dot{\mathbb{J}} = \phi \tag{6.5}$$ {{complexflow}}

By complexifying and decomposing $\wedge^2 E^*_{\mathbb{C}} = \wedge^2 L^* \oplus L^* \otimes \overline{L}^* \oplus \wedge^2 \overline{L}^*$, we see that the condition $\mathbb{J}^2 = -1$ is preserved by eq. (6.5) precisely when ϕ is a section of $\mathrm{Re}(\wedge^2 L^* \oplus \wedge^2 \overline{L}^*)$. Thus we write $\phi = \varepsilon + \overline{\varepsilon}$ where ε is a section of $\wedge^2 L^*$. The following proposition characterizes those deformations that preserve integrability of $\mathbb{J}$.

Proposition 13. [22, 36] Suppose that $\mathbb{J}_t$ is a continuous family of sections of $\mathfrak{so}(E)$ over some interval containing the origin and that $\mathbb{J}_0$ is a generalized almost complex structure. If

$$\dot{\mathbb{J}}_t = \varepsilon_t + \overline{\varepsilon}_t$$

where ε_t is a family of sections of $\wedge^2 L^* \subseteq \mathfrak{so}(E_{\mathbb{C}})$, then $\mathbb{J}_t$ is generalized almost complex for all t. If, in addition, $\mathbb{J}_0$ is integrable, then $\mathbb{J}_t$ is integrable for all t if and only if $\partial_{L_t}\varepsilon_t = 0$ for all t.

Proof. By theorem 3 it suffices to show that the quantity $[[\Theta, \mathbb{J}], \mathbb{J}] + \Theta$ is constant. We compute

$$\begin{aligned}
\frac{d}{dt}\left([[\Theta, \mathbb{J}], \mathbb{J}] + \Theta\right) &= [[\Theta, \varepsilon + \overline{\varepsilon}], \mathbb{J}] + [[\Theta, \mathbb{J}], \varepsilon + \overline{\varepsilon}] \\
&= [\Theta, [\varepsilon + \overline{\varepsilon}, \mathbb{J}]] + 2[[\Theta, \mathbb{J}], \varepsilon + \overline{\varepsilon}] \\
&= d_E(2i\overline{\varepsilon} - 2i\varepsilon) + 2d^c_{\mathbb{J}}(\varepsilon + \overline{\varepsilon}) \\
&= 4i(\overline{\partial_L \varepsilon} - \partial_L \varepsilon)
\end{aligned}$$

which vanishes precisely when $\partial_L \varepsilon$ does. . ☐

Summarizing, we have an identification $T_{\mathbb{J}}\mathcal{GC} = \Omega^{2,cl}_L$. This identification makes clear the (formal) complex structure on $\mathcal{GC}$. Indeed, in real terms, this complex structure takes the form $\phi \mapsto \mathbb{J}\phi$.

It is possible to obtain tangent vectors in $T_{\mathbb{J}}\mathcal{CG}$ from symmetries of the Courant bracket. Indeed, a symmetry of D is nothing but a d_E-closed element of $C^2(E)$. The projection of D onto $C^2_1(E)$ then gives a closed element of Ω^2_L. It is natural to express such a flow

$$\dot{\mathbb{J}} = [D, \mathbb{J}]. \tag{6.6}$$

{{eq:trivialflow}

the interpretation being that $\mathbb{J}$ is being deformed by an automorphism of the Courant algebroid. Thus we say that a deformation of this form is trivial. It is important to note however, that using the complex structure on $T_{\mathbb{J}}\mathcal{GC}$ we obtain another deformation

$$\dot{\mathbb{J}} = \mathbb{J}[D, \mathbb{J}] = \frac{1}{2}[[D, \mathbb{J}], \mathbb{J}] \tag{6.7}$$

{{eq:Dflow}}

which is, in general, nontrivial.

6.5 Generalized Chern connections

{sec:gchern}

In this section we investigate the generalized Chern connection of [20].

Let $\mathbb{J}$ be a generalized complex structure on an exact Courant algebroid E with $+i$-eigenbundle L. We say that vector bundle V is generalized holomorphic, or more precisely $\mathbb{J}$-holomorphic, when it is equipped with a flat L-connection $\overline{\partial}$. When V is also equipped with a Hermitian metric h, there is a unique E-connection $\mathbb{D}$ on V for which:

- $\mathbb{D}h = 0$ and,
- $\mathbb{D}_u = \overline{\partial}_u$ for any $u \in \Gamma(L)$

This E-connection is known as the generalized Chern connection. The curvature iR of $\mathbb{D}$ is then a 2-cocycle of $C^2(E)$ which preserves $\mathbb{J}$:

Proposition 14. Let V be a $\mathbb{J}$-holomorphic, Hermitian vector bundle. The generalized Chern curvature R of V has degree 0 with respect to $\mathbb{J}$, that is, $R \in C_0^{2,cl}(E; \mathfrak{u}(V)) := C_0^{2,cl}(E) \otimes_{C^\infty(M)} \Gamma(\mathfrak{u}(V))$.

Proof. It suffices to show that $R(u, v) = 0$ for any sections u and v of L. The restriction of $\mathbb{D}$ to L is the generalized Dolbeault operator on V, $\overline{\partial}$. The result follows since this L-connection is flat. ☐

In particular, if V is a line bundle then $\mathfrak{u}(V) = i\mathbb{R}$ so the curvature is i times a real symmetry of $\mathbb{J}$. Moreover, given a local section ϕ, we can produce an explicit formula for the curvature as follows.

`{prop:Poincare-Le`

Proposition 15 (Generalized Poincaré-Lelong formula)**.** Let $(V, h, \overline{\partial})$ be a generalized holomorphic Hermitian line bundle. A local nonvanishing section ϕ of V induces a trivialization of the generalized Chern connection, $\mathbb{D} = d_E + a$, where

$$a = d_E \log|\phi| - i d^c_{\mathbb{J}} \log|\phi| + 2i \operatorname{Im} e \in C^1(E_{\mathbb{C}}) \tag{6.8}$$

`{{eq:poincarelelo`

and e is the section of $L^* = \overline{L}$ for which $\overline{\partial}\phi = e \otimes \phi$. In particular, the curvature of V is then given (locally) by

$$iR = -i d_E d^c_{\mathbb{J}} \log|\phi| + 2i \operatorname{Im} d_E e$$

Proof. Compatibility with the Hermitian metric is equivalent to the relation

$$\operatorname{Re} a = d_E \log|\phi|.$$

To verify that eq. (6.8) gives the generalized Chern connection, it suffices to show that the $\overline{L}$ component of a agrees with the e . This follows from the fact that $d_E - i d^c_{\mathbb{J}} = 2\overline{d}_L$ which maps functions to sections of $\overline{L}^* = L$. ☐

Remark 5. This is a slight generalization of the Poincaré-Lelong formula given in [20] which applies only to holomorphic trivializations.

The main source of examples of a generalized holomorphic bundle comes from the canonical bundle of a generalized complex structure. Indeed if $K = U_n$ is the canonical bundle of a complex structure $\mathbb{J}$ then the operator $\overline{\partial}_L : U_n \to U_{n-1}$ may be interpreted as a flat L-module on K.

It turns out that a choice of Hermitian metric $h : K \otimes \overline{K} \to \mathbb{R}$ is equivalent to a choice of volume form on M. Indeed by proposition 12 the Chevalley pairing gives an isomorphism $K \otimes \overline{K} \to \det T^*M$. In particular, a generalized complex structure on M induces an orientation on M.

The vector component of the resulting generalized Chern connection is dependent only on the chosen volume form and the Poisson structure $\pi = \mathsf{a}\mathbb{J}\mathsf{a}^t$ corresponding to $\mathbb{J}$. In particular, any Poisson structure π on a manifold with volume form μ induces a vector field known as the *modular vector field* [34], X_π of π. There are many equivalent ways to define the modular vector field of a Poisson structure (see e.g. [48, 13, 33]). We make use of the following definition which gives X_π in terms of its action on a function f.

$$X_\pi \cdot f := \frac{\mathcal{L}_{\pi(df)}\mu}{\mu} \tag{6.9}$$

`{{eq:modularvf}}`

`{thm:chernmodular`

Theorem 5. Let $\mathbb{J}$ be a generalized complex structure and K its canonical bundle. Any hermitian metric h on K may be expressed in the form

$$h(\phi,\psi) = i^n \frac{(\phi,\overline{\psi})}{\mu} \tag{6.10}$$

`{{eq:hermmetric}}`

for μ a volume form defining the same orientation as $\mathbb{J}$. Let $\pi = \mathsf{a}\mathbb{J}\mathsf{a}^t$ be the Poisson bivector corresponding to $\mathbb{J}$ and let $\mathbb{D}$ be the generalized Chern connection of K with respect to h. The vector component of $\mathbb{D}$ is given by

$$\mathbb{D}_{\mathsf{a}^t(\xi)} = -\frac{i}{2}\xi(X_\pi)$$

where X_π is the modular vector field of π with respect to μ.

Proof. Choose a splitting for the Courant algebroid so that $\mathbb{D} = \nabla - iX_{\mathbb{J}}$ where ∇ is a connection and $X_{\mathbb{J}}$ is a real vector field. Then, since ∇ preserves the Hermitian metric, note that for f a real function,

$$\mathcal{L}_{\pi(df)}(\phi,\overline{\phi}) = i^{-n}\left(\mathcal{L}_{\pi(df)}|\phi|^2\right)\mu + (X_\pi \cdot f)(\phi,\overline{\phi}) = (\nabla_{\pi(df)}\phi,\overline{\phi}) + (\phi,\overline{\nabla_{\pi(df)}\phi}) + (X_\pi \cdot f)(\phi,\overline{\phi})$$

On the other hand, by Lemma 3,

$$\begin{aligned}
\mathcal{L}_{\pi(df)}\left(\phi,\overline{\phi}\right) &= \left(\overline{\partial}_u\phi,\overline{\phi}\right) + \left(\phi,\partial_v\overline{\phi}\right) \\
&= (\mathbb{D}_{\pi(df)+\alpha}\phi,\overline{\phi}) + (\phi,\overline{\mathbb{D}_{\pi(df)+\overline{\beta}}\phi}) \\
&= (\nabla_{\pi(df)}\phi,\overline{\phi}) - i\iota_{X_{\mathbb{J}}}\alpha(\phi,\overline{\phi}) + (\phi,\overline{\nabla_{\pi(df)}\phi}) + i\iota_{X_{\mathbb{J}}}\beta(\phi,\overline{\phi}) \\
&= (\nabla_{\pi(df)}\phi,\overline{\phi}) + (\phi,\overline{\nabla_{\pi(df)}\phi}) + 2(X_{\mathbb{J}}\cdot f)(\phi,\overline{\phi})
\end{aligned}$$

since $\alpha - \beta = 2idf$. Combining, we get $X_{\mathbb{J}} = \frac{1}{2}X_\pi$. □

Chapter 7

Spinors and curvature in generalized Kähler geometry

{chpt:spinorsandc

Many curvature quantities appear in generalized Kähler geometry. In this section we explain their appearance in terms of the bundle of spinors. This gives convenient relationships between the various connections and curvatures. The key diagram is the decomposition of spinors (differential forms) coming from the spin action of $\mathbb{J}_A$ and $\mathbb{J}_B$.

7.1 Generalized Kähler geometry

{sec:gkgeom}

A generalized Kähler structure on an exact Courant algebroid $\mathbb{T}M$ with H-twisted Courant bracket:

$$[\![X+\xi, Y+\eta]\!]_H = [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi + \iota_Y\iota_X H$$

is a pair $(\mathbb{J}_A, \mathbb{J}_B)$ of commuting generalized complex structures such that $\mathbb{G} = -\mathbb{J}_A\mathbb{J}_B$ is a generalized metric. In [24], Gualtieri showed that a generalized Kähler structure is equivalent to a bi-Hermitian structure $(g, I_\pm, b)$ in the sense of [16], that is, a pair of Hermitian structures $I_\pm$, sharing the same Riemannian metric g and a 2-form b, such that the Hermitian forms $\omega_\pm = gI_\pm$ satisfy

$$-d^c_+\omega_+ = d^c_-\omega_- =: H + db, \qquad\qquad dH = 0. \tag{7.1}$$

{{eq:bihermitiani

The equivalence of Gualtieri is given explicitly by the formula

$$\mathbb{J}_A = \frac{1}{2}e^b\begin{pmatrix} -(I_+ + I_-) & -(I_+ - I_-)g^{-1} \\ -g(I_+ - I_-) & I_+^* + I_-^* \end{pmatrix}e^{-b}$$

$$\mathbb{J}_B = \frac{1}{2}e^b\begin{pmatrix} -(I_+ - I_-) & -(I_+ + I_-)g^{-1} \\ -g(I_+ + I_-) & I_+^* - I_-^* \end{pmatrix}e^{-b}$$

Example 7. If (M, I, ω) is a Kähler manifold then $(\mathbb{J}_\omega, \mathbb{J}_I)$ is a generalized Kähler structure on

$\mathbb{T}M$. The generalized metric is given by

$$\mathbb{G} = \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix}$$

on $TM \oplus T^*M$. In this case, the complex structures I_+ and I_- coincide with I and thus the Poisson structures $Q, \sigma_\pm$ vanish.

7.2 The decomposition of the Roytenberg algebra

{sec:gkroytenberg

In this section we describe the structure of the Roytenberg algebra $C^\bullet(E)$ of a Courant algebroid E in the presence of a generalized Kähler structure $(\mathbb{J}_A, \mathbb{J}_B)$. Since $\mathbb{J}_A$ and $\mathbb{J}_B$ commute, their actions on $C^\bullet(E_\mathbb{C})$ also commute and by the results of section 6.2 they have a common eigendecomposition:

Proposition 16. The Roytenberg algebra has decomposition

$$C^k(E_\mathbb{C}) = \bigoplus_{r,s=-k}^{k} C^k_{r,s}(E_\mathbb{C})$$

making it into a $\mathbb{Z}^3$-graded algebra. That is,

$$C^k_{r,s}(E_\mathbb{C}) \cdot C^l_{p,q}(E_\mathbb{C}) \subseteq C^{k+l}_{r+p,s+q}(E_\mathbb{C})$$

for any $k, r, s, l, p, q \in \mathbb{Z}$. The generalized complex structures $\mathbb{J}_A$ and $\mathbb{J}_B$ act by ir and is respectively on $C^k_{r,s}(E_\mathbb{C})$. This decomposition gives rise to a decomposition of the differential

$$d_E = d_+ + d_- + \overline{d}_+ + \overline{d}_-$$

where $d_\pm = \pi^{k+1}_{r+1,s\pm 1} \circ d_E : C^k_{r,s} \to C^{k+1}_{r+1,s\pm 1}$. The Dolbeault operators for $\mathbb{J}_A$ and $\mathbb{J}_B$ are then given by $d_A = d_+ + d_-$ and $d_B = d_+ + \overline{d}_-$. In addition to squaring to zero, these operators satisfy the relations

$$\begin{aligned} [d_\pm, d_\mp] &= 0 \\ [d_\pm, \bar{d}_\mp] &= 0 \\ [\bar{d}_\pm, \bar{d}_\mp] &= 0 \\ [d_+, \bar{d}_+] + [d_-, \bar{d}_-] &= 0 \end{aligned}$$

The picture that the reader should have in mind is of a pyramid

$$\begin{tikzcd}
C^2_{-2,-2} & C^2_{-2,0} & C^2_{-2,2} & C^2_{0,-2} & C^2_{0,0} & C^2_{0,2} & C^2_{2,-2} & C^2_{2,0} & C^2_{2,2} \\
& C^1_{-1,-1} \arrow[ul] \arrow[u] \arrow[urr] \arrow[urrr] & & C^1_{-1,1} \arrow[ull] \arrow[ul] \arrow[ur] \arrow[urr] & & C^1_{1,-1} \arrow[ull] \arrow[ul] \arrow[ur] \arrow[urr] & & C^1_{1,1} \arrow[ulll] \arrow[ull] \arrow[u] \arrow[ur] & \\
& & & & C^0_{0,0} \arrow[ulll] \arrow[ul] \arrow[ur] \arrow[urrr] & & & &
\end{tikzcd}$$

where the exterior faces are the de Rham complexes for the Lie algebroids L_A, $\overline{L}_A$, L_B and $\overline{L}_B$ and the exterior edges are the de Rham complexes for $\ell_+ := L_A \cap L_B$, $\ell_- := L_A \cap \overline{L}_B$ and their complex conjugates. The following Lemma clarifies the role played by these intersections.

{lem:simeigen}

Lemma 4. The simultaneous eigenbundles of $\mathbb{J}_A$ and $\mathbb{J}_B$ are given by

$$\ell_\pm = e^{b \pm i\omega_\pm} T_{0,1} X_\pm = \{X + (b + i\omega_\pm)X \mid X \in T_{0,1}X_\pm\}$$

where $X_\pm = (M, I_\pm)$.

As a result, the edges of the above diagram may interpreted as Dolbeault complexes for the complex structures $I_\pm$.

In addition to the operators $d^c_A = i(\bar{d}_A - d_A) = i(\bar{d}_+ + \bar{d}_- - d_+ - d_-)$ and $d^c_B = i(\bar{d}_B - d_B) =$ there is a real operator

$$d_{AB} = d_+ + \bar{d}_+ - d_- - \bar{d}_-.$$

which is the Hamiltonian vector field of $-[\mathbb{J}_A, [\mathbb{J}_B, \Theta]]$.

7.3 The spinor diagram

{sec:gkspinors}

We begin by reviewing the decomposition of the spinors described by Gualtieri [23]. This decomposition has the utility of giving relations between the various canonical bundles, to which we will often refer.

The canonical bundles of a generalized Kähler structure $(\mathbb{J}_A, \mathbb{J}_B)$ on a Courant algebroid $(\mathbb{T}M, H)$ are the spinor subbundles K_A, $K_B \subseteq \Omega^\bullet(M)$ which are annihilated by the Clifford action of L_A and L_B respectively. In addition to these, there are also canonical bundles $K_\pm = \Omega^{n,0}_\pm$ coming from the complex structures $I_\pm$. These complex structures are specified by lemma 4 or equivalently, by $T^{0,1}_\pm = \mathsf{a}(\ell_\pm)$, where $\mathsf{a} : \mathbb{T}M \to TM$ is the projection onto TM known as the anchor map. Thus the anchor map naturally identifies $K_\pm$ with $\wedge^n \overline{\ell}^*_\pm = \wedge^n \ell_\pm$ where the latter equality uses the pairing on E.

The canonical bundles of $\mathbb{J}_A$, $\mathbb{J}_B$ and $I_\pm$ are related as follows. The Clifford action of $\wedge^n \ell_- \subset Cl(E)$ on K_B gives a new spinor line which is annihilated by both components of $L_A = \ell_+ \oplus \ell_-$. Thus $K_A = K_- \otimes K_B$ and by a similar argument $K_A = K_+ \otimes \overline{K}_B$. We summarize these relationships with the following diagram

$$\begin{tikzcd}
 & K_B \arrow[dr, "K_-"] & \\
\overline{K}_A \arrow[ur, "K_+"] \arrow[dr, "K_-"'] & & K_A \\
 & \overline{K}_B \arrow[ur, "K_+"'] &
\end{tikzcd} \tag{7.2}$$

{{eq:canonicalbdl

where the arrows represent tensoring (or Clifford action) by the indicated line bundle.

This diamond can be completed to include all of the spinors in the following way: since the spinor module is an irreducible Clifford module we have

$$\Omega_M = Cl(E) \cdot K_A = Cl(E) \cdot K_B. \tag{7.3}$$

{{eq:spinorgen}}

The algebras $\wedge \ell_\pm$ and $\wedge \overline{\ell}_\pm$ embed as subalgebras of the Clifford algebra and the multiplication map gives an isomorphism of graded vector spaces

$$\wedge \overline{\ell}_+ \otimes \wedge \overline{\ell}_- \otimes \wedge \ell_+ \otimes \wedge \ell_- \to Cl(E).$$

Since ℓ_+ and ℓ_- act trivially on K_A, the relation (7.3) may be reduced to

$$\Omega_M = \wedge \overline{\ell}_+ \otimes \wedge \overline{\ell}_- \otimes K_A = \wedge \overline{\ell}_+ \otimes \wedge \ell_- \otimes K_B$$

This description provides a bigrading on the spinors. In fact, this bigrading is the eigendecomposition of the actions of $\mathbb{J}_A$ and J_B on the spinors. Indeed, K_A has eigenvalue in for the action of $\mathbb{J}_A$ and Clifford action by an element of $\overline{\ell}_-$ has the effect of raising the eigenvalue by i, so it follows that K_A is in the kernel of $\mathbb{J}_B$, that is, K_A is type $(n, 0)$. From here we may determine the eigenvalues of all other components using the fact that action by ℓ_+ has degree $(1, 1)$ and action by ℓ_- has degree $(1, -1)$.

Summarizing, we have a $\mathbb{Z} \times \mathbb{Z}$ decomposition of the spinors.

$$\Omega_M = \bigoplus_{p,q \in \mathbb{Z}} U_{p,q}$$

where $U_{p,q}$ is the intersection of the ip and iq eigenspaces for $\mathbb{J}_A$ and $\mathbb{J}_B$ respectively.

$$\begin{tikzcd}
 & & & & U_{0,n} & & & & \\
 & & & U_{-1,n-1} & & U_{1,n-1} & & & \\
 & & \iddots & & U_{0,n-2} & & \ddots & & \\
 & U_{1-n,1} & & & \vdots & & & U_{n-1,1} & \\
U_{-n,0} & & U_{2-n,0} & \cdots & & \cdots & U_{n-2,0} & & U_{n,0} \\
 & U_{1-n,-1} & & & \vdots & & & U_{n-1,-1} & \\
 & & \ddots & & U_{0,2-n} & & \iddots & & \\
 & & & U_{-1,1-n} & & U_{1,1-n} & & & \\
 & & & & U_{0,-n} & & & &
\end{tikzcd}$$

In terms of the canonical bundles, the decomposition has the following explicit description:

{prop:gkspinordec

Proposition 17. Via the Clifford action the (p,q) component of the bigrading may be expressed as

$$\begin{aligned} U_{p,q} &= \wedge^{\frac{n-p-q}{2}} \overline{\ell}_+ \otimes \wedge^{\frac{n-p+q}{2}} \overline{\ell}_- \otimes K_A \\ &= \wedge^{\frac{n-p-q}{2}} \overline{\ell}_+ \otimes \wedge^{\frac{n+p-q}{2}} \ell_- \otimes K_B \end{aligned}$$

wherever the exterior powers are integral. Otherwise, the component $U_{p,q}$ is 0.

7.4 Hermitian metric

{sec:gkchern}

In this section we describe a collection of holomorphic structures that appear in certain loci of the spinor diagram. Together with a natural Hermitian metric, these give rise to Chern connections and curvatures, which are related to the Bismut-Ricci forms for the Hermitian structures $I_\pm$.

The bundle of spinors Ω_M is equipped with a Hermitian metric in the following way. The generalized Kähler structure has a *generalized metric* $\mathbb{G} = -\mathbb{J}_A \mathbb{J}_B$ which squares to id and has ± 1-eigenspaces $V_\pm$. The *chirality element* of the generalized metric is given by $* = a_1 \dots a_{2n} \in Cl(V_+) \subseteq Cl(E)$ for $a_1, \dots, a_{2n}$ an oriented orthonormal basis of V_+. It is an element of the Spin group and has the important feature that it projects, via the twisted adjoint representation to $-\mathbb{G} = \mathbb{J}_A \mathbb{J}_B$. Then, since $\mathbb{J}_{A/B} = e^{\frac{\pi}{2}\mathbb{J}_{A/B}}$ it follows that the chirality element acts by

The Hermitian metric is then given by

$$h(\alpha, \beta) = \frac{(*\alpha, \overline{\beta})}{\mathrm{vol}_g}$$

where $(\cdot,\cdot)$ is the Chevalley pairing. If the splitting of E is the canonical one coming from the generalized metric, then this Hermitian metric is identified with the natural Hermitian metric on differential forms coming from the Riemannian metric g. Indeed, $a_i = X_i + g(X_i)$ for $X_1, \dots, X_{2n}$ an oriented orthonormal basis for TM. Then

$$(*\alpha, \overline{\beta}) = (\alpha, *^T\overline{\beta}) = [\alpha \wedge \star\overline{\beta}]_{2n}$$

where $\star$ is the usual Hodge star operator on differential forms. The Hermitian metric h restricts to a collection of Hermitian metrics on the corners of the diamond: K_A, K_B, $\overline{K}_A$, and $\overline{K}_B$. The line bundles $K_\pm$ are also equipped with a Hermitian metric, which comes from the Riemannian metric on TM. We claim that the identifications in eq. (7.2) preserve these metrics. To see this, let $e_1, \dots, e_n$ be an orthonormal basis for ℓ_- so that any $\kappa \in K_-$ may be expressed as $fe_1 \wedge \cdots \wedge e_n$ for some function f. Since by **??** the operator $*$ acts by i^{-n} on K_A and K_B, it suffices to show that $(\kappa\beta, \overline{\kappa\beta}) = |f|^2(\beta, \overline{\beta})$ for β a section of K_B.

Using (4.3) we compute

$$\begin{aligned}(\kappa\beta, \overline{\kappa\beta}) &= (\beta, \kappa^T\overline{\kappa}\overline{\beta})\\ &= |f|^2(\beta, e_n \cdots e_1\overline{e}_1 \cdots \overline{e}_n\overline{\beta})\\ &= |f|^2(\beta, \overline{\beta})\end{aligned}$$

where the last equality follows from repeated application of the Clifford identity $e_i\overline{e}_j + \overline{e}_je_i = \delta_{ij}$ along with the fact that $e_i\overline{\beta} = 0$. An analogous calculation shows that the identity $K_A = K_+ \otimes \overline{K}_B$ preserves the Hermitian metrics.

7.5 Holomorphic structures and Chern connections

{sec:chernconnect

The H-twisted differential has a decomposition into 4 components of degree ±1 with respect to the bigrading:

$$\begin{tikzcd}
U_{p-1,q+1} & & U_{p+1,q+1} \\
& U_{p,q} \arrow[ul, "\overline{\delta}_-"'] \arrow[ur, "\delta_+"] \arrow[dl, "\overline{\delta}_+"] \arrow[dr, "\delta_-"'] & \\
U_{p-1,q-1} & & U_{p+1,q-1}
\end{tikzcd} \tag{7.4}$$

Because $\delta_\pm^2 = 0 = \overline{\delta}_\pm^2$, various homolomorphic structures can be derived from this diagram. For convenience, we make the following definitions

$$\begin{aligned}U_+ &:= \bigoplus_{p+q=n} U_{p,q}\\ U_- &:= \bigoplus_{p-q=n} U_{p,q}\end{aligned}$$

so that U_- and U_+ are the top and bottom right-most diagonals of the spinor diamond respectively. Equivalently,

$$U_\pm = \{\phi \in \Omega_M \mid \ell_\pm \cdot \phi = 0\}.$$

For these bundles, the $\overline{\delta}_\pm$ operator endows $U_{p,q}$ with an $I_\pm$-holomorphic structure since $U_{p+1,q\pm1} = \overline{\ell}_\pm \otimes U_{p,q} = \Omega^{0,1}_\pm(U_{p,q})$. We may express this $\ell_\pm$-module structure more explicitly by saying that the differentiation of ϕ in the direction of $u \in \ell_\pm$ is given by $u\overline{\delta}_\pm\phi$.

The bundles $V_\mp$ also have flat $\ell_\pm$-connections, coming from the Courant bracket:

$$\overline{\partial}_u v = [\![u, v]\!]_\mp$$

for $u \in \ell_\pm$ and $v \in V_\mp$. Now, we claim that $Cl(V_\mp)$ acts $I_\pm$-holomorphically on $U_\pm$. This follows the derived bracket relation for $\phi \in U_\pm$:

$$\begin{aligned} [\![u, v]\!]_\mp \cdot \phi &= [\![u, v]\!] \cdot \phi \\ &= [[d^H, u], v]\phi \\ &= ud^H v\phi - vud^H\phi \\ &= u\overline{\delta}_\pm v\phi - vu\overline{\delta}_\pm\phi \end{aligned}$$

The above discussion demonstrates, in particular, that K_A is $I_\pm$-holomorphic and K_B is $\pm I_\pm$-holomorphic. Moreover $K_\mp$ is $I_\pm$-holomorphic and the relations in 7.2 preserve these holomorphic structures:

{spinorsummary}

Proposition 18. The Clifford multiplication gives isomorphisms of Hermitian vector bundles

$$K_A = K_- \otimes K_B \qquad\qquad K_A = K_+ \otimes \overline{K}_B$$

which are isomorphisms of I_+-holomorphic bundles and I_--holomorphic bundles respectively.

As K_A is holomorphic with respect to both the I_+ and I_- complex structures, and has a Hermitian metric, it inherits corresponding Chern connections $\nabla^\pm_A$ with curvatures $i\rho^\pm_A$. Similarly, using the $\pm I_\pm$ holomorphic structures we get Chern connections $\nabla^\pm_B$ on K_B with curvatures $i\rho^\pm_B$. Similarly, K_+ and K_- each have a pair of Chern connections, coming from the $I_\pm$-holomorphic structures on each. These come from what are, somewhat confusingly, known as the Chern and Bismut connections on $T_{1,0}X_\pm$.

Proposition 18 has the following corollary.

{cor:chernbismutr

Corollary 4. The curvatures of K_A, K_B and $K_\pm$ are related in the following ways:

$$\rho^+_A = \rho^+_B - \rho_{Bis-} \qquad\qquad \rho^-_A = -\rho^-_B - \rho_{Bis+}$$

Proof. Since the curvature of the $I_\pm$-Chern connection on $K_\mp$ is $-i$ times the Bismut-Ricci form $-i\rho_{Bis\mp}$, this follows directly from proposition 18. □

Remark 6. Though these Chern connections have not previously been identified in the literature, using the Poincaré-Lelong formula these relations have been observed in the case where K_A and K_B have global d_H-closed pure spinors [2, 3].

7.6 Generalized Chern connections of canonical line bundles

In this section we compute the generalized connections of the canonical line bundles of $\mathbb{J}_A$ and $\mathbb{J}_B$. This has a number of useful corollaries, in particular giving a relation between the classical curvatures described in section 7.5.

The holomorphic structures on K_A and K_B combine to give flat L_A and L_B connections respectively and as a result there are a pair of generalized Chern connections $\mathbb{D}_A$ and $\mathbb{D}_B$. In the following proposition we explain the relationship between these generalized Chern curvatures and their classical counterparts in section 7.5.

{thm:cherncomp}

Theorem 6. The generalized Chern connections $\mathbb{D}_A$ and $\mathbb{D}_B$ have decompositions

$$\mathbb{D}_A = (\nabla_A - i\iota_{X_A} b) - iX_A \qquad\qquad \mathbb{D}_B = (\nabla_B - i\iota_{X_B} b) - iX_B$$

where

$$\nabla_A = \frac{1}{2}\left(\nabla_A^+ + \nabla_A^-\right) \qquad\qquad \nabla_B = \frac{1}{2}\left(\nabla_B^+ + \nabla_B^-\right)$$
$$g(X_A) = \frac{1}{2i}\left(\nabla_A^- - \nabla_A^+\right) \qquad\qquad g(X_B) = \frac{1}{2i}\left(\nabla_B^- - \nabla_B^+\right)$$

Proof. We will perform the calculation for $\mathbb{D}^A$, leaving the calculation for $\mathbb{D}^B$ to the reader. Note that for any vector field X and 1-form,

$$\begin{aligned}(\nabla_A)_X &= (\mathbb{D}_A)_{X+bX} \\ &= \frac{1}{2}\left((\mathbb{D}_A)_{X+(b+g)X} + (\mathbb{D}_A)_{X+(b-g)X}\right)\end{aligned} \qquad \begin{aligned}g(X_A, X) &= i(\mathbb{D}_A)_{gX} \\ &= \frac{1}{2i}\left((\mathbb{D}_A)_{X+(b-g)X} - (\mathbb{D}_A)_{X+(b+g)X}\right)\end{aligned}$$

It suffices then, to show that $(\nabla_A^\pm)_X = (\mathbb{D}_A)_{X+(b\pm g)X}$. To do so we first observe that

$$(\mathbb{D}_A)_{X+(b\pm g)X}(f\phi) = (X \cdot f)\phi + f(\mathbb{D}_A)_{X+(b\pm g)X}\phi$$

so that the expression $(\widetilde{\nabla}_A^\pm)_X \coloneqq \mathbb{D}^A_{X+(b\pm g)X}$ defines a pair of connections. Moreover by lemma 4, $\ell_\pm = \{X + (b \pm g)X \mid X \in T_{0,1}X_\pm\}$ so for $X \in T_{0,1}X_\pm$, we have

$$(\mathbb{D}_A)_{X+(b\pm g)X}\phi = (X + (b \pm g)X) \cdot d^H\phi = \left(\overline{\partial}_\pm\right)_X \phi$$

so that $\widetilde{\nabla}_A^\pm$ is compatible with the holomorphic structure $\overline{\partial}_\pm$ on K_A. Since $\widetilde{\nabla}_A^\pm$ preserves the Hermitian metric, it must coincide with the Chern connection $\nabla_A^\pm$. ☐

As an immediate corollary, we can compute the curvatures of $\mathbb{D}_A$ and $\mathbb{D}_B$, using 5.1.

{cor:cherncomp}

Corollary 5. The curvatures iR_A and iR_B of $\mathbb{D}^A$ and $\mathbb{D}^B$ are given by

$$R_A = (X_A, \frac{1}{2}(\rho_A^+ + \rho_A^-) - d\iota_{X_A} b + \iota_{X_A} H) \qquad R_B = (X_B, \frac{1}{2}(\rho_B^+ + \rho_B^-) - d\iota_{X_B} b + \iota_{X_B} H)$$

By theorem 5, we know that the vector components of these Chern connections are proportional to the modular vector fields of π_A and π_B. More precisely, since the Hermitian metric on K_A (or

K_B) is given by

$$h(\phi,\psi)=\frac{(*\phi,\overline{\psi})}{\mathrm{vol}_g}=i^n\frac{(\phi,\overline{\psi})}{\mathrm{vol}_g}$$

we must compute the modular vector field with respect to vol_g.

{lem:gkmodularvf}

Lemma 5. With respect to the Riemannian volume form vol_g the modular vector fields X_{π_A} and X_{π_B}, of $\pi_A=\frac{1}{2}g^{-1}(I_+-I_-)^*$ and $\pi_B=\frac{1}{2}g^{-1}(I_++I_-)^*$ are given respectively by

$$g(X_{\pi_A})=\frac{1}{2}d^*(\omega_+-\omega_-)\qquad\qquad g(X_{\pi_B})=\frac{1}{2}d^*(\omega_++\omega_-)$$

Equivalently,

$$X_{\pi_A}=-\frac{1}{2}(I_+\theta_+^\sharp-I_-\theta_-^\sharp)\qquad\qquad X_{\pi_B}=-\frac{1}{2}(I_+\theta_+^\sharp+I_-\theta_-^\sharp)$$

where $\theta_\pm=-I_\pm^*d^*\omega_\pm$ are the Lee forms.

Proof. We will calculate the modular vector field for π_A, leaving the calculation for π_B to the reader. Using eq. (2.4), we compute

$$\begin{aligned}
\mathcal{L}_{\pi_A(df)}\,\mathrm{vol}_g &= \frac{1}{2}d\left(\iota_{g^{-1}(I_+-I_-)^*df}\star 1\right)\\
&=\frac{1}{2}d(\star(I_+-I_-)^*df)\\
&=-\frac{1}{2}d\left(\star\iota_{g^{-1}df}(\omega_+-\omega_-)\right)\\
&=\frac{1}{2}d\left(df\wedge\star(\omega_+-\omega_-)\right)\\
&=\frac{1}{2}df\wedge\star d^*(\omega_+-\omega_-)\\
&=\frac{1}{2}\langle df,d^*(\omega_+-\omega_-)\rangle\,\mathrm{vol}_g\\
&=\frac{1}{2}\left(\mathcal{L}_{g^{-1}d^*(\omega_+-\omega_-)}f\right)\mathrm{vol}_g
\end{aligned}$$

completing the proof. □

In conjunction with theorem 5, lemma 5 allows us to compute X_A and X_B.

{prop:gkgchernvec

Proposition 19. The vector components of $\mathbb{D}_A$ and $\mathbb{D}_B$ are given by

$$\begin{aligned}
X_A&=\frac{1}{4}g^{-1}d^*\left(\omega_+-\omega_-\right) & X_B&=\frac{1}{4}g^{-1}d^*\left(\omega_++\omega_-\right)\\
&=-\frac{1}{4}\left(I_+\theta_+^\sharp-I_-\theta_-^\sharp\right) & &=-\frac{1}{4}\left(I_+\theta_+^\sharp+I_-\theta_-^\sharp\right)
\end{aligned}$$

The first corollary of the above discussion is that we have a complete characterization of the curvatures $\rho_A^\pm$ and $\rho_B^\pm$ defined in section 7.5 in terms of bi-Hermitian data.

{cor:chernforms}

Corollary 6. The classical Chern curvatures of K_A and K_B are given by

$$\rho_A^+ = -\frac{1}{2}(\rho_{Bis+} + \rho_{Bis-} + dd^*\omega_+) \qquad \rho_B^+ = -\frac{1}{2}(\rho_{Bis+} - \rho_{Bis-} + dd^*\omega_+) \tag{7.5}$$

$$\rho_A^- = -\frac{1}{2}(\rho_{Bis+} + \rho_{Bis-} + dd^*\omega_-) \qquad \rho_B^- = -\frac{1}{2}(\rho_{Bis+} - \rho_{Bis-} - dd^*\omega_-) \tag{7.6}$$

Proof. By theorem 6, the difference between ∇_A^+ and ∇_A^- is $-2ig(X_A)$. On the other hand, by theorem 5 and lemma 5, $g(X_A) = \frac{1}{4}d^*(\omega_+ - \omega_-)$ and $g(X_B) = \frac{1}{4}d^*(\omega_+ + \omega_-)$. Thus we arrive at the relations

$$\rho_A^+ = \rho_A^- - \frac{1}{2}dd^*(\omega_+ - \omega_-) \qquad \rho_B^+ = \rho_B^- - \frac{1}{2}dd^*(\omega_+ + \omega_-).$$

Combining these with corollary 4 allows us to extract the desired expressions for $\rho_A^\pm$ and $\rho_B^\pm$. □

Corollary 7. With respect to the canonical gauge (where $b = 0$), the curvatures iR_A and iR_B of $\mathbb{D}_A$ and $\mathbb{D}_B$ are given by

$$\begin{aligned} R_A &= \frac{1}{4}\left(g^{-1}d^*(\omega_+ - \omega_-), -2(\rho_{Bis+} + \rho_{Bis-}) - dd^*(\omega_+ + \omega_-) - d\iota_{X_A}b + \iota_{g^{-1}d^*(\omega_+ - \omega_-)}H\right) \\ R_B &= \frac{1}{4}\left(g^{-1}d^*(\omega_+ + \omega_-), -2(\rho_{Bis+} - \rho_{Bis-}) - dd^*(\omega_+ - \omega_-) - d\iota_{X_B}b + \iota_{g^{-1}d^*(\omega_+ + \omega_-)}H\right) \end{aligned} \tag{7.7}$$

{eq:RARB}

Proof. This follows directly from corollary 5 by substituting the results of proposition 19 and corollary 6. □

The expressions for $R_A \pm R_B$ take a slightly simpler form:

$$\begin{aligned} R_A + R_B &= \frac{1}{2}(-I_+\theta_+^\sharp,\ -2\rho_{Bis+} - dd^*\omega_+ - \iota_{I_+\theta_+^\sharp}H) \\ R_A - R_B &= \frac{1}{2}(I_-\theta_-^\sharp, -2\rho_{Bis-} - dd^*\omega_- + \iota_{I_-\theta_-^\sharp}H) \end{aligned}$$

From here, using the fact that $R_{A/B}$ is a symmetry of $\mathbb{J}_{A/B}$ we can derive a number of useful formulae. For example:

$$\begin{aligned} &\mathcal{L}_{I_+\theta_+^\sharp}I_+ - \mathcal{L}_{I_-\theta_-^\sharp}I_- = I_+g^{-1}(2\rho_{Bis+} + dd^*\omega_+ + \iota_{I_+\theta_+^\sharp}H) - I_-g^{-1}(2\rho_{Bis-} + dd^*\omega_- - \iota_{I_-\theta_-^\sharp}H) \\ &\mathcal{L}_{I_+\theta_+^\sharp}\omega_+^{-1} + \mathcal{L}_{I_-\theta_-^\sharp}\omega_-^{-1} = 0 \end{aligned}$$

Example 8. Generalized scalar curvature and symplectic type generalized Kähler. Suppose that $\mathbb{J}_A$ is of symplectic type, that is,

$$\mathbb{J}_A = e^b \begin{pmatrix} & F^{-1} \\ -F & \end{pmatrix} e^{-b}$$

for some 2-form b and symplectic form F. Then by comparing with the bi-Hermitian data, we find that $I_+ - I_-$ must be invertible and $F = -2g(I_+ - I_-)^{-1}$. We also find that $b = \frac{1}{2}F(I_+ - I_-) = -g(I_+ - I_-)^{-1}(I_+ + I_-)$.

Chapter 8

Holomorphic Poisson geometry

{chpt:holomorphic

8.1 Manin triples and generalized Kähler geometry

We may associate to any generalized Kähler structure $(g, b, I\pm)$ a pair of complex Courant algebroids $(\mathbb{T}M \otimes \mathbb{C}, [\![\cdot,\cdot]\!]_{\mathcal{H}_\pm})$ where

$$\mathcal{H}_\pm = \pm 2i\partial_\pm \omega_\pm - 2db_\pm^{2,0}$$

which are gauge equivalent to the H-twisted Courant algebroid via the following diagram of equivalences:

$$\begin{tikzcd}
\overline{\mathcal{H}}_- & & \mathcal{H}_+ \\
& H \arrow[ul, "i\omega_-"'] \arrow[ur, "b_+^{1,1} - b_+^{2,0} + b_+^{0,2} + i\omega_+"'] \arrow[dl, "-i\omega_+"'] \arrow[dr, "-i\omega_-"] & \\
\overline{\mathcal{H}}_+ & & \mathcal{H}_-
\end{tikzcd}$$

In fact, since $\mathcal{H}_\pm$ are of type $(2,1)$ with respect to $I_\pm$, they determine holomorphic Courant algebroids in the following way. On the vector bundle $\mathbb{T}_{1,0}X_\pm \coloneqq T_{1,0}X_\pm \oplus T_{1,0}^*X_\pm$ we define a Dolbeault operator:

$$\overline{D}_X = \begin{pmatrix} \overline{\partial} & \\ \mathcal{H} & \overline{\partial} \end{pmatrix},$$

and a Courant bracket on smooth sections of $\mathbb{T}_{1,0}X_\pm$

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + [\partial, \iota_X]\eta - \iota_Y \partial\xi$$

which descends to $\overline{D}$-holomorphic sections. We will frequently make use of the relations

$$\ell_\pm = e^{\pm i\omega_\pm} T_{0,1}X_\pm \tag{8.1}$$ {{eq:ellpm}}

which demonstrate how to recover the complex structures $I_\pm$ from the generalized complex structures $L_A = \ell_+ \oplus \ell_-$ and $L_B = \ell_+ \oplus \overline{\ell}_-$. In particular, we consider the Dirac structures

$$A_+ := e^{-i\omega_+} L_A \qquad\qquad A_- := e^{i\omega_-} L_A \tag{8.2}$$

{{eq:gaugeequiv}}

$$B_+ := e^{-i\omega_+} L_B \qquad\qquad B_- := e^{i\omega_-} \overline{L}_B \tag{8.3}$$

Notice, for example, that by eq. (8.1),

$$\begin{aligned} A_+ &= T_{0,1}X_+ \oplus e^{-i(\omega_+ + \omega_-)} T_{0,1}X_- \\ &= T_{0,1}X_+ \oplus (\mathrm{id} - 2gP^+_{0,1}) T_{0,1}X_- \\ &= T_{0,1}X_+ \oplus P^+_{1,0}(\mathrm{id} - 2g) T_{0,1}X_- \end{aligned}$$

so that $A_+ = (A_+ \cap \mathbb{T}_{1,0}X_+) \oplus (A_+ \cap \mathbb{T}_{0,1}X_+)$. This suffices to show that A_+ is the matched pair of a holomorphic Dirac structure $\mathcal{A}_+ \subset \mathcal{E}_+$. Similarly, A_- and $B_\pm$ are matched pairs of holomorphic Dirac structures:

$$\begin{aligned} \mathcal{A}_+ &= \mathbb{T}_{1,0}X_+ \cap A_+ & \qquad \mathcal{A}_- &= \mathbb{T}_{1,0}X_- \cap A_- \\ \mathcal{B}_+ &= \mathbb{T}_{1,0}X_+ \cap B_+ & \qquad \mathcal{B}_- &= \mathbb{T}_{1,0}X_- \cap B_- \end{aligned}$$

and the above calculation (and its counterparts) also show that

$$\begin{aligned} \mathcal{A}_+ &= P^+_{1,0}(\mathrm{id} - 2g) T_{0,1}X_- & \qquad \mathcal{A}_- &= P^-_{1,0}(\mathrm{id} + 2g) T_{0,1}X_+ \\ \mathcal{B}_+ &= P^+_{1,0}(\mathrm{id} - 2g) T_{1,0}X_- & \qquad \mathcal{B}_- &= P^-_{1,0}(\mathrm{id} + 2g) T_{1,0}X_+ \end{aligned}$$

From these equations, it is clear that $\mathcal{A}_+$ and $\mathcal{B}_+$ are complementary and thus make up a Manin triple $\mathcal{E}_+ = \mathcal{A}_+ \oplus \mathcal{B}_+$. Similarly, we have a Manin triple $\mathcal{E}_- = \mathcal{A}_- \oplus \mathcal{B}_-$.

The imaginary parts of these Manin triples are given by

$$\mathrm{Im}(\mathcal{E}_\pm, \mathcal{A}_\pm, \mathcal{B}_\pm) = e^{\mp\omega_\pm}(\mathbb{T}M, \Gamma_{\pi_A}, \Gamma_{\pm\pi_B})$$

$$L_{\sigma_-} = \frac{1}{2i}(\overline{L}_A - L_B) = \frac{1}{2i}(e^{i\omega_+}A_+ - e^{-i\omega_+}\overline{B}_+) = e^{\omega_+}\frac{1}{2i}(A_+ - \overline{B}_+)$$

In this chapter we examine the presence of holomorphic Poisson modules in generalized Kähler geometry. There are many natural examples, coming from the various canonical bundles. Using their generalized Chern connections we produce generalized Chern curvatures and give characterizations of these in terms of classical, bi-Hermitian quantities. As a result we are able to describe the generalized Kähler-Ricci flow as the flow by one of these generalized Chern curvatures.

8.2 Holomorphic Poisson modules on generalized Kähler manifolds

An important insight of Hitchin [27] is that on a generalized Kähler manifold, each of the complex manifolds $X_\pm = (M, I_\pm)$ is equipped with a holomorphic Poisson structure $\sigma_\pm = -\frac{1}{4}(I_\pm Q + iQ)$

where $Q = \frac{1}{2}[I_+, I_-]g^{-1}$. An elegant reformulation of these was given by Gualtieri [24] using the Dirac calculus:

$$L_{\sigma_+} = \frac{1}{2i}\left(L_A - L_B\right) \qquad\qquad L_{\sigma_-} = \frac{1}{2i}\left(L_A - \overline{L}_B\right)$$

where $L_{\sigma_\pm} = \Gamma_{\sigma_\pm} \oplus T_{0,1}X_\pm$ is the i-eigenbundle of the generalized complex structure

$$\mathbb{J}_{\sigma_\pm} = \begin{pmatrix} -I_\pm & Q \\ & I_\pm^* \end{pmatrix}.$$

Thus, by definition, there are pullback Lie algebroid diagrams

$$\begin{tikzcd}
& L_{\sigma_+} \arrow[dl, "\Phi_A^+"'] \arrow[dr, "\Phi_B^+"] & & & L_{\sigma_-} \arrow[dl, "\Phi_A^-"'] \arrow[dr, "\Phi_B^-"] & \\
L_A \arrow[dr] & & L_B \arrow[dl] & L_A \arrow[dr] & & \overline{L}_B \arrow[dl] \\
& T_{\mathbb{C}}M & & & T_{\mathbb{C}}M &
\end{tikzcd} \tag{8.2.1}$$

{diagram:poissonc

whose defining feature is that $\Phi_A^\pm - \Phi_B^\pm$ is $2i$ times the projection onto $T_{\mathbb{C}}^*M$. As a result, any L_A-module inherits flat Lie algebroid connections for $L_{\sigma_\pm}$ while any L_B-module inherits a flat connections for L_{σ_+} and $\overline{L}_{\sigma_-}$. In this section we give explicit formulae for these morphisms, and use this to relate resulting Chern connections in the presence of a Hermitian metric.

We begin by observing that the generalized complex structures $\mathbb{J}_{\sigma_\pm}$ share a common Poisson tensor $Q = -4\operatorname{Im}\sigma_\pm$, since

$$\frac{1}{2i}\left(L_{\sigma_+} - \overline{L}_{\sigma_+}\right) = -\frac{1}{4}\left(L_A - L_B - \overline{L}_B + \overline{L}_A\right) = \frac{1}{2i}\left(L_{\sigma_-} - \overline{L}_{\sigma_-}\right).$$

In terms of the bi-Hermitian data, this tensor is expressed as $Q = \frac{1}{2}[I_+, I_-]g^{-1}$ (see []).

Theorem 7. The maps $\Phi_A^\pm : \mathbb{T}M \to \mathbb{T}M$ and $\Phi_B^\pm : \mathbb{T}M \to \mathbb{T}M$ defined by

$$\begin{aligned} \Phi_A^+ &= \begin{pmatrix} \mathrm{id} & \\ g & I_+^* + I_-^* \end{pmatrix} & \Phi_A^- &= \begin{pmatrix} \mathrm{id} & \\ -g & I_+^* + I_-^* \end{pmatrix} \\ \Phi_B^+ &= \begin{pmatrix} \mathrm{id} & \\ g & -I_+^* + I_-^* \end{pmatrix} & \Phi_B^- &= \begin{pmatrix} \mathrm{id} & \\ -g & I_+^* - I_-^* \end{pmatrix} \end{aligned} \tag{8.4}$$

{{eq:realmorph}}

restrict to the morphisms in diagram diagram 8.2.1.

Proof. We will exhibit the proof for the diagram involving σ_+ leaving the other diagram to the reader. First, a straightforward calculation shows that Φ_A^+ and Φ_B^+ intertwine generalized complex structures in the following way:

$$\mathbb{J}_A\Phi_A^+ = \Phi_A^+\mathbb{J}_{\sigma_+} \qquad\qquad \mathbb{J}_A\Phi_A^- = \Phi_B^-\mathbb{J}_{\sigma_-} \tag{8.5}$$

$$\mathbb{J}_B\Phi_B^+ = \Phi_B^+\mathbb{J}_{\sigma_+} \qquad\qquad \mathbb{J}_B\Phi_B^- = -\Phi_B^-\mathbb{J}_{\sigma_-} \tag{8.6}$$

Thus, Φ_A^+ and Φ_B^+ restrict to morphisms $L_{\sigma_+} \to L_A$ and $L_{\sigma_+} \to L_B$. Moreover, it is clear that the

Figure 8.1: Generalized Chern connections on canonical bundles

{fig:GChern}

	L_A	L_B	L_{σ_+}	L_{σ_-}	$\overline{L}_{\sigma_-}$
K_A	$\mathbb{D}_A$		$\mathbb{D}_A^+$	$\mathbb{D}_A^-$	
K_B		$\mathbb{D}_B$	$\mathbb{D}_B^+$		$\mathbb{D}_B^-$
K_+			$\mathbb{D}_+^+$	$\mathbb{D}_+^-$	
K_-			$\mathbb{D}_-^+$	$\mathbb{D}_-^-$	

{fig:my-tikz-char

difference $\Phi_A^+ - \Phi_B^+$ restricted to L_{σ_+} is 2i times the projection onto $T^*_{\mathbb{C}}M$, and that $\mathsf{a} \circ \Phi_A^+ = \mathsf{a} = \mathsf{a} \circ \Phi_B^+$. □

Remark 7. It is also possible to write down explicit expressions for the maps in diagram 8.2.1. However, since these expressions are lengthy and not particularly informative, we omit them. As we will see, it is their real parts given by eq. (8.4) that we will use in the following section.

8.3 Generalized Chern curvatures of holomorphic Poisson modules

In this section we apply the results of the previous section to the canonical bundles K_A and K_B. Since these are L_A and L_B-modules respectively, they each inherit a triple of generalized Chern connections, as described in fig. 8.1. Moreover, using eq. (7.2), the canonical bundles K_+ and K_- each obtain a pair of generalized Chern connections.

We begin with the following observation about Poisson modules on a generalized Kähler manifold. If V is an L_A-module with Hermitian metric h, then it is equipped with a generalized Chern connection $\mathbb{D}_A$. On the other hand, as a result of diagram 8.2.1, it is also an $L_{\sigma_\pm}$-module and thus has two more generalized Chern connections $\mathbb{D}_A^\pm$. Similarly, an L_B-module with Hermitian metric h has a generalized Chern connection $\mathbb{D}_B$ coming from the flat L_B-connection, but also a pair of generalized Chern connections compatible with L_{σ_+} and L_{σ_-} connections.

{cor:gcrels}

Lemma 6. The Chern connections $\mathbb{D}_A^\pm$ and $\mathbb{D}_B^\pm$ may be recovered from $\mathbb{D}_A$ and $\mathbb{D}_B$ in the following way:

$$\begin{aligned} \left(\mathbb{D}_A^+\right)_{X+\xi} &= (\mathbb{D}_A)_{\Phi_A^+(X+\xi)} & \left(\mathbb{D}_A^-\right)_{X+\xi} &= (\mathbb{D}_A)_{\Phi_A^-(X+\xi)} \\ \left(\mathbb{D}_B^+\right)_{X+\xi} &= (\mathbb{D}_B)_{\Phi_B^+(X+\xi)} & \left(\mathbb{D}_B^-\right)_{X+\xi} &= (\mathbb{D}_B)_{\Phi_B^-(X+\xi)} \,. \end{aligned}$$

In particular, if $\mathbb{D}_A = \nabla_A - i\mathcal{X}_A$ and $\mathbb{D}_B = \nabla_B - i\mathcal{X}_B$, then

$$\mathbb{D}_A^\pm = \nabla_A^\pm - i(I_+ + I_-)\mathcal{X}_A \qquad \mathbb{D}_B^\pm = \nabla_B^\pm \pm i(I_+ - I_-)\mathcal{X}_B \tag{8.7}$$

where $\left(\nabla_A^\pm\right)_X = (\mathbb{D}_A)_{X \pm gX}$ and $\left(\nabla_B^\pm\right)_X = (\mathbb{D}_B)_{X \pm gX}$.

We will decorate Chern connections and curvatures in the same way, so that, for example, the curvature of $\mathbb{D}_B^-$ on K_B is R_B^-. As an immediate application of **??** and **??**, we can compute the connection and vector components of $\mathbb{D}_A^\pm$ and $\mathbb{D}_B^\pm$:

Theorem 8. If $\mathbb{D}^{\pm}_A = \nabla^{\pm}_A - i\mathcal{X}^{\pm}_A$ and $\mathbb{D}^{\pm}_B = \nabla^{\pm}_B - i\mathcal{X}^{\pm}_B$ then $\nabla^{\pm}_A$ and $\nabla^{\pm}_B$ are the $I_{\pm}$-Chern connections on K_A and K_B respectively.

Since $K_A = K_- \otimes K_B$ and both K_A and K_B are σ_+-modules, it follows that there is a σ_+-module on K_- and a corresponding generalized Chern connection $\mathbb{D}^+_-$. The curvature iR^+_- of $\mathbb{D}^+_-$ is then an imaginary valued symmetry of $\mathbb{J}_{\sigma_+}$. Similarly, the relation $K_A = K_+ \otimes \overline{K}_B$ gives a σ_--module structure on K_+ and a generalized Chern connection $\mathbb{D}^-_+$ whose curvature iR^-_+ is a symmetry of $\mathbb{J}_{\sigma_-}$. The cumulative result of our calculations gives expression for these curvatures in terms of the Bi-Hermitian data.

Theorem 9. With respect to the generalized complex structures $\mathbb{J}_{\sigma_\pm}$, the generalized Chern curvatures of the line bundles K_A and K_B are given by

$$R^{\pm}_A = \left(\ \frac{1}{4}(\theta^{\sharp}_+ - \theta^{\sharp}_- + I_+I_-\theta^{\sharp}_- - I_-I_+\theta^{\sharp}_+), \ \rho^{\pm}_A \right) \tag{8.8}$$

$$R^{\pm}_B = \left(\mp\frac{1}{4}(\theta^{\sharp}_+ - \theta^{\sharp}_- - I_+I_-\theta^{\sharp}_- + I_-I_+\theta^{\sharp}_+), \ \rho^{\pm}_B \right) \tag{8.9}$$

while the generalized Chern curvatures of the line bundles K_+ and K_- are given by

$$R^+_+ = \left(\frac{1}{2}(I_+I_-\theta^{\sharp}_- - I_-I_+\theta^{\sharp}_+), \ -\rho_{Bis+} - dd^*\omega_+ \right) \quad R^+_- = \left(\frac{1}{2}(\theta_+ - \theta_-)^{\sharp}, \ -\rho_{Bis-} \right) \tag{8.10}$$

$$R^-_+ = \left(\frac{1}{2}(\theta_+ - \theta_-)^{\sharp}, \ -\rho_{Bis+} \right) \quad R^-_- = \left(\frac{1}{2}(I_+I_-\theta^{\sharp}_- - I_-I_+\theta^{\sharp}_+), \ -\rho_{Bis-} - dd^*\omega_- \right) \tag{8.11}$$

Proof. To compute $R^{\pm}_A$ and $R^{\pm}_B$ we apply lemma 6 to our computation of R_A and R_B in **????**. For example, we find that the vector component of $R^{\pm}_A$ is given by

$$\mathcal{X}^{\pm}_A = (I_+ + I_-)\mathcal{X}_A = -\frac{1}{4}(I_+ + I_-)(I_+\theta^{\sharp}_+ - I_-\theta^{\sharp}_-) = \frac{1}{4}(\theta^{\sharp}_+ - \theta^{\sharp}_- + I_+I_-\theta^{\sharp}_- - I_-I_+\theta^{\sharp}_+)$$

while the 2-form component is given by the curvature of $\nabla^{\pm}_A$. To compute $R^{\mp}_{\pm}$ and $R^{\pm}_{\pm}$ we use the relations:

$$\begin{aligned} R^+_+ &= R^+_A + R^+_B & \qquad R^+_- &= R^+_A - R^+_B \\ R^-_+ &= R^-_A + R^-_B & \qquad .R^-_- &= R^-_A - R^-_B \end{aligned}$$

which follow from eq. (7.2). The 2-form component is then computed using corollary 6. □

Remark 8. Since these are generalized Chern curvatures for either $\mathbb{J}_{\sigma_+}$ or $\mathbb{J}_{\sigma_-}$, they are symmetries of these generalized complex structures. This gives rise to many useful equations. For example, the fact that $R^{\mp}_{\pm} = (\frac{1}{2}(\theta^{\sharp}_+ - \theta^{\sharp}_-), -\rho_{Bis\pm})$ is a symmetry of $\mathbb{J}_{\sigma_\mp}$ is equivalent to the set of equations:

$$\begin{aligned} \rho_{Bis\pm}I_{\mp} + I^*_{\mp}\rho_{Bis\pm} &= 0 \\ \mathcal{L}_{\frac{1}{2}(\theta^{\sharp}_+ - \theta^{\sharp}_-)}I_{\mp} &= Q\rho_{Bis\pm} \\ \mathcal{L}_{\frac{1}{2}(\theta^{\sharp}_+ - \theta^{\sharp}_-)}Q &= 0 \end{aligned} \tag{8.12}$$

`{{eq:holpoissymm1`

where $Q = \frac{1}{2}[I_+, I_-]g^{-1}$. The first of these says that $\rho_{Bis\pm}$ is of type $(1,1)$ with respect to $I_\mp$, which also follows from the fact that it is the curvature of a Chern connection with respect to an $I_\mp$-holomorphic structure. Similarly, the equation $[R^\pm_\pm, \mathbb{J}_{\sigma_\pm}] = 0$ is equivalent to the set of equations:

$$\begin{aligned}(\rho_{Bis\pm} + dd^*\omega_\pm)I_\pm + I^*_\pm(\rho_{Bis\pm} + dd^*\omega_\pm) = 0\\ \mathcal{L}_{\frac{1}{2}(I_+I_-\theta^\sharp_- - I_-I_+\theta^\sharp_+)}I_\pm = Q(\rho_{Bis\pm} + dd^*\omega_\pm)\\ \mathcal{L}_{\frac{1}{2}(I_+I_-\theta^\sharp_- - I_-I_+\theta^\sharp_+)}Q = 0\end{aligned} \tag{8.13}$$

Lemma 7. [17] Let (M, g, I) be a pluriclosed Hermitian manifold. The $(1,1)$-component of the Ricci-Bismut 2-form is determined by {lem:pluriclosedb

$$I^*(\rho_{Bis})^{1,1} = \mathrm{Rc} - \frac{1}{4}H^2 + \frac{1}{2}\mathcal{L}_{\theta^\sharp}g$$

where Rc is the Ricci tensor, $H^2(X,Y) = \langle \iota_X H, \iota_Y H\rangle$ and $\theta = -d^*\omega \circ I$ is the Lee form.

Applying this lemma to the two pluriclosed structures associated to a generalized Kähler manifold, we find

$$\begin{aligned}\mathcal{L}_{\frac{1}{2}(\theta^\sharp_+ - \theta^\sharp_-)}g &= I^*_+\left(\rho_{Bis+}\right)^{1,1}_+ - I^*_-\left(\rho_{Bis-}\right)^{1,1}_-\\ &= \frac{1}{2}\left(I^*_+\rho_{Bis+} - \rho_{Bis+}I_+\right) - \frac{1}{2}\left(I^*_-\rho_{Bis-} - \rho_{Bis-}I_-\right)\\ &= \frac{1}{2}(I^*_+ - I^*_-)(\rho_{Bis-} + \rho_{Bis+}) + \frac{1}{2}(\rho_{Bis-} - \rho_{Bis+})(I_+ + I_-)\end{aligned} \tag{8.14}$$ {{eq:gliederiv}}

{prop:pisymm}

Proposition 20. The derivations $\left(\frac{1}{2}(\theta_+ - \theta_-), -\rho_{Bis+} - \rho_{Bis-}\right)$ and $\left(\frac{1}{2}(\theta_+ - \theta_-), -\rho_{Bis+} + \rho_{Bis-}\right)$ are symmetries of Γ_{π_A} and Γ_{π_B} respectively. Equivalently,

$$\mathcal{L}_{\frac{1}{2}(\theta^\sharp_+ - \theta^\sharp_-)}\pi_A + \pi_A(\rho_{Bis+} + \rho_{Bis-})\pi_A = 0 \qquad \mathcal{L}_{\frac{1}{2}(\theta^\sharp_+ - \theta^\sharp_-)}\pi_B + \pi_B(\rho_{Bis+} - \rho_{Bis-})\pi_B = 0.$$

Proof. We compute, using eq. (8.14) and eq. (8.12)

$$\begin{aligned}\mathcal{L}_{\frac{1}{2}(\theta^\sharp_+ - \theta^\sharp_-)}\pi_A &= \frac{1}{2}\left(\mathcal{L}_{\frac{1}{2}(\theta^\sharp_- - \theta^\sharp_+)}(I_+ - I_-)\right)g^{-1} + \frac{1}{2}(I_+ - I_-)g^{-1}\left(\mathcal{L}_{\frac{1}{2}(\theta^\sharp_- - \theta^\sharp_+)}g\right)g^{-1}\\ &= \frac{1}{2}Q(\rho_{Bis+} - \rho_{Bis-})g^{-1} - \pi_A((\rho_{Bis-} - \rho_{Bis+})\pi_B g - g\pi_A(\rho_{Bis-} + \rho_{Bis+}))g^{-1}\\ &= \pi_A\left(g\pi_B(\rho_{Bis+} - \rho_{Bis-}) + (\rho_{Bis-} - \rho_{Bis+})\pi_B g - g\pi_A(\rho_{Bis-} + \rho_{Bis+})\right)g^{-1}\\ &= \pi_A\left(\frac{1}{2}(I^*_+ + I^*_-)(\rho_{Bis+} - \rho_{Bis-}) - \frac{1}{2}(\rho_{Bis-} - \rho_{Bis+})(I_+ + I_-) - \frac{1}{2}(I^*_+ - I^*_-)(\rho_{Bis-} + \rho_{Bis+})\right)g^{-1}\\ &= -\pi_A\left(\rho_{Bis+} + \rho_{Bis-}\right)\pi_A\end{aligned}$$

The proof of the other equation is similar. □

8.4 Generalized Kähler-Ricci flow

{sec:gkrf}

As we have seen, a generalized Kähler manifold determines a pair of real Poisson structures π_A and π_B as well as a pair of holomorphic Poisson structures $(I_\pm, \sigma_\pm)$. In fact, the data of both real

Poisson structures and either one of the holomorphic Poisson structures is sufficient to recover all the remaining data of the bi-Hermitian manifold, since

$$\omega_+ = (\pi_A + \pi_B)^{-1} \qquad\qquad \omega_- = (\pi_B - \pi_A)^{-1} \tag{8.15}$$

{{eq:piomega}}

In this section we use this observation to describe the generalized Kähler-Ricci flow as the flow by a generalized curvature.

The generalized Kähler-Ricci flow is given by the following system of equations:

$$\begin{aligned} \dot{g} &= -\operatorname{Rc} + \frac{1}{4}H^2 \\ \dot{I}_\pm &= \frac{1}{2}\mathcal{L}_{\theta^\sharp_\pm} I_\pm \end{aligned} \tag{8.16}$$

{{eq:gkrf}}

where $H^2(X,Y) = \langle \iota_X H, \iota_Y H\rangle_g$ and $\theta_\pm$ is the Lee form for $I_\pm$. An important theorem of Streets and Tian shows that solutions to the generalized Kähler-Ricci flow exist for short time and preserve the integrability condition (eq. (7.1)).

Proposition 21. [46, 45] Let (g, I_+, I_-) be a generalized Kähler structure on a compact manifold M. There exists $T \in [0,\infty]$ such that eq. (8.16) has a unique maximal solution on $[0,T]$. Moreover that solution remains generalized Kähler for all $t \in [0,T]$.

The key observation that is used in the proof of short-time existence is that solutions to eq. (8.16) may be obtained from solutions to the pluriclosed flow (c.f. eq. (2.11)). Indeed, if (g, I_+, I_-) is a generalized Kähler structure and $(g^\pm_t, I_\pm)$ are the unique solutions to the pluriclosed flow with initial conditions given by the two Hermitian structures $(g, I_\pm)$, and $\phi^\pm_t$ is the 1-parameter family of diffeomorphisms integrating $\frac{1}{2}\theta^\sharp_\pm$, then $(\phi^+_t)^* g^+_t = (\phi^-_t)^* g^-_t =: g_t$ and $(g_t, (\phi^+_t)^* I_+, (\phi^-_t)^* I_-)$ is a solution to eq. (8.16). As a result there are a pair of flows :

$$\begin{cases} \dot{g} = -\operatorname{Rc} + \frac{1}{4}H^2 - \frac{1}{2}\mathcal{L}_{\theta^\sharp_+} g \\ \dot{I}_+ = 0 \\ \dot{I}_- = \frac{1}{2}\mathcal{L}_{\theta^\sharp_- - \theta^\sharp_+} I_- \end{cases} \qquad\qquad \begin{cases} \dot{g} = -\operatorname{Rc} + \frac{1}{4}H^2 - \frac{1}{2}\mathcal{L}_{\theta^\sharp_-} g \\ \dot{I}_+ = \frac{1}{2}\mathcal{L}_{\theta^\sharp_+ - \theta^\sharp_-} I_+ \\ \dot{I}_- = 0 \end{cases} \tag{8.17}$$

known as the $I_\pm$-gauge-fixed generalized Kähler-Ricci flows respectively.

It is an important observation of Gibson and Streets that the generalized Kähler-Ricci flow can be recovered from ρ_{Bis+}:

Theorem 10. [17] The I_+-gauge-fixed Generalized Kähler-Ricci flow is given by {thm:gibsonstreet

$$\begin{aligned} \dot{\mathbb{J}}_A &= [\mathbb{J}_A, e^{-\rho_{Bis+}}\mathbb{J}_A] \\ \dot{\mathbb{J}}_B &= [\mathbb{J}_B, e^{-\rho_{Bis+}}\mathbb{J}_B] \end{aligned} \tag{8.18}$$

{{eq:gsgkrf}}

On can verify that the equations eq. (8.18) are equivalent to the infinitesimal action of the imaginary B-field $-i\rho_{Bis}$ on L_A and L_B. That is, the generalized Kähler-Ricci flow in I_+-fixed gauge is given by

$$\begin{aligned} \dot{L}_A &= -i\rho_{Bis} \\ \dot{L}_B &= -i\rho_{Bis} \end{aligned} \tag{8.19}$$

{{eq:gkrf4}}

Theorem 11. The generalized Kähler-Ricci flow in I_+-fixed gauge is given by the action of the derivation R_-^+ on the triple of Dirac structures $(\Gamma_{\pi_A}, \Gamma_{-\pi_B}, L_{\sigma_+})$. Similarly, the action of R_+^- on $(\Gamma_{\pi_A}, \Gamma_{\pi_B}, L_{\sigma_-})$ gives the generalized Kähler-Ricci flow in the I_--fixed gauge.

Proof. By theorem 10 the generalized Kähler-Ricci flow in I_+-fixed gauge is given as the action of the imaginary B-field $-i\rho_{Bis+}$ on L_A and L_B. Since

$$\begin{aligned} \Gamma_{\pi_A} &= \frac{1}{2i}(L_A - \overline{L}_A) \\ \Gamma_{-\pi_B} &= \frac{1}{2i}(\overline{L}_B - L_B) \\ L_{\sigma_+} &= \frac{1}{2i}(L_A - L_B) \end{aligned}$$

it follows that the generalized Kähler-Ricci flow can be reconstituted as the flow

$$\begin{aligned} \dot{\Gamma}_{\pi_A} &= -\pi_A\rho_{Bis+}\pi_A \\ \dot{\Gamma}_{-\pi_B} &= \pi_B\rho_{Bis+}\pi_B \\ \dot{L}_{\sigma_+} &= 0 \end{aligned}$$

On the other hand, the deformation of Γ_{π_A} generated by $R_-^+ = (\frac{1}{2}(\theta_+^\sharp - \theta_-^\sharp), -\rho_{Bis-})$ is given by

$$\begin{aligned} R_-^+(\pi_A\xi + \xi, \pi_A\eta + \eta) &= (\mathcal{L}_{\frac{1}{2}(\theta_+^\sharp - \theta_-^\sharp)}\pi_A + \pi_A\rho_{Bis-}\pi_A)(\xi, \eta) \\ &= -\pi_A\rho_{Bis+}\pi_A \end{aligned}$$

where the last line follows by proposition 20. Similarly,

$$\begin{aligned} R_-^+(-\pi_B\xi + \xi, -\pi_B\eta + \eta) &= (-\mathcal{L}_{\frac{1}{2}(\theta_+^\sharp - \theta_-^\sharp)}\pi_B + \pi_B\rho_{Bis-}\pi_B)(\xi, \eta) \\ &= \pi_B\rho_{Bis+}\pi_B \end{aligned}$$

Finally, since R_-^+ is a symmetry of $\mathbb{J}_{\sigma_+}$ we have the correct evolution equations for the triple.

We give an alternate, more direct derivation of the flow equations for the action of R_+^- on $(\Gamma_{\pi_A}, \Gamma_{\pi_B}, L_{\sigma_-})$. Since, by proposition 20,

$$\begin{aligned} R_+^-(\pi_A\xi + \xi, \pi_A\eta + \eta) &= \left(\mathcal{L}_{\frac{1}{2}(\theta_+^\sharp - \theta_-^\sharp)}\pi_A + \pi_A\rho_{Bis+}\pi_A\right)(\xi, \eta) = -(\pi_A\rho_{Bis-}\pi_A)(\xi, \eta) \\ R_+^-(\pi_B\xi + \xi, \pi_B\eta + \eta) &= \left(\mathcal{L}_{\frac{1}{2}(\theta_+^\sharp - \theta_-^\sharp)}\pi_B + \pi_B\rho_{Bis+}\pi_B\right)(\xi, \eta) = (\pi_B\rho_{Bis-}\pi_B)(\xi, \eta) \end{aligned}$$

the flow equations in question take the form

$$\dot{\pi}_A = -\pi_A\rho_{Bis-}\pi_A \qquad\qquad \dot{\pi}_B = \pi_B\rho_{Bis-}\pi_B. \tag{8.20}$$

Using the bi-Hermitian expressions for π_A and π_B we find

$$\begin{aligned}
\dot\pi_A + \dot\pi_B &= -\frac{1}{2}g^{-1}\left(I_+^*\rho_{Bis-}I_- + I_-^*\rho_{Bis-}I_+\right)g^{-1}\\
\dot\pi_B - \dot\pi_A &= -\frac{1}{2}g^{-1}\left(I_+^*\rho_{Bis-}I_+ + I_-^*\rho_{Bis-}I_-\right)g^{-1} = -\frac{1}{2}g^{-1}\left(\rho_{Bis-} + I_-^*\rho_{Bis-}I_-\right)g^{-1}
\end{aligned}$$

where the final equality follows from the fact that $\rho_{Bis-} \in \Omega_+^{1,1}$. Then, using eq. (8.15) we find

$$\begin{aligned}
\dot\omega_+ &= -\omega_+(\dot\pi_A + \dot\pi_B)\omega_+ = \frac{1}{2}(\rho_{Bis-}I_-I_+ + I_+^*I_-^*\rho_{Bis-}) && (8.21)\\
\dot\omega_- &= -\omega_-(\dot\pi_B - \dot\pi_A)\omega_- = -\frac{1}{2}(\rho_{Bis-} + I_-^*\rho_{Bis-}I_-) = -\left(\rho_{Bis-}\right)_-^{1,1} && (8.22)
\end{aligned}$$

`{{eq:omegaplusdot`

We then use $\dot I_- = 0$ to compute

$$\dot g = -\dot\omega_- I_- = \left(\rho_{Bis-}\right)_-^{1,1} I_- = -\operatorname{Rc} + \frac{1}{4}H^2 - \frac{1}{2}\mathcal{L}_{\theta_-^\sharp}g$$

where the last line follows from lemma 7. It remains then, to compute the flow equation for I_+. Using eq. (8.21) and eq. (8.14) we find

$$\begin{aligned}
\dot I_+ &= g^{-1}(\dot\omega_+ - \dot g I_+)\\
&= g^{-1}\left(\frac{1}{2}\left(\rho_{Bis-}I_-I_+ + I_+^*I_-^*\rho_{Bis-}\right) - \left(\rho_{Bis-}\right)_-^{1,1}I_-I_+\right)\\
&= \frac{1}{2}g^{-1}\left(\rho_{Bis-}I_-I_+ + I_+^*I_-^*\rho_{Bis-} - \rho_{Bis-}I_-I_+ + I_-^*\rho_{Bis-}I_+\right)\\
&= Q\rho_{Bis-}\\
&= \mathcal{L}_{\frac{1}{2}(\theta_+^\sharp - \theta_-^\sharp)}I_+
\end{aligned}$$

where the last line follows from eq. (8.12). □

8.5 The flow construction

In [25], Gualtieri describes a method of deforming generalized Kähler manifolds using B-fields. In this section we review this construction and compare it to the generalized Kähler-Ricci flow.

The starting point for this construction is the observation that for a generalized complex structure $\mathbb{J}$ with $+i$-eigenbundle L and a real , closed 2-form F, the Dirac structure $e^{iF}L$ defines a new generalized complex structure, provided that $e^F\Gamma_\pi$ remains Poisson, for $\pi = \mathsf{a}\mathbb{J}\mathsf{a}^t$.

Theorem 12. [25] Given a derivation (X, ρ) of $\mathbb{T}M$ which is a symmetry of $\mathbb{J}_{\sigma_+}$, the pair

$$(L_A(t), L_B(t)) = (e^{iF_t}L_A, e^{iF_t}L_B) \tag{8.23}$$

is generalized Kähler for sufficiently small t.

Proof. It suffices to show that $\mathbb{J}_A(t)$ and $\mathbb{J}_B(t)$ have a common eigendecomposition $\mathbb{T}M = \ell_+(t) \oplus \ell_-(t) \oplus \overline{\ell}_+(t) \oplus \overline{\ell}_-(t)$. Since $L_A(t) \cap L_B(t) = (e^{iF_t}L_A) \cap (e^{iF_t}L_B) = e^{iF_t}\ell_+$ and $L_A(t) \cap \overline{L_B(t)} = \left(e^{iF_t}L_A\right) \cap \left(e^{-iF_t}\overline{L}_B\right) =$ □

Equation (8.19) allows us to reformulate a solution to generalized Kähler-Ricci flow in a manner which bears striking similarity to Gualtieri's flow construction.

Theorem 13. Give a solution $(\mathbb{J}_A(t), \mathbb{J}_B(t))$ to the I_+-fixed generalized Kähler-Ricci flow, the corresponding curvature $R_+^-(t) = \left(\frac{1}{2}\left(\theta_+^\sharp(t) - \theta_-^\sharp(t)\right), -\rho_{Bis+}(t)\right)$ gives a symmetry of $L_{\sigma_-(t)}$. Moreover,

$$(L_A(t), L_B(t)) = \left(e^{-iF_t} L_A, e^{-iF_t} L_B\right) \tag{8.24}$$

`{{eq:gkrfasflowco`

where $F_t = \int_0^t \rho_{Bis+}(s)ds$.

Proof. By uniqueness it suffices to show that eq. (8.24) satisfies the generalized Kähler-Ricci flow. This is easily verified by comparing with the form the generalized Kähler-Ricci flow given by eq. (8.19). □

Thus we see that the generalized Kähler-Ricci flow is given as a time dependent version of Gualtieri's flow construction.

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