On Density and Equidistribution of Stationary Geodesic Nets

by

Bruno Staffa

A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy

Department of Mathematics
University of Toronto

On Density and Equidistribution of Stationary Geodesic Nets

Bruno Staffa
Doctor of Philosophy

Department of Mathematics
University of Toronto
2025

Abstract

Stationary geodesic nets are embedded graphs in a Riemannian manifold (M^n, g) which are stationary with respect to the length functional. In this thesis, we study the distribution of closed geodesics and stationary geodesic nets in Riemannian manifolds. We prove that for a generic set of metrics on a closed manifold M^n, $n \geq 2$, the union of all the embedded stationary geodesic nets in (M^n, g) forms a dense subset of M^n. For $n = 2$, we prove that for generic metrics on M^2 we can obtain an equidistributed sequence of closed geodesics. This means that there exists a sequence of closed geodesics $\{\gamma_i\}_{i\in\mathbb{N}}$ such that for every open subset U of M^2,

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k} \mathrm{L}_g(\gamma_i \cap U)}{\sum_{i=1}^{k} \mathrm{L}_g(\gamma_i)} = \frac{\mathrm{Vol}_g(U)}{\mathrm{Vol}_g(M)}.$$

We show that the previous equidistribution result also holds for $n \geq 3$ but replacing closed geodesics by stationary geodesic nets. The main tool that we use is Almgren-Pitts Min-Max Theory, in particular the Weyl law for the volume spectrum. We also prove a Structure Theorem for stationary geodesic nets analogous to that of Brian White for minimal submanifolds, which is used to prove the density and equidistribution results. The density result was obtained in collaboration with Yevgeny Liokumovich, and the equidistribution result in dimensions 2 and 3 is joint work with Xinze Li.

To my parents, Andrea and Alejandro, and to my brother Luca.

Acknowledgements

I want to thank Yevgeny for being an extremely dedicated and generous supervisor. Thank you for sharing your mathematical knowledge and intuition with me, for suggesting great problems to work in, for always making time to meet when needed, for your valuable career advice, for providing funding to attend mathematical events and to alleviate my teaching load, for helping me disseminate my research and get in contact with some of the most important mathematicians in our area. Thank you for your enormous help during my postdoc applications, and for everything you have done before so that I could get there well prepared. Being your (first) PhD student has been a pleasure.

I want to thank Alex and Bob, who were the members of my PhD Supervisory Committee together with Yevgeny. Thank you for your feedback on my PhD progress and on my talks (it helped me improve them!), and for your help in my postdoc applications.

I want to thank the professors at the University of Toronto who have inspired me as a math researcher and teacher. Thank you Florian Herzig, Vitali Kapovitch, Edward Bierstone, James Arthur, Boris Khesin, Yael Karshon, Jacopo de Simoi, Regina Rotman, Fedor Manin, Spyros Alexakis, Stefanos Aretakis, Robert McCann, Mary Pugh and Cindy Blois.

I want to thank Xinze Li. Our collaboration on the equidistribution of stationary geodesic nets is a central part of this thesis.

I want to thank my parents and my brother, for all their love and support. Thank you for always being there and for making it feel like we were close, despite the distance. Thank you for always encouraging me to go on, especially during the hardest moments. Thank you for all your advice and help with the various situations and struggles of life I've had to deal with. Our regular video calls were key to make it through my doctoral studies and to have a more smooth and enjoyable experience. I'm also very grateful for everything you have done in the past which was crucial for me to get here, from your support during my participation in mathematical olympiads to funding my undergraduate studies in Buenos Aires.

I want to thank the rest of my family as well, for their warmth and constant interest in my life experience abroad.

I want to thank Fran, Gastón and Pola for all the support during my first year of PhD. That was a tough one for me: lots of changes, I was missing Argentina a lot, the pandemic came, I didn't have many friends in Toronto yet... and I mostly talked to you three. Thank you for bearing with me and for being amazing friends.

I want to thank the many friends I have made at UofT. Thank you Virginia, my Argentinean friend in Toronto, for all we shared, for being so welcoming and for your Argentinianness. Thank you Ángel for our long walks around Toronto, for all the laughs and great conversations. Thank you Herng Yi, for our many memories as roommates, conference travelers and researchers. It's always a pleasure to discuss math (and life in general) with you. Thank you Amaranta, for all the nice moments we shared together, for always listening to me, for your warmth and tenderness. Thank you Eva and Valia for the amazing movie nights, parties, and time together. Thank you Wenkui for your friendship and for all our interesting conversations, both mathematical and non mathematical. Thank you Peter, Dinushi, Lemonte, Vicente, Luciano, Joaquín, Sebastian, Saeyon, Isabel, Tona, Ramón, Malors, Nick, Ethan, Vanya, Reila, Nischay, Janane, Matías, Heejong, Talant and Brendan for accompanying me during this journey.

I want to thank Gabriel, for becoming a close friend during this last year, with whom I could discuss the ups and downs of my research and my life.

I want to thank Chris, my non mathematical friend in Toronto, for all the support and the nice moments together. Thank you for being so warm and welcoming, and for helping me enjoy Toronto besides UofT.

I want to thank Azul, Matías and Kevin for always being present, for being so supportive and for keeping me part of your lifes despite the distance.

I want to thank the staff of the Mathematics Department at UofT for making the administrative side of the PhD experience smooth. I'm especially grateful to Jemima and Sonja, for their helpfulness and kindness, and to Coco for her work regarding my thesis defense.

Finally, I want to thank the Hausdorff Research Institute for Mathematics. Part of this thesis was written during my participation at the "Metric Analysis" trimester program held there. I'm very grateful for their hospitality and for the mathematical excellence of the program they organized.

Contents

Chapter 1

Introduction

In the 1960's Almgren started to develop a Morse theory on the space of Lipschitz k-cycles $\mathcal{Z}_k(M;G)$ on an n-dimensional Riemannian manifold M with respect to the (k-dimensional) volume functional. Here G denotes an abelian group, during this thesis we will consider the case $G=\mathbb{Z}_2$ (for a detailed discussion on the space $\mathcal{Z}_k(M;G)$ of flat k-cycles with coefficients in G, its topology and the volume (or mass) functional on it, see [10], [11], [1] and [2]). The critical points of that functional correspond to possibly singular closed minimal submanifolds of M. The topology of $\mathcal{Z}_k(M;\mathbb{Z}_2)$ is exploited to find critical points of the volume functional, which are constructed via a mountain-pass argument. The latter functional has very low regularity, as it is only lower semicontinuous. Hence proving existence and regularity of critical points is much more delicate than in the classical setting. In the 1980s Pitts and Schoen-Simon, building on Almgren's work, made significant progress in the codimension-1 case. The last decade saw a renaissance in development of this theory, initiated by Marques and Neves. Major achievements include the solution of the Willmore Conjecture by Marques-Neves [29], generic density of closed embedded minimal hypsurfaces by Irie-Marques-Neves [22] and generic equidistribution by Marques-Neves-Song [31], Song's solution of the Yau conjecture [41], the proof of the Weyl law for the volume spectrum by Liokumovich-Marques-Neves [26] for codimension-1 cycles and by Guth-Liokumovich [16] for 1-cycles in 3-manifolds, the Allen-Cahn Min-Max Theory initiated by Gaspar-Guaraco [12], the proof of the multiplicity 1 conjecture in 3-manifolds by Chodosh-Mantoulidis [8] then extended by Zhou [50] for dimensions $3\leq n\leq 7$, Zhou-Zhu Min-Max theory for CMC hypersurfaces [51], Chodosh-Mantoulidis results on the p-widths of a surface [7], the solution of the Pitts-Rubinstein Conjecture by Ketover-Liokumovich-Song [24], the PDE proof of existence of codimension 2 stationary integral varifolds by Pigati-Stern [35], among others.

Most of the previous results are about closed embedded minimal hypersurfaces. Extending them to higher codimension cycles (or to ambient manifolds of dimension higher than 7 in the case of codimension-1 cycles) turned out to be challenging, mostly because the stationary objects provided by Almgren-Pitts Theory may have lower regularity. For 1-cycles, the latter theory produces stationary geodesic nets. Let us introduce them.

Let Γ be a weighted multigraph, which is a graph with multiplicities assigned to each of its edges. Let $\mathscr{E}$ be the set of edges of Γ, $\mathscr{V}$ the set of vertices and for each $E\in\mathscr{E}$ let $n(E)\in\mathbb{N}$ be its multiplicity. Consider the space

$$\Omega(\Gamma, M) = \{f : \Gamma \to M : f \text{ is continuous and } f|_E \text{ is a } C^2 \text{ immersion } \forall E \in \mathscr{E}\}.$$

We say that $f_0 \in \Omega(\Gamma, M)$ is a stationary geodesic network with respect to a metric $g \in \mathcal{M}^k$ if it is a critical point of the length functional $\mathrm{L}_g : \Omega(\Gamma, M) \to \mathbb{R}$ defined as

$$\mathrm{L}_g(f) = \int_\Gamma \sqrt{g_{f(t)}(\dot f(t), \dot f(t))}dt = \sum_{E \in \mathscr{E}} n(E) \int_E \sqrt{g_{f(t)}(\dot f(t), \dot f(t))}dt.$$

In other words, f_0 is stationary with respect to g if for every one parameter family $f : (-\delta, \delta) \times \Gamma \to M$ of Γ-nets with $f(0, \cdot) = f_0$ we have

$$\frac{d}{ds}\bigg|_{s=0} \mathrm{L}_g(f_s) = 0$$

where $f_s = f(s, \cdot)$.

In Section 2.2 we derive first and second variation formulas for stationary geodesic nets. The first one implies that $f : \Gamma \to (M, g)$ is stationary if and only if each edge of Γ is mapped to a geodesic segment and at every vertex the sum of the inward pointing unit tangent vectors (with multiplicity) is zero. In Figure 1.1, we provide two examples of stationary geodesic nets in the round sphere S^2. The first one in a stationary θ-graph (a graph with two vertices and three different edges connecting them). The vertices are mapped to two antipodal points, and the edges to three geodesic segments meeting at 120°angles. The second one corresponds to a complete graph Γ with 4 vertices, which are mapped to the vertices of an equilateral tetrahedron inscribed in the sphere. If we regard Γ as the 1-skeleton of that tetrahedron, the edges of the later are mapped to the geodesic segments connecting them, meeting at 120°angles. For background and open problems on stationary geodesic nets, see [33] and [17].

This thesis focuses on understanding the distribution of closed geodesics and stationary geodesic nets on Riemannian manifolds, by applying Almgren-Pitts Min-Max theory. In collaboration with Yevgeny Liokumovich [27], we proved that for a Baire-generic set of metrics in a closed manifold M, the union of all stationary geodesic nets forms a dense subset of M. Later with Xinze Li [25] we proved generic equidistribution of closed geodesics on surfaces and of stationary geodesic nets in 3-manifolds. The subsequent works of the author [44] and [45] proving the Weyl law for 1-cycles in n-manifolds for $n \geq 4$ allowed to extend the generic equidistribution result for geodesic nets to closed manifolds M of dimension $n \geq 4$. In [46], I proved a Structure Theorem for stationary geodesic nets analogous to that of Brian White for smooth minimal submanifolds (see [47],[48]). This result was used to obtain generic density and equidistribution of stationary geodesic nets. Before stating precisely the main results of this thesis, which are the Structure Theorem for stationary geodesic nets and the density and equidistribution results for closed geodesics and for stationary geodesic nets previously mentioned, let me introduce the volume spectrum of a Riemannian manifold M, which is a fundamental invariant of M constructed using Almgren-Pitts Min-Max Theory.

The volume spectrum of a Riemannian manifold is a sequence of numbers $(\omega_p^k(M))_{p\in\mathbb{N}}$ which correspond to the volumes of certain (possibly singular) k-dimensional minimal submanifolds of M, $1 \leq k \leq n-1$, $\dim(M) = n$. They are constructed via a min-max procedure in $\mathcal{Z}_k(M; \mathbb{Z}_2)$ and $\omega_p^k(M)$ is called the (k-dimensional) p-width of M (see Section 3.2 and [15] for more details about

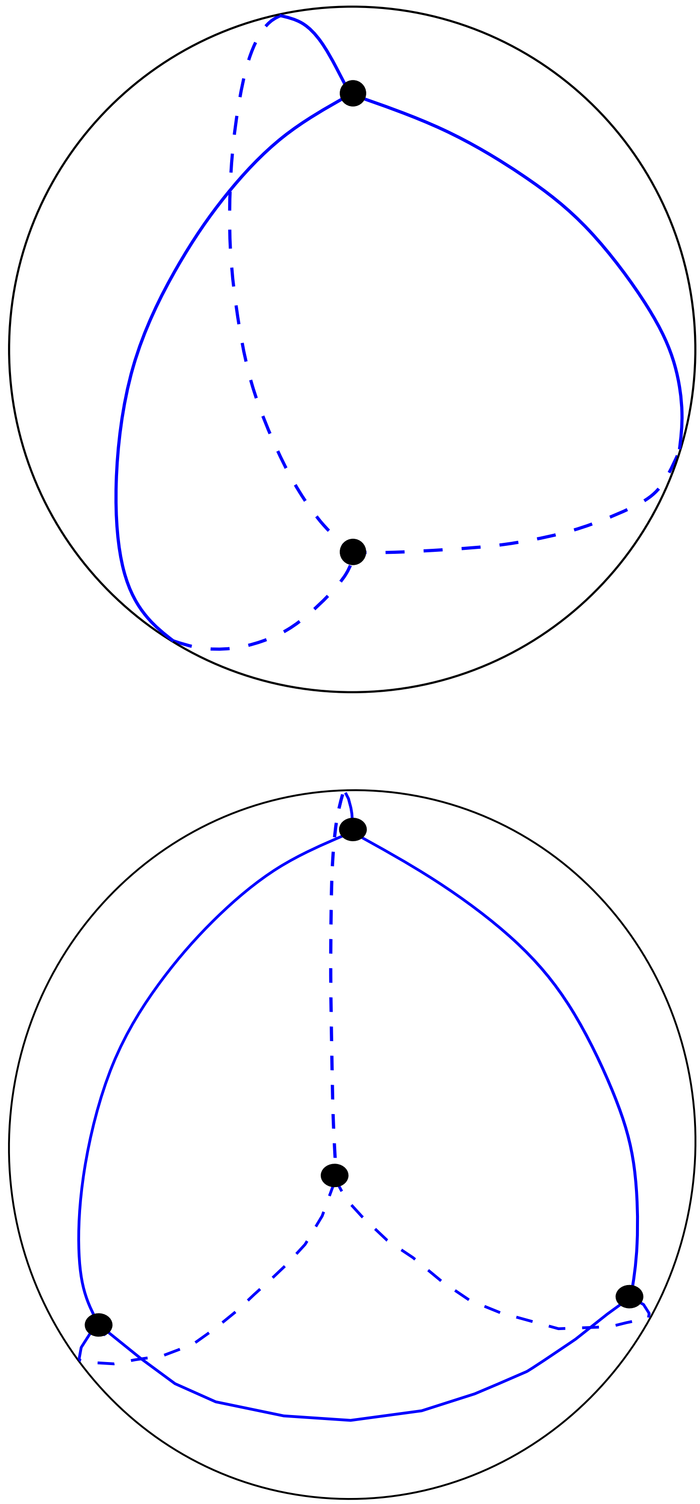

Figure 1.1: Stationary geodesic networks in the sphere.

this construction). In the 1980's, Gromov suggested to think of the p-widths as non-linear analogs of the eigenvalues of the Laplacian on M. He conjectured that they should satisfy the following Weyl law

$$\lim_{p\to\infty} \omega_p^k(M)p^{-\frac{n-k}{n}} = \alpha(n,k)\,\mathrm{Vol}(M)^{\frac{k}{n}} \tag{1.1}$$

for a certain universal constant $\alpha(n,k)$. The conjecture was resolved by Liokumovich, Marques and Neves for $k=n-1$ and n arbitrary, and later by Guth and Liokumovich for $k=1$, $n=3$. In [25], we used the previous cases of the Weyl law to derive equidistribution of closed geodesics in dimension 2 and of stationary geodesic nets in dimension 3. As mentioned before, the extension of the Weyl law for 1-cycles to manifolds of dimension $n\geq 4$ [45] allowed me to extend the equidistribution result to such manifolds as well.

1.1 Structure Theorem for Geodesic Nets

The Structure Theorem of Brian White [47] was used by Irie-Marques-Neves [22] and Marques-Neves-Song [31] to prove generic density and equidistribution of minimal hypersurfaces on compact manifolds M^n, $3\leq n\leq 7$. To extend their results to 1-cycles in n-dimensional manifolds, we had to deal with the fact that the corresponding widths are realized by stationary geodesic nets, as proved by Pitts in [36]. Therefore, it was necessary to prove a structure theorem for this type of objects, as stated below.

Theorem 1.1.1 (Structure theorem for geodesic nets)**.** *Let Γ be a weighted multigraph and $k\geq 3$. Let $\mathcal{M}^k$ be the space of C^k Riemannian metrics on M. Consider the space*

$$\mathcal{S}^k(\Gamma)=\{(g,f): g\in\mathcal{M}^k, f:\Gamma\to M \text{ is stationary with respect to } g\}.$$

Then

1. *$\mathcal{S}^k(\Gamma)$ has a C^{k-2} Banach manifold structure.*
2. *The projection map $\Pi:\mathcal{S}^k(\Gamma)\to\mathcal{M}^k$ onto the first coordinate is Fredholm of index 0.*
3. *Given $(g,f)\in\mathcal{S}^k(\Gamma)$, f is nondegenerate with respect to g if and only if $D\Pi_{(g,f)}:T_{(g,f)}\mathcal{S}^k(\Gamma)\to T_g\mathcal{M}^k$ is an isomorphism.*

In order to prove the theorem, it was necessary to find a Banach Manifold structure for the space $\Omega^{emb}(\Gamma,M)$ of embeddings of the graph Γ into M modulo reparametrization. In the case of smooth submanifolds, that space is locally modeled by the space of normal vector fields along a fixed embedded submanifold $N_0\subseteq M$. Nevertheless, for embedded graphs it is unclear which is the space of vector fields to consider, due to the singularities at the vertices. To be normal along each of the edges may imply to vanish at the vertices (as it happens when $\dim(M)=2$ at a triple junction), and we want to consider all possible variations to define our stationary objects, including those which move the vertices in any possible direction. But at the same time, we want to have an injective parametrization of $\Omega^{emb}(\Gamma,M)$ and hence we can not consider the space of all vector fields along a fixed $\gamma_0\in\Omega^{emb}(\Gamma,M)$, as for example all parallel vector fields would correspond to the same object γ_0. The solution I proposed was to view $\Omega^{emb}(\Gamma,M)$ as a subspace of the product over the edges E

of Γ of $\Omega^{emb}(E, M)$, which is the space of immersions $f_E : E \to M$ modulo reparametrization. It was possible to provide each $\Omega^{emb}(E, M)$ with a Banach manifold structure and prove that $\Omega^{emb}(\Gamma, M)$ is an embedded submanifold of the previously mentioned product.

A corollary of the Structure Theorem is the following Bumpy Metrics Theorem, which says that for a generic metric g on a manifold the length functional with respect to g is Morse.

Theorem 1.1.2 (Bumpy metrics theorem for geodesic nets)**.** *Let M be a smooth manifold. For a generic (in the Baire sense) set of metrics g on M, every stationary geodesic net with respect to g is nondegenerate.*

1.2 Generic density and equidistribution of stationary geodesic nets

In [22], Irie, Marques and Neves proved that for a generic set of metrics in a closed manifold M^n, $3 \leq n \leq 7$, the union of all the closed embedded minimal hypersurfaces is a dense subset of M. They used the Weyl law for codimension-1 cycles [26], the Structure Theorem of Brian White [47] and the regularity theory of Pitts-Schoen-Simon. In our work [27] together with Yevgeny Liokumovich, we applied the strategy from [22] to show the following result.

Theorem 1.2.1 (Generic density of stationary geodesic nets)**.** *Let M^n, $n \geq 2$, be a closed manifold and let $\mathcal{M}^k$ be the space of C^k Riemannian metrics on M, $3 \leq k \leq \infty$. For a generic (in the Baire sense) subset of $\mathcal{M}^k$ the union of the images of all embedded stationary geodesic nets in (M, g) is dense.*

The proof is based in the fact that due to the work of Pitts in [36] and [37], the p-widths $\omega_p^1(M, g)$ associated to sweepouts of M by 1-cycles are realized by stationary geodesic nets. The argument proceeds by contradiction, and starts by assuming that there exists a certain open set $U \subseteq M$ and an open set of metrics $\mathcal{V} \subseteq \mathcal{M}^k$ such that no metric $g \in \mathcal{V}$ admits a nondegenerate stationary geodesic net intersecting U. By picking a bumpy metric $g_0 \in \mathcal{V}$ (which exists by [46]), doing a conformal perturbation of g_0 and making use of the asymptotic behavior of the p-widths $\omega_p^1(M, g)$, it is possible to obtain a contradiction. In [22], the Weyl law for codimension-1 cycles is used in this part of the argument. That result was unknown for 1-cycles by the time we wrote [27] (except for the overlapping case $n = 2$). What we did instead was to apply an argument of Antoine Song [42], which shows that in fact it is enough to use the sublinear growth of the widths

$$\omega_p^1(M, g) \leq C(M, g) p^{\frac{n-1}{n}}$$

to prove density. Such sublinear bounds were proved by Gromov and Guth (see [15]).

Later, together with Xinze Li [25] we obtained a quantitative version of the previous result for $n = 2$ and $n = 3$, which builds in the work of Marques, Neves and Song [31]. More recently, I could extend the result to $n \geq 4$ by proving the Weyl law for 1-cycles [45].

Theorem 1.2.2. *Let M be a closed n-manifold, $n \geq 2$. For a Baire-generic set of C^∞ Riemannian metrics g on M, there exists a sequence of stationary geodesic nets $\{\gamma_i : \Gamma_i \to M\}$ that is equidistributed in M. Specifically, for every g in the generic set and for every C^∞ function $f : M \to \mathbb{R}$ we* *have*

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k} \int_{\gamma_i} f \, \mathrm{dL}_g}{\sum_{i=1}^{k} \mathrm{L}_g(\gamma_i)} = \frac{\int_M f \, \mathrm{dVol}_g}{\mathrm{Vol}(M,g)}.$$

When $n = 2$, we can replace "stationary geodesic net" by "immersed closed geodesic".

This was the first result on equidistribution of possibly singular minimal submanifolds in codimension higher than 1 on Riemannian manifolds without any curvature restrictions. In [25], the ambient manifold dimension was restricted to 2 or 3 because those were the only known cases of the Weyl law for 1-cycles, which were obtained in [26] and [16] respectively. Now that I could extend the Weyl law for 1-cycles to manifolds M of any dimension, Theorem 1.2.2 can be extended to n arbitrary as stated above. When $n = 2$, the regularity can be improved to immersed closed geodesics due to the work of Chodosh and Mantoulidis in [7], where they showed that the p-widths on surfaces are realized by collections of immersed closed geodesics.

To obtain equidistribution, we used a conformal perturbation argument which involved Theorem 1.1.1 and is more delicate than the one to prove density. It required to show ellipticity of the Jacobi operator along a stationary geodesic net. That is a well known fact in the case of the stability operator on a smooth minimal submanifold, but is not straightforward in the case of geodesic nets due to their singularities (which make the difficulties explained in Section 1.1 to arise).

The thesis is organized as follows. Chapter 2 focuses on the proof of Theorems 1.1.1 and 1.1.2, Chapter 3 on the proof of Theorem 1.2.1 and Chapter 4 on the proof of Theorem 1.2.2.

Chapter 2

Structure Theorem for Stationary Geodesic Nets

2.1 Summary of the chapter

Let M be an n-dimensional smooth manifold and let Γ be a weighted multigraph. Let $\mathscr{E}$ be the set of edges of Γ, $\mathscr{V}$ the set of vertices and for each $E \in \mathscr{E}$ let $n(E) \in \mathbb{N}$ be its multiplicity. Recall that we denote

$$\begin{aligned} \mathcal{M}^k &= \{g : g \text{ is a } C^k \text{ Riemannian metric on } M\}, \\ \Omega(\Gamma, M) &= \{f : \Gamma \to M : f \text{ is continuous and } f|_E \text{ is a } C^2 \text{ immersion } \forall E \in \mathscr{E}\}. \end{aligned}$$

and that $f \in \Omega(\Gamma, M)$ is a stationary geodesic network with respect to a metric $g \in \mathcal{M}^k$ if it is a critical point of the length functional $\mathrm{L}_g : \Omega(\Gamma, M) \to \mathbb{R}$.

Given a stationary geodesic network f_0 with respect to $g \in \mathcal{M}^k$, we can consider the Hessian of L_g at f_0:

$$\mathrm{Hess}_{f_0} \mathrm{L}_g(X, Y) = \left.\frac{\partial^2}{\partial x \partial s}\right|_{(0,0)} \mathrm{L}_g(f(x, s))$$

where X, Y are C^2 vector fields along f_0 and $f : (-\varepsilon, \varepsilon)^2 \to \Omega(\Gamma, M)$ is a two parameter family with $f(0, 0) = f_0$ verifying $\frac{\partial f}{\partial s}(0, 0, t) = X(t)$ and $\frac{\partial f}{\partial x}(0, 0, t) = Y(t)$. A vector field J along f_0 is said to be Jacobi if $\mathrm{Hess}_{f_0} \mathrm{L}_g(X, J) = 0$ for every vector field X along f_0. It is easy to check that every parallel vector field along f_0 (i.e. any continuous vector field J which is parallel and C^2 when restricted to each edge E of Γ) is Jacobi. Therefore we say that f_0 is a nondegenerate stationary geodesic network with respect to g if every Jacobi field along f_0 is parallel (notice that this is analogous to the notion of nondegeneracy for minimal submanifolds). We say that a metric $g \in \mathcal{M}^k$ is bumpy if every embedded stationary geodesic network with respect to g whose domain is a good weighted multigraph Γ is nondegenerate (see Section 2.2 for the definition of good weighted multigraph).

Our goal is to prove that for each $k \in \mathbb{N}_{\geq 3} \cup \{\infty\}$ the set of bumpy metrics is "big", in the sense it is Baire-generic in $\mathcal{M}^k$. To achieve that, we will study the space of stationary geodesic networks for varying Riemannian metrics on M. We will follow the ideas of [47], where this problem is studied for embedded minimal submanifolds, and adapt the arguments developed there to our setting. The main difference with the minimal submanifold problem is that our objects (stationary geodesic networks) are not everywhere smooth. Therefore, when we want to model a neighborhood of some $f_0 \in \Omega(\Gamma, M)$ we have to consider two degrees of freedom that determine a nearby $f \in \Omega(\Gamma, M)$: one is related to the image of the vertices and the other with the map along the edges. In order to have an injective parametrization of these geometric objects, we will mod out by reparametrizations and work with the quotient space $\hat{\Omega}(\Gamma, M) = \Omega(\Gamma, M)/\sim$ where $f \sim g$ if and only if there exists a homeomorphism $\tau : \Gamma \to \Gamma$ such that τ fixes the vertices of the graph, $\tau(E) = E$ for all $E \in \mathscr{E}$ and $\tau|_E : E \to E$ is a C^2 diffeomorphism for all $E \in \mathscr{E}$. In Section 2.3 of this chapter we study $\hat{\Omega}([0,1], M)$ (i.e. the space of immersed paths on M under reparametrization) and show that any $[f]$ close to $[f_0] \in \hat{\Omega}([0,1], M)$ can be obtained by a composition of a horizontal displacement (moving the vertices along an extension of the smooth curve $f_0 : [0,1] \to M$) and a normal one (moving in the direction of a normal vector field along f_0 with respect to a background metric γ_0). Therefore $\hat{\Omega}([0,1], M)$ is modeled by the Banach space $\mathbb{R}^2 \times \mathrm{Sect}(N_{f_0})$ where $\mathrm{Sect}(N_{f_0})$ denotes the space of C^2 sections of the normal bundle N_{f_0} along $f_0 : [0,1] \to M$ with respect to the background metric γ_0. Here we see the difference with the closed submanifold case analysed in [47], where the space of normal vector fields along a minimal submanifold $f_0 : N \to M$ models a neighborhood of $[f_0]$; while for paths we have an additional $\mathbb{R}^2$ factor because there is an extra degree of freedom for each vertex. Those extra degrees of freedom will also be present in the spaces $\hat{\Omega}(\Gamma, M)$ we are interested in, as it is shown in Section 2.4 where a C^0 (but not differentiable) Banach manifold structure is given to those spaces.

Once we have such structure for $\hat{\Omega}(\Gamma, M)$, it is possible to derive the first and second variation formulas for the length functional in local coordinates. We obtain expressions analogous to those derived in [47] but with additional terms corresponding to the vertices. This allows us to understand the space

$$\mathcal{S}_0^k(\Gamma) = \{(g, f) \in \mathcal{M}^k \times \hat{\Omega}(\Gamma, M) : f \text{ is stationary with respect to } g\}$$

locally as the set of zeros of a mean curvature map $H : \mathcal{M}^k \times C_0 \to \mathcal{Y}$ where C_0 is a Banach manifold which is the image of $\hat{\Omega}(\Gamma, M)$ under a chart, $\mathcal{Y}$ is a suitable Banach space that is defined in Section 2.4 and H is a C^{k-2} map between Banach manifolds. We use [47, Theorem 1.2] to give a Banach manifold structure to an open subset $\mathcal{S}^k(\Gamma) \subseteq \mathcal{S}_0^k(\Gamma)$. In order to do that, we prove in Section 4 that $D_2 H$ is Fredholm of index 0. Additionally, to satisfy condition (C) of [47, Theorem 1.2], we restrict our attention to good weighted multigraphs Γ and to embedded Γ-nets as defined in Section 2.2. We denote

$$\Omega^{emb}(\Gamma, M) = \{f \in \Omega(\Gamma, M) : f \text{ is embedded}\}.$$

As we show in Section 2.4, this technical condition rules out the possibility of having parallel Jacobi fields along $[f] \in \hat{\Omega}(\Gamma, M)$ and allows us to give a Banach manifold structure to

$$\mathcal{S}^k(\Gamma) = \{(g, f) \in \mathcal{M}^k \times \hat{\Omega}^{emb}(\Gamma, M) : f \text{ is stationary with respect to } g\} \subseteq \mathcal{S}_0^k(\Gamma).$$

Remark 2.1.1. *It is proved in Lemma 2.2.14 that given a stationary geodesic net $f : \Gamma \to M$ (with respect to a metric g), there exist $\{f_i : \Gamma_i \to M\}$ where each Γ_i is a good weighted multigraph and each $f_i : \Gamma_i \to M$ is an embedded stationary geodesic net such that their union has the same image and multiplicity at every point as f. Hence we do not loose much generality by restricting our attention to good multigraphs and embedded stationary geodesic nets.*

Having the previous considerations in mind and applying [47, Theorem 1.2] as mentioned before, we prove in Section 2.6 that $\mathcal{S}^k(\Gamma)$ is a C^{k-2} Banach manifold and that the projection $\Pi : \mathcal{S}^k(\Gamma) \to \mathcal{M}^k$, $(g, f) \mapsto g$ is Fredholm of index 0. This can be summarized in the following structure theorem.

Theorem 2.1.2 (Structure theorem for geodesic nets)**.** *Let Γ be a good weighted multigraph and $k \in \mathbb{N}_{\geq 3}$. Then*

1. *The space*

$$\mathcal{S}^k(\Gamma) = \{(g, f) \in \mathcal{M}^k \times \hat{\Omega}^{emb}(\Gamma, M) : f \text{ is stationary with respect to } g\}$$

   has a C^{k-2} Banach manifold structure.

2. *The projection map $\Pi : \mathcal{S}^k(\Gamma) \to \mathcal{M}^k$ onto the first coordinate is Fredholm of index 0.*

3. *Given $(g, f) \in \mathcal{S}^k(\Gamma)$, f is nondegenerate with respect to g if and only if $D\Pi_{(g,f)} : T_{(g,f)}\mathcal{S}^k(\Gamma) \to T_g\mathcal{M}^k$ is an isomorphism.*

The previous theorem together with Smale's version of Sard's theorem for Banach spaces from [40] implies

Theorem 2.1.3 (Bumpy metrics theorem for stationary geodesic nets)**.** *Given $k \in \mathbb{N}_{\geq 3} \cup \{\infty\}$ the subset $\mathcal{N}^k \subseteq \mathcal{M}^k$ of bumpy metrics is generic in the Baire sense.*

To be precise, Theorem 2.1.2 and Theorem 2.1.3 for $k \in \mathbb{N}_{\geq 3}$ are proved in Section 2.6 using the fact that C^k spaces have a Banach manifold structure. Although the same reasoning does not hold immediately for C^∞ spaces because they only have Frechet structures, in Section 2.7 we extend Theorem 2.1.3 to C^∞ metrics.

Remark 2.1.4. *Observe that our result does not provide nondegeneracy for not embedded stationary geodesic networks $f : \Gamma \to M$. In particular, we do not rule out the possibility of having a sequence of non-smooth stationary geodesic nets $f_n : \Gamma \to M$ converging to a stationary geodesic net $f_0 : \Gamma \to M$ which represents a closed geodesic loop with certain multiplicity (for example, a sequence of stationary figure eights which converges to a simple closed geodesic with multiplicity 2).*

As discussed in Chapter 3, Theorem 2.1.2 allowed Yevgeny Liokumovich and the author to prove that for a generic metric in a closed manifold M, the union of all stationary geodesic nets forms a dense subset of M (see the work [27]). More recently, Theorem 2.1.2 was used in [25] to prove that for a generic Riemannian metric g in a closed 2-manifold (respectively n-manifold for $n \geq 3$), there exists a sequence of closed geodesics (respectively of embedded stationary geodesic networks) which is equidistributed in (M, g), as explained in Chapter 4.

Remark 2.1.5. *Otis Chodosh and Christos Mantoulidis have independently proved a different Bumpy Metrics Theorem for stationary geodesic networks in 2-manifolds as part of their work [7], where they proved several remarkable results including the computation of the Weyl law constant for surfaces and the fact that the p-widths of surfaces can be realized as unions of immersed closed geodesics.*

All the content of this chapter is based on the article [46].

2.2 Set up

Definition 2.2.1. *A weighted multigraph is a graph* $\Gamma = (\mathscr{E}, \mathscr{V}, \{\pi_E\}_{E\in\mathscr{E}}, \{n(E)\}_{E\in\mathscr{E}})$ *consisting of:*

1. *A set of edges* $\mathscr{E}$*. For each* $E \in \mathscr{E}$*, we fix an homeomorphism* $E \cong [0, 1]$.
2. *A set of vertices* $\mathscr{V}$*.*
3. *For each* $E \in \mathscr{E}$*, a map* $\pi_E : \{0, 1\} \to \mathscr{V}$ *which sends each of the boundary points of the edge* E *(identified with* 0 *and* 1*) to their corresponding vertex* v*.*
4. *A multiplicity* $n(E) \in \mathbb{N}$ *assigned to each edge* $E \in \mathscr{E}$*.*

We will also denote by Γ *the one-dimensional simplicial complex* $\mathscr{E} \times [0, 1]/\sim$ *where* $(E, s) \sim (E', s')$ *if and only if* $s, s' \in \{0, 1\}$ *and* $\pi_E(s) = \pi_{E'}(s')$*.*

Definition 2.2.2. *Let* Γ *be a weighted multigraph. Given a vertex* $v \in \mathscr{V}$*, an incoming edge at* v *is a pair* $(E, i) \in \mathscr{E} \times \{0, 1\}$ *such that* $\pi_E(i) = v$*. We will assume that every vertex of* Γ *has at least two different incoming edges.*

Remark 2.2.3. *Notice that if we consider the simplicial complex associated to* Γ*, each loop edge at* v *appears two times as an incoming edge at* v *(as* $(E, 0)$ *and as* $(E, 1)$*) and all the other edges appear exactly once (either as* $(E, 0)$ *or as* $(E, 1)$*).*

Definition 2.2.4. *A weighted multigraph* Γ *is good* if the underlying one-dimensional simplicial complex is connected and each vertex* $v \in \mathscr{V}$ *has at least three different incoming edges. A weighted multigraph is good if either it is good* or it is a simple loop with multiplicity.*

Definition 2.2.5. *A* Γ*-net* f *on* M *is a continuous map* $f : \Gamma \to M$ *which is a* C^2 *immersion when restricted to the edges of* Γ*. The previous means that for each* $E \in \mathscr{E}$ *the map*

$$f_E : [0, 1] \xrightarrow{\iota_E} \mathscr{E} \times [0, 1] \xrightarrow{q} \Gamma \xrightarrow{f} M$$

is a C^2 *immersion (here* $\iota_E(t) = (E, t)$ *and* q *is the quotient map* $q : \mathscr{E} \times [0, 1] \to \Gamma = \mathscr{E} \times [0, 1]/\sim$*). We think of* f_E *as the restriction of* f *to the edge* E *and sometimes regard its domain as* E *under the identification* $E \cong [0, 1]$*. We denote* $\Omega(\Gamma, M)$ *the space of* Γ*-nets on* M*.*

Definition 2.2.6. *Given* $f \in \Omega(\Gamma, M)$ *and* $k \geq 0$*, we denote* $\mathfrak{X}^k(f)$ *the space of continuous vector fields along* f *which are of class* C^k *along each edge of* Γ *(observe that* $\mathfrak{X}^k(f_0)$ *is always well defined for* $k \leq 2$ *and could be defined for bigger values of* k *provided the restrictions of* f *to the edges have enough regularity).*

Notation 2.2.7. *Given* $f \in \Omega(\Gamma, M)$*,* $X \in \mathfrak{X}^2(f)$*,* $E \in \mathscr{E}$ *and* $t \in E$ *we will denote* $\dot{X}_E(t)$ *the covariant derivative of the vector field* X *along* f_E *at* t *(with respect to a Riemannian metric to be specified at each case). Notice that when* t *is a vertex of* Γ *this definition depends on* E*. We will omit the subscript* E *when it is implicit which edge are we differentiating along.*

Definition 2.2.8. *We say that a* Γ*-net* f *is embedded if the map* $f : \Gamma \to M$ *is injective (notice that by the compactness of* Γ *this is equivalent to say that the map* $f : \Gamma \to M$ *is a homeomorphism onto its image). We denote*

$$\Omega^{emb}(\Gamma, M) = \{f \in \Omega(\Gamma, M) : f \text{ is embedded}\}.$$

The spaces $\Omega(\Gamma, M)$ and $\Omega^{emb}(\Gamma, M)$ have natural Banach manifold structures with the C^2 topology (both are open subspaces of the space $C^2(\Gamma, M)$ of continuous maps $f : \Gamma \to M$ which are of class C^2 along each edge). Let $\mathcal{M}^k$ be the space of C^k Riemannian metrics on M. In the following we will omit the superscript k for simplicity, assuming it is fixed. Given $g \in \mathcal{M}$ and $f \in \Omega(\Gamma, M)$, we define the g-length of f by

$$\mathrm{L}_g(f) = \int_\Gamma \sqrt{g_{f(t)}(\dot{f}(t), \dot{f}(t))}dt$$

where given a measurable function $h : \Gamma \to \mathbb{R}$ which is integrable along each edge $E \in \mathscr{E}$, we define

$$\int_\Gamma h(t)dt = \sum_{E\in\mathscr{E}} n(E) \int_E h(t)dt.$$

Definition 2.2.9. *A Γ-net $f_0 \in \Omega(\Gamma, M)$ is a stationary geodesic network with respect to the metric $g \in \mathcal{M}$ if it is a critical point of the length functional $\mathrm{L}_g : \Omega(\Gamma, M) \to \mathbb{R}$. The previous holds if for every one parameter family $f : (-\delta, \delta) \times \Gamma \to M$ of Γ-nets with $f(0, \cdot) = f_0$ we have*

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{L}_g(f_s) = 0$$

where $f_s = f(s, \cdot)$.

In order to give a more precise description of this condition, and to define what it means for a stationary geodesic network to be nondegenerate, we derive the first and second variation formulas for the length functional on $\Omega(\Gamma, M)$.

Let $f : (-\varepsilon, \varepsilon) \times \Gamma \to M$ be a one parameter family of Γ-nets through $f_0 = f(0, \cdot)$ and let $X(t) = \frac{\partial f}{\partial s}(0, t)$ be the corresponding variational vector field along f_0. Then

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{L}_g(f_s) = \int_\Gamma \frac{g_{f_0(t)}(\dot{X}(t), \dot{f}_0(t))}{\sqrt{g_{f_0(t)}(\dot{f}_0(t), \dot{f}_0(t))}}dt. \tag{2.1}$$

To simplify the computation we will assume that each edge of f_0 is parametrized with constant speed (we don't loose generality by doing so because every Γ-net can be reparametrized with constant speed in a unique way), being $\sqrt{g_{f_0(t)}(\dot{f}_0(t), \dot{f}_0(t))} = \mathrm{L}_g(f_0(E))$ for all $t \in E$. Denoting $\mathrm{L}_g(f_0(E)) = l(E)$ for simplicity, we get

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{L}_g(f_s) = \sum_{E\in\mathscr{E}} \frac{n(E)}{l(E)} \int_E g_{f_0(t)}(\dot{X}(t), \dot{f}_0(t))dt.$$

Integrating by parts we obtain

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{L}_g(f_s) = -\sum_{E\in\mathscr{E}} \frac{n(E)}{l(E)} \int_E g_{f_0(t)}(X(t), \ddot{f}_0(t))dt + \sum_{v\in\mathscr{V}} g_{f_0(v)}(X(v), V(f_0)(v))$$

where

$$V(f_0)(v) := \sum_{(E,i):\pi_E(i)=v} (-1)^{i+1} n(E) \frac{\dot{f}_{0,E}(i)}{|\dot{f}_{0,E}(i)|}$$

and $f_{0,E} = (f_0)_E$.

From the previous computation, we see that a constant speed parametrized Γ-net f_0 is stationary with respect to L_g if and only if:

1. $\ddot{f}_0(t) = 0$ along each edge $E \in \mathscr{E}$ (i.e. the edges of Γ are mapped to geodesic segments).

2. $V(f_0)(v) = 0$ for all $v \in \mathscr{V}$. This means that the sum with multiplicity of the inward unit tangent vectors to the edges concurring at each vertex v must be 0.

Now assume f_0 is parametrized with constant speed and stationary. We want to define a continuous bilinear map $\mathrm{Hess}_{f_0} \mathrm{L}_g : \mathfrak{X}^2(f_0) \times \mathfrak{X}^2(f_0) \to \mathbb{R}$ which will be the Hessian of L_g at the critical point f_0 in the following way. Consider a two parameter variation $f : (-\varepsilon, \varepsilon)^2 \times \Gamma \to M$ with $f(0,0) = f_0$. Let $X(t) = \frac{\partial f}{\partial s}(0,0,t)$ and $Y(t) = \frac{\partial f}{\partial x}(0,0,t)$. We set $\mathrm{Hess}_{f_0}(X,Y) = \frac{\partial^2}{\partial x \partial s}\Big|_{(0,0)} \mathrm{L}_g(f(x,s))$. Next we will compute that expression and show that it is well defined (i.e. that it is independent of the two parameter family $f(x,s)$). From (2.1),

$$
\begin{aligned}
\mathrm{Hess}_{f_0} \mathrm{L}_g(X,Y) =& \frac{d}{dx}\Big|_{x=0} \sum_{E\in\mathscr{E}} n(E) \int_E g_{f_{x0}(t)}\Big(\frac{D}{dt}\frac{\partial f}{\partial s}(x,0,t), \frac{\frac{\partial f}{\partial t}(x,0,t)}{|\frac{\partial f}{\partial t}(x,0,t)|}\Big)dt \\
=& \sum_{E\in\mathscr{E}} n(E) \int_E g_{f_0(t)}\Big(\frac{D}{dx}\frac{D}{dt}\frac{\partial f}{\partial s}(x,0,t)\Big|_{(0,0,t)}, \frac{\dot{f}_0(t)}{|\dot{f}_0(t)|}\Big)dt \\
&+ \sum_{E\in\mathscr{E}} n(E) \int_E g_{f_0(t)}\Big(\dot{X}(t), \frac{D}{dx}\frac{\frac{\partial f}{\partial t}(x,0,t)}{|\frac{\partial f}{\partial t}(x,0,t)|}\Big|_{(0,0,t)}\Big)dt.
\end{aligned}
$$

Computing each sum separately we get

$$
\begin{aligned}
\mathrm{Hess}_{f_0} \mathrm{L}_g(X,Y) =& \sum_{E\in\mathscr{E}} \frac{n(E)}{l(E)} \Big[\int_E g(\dot{X}(t), \dot{Y}(t)) - g(\dot{Y}(t), \frac{\dot{f}_0(t)}{|\dot{f}_0(t)|}) g(\dot{X}(t), \frac{\dot{f}_0(t)}{|\dot{f}_0(t)|}) \\
&- g(R(\dot{f}_0(t), Y(t))\dot{f}_0(t), X(t))dt\Big] + n(E) g(\frac{D}{dx}\frac{\partial f}{\partial s}|_{(0,0,\pi_E(i))}, \frac{\dot{f}_{0,E}(i)}{|\dot{f}_{0,E}(i)|})\Big|_0^1.
\end{aligned}
\tag{2.2}
$$

Observe that

$$
\begin{aligned}
&\sum_{E\in\mathscr{E}} n(E) g(\frac{D}{dx}\frac{\partial f}{\partial s}|_{(0,0,\pi_E(i))}, \frac{\dot{f}_{0,E}(i)}{|\dot{f}_{0,E}(i)|})\Big|_0^1 \\
&= \sum_{v\in\mathscr{V}} \sum_{(E,i):\pi_E(i)=v} (-1)^{i+1} n(E) g(\frac{D}{dx}\frac{\partial f}{\partial s}|_{(0,0,v)}, \frac{\dot{f}_{0,E}(i)}{|\dot{f}_{0,E}(i)|}) \\
&= \sum_{v\in\mathscr{V}} g(\frac{D}{dx}\frac{\partial f}{\partial s}\Big|_{(0,0,v)}, \sum_{(E,i):\pi_E(i)=v} (-1)^{i+1} n(E) \frac{\dot{f}_{0,E}(i)}{|\dot{f}_{0,E}(i)|}) \\
&= 0
\end{aligned}
$$

because $V(f_0)(v) = 0$ for all $v \in \mathscr{V}$. Using this and integrating by parts the first two terms of (2.2) we get

$$
\begin{aligned}
&\operatorname{Hess}_{f_0} \mathrm{L}_g(X,Y) = \\
&\qquad \sum_{E\in\mathscr{E}} \frac{n(E)}{l(E)} \Bigg[\int_E g(-\ddot{Y}(t) - R(\dot{f}_0(t), Y(t))\dot{f}_0(t) + g(\ddot{Y}(t), \frac{\dot{f}_0(t)}{|\dot{f}_0(t)|})\frac{\dot{f}_0(t)}{|\dot{f}_0(t)|}, X(t))dt \\
&\qquad\qquad + g(\dot{Y}_E(i) - g(\dot{Y}_E(i), \frac{\dot{f}_{0,E}(i)}{|\dot{f}_{0,E}(i)|})\frac{\dot{f}_{0,E}(i)}{|\dot{f}_{0,E}(i)|}, X(\pi_E(i)))\Big|_0^1 \Bigg].
\end{aligned}
$$

Therefore we can define a second order differential operator A_E along the edge E as

$$
\begin{aligned}
A_E(Y) &= \frac{n(E)}{l(E)}\Bigg[-\ddot{Y}(t) - R(\dot{f}_0(t), Y(t))\dot{f}_0(t) + g(\ddot{Y}(t), \frac{\dot{f}_0(t)}{|\dot{f}_0(t)|})\frac{\dot{f}_0(t)}{|\dot{f}_0(t)|}\Bigg] \\
&= -\frac{n(E)}{l(E)}\Big[\ddot{Y}^\perp + R(\dot{f}_0(t), Y(t)^\perp), \dot{f}_0(t)\Big]
\end{aligned}
$$

and an operator $B_v : \mathfrak{X}^2(f_0) \to T_{f(v)}M$ at each vertex $v \in \mathscr{V}$ as

$$
\begin{aligned}
B_v(Y) &= \sum_{(E,i):\pi_E(i)=v} (-1)^{i+1}\frac{n(E)}{l(E)}\left(\dot{Y}_E(i) - g(\dot{Y}_E(i), \frac{\dot{f}_{0,E}(i)}{|\dot{f}_{0,E}(i)|})\frac{\dot{f}_{0,E}(i)}{|\dot{f}_0(i)|}\right) \\
&= \sum_{(E,i):\pi_E(i)=v} (-1)^{i+1}\frac{n(E)}{l(E)}\dot{Y}_E(i)^\perp
\end{aligned}
$$

where given $V \in T_{f_0(t)}M$ we denote $V^\perp$ the projection of V onto the orthogonal complement of the subspace $\langle \dot{f}_0(t)\rangle$. Thus we have the second variation formula

$$
\operatorname{Hess}_{f_0} \mathrm{L}_g(X,Y) = \sum_{E\in\mathscr{E}} \int_E g(A_E(Y)(t), X(t))dt + \sum_{v\in\mathscr{V}} g(B_v(Y), X(v)).
$$

We say that a vector field J along f_0 is Jacobi if $\operatorname{Hess}_{f_0} \mathrm{L}_g(J,X) = 0$ for all vector fields X along f_0. By the second variation formula, J is Jacobi along f_0 if and only if

1. J verifies the Jacobi equation $\ddot{J}^\perp + R(\dot{f}_0(t), J(t)^\perp)\dot{f}_0(t) = 0$ along each $E \in \mathscr{E}$.
2. $B_v(J) = 0$ for all $v \in \mathscr{V}$.

Definition 2.2.10. *We say that a vector field $X \in \mathfrak{X}^2(f_0)$ is parallel if its restriction to each edge $E \in \mathscr{E}$ is a parallel vector field along the corresponding geodesic segment.*

Remark 2.2.11. *By the second variation formula, any parallel vector field along f_0 is automatically Jacobi.*

Definition 2.2.12. *A stationary geodesic network $f_0 \in \Omega(\Gamma, M)$ with respect to a metric $g \in \mathcal{M}$ is nondegenerate if every Jacobi field J along f_0 is parallel.*

Definition 2.2.13. *Given a weighted multigraph Γ and a Riemannian metric $g \in \mathcal{M}^k$, g is said to be bumpy with respect to Γ if every stationary geodesic network $f \in \Omega^{emb}(\Gamma, M)$ with respect to g is nondegenerate. A Riemannian metric $g \in \mathcal{M}^k$ is said to be bumpy if it is bumpy with respect to Γ for every good weighted multigraph Γ.*

We end this section with the following lemma from [27], which tells us that every stationary geodesic net can be reparametrized as an embedded one (by possibly changing its domain).

Lemma 2.2.14. *Let $f : \Gamma \to (M, g)$ be a stationary geodesic net. Then there exist an embedded stationary geodesic net $\tilde{f} : \tilde{\Gamma} \to (M, g)$ which has the same image with multiplicity as f and the property that each connected component of $\tilde{\Gamma}$ is good. In particular, it holds $\mathrm{L}_g(f) = \mathrm{L}_g(\tilde{f})$.*

Proof. First of all, we can find an injective stationary geodesic net $f' : \Gamma' \to M$ which has the same image with multiplicity as f. This can be done as follows.

1. Firstly, we replace the weighted multigraph Γ by a new one such that for every edge E, the map $f|_E$ does not have any self-intersections. This is done by subdividing each edge E in equal parts $E_1, ..., E_l$ so that the length of $f(E_i)$ is not bigger than the injectivity radius of (M, g) for every $1 \leq i \leq l$.

2. Once the previous is done, suppose we have two different edges E_1 and E_2 with multiplicities n_1 and n_2 respectively whose interiors overlap non-transversally. Assume $f(E_1) \cap f(E_2)$ is connected and that their symmetric difference is non-empty. The cases when $f(E_1) \cap f(E_2)$ has two components or $f(E_i) \subset f(E_j)$ are treated similarly.

   Let v_{11}, v_{12} be the vertices of E_1 and v_{21}, v_{22} be the vertices of E_2. Then we can remove E_1 and E_2, and replace them by three new edges: E_3 which has vertices v_{11} and v_{21}, multiplicity n_1 and represents the part of E_1 where there is no overlap with E_2; E_4 which has vertices v_{21} and v_{12}, multiplicity $n_1 + n_2$ and represents the overlap between E_1 and E_2; and E_5 which has vertices v_{12} and v_{22}, multiplicity n_2 and represents the part of E_2 where there is no overlap with E_1. Observe that after applying this procedure, the edges of the new graph are still mapped to geodesic segments of length bounded by the injectivity radius of (M, g), and therefore such curves do not have any self intersections. As each time we do this operation the number of pairs of edges whose interiors intersect non-transversally at some point decreases, eventually we will get a new weighted multigraph such that if two edges intersect at an interior point, then the intersection is transverse.

3. After the previous step, if $f(E_1)$ intersects $f(E_2)$, then $E_1 \neq E_2$ and the intersection is transverse. Consider an intersection point P between $f(E_1)$ and $f(E_2)$, $E_1 \neq E_2$ edges. Let v_{11}, v_{12} and v_{21}, v_{22} be the vertices of E_1 and E_2 respectively. We can introduce a new vertex v which will be mapped to P and replace E_1, E_2 by E_3, E_4, E_5, E_6 where E_3, E_4 are obtained by the subdivision of E_1 induced by P, and E_5, E_6 are obtained by the subdivision of E_2 induced by P. After doing this operation with each intersection point P of the images of different edges, we will obtain a geodesic net $f : \Gamma \to M$ such that given any two different edges E_1, E_2, $f(E_1)$ and $f(E_2)$ do not overlap at any interior point and no edge self-intersects.

4. At this point, if $f(t_1) = f(t_2)$ for some $t_1 \neq t_2$, then both t_1 and t_2 must be vertices. Denote $v_j = t_j$ for $j = 1, 2$. If we replace Γ by the quotient graph obtained by identifying v_1 and v_2, and iterate this procedure each time it is possible, we obtain an injective stationary geodesic net $f : \Gamma \to M$.

Now we perform some changes to ensure that each connected component of Γ' is good. We do this component by component, so we can assume that we start from an embedded stationary geodesic net $f' : \Gamma' \to M$ where Γ' is connected. In such situation, consider a vertex v, such that all edges adjacent to v have colinear tangent vectors at v. We assume that Γ' is not a simple loop with multiplicity, as in that case we are done. Since the vertex is balanced, there exist edges E_1 with multiplicity n_1 (with vertices v_1 and v) and E_2 with multiplicity n_2 (with vertices v and v_2) with opposite inward tangent vectors at v and $v_i \neq v$ for $i = 1, 2$. As the map f' is injective, it must be $n_1 = n_2$ and E_1, E_2 should be the only edges at v (if not, there would be another edge E_3 concurring at v with the same inward tangent vector as E_i for some $i \in \{1, 2\}$, and as E_3, E_i are mapped to geodesics, their images would coincide along an interval). Thus if $v_1 \neq v_2$, we can define a new graph Γ' by deleting v, E_1 and E_2, and adding an edge E connecting v_1 and v_2 with multiplicity $n_1 = n_2$ and image $f'(E_1) \cup f'(E_2)$. This operation keeps Γ' connected and f' injective. If $v_1 = v_2$, the previous construction gives us a simple geodesic loop with multiplicity $n_1 = n_2$. Iterating this construction, we eventually obtain a new $\tilde{f} : \tilde{\Gamma} \to M$ such that $\tilde{\Gamma}$ is either a simple loop with multiplicity or it satisfies that each of its vertices v admits two incoming edges E_1, E_2 such that $\tilde{f}(E_1)$ and $\tilde{f}(E_2)$ have different tangent lines at $\tilde{f}(v)$. In the latter case, the condition that the sum of the unit inward tangent vectors at v should be 0 forces there to be at least three different incoming edges at v making that component of $\tilde{\Gamma}$ a good* weighted multigraph. This completes the proof. $\square$

2.3 C^0 Banach manifold structure for the space of immersed paths under reparametrizations

Consider the space $\Omega([0,1], M)$ of C^2 immersions $f : [0,1] \to M$, where M is an n-dimensional smooth manifold provided with an auxiliary smooth Riemannian metric γ_0. Denote

$$\mathrm{Diff}_2([0,1]) = \{\tau : [0,1] \to [0,1] : \tau \text{ is a } C^2 \text{ diffeomorphism}, \tau(0) = 0, \tau(1) = 1\}$$

Define an equivalence relation $\sim$ on $\Omega([0,1], M)$ as $f \sim g$ if and only if there exists $\tau \in \mathrm{Diff}_2([0,1])$ such that $f = g \circ \tau$. If that happens we will say that f is a reparametrization of g. Let $\hat{\Omega}([0,1], M) = \Omega([0,1], M)/\sim$ be the quotient space by the equivalence relation $\sim$ with the quotient topology. The aim of this section is to give a C^0 Banach manifold structure for $\hat{\Omega}([0,1], M)$ (i.e. an atlas consisting of charts with values in a fixed Banach space whose transition maps are continuous). Our constructions would also work if we replaced C^2 regularity by C^k regularity for any $k \geq 1$, but we will focus on the case $k = 2$ because that is what we are using in the rest of the chapter.

Remark 2.3.1. *We only get a C^0 Banach manifold structure (and not C^j for any $j \geq 1$) due to the fact that, as it is shown below, the transition maps involve taking compositions and inverses of C^2 functions; and the operators $C^k(B,C) \times C^k(A,B) \to C^k(A,C)$, $(g,f) \mapsto g \circ f$ and $\mathrm{Diff}^k(A) \to \mathrm{Diff}^k(A)$, $f \mapsto f^{-1}$ (where A,B,C are smooth manifolds) are continuous but not differentiable in the C^k topology for $k < \infty$ (see for example [20, p. 2]).*

Let us fix $f_0 \in \Omega([0,1], M)$. By density of the C^∞ immersions, we can assume without loss of generality that f_0 is C^∞ (any $[f_0]$ will be in the domain of a chart of $\hat{\Omega}(\Gamma, M)$ centered at $[\tilde{f}_0]$ for some $\tilde{f}_0$ of class C^∞). This will allow us to apply the Tubular Neighborhood Theorem and have a C^∞ exponential map. We want to describe a neighborhood of $[f_0]$ in $\hat{\Omega}([0,1], M)$. Take $\eta > 0$

small so that f_0 can be extended to a C^∞ immersion $f_0 : (-\eta, 1+\eta) \to M$. Denote by N_{f_0} the normal bundle along $f_0 : (-\eta, 1+\eta) \to M$ and given $s > 0$ let $N^s_{f_0} = \{v \in N_{f_0} : |v|_{\gamma_0} < s\}$ and $U^s_{f_0} = \mathcal{E}(N^s_{f_0}) \subseteq M$ (where $\mathcal{E} : TM \to M$ is the exponential map with respect to the auxiliary metric γ_0). By the Tubular Neighborhood Theorem, there exists $r > 0$ such that $\mathcal{E} : N^r_{f_0} \to U^r_{f_0}$ is a local diffeomorphism (it is actually a diffeomorphism if f_0 is an embedding).

Lemma 2.3.2. *There exist a neighborhood W_1 of f_0 in $\Omega([0,1], M)$ and a neighborhood $\overline{W}_2$ of $h_0(t) = (t, 0)$ in $\Omega([0,1], N_{f_0})$ such that the map $\overline{\Theta} : \overline{W}_2 \to W_1$ defined as $\overline{\Theta}(h) = \mathcal{E} \circ h$ is a diffeomorphism of Banach manifolds.*

Proof. In case $\mathcal{E} : N^r(f_0) \to U^r_{f_0}$ is a diffeomorphism, we can define

$$W_1 = \{f \in \Omega([0,1], M) : \mathrm{Im}(f) \subseteq U^r_{f_0}\} = \Omega([0,1], U^r_{f_0}),$$
$$\overline{W}_2 = \{v \in \Omega([0,1], N_{f_0}) : \mathrm{Im}(v) \subseteq N^r_{f_0}\} = \Omega([0,1], N^r_{f_0})$$

and a map $\overline{\Theta}' : W_1 \to \overline{W}_2$ as $\overline{\Theta}'(f) = \mathcal{E}^{-1} \circ f$. Both $\overline{\Theta}$ and $\overline{\Theta}'$ are smooth maps of Banach manifolds, and inverses of each other so we get the desired result. When the immersion f_0 is not injective (for example when it parametrizes a closed curve, case that will be relevant later in this chapter as it will correspond to loop edges in the multigraph Γ), $\overline{\Theta}'(f) = \mathcal{E}^{-1} \circ f$ is not well defined globally, but we can define it locally over a finite collection of intervals $\{I_i\}_{1 \le i \le K}$ covering $[0,1]$ such that f_0 is injective along I_i and $\mathcal{E}$ is a diffeomorphism when restricted to $N^r_{f_0}|_{I_i}$ for each $i \le i \le K$ and some $r > 0$. By a gluing argument, we can construct a smooth inverse $\overline{\Theta}'$ for $\overline{\Theta}$. □

Corollary 2.3.3. *Let $\phi : G = (-\eta, 1+\eta) \times \mathbb{R}^{n-1} \to N_{f_0}$ be a trivialization of the normal bundle N_{f_0}. Let $v_0 : [0,1] \to G$ be the map $v_0(t) = (t, 0)$. Then taking $W_1 \subseteq \Omega([0,1], M)$ from the previous lemma, there exists a neighborhood $W_2 \subseteq \Omega([0,1], G)$ of v_0 such that $\Theta : W_2 \to W_1$ given by $\Theta(v) = \mathcal{E} \circ \phi \circ v$ is a diffeomorphism of Banach manifolds.*

Therefore it is enough to model a neighborhood of $[v_0] \in \hat{\Omega}([0,1], G)$ as an open subset of a Banach manifold.

Proposition 2.3.4. *Given $v \in \Omega([0,1], G)$ let us denote $a_v = \pi(v(0))$ and $b_v = \pi(v(1))$, where $\pi : G = (-\eta, 1+\eta) \times \mathbb{R}^{n-1} \to (-\eta, 1+\eta)$ is the projection onto the first coordinate. There exists an open neighborhood $v_0 \in W_3 \subseteq W_2 \subseteq \Omega([0,1], G)$ with the following property: for every $v \in W_3$ there exists a section $\tilde{v}$ of $G|_{[a_v, b_v]}$ such that the map $\tilde{v} : [a_v, b_v] \to G$ is a reparametrization of v.*

Proof. Take $\delta < \eta$ such that $W_3 := \{v \in \Omega([0,1], G) : \|v - v_0\|_2 < \delta\}$ is contained in W_2 and each $v \in W_3$ is an embedding. Assume also that $\delta < \frac{1}{7}$. Pick $v \in W_3$. First, we want to prove that $\pi(v([0,1])) = [a_v, b_v]$. Notice that it suffices to show $\pi(v([0,1])) \subseteq [a_v, b_v]$. Define $v_1 : [0,1] \to G$ by $v_1(t) = ((1-t)a_v + tb_v, 0)$. Consider the map $w = \pi \circ v : [0,1] \to \mathbb{R}$ which is just the first component of v. We claim that $w'(t) \ge 0$ for all $t \in [0,1]$. Suppose not. Then there exists $t_0 \in [0,1]$ such that $w'(t_0) < 0$. Therefore,

$$|v'(t_0) - v_1'(t_0)| \ge |\pi(v'(t_0) - v_1'(t_0))| = |w'(t_0) - (b_v - a_v)| \ge b_v - a_v > 1 - 2\delta.$$

On the other hand, it is easy to see that $\|v_0 - v_1\|_2 < 2(|a_v| + |b_v - 1|) < 4\delta$, then from the previous

$$\|v - v_0\|_2 \ge \|v - v_1\|_2 - \|v_1 - v_0\|_2 > (1 - 2\delta) - 4\delta = 1 - 6\delta > \delta$$

as $\delta < \frac{1}{7}$, which is a contradiction because we assumed $\|v - v_0\|_2 < \delta$. Then, $w'(t) \geq 0$ for all $t \in [0,1]$ and as $w(0) = a_v$ we deduce $w(t) \geq a_v$ for all $t \in [0,1]$. Analogously, $w(t) \leq b_v$ for all $t \in [0,1]$ and hence $\pi(v([0,1])) \subseteq [a_v, b_v]$ as desired.

Given $a, b \in (-\eta, 1+\eta)$ with $a < b$ define $\tau_{ab} : [a,b] \to [0,1]$ as $\tau_{ab}(t) = \frac{t-a}{b-a}$. Notice that τ_{ab} is the inverse of $\chi_{ab} : [0,1] \to [a,b]$ given by $\chi_{ab}(t) = (1-t)a + tb$. By the previous, each $v \in W_3$ induces a smooth function $\theta_v := \tau_{a_v b_v} \circ \pi \circ v : [0,1] \to [0,1]$. Explicitly, $\theta_v(t) = \frac{\pi(v(t)) - \pi(v(0))}{\pi(v(1)) - \pi(v(0))}$. As $\theta_{v_0} = id$, shrinking δ again if necessary we can assume that $v \in W_3$ implies $\theta_v : [0,1] \to [0,1]$ is a C^2 diffeomorphism fixing 0 and 1. In that case, $\pi \circ v$ and hence $\pi : v([0,1]) \to [a_v, b_v]$ are diffeomorphisms. If we denote $\tilde{v} : [a_v, b_v] \to v([0,1])$ the inverse of $\pi : v([0,1]) \to [a_v, b_v]$, then $\tilde{v}$ is a section of $G|_{[a_v, b_v]}$ and we have $v = \tilde{v} \circ \chi_{a_v b_v} \circ \theta_v$ being v a reparametrization of $\tilde{v}$. □

The previous tells us that if we take $a \in (-\delta, \delta)$, $b \in (1-\delta, 1+\delta)$ and $u \in C^2([0,1], \mathbb{R}^{n-1})$ where δ is as small as indicated above, we can define a map $v_{abu} : [0,1] \to G$ as $v_{abu}(t) = ((1-t)a + tb, u(t))$ so that every $v \in W_3$ is a reparametrization of some v_{abu}. Specifically, given $v \in W_3$ if $v = \tilde{v} \circ \chi_{a_v b_v} \circ \theta_v$ as above and $\tilde{v}(s) = (s, \tilde{u}(s))$ then we must choose $a = a_v$, $b = b_v$ and $u = \tilde{u} \circ \chi_{a_v b_v}$. Consider the map $\Xi : (-\delta, \delta) \times (1-\delta, 1+\delta) \times C^2([0,1], \mathbb{R}^{n-1}) \to \Omega([0,1], G)$ given by $\Xi(a,b,u) = v_{abu}$. Denote $p : \Omega([0,1], G) \to \hat{\Omega}([0,1], G)$ the projection map.

Lemma 2.3.5. *The map $\Xi' : W_3 \to (-\delta, \delta) \times (1-\delta, 1+\delta) \times C^2([0,1], \mathbb{R}^{n-1})$ given by $\Xi'(v) = (a_v, b_v, u_v)$ with $u_v = \tilde{u} \circ \chi_{a_v b_v}$ as described before is continuous.*

Proof. It is enough to show that $\Xi_3' : W_3 \to C^2([0,1], \mathbb{R}^{n-1})$ defined as $\Xi_3'(v) = u_v$ is continuous. Let $\tilde{\pi} : G \to \mathbb{R}^{n-1}$ be the projection onto the last $n-1$ coordinates so that $\tilde{u} = \tilde{\pi} \circ \tilde{v}$. We have

$$u_v = \tilde{u} \circ \chi_{a_v b_v} = \tilde{\pi} \circ \tilde{v} \circ \chi_{a_v b_v} = \tilde{\pi} \circ v \circ \theta_v^{-1} \circ \tau_{a_v b_v} \circ \chi_{a_v b_v} = \tilde{\pi} \circ v \circ \theta_v^{-1}.$$

But $v \mapsto \theta_v^{-1}$ is continuous because so is $v \mapsto \theta_v$ and $\theta \mapsto \theta^{-1}$ for $\theta \in \mathrm{Diff}_2([0,1])$. Therefore by continuity of the composition, Ξ' is continuous. Notice that the differentiability fails as we are precomposing with and taking inverses of C^2 maps, which are not differentiable operations on spaces of C^2 functions ([20, p. 2]). □

Remark 2.3.6. *Given $v \in W_3$ we have $p \circ \Xi \circ \Xi'(v) = p(v)$.*

Lemma 2.3.7. *Define $W_4 = \Xi^{-1}(W_3) \subseteq \mathbb{R}^2 \times C^2([0,1], \mathbb{R}^{n-1})$. Then $p \circ \Xi : W_4 \to \hat{\Omega}([0,1], G)$ is injective.*

Proof. Suppose $p \circ \Xi(a_1, b_1, u_1) = p \circ \Xi(a_2, b_2, u_2)$ for some $(a_1, b_1, u_1), (a_2, b_2, u_2) \in U$. Denote $v_i = v_{a_i b_i u_i} \in W_3$ for $i = 1, 2$; being $[v_1] = [v_2]$. Then v_1 and v_2 have the same image, so $\pi \circ v_1([0,1]) = \pi \circ v_2([0,1])$ which means $[a_1, b_1] = [a_2, b_2]$ hence $a_1 = b_1$ and $a_2 = b_2$. Therefore, $v_1(t) = (a_1(1-t) + b_1 t, u_1(t))$ is a reparametrization of $v_2(t) = (a_1(1-t) + b_1 t, u_2(t))$. By looking at the first coordinate we deduce that the reparametrization must be just composing with the identity, and hence $u_1 = u_2$. □

Lemma 2.3.8. *There exists an open neighborhood $W_5 \subseteq W_4 \subseteq (-\delta, \delta) \times (1-\delta, 1+\delta) \times C^2([0,1], \mathbb{R}^{n-1})$ of $(0,1,0)$ such that $\Xi' \circ \Xi(a,b,u) = (a,b,u)$ for all $(a,b,u) \in W_5$.*

Proof. First of all observe that $\Xi(0,1,0) = v_0 : t \mapsto (t, 0)$ and by definition $\Xi'(v_0) = (0,1,0)$ so $\Xi' \circ \Xi(0,1,0) = (0,1,0)$. Set $W_5 = W_4 \cap (\Xi' \circ \Xi)^{-1}(W_4)$, by continuity of Ξ and Ξ' and the previous observation W_5 is an open neighborhood of $(0,1,0)$. Given $(a,b,u) \in W_5$ let $(\tilde{a},\tilde{b},\tilde{u}) = \Xi' \circ \Xi(a,b,u) \in W_4$. Then $p \circ \Xi(\tilde{a},\tilde{b},\tilde{u}) = p \circ \Xi \circ \Xi' \circ \Xi(a,b,u) = p \circ \Xi(a,b,u)$ because of Remark 2.3.6 and the fact that $\Xi(a,b,u) \in W_3$. As $(a,b,u),(\tilde{a},\tilde{b},\tilde{u}) \in W_4$ and $p \circ \Xi|_{W_4}$ is injective we deduce $(a,b,u) = (\tilde{a},\tilde{b},\tilde{u})$. □

Let us provide $\mathbb{R} \times \mathbb{R} \times C^2([0,1],\mathbb{R}^{n-1})$ with the norm $\|(a,b,u)\| = |a| + |b| + \|u\|_2$ making it a Banach space. Notice that $\Xi : \mathbb{R} \times \mathbb{R} \times C^2([0,1],\mathbb{R}^{n-1}) \to C^2([0,1],\mathbb{R}^n)$ is linear and

$$\frac{1}{3}\|(a,b,u)\| \leq \|v_{abu}\|_2 = \|\Xi(a,b,u)\|_2 \leq 2\|(a,b,u)\|$$

therefore $C^2_*([0,1],\mathbb{R}^n) := \mathrm{Im}(\Xi) \subseteq C^2([0,1],\mathbb{R}^n)$ is a closed subspace of $C^2([0,1],\mathbb{R}^n)$ and by the Open Mapping Theorem $\Xi : \mathbb{R} \times \mathbb{R} \times C^2([0,1],\mathbb{R}^{n-1}) \to C^2_*([0,1],\mathbb{R}^n)$ is an isomorphism of Banach spaces.

Theorem 2.3.9. *The subset $\hat{W}_5 := p \circ \Xi(W_5) \subseteq \hat{\Omega}([0,1],G)$ is open and $p \circ \Xi : W_5 \to \hat{W}_5$ is a homeomorphism.*

Proof. Let us start by showing that $p \circ \Xi : W_5 \to \hat{\Omega}([0,1],G)$ is an open map. Let $V \subseteq W_5$ be an open subset. Then $V' := \Xi(V) \subseteq C^2_*([0,1],\mathbb{R}^n) \cap W_3$ is an open subset of $C^2_*([0,1],\mathbb{R}^n)$. Define $W' = (\Xi \circ \Xi')^{-1}(V') \cap W_3 \subseteq (\Xi \circ \Xi')^{-1}(W_3) \cap W_3$ which is an open subset of $\Omega([0,1],G) \subseteq C^2([0,1],G)$. If $v \in V'$ then $v = \Xi(a,b,u)$ for some $(a,b,u) \in W_5$ therefore $\Xi \circ \Xi'(v) = \Xi \circ \Xi' \circ \Xi(a,b,u) = \Xi(a,b,u) = v$ by Lemma 2.3.8 and hence $v \in (\Xi \circ \Xi')^{-1}(V') \cap W_3 = W'$. This means that $V' \subseteq W'$ and hence $p(V') \subseteq p(W')$. But conversely, given $v \in W'$ by definition $v \in W_3$ so $p(v) = p \circ \Xi \circ \Xi'(v) \in p(V')$ because $\Xi \circ \Xi'(v) \in V'$. Therefore $p(V') = p(W')$ and as p is an open map we deduce that $p(V') = p \circ \Xi(V)$ is open, as desired.

Therefore, as $p \circ \Xi : W_5 \to \hat{\Omega}([0,1],G)$ is continuous, open and injective, it is a homeomorphism onto its image $\hat{W}_5$ as we wanted. □

We can use the results of this section to construct an atlas for $\hat{\Omega}([0,1],M)$ with charts of the form $(\hat{W}_5, (p \circ \Theta \circ \Xi)^{-1})$ centered at C^∞ immersions $[f_0]$ (with $\hat{W}_5$ as in Theorem 2.3.9), which yields a C^0 Banach manifold structure modeled by $\mathbb{R} \times \mathbb{R} \times C^2([0,1],\mathbb{R}^{n-1})$.

2.4 The length functional on the space $\hat{\Omega}(\Gamma, M)$

Let us fix a good* weighted multigraph Γ (i.e. Γ is connected and every vertex v of Γ has at least three different incoming edges). We define an equivalence relation $\sim$ in $\Omega(\Gamma,M)$ as follows: $f_0 \sim f_1$ if and only if there exists a homeomorphism $\theta : \Gamma \to \Gamma$ such that

1. $\theta(v) = v$ for all $v \in \mathscr{V}$.

2. $\theta(E) = E$ for all $E \in \mathscr{E}$ and moreover $\theta_E := \theta|_E : E \to E$ is a C^2 diffeomorphism.

3. $f_1 = f_0 \circ \theta$.

We consider the quotient space $\hat{\Omega}(\Gamma,M) = \Omega(\Gamma,M)/\sim$ with the quotient topology. Define the space $\Omega(\mathscr{E},M) = \prod_{E \in \mathscr{E}} \Omega(E,M)$ ($\Omega(E,M) \cong \Omega([0,1],M)$ by identifying $E \cong [0,1]$) being the map $\iota : \Omega(\Gamma,M) \to \Omega(\mathscr{E},M)$ defined as $\iota(f) = (f_E)_{E \in \mathscr{E}}$ a subspace map. We can also consider an equivalence relation $\sim$ in $\Omega(\mathscr{E},M)$ as follows: $f=(f_E)_{E\in\mathscr{E}}\sim g=(g_E)_{E\in\mathscr{E}}$ if there exists $\theta=(\theta_E)_{E\in\mathscr{E}}\in\prod_{E\in\mathscr{E}}\mathrm{Diff}_2(E)$ such that θ_E fixes the vertices of E and $g_E=f_E\circ\theta_E$ for all $E\in\mathscr{E}$. As before, we define the quotient space $\hat\Omega(\mathscr{E},M)=\Omega(\mathscr{E},M)/\sim\cong\prod_{E\in\mathscr{E}}\hat\Omega(E,M)$ with the quotient topology. It is clear that ι descends to a subspace map $\hat\iota:\hat\Omega(\Gamma,M)\to\hat\Omega(\mathscr{E},M)$.

Observe that the space $\hat\Omega(\mathscr{E},M)$ has a product C^0 manifold structure modeled on the Banach space $\mathcal{B}=\prod_{E\in\mathscr{E}}\mathbb{R}^2\times C^2(E,\mathbb{R}^{n-1})$. We proceed to describe the atlas induced by this product structure. Let $f=(f_E)_{E\in\mathscr{E}}\in\Omega(\mathscr{E},M)$ be such that f_E is C^∞ for every $E\in\mathscr{E}$. For each f_E we do the constructions of the previous section, i.e. we consider:

1. A trivialization $\phi_E:(-\eta_E,1+\eta_E)\times\mathbb{R}^{n-1}\to N_{f_E}$ of $N_{f_E}|_{(-\eta_E,1+\eta_E)}$.
2. Open sets $f_E\in W_1(f_E)\subseteq\Omega(E,M)$, $W_3(f_E)\subseteq W_2(f_E)\subseteq\Omega([0,1],G)$ and $W_5(f_E)\subseteq W_4(f_E)\subseteq(-\eta_E,\eta_E)\times(1-\eta_E,1+\eta_E)\times C^2([0,1],\mathbb{R}^{n-1})$ with the properties described in the previous section. In particular, $p\circ\Theta_E\circ\Xi:W_5(f_E)\to\hat W_5(f_E)=p\circ\Theta_E\circ\Xi(W_5(f_E))$ is a homeomorphism.
3. A real number $\delta_E>0$ such that $U_E:=(-\delta_E,\delta_E)\times(1-\delta_E,1+\delta_E)\times C^2([0,1],\mathbb{R}^{n-1})_{\delta_E}\subseteq W_5(f_E)$.

In the previous, we used the notation

$$C^2([0,1],\mathbb{R}^{n-1})_\alpha:=\{u\in C^2([0,1],\mathbb{R}^{n-1}):\|u\|_2<\alpha\}.$$

Denote $\hat U_E=p\circ\Theta_E\circ\Xi(U_E)\subseteq\hat\Omega(E,M)$, $U=\prod_{E\in\mathscr{E}}U_E\subseteq\mathcal{B}$ and $\hat U=\prod_{E\in\mathscr{E}}\hat U_E\subseteq\hat\Omega(\mathscr{E},M)$. From the previous section, we have homeomorphisms $\Lambda_E:U_E\to\hat U_E$ defined as $\Lambda_E(u)=p(\Theta_E(\Xi(u)))$ and they induce a homeomorphism $\Lambda=\prod_{E\in\mathscr{E}}\Lambda_E:U\to\hat U$. We define $\tilde\Lambda_E:U_E\to\Omega(E,M)$ as $\tilde\Lambda_E(u)=\Theta_E\circ\Xi(u)$ and $\tilde\Lambda:U\to\prod_{E\in\mathscr{E}}\Omega(E,M)$ as $\tilde\Lambda(u)=(\tilde\Lambda(u_E))_{E\in\mathscr{E}}$. Denote $\Sigma=\Lambda^{-1}$. Then $(\hat U,\Sigma)$ is a chart of $\hat\Omega(\mathscr{E},M)$ at f for the product structure we are considering, and the collection of all such charts is a C^0 atlas of $\hat\Omega(\mathscr{E},M)$.

Proposition 2.4.1. *$\hat\Omega(\Gamma,M)\subseteq\hat\Omega(\mathscr{E},M)$ is an embedded C^0 Banach submanifold, and its image under any chart $(\hat U,\Sigma)$ as constructed above is a smooth Banach submanifold of $\mathcal{B}$.*

We introduce the following notation which will be useful in the proof of the proposition and in the rest of the section.

Notation 2.4.2. *Given an edge $E\in\mathscr{E}$ we denote $c_0(E)=a_E$ and $c_1(E)=b_E$.*

Definition 2.4.3. *Given a vertex $v\in\mathscr{V}$, we denote $m(v)$ the number of incoming edges of the graph Γ at v, i.e. the number of pairs $(E,i)\in\mathscr{E}\times\{0,1\}$ such that $\pi_E(i)=v$ (as in Definition 2.2.2, notice that loops at v count twice as incoming edges). For each vertex v, we choose a preferred pair $(E_v,i_v)\in\mathscr{E}\times\{0,1\}$ such that $\pi_{E_v}(i_v)=v$.*

Remark 2.4.4. *Notice that $\sum_{v\in\mathscr{V}}m(v)=2|\mathscr{E}|$.*

Proof of Proposition 2.4.1. Let $f_0\in\Omega(\Gamma,M)$ be a Γ-net which is C^∞ along the edges. Consider a chart $(\hat U,\Sigma)$ at $[f_0]$ of the product manifold $\hat\Omega(\mathscr{E},M)=\prod_{E\in\mathscr{E}}\hat\Omega(E,M)$ as constructed before. We are going to describe $\Sigma(\hat\Omega(\Gamma,M)\cap\hat U)$ as an embedded Banach submanifold of U.

In order to do that, we need to understand which elements $u = (a_E, b_E, u_E)_{E\in\mathscr{E}}$ verify $\tilde{\Lambda}(u) \in \Omega(\Gamma, M)$. Notice that $(c_i(E), u_E(i))$ is equal to $(a_E, u_E(0))$ if $i = 0$ and to $(b_E, u_E(1))$ if $i = 1$. Observe that $u \in U$ represents a map which is continuous at v if and only if given $(E, i) \in \mathscr{E} \times \{0, 1\}$ such that $\pi_E(i) = v$ we have

$$\mathcal{E} \circ \phi_E(c_i(E), u_E(i)) = \mathcal{E} \circ \phi_{E_v}(c_{i_v}(E_v), u_{E_v}(i_v)).$$

We know that the map $\mathcal{E} \circ \phi_{E_v}$ is a diffeomorphism in a small neighborhood of $(i_v, 0)$. Let us denote its inverse as $(\mathcal{E} \circ \phi_{E_v})^{-1}$. Define $C_v : U \to (\mathbb{R}^n)^{m(v)-1}$ as

$$C_v(u) = \left((\mathcal{E} \circ \phi_{E_v})^{-1} \circ \mathcal{E} \circ \phi_E(c_i(E), u_E(i)) - (c_{i_v}(E_v), u_{E_v}(i_v))\right)_{(E,i)\neq(E_v,i_v):\pi_E(i)=v}.$$

Then u represents a map which is continuous at v if and only if $C_v(u) = 0$. Denote $C : U \to \prod_{v\in\mathscr{V}} (\mathbb{R}^n)^{m(v)-1} = (\mathbb{R}^n)^{2|\mathscr{E}|-|\mathscr{V}|}$ the smooth map defined as $C(u) = (C_v(u))_{v\in\mathscr{V}}$. From the previous we see that $\Lambda(u) \in \hat{\Omega}(\Gamma, M)$ if and only if $C(u) = 0$.

Lemma 2.4.5. *$C^{-1}(0)$ is a smooth embedded Banach submanifold of U.*

Proof of the lemma. Let $u \in U$ be such that $C(u) = 0$. Let $C_v^{(E,i)}$ be the component of C_v corresponding to (E, i). Denote $\{e_j\}_{1\le j\le n}$ the canonical basis of $\mathbb{R}^n$. Consider the basis $\{e_j^{v,(E,i)} : 1 \le j \le n, v \in \mathscr{V}, (E, i) \neq (E_v, i_v) : \pi_E(i) = v\}$ of $\prod_{v\in\mathscr{V}} (\mathbb{R}^n)^{m(v)-1}$. Given $v \in \mathscr{V}$, $(E, i) \in \mathscr{E} \times \{0, 1\}$ such that $\pi_E(i) = v$ and $(E, i) \neq (E_v, i_v)$, and $1 \le j \le n$ we will construct a one parameter family $\{u_s\}_{s\in(-1,1)}$ in U such that $u_0 = u$, $\frac{d}{ds}|_{s=0} C_v^{(E,i)}(u_s) = e_j$ and $\frac{d}{ds}|_{s=0} C_{v'}^{(E',i')}(u_s) = 0$ for all $(v', (E', i')) \neq (v, (E, i))$. Therefore if $X(v, (E, i), j) = \frac{d}{ds}|_{s=0} u_s$ by definition we will have $DC_u(X(v, (E, i), j)) = e_j^{v,(E,i)}$.

The construction is as follows. Let ρ be a bump function on E which is zero outside a small interval I around i and which takes the value 1 at i. Let $a \in \mathbb{R}$ and $w \in \mathbb{R}^{n-1}$ be such that $D((\mathcal{E} \circ \phi_{E_v})^{-1} \circ \mathcal{E} \circ \phi_E)_{(c_i(E),u_E(i))}(a, w) = e_j$. Define

$$u_{s,E}(t) = u_E(t) + s\rho(t)w$$

and $u_{s,E'}(t) = u_{E'}(t)$ for all $E' \neq E$. Define $c_{s,i}(E) = c_i(E) + sa$ and $c_{s,i'}(E') = c_{i'}(E')$ for all $(E', i') \neq (E, i)$. Then $u_s = (a_{s,E}, b_{s,E}, u_{s,E})_{E\in\mathscr{E}}$ is a smooth one parameter family with $u_0 = u$ and

$$\left.\frac{d}{ds}\right|_{s=0} C_v^{(E,i)}(u_s) = D((\mathcal{E} \circ \phi_{E_v})^{-1} \circ \mathcal{E} \circ \phi_E)_{(c_i(E),u_E(i))}(a, w) = e_j.$$

Also as $C_{v'}^{(E',i')}(u_s) = C_{v'}^{(E',i')}(u)$ for all $(v', (E', i')) \neq (v, (E, i))$ we deduce $\frac{d}{ds}|_{s=0} C_{v'}^{(E',i')}(u_s) = 0$ in that case.

Therefore we have a collection of vectors $\{X(v, (E, i), j)\}$ such that $DC_u(X(v, (E, i), j)) = e_j^{v,(E,i)}$. Observe that their images under DC_u form a basis of $\prod_{v\in\mathscr{V}} (\mathbb{R}^n)^{m(v)-1}$ and hence DC_u is surjective. Denote $S \subseteq \mathcal{B} = \prod_{E\in\mathscr{E}} \mathbb{R}^2 \times C^2(E, \mathbb{R}^{n-1})$ the span of $\{X(v, (E, i), j)\}$. Then S is finite dimensional and hence closed, and by linear algebra $\ker(DC_u) \oplus S = \mathcal{B}$. Thus 0 is a regular value of C and we can apply the Implicit Function Theorem to deduce that $\Sigma(\hat{\Omega}(\Gamma, M) \cap \hat{U}) = C^{-1}(0)$ is a smooth Banach submanifold of U. $\square$

So far we have shown that $\hat{\Omega}(\Gamma, M) \cap \hat{U}$ is an embedded Banach submanifold of $\hat{U}$ modeled in the space $\ker(DC_{u_0})$ where $\Lambda(u_0) = [f_0] \in \hat{\Omega}(\Gamma, M) \cap \hat{U}$. As $\ker(DC_{u_0})$ has codimension $n(\sum_{v \in \mathscr{V}} m(v) - 1) = n(2|\mathscr{E}| - |\mathscr{V}|)$ for all u_0, we deduce that the space modeling $\hat{\Omega}(\Gamma, M)$ locally is independent of the chart $(\hat{U}, \Sigma)$ (this is because two closed subspaces of a Banach space with the same finite codimension are isomorphic). Therefore, $\hat{\Omega}(\Gamma, M) \subseteq \hat{\Omega}(\mathscr{E}, M)$ is a C^0 Banach submanifold whose image under any chart $(\hat{U}, \Sigma)$ as constructed above is a smooth Banach submanifold of $\mathcal{B}$. □

Definition 2.4.6. *Following the constructions in the previous proof, we will denote $C_0 = C^{-1}(0) = \Sigma(\hat{\Omega}(\Gamma, M) \cap \hat{U})$, being $C_0 \subseteq U$ a Banach submanifold.*

Remark 2.4.7. *All the Banach manifolds previously defined are second countable. This is because they can be obtained from $C^2([0,1], M)$ and $\mathbb{R}$ by taking products, quotients and topological subspaces. The same holds for the Banach manifold $\mathcal{M}^k$ of C^k Riemannian metrics on M. These facts are consequences of the following: given a compact manifold M_1, a smooth manifold M_2 and a natural number $k \geq 1$; the space $C^k(M_1, M_2)$ with the C^k compact-open topology is metrizable and has a countable base, as it is explained in [19, p. 35].*

We will use the C^0 Banach submanifold structure of $\hat{\Omega}(\Gamma, M)$ in $\hat{\Omega}(\mathscr{E}, M)$ and the particular adapted charts under the atlas $\{(\hat{U}, \Sigma)\}$ described above to derive the first and second variation formulas for the length functional in local coordinates, as in [47]. The formulas that we will obtain will be analogous to those presented in Section 2.2, the advantage of this approach is that it allows us to use techniques from Differential Equations and Functional Analysis to give a geometric structure to the space of stationary geodesic networks for varying Riemannian metrics. Fix $f_1 \in \Omega(\Gamma, M)$ and a chart $(\hat{U}, \Sigma)$ of $\hat{\Omega}(\mathscr{E}, M)$ centered at $[f_1]$ as constructed above. Let g be a Riemannian metric on M. Given $u \in U$ we define $\mathrm{L}_g(u) = \mathrm{L}_g(\Lambda(u)) = \mathrm{L}_g(f)$. Then by definition of g-length,

$$\mathrm{L}_g(u) = \int_\Gamma \sqrt{g_{f(t)}(\dot{f}(t), \dot{f}(t))} dt = \sum_{E \in \mathscr{E}} n(E) \int_E \sqrt{g_{f(t)}(\dot{f}_E(t), \dot{f}_E(t))} dt$$

and by definition of Λ,

$$f_E(t) = \mathcal{E} \circ \phi_E(a_E(1-t) + b_E t, u_E(t)),$$
$$\dot{f}_E(t) = D(\mathcal{E} \circ \phi_E)_{v_E(t)}(b_E - a_E, \dot{u}_E(t))$$

where $v_E(t) = (a_E(1-t) + b_E t, u_E(t))$. Therefore if we define $F_g^E : [(-\delta_E, 1+\delta_E) \times \mathbb{R}^{n-1}] \times \mathbb{R}^n \to \mathbb{R}$ as

$$F_g^E(v, w) = \sqrt{g_{\mathcal{E} \circ \phi_E(v)}(D(\mathcal{E} \circ \phi_E)_v w, D(\mathcal{E} \circ \phi_E)_v w)}$$

and $\rho : E \times (-\delta_E, \delta_E) \times (1 - \delta_E, 1 + \delta_E) \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to [(-\delta_E, 1+\delta_E) \times \mathbb{R}^{n-1}] \times \mathbb{R}^n$ as

$$\rho(t, a, b, u, w) = ((a(1-t) + bt), u), (b - a, w))$$

then if $L_g^E = F_g^E \circ \rho$ it is clear that

$$\mathrm{L}_g(u) = \sum_{E \in \mathscr{E}} n(E) \int_E L_g^E(t, a_E, b_E, u_E(t), \dot{u}_E(t)) dt.$$

Now if we have a one parameter family $f : (-\varepsilon, \varepsilon) \to \hat{\Omega}(\Gamma, M) \cap \hat{U}$ and consider $u_s = \Lambda^{-1}(f(s)) = (a_E(s), b_E(s), u_{s,E})_{E\in\mathscr{E}}$, then

$$\begin{aligned}
\frac{d}{ds}\bigg|_{s=0} \mathrm{L}_g(u_s) &= \sum_{E\in\mathscr{E}} n(E) \int_E \frac{d}{ds}\bigg|_{s=0} L_g^E(t, a_E(s), b_E(s), u_{s,E}(t), \dot{u}_{s,E}(t))dt \\
&= \sum_{E\in\mathscr{E}} n(E) \int_E \frac{\partial L_g^E}{\partial a}(t, a_E(0), b_E(0), u_{0,E}(t), \dot{u}_{0,E}(t))a'_E(0)dt \\
&\quad + \sum_{E\in\mathscr{E}} n(E) \int_E \frac{\partial L_g^E}{\partial b}(t, a_E(0), b_E(0), u_{0,E}(t), \dot{u}_{0,E}(t))b'_E(0)dt \\
&\quad + \sum_{E\in\mathscr{E}} n(E) \int_E \nabla_u L_g^E(t, a_E(0), b_E(0), u_{0,E}(t), \dot{u}_{0,E}(t)) \cdot \frac{\partial u_E}{\partial s}(0,t)dt \\
&\quad + \sum_{E\in\mathscr{E}} n(E) \int_E \nabla_w L_g^E(t, a_E(0), b_E(0), u_{0,E}(t), \dot{u}_{0,E}(t)) \cdot \frac{\partial^2 u_E}{\partial s \partial t}(0,t)dt
\end{aligned}$$

where $(t, a, b, u, w) \in E \times (-\delta_E, \delta_E) \times (1-\delta_E, 1+\delta_E) \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ are the 5 variables on which the function L_g^E depends. Omitting those variables in the notation and integrating by parts we obtain

$$\begin{aligned}
\frac{d}{ds}\bigg|_{s=0} \mathrm{L}_g(u_s) = \sum_{E\in\mathscr{E}} n(E) \int_E \frac{\partial L_g^E}{\partial a} a'_E(0) + \frac{\partial L_g^E}{\partial b} b'_E(0) + (\nabla_u L_g^E - \frac{d}{dt}\nabla_w L_g^E) \cdot \frac{\partial u_E}{\partial s}(0,t)dt \\
+ \sum_{E\in\mathscr{E}} n(E) \nabla_w L_g^E(i, a_E(0), b_E(0), u_{0,E}(i), \dot{u}_{0,E}(i)) \cdot \frac{\partial u_E}{\partial s}(0,i)\bigg|_0^1.
\end{aligned} \tag{2.3}$$

Denote $X(t) = (a'_E(0), b'_E(0), \frac{\partial u_E}{\partial s}(0,t))_{E\in\mathscr{E}} \in T_{u_0}C_0 = \ker(DC_{u_0}) \subseteq \mathcal{B}$. Define $H_g^{1,E} : U \to C^0(E, \mathbb{R}^{n-1})$ and $A_g^{0,E}, A_g^{1,E} : U \to \mathbb{R}^n$ as

$$\begin{aligned}
H_g^{1,E}(u)(t) &= n(E)\Big(\nabla_u L_g^E(t, a_E, b_E, u_E(t), \dot{u}_E(t)) - \frac{d}{dt}\Big[\nabla_w L_g^E(t, a_E, b_E, u_E(t), \dot{u}_E(t))\Big]\Big), \\
A_g^{0,E}(u) &= n(E)\Big(\int_E \frac{\partial L_g^E}{\partial a}(t, a_E, b_E, u_E(t), \dot{u}_E(t))dt, -\nabla_w L_g^E(0, a_E, b_E, u_E(0), \dot{u}_E(0))\Big), \\
A_g^{1,E}(u) &= n(E)\Big(\int_E \frac{\partial L_g^E}{\partial b}(t, a_E, b_E, u_E(t), \dot{u}_E(t))dt, \nabla_w L_g^E(1, a_E, b_E, u_E(1), \dot{u}_E(1))\Big).
\end{aligned}$$

According to (2.3), an element $u_0 \in C_0$ represents a stationary geodesic network if and only if for every $X = (c_0(E), c_1(E), u_E)_{E\in\mathscr{E}} \in \ker(DC_{u_0})$ it holds

$$\sum_{E\in\mathscr{E}} \int_E H_g^{1,E}(u_0)(t) \cdot u_E(t)dt + \sum_{E\in\mathscr{E}} \sum_{i=0}^{1} A_g^{i,E}(u_0) \cdot (c_i(E), u_E(i)) = 0. \tag{2.4}$$

Now observe that the condition $DC_{u_0}(X) = 0$ implies that given $(E, i) \in \mathscr{E} \times \{0, 1\}$ with $\pi_E(i) = v$, there exists a linear transformation $T_v^{(E,i)}(u_0) : \mathbb{R}^n \to \mathbb{R}^n$ such that $(c_i(E), u_E(i)) = T_v^{(E,i)}(u_0)(c_{i_v}(E_v), u_{E_v}(i_v))$. Moreover, $DC_{u_0}(X) = 0$ if and only if $(c_i(E), u_E(i)) = T_v^{(E,i)}(u_0)(c_{i_v}(E_v), u_{E_v}(i_v))$ for every

$(E,i) \in \mathscr{E} \times \{0,1\}$. Thus we can rewrite

$$\begin{aligned}
&\sum_{E\in\mathscr{E}}\sum_{i=0}^{1} A_g^{i,E}(u_0)\cdot(c_i(E),u_E(i)) \\
&= \sum_{v\in\mathscr{V}}\sum_{(E,i):\pi_E(i)=v} A_g^{i,E}(u_0)\cdot T_v^{(E,i)}(u_0)(c_{i_v}(E_v),u_{E_v}(i_v)) \\
&= \sum_{v\in\mathscr{V}}\sum_{(E,i):\pi_E(i)=v} T_v^{(E,i)}(u_0)^*(A_g^{i,E}(u_0))\cdot(c_{i_v}(E_v),u_{E_v}(i_v))
\end{aligned}$$

where $T_v^{(E,i)}(u_0)^*$ denotes the adjoint of the linear operator $T_v^{(E,i)}(u_0) : \mathbb{R}^n \to \mathbb{R}^n$ with respect of the Euclidean inner product on $\mathbb{R}^n$. Define $H_g^{2,v} : U \to \mathbb{R}^n$ as

$$H_g^{2,v}(u) = \sum_{(E,i):\pi_E(i)=v} T_v^{(E,i)}(u_0)^*(A_g^{i,E}(u))$$

and $H_g^2 : U \to (\mathbb{R}^n)^{|\mathscr{V}|}$ as $H_g^2(u) = (H_g^{2,v}(u))_{v\in\mathscr{V}}$. Then (2.4) can be rewritten as

$$\sum_{E\in\mathscr{E}}\int_E H_g^{1,E}(u_0)(t)\cdot u_E(t)dt + \sum_{v\in\mathscr{V}} H_g^{2,v}(u_0)\cdot(c_{i_v}(E_v),u_{E_v}(i_v)) = 0. \tag{2.5}$$

Define $H_g^1 : U \to \prod_{E\in\mathscr{E}} C^0(E,\mathbb{R}^{n-1})$ as $H_g^1(u) = (H_g^{1,E}(u_E))_{E\in\mathscr{E}}$.

Proposition 2.4.8. *Let $u \in U$. Then $\Lambda(u)$ is a stationary geodesic network with respect to $g \in \mathcal{M}^k$ if and only if $H_g^1(u) = H_g^2(u) = C(u) = 0$.*

Proof. From (2.5), it is clear that if $u_0 \in U$ verifies $C(u_0) = H_g^1(u_0) = H_g^2(u_0) = 0$ then $\Lambda(u_0)$ is a stationary geodesic network. We want to see that the converse is also true. Assume $\Lambda(u_0)$ is stationary. Then $\Lambda(u_0) \in \hat{\Omega}(\Gamma, M)$ and hence $C(u_0) = 0$. We also know that (2.5) should hold for every $X = (c_0(E), c_1(E), u_E)_{E\in\mathscr{E}} \in \ker(DC_{u_0})$.

Suppose $H_g^{1,E}(u_0)$ is not identically zero for some $E \in \mathscr{E}$. Let $t_0 \in \mathrm{int}(E)$ be such that $H_g^{1,E}(u_0)(t_0) = w \neq 0$. Let $u_E : E \to \mathbb{R}^{n-1}$ be given by $u_E(t) = \rho_E(t)w$, where $\rho_E : E \to \mathbb{R}_{\geq 0}$ is a C^2 function such that $\rho_E(t_0) = 1$ and ρ_E is identically zero outside a small interval I around t_0 where $H_g^{1,E}(u_0)(t)\cdot w > 0$. Let $u_{E'}$ be the identically zero function for all $E' \neq E$. Define $X = (0,0,u_{E'})_{E'\in\mathscr{E}}$. Then as $u_{E'}(0) = u_{E'}(1) = a_{E'} = b_{E'} = 0$ for all $E' \in \mathscr{E}$, $X \in \ker(DC_{u_0})$. If we plug in X in (2.5), the second term vanishes and the first term is equal to

$$\int_E H_g^{1,E}(u_0)(t)\cdot u_E(t)dt > 0$$

which is a contradiction. Therefore, $H_g^1(u_0)$ must be identically zero. Thus we know that for all $X \in \ker(DC_{u_0})$

$$\sum_{v\in\mathscr{V}} H_g^{2,v}(u_0)\cdot(c_{i_v}(E_v),u_{E_v}(i_v)) = 0.$$

As given any vector $(c_v,u_v)_{v\in\mathscr{V}} \in (\mathbb{R}^n)^{|\mathscr{V}|}$ there exists $X \in \ker(DC_{u_0})$ such that $(c_{i_v}(E_v),u_{E_v}(i_v)) = (c_v,u_v)$ for all $v \in \mathscr{V}$ we deduce that $H_g^2(u_0) = 0$. □

Let us define $H : \mathcal{M}\times C_0 \to \mathcal{Y} = \left[\prod_{E\in\mathscr{E}} C^0(E,\mathbb{R}^{n-1})\right] \times (\mathbb{R}^n)^{|\mathscr{V}|}$ as $H(g,u) = (H_g^1(u), H_g^2(u))$. Then H is of class C^{k-2} if $\mathcal{M} = \mathcal{M}^k$ and the previous proposition implies that given $u \in C_0$, u is stationary with respect to g if and only if $H(g,u)=0$. Thus if

$$\mathcal{S}_0^k(\Gamma)=\{(g,f)\in\mathcal{M}^k\times\hat{\Omega}(\Gamma,M): f \text{ is stationary with respect to } g\}$$

then for any chart $(\hat{U},\Sigma)$ we have $\Sigma(\mathcal{S}_0^k(\Gamma)\cap\hat{U})=H^{-1}(0)$, hence we want to study $H^{-1}(0)$. For technical reasons that will become evident in the subsequent proofs, we will restrict our attention to embedded Γ-nets (and consider only good* weighted multigraphs as stated at the beginning of the section). Therefore we define

$$\mathcal{S}^k(\Gamma)=\{(g,f)\in\mathcal{M}\times\hat{\Omega}^{emb}(\Gamma,M): f \text{ is stationary with respect to } g\}\subseteq\mathcal{S}_0^k(\Gamma)$$

and we assume that all charts $(\hat{U},\Sigma)$ considered verify $\hat{U}\cap\hat{\Omega}(\Gamma,M)\subseteq\hat{\Omega}^{emb}(\Gamma,M)$. We are going to show that under the previous conditions 0 is a regular value for H. In order to do that, we need to study D_2H which is related to the Hessian of the length functional. For that purpose, in the remainder of this section we derive the second variation formula, define the notion of Jacobi field and discuss the relation between these definitions in local coordinates and the intrinsic ones given in Section 2.2.

Let $f:(-\varepsilon,\varepsilon)^2\to\hat{\Omega}(\Gamma,M)\cap\hat{U}$ be a two parameter family and denote $u_{xs}=u(x,s)=\Sigma(f(x,s))$ the corresponding two parameter family in C_0. Assume u_{00} is stationary and denote $X(t)=\frac{\partial u}{\partial s}(0,0,t)=(a_E^X,b_E^X,u_E^X(t))_{E\in\mathscr{E}}$ and $Y(t)=\frac{\partial u}{\partial x}(0,0,t)$. We know that $X,Y\in T_{u_0}C_0=\ker(DC_{u_0})$. Using (2.5), we have

$$\begin{aligned}\frac{\partial^2}{\partial x\partial s}\bigg|_{(0,0)}\mathrm{L}_g(u(x,s))=&\frac{d}{dx}\bigg|_{x=0}\Bigg[\sum_{E\in\mathscr{E}}\int_E H_g^{1,E}(u_{x0})(t)\cdot\frac{\partial u_E}{\partial s}(x,0,t)dt\\&+\sum_{v\in\mathscr{V}}H_g^{2,v}(u_{x0})\cdot(\frac{\partial c_{i_v}(E_v)}{\partial s}(x,0),\frac{\partial u_{E_v}}{\partial s}(x,0,i_v)\Bigg]\\=&\sum_{E\in\mathscr{E}}\int_E DH_g^{1,E}(u_{00})(Y)(t)\cdot u_E^X(t)dt\\&+\sum_{v\in\mathscr{V}}DH_g^{2,v}(u_{00})(Y)\cdot(c_{i_v}^X(E_v),u_{E_v}^X(i_v)).\end{aligned}$$

The Hessian $\mathrm{Hess}_{u_0}\mathrm{L}_g$ of the length functional at a critical point $u_0\in C_0$ is the continuous bilinear map $\mathrm{Hess}_{u_0}\mathrm{L}_g: T_{u_0}C_0\times T_{u_0}C_0\to\mathbb{R}$ given by $\mathrm{Hess}_{u_0}\mathrm{L}_g(X,Y)=\frac{\partial^2}{\partial x\partial s}|_{(0,0)}\mathrm{L}_g(u(x,s))$ where $u(x,s)$ is a two parameter family in C_0 such that $X(t)=\frac{\partial u}{\partial s}(0,0,t)$ and $Y(t)=\frac{\partial u}{\partial x}(0,0,t)$. The previous computation shows that the Hessian is well defined and

$$\begin{aligned}\mathrm{Hess}\,\mathrm{L}_g(u_0)(X,Y)=&\sum_{E\in\mathscr{E}}\int_E DH_g^{1,E}(u_{00})(Y)(t)\cdot u_E^X(t)dt\\&+\sum_{v\in\mathscr{V}}DH_g^{2,v}(u_{00})(Y)\cdot(c_{i_v}^X(E_v),u_{E_v}^X(i_v)).\end{aligned}$$

A vector field $X\in T_{u_0}C_0$ is said to be Jacobi along u_0 if it is a null vector for $\mathrm{Hess}_{u_0}\mathrm{L}_g$ i.e. if for every $Y\in T_{u_0}C_0$ we have $\mathrm{Hess}\,\mathrm{L}_g(u_0)(X,Y)=0$. Arguing as we did before with the first variation formula, it is clear that $X\in T_{u_0}C_0$ is Jacobi along u_0 if and only if $DH_g^1(u_0)(X)=DH_g^2(u_0)(X)=0$.

Definition 2.4.9. *Given a critical point u_0 of the length functional L_g, we say that u_0 is nondegenerate if the only Jacobi field X along u_0 is the zero vector field.*

Now we can study the relation between this notion of Jacobi field and nondegeneracy and that presented in Section 2.2. Denote $W_E \subseteq \Omega(E, M)$ the image of $\Theta_E : W_3(f_1|_E) \to \Omega(E, M)$ and $W = \prod_{E\in\mathscr{E}} W_E$. Consider the map $\tilde{\Sigma} : W \to U = \prod_{E\in\mathscr{E}} U_E$ defined as $\tilde{\Sigma}(v) = (\Xi'(\Theta_E^{-1}(v_E)))_{E\in\mathscr{E}}$. We can establish a correspondence between vector fields $X \in T_{u_0}C_0$ along u_0 and vector fields J along $f_0 = \tilde{\Lambda}(u_0)$ by $J = D\tilde{\Lambda}_{u_0}(X)$ and $X = D\tilde{\Sigma}_{f_0}(J)$. Notice that we can assume J is at least C^3 as we are working with Jacobi fields along a geodesic net with respect to a C^k metric with $k \geq 3$, and therefore $\tilde{\Sigma} : \mathfrak{X}^3(f_0) \to U \subseteq \prod_{E\in\mathscr{E}} \mathbb{R}^2 \times C^2(E, \mathbb{R}^{n-1})$ is differentiable. As $\tilde{\Sigma}$ is not exactly the inverse of $\tilde{\Lambda}$, one would not expect this correspondence to be bijective. However, we have the following characterization if we restrict to the space of embedded Γ-nets.

Proposition 2.4.10. *Let $f_0 \in \Omega^{emb}(\Gamma, M)$ where Γ is a good* weighted multigraph and let $u_0 \in C_0$ be such that $\tilde{\Lambda}(u_0) = f_0$. Let $X \in T_{u_0}C_0$ and let J be a vector field along f_0. Assume f_0 is stationary with respect to a metric g. Then*

1. *If $J = D\tilde{\Lambda}_{u_0}(X)$ is parallel along f_0 then $X = J = 0$.*
2. *$D\tilde{\Sigma}_{f_0}(D\tilde{\Lambda}_{u_0}(X)) = X$.*
3. *$D\tilde{\Lambda}_{u_0}(D\tilde{\Sigma}_{f_0}(J)) = J + K$ where K is a parallel vector field along f_0.*
4. *If X is Jacobi along u_0 then $J = D\tilde{\Lambda}_{u_0}(X)$ is Jacobi along f_0.*
5. *If J is Jacobi along f_0 then $X = D\tilde{\Sigma}_{f_0}(J)$ is Jacobi along u_0.*

Proof. First, let us show that if $J = D\tilde{\Lambda}_{u_0}(X)$ is parallel along f_0 then $X = J = 0$. Given $v \in \mathscr{V}$, as there are at least three different incoming edges at v, the tangent lines to two of them at v should be different because otherwise two incoming edges would have the same inward tangent unit vector and by uniqueness of the solutions of the geodesic equation f_0 would not be embedded, absurd. Therefore, $J(v) = 0$ for all $v \in \mathscr{V}$ (as $J(v)$ has to be colinear with all the inward tangent vectors to the edges at v). If $X = (c_E, d_E, w_E)_{E\in\mathscr{E}}$, $u_0 = (a_E, b_E, u_{0,E})_{E\in\mathscr{E}}$ and $v_{0,E} = \Xi(a_E, b_E, u_{0,E})$, then $K_E = D\Xi_{u_{0,E}}(X)$ is parallel along $v_{0,E}$ and $K_E(0) = K_E(1) = 0$ for all $E \in \mathscr{E}$ (as $f_0|_E(t) = \mathcal{E} \circ \phi_E(v_{0,E}(t))$ and $J_E = D(\mathcal{E} \circ \phi_E)_{v_{0,E}}(K_E)$). But we know that $K_E(t) = (c_E(1-t) + d_E t, w_E(t))$ thus $c_E = d_E = 0$. Then there exists a C^2 function $h : [0, 1] \to \mathbb{R}$ such that

$$K_E(t) = (0, w_E(t)) = h(t)\dot{v}_{0,E}(t) = h(t)(b_E - a_E, \dot{u}_{0,E}(t)).$$

This implies that $h \equiv 0$ and hence $X = J = 0$ as claimed in *(1)*.

Now take $X \in T_{u_0}C_0$ and a one parameter family $u : (-\varepsilon, \varepsilon) \to U$ such that $\frac{d}{ds}|_{s=0} u_s = X$. Then

$$D\tilde{\Sigma}_{f_0}(D\tilde{\Lambda}_{u_0}(X)) = \left.\frac{d}{ds}\right|_{s=0} \tilde{\Sigma}(\tilde{\Lambda}(u_s)) = \left.\frac{d}{ds}\right|_{s=0} u_s = X$$

as $\tilde{\Sigma} \circ \tilde{\Lambda} = id_U$ because of Lemma 2.3.8. This proves *(2)*.

On the other hand, if we take $J \in T_{f_0}\Omega(\Gamma, M)$ and a one parameter family $f : (-\varepsilon, \varepsilon) \to U$ such that $\frac{d}{ds}|_{s=0} f_s = J$, we have that $\tilde{\Lambda}(\tilde{\Sigma}(f_s))$ is a reparametrization of f_s for all $s \in (-\varepsilon, \varepsilon)$ because of Remark 2.3.6. Writing $\tilde{\Lambda}(\tilde{\Sigma}(f_s))(t) = f(s, \theta(s,t))$ and using that $\theta(0,t) = t$ for every $t \in \Gamma$ as $\tilde{\Lambda}(u_0) = f_0$, we can see

$$D\tilde{\Lambda}_{u_0}(D\tilde{\Sigma}_{f_0}(J)) = \frac{d}{ds}\bigg|_{s=0} \tilde{\Lambda}(\tilde{\Sigma}(f_s)) = \frac{d}{ds}\bigg|_{s=0} f(s,\theta(s,t)) = J(t) + \frac{\partial\theta}{\partial s}(0,t)\dot{f}_0(t)$$

so we get *(3)* by defining $K(t) = \frac{\partial\theta}{\partial s}(0,t)\dot{f}_0(t)$.

Now let X be a Jacobi field along u_0 (which is assumed to be stationary). Let $J = D\tilde{\Lambda}_{u_0}(X)$ and let $f(-\varepsilon,\varepsilon)^2 \to \Omega(\Gamma, M)$ be a two parameter family such that $J(t) = \frac{\partial f}{\partial s}(0,0,t)$. Consider the two parameter family $u(x,s) = \tilde{\Sigma}(f(x,s))$ through u_0. Then

$$D\tilde{\Lambda}_{u_0}(\frac{\partial u}{\partial s}(0,0)) = D\tilde{\Lambda}_{u_0}(D\tilde{\Sigma}_{f_0}(J)) = J + K$$

for some parallel vector field K along f_0 because of *(3)* which we have just proved. Therefore $K = D\tilde{\Lambda}_{u_0}(\frac{\partial u}{\partial s}(0,0) - X)$ and due to *(1)* it must be $K = 0$. Therefore $D\tilde{\Lambda}_{u_0}(\frac{\partial u}{\partial s}(0,0)) = J = D\tilde{\Lambda}_{u_0}(X)$ and hence $X = \frac{\partial u}{\partial s}(0,0)$ because $D\tilde{\Lambda}_{u_0}$ is a monomorphism. As X is Jacobi and $\tilde{\Lambda}(u(x,s))$ is a reparametrization of $f(x,s)$ for all $(x,s) \in (-\varepsilon,\varepsilon)^2$, this implies that

$$\frac{\partial^2}{\partial x \partial s}\bigg|_{(0,0)} \mathrm{L}_g\, f(x,s) = \frac{\partial^2}{\partial x \partial s}\bigg|_{(0,0)} \mathrm{L}_g\, u(x,s) = 0$$

and hence J is Jacobi along f_0. This proves *(4)*.

Let J be a Jacobi field along the stationary geodesic net f_0. Let $X = D\tilde{\Sigma}_{f_0}(J)$ and let $u : (-\varepsilon,\varepsilon)^2 \to U$ be a two parameter family with $u(0,0) = u_0$ and $\frac{\partial u}{\partial s}(0,0) = X$. Then if $f(x,s) = \tilde{\Lambda}(u(x,s))$

$$\frac{\partial f}{\partial s}(0,0) = D\tilde{\Lambda}_{u_0}(X) = D\tilde{\Lambda}_{u_0}(D\tilde{\Sigma}_{f_0}(J)) = J + K$$

for some parallel vector field K along f_0 because of *(3)*. As both J and K are Jacobi along f_0, so is $J + K$ and hence

$$\frac{\partial^2}{\partial x \partial s}\bigg|_{(0,0)} \mathrm{L}_g\, u(x,s) = \frac{\partial^2}{\partial x \partial s}\bigg|_{(0,0)} \mathrm{L}_g\, f(x,s) = 0$$

so we can deduce that X is Jacobi along u_0, which completes the proof of *(5)*. □

From the proposition we can see that given a critical point u_0 of the length functional with respect to g, u_0 is nondegenerate in the sense that it does not admit any nonzero Jacobi field if and only if $f_0 = \tilde{\Lambda}(u_0)$ is nondegenerate in the sense that every Jacobi field is parallel. Hence the two notions of nondegeneracy are equivalent.

2.5 D_2H is Fredholm of index 0

Fix a good* weighted multigraph Γ. Let us continue working in local coordinates $(\hat{U}, \Sigma)$ verifying $\hat{U} \cap \hat{\Omega}(\Gamma, M) \subseteq \hat{\Omega}^{emb}(\Gamma, M)$ as we have been doing previously. The goal of this section is to prove the following result.

Proposition 2.5.1. *Given $u_0 \in C_0$ and $g \in \mathcal{M}$ such that $H(g, u_0) = 0$, the operator $D_2H_{(g,u_0)} : T_{u_0}C_0 \to \left[\prod_{E\in\mathscr{E}} C^0(E, \mathbb{R}^{n-1})\right] \times (\mathbb{R}^n)^{|\mathscr{V}|}$ is Fredholm of index 0.*

We will need to introduce some notation and use the subsequent two lemmas. Their proofs are elementary using the definitions and results discussed in [28].

Definition 2.5.2. *Given a Riemannian metric h on M, a Γ-net $f \in \Omega(\Gamma, M)$ and an integer $k \geq 0$, we denote*

$$\begin{aligned}
\mathfrak{X}_0^k(f) &= \{Z \in \mathfrak{X}^k(f) : Z(v) = 0 \ \forall v \in \mathscr{V}\},\\
\mathfrak{X}^k(f)^{\|} &= \{Z \in \mathfrak{X}^k(f) : Z_E \text{ is parallel along } f_E \ \forall E \in \mathscr{E}\},\\
\mathfrak{X}^k(f)_h^{\perp} &= \prod_{E\in\mathscr{E}} \{Z_E \in \mathfrak{X}^k(f_E) : Z_E \text{ is normal along } f_E \text{ with respect to } h\},\\
\mathfrak{X}_0^k(f)_h^{\perp} &= \{Z \in \mathfrak{X}^k(f)_h^{\perp} : Z_E(0) = Z_E(1) = 0 \ \forall E \in \mathscr{E}\}.
\end{aligned}$$

Given $t \in \Gamma$ and $v \in T_{f(t)}M$, we denote $v_h^{\perp}$ the projection of v to the orthogonal complement of $\langle \dot f(t)\rangle$ with respect to h.

Remark 2.5.3. *Observe that $\mathfrak{X}^k(f)_h^{\perp} \not\subseteq \mathfrak{X}^k(f)$ but $\mathfrak{X}_0^k(f)_h^{\perp} \subseteq \mathfrak{X}_0^k(f)$.*

Lemma 2.5.4. *Let h_1 and h_2 be two Riemannian metrics on M and $k \geq 0$. Then the map $O : \mathfrak{X}^k(f)_{h_1}^{\perp} \to \mathfrak{X}^k(f)_{h_2}^{\perp}$, $Z \mapsto ((Z_E)_{h_2}^{\perp})_{E\in\mathscr{E}}$ is a continuous linear isomorphism. The same holds changing $\mathfrak{X}^k(f)_{h_i}^{\perp}$ by $\mathfrak{X}_0^k(f)_{h_i}^{\perp}$.*

Lemma 2.5.5. *Let $\mathcal{F}, \mathcal{G}$ be Banach spaces and $L : \mathcal{F} \to \mathcal{G}$ be a linear and continuous map. Let $\mathcal{F}_0, \mathcal{F}_1 \subseteq \mathcal{F}$ and $\mathcal{G}_0, \mathcal{G}_1 \subseteq \mathcal{G}$ be closed subspaces with $\dim(\mathcal{F}_1) = \dim(\mathcal{G}_1) < \infty$ such that $\mathcal{F}_0 \bigoplus \mathcal{F}_1 = \mathcal{F}$ and $\mathcal{G}_0 \bigoplus \mathcal{G}_1 = \mathcal{G}$. Let $L_{ij} : \mathcal{F}_i \to \mathcal{G}_j$, $i, j \in \{0, 1\}$ be such that $L(f_0, f_1) = (L_{00}(f_0) + L_{10}(f_1), L_{01}(f_0) + L_{11}(f_1))$ for each $f_0 \in \mathcal{F}_0$ and $f_1 \in \mathcal{F}_1$. Assume $L_{11} : \mathcal{F}_0 \to \mathcal{G}_0$ is Fredholm of index 0. Then $L : \mathcal{F} \to \mathcal{G}$ is Fredholm of index 0.*

Proof of Proposition 2.5.1. Let $g \in \mathcal{M}$ and $u_0 \in C_0$ verifying $H(g, u_0) = 0$. Consider the spaces

$$\begin{aligned}
C_0^2(E, \mathbb{R}^{n-1}) &= \{u \in C^2(E, \mathbb{R}^{n-1}) : u(0) = u(1) = 0\},\\
\mathcal{F} &= T_{u_0} C_0,\\
\mathcal{F}_0 &= \prod_{E\in\mathscr{E}} \{0\} \times C_0^2(E, \mathbb{R}^{n-1}),\\
\mathcal{G} &= \Big[\prod_{E\in\mathscr{E}} C^0(E, \mathbb{R}^{n-1})\Big] \times (\mathbb{R}^n)^{|\mathscr{V}|},\\
\mathcal{G}_0 &= \Big[\prod_{E\in\mathscr{E}} C^0(E, \mathbb{R}^{n-1})\Big] \times \{0\},\\
\mathcal{G}_1 &= \{0\} \times (\mathbb{R}^n)^{|\mathscr{V}|}.
\end{aligned}$$

Observe that as $\mathcal{F}_0 \subseteq \mathcal{F} \subseteq \mathcal{B}$,

$$\operatorname{codim}_{\mathcal{F}}(\mathcal{F}_0) = \operatorname{codim}_{\mathcal{B}}(\mathcal{F}_0) - \operatorname{codim}_{\mathcal{B}}(\mathcal{F}) = 2n|\mathscr{E}| - n(2|\mathscr{E}| - |\mathscr{V}|) = n|\mathscr{V}|$$

and hence there exists a subspace $\mathcal{F}_1 \subseteq \mathcal{F}$ of dimension $n|\mathscr{V}|$ such that $\mathcal{F}_0 \bigoplus \mathcal{F}_1 = \mathcal{F}$. As $\mathcal{G}_0 \bigoplus \mathcal{G}_1 = \mathcal{G}$ and $\dim(\mathcal{G}_1) = n|\mathscr{V}| = \dim(\mathcal{F}_1)$, by Lemma 2.5.5 it suffices to show that $D_2 H^1_{(g,u_0)} : \mathcal{F}_0 \to \mathcal{G}_0$ is Fredholm of index 0.

Let $X, Y \in \mathcal{F}_0$ with $X = (0,0,u_E)_{E\in\mathscr{E}}$ and $Y = (0,0,v_E)_{E\in\mathscr{E}}$, $u_E, v_E \in C_0^2(E,\mathbb{R}^{n-1})$ $\forall E \in \mathscr{E}$. By the second variation formula, we have

$$\frac{\partial^2}{\partial x \partial s}\bigg|_{(0,0)} \mathrm{L}_g(u_{xs}) = \sum_{E\in\mathscr{E}} \int_E D_2 H^{1,E}_{(g,u_0)}(Y)(t) \cdot u_E(t) dt \tag{2.6}$$

where $u_{x,s} = u_0 + sX + xY$. On the other hand, if $f(x,s) = \tilde{\Lambda}(u_{xs})$, then as the vector fields $\tilde{X} = D\tilde{\Lambda}_{u_0}(X)$ and $\tilde{Y} = D\tilde{\Lambda}_{u_0}(Y)$ vanish at the vertices of Γ we have

$$\frac{\partial^2}{\partial x \partial s}\bigg|_{(0,0)} \mathrm{L}_g(f_{xs}) = \sum_{E\in\mathscr{E}} -\frac{n(E)}{l(E)} \int_E g(\ddot{\tilde{Y}}_g^{\perp} + R(\dot{f}_0, \dot{\tilde{Y}}_g^{\perp})\dot{f}_0, \tilde{X}(t)_g^{\perp}) dt \tag{2.7}$$

being $f_0 = f_{00}$. Notice that as $f_0 = \tilde{\Lambda}(u_0)$ is embedded with domain a good* weighted multigraph, $\mathfrak{X}^2(f_0)^{\parallel} \subseteq \mathfrak{X}_0^2(f_0)$. Observe that $D\tilde{\Lambda}_{u_0}(\mathcal{F}_0) = \mathfrak{X}_0^2(f_0)^{\perp}_{\gamma_0}$, where γ_0 is the the auxiliary Riemannian metric used to construct the atlas of $\hat{\Omega}(\Gamma, M)$ in Section 2.4. Then as $D\tilde{\Lambda}_{u_0}$ is a monomorphism, by the Open Mapping Theorem it is an isomorphism between the spaces $\mathcal{F}_0$ and $\mathfrak{X}_0^2(f_0)^{\perp}_{\gamma_0}$. Similarly, $D\tilde{\Lambda}_{u_0} : \mathcal{G}_0 \to \mathfrak{X}^0(f_0)^{\perp}_{\gamma_0}$ is a linear isomorphism. Let us define the operators $M : \mathcal{F}_0 \to \mathfrak{X}_0^2(f_0)^{\perp}_g$, $M = O \circ D\tilde{\Lambda}_{u_0}$ and $N : \mathcal{G}_0 \to \mathfrak{X}^0(f_0)^{\perp}_g$, $N = O \circ D\tilde{\Lambda}_{u_0}$ (O is as in Lemma 2.5.4 with respect to the metrics $h_1 = \gamma_0$ and $h_2 = g$). By Lemma 2.5.4, M and N are isomorphisms.

Let $L : \mathfrak{X}^2(f_0) \to \mathfrak{X}^0(f_0)^{\perp}$ be the operator

$$L(Z) = -\left(\frac{n(E)}{l(E)}\left[(\ddot{Z}_E)_g^{\perp} + R(\dot{f}_0, (Z_E)_g^{\perp})\dot{f}_0\right]\right)_{E\in\mathscr{E}}.$$

It holds $L(O(Z)) = L(Z)$ for every $Z \in \mathfrak{X}^2(f_0)$ (as L vanishes over parallel vector fields). In particular, $L(\tilde{Y}) = L(O(\tilde{Y})) = L(M(Y))$. Also $L : \mathfrak{X}_0^2(f_0)^{\perp}_g \to \mathfrak{X}^0(f_0)^{\perp}_g$ is Fredholm of index 0, as so are the Jacobi operators $L_E : \mathfrak{X}_0^2(f_{0,E})^{\perp}_g \to \mathfrak{X}^0(f_{0,E})^{\perp}_g$, $L_E(Z) = \ddot{Z}_g + R(\dot{f}_0, Z)\dot{f}_0$ (because they are elliptic). Therefore, by (2.6), (2.7) and the fact that $\mathrm{L}_g(f_{xs}) = \mathrm{L}_g(u_{xs})$ we get

$$\sum_{E\in\mathscr{E}} \int_E D_2 H^{1,E}_{(g,u_0)}(Y)(t) \cdot u_E(t) dt = \sum_{E\in\mathscr{E}} \int_E g(L(\tilde{Y})(t), \tilde{X}(t)_g^{\perp}) dt. \tag{2.8}$$

Let $\langle,\rangle_1$ and $\langle,\rangle_2$ denote the inner products in $\mathfrak{X}^0(f_0)^{\perp}_g$ and $\mathcal{G}_0$ respectively given by

$$\langle Z_1, Z_2\rangle_1 = \sum_{E\in\mathscr{E}} \int_E g(Z_1(t), Z_2(t)) dt,$$
$$\langle X_1, X_2\rangle_2 = \sum_{E\in\mathscr{E}} \int_E u_1(t) \cdot u_2(t) dt$$

where $X_i = (0,0,u_E^i)_{E\in\mathscr{E}}$, $i = 1,2$. Let N^* denote the adjoint of N with respect to these inner products, i.e. the map $N^* : \mathfrak{X}^0(f_0)^{\perp}_g \to \mathcal{G}_0$ such that for every $X \in \mathcal{G}_0$ and every $Z \in \mathfrak{X}^0(f_0)^{\perp}_g$ it holds

$$\langle Z, N(X)\rangle_1 = \langle N^*(Z), X\rangle_2.$$

Then

$$\sum_{E\in\mathscr{E}}\int_E g(L(\tilde{Y})(t),\tilde{X}(t)^{\perp})dt = \langle L(\tilde{Y}),O(\tilde{X})\rangle_1$$

$$= \langle L(M(Y)),N(X)\rangle_1$$
$$= \langle N^*(L(M(Y))),X\rangle_2$$

and by (2.8) we deduce that

$$\langle D_2H^1_{(g,u_0)}(Y),X\rangle_2 = \langle N^*(L(M(Y))),X\rangle_2$$

holds for every $X,Y\in\mathcal{F}_0$. Hence, $D_2H^1_{(g,u_0)} = N^*\circ L\circ M$ and as both M,N^* are linear isomorphisms this yields that $D_2H^1_{(g,u_0)}:\mathcal{F}_0\to\mathcal{G}_0$ is Fredholm of index 0, as desired. $\square$

We finish this section by proving the following lemma which will be used in [27].

Lemma 2.5.6. *Let Γ be a good* weighted multigraph and $f_0:\Gamma\to M$ be an embedded non-degenerate stationary geodesic net with respect to a C^k metric g_0, $k\geq 3$. Then there exists a neighborhood W of g_0 in $\mathcal{M}^k$ and a differentiable map $\Delta:W\to\Omega(\Gamma,M)$ such that $\Delta(g)$ is a non-degenerate stationary geodesic net with respect to g for every $g\in W$.*

Proof. Let $f_0:\Gamma\to M$ be as in the statement of the lemma. Take a chart $(\hat{U},\Sigma)$ of $\hat{\Omega}(\mathscr{E},M)$ containing $[f_0]$ as constructed in the previous section. Then we know that there exists a differentiable map $H:\mathcal{M}\times C_0\to\mathcal{Y}$ such that $[f]\in\hat{U}\cap\hat{\Omega}(\Gamma,M)$ is stationary with respect to g if and only if $H(g,\Sigma([f]))=0$. As f_0 is nondegenerate and embedded, $u_0=\Sigma([f_0])$ is nondegenerate and hence we know that $D_2H(g_0,u_0)$ is an isomorphism (here we are using that $D_2H(g_0,u_0)$ is Fredholm of index 0 and Proposition 2.4.10). Applying the Implicit Function Theorem to the map H at the point (g_0,u_0), we get that there is a neighborhood W of g_0 in $\mathcal{M}^k$ and a differentiable map $\Delta:W\to\Omega(\Gamma,M)$ with $\Delta(g_0)=f_0$ such that $\Delta(g)$ is stationary with respect to g for all $g\in W$. By continuity of the Hessian with respect to the metric, shrinking W if necessary we can guarantee that $\Delta(g)$ is nondegenerate. $\square$

2.6 Proofs of the Structure Theorem and of the Bumpy Metrics Theorem in the case $k<\infty$

Let Γ be a good* weighted multigraph. Recall that

$$\mathcal{S}^k(\Gamma)=\{(g,f)\in\mathcal{M}^k\times\hat{\Omega}^{emb}(\Gamma,M): f \text{ is stationary with respect to } g\}.$$

We are going to prove that given a chart $(\hat{U},\Sigma)$ as described in the previous sections, $id\times\Sigma(\mathcal{S}^k(\Gamma)\cap\mathcal{M}^k\times\hat{U})$ is a C^{k-2} Banach submanifold of $\mathcal{M}^k\times C_0$. Then we will use this to construct an atlas for $\mathcal{S}^k(\Gamma)$. We know that

$$id\times\Sigma(\mathcal{S}^k(\Gamma)\cap(\mathcal{M}^k\times\hat{U}))=\{(g,u)\in\mathcal{M}^k\times C_0: H(g,u)=0\}=H^{-1}(0)$$

so our strategy will be to prove that 0 is a regular value of H. For that purpose we will need [47, Theorem 1.2], which we state below.

Theorem 2.6.1. *Let $\mathcal{M}$, X and Y be Banach spaces and $\mathcal{H}$ be a Hilbert Space with $X \subseteq Y \subseteq \mathcal{H}$. Let $L : \mathcal{M} \times X \to \mathbb{R}$ be a C^2 function and suppose there is a C^q map $H : \mathcal{M} \times X \to Y$ such that*

$$\left.\frac{d}{dt}\right|_{t=0} L(g, u + tv) = \langle H(g, u), v\rangle$$

for all $g \in \mathcal{M}$ and all $u, v \in X$. Suppose also that $D_2H(g_0, u_0) : X \to Y$ is a Fredholm map of Fredholm index 0 and that for every nonzero $\kappa \in K = \ker(D_2H(g_0, u_0))$ there exists a one parameter family $g(s) \in \mathcal{M}$ with $g(0) = g_0$ such that

$$\left.\frac{\partial^2}{\partial s \partial t}\right|_{s=t=0} L(g(s), u_0 + t\kappa) \neq 0. \tag{C}$$

Then:

1. *The map $H : \mathcal{M} \times X \to Y$ is a submersion near (g_0, u_0), so there exists a neighborhood W of (g_0, u_0) such that*
$$\mathcal{S} = \{(g, u) \in W : H(g, u) = 0\}$$
is a C^q Banach submanifold of $\mathcal{M} \times X$ and
$$T_{(g,u)}\mathcal{S} = \ker(DH_{(g,u)})$$
for all $(g, u) \in \mathcal{S}$.
2. *The projection $\Pi : \mathcal{S} \to \mathcal{M}$, $\Pi(g, u) = g$ is a C^q Fredholm map of index 0.*

We want to apply the previous theorem for $\mathcal{M} = \mathcal{M}^k$, $X = \Sigma(\hat{U} \cap \hat{\Omega}(\Gamma, M)) = C_0$ which is a Banach submanifold of $U = \Sigma(\hat{U})$ modeled in the Banach space $\mathcal{X}_0 = \ker(DC_{u_0}) \subseteq \mathcal{B} = \prod_{E\in\mathscr{E}} \mathbb{R}^2 \times C^2(E, \mathbb{R}^{n-1})$ (notice that the theorem still works if X is assumed to be a Banach manifold instead of a Banach space, as we are focusing on local structure), $Y = \mathcal{Y} = [\prod_{E\in\mathscr{E}} C^0(E, \mathbb{R}^{n-1})] \times (\mathbb{R}^n)^{|\mathscr{V}|}$, $\mathcal{H} = [\prod_{E\in\mathscr{E}} L^2(E, \mathbb{R}^{n-1})] \times (\mathbb{R}^n)^{|\mathscr{V}|}$, where the inner product of two elements $u_i = ((u_{i,E})_{E\in\mathscr{E}}, (a_{i,v})_{v\in\mathscr{V}})$ of
$[\prod_{E\in\mathscr{E}} L^2(E, \mathbb{R}^{n-1})] \times (\mathbb{R}^n)^{|\mathscr{V}|}$ $(i = 1, 2)$ is given by

$$\langle u_1, u_2\rangle = \sum_{E\in\mathscr{E}} \int_E u_{1,E}(t) \cdot u_{2,E}(t)dt + \sum_{v\in\mathscr{V}} a_{1,v} \cdot a_{2,v}$$

where $\cdot$ denotes the Euclidean inner product in $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$. We consider the inclusion $\iota : \mathcal{X}_0 \to \mathcal{Y}$ given by $\iota(u) = ((u_E)_{E\in\mathscr{E}}, (c_{i_v}(E_v), u_{E_v}(i_v))_{v\in\mathscr{V}})$. The map $L : \mathcal{M} \times X \to \mathbb{R}$ is given by $L(g, u) = \mathrm{L}_g(u)$ and $H : \mathcal{M} \times X \to Y$ is the previously defined map. Notice that H is of class C^q for $q = k - 2 \geq 1$. By the first variation formula, we have

$$\left.\frac{d}{dt}\right|_{t=0} L(g, u + tv) = \langle H(g, u), \iota(v)\rangle.$$

As we proved in the previous section that $D_2H_{(g_0,u_0)}$ is Fredholm of index 0, in order to apply the theorem it only remains to show that condition (C) holds.

Proof that Condition (C) holds. Let us take $\kappa \in \ker(D_2 H(g_0, u_0)) \setminus \{0\}$. Let $u : (-\alpha, \alpha) \to U$ be a one parameter family in C_0 with $u(0, \cdot) = u_0$ and $\frac{d}{ds}|_{s=0} u_s = \kappa$. Write $u_s = (a_E(s), b_E(s), u_{s,E})_{E \in \mathscr{E}}$. Consider the corresponding one parameter family $f_s = \tilde{\Lambda}(u_s)$ and the associated vector field $J = \frac{d}{ds}|_{s=0} f_s = D\tilde{\Lambda}(\kappa)$. By Proposition 2.4.10 we know that J is Jacobi along f_0. We want to construct a one parameter family $g(x)$ of metrics with $g(0) = g_0$ such that

$$\frac{\partial^2}{\partial x \partial s}\bigg|_{x=s=0} L(g(x), u_s) \neq 0.$$

By definition of $L(g, u)$, this is the same as finding $g(x)$ such that

$$\frac{\partial^2}{\partial x \partial s}\bigg|_{x=s=0} \sum_{E \in \mathscr{E}} \int_E L^E_{g(x)}(t, a_E(s), b_E(s), u_{s,E}(t), \dot{u}_{s,E}(t)) dt \neq 0.$$

We will follow the reasoning from [47]. Consider a one parameter family of metrics $g_x(z) = (1 + xh(z))g_0(z)$ conformal to g_0 (here $h : M \to \mathbb{R}$ is a smooth function). Then given $u = (a_E, b_E, u_E)_{E \in \mathscr{E}}$ and $E \in \mathscr{E}$,

$$L^E_{g(x)}(t, a_E, b_E, u_E(t), \dot{u}_E(t)) = \sqrt{1 + xh(f_E(t))} L_{g_0}(t, a_E, b_E, u_E(t), \dot{u}_E(t))$$

where $f_E(t) = \mathcal{E} \circ \phi_E(a_E(1-t) + b_E t, u_E(t))$. Suppose that h vanishes along f_0. Denote $f_E(s, t) = \mathcal{E} \circ \phi_E(a_E(s)(1-t) + b_E(s)t, u_{s,E}(t))$ the restriction of the previously defined f_s to the edge E. Then

$$\begin{aligned}
&\frac{\partial^2}{\partial x \partial s}\bigg|_{x=s=0} \sum_{E \in \mathscr{E}} \int_E L^E_{g(x)}(t, a_E(s), b_E(s), u_{s,E}(t), \dot{u}_{s,E}(t)) dt \\
&= \frac{d}{ds}\bigg|_{s=0} \sum_{E \in \mathscr{E}} \int_E \frac{1}{2} h(f_E(s,t)) L^E_{g_0}(t, a_E(s), b_E(s), u_s(t), \dot{u}_s(t)) dt \\
&= \frac{1}{2} \sum_{E \in \mathscr{E}} \int_E \big[\frac{d}{ds}\bigg|_{s=0} h(f_E(s,t))\big] L^E_{g_0}(t, a_E(0), b_E(0), u_{0,E}(t), \dot{u}_{0,E}(t)) dt \\
&\quad + \frac{1}{2} \sum_{E \in \mathscr{E}} \int_E h(f_0(t)) \big[\frac{d}{ds}\bigg|_{s=0} L^E_{g_0}(t, a_E(s), b_E(s), u_s(t), \dot{u}_s(t))\big] dt \\
&= \frac{1}{2} \sum_{E \in \mathscr{E}} \int_E \langle \nabla h_{f_0(t)}, J(t) \rangle_{\gamma_0} L^E_{g_0}(t, a_E(0), b_E(0), u_{0,E}(t), \dot{u}_{0,E}(t)) dt
\end{aligned}$$

where we used that $h(f_0(t)) = 0$ for all $t \in \Gamma$ because h vanishes along f_0 and that $\frac{\partial f}{\partial s}(0, t) = D\tilde{\Lambda}_{u_0}(\kappa) = J$.

By Proposition 2.4.10, $J = D\tilde{\Lambda}_{u_0}(\kappa)$ is not a parallel Jacobi field along $f_0 = \Lambda(u_0)$ because $\kappa \neq 0$. Therefore, there must exist an edge $E_0 \in \mathscr{E}$ and an interior point $t_0 \in \mathrm{int}(E_0)$ such that $J(t_0)$ is not parallel to $\dot{f}_0(t_0)$. It is possible to define the smooth function $h : M \to \mathbb{R}$ with the following properties:

1. h has support in a small ball around $f_0(t_0)$ which does not intersect $f_0(E)$ for any $E \in \mathscr{E} \setminus \{E_0\}$.

2. $h(f_0(t)) = 0$ for all $t \in \Gamma$.

3. $\langle \nabla h_{f_0(t)}, J(t)\rangle_{\gamma_0} \geq 0$ for all $t \in \Gamma$ and $\langle \nabla h_{f_0(t_0)}, J(t_0)\rangle_{\gamma_0} > 0$.

As $L_{g_0}(t, a_E(0), b_E(0), u_{0,E}(t), \dot{u}_{0,E}(t)) > 0$ for all $t \in \Gamma$ we deduce

$$\sum_{E \in \mathscr{E}} \int_E \langle \nabla h_{f_0(t)}, J(t)\rangle_{\gamma_0} L_{g_0}(t, a_E(0), b_E(0), u_{0,E}(t), \dot{u}_{0,E}(t))dt > 0$$

and hence condition (C) is satisfied and Theorem 2.6.1 can be applied. □

The previous shows that $\mathcal{S}^k(\Gamma) \subseteq \mathcal{M}^k \times \hat{\Omega}^{emb}(\Gamma, M)$ is a C^0 embedded Banach submanifold (recall that the charts of $\hat{\Omega}(\Gamma, M)$ do not have differentiable transition maps). We will prove that in fact the transition maps on the induced atlas for $\mathcal{S}^k(\Gamma)$ are of class C^{k-2} (i.e. as regular as they can be), using an argument of Brian White (see [47, p. 179]).

The idea will be to use that $\Pi : \mathcal{S} \to \mathcal{M}^k$ is Fredholm of index 0 (as shown above) to prove that the Banach space modelling $\mathcal{S}^k(\Gamma)$ is the one which models $\mathcal{M}^k$. This is clear when $\ker(D\Pi_{(g_0,u_0)}) = 0$ by the Inverse Function Theorem. In general, we can do the following construction.

Let $(g_0, f_0) \in \mathcal{S}^k(\Gamma)$ and consider two charts $(\hat{U}_1, \Sigma_1)$ and $(\hat{U}_2, \Sigma_2)$ containing f_0. Denote $\mathcal{S}_i = id \times \Sigma_i((\mathcal{M}^k \times \hat{U}_1 \cap \hat{U}_2) \cap \mathcal{S}^k(\Gamma)) \subseteq \mathcal{M}^k \times \mathcal{B}$ and $u_i = \Sigma_i(f_0)$. We know that $K_i = \ker(D\Pi_{(g_0,u_i)})$ is finite dimensional. Define a map $\Psi : \mathcal{S}^k(\Gamma) \to \mathcal{M}^k \times \mathbb{R}^Q$ as

$$\Psi(g, f) = (g, \int_f \omega_1, ..., \int_f \omega_Q)$$

where $\omega_1, ..., \omega_Q$ are smooth 1-forms on M to be chosen. The idea is to choose these differential forms so that the maps $\Psi \circ \Lambda_i : \mathcal{S}_i \to \mathcal{M}^k \times \mathbb{R}^Q$ are C^{k-2} embeddings in a small neighborhood of u_i. Notice that for any choice of the ω_j's, $\Psi \circ \Lambda_i$ is C^{k-2} and has finite dimensional kernel and cokernel, hence it suffices to do the choices so that their differentials are injective. Denote $F^\omega(g, u) = \int_{\Lambda_1(u)} \omega$, $F^\omega : \mathcal{S}_1 \to \mathbb{R}$. We will choose $\{\omega_1, ..., \omega_r\}$ so that for every $\kappa \in K_1 \setminus \{0\}$ there exists $1 \leq j \leq r$ so that $DF^{\omega_j}_{(g_0,u_1)}(\kappa) \neq 0$. Then we will do the same for K_2 by choosing the forms $\omega_{r+1}, ..., \omega_Q$ in an analogous way. This will guaranty that $D(\Psi \circ \Lambda_i)_{(g_0,u_i)}$ is a monomorphism for $i = 1, 2$.

Given $\kappa \in K_1 \setminus \{0\}$, if $\{\phi_s\}_s$ is the one-parameter family of diffeomorphisms generated by $J = D\Lambda_{u_0}(\kappa)$, we have

$$\begin{aligned}
DF^\omega_{(g_0,u_1)}(\kappa) &= \frac{d}{ds}\bigg|_{s=0} \int_{\Lambda(u_1 + s\kappa_i)} \omega \\
&= \frac{d}{ds}\bigg|_{s=0} \int_{f_0} \phi_s^* \omega \\
&= \int_{f_0} \frac{d}{ds}\bigg|_{s=0} \phi_s^* \omega \\
&= \int_{f_0} \mathcal{L}_J \omega \\
&= \int_{f_0} d\iota_J \omega + \iota_J d\omega
\end{aligned}$$

by Cartan's magic formula. Thus given $\kappa \in K_1 \setminus \{0\}$, as J is a nonparallel Jacobi field along f_0, we can pick a point t_0 in the interior of an edge such that $J(t_0) \notin \langle \dot{f}_0(t_0)\rangle$. Then we can define a smooth function h with support in a small ball around $f_0(t_0)$, which vanishes along f_0 and such that $\langle J(t), \nabla h_{f_0(t)}\rangle_{\gamma_0} \geq 0$ for all t, with strict inequality at $t = t_0$ as described above. Then if η

is a 1-form on M verifying $\eta(\dot{f}_0(t)) = 0$ for all $t \in \Gamma$, the form $\omega = h\eta$ will work. This is because $\int_{f_0} d\iota_J \omega$ vanishes as $\iota_J\omega$ vanishes at all the vertices of Γ, and as h vanishes along f_0,

$$\int_{f_0} \iota_J d\omega = \int_\Gamma dh \wedge \eta(J(t), \dot{f}_0(t))dt = \int_\Gamma dh(J(t))\eta(\dot{f}_0(t))dt = \int_\Gamma \langle \nabla h_{f_0(t)}, J(t)\rangle_{\gamma_0} dt$$

because $dh(\dot{f}_0(t)) = 0$. The previous quantity is strictly positive by construction of h, being $DF^\omega_{(g_0,u_1)}(\kappa) \neq 0$. Although this ω is a priori only C^k, we can perturb it slightly so that it becomes smooth but still verifies $DF^\omega_{(g_0,u_1)}(\kappa) \neq 0$.

On the other hand, given a smooth 1-form ω, the set $A^\omega := \{\kappa \in K_1 : DF^\omega_{(g_0,u_1)}(\kappa) \neq 0\}$ is open. Therefore, $\{A^\omega : \omega \text{ 1-form on } M\}$ is an open cover of $K_1 \setminus \{0\}$. Take $\{\omega_1, ..., \omega_r\}$ such that $\{A^{\omega_j} : 1 \leq j \leq r\}$ is a finite subcover of $\{\kappa \in K_1 : |\kappa| = 1\}$ (which is compact independently of the norm $|\cdot|$ we choose for K_1 as it is finite dimensional). It follows that if $F = (F^{\omega_1}, ..., F^{\omega_r})$ then $DF_{(g_0,u_1)}(\kappa) \neq 0$ for every $\kappa \in K_1 \setminus \{0\}$. Proceeding equally for K_2, we obtain that $\Psi \circ \Lambda_i$ is an immersion near (g_0, u_i) for $i = 1, 2$. Then the transition map $\Lambda_2^{-1} \circ \Lambda_1 = (\Psi \circ \Lambda_2)^{-1} \circ (\Psi \circ \Lambda_1)$ is a C^{k-2} diffeomorphism of Banach manifolds near (g_0, u_1), as desired.

Observe that Proposition 2.4.10 implies that if $(g, [f]) \in \mathcal{S}^k(\Gamma)$, f is nondegenrerate if and only if given a chart $(\hat{U}, \Sigma)$ containing f with $\Sigma([f]) = u$ we have that u is nondegenerate as defined in Section 2.4. But the following 4 conditions are equivalent:

1. $D\Pi_{(g,u)}$ is an epimorphism.

2. $D\Pi_{(g,u)}$ is injective.

3. $D_2H_{(g,u)}$ is injective.

4. u is nondegenerate with respect to g

as $\ker(D\Pi_{(g,u)}) = \{0\} \times \ker D_2H_{(g,u)}$. This completes the proof of Theorem 2.1.2 for good* weighted multigraphs. The theorem for closed loops with multiplicity is a particular case of the Structure Theorem of Brian White proved in [47], and this covers all the cases of Theorem 2.1.2.

On the other hand, by Smale's version of Sard's Theorem for Banach spaces proved in [40], for each good weighted multigraph Γ the subset $\mathcal{N}^k(\Gamma) \subseteq \mathcal{M}^k$ of regular values of $\Pi : \mathcal{S}^k(\Gamma) \to \mathcal{M}^k$ is generic in the Baire sense. Observe that parts *(2)* and *(3)* of Theorem 2.1.2 imply that $g \in \mathcal{N}^k(\Gamma)$ if and only if g is bumpy with respect to Γ. Considering that the collection $\{\Gamma : \Gamma \text{ is a good weighed multigraph}\}$ is countable, $\mathcal{N}^k := \bigcap_\Gamma \mathcal{N}^k(\Gamma)$ is also generic in the Baire sense and is by definition the set of bumpy C^k metrics. This proves Theorem 2.1.3 in the case $k < \infty$.

2.7 C^∞ case

In this section, we are going to discuss how to extend Theorem 2.1.3 to C^∞ Riemannian metrics (the analog result for minimal submanifolds is stated in [49]). Denote $\mathcal{M}^\infty = \bigcap_{k\in\mathbb{N}} \mathcal{M}^k$ the space of C^∞ Riemannian metrics on M equipped with the C^∞ topology, which admits a natural Frechet manifold structure.

Theorem 2.7.1. *The subset $\mathcal{N}^\infty \subseteq \mathcal{M}^\infty$ of bumpy C^∞ metrics is generic in the Baire sense with respect to the C^∞ topology.*

In order to prove the theorem, we will need the following lemma.

Lemma 2.7.2. *Let $\mathcal{N}^k \subseteq \mathcal{M}^k$ be a generic subset in the Baire sense with respect to the C^k topology for each $k \in \mathbb{N}_{\geq 3}$. Assume that if $k' \geq k$ then $\mathcal{N}^{k'} = \mathcal{N}^k \cap \mathcal{M}^{k'}$. Then $\mathcal{N}^\infty = \bigcap_{k\in\mathbb{N}} \mathcal{N}^k \subseteq \mathcal{M}^\infty$ is generic in the Baire sense with respect to the C^∞ topology.*

Proof. Let us write $\mathcal{N}^3 = \cap_{l\in\mathbb{N}} \mathcal{N}^{3,l}$ where each $\mathcal{N}^{3,l}$ is open and dense in $\mathcal{M}^3$ with the C^3 topology. For each $k \geq 3$ define $\mathcal{N}^{k,l} = \mathcal{N}^{3,l} \cap \mathcal{M}^k$. Observe that given $k \geq 3$,

$$\bigcap_{l\in\mathbb{N}} \mathcal{N}^{k,l} = (\bigcap_{l\in\mathbb{N}} \mathcal{N}^{3,l}) \cap \mathcal{M}^k = \mathcal{N}^3 \cap \mathcal{M}^k = \mathcal{N}^k.$$

As by hypothesis $\mathcal{N}^k \subseteq \mathcal{M}^k$ is generic, by the Baire Category Theorem it is dense and therefore each $\mathcal{N}^{k,l} \subseteq \mathcal{M}^k$ is dense and also open (because the C^k topology is finer than the C^3 topology for every $k \geq 3$). Define

$$\mathcal{N}^{\infty,l} = \mathcal{N}^{3,l} \cap \mathcal{M}^\infty = \mathcal{N}^{3,l} \cap \bigcap_{k\geq 3} \mathcal{M}^k = \bigcap_{k\geq 3} \mathcal{N}^{k,l}.$$

Let us show that $\mathcal{N}^{\infty,l} \subseteq \mathcal{M}^\infty$ is dense. Pick $g_0 \in \mathcal{M}^\infty$ and an open neighborhood W of g_0 in $\mathcal{M}^\infty$. Let $k \in \mathbb{N}_{\geq 3}$ and $\delta > 0$ be such that $\{g \in \mathcal{M}^\infty : d_k(g, g_0) < \delta\} \subseteq W$ where d_k is a metric which induces the C^k topology on $\mathcal{M}^k$. By density of $\mathcal{N}^{k,l}$ in $\mathcal{M}^k$, there exists $g_1 \in \mathcal{N}^{k,l}$ such that $d_k(g_1, g_0) < \frac{\delta}{2}$. On the other hand, as $\mathcal{M}^\infty \subseteq \mathcal{M}^k$ is dense with the C^k topology and $\mathcal{N}^{k,l} \subseteq \mathcal{M}^k$ is open, there exists $g_2 \in \mathcal{M}^\infty \cap \{g \in \mathcal{M}^k : d_k(g, g_1) < \frac{\delta}{2}\} \cap \mathcal{N}^{k,l}$. Therefore by triangle inequality $g_2 \in \{g \in \mathcal{M}^\infty : d_k(g, g_0) < \delta\} \cap \mathcal{N}^{\infty,l} \subseteq W \cap \mathcal{N}^{\infty,l}$ so $\mathcal{N}^{\infty,l} \subseteq \mathcal{M}^\infty$ is dense. It is also open as the C^∞ topology is finer than the C^3 one. Additionally,

$$\bigcap_{l\in\mathbb{N}} \mathcal{N}^{\infty,l} = (\bigcap_{l\in\mathbb{N}} \mathcal{N}^{3,l}) \cap \mathcal{M}^\infty = \mathcal{N}^3 \cap (\bigcap_{k\geq 3} \mathcal{M}^k) = \bigcap_{k\geq 3} \mathcal{N}^k = \mathcal{N}^\infty.$$

This means that $\mathcal{N}^\infty \subseteq \mathcal{M}^\infty$ is generic with respect to the C^∞ topology, as desired. $\square$

Proof of Theorem 2.7.1. For each $k \in \mathbb{N}_{\geq 3}$, define $\mathcal{N}^k \subseteq \mathcal{M}^k$ as the set of C^k bumpy metrics. By Theorem 2.1.3 in the case $k < \infty$ (which was already proved), $\mathcal{N}^k \subseteq \mathcal{M}^k$ is generic with respect to the C^k topology for every $k \in \mathbb{N}_{\geq 3}$ and it clearly holds that $\mathcal{N}^{k'} = \mathcal{N}^k \cap \mathcal{M}^{k'}$ whenever $k' \geq k$. Therefore we can apply the lemma and deduce that $\mathcal{N}^\infty = \cap_{k\in\mathbb{N}} \mathcal{N}^k$ is generic in the space $\mathcal{M}^\infty$ of C^∞ metrics. As $\mathcal{N}^\infty$ is precisely the set of C^∞ bumpy metrics, this completes the proof of the theorem. $\square$

Chapter 3

Generic Density of Stationary Geodesic Nets

3.1 Summary of the chapter

This chapter is based on the article [27]. We prove the following result.

Theorem 3.1.1. *Let M^n, $n \geq 2$, be a closed manifold and let $\mathcal{M}^k$ be the space of C^k Riemannian metrics on M, $3 \leq k \leq \infty$. For a generic (in the Baire sense) subset of $\mathcal{M}^k$ the union of the images of all embedded stationary geodesic nets in (M, g) is dense.*

An analogous density result for closed geodesics on surfaces was proved by Irie [23]. For minimal hypersurfaces in Riemannian manifolds of dimension $3 \leq n \leq 7$ a generic density result was proved by Irie-Marques-Neves [22].

Regarding stationary geodesic networks, we use the notation and results from Chapter 2, in particular, the Structure Theorem for Stationary Geodesic Nets (Theorem 2.1.2). We also use the following result which was proved for embedded Γ-nets when Γ is good* in Chapter 2 (Lemma 2.5.6). The same argument can be adapted to closed geodesics using the Structure Theorem of Brian White [47] instead of Theorem 2.1.2. A more elementary proof can be obtained considering the finite dimensional models of the spaces of geodesic nets (instead of working with the infinite dimensional $\Omega(\Gamma, M)$ as in [46]).

Lemma 3.1.2. *Let Γ be a good weighted multigraph and $f_0 : \Gamma \to M$ be an embedded non-degenerate stationary geodesic net with respect to a C^k metric g_0, $k \geq 3$. Then there exists a neighborhood W of g_0 in $\mathcal{M}^k$ and a differentiable map $\Delta : W \to \Omega(\Gamma, M)$ such that $\Delta(g)$ is a non-degenerate stationary geodesic net with respect to g for every $g \in W$.*

3.2 Min-max constructions

Stationary geodesic nets arise from Almgren-Pitts Morse theory on the space of 1-cycles.

By Almgren isomorphism theorem ([1], [2], [16]) the space of mod 2 k-cycles on the n-sphere $\mathcal{Z}_k(S^n, \mathbb{Z}_2)$ is weakly homotopy equivalent to the Eilenberg-MacLane space $K(\mathbb{Z}_2, n-k)$. Let $\overline{\lambda}$

denote the non-trivial element of $H^{n-k}(\mathcal{Z}_k(S^n,\mathbb{Z}_2);\mathbb{Z}_2)$. Note that all cup powers of $\overline{\lambda}$ are non-trivial and the cohomology ring of $\mathcal{Z}_k(S^n,\mathbb{Z}_2)$ is generated by the cup powers and Steenrod squares of $\overline{\lambda}$ ([18]).

Given a closed n-dimensional Riemannian manifold (M,g) consider $\phi : M \to S^n$ that maps a small open ball $B \subset M$ diffeomorphically onto $S^n \setminus \{p\}$ and sends the rest of M to point $\{p\}$. For the corresponding map on the space of cycles $\Phi : \mathcal{Z}_k(M,\mathbb{Z}_2) \to \mathcal{Z}_k(S^n,\mathbb{Z}_2)$ the pull-back $\lambda = \Phi^*(\overline{\lambda}) \neq 0$. The cohomology class $\lambda \in H^{n-k}(\mathcal{Z}_k(M;\mathbb{Z}_2))$ is the fundamental cohomology class of $\mathcal{Z}_k(M;\mathbb{Z}_2)$ and can be constructed intrinsically on each compact manifold M using Almgren's gluing homomorphism associated to a continuous family of cycles $F : X \to \mathcal{Z}_k(M)$. This was done by Larry Guth in [15][Section 1] for families in $(D^n, \partial D^n)$. The same construction can be performed on any compact manifold with boundary, as stated in [15][Appendix 3].

Given a cubical complex X we say that $F : X \to \mathcal{Z}_k(M,\mathbb{Z}_2)$ is a p-sweepout if $F^*(\lambda^p) \neq 0 \in H^{p(n-k)}(X;\mathbb{Z}_2)$ and F satisfies a no-concentration of mass property (cf. [29], [26]). We define the k-dimensional p-width $\omega_p^k(M,g)$ by

$$\omega_p^k(M,g) = \inf\{\sup_{x\in X} \mathbf{M}(F(x)) : F \text{ is a } p\text{-sweepout of } M\}$$

Using arguments of [13], [14, Section 8], [15] we obtain the following upper bounds for the widths $\omega_p^k(M,g)$.

Proposition 3.2.1. *Let (M,g) be a closed n-dimensional Riemannian manifold. There exists a constant $C = C(M,g)$ such that $\omega_p^k(M,g) \leq Cp^{\frac{n-k}{n}}$.*

Proof. The case of $k = n-1$ was proved in [30, Theorem 5.1]. Assume $1 \leq k \leq n-2$. Let $\mathrm{Sym}_p\, S^{n-k}$ denote the symmetric product of spheres $\mathrm{Sym}_p\, S^{n-k} = \{(x_1,...,x_p) : x_i \in S^{n-k}\}/\operatorname{Per}(p)$, where $\operatorname{Per}(p)$ is the group of permutations of p elements. For $1 \leq j \leq p$ we have that $H^{j(n-k)}(\mathrm{Sym}_p\, S^{n-k}) = \langle \alpha^j \rangle$, where α is the non-trivial cohomology class in $H^{n-k}(\mathrm{Sym}_p\, S^{n-k})$ (we are considering cohomology with $\mathbb{Z}_2$ coefficients, see [34]). In [15] Guth constructed p-sweepouts $F_p : \mathrm{Sym}_p\, S^{n-k} \to \mathcal{Z}_k(B,\partial B;\mathbb{Z}_2)$ of the Euclidean unit ball $B \subset \mathbb{R}^n$ by piecewise linear relative k-cycles satisfying

$$\sup\{\mathbf{M}(F_p(x)) : x \in \mathrm{Sym}_p\, S^{n-k}\} \leq C_n p^{\frac{n-k}{n}}$$

Fix a fine triangulation and PL structure on M that is bilipschitz equivalent to the original metric g, and let $\Phi : M \to \mathbb{R}^n$ be a PL map, such that each simplex Δ is bilipschitz to $\Phi(\Delta)$. After scaling we may assume that $\Phi(M) \subset \operatorname{int}(B)$. If z is a piecewise linear relative cycle in B, then $\Phi^{-1}(z)$ is a k-cycle in M. The map $F'_p : \mathrm{Sym}_p\, S^{n-k} \to \mathcal{Z}_k(M;\mathbb{Z}_2)$ defined as $F'_p(x) = \Phi^{-1}(F_p(x))$ satisfies the desired mass bound. To see that this is a p-sweepout consider the restriction of F'_p to $\{[x,0,...,0] : x \in S^{n-k}\} \subset \mathrm{Sym}_p\, S^{n-k}$. It is straightforward to check that Almgren gluing map ([1]) maps this family to the fundamental homology class of M, so $(F'_p)^*(\lambda) = \alpha \in H^{(n-k)}(\mathrm{Sym}_p\, S^{n-k})$. $\square$

Almgren showed that widths correspond to volumes of stationary integral varifolds. For 1-dimensional widths a stronger regularity result is known (see [1], [2], [5], [32], [36], [37]), namely, that the stationary integral 1-varifolds are, in fact, stationary geodesic nets. Combining this result with Lemma 2.2.14 we obtain the following.

Proposition 3.2.2. *The width $\omega_p^1(M,g) = \sum_{i=1}^P \mathrm{L}_g(\gamma_i)$, where $\gamma_i : \Gamma_i \to M$ is an embedded stationary geodesic net and Γ_i is a good weighted multigraph for each $1 \leq i \leq P$.*

In [22] density of minimal hypersurfaces was proved using a Weyl law for $(n-1)$-dimensional p-widths. The Weyl law was proved for $(n-1)$-cycles in arbitrary compact manifolds and for k-cycles in Euclidean domains in [26]. However, it is not known in general for $k < n-1$, although the special case of 1-cycles in 3-manifolds was resolved in [16] shortly before our work [27] was posted.

In [43] Song observed that the full strength of the Weyl law is not needed to prove density of minimal hypersurfaces for generic metrics. (It does, however, seem that the Weyl law is necessary to prove a stronger equidistribution result in [31]). The idea of Song allows us to circumvent the use of Weyl law to prove density of stationary geodesic nets.

Lemma 3.2.3. *Let g_1 and g_2 be two metrics on M with $g_2 \geq g_1$ and $g_2(x_0) > g_1(x_0)$ for some $x_0 \in M$. Then there exists $p \geq 1$, such that $\omega_p^k(M,g_2) > \omega_p^k(M,g_1)$.*

Proof. Let $B_r(x_0)$ be a small closed ball such that $g_2 > g_1$ on $B_r(x_0)$. Fix $\varepsilon > 0$, such that for every k-cycle z with g_2-mass $\mathbf{M}_{g_2}(z \llcorner B_r(x_0)) > \frac{1}{2}\omega_1^k(B_r(x_0), g_2)$ we have $\mathbf{M}_{g_2}(z) - \mathbf{M}_{g_1}(z) > \varepsilon$.

By Proposition 3.2.1 we have $\omega_p^k(M,g_1) \leq Cp^{\frac{n-k}{n}}$ for some constant $C > 0$. In particular, we can find $p > 0$ such that $\omega_p^k(M,g_1) - \omega_{p-1}^k(M,g_1) < \varepsilon/4$. Let $F : X \to Z_k(M;\mathbb{Z}_2)$ be a p-sweepout of (M,g_2) such that $\mathbf{M}_{g_2}(F(x)) \leq \omega_p^k(M,g_2) + \varepsilon/4$ for all $x \in X$. By [26, Lemma 2.15] we can assume that the map F is continuous in the mass norm.

Recall that if two manifolds are bilipschitz diffeomorphic, then the corresponding spaces of cycles are homeomorphic. In particular, a p-sweepout of one induces a p-sweepout of the other. Let $X_1 = \{x \in X : \mathbf{M}_{g_2}(F(x) \llcorner B_r(x_0)) > \frac{1}{2}\omega_1^k(B_r(x_0), g_2)\}$ be an open subset of X. We claim that the restriction of F to X_1 is a $(p-1)$-sweepout of M (with respect to both g_1 and g_2 as (M,g_1) and (M,g_2) are bilipschitz diffeomorphic). Indeed, let $\lambda \in H^{n-k}(\mathcal{Z}_k(M,\mathbb{Z}_2))$ be the fundamental cohomology class. Then λ vanishes on $X \setminus X_1$ because $F|_{X\setminus X_1} \llcorner B_r(x_0)$ is not a sweepout of $B_r(x_0)$ and hence $F|_{X\setminus X_1}$ can not be a sweepout of M. If λ^{p-1} vanishes on X_1, then λ^p vanishes on $X_1 \cup (X \setminus X_1) = X$, which contradicts the definition of p-sweepout.

It follows that $\{F(x)\}_{x\in X_1}$ is a $(p-1)$-sweepout of M and

$$\begin{aligned}
\omega_{p-1}^k(M,g_1) &\leq \sup\{\mathbf{M}_{g_1}(F(x)) : x \in X_1\} \\
&\leq \sup\{\mathbf{M}_{g_2}(F(x)) : x \in X_1\} - \varepsilon \\
&\leq \omega_p^k(M,g_2) - 3/4\varepsilon
\end{aligned}$$

If $\omega_p^k(M,g_2) = \omega_p^k(M,g_1)$ then our choice of p leads to a contradiction. $\square$

The next Lemma follows as in [31, Lemma 1].

Lemma 3.2.4. *Let M be a closed manifold. Then the k-dimensional p-width $\omega_p^k(g)$ is a locally Lipschitz function of the metric g in the space $\mathcal{M}^0$ of C^0 metrics.*

Proof. First we need to give a metric space structure to the set $\mathcal{M}^0$. Observe that each $g \in \mathcal{M}^0$ induces a metric d_g in $\mathcal{M}^0$ defined as

$$d_g(g_1,g_2) = \sup_{v\neq 0} \frac{|g_1(v,v) - g_2(v,v)|}{g(v,v)}$$

It is easy to show that as M is compact, given $g, g' \in \mathcal{M}^0$ the induced metrics d_g and $d_{g'}$ are equivalent. Therefore we can pick an arbitrary $g_0 \in \mathcal{M}^0$ and fix d_{g_0} as our metric.

Now in order to prove the lemma, fix a metric $g \in \mathcal{M}^0$ and suppose g_1, g_2 satisfy $g/C_1 \leq g_i \leq C_1 g$ for $i = 1, 2$ and some $C_1 > 1$. For some constant $C = C(g) > 0$ we have $\omega_p^k(M, g) \leq C p^{\frac{n-k}{n}}$ by Proposition 3.2.1.

Given a k-cycle $z \in \mathcal{Z}_k(M; \mathbb{Z}_2)$ we have

$$\begin{aligned}
\mathbf{M}_{g_1}(z) - \mathbf{M}_{g_2}(z) &\leq \Big(\big(\sup_{v \neq 0} \frac{g_1(v,v)}{g_2(v,v)} \big)^{\frac{k}{2}} - 1 \Big) \mathbf{M}_{g_2}(z) \\
&\leq \Big(\big(1 + \sup_{v \neq 0} \frac{|g_1(v,v) - g_2(v,v)|}{g_2(v,v)} \big)^{\frac{k}{2}} - 1 \Big) \mathbf{M}_{g_2}(z) \\
&\leq \Big(\big(1 + C_1 d_g(g_1, g_2) \big)^{\frac{k}{2}} - 1 \Big) \mathbf{M}_{g_2}(z) \\
&\leq C_1 k d_g(g_1, g_2) \mathbf{M}_{g_2}(z)
\end{aligned}$$

for small $d_g(g_1, g_2)$.

Then for g_1, g_2 near g we have

$$\begin{aligned}
|\omega_p^k(M, g_1) - \omega_p^k(M, g_2)| &\leq C_1 k d_g(g_1, g_2) \omega_p^k(M, g_2) \\
&\leq C_1^{1+\frac{k}{2}} k C p^{\frac{n-k}{n}} d_g(g_1, g_2)
\end{aligned}$$

As d_g is equivalent to d_{g_0} we get the desired result. □

3.3 Proof of the main theorem

Fix a manifold M and an open subset $U \subset M$. Let $\mathcal{M}^k(U) \subset \mathcal{M}^k$ denote the set of C^k metrics g such that there exists an embedded non-degenerate stationary geodesic net in (M, g) intersecting U whose domain is a good weighted multigraph. First we will analyse the case $3 \leq k < \infty$.

By Lemma 3.1.2 we have that $\mathcal{M}^k(U)$ is open. Now we will show that $\mathcal{M}^k(U)$ is dense. Let $V \subseteq \mathcal{M}^k$ be an open subset. We have to show that there exists some $g \in V \cap \mathcal{M}^k(U)$.

Let $\{\Gamma_m\}_{m \in \mathbb{N}}$ be the countable collection of all good weighted multigraphs. Let $\mathcal{C}_m = \mathcal{S}^k(\Gamma_m)$. We have that the projection map $\Pi_m : \mathcal{C}_m \to \mathcal{M}^k$ is a Fredholm map of index 0 by Theorem 2.1.2. Let $Reg_m \subset \mathcal{M}^k$ denote the set of regular values of Π_m and $R = \bigcap_{m \geq 0} Reg_m$. By Sard-Smale theorem the set R is comeager, so we can find a metric $g_0 \in V \cap R$. If $g_0 \in \mathcal{M}^k(U)$ we are done, so let us assume the contrary. Then all embedded stationary geodesic nets of (M, g_0) with domain a good weighted multigraph are non-degenerate and do not intersect U. Let L_0 denote the (countable) set of lengths of such geodesic networks. By Lemma 2.2.14, the set L_1 of lengths of all stationary geodesic nets for the metric g_0 is the set of finite sums of elements in L_0, and hence it is also countable.

Let $\phi : M \to \mathbb{R}$ be a non-negative smooth bump function supported in U with $\phi(x_0) > 0$ for some $x_0 \in U$. Define $g_t(x) = (1 + t\phi(x))g_0(x)$. For some sufficiently small $\varepsilon > 0$ we have that $g_t \in V$ for all $t \in [0, \varepsilon]$. By Lemma 3.2.3 there exists $p > 0$, such that $\omega_p^1(g_\varepsilon) > \omega_p^1(g_0)$.

By Smale's transversality theorem from [40], there exists a sequence of embeddings $g_i : [0, \varepsilon] \to \mathcal{M}^k$ converging to g, such that each g_i is transverse to the maps $\Pi_m : \mathcal{C}_m \to \mathcal{M}^k$ for all $m \geq 0$. Moreover, using [40, Theorem 3.3] we have that $I_{i,m} = \Pi_m^{-1}(g_i([0, \varepsilon]))$ is a 1-dimensional submanifold of $\mathcal{C}_m$ for each $(i, m) \in \mathbb{N} \times \mathbb{N}_0$. Notice that by transversality, if t is a regular value of $(g_i)^{-1} \circ \Pi_m|_{I_{i,m}}$, then $g_i(t) \in Reg_m$ (cf. [31, Lemma 2]). By the finite-dimensional Sard's lemma applied to $(g_i)^{-1} \circ \Pi_m|_{I_{i,m}}$ we have that $C_i = \bigcap_{m \geq 0} \{t : g_i(t) \in Reg_m\} \subset [0, \varepsilon]$ is a subset of full measure.

Note that $\omega_p^1(g_i([0,\varepsilon])) \to \omega_p^1(g([0,\varepsilon]))$ as $i \to \infty$ and without any loss of generality we may assume that there is an interval $[a, b] \subset \omega_p^1(g_i([0, \varepsilon]))$ for all i. By Lemma 3.2.4 we have that $C = \bigcap_{i=1}^{\infty} \omega_p^1(g_i(C_i)) \cap [a, b] \setminus L_1$ is non-empty (because L_1 is countable and $\omega_p^1(g_i(C_i)) \cap [a, b]$ is a full measure subset of $[a, b]$ for every $i \in \mathbb{N}$). Let $l \in C$. By Proposition 3.2.2, for each $i \in \mathbb{N}$ we have that $l = \sum_{j=1}^{P_i} \mathrm{L}_{g_i(t_i)}(\gamma_j^i)$, where each γ_j^i is an (non-degenerate) embedded stationary geodesic net in $(M, g_i(t_i))$ whose domain is a good weighted multigraph, for some $t_i \in (\omega_p^1 \circ g_i)^{-1}(l)$. Passing to a subsequence if necessary, we can assume that there exists $t' = \lim_{i\to\infty} t_i \in [0, \varepsilon]$ and that the sequence $\gamma^i = \bigcup_j \gamma_j^i$ converges to a stationary geodesic net γ in $(M, g_{t'})$. However, since $\mathrm{L}_{g'}(\gamma) = l \notin L_1$, γ is not a stationary geodesic net for g_0 and hence it must intersect U. As $\lim_{i\to\infty} \gamma^i = \gamma$, there exists $i_1 \in \mathbb{N}$ such that γ^i intersects U for all $i \geq i_1$. On the other hand, as $\lim_{i\to\infty} g_i(t_i) = g_{t'} \in V$, there exists $i_2 \in \mathbb{N}$ such that $i \geq i_2$ implies $g_i(t_i) \in V$. Thus if $i \geq \max\{i_1, i_2\}$, the metric $g_i(t_i)$ is in V and one component γ_j^i of γ^i is an embedded stationary geodesic net intersecting U whose domain is a good weighted multigraph. As $g_i(t_i)$ is bumpy, we deduce that $g_i(t_i) \in \mathcal{M}^k(U)$ and hence $V \cap \mathcal{M}^k(U) \neq \emptyset$.

So far we have proved that for $3 \leq k < \infty$, $\mathcal{M}^k(U) \subseteq \mathcal{M}^k$ is open and dense for every open subset $U \subseteq M$. Taking a countable basis $\{U_m\}_{m\in\mathbb{N}}$ for the topology of M and setting $\mathcal{N}^k = \bigcap_{m\in\mathbb{N}} \mathcal{M}^k(U_m)$ we see that $\mathcal{N}^k \subseteq \mathcal{M}^k$ is generic and $g \in \mathcal{N}^k$ if and only if the union of the images of all nondegenerate embedded stationary geodesic networks with respect to g whose domain is a good weighted multigraph is dense in M. This proves Theorem 3.1.1 in the case $3 \leq k < \infty$. For the case $k = \infty$, we can define $\mathcal{N}^\infty$ to be the set of C^∞ metrics for which the union of the images of all nondegenerate embedded stationary geodesic nets whose domain is a good weighted multigraph is dense in M. Thus it is clear that $\mathcal{N}^\infty = \bigcap_{k\geq 3} \mathcal{N}^k$ and that if $k' \geq k$ then $\mathcal{N}^{k'} = \mathcal{N}^k \cap \mathcal{M}^{k'}$; so by Lemma 2.7.2 we deduce that $\mathcal{N}^\infty$ is a generic subset of $\mathcal{M}^\infty$ (see also a similar argument in [49] and [7, Corollary 5.14]).

Chapter 4

Generic Equidistribution of Stationary Geodesic Nets

4.1 Summary of the chapter

Marques, Neves and Song proved in [31] that for a generic set of Riemannian metrics in a closed manifold M^n, $3 \leq n \leq 7$, there exists a sequence of closed, embedded, connected minimal hypersurfaces which is equidistributed in M. In this chapter we study the equidistribution of closed geodesics and stationary geodesic nets on a Riemannian manifold (M^n, g), $n \geq 2$. It is based on the article [25], which is joint work with Xinze Li. There we proved the following two results, for dimensions 2 and 3 of the ambient manifold respectively:

Theorem 4.1.1. *Let M be a closed 2-manifold. For a Baire-generic set of C^∞ Riemannian metrics g on M, there exists a set of closed geodesics that is equidistributed in M. Specifically, for every g in the generic set, there exists a sequence $\{\gamma_i : S^1 \to M\}$ of closed geodesics in (M, g), such that for every C^∞ function $f : M \to \mathbb{R}$ we have*

$$\lim_{k\to\infty} \frac{\sum_{i=1}^k \int_{\gamma_i} f \, \mathrm{dL}_g}{\sum_{i=1}^k \mathrm{L}_g(\gamma_i)} = \frac{\int_M f \, \mathrm{dVol}_g}{\mathrm{Vol}(M, g)}.$$

Theorem 4.1.2. *Let M be a closed 3-manifold. For a Baire-generic set of C^∞ Riemannian metrics g on M, there exists a set of connected embedded stationary geodesic nets that is equidistributed in M. Specifically, for every g in the generic set , there exists a sequence $\{\gamma_i : \Gamma_i \to M\}$ of connected embedded stationary geodesic nets in (M, g), such that for every C^∞ function $f : M \to \mathbb{R}$ we have*

$$\lim_{k\to\infty} \frac{\sum_{i=1}^k \int_{\gamma_i} f \, \mathrm{dL}_g}{\sum_{i=1}^k \mathrm{L}_g(\gamma_i)} = \frac{\int_M f \, \mathrm{dVol}_g}{\mathrm{Vol}(M, g)}.$$

Remark 4.1.3. *We have an equivalent notion of equidistribution for a sequence of closed geodesics or geodesic nets: we say that $\{\gamma_i\}_{i\in\mathbb{N}}$ is equidistributed in (M, g) if for every open subset $U \subseteq M$ it holds*

$$\lim_{k\to\infty} \frac{\sum_{i=1}^k \mathrm{L}_g(\gamma_i \cap U)}{\sum_{i=1}^k \mathrm{L}_g(\gamma_i)} = \frac{\mathrm{Vol}_g U}{\mathrm{Vol}_g M}.$$

Theorem 4.1.2 is, as far as the authors know, the first result on equidistribution of k-stationary varifolds in Riemannian n-manifolds for $k < n-1$ (i.e. in codimension greater than 1). Regarding Theorem 4.1.1, similar equidistribution results for closed geodesics have been proved for compact hyperbolic manifolds in [4] in 1972 and for compact surfaces with constant negative curvature in [38] in 1985. More recently, those results were extended to non-compact manifolds with negative curvature in [39] and to surfaces without conjugate points in [9]. The four previous works have in common that they approach the problem from the dynamical systems point of view. In the present chapter, we approach it using Almgren-Pitts min-max theory (as it was done in [31] for minimal hypersurfaces). Additionally, Theorem 4.1.1 is the first equidistribution result for closed geodesics on closed surfaces that is proved for generic metrics, without any restriction regarding the curvature of the metric or the presence of conjugate points.

Our proof is inspired by the ideas in [31]. There are two key results used in [31] to prove equidistribution of minimal hypersurfaces for generic metrics: the Bumpy Metrics Theorem of Brian White [47] and the Weyl Law for the Volume Spectrum proved by Liokumovich, Marques and Neves in [26]: given a compact Riemannian manifold (M^n, g) with $n \geq 2$ (possibly with boundary), we have

$$\lim_{p\to\infty} \omega_p^{n-1}(M,g)p^{-\frac{1}{n}} = \alpha(n)\operatorname{Vol}(M,g)^{\frac{n-1}{n}}$$

for some constant $\alpha(n) > 0$. Here, given $1 \leq k \leq n-1$ we denote by $\omega_p^k(M,g)$ the k-dimensional p-width of M with respect to the metric g (for background on this, see [15], [27], [30] [22]). It was conjectured by Gromov (see [14, section 8.4]) that the Weyl law can be extended to other dimensions and codimensions. In this work, we are interested in the case of 1-dimensional cycles. The following is the Weyl law for 1-cycles which was conjectured by Gromov.

Conjecture 4.1.4. *Let (M^n, g) be a closed n-dimensional manifold, $n \geq 2$. Then there exists a constant $\alpha(n,1) > 0$ such that*

$$\lim_{p\to\infty} \omega_p^1(M^n,g)p^{-\frac{n-1}{n}} = \alpha(n,1)\operatorname{Vol}(M^n,g)^{\frac{1}{n}}.$$

By the time our article [25] was published, Conjecture 4.1.4 had only been proved for $n = 2$ as a particular case of [26] and for $n = 3$ by Guth and Liokumovich in their work [16]. In 2024, it was proved by the author for $n \geq 4$ ([44], [45]). In [25], we used those first two versions of the Weyl law to prove Theorem 4.1.1 and Theorem 4.1.2; and we also used the Structure Theorem for Stationary Geodesic Networks [46] discussed in Chapter 2 of this thesis, and the Structure Theorem of White ([47]) for the case of embedded closed geodesics. The work of Chodosh and Mantoulidis in [7] was used to upgrade the equidistribution result for stationary geodesic networks to closed geodesics in dimension 2. The only obstruction to extend our proof of the equidistribution of stationary geodesic nets to arbitrary dimensions of the ambient manifold M was that Conjecture 4.1.4 had not been proved yet if $n > 3$. As currently the Weyl law for 1-cycles holds in all its generality, this chapter will be focused on proving the following result and then adapting the proof in the case $n = 2$ to obtain closed geodesics.

Theorem 4.1.5. *Let M^n, $n \geq 2$ be a closed manifold. For a Baire-generic set of C^∞ Riemannian metrics g on M, there exists a set of connected embedded stationary geodesic nets that is equidistributed in M. Specifically, for every g in the generic set , there exists a sequence $\{\gamma_i : \Gamma_i \to M\}$ of* *connected embedded stationary geodesic nets in* (M,g)*, such that for every* C^∞ *function* $f : M \to \mathbb{R}$ *we have*

$$\lim_{k\to\infty} \frac{\sum_{i=1}^k \int_{\gamma_i} f \, \mathrm{dL}_g}{\sum_{i=1}^k \mathrm{L}_g(\gamma_i)} = \frac{\int_M f \, \mathrm{dVol}_g}{\mathrm{Vol}(M,g)}.$$

In order to simplify the exposition, we consider integrals of C^∞ functions instead of the more general traces of 2-tensors discussed in [31]. Next we proceed to describe the intuition behind the proof, the technical issues which appear when one tries to carry on that intuition and how to sort them.

Let g be a Riemannian metric on M. We want to do a very small perturbation of g to obtain a new metric $\hat{g}$ which admits a sequence of equidistributed stationary geodesic networks. Let $f : M \to \mathbb{R}$ be a smooth function. Consider a conformal perturbation $\hat{g} : (-\delta, \delta) \to \mathcal{M}^\infty$ (for some $\delta > 0$ small) defined as

$$\hat{g}(t) = e^{2tf} g.$$

By [27, Lemma 3.4] the normalized p-widths $t \mapsto p^{-\frac{n-1}{n}} \omega_p^1(M, \hat{g}(t))$ are uniformly locally Lipschitz. This combined with the Weyl Law implies that the sequence of functions $h_p : (-\delta, \delta) \to \mathbb{R}$

$$h_p(t) = \frac{p^{-\frac{n-1}{n}} \omega_p^1(M, \hat{g}(t))}{\mathrm{Vol}(M, \hat{g}(t))^{\frac{1}{n}}}$$

converges uniformly to the constant $\alpha(n,1)$. Considering

$$\tilde{h}_p(t) = \log(h_p(t)) = -\frac{n-1}{n} \log(p) + \log(\omega_p(M, \hat{g}(t))) - \frac{1}{n} \log(\mathrm{Vol}(M, \hat{g}(t)))$$

we have that $\tilde{h}_p$ converges uniformly to the constant $\log(\alpha(n,1))$. On the other hand, Almgren showed that there is a correspondence between 1-widths and the volumes of stationary varifolds (see [1], [2], [5], [32], [36], [37]) such that for each $p \in \mathbb{N}$ and $t \in (-\delta, \delta)$ there exists a (possibly non unique) stationary geodesic network $\gamma_p(t)$ such that

$$\mathrm{L}_{\hat{g}(t)}(\gamma_p(t)) = \omega_p^1(\hat{g}(t)). \tag{4.1}$$

Assume that the $\gamma_p(t)$'s can be chosen so that all of them are parametrized by the same graph Γ and the maps $(-\delta, \delta) \to \Omega(\Gamma, M)$, $t \mapsto \gamma_p(t)$ are differentiable (this is a very strong assumption and doesn't necessarily hold, as the map $t \mapsto \omega_p^1(\hat{g}(t))$ may not be differentiable; a counterexample is shown below). In that case we can differentiate $\tilde{h}_p$ and obtain

$$\begin{aligned}
\frac{d}{dt}\tilde{h}_p(t) &= \frac{1}{\omega_p(M, \hat{g}(t))} \frac{d}{dt} \omega_p^1(M, \hat{g}(t)) - \frac{1}{n \,\mathrm{Vol}(M, \hat{g}(t))} \frac{d}{dt} \mathrm{Vol}(M, \hat{g}(t)) \\
&= \frac{1}{L_{\hat{g}(t)}(\gamma_p(t))} \int_{\gamma_p(t)} f \, \mathrm{dL}_{\hat{g}(t)} - \frac{1}{n \,\mathrm{Vol}(M, \hat{g}(t))} \int_M nf \, \mathrm{dVol}_{\hat{g}(t)} \\
&= \fint_{\gamma_p(t)} f \, \mathrm{dL}_{\hat{g}(t)} - \fint_M f \, \mathrm{dVol}_{\hat{g}(t)} .
\end{aligned}$$

As $\{\tilde{h}_p\}_p$ converges uniformly to a constant, we could expect that the sequence $\{\tilde{h}'_p(t)\}_p$ converges to 0 for some values of t. If that was the case, the sequence $\{\gamma_p(t)\}_p$ would verify the equidistribution formula for the function f with respect to the metric $\hat{g}(t)$. Nevertheless, this does not have to be true, because of two reasons. The first one is that the uniform convergence of a sequence of functions to a constant does not imply convergence of the derivatives to 0 at any point. Indeed, we can construct a sequence of zigzag functions which converges uniformly to 0 but $h_p'(t)$ does not converge to 0 for any t. The second one is that the differentiability of $t \mapsto \gamma_p(t)$ could fail, a counterexample is shown in the next paragraph. And even if that reasoning was true and such t existed, the sequence $\{\gamma_p(t)\}_p$ constructed would only give an equidistribution formula for the function f (which is used to construct the sequence) instead of for all C^∞ functions at the same time; and with respect to a metric $\hat{g}(t)$ which could also vary with f.

An example when $t \mapsto \omega_p^1(\hat{g}(t))$ is not differentiable is the following. Let us consider a dumbbell metric g on S^2 obtained by constructing a connected sum of two identical round 2-spheres S_1^2 and S_2^2 of radius 1 by a thin neck. Define a 1-parameter family of metrics $\{\hat{g}(t)\}_{t\in(-1,1)}$ such that $\hat{g}(t) = (1+t)^2 g$ along S_1^2, $\hat{g}(t) = (1-t)^2 g$ along S_2^2 (interpolating along the neck so that it is still very thin). It is clear than for $t \geq 0$, the 1-width is realized by a great circle in S_1^2 with length $1+t$, and for $t \leq 0$ it is realized by a great circle in S_2^2 of length $1-t$. Therefore

$$\omega_1^1(\hat{g}(t)) = \begin{cases} 1-t & \text{if } t \leq 0 \\ 1+t & \text{if } t \geq 0 \end{cases}$$

and hence it is not differentiable at 0.

To fix the previous issue (differentiability of $\omega_p^1(g(t))$), we prove Proposition 4.4.1 which is a version for stationary geodesic networks of [31, Lemma 2]. Regarding the convergence of $h_p'(t)$ to 0 for certain values of t, we use Lemma 4.4.6 which is exactly [31, Lemma 3]. To obtain a sequence of stationary geodesic networks that verifies the equidistribution formula for all C^∞ functions (and not only for a particular one as above), we carry on a construction described in Section 4.5 using certain stationary geodesic nets which realize the p-widths in a similar way as the $\gamma_p(t)$'s above. The key idea here is that the integral of any C^∞ function f over M can be approximated by Riemann sums along small regions with piecewise smooth boundary where f is almost constant. Therefore, if we have an equidistribution formula for the characteristic functions of those regions (or some suitable smooth approximations), then we will be able to deduce it for an arbitrary $f \in C^\infty(M,\mathbb{R})$. The advantage of doing this is that we reduce the problem to a countable family of functions. This argument is also inspired by [31].

The chapter is structured as follows. In Section 4.2, we introduce the set up and necessary preliminaries. In Section 4.3, we define the Jacobi Operator along a stationary geodesic net and show that it has all the nice properties that an elliptic operator has (mainly, admitting an orthonormal basis of eigenfunctions and therefore having a min-max characterization for its eigenvalues). This is crucial to prove Proposition 4.4.5. In Section 4.4, the technical propositions necessary to prove Theorem 4.1.5 are discussed. In Section 4.5 we prove Theorem 4.1.5. In Section 4.6, we combine the proof of Theorem 4.1.5 with the work of Chodosh and Mantoulidis in [7] (where it is shown that the p-widths on a surface are realized by finite unions of closed geodesics) to prove Theorem 4.1.1.

Remark 4.1.6. *Rohil Prasad pointed out that an alternative proof of Theorem 4.1.1 could be obtained using the methods of Irie in [21]. Given a closed Riemannian 2-manifold (M,g), its unit cotangent bundle U_g^*M is a closed 3-manifold equipped with a natural contact structure induced by the contact form λ_g which is the restriction of the Liouville form λ on T^*M to U_g^*M. It is a well known fact* *that the Reeb vector field associated to λ_g generates the geodesic flow of (M, g). Additionally, given a function $f : M \to \mathbb{R}$, the Riemannian metric $g' = e^f g$ corresponds to the conformal perturbation $\lambda_{g'} = e^{\frac{f\circ\pi}{2}}\lambda_g$ of the contact form in U_g^*M (here $\pi : U_g^*M \to M$ is the projection map); and both λ_g and $\lambda_{g'}$ are compatible with the same contact structure on U_g^*M. Thus one would like to apply [21, Corollary 1.4] to U_g^*M with the contact structure induced by λ_g. However, that result is about generic perturbations of the contact form of the type $e^{\tilde{f}}\lambda_g$, where $\tilde{f} : U_g^*M \to \mathbb{R}$ and we only want to consider perturbations $\tilde{f} = f \circ \pi$ which are liftings to U_g^*M of maps $f : M \to \mathbb{R}$ so some work should be done here in order to apply Irie's result in our setting. This issue was pointed out in [6, Remark 2.3], where a similar problem is studied for Finsler metrics and a solution is given for that class of metrics. Additionally, Irie's theorem would give us an equidistributed sequence for a generic conformal perturbation of each metric g. This immediately implies that for a dense set of Riemannian metrics such an equidistribution result holds, but some additional arguments are needed to prove it for a Baire-generic metric. It is important to point out that the result in [21] uses the ideas of [31] but in the different setting of contact geometry, applying results of Embedded Contact Homology with the purpose of finding closed orbits of the Reeb vector field; while in [31] Almgren-Pitts theory is used to find closed minimal surfaces.*

4.2 Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold. We will work with the same definitions regarding stationary geodesic nets on M as in Chapter 2. We recall some of them because of their importance in this chapter and introduce some new ones.

Definition 4.2.1. *We say that two Γ-nets γ_1 and γ_2 are equivalent if for every edge E of Γ the map $\gamma_1|_E$ is a C^2 reparametrization of $\gamma_2|_E$ fixing the endpoints. This defines an equivalence relation $\sim$ in $\Omega(\Gamma, M)$. We denote $\hat{\Omega}(\Gamma, M) = \Omega(\Gamma, M)/\sim$ the quotient space. Given $\gamma \in \hat{\Omega}(\Gamma, M)$ we will often denote a representative of the equivalence class γ also by γ, and regard different representatives as different parametrizations of the geometric object $\gamma \in \hat{\Omega}(\Gamma, M)$.*

Notation 4.2.2. *Given a Γ-net γ and an edge $E \in \mathscr{E}$, we denote γ_E the restriction of γ to E. We also define $\gamma_E(0) := \gamma_E(\pi_E(0))$ and $\gamma_E(1) := \gamma_E(\pi_E(1))$.*

Definition 4.2.3. *Let $\gamma \in \hat{\Omega}(\Gamma, M)$ and let h be a continuous function defined in $\mathrm{Im}(\gamma) \subseteq M$. Given a metric $g \in \mathcal{M}^q$ we define*

$$\int_\gamma h \, \mathrm{dL}_g = \sum_{E\in\mathscr{E}} n(E) \int_E h \circ \gamma(u)\sqrt{g_{\gamma(u)}(\dot{\gamma}(u), \dot{\gamma}(u))}du.$$

Observe that the right hand side is independent of the parametrization we choose and therefore $\int_\gamma h \, \mathrm{dL}_g$ is well defined.

Definition 4.2.4 (Stationary Geodesic Network)**.** *We say that $\gamma \in \Omega(\Gamma, M)$ is a stationary geodesic network with respect to a metric $g \in \mathcal{M}^q$ $(q \geq 2)$ if it is a critical point of the length functional $\mathrm{L}_g : \Omega(\Gamma, M) \to \mathbb{R}$. This means that given any smooth one parameter family $\tilde{\gamma} : (-\delta, \delta) \to \Omega(\Gamma, M)$*

with $\tilde{\gamma}(0) = \gamma$ we have

$$\frac{d}{ds}\bigg|_{s=0} \mathrm{L}_g(\tilde{\gamma}(s)) = 0.$$

Assuming that the edges of γ are parametrized by constant speed, if $X(t) = \frac{\partial \tilde{\gamma}}{\partial s}(0,t)$ (here we regard $\tilde{\gamma} : (-\delta, \delta) \times \Gamma \to M$) then

$$\frac{d}{ds}\bigg|_{s=0} \mathrm{L}_g(\tilde{\gamma}(s)) = -\sum_{E \in \mathscr{E}} \frac{n(E)}{l(E)} \int_E \langle \ddot{\gamma}(t), X(t) \rangle_g dt + \sum_{v \in \mathscr{V}} \langle V_v(\gamma), X(v) \rangle_g \tag{4.2}$$

where $l(E) = \mathrm{L}_g(\gamma_E)$ and

$$V_v(\gamma) = \sum_{(E,i):\pi_E(i)=v} (-1)^{i+1} n(E) \frac{\dot{\gamma}_E(i)}{|\dot{\gamma}_E(i)|}.$$

Equation (4.2) is called the First Variation Formula and was computed in [46, Section 1]. It implies that $\gamma : \Gamma \to M$ is stationary with respect to g if and only if each edge is mapped to a geodesic segment in (M, g) and the stability condition at the vertices $V_v(\gamma) = 0$ is verified. The latter means that for each $v \in \mathscr{V}$, the sum of the inward pointing unit tangent vectors to each edge at v is 0.

Definition 4.2.5. *We say that $\gamma \in \hat{\Omega}(\Gamma, M)$ is a stationary geodesic network if every representative $\tilde{\gamma} \in \Omega(\Gamma, M)$ of γ is a stationary geodesic network.*

Definition 4.2.6. *We denote $C^2(\gamma)$ the space of continuous vector fields along γ whose restriction to each edge is of class C^2.*

Remark 4.2.7. *If $g \in \mathcal{M}^q$, $q \geq 2$ and $\gamma \in \Omega(\Gamma, M)$ is stationary with respect to g then by the regularity of the solutions of an ODE, γ_E is of class C^q for every $E \in \mathscr{E}$. This is why we only ask C^2 regularity to Γ-nets and vector fields along them.*

Assume $\gamma \in \Omega(\Gamma, M)$ is a stationary geodesic net with respect to a C^q metric with $q \geq 3$ (so that the Riemann curvature tensor is of class C^1). Let $\tilde{\gamma} : (-\delta, \delta)^2 \to \Omega(\Gamma, M)$ be a smooth 2-parameter family of Γ-nets with $\tilde{\gamma}(0,0) = \gamma$. Let $X(t) = \frac{\partial \tilde{\gamma}}{\partial x}(0,0,t)$ and $Y(t) = \frac{\partial \tilde{\gamma}}{\partial s}(0,0,t)$. We define the Hessian $\mathrm{Hess}_\gamma \mathrm{L}_g : C^2(\gamma) \times C^2(\gamma) \to \mathbb{R}$ of the length functional at γ as the bilinear form

$$\mathrm{Hess}_\gamma \mathrm{L}_g(X, Y) = \frac{\partial^2}{\partial x \partial s}\bigg|_{(0,0)} \mathrm{L}_g(\tilde{\gamma}(x, s)).$$

In [46, Section 2] it was shown that $\mathrm{Hess}_\gamma \mathrm{L}_g$ is well defined (i.e. it does not depend on which two parameter variation $\tilde{\gamma}$ with directional derivatives X and Y we choose) and in fact it holds

$$\mathrm{Hess}_\gamma \mathrm{L}_g(X, Y) = \sum_{E \in \mathscr{E}} \int_E \langle A_E(X)(t), Y(t) \rangle_g dt + \sum_{v \in \mathscr{V}} \langle B_v(X), Y(v) \rangle_g \tag{4.3}$$

where

$$A_E(X) = -\frac{n(E)}{l(E)} [\ddot{X}_E^\perp + R(\dot{\gamma}, X_E^\perp)\dot{\gamma}]$$

$$B_v(X) = \sum_{(E,i):\pi_E(i)=v} (-1)^{i+1} \frac{n(E)}{l(E)} \dot{X}_E^\perp(i)$$

being X_E the restriction of the vector field X to the edge E and $X_E^{\perp}$ the component of X_E orthogonal to γ_E. Observe that A_E is (up to a positive constant) the Jacobi operator along γ_E. Equation (4.3) is the Second Variation Formula.

Definition 4.2.8. *We say that a vector field $J \in C^2(\gamma)$ is Jacobi if it is a null vector of* $\mathrm{Hess}_\gamma \mathrm{L}_g$*, i.e. if* $\mathrm{Hess}_\gamma \mathrm{L}_g(J, X) = 0$ *for every $X \in C^2(\gamma)$. By the Second Variation Formula, J is Jacobi along γ if and only if $A_E(J) = 0$ for every $E \in \mathscr{E}$ and $B_v(J) = 0$ for every $v \in \mathscr{V}$.*

Definition 4.2.9. *A vector field $X \in C^2(\gamma)$ is said to be parallel if X_E is parallel along γ_E for every $E \in \mathscr{E}$.*

Remark 4.2.10. *Observe that every parallel vector field J along γ is Jacobi.*

Definition 4.2.11. *A stationary geodesic net $\gamma : \Gamma \to (M, g)$ is nondegenerate if every Jacobi field along γ is parallel.*

Remark 4.2.12. *In [27, Lemma 2.5], it was shown that every stationary geodesic network with respect to a metric g can be represented by a map $\gamma : \Gamma \to M$, where $\Gamma = \bigcup_{i=1}^P \Gamma_i$ is the finite union of the good weighted multigraphs $\{\Gamma_i\}_{1\le i\le P}$ and $\gamma|_{\Gamma_i}$ is an embedded stationary geodesic network for each $1 \le i \le P$ (moreover, the map $\gamma : \Gamma \to M$ is a topological embedding).*

Definition 4.2.13. *Given a stationary geodesic network $\gamma : \Gamma \to (M, g)$, we say that its connected components are nondegenerate if*

1. *We can express $\Gamma = \bigcup_{i=1}^P \Gamma_i$ as a disjoint union of good weighted multigraphs.*
2. *$\gamma|_{\Gamma_i}$ is an embedded nondegenerate stationary geodesic network for every $1 \le i \le P$.*

Definition 4.2.14. *A primitive closed geodesic in a Riemannian manifold (M, g) is a closed geodesic $\gamma : S^1 \to (M, g)$ traversed with multiplicity one.*

Notation 4.2.15. *Given a symmetric 2-tensor T, a metric $g \in \mathcal{M}^q$, a stationary geodesic network $\gamma : \Gamma \to M$ on (M, g) and $t \in \Gamma$, we denote*

$$\mathrm{tr}_{\gamma,g} T(t) = T\Big(\frac{\dot\gamma(t)}{|\dot\gamma(t)|_g}, \frac{\dot\gamma(t)}{|\dot\gamma(t)|_g}\Big)$$

which is the trace of the tensor T along γ with respect to the metric g.

Definition 4.2.16 (Average integral along γ)**.** *Let Γ be a weighted multigraph. Given $\gamma \in \Omega(\Gamma, M)$, a metric $g \in \mathcal{M}^q$ and a continuous function h defined in* $\mathrm{Im}(\gamma)$*, we define the average integral of h with respect to metric g as*

$$⨍_\gamma h \, \mathrm{dL}_g := \frac{1}{\mathrm{L}_g(\gamma)} \int_\gamma h \, \mathrm{dL}_g \,.$$

4.3 The Jacobi Operator

In this section we will study some properties of the Jacobi operator of an embedded stationary geodesic network $\gamma : \Gamma \to (M, g)$, where Γ is a good weighted multigraph and $g \in \mathcal{M}^q$, $q \ge 3$. We will focus on the case when Γ is good* (i.e. every vertex has at least three different incoming edges), because when Γ is a loop with multiplicity what we get is the Jacobi operator along an embedded closed geodesic acting on normal vector fields, which is known to be elliptic; and hence it has all the nice properties that we will describe below. We first introduce some notation. Let

$$\begin{aligned} C^2(\gamma) &= \{X \text{ continuous vector field along } \gamma : X_E \text{ is } C^2 \ \forall E \in \mathscr{E}\} \\ C^2(\gamma)^{\parallel} &= \{X \in C^2(\gamma) : X \text{ is parallel along } \gamma\} \\ C^2_0(\gamma)^{\perp} &= \{X \in C^2(\gamma) : X(t) \perp \dot\gamma(t) \ \forall t \in \Gamma \setminus \mathscr{V} \text{ and } X(v) = 0 \ \forall v \in \mathscr{V}\} \\ C^2(E)^{\perp} &= \{X \in C^2(E) : X(t) \perp \dot\gamma(t) \ \forall E \in \mathscr{E}\} \\ C^2(\mathscr{E})^{\perp} &= \prod_{E \in \mathscr{E}} C^2(E)^{\perp}. \end{aligned}$$

Observe that as Γ is good*, if $X \in C^2(\gamma)^{\parallel}$ then $X(v) = 0$ for every $v \in \mathscr{V}$. Denote

$$T\mathscr{V} = \prod_{v \in \mathscr{V}} T_{\gamma(v)} M.$$

By the second variation formula (4.3), we can define the Jacobi operator $L : C^2(\gamma) \to C^0(\mathscr{E})^{\perp} \times T\mathscr{V}$ as

$$L(J) = ((-\frac{n(E)}{l(E)}(\ddot J_E^{\perp} + R(\dot\gamma, J_E^{\perp})\dot\gamma))_{E \in \mathscr{E}}, (B_v(J))_{v \in \mathscr{V}}). \tag{4.4}$$

We know that each $X \in C^2(\gamma)^{\parallel}$ is Jacobi (i.e. it verifies $L(J) = 0$). We want to construct a complement of $C^2(\gamma)^{\parallel}$ in $C^2(\gamma)$, and show that when we restrict L to that complement it behaves like an elliptic operator (this complement will play the role of the space of normal Jacobi fields along a minimal submanifold in the smooth case, when it is known that the stability operator is elliptic).

To do this, we will need to define a finite dimensional subspace $S^2(\gamma) \subseteq C^2(\gamma)$ such that the evaluation map $\mathrm{ev} : S^2(\gamma) \to T\mathscr{V},\ J \mapsto (J(v))_{v \in \mathscr{V}}$ is a linear isomorphism. This can be done by taking a basis $\mathcal{B}_v$ of $T_{\gamma(v)}M$ for each $v \in \mathscr{V}$, and for each pair (v, w) with $v \in \mathscr{V}$ and $w \in \mathcal{B}_v$ defining a vector field $J_{(v,w)} \in C^2(\gamma)$ such that $J_{(v,w)}(v) = w$ and $J_{(v,w)}(v') = 0$ for every $v' \neq v$. Then we can define $S^2(\gamma) = \langle J_{(v,w)} : v \in \mathscr{V}, w \in \mathcal{B}_v \rangle$. Of course the choice of $S^2(\gamma)$ is not canonical, but we fix one choice and work with it for the rest of the section (it will be deduced from the arguments below that the results that we prove hold independently of the choice of $S^2(\gamma)$). It is clear that

$$C^2(\gamma) = C^2(\gamma)^{\parallel} \oplus C^2_0(\gamma)^{\perp} \oplus S^2(\gamma).$$

Denote $C^2(\gamma)^C = C^2_0(\gamma)^{\perp} \oplus S^2(\gamma)$ which is a complement of the space of parallel vector fields along γ. Same as in the theory of elliptic operators, we can extend the Jacobi operator to Sobolev spaces of vector fields along γ once we have a suitable definition of them. Denote

$$\begin{aligned}
H^2_0(E) &= \{X \text{ normal vector field of class } H^2_0 \text{ along } E\} \\
H^2_0(\gamma) &= \prod_{E\in\mathscr{E}} H^2_0(E) \\
H^2(\gamma) &= H^2_0(\gamma) \oplus S^2(\gamma) \\
L^2(E) &= \{X \text{ normal vector field of class } L^2 \text{ along } E\} \\
L^2(\mathscr{E}) &= \prod_{E\in\mathscr{E}} L^2(E) \\
L^2(\gamma) &= L^2(\mathscr{E}) \oplus T\mathscr{V}.
\end{aligned}$$

Notice that $H^2(\gamma)$ is the H^2-version of $C^2(\gamma)^C$ and will be the domain of the Jacobi operator we will work with (as that operator vanishes on $C^2(\gamma)^{\|}$). The previous spaces are defined in analogy with the spaces of C^2, H^2 and L^2 normal vector fields along a smooth closed submanifold which appear when studying the ellipticity of its Jacobi operator. The space $L^2(\gamma)$ is a Hilbert space with the inner product

$$\langle ((X_E)_E,(u_v)_v),((Y_E)_E,(w_v)_v)\rangle = \sum_{E\in\mathscr{E}}\int_E \langle X_E(t),Y_E(t)\rangle_g dt + \sum_{v\in\mathscr{V}}\langle u_v,w_v\rangle_g$$

and we have a monomorphism $\iota : H^2(\gamma) \to L^2(\gamma)$ with dense image given by

$$\iota(J) = ((J_E)^{\perp}_{E\in\mathscr{E}}, (J(v))_{v\in\mathscr{V}})$$

which allows us to write the Hessian $\operatorname{Hess}_\gamma \mathrm{L}_g : H^2(\gamma)\times H^2(\gamma) \to \mathbb{R}$ as

$$\operatorname{Hess}_\gamma \mathrm{L}_g(J,\tilde{J}) = \langle L(J), \iota(\tilde{J})\rangle$$

where $\langle , \rangle$ is the inner product in $L^2(\gamma)$. Here we considered $L : H^2(\gamma) \to L^2(\gamma)$ given by (4.4) which is a bounded linear operator. As in the smooth case, we can also regard L as an unbounded operator $L : L^2(\gamma) \to L^2(\gamma)$ whose domain is the dense linear subspace $H^2(\gamma)$. We would therefore expect that for a certain $\lambda \in \mathbb{R}$ the operator $L - \lambda\iota : H^2(\gamma) \to L^2(\gamma)$ has a compact inverse, and from that get an orthonormal basis of $L^2(\gamma)$ consisting of eigenvectors of L. This indeed holds, as it is shown in the following proposition.

Proposition 4.3.1. *For every $\lambda \in \mathbb{R}$, the operator $L-\lambda\iota : H^2(\gamma) \to L^2(\gamma)$ defined as $(L-\lambda\iota)(J) = L(J) - \lambda\iota(J)$ is Fredholm of index 0. The spectrum of L consists of an increasing sequence of eigenvalues $\lambda_1 \le \lambda_2 \le \ldots$ with $\lim_{i\to\infty}\lambda_i = +\infty$ (i.e. $L-\lambda\iota$ has nontrivial kernel if and only if $\lambda \in \{\lambda_i\}_{i\in\mathbb{N}}$ and has a continuous inverse $(L-\lambda\iota)^{-1} : L^2(\gamma) \to H^2(\gamma)$ otherwise). In addition, there exists sequence $\{J_i\}_{i\in\mathbb{N}}$ in $H^2(\gamma)$ such that $\{\iota(J_i)\}_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(\gamma)$ and $L(J_i) = \lambda_i\iota(J_i)$ for each $i\in\mathbb{N}$. Therefore, we have the following min-max characterization of the eigenvalues of L*

$$\lambda_i = \min_W \max_{J\in W\setminus\{0\}} \frac{\langle L(J),\iota(J)\rangle}{\langle \iota(J),\iota(J)\rangle}$$

where the minimum is taken over all i-dimensional subspaces $W \subseteq H^2(\gamma)$.

Proof. Let $\lambda \in \mathbb{R}$. Then if $J \in H_0^2(\gamma)$ and $\tilde{J} \in S^2(\gamma)$,

$$(L - \lambda\iota)(J + \tilde{J}) = ((L_E(J_E) - \lambda J_E)_E + (L_E(\tilde{J}_E^{\perp}) - \lambda \tilde{J}_E^{\perp})_E, (B_v(J + \tilde{J}) - \lambda \tilde{J}(v))_v)$$

where $L_E : H_0^2(E) \to L^2(E)$ is (a constant multiple of) the Jacobi operator along γ_E given by $J \mapsto -\frac{n(E)}{l(E)}(\ddot{J} + R(\dot{\gamma}, J)\dot{\gamma})$. We know that each L_E is elliptic, and therefore $L_E - \lambda$ is Fredholm of index 0 for every $\lambda \in \mathbb{R}$. This implies that the product operator $\tilde{L} : H_0^2(\gamma) \to L^2(\mathscr{E})$, $\tilde{L} = (L_E)_E$ verifies that $\tilde{L} - \lambda$ is Fredholm of index 0 for every $\lambda \in \mathbb{R}$. Thus the fact that $L - \lambda\iota$ is always Fredholm of index 0 can be deduced from the following lemma.

Lemma 4.3.2. *Let E_1, E_2, $\overline{E}_1$, $\overline{E}_2$ be Banach spaces with $\dim(E_2) = \dim(\overline{E}_2) < \infty$. Let $L : E_1 \oplus E_2 \to \overline{E}_1 \oplus \overline{E}_2$ be a continuous linear map, and write $L(e_1, e_2) = (L_{11}(e_1) + L_{21}(e_2), L_{12}(e_1) + L_{22}(e_2))$ with $L_{ij} : E_i \to \overline{E}_j$. Assume L_{11} is Fredholm of index 0. Then L is Fredholm of index 0.*

Proof of Lemma 4.3.2. Let $\tilde{L} : E_1 \oplus E_2 \to \overline{E}_1 \oplus \overline{E}_2$ be the operator $\tilde{L}(e_1, e_2) = (L_{11}(e_1), 0)$. Because L_{11} is Fredholm of index 0 and $\dim(E_2) = \dim(\overline{E}_2)$, we see that $\tilde{L}$ is also Fredholm of index 0. As $L = \tilde{L} + F$ with $F(e_1, e_2) = (L_{21}(e_2), L_{12}(e_1) + L_{22}(e_2))$ compact because of the finite dimensionality of E_2 and $\overline{E}_2$, by [28, Theorem 12-5.13] we deduce that L is also Fredholm of index 0. $\square$

Now we are going to show that the quadratic form $\operatorname{Hess}_\gamma \mathrm{L}_g : H^2(\gamma) \times H^2(\gamma) \to \mathbb{R}$ is bounded from below. We know

$$\operatorname{Hess}_\gamma \mathrm{L}_g(J, \tilde{J}) = \sum_{E \in \mathscr{E}} \int_E \langle L_E(J_E^{\perp})(t), \tilde{J}_E^{\perp}(t)\rangle_g dt + \sum_{v \in \mathscr{V}} \langle B_v(J), \tilde{J}(v)\rangle_g.$$

Denote by $C : H^2(\gamma) \times H^2(\gamma) \to \mathbb{R}$ the form $C(J, \tilde{J}) = \sum_{v \in \mathscr{V}} \langle B_v(J), \tilde{J}(v)\rangle_g$. C is symmetric because so are $\operatorname{Hess}_\gamma \mathrm{L}_g$ and L_E for each $E \in \mathscr{E}$. If we endow $S^2(\gamma)$ with the inner product

$$\langle J, \tilde{J}\rangle = \sum_{v \in \mathscr{V}} \langle J(v), \tilde{J}(v)\rangle_g$$

then as $\dim(S^2(\gamma)) < \infty$, we can see that there exists some constant $\alpha > 0$ such that

$$|C(J, J)| \leq \alpha \sum_{v \in \mathscr{V}} \langle J(v), J(v)\rangle_g \tag{4.5}$$

for every $J \in S^2(\gamma)$. But then as C vanishes on $H_0^2(\gamma)$, by its bilinearity and symmetry we can see that in fact (4.5) is valid for every $J \in H^2(\gamma)$.

On the other hand, using that each L_E is elliptic, for each $E \in \mathscr{E}$ there exists $\beta_E \in \mathbb{R}$ such that

$$\int_E \langle L_E(J_E^{\perp})(t), J_E^{\perp}(t)\rangle_g dt \geq \beta_E \int_E \langle J_E^{\perp}(t), J_E^{\perp}(t)\rangle_g dt. \tag{4.6}$$

Thus if $\beta = \min\{\beta_E : E \in \mathscr{E}\}$ and $\gamma = \min\{\beta, -\alpha\}$, from (4.5) and (4.6) we deduce

$$\operatorname{Hess}_\gamma \mathrm{L}_g(J, J) \geq \beta \sum_{E \in \mathscr{E}} \int_E \langle J_E^{\perp}(t), J_E^{\perp}(t)\rangle_g dt - \alpha \sum_{v \in \mathscr{V}} \langle J(v), J(v)\rangle_g \geq \gamma\langle\iota(J), \iota(J)\rangle$$

which considering that $\operatorname{Hess}_\gamma \mathrm{L}_g(J,J) = \langle L(J), \iota(J)\rangle$ implies that for every $\lambda \in \mathbb{R}$ it holds

$$\langle (L+\lambda\iota)(J), \iota(J)\rangle \geq (\lambda+\gamma)\langle \iota(J), \iota(J)\rangle$$

and in particular if $\lambda > -\gamma$ implies that $L+\lambda\iota$ is a monomorphism. Because we also know that these operators are Fredholm of index 0, by the Open Mapping Theorem we conclude that $L+\lambda\iota : H^2(\gamma) \to L^2(\gamma)$ is a continuous linear isomorphism for every $\lambda > -\gamma$.

Fix $\lambda > -\gamma$. We will now show that $\iota \circ (L+\lambda\iota)^{-1} : L^2(\gamma) \to L^2(\gamma)$ is compact. Let $(X_i)_{i\in\mathbb{N}}$ be a bounded sequence in $L^2(\gamma)$ and define $(J^i, \tilde{J}^i) = (L+\lambda\iota)^{-1}(X_i)$ with $J^i \in H^2_0(\gamma)$ and $\tilde{J}^i \in S^2(\gamma)$. As $(L+\lambda\iota)^{-1}$ is bounded, $(J^i, \tilde{J}^i)$ is a bounded sequence in $H^2(\gamma)$. Therefore, for each $E \in \mathscr{E}$ the sequence of normal vector fields $(J^i_E)_{i\in\mathbb{N}}$ along γ_E is bounded in $H^2_0(E)$ and therefore in $H^1_0(E)$. Hence, by the Rellich-Kondrachov Compactness Theorem we can find a subsequence $(i_k)_{k\in\mathbb{N}}$ such that $(J^{i_k}_E)_{k\in\mathbb{N}}$ converges in $L^2(E)$ for every $E \in \mathscr{E}$. On the other hand, using that $S^2(\gamma)$ is finite dimensional, we can extract a further subsequence $(i_{k_l})_l$ to have the additional property that $(\tilde{J}^{i_{k_l}})_{l\in\mathbb{N}}$ converges in $S^2(\gamma)$. This implies that the sequence of general term $\iota\circ(L+\lambda\iota)^{-1}(X_{i_{k_l}}) = \iota(J^{i_{k_l}}, \tilde{J}^{i_{k_l}})$ converges in $L^2(\gamma)$, and this completes the proof that $\iota\circ(L+\lambda\iota)^{-1}$ is compact.

The symmetry of $\operatorname{Hess}_\gamma \mathrm{L}_g(J,\tilde{J}) = \langle L(J), \iota(\tilde{J})\rangle$ implies that $\iota\circ(L+\lambda\iota)^{-1}$ is self-adjoint, which together with its compactness implies the existence of an orthonormal basis $\{X_i\}_{i\in\mathbb{N}}$ of $L^2(\gamma)$ such that $\iota\circ(L+\lambda\iota)^{-1}X_i = \delta_i X_i$ for some decreasing sequence $\delta_i \to 0^+$ (because by our choice of λ, $\iota\circ(L+\lambda\iota)^{-1} \geq 0$). But we claim that $X \in L^2(\gamma)$ is an eigenvector of $\iota\circ(L+\lambda\iota)^{-1}$ of eigenvalue $\delta\in\mathbb{R}$ if and only if $X = \iota(J)$ for some $J \in H^2(\gamma)$ such that $L(J) = (\delta^{-1}-\lambda)\iota(J)$. This is because $\iota\circ(L+\lambda\iota)^{-1}(X) = \delta X$ if and only if there exists $J \in H^2(\gamma)$ with $\iota(J) = X$ which verifies any of the the following equivalent conditions:

$$\begin{aligned}
\iota\circ(L+\lambda\iota)^{-1}\circ\iota(J) &= \delta\iota(J)\\
(L+\lambda\iota)^{-1}\circ\iota(J) &= \delta J\\
\iota(J) &= \delta(L+\lambda\iota)(J)\\
L(J) &= (\delta^{-1}-\lambda)\iota(J).
\end{aligned}$$

From the previous, we conclude that if $\lambda_i := \delta_i^{-1} - \lambda$ then $\operatorname{spec}(L) = \{\lambda_i\}_n$, $\lim_{i\to\infty}\lambda_i = +\infty$ and $L(J_i) = \lambda_i\iota(J_i)$. This implies the min-max theorem for the eigenvalues holds for L, which completes the proof. □

4.4 Some auxiliary results

Proposition 4.4.1. *Let $g : I^N \to \mathcal{M}^q$ be a smooth embedding, $N \in \mathbb{N}$, $I = (-1,1)$. If $q \geq N+3$, there exists an arbitrarily small perturbation in the C^∞ topology $g' : I^N \to \mathcal{M}^q$ such that there is a full measure subset $\mathcal{A} \subseteq I^N$ with the following properties: for any $p \in \mathbb{N}$ and any $t \in \mathcal{A}$, the function $s \mapsto \omega^1_p(g'(s))$ is differentiable at t, and there exists a (possibly disconnected) weighted multigraph Γ and a stationary geodesic network $\gamma_p = \gamma_p(t) : \Gamma \to (M, g'(t))$ such that the following two conditions hold*

1. $\omega^1_p(g'(t)) = L_{g'(t)}(\gamma_p(t))$.

2. $\frac{\partial}{\partial v}(\omega_p^1 \circ g')\big|_{s=t} = \frac{1}{2}\int_{\gamma_p(t)} \mathrm{tr}_{\gamma_p(t),g'(t)} \frac{\partial g'}{\partial v}(t)\,\mathrm{dL}_{g'(t)}$ *for every* $v \in \mathbb{R}^N$.

To prove the proposition, we will need to have a condition for a sequence of embedded stationary geodesic nets $\gamma_n : \Gamma \to (M, g_n)$ converging to some $\gamma : \Gamma \to (M, g)$ that guarantees that γ is also embedded. The condition we will work with can be expressed as a collection of lower and upper bounds of certain functionals defined for pairs (g, γ) where γ is stationary with respect to g. We proceed to describe those functionals.

The first one is

$$F_1(g, \gamma) = \min\{|\dot\gamma_E(t)|_g : t \in E, E \in \mathscr{E}\}.$$

A lower bound for this functional will imply that the limit net is an immersion along each edge.

Then we have a family of functionals $F_2^{(E_1,i_1),(E_2,i_2)}$ defined for each pair $((E_1, i_1), (E_2, i_2)) \in (\mathscr{E} \times \{0, 1\})^2$ such that $\pi_{E_1}(i_1) = \pi_{E_2}(i_2)$ (see Section 4.2 for the notation) as follows

$$F_2^{(E_1,i_1),(E_2,i_2)}(g, \gamma) = (-1)^{i_1+i_2} \frac{\langle \dot\gamma_{E_1}(i_1), \dot\gamma_{E_2}(i_2)\rangle_g}{|\dot\gamma_{E_1}(i_1)|_g |\dot\gamma_{E_2}(i_2)|_g}.$$

Notice that $(-1)^{i_j} \frac{\dot\gamma_{E_j}(i_j)}{|\dot\gamma_{E_j}(i_j)|_g}$ is the unit inward pointing tangent vector to γ at $v = \pi_{E_j}(i_j)$ along E_j, $j = 1, 2$ (and observe that in case E is a loop at v, there are two inward tangent vectors to γ along E at v corresponding to the pairs $(E, 0)$ and $(E, 1)$). The condition $F_2^{(E_1,i_1),(E_2,i_2)}(g_n, \gamma_n) \leq 1 - \delta$ for some $\delta > 0$ and for every possible choice $(E_1, i_1) \neq (E_2, i_2)$ with $\pi_{E_1}(i_1) = \pi_{E_2}(i_2)$ implies that the limit (g, γ) has the property that given $v \in \mathscr{V}$, there exists an open neighborhood U_v of v in Γ such that $\gamma : U_v \to \gamma(U_v)$ is a homeomorphism (i.e. no two edges of Γ at v are mapped to the same geodesic segment). Explicitly,

$$U_v = \bigcup_{(E,i):\pi_E(i)=v} \{t \in E : |t - i| < \min\{\frac{\mathrm{injrad}(g)}{\mathrm{L}_g(\gamma_E)}, \frac{1}{2}\}\}$$

where injrad : $\mathcal{M}^q \to \mathbb{R}_{>0}$, $g \mapsto \mathrm{injrad}(g)$ is a continuous choice of the injectivity radius for each C^q Riemannian metric g. This is because if we consider U_v as a graph obtained by gluing at v one edge for each pair $(E, i) \in \mathscr{E} \times \{0, 1\}$ such that $\pi_E(i) = v$, this graph is mapped by γ into a geodesic ball centered at $\gamma(v)$ of radius injrad(g) and the image of each incoming edge at v has a different inward tangent vector at $\gamma(v)$.

To ensure injectivity along the edges, we define for each edge $E \in \mathscr{E}$ a function

$$d^E_{(g,\gamma)}(t) = \min\{d_g(\gamma(t), \gamma(s)) : s \in E, |t - s| \geq \frac{\mathrm{injrad}(g)}{\mathrm{L}_g(\gamma_E)}\}.$$

In case $\pi_E(0) = \pi_E(1)$, the distance $|s - t|$ between two points $s, t \in E$ is measured with respect of the length of $S^1 = E/0 \sim 1$.

To ensure that the images of different edges under γ do not overlap, we define for each pair

$E, E' \in \mathscr{E}$, $E \neq E'$ a function $d^{E,E'}_{(g,\gamma)} : E \to \mathbb{R}_{\geq 0}$ as

$$d^{E,E'}_{(g,\gamma)}(t) = \min\{d_g(\gamma(t), \gamma(s)) : s \in E', |s - i| \geq \frac{\mathrm{injrad}(g)}{\mathrm{L}_g(\gamma_{E'})} \text{ for each } i \in \{0,1\}$$
$$\text{s.t. } \exists j \in \{0,1\} \text{ with } \pi_{E'}(i) = \pi_E(j) \text{ and } |t - j| \leq \frac{\mathrm{injrad}(g)}{\mathrm{L}_g(\gamma_E)}\}.$$

Let us fix an auxiliary isometric embedding $\psi : M \to \mathbb{R}^l$ and identify from now on our manifold M with the submanifold $\psi(M) \subseteq \mathbb{R}^l$. Given a multigraph Γ and a continuous map $\gamma : \Gamma \to M$ which is C^3 when restricted to each edge, we can consider

$$\|\gamma\|_3 = \|\gamma\|_0 + \|\dot{\gamma}\|_0 + \|\ddot{\gamma}\|_0 + \|\dddot{\gamma}\|_0$$

where given a collection $u = (u_E)_{E \in \mathscr{E}}$ of continuous functions along the edges of Γ, we define

$$\|u\|_0 = \max\{|u_E(t)| : t \in E, E \in \mathscr{E}\}$$

being $|\cdot|$ the Euclidean norm in $\mathbb{R}^l$. We have the following compactness result.

Lemma 4.4.2. *Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of C^3 Riemannian metrics converging to some metric $g \in \mathcal{M}^3$. Let $\gamma_n : \Gamma \to (M, g_n)$ be a sequence of stationary geodesic networks. Assume $\|\gamma_n\|_3 \leq M$ for some $M \in \mathbb{R}_{>0}$. Then there exists a subsequence $(\gamma_{n_k})_k$ and $\gamma \in \Omega(\Gamma, M)$ such that $\lim_{k\to\infty} \gamma_{n_k} = \gamma$ in $\Omega(\Gamma, M)$ and $\gamma : \Gamma \to (M, g)$ is stationary.*

Proof. The Arzela-Ascoli Theorem gives a subsequence $\gamma_{n_k} \to \gamma$ in $\Omega(\Gamma, M)$. The fact that γ is stationary with respect to g comes from the continuity of the operator H defined in [46] (which plays the role of the mean curvature operator on minimal surfaces) which vanishes in a pair $(g, [\gamma])$ if and only if γ is stationary with respect to g. $\square$

We will also need the following two lemmas.

Lemma 4.4.3. *Let $F : \mathbb{R}^n \to \mathbb{N}$ be a function. Then there exists $m \in \mathbb{N}$ and a basis $\{v_1, ..., v_n\}$ of $\mathbb{R}^n$ such that $F(v_i) = m$ for all $1 \leq i \leq n$.*

Proof. Observe that $\mathbb{R}^n = \bigcup_{m \in \mathbb{N}} F^{-1}(m)$ and therefore $\mathbb{R}^n = \bigcup_{m \in \mathbb{N}} \langle F^{-1}(m) \rangle$ where given $A \subseteq \mathbb{R}^n$ we denote $\langle A \rangle$ the subspace spanned by A. If $F^{-1}(m)$ did not contain a basis of $\mathbb{R}^n$ for every $m \in \mathbb{N}$, $\langle F^{-1}(m) \rangle$ would a proper subspace for every m. Therefore, $\mathbb{R}^n$ would be a countable union of closed subspaces with empty interior, which leads to a contradiction due to the Baire Category Theorem. $\square$

Lemma 4.4.4. *Let $\gamma : (-1,1)^N \to \Omega(\Gamma, M)$ and $g : (-1,1)^N \to \mathcal{M}^q$ be smooth maps. Assume that $\gamma(s)$ is stationary with respect to $g(s)$ for every $s \in (-1,1)^N$. Then for every $t \in (-1,1)^N$ and every $v \in \mathbb{R}^N$*

$$\frac{\partial}{\partial v}\Big|_{s=t} \mathrm{L}(g(s), \gamma(s)) = \frac{1}{2} \int_{\gamma(t)} \mathrm{tr}_{\gamma(t), g(t)} \frac{\partial g}{\partial v}(t) \, \mathrm{dL}_{g(t)} \, .$$

Proof. Using that the length functional is a differentiable function $\mathrm{L} : \mathcal{M}^q \times \Omega(\Gamma, M) \to \mathbb{R}$ and the chain rule, we get

$$\begin{aligned}
\frac{\partial}{\partial v}\Big|_{s=t} \mathrm{L}(g(s),\gamma(s)) &= D\,\mathrm{L}_{(g(t),\gamma(t))}(D(g\times\gamma)_t(v)) \\
&= D\,\mathrm{L}_{(g(t),\gamma(t))}(\frac{\partial g}{\partial v}(t), \frac{\partial \gamma}{\partial v}(t)) \\
&= D_1\,\mathrm{L}_{(g(t),\gamma(t))}(\frac{\partial g}{\partial v}(t)) + D_2\,\mathrm{L}_{g(t),\gamma(t))}(\frac{\partial \gamma}{\partial v}(t)) \\
&= D_1\,\mathrm{L}_{(g(t),\gamma(t))}(\frac{\partial g}{\partial v}(t)).
\end{aligned}$$

The second term in the penultimate equation vanishes because $\gamma(t)$ is stationary with respect to $g(t)$. Hence

$$\begin{aligned}
\frac{\partial}{\partial v}\Big|_{s=t} \mathrm{L}(g(s),\gamma(s)) &= \frac{d}{ds}\Big|_{s=0} \mathrm{L}(g(t+sv),\gamma(t)) \\
&= \frac{d}{ds}\Big|_{s=0} \sum_{E\in\mathscr{E}} n(E) \int_E \sqrt{g_{t+sv}(\dot\gamma_t(u),\dot\gamma_t(u))}du \\
&= \sum_{E\in\mathscr{E}} n(E) \int_E \frac{d}{ds}\Big|_{s=0} \sqrt{g_{t+sv}(\dot\gamma_t(u),\dot\gamma_t(u))}du \\
&= \sum_{E\in\mathscr{E}} n(E) \int_E \frac{\frac{\partial g}{\partial v}(t)(\dot\gamma_t(u),\dot\gamma_t(u))}{2\sqrt{g_t(\dot\gamma_t(u),\dot\gamma_t(u))}}du \\
&= \frac{1}{2}\sum_{E\in\mathscr{E}} n(E) \int_E \frac{\frac{\partial g}{\partial v}(t)(\dot\gamma_t(u),\dot\gamma_t(u))}{g_t(\dot\gamma_t(u),\dot\gamma_t(u))}\sqrt{g_t(\dot\gamma_t(u),\dot\gamma_t(u))}du \\
&= \frac{1}{2}\sum_{E\in\mathscr{E}} n(E) \int_{\gamma(t)_E} \mathrm{tr}_{\gamma(t),g(t)} \frac{\partial g}{\partial v}(t)\,\mathrm{dL}_{g(t)} \\
&= \frac{1}{2}\int_{\gamma(t)} \mathrm{tr}_{\gamma(t),g(t)} \frac{\partial g}{\partial v}(t)\,\mathrm{dL}_{g(t)}\,.
\end{aligned}$$

$\square$

Proof of Proposition 4.4.1. Notice that it suffices to show that for each $p\in\mathbb{N}$, there exists a full measure subset $\mathcal{A}(p)\subseteq I^N$ where *(1)* and *(2)* hold, because in that case $\mathcal{A}=\bigcap_{p\in\mathbb{N}}\mathcal{A}(p)$ will have the desired property. Therefore we will assume $p\in\mathbb{N}$ is fixed.

Let $g: I^N\to\mathcal{M}^q$ be a smooth embedding. Let $\{\Gamma_i\}_{i\geq 1}$ be a sequence enumerating the countable collection of all good weighted multigraphs. For each $i\geq 1$, let $\mathcal{S}^q(\Gamma_i)$ be the space of pairs $(g,[\gamma])$ where $g\in\mathcal{M}^q$, $\gamma:\Gamma_i\to(M,g)$ is an embedded stationary geodesic net and $[\gamma]$ denotes its class modulo reparametrization as defined in [46] for connected multigraphs with at least three incoming edges at each vertex and in [47] for embedded closed geodesics. By the structure theorems proved in [46] and [47], each $\mathcal{S}^q(\Gamma_i)$ is a second countable Banach manifold and the projection map $\Pi_i:\mathcal{S}^q(\Gamma_i)\to\mathcal{M}^q$ mapping $(g,[\gamma])\mapsto g$ is Fredholm of index 0. A pair $(g,[\gamma])\in\mathcal{S}_q(\Gamma_i)$ is a critical point of Π_i if and only if γ admits a nontrivial Jacobi field with respect to the metric g.

By Smale's transversality theorem, we can perturb $g: I^N\to\mathcal{M}^q$ slightly in the C^∞ topology to a C^∞ embedding $g': I^N\to\mathcal{M}^q$ which is transversal to $\Pi_i:\mathcal{S}^q(\Gamma_i)\to\mathcal{M}^q$ for every $i\in\mathbb{N}$. Transversality implies that $M_i=\Pi_i^{-1}(g'(I^N))$ is an N-dimensional embedded submanifold of $\mathcal{S}^q(\Gamma_i)$ for every $i\in\mathbb{N}$. Let $\pi_i=(g')^{-1}\circ\Pi_i\big|_{M_i}: M_i\to I^N$. Let $\tilde{\mathcal{A}}_i\subseteq I^n$ be the set of regular values of π_i, which is a set of full measure by Sard's theorem. Let $\tilde{\mathcal{A}}_0\subseteq I^N$ be the set of points for which the

Lipschitz function $s \mapsto \omega_p^1(g'(s))$ is differentiable. Observe that $\tilde{\mathcal{A}}_0$ has full measure by Rademacher's theorem. Therefore, $\tilde{\mathcal{A}} = \bigcap_{i \geq 0} \tilde{\mathcal{A}}_i$ is a full measure subset of I^N. Notice that by transversality, if $t \in \tilde{\mathcal{A}}$ then $g'(t)$ is a bumpy metric, i.e. all embedded stationary geodesic nets with respect to $g'(t)$ and with domain a good weighted multigraph are nondegenerate; and also the map $s \mapsto \omega_p^1(g'(s))$ is differentiable at $s = t$.

Given a weighted multigraph $\Gamma = \bigcup_{i=1}^P \Gamma_i$ whose connected components Γ_i are good and a natural number $M \in \mathbb{N}$, we define $\mathcal{B}_{\Gamma,M}$ as the set of all $t \in I^N$ such that there exists a stationary geodesic network $\gamma : \Gamma \to (M, g'(t))$ verifying

1. $\gamma_i = \gamma|_{\Gamma_i}$ is an embedding for each $1 \leq i \leq P$.
2. $\|\gamma_i\|_3 \leq M$ for every $1 \leq i \leq P$.
3. $F_1(g'(t), \gamma_i) \geq \frac{1}{M}$ for every $1 \leq i \leq P$.
4. $F_2^{(E_1,i_1),(E_2,i_2)}(g'(t), \gamma_i) \leq 1 - \frac{1}{M}$ for every $1 \leq i \leq P$ and every pair $(E_1, i_1) \neq (E_2, i_2)$ in $\mathscr{E}_i \times \{0, 1\}$ such that $\pi_{E_1}(i_1) = \pi_{E_2}(i_2)$.
5. $d^E_{(g'(t),\gamma_i)}(s) \geq \frac{1}{M}$ for every $1 \leq i \leq P$, $E \in \mathscr{E}_i$ and $s \in E$.
6. $d^{E,E'}_{(g'(t),\gamma_i)}(s) \geq \frac{1}{M}$ for every $1 \leq i \leq P$, $E \neq E' \in \mathscr{E}_i$ and $s \in E$.
7. $\omega_p^1(g'(t)) = \mathrm{L}_{g'(t)}(\gamma)$.

where $\mathscr{E}_i$ denotes the set of edges of Γ_i. Observe that $I^N = \bigcup_{\Gamma,M} \mathcal{B}_{\Gamma,M}$ because of (4.1) and Remark 4.2.12. We claim that each $\mathcal{B}_{\Gamma,M} \subseteq I^N$ is closed.

Indeed, suppose we have a sequence $\{t_j\}_{j\in\mathbb{N}} \subseteq \mathcal{B}_{\Gamma,M}$ converging to some $t \in I^N$. Let γ^j be the stationary geodesic network corresponding to $g'(t_j)$ and verifying properties (1) to (7) above. By property (2) and Lemma (4.4.2), passing to a subsequence we have that if $\gamma_i^j = \gamma^j\big|_{\Gamma_i}$ then there exists $\gamma_i : \Gamma_i \to M$ such that $\lim_{j\to\infty} \gamma_i^j = \gamma_i$ in $\Omega(\Gamma_i, M)$ and γ_i is stationary with respect to $g'(t)$ for each $1 \leq i \leq P$. Observe also that if $\gamma = \bigcup_j \gamma_i$

$$L_{g'(t)}(\gamma) = \lim_{j\to\infty} L_{g'(t_j)}(\gamma^j) = \lim_{j\to\infty} \omega_p(t_j) = \omega_p(t).$$

Properties (2) to (6) are preserved when we take the limit of the sequence γ^j, so it suffices to show that $\gamma|_{\Gamma_i}$ is embedded for each $1 \leq i \leq P$. Fix such i. Properties (3), (4) and (5) imply that γ_i is injective along the edges and property (6) combined with property (4) imply that the images of different edges do not intersect (except at the common vertices).

As each $\mathcal{B}_{\Gamma,M}$ is closed, they are measurable and therefore so are the sets $\tilde{\mathcal{A}}_{\Gamma,M} = \tilde{\mathcal{A}} \cap \mathcal{B}_{\Gamma,M}$ (whose union is $\tilde{\mathcal{A}}$). Let $\mathcal{A}_{\Gamma,M}$ be the set of points $t \in \tilde{\mathcal{A}}_{\Gamma,M}$ where the Lebesgue density of $\tilde{\mathcal{A}}_{\Gamma,M}$ at t is 1. By the Lebesgue Differentiation Theorem, $\tilde{\mathcal{A}}_{\Gamma,M} \setminus \mathcal{A}_{\Gamma,M}$ has Lebesgue measure 0 for each pair (Γ, M). Let us define $\mathcal{A} = \bigcup_{\Gamma,M} \mathcal{A}_{\Gamma,M}$, observe that as $\tilde{\mathcal{A}} \setminus \mathcal{A}$ has measure 0, $\mathcal{A} \subseteq I^N$ has full measure.

Fix $t \in \mathcal{A}$. Let (Γ, M) be such that $t \in \mathcal{A}_{\Gamma,M}$. As the density of $\tilde{\mathcal{A}}_{\Gamma,M}$ at t is 1, given $v \in \mathbb{R}^N$ with $|v| = 1$ we can find a sequence $\{t_m(v)\}_{m\in\mathbb{N}} \subseteq \tilde{\mathcal{A}}_{\Gamma,M}$ such that $\lim_{m\to\infty} t_m(v) = t$ and $\lim_{m\to\infty} \frac{t - t_m(v)}{|t - t_m(v)|} = v$. Denoting $\omega_p(s) = \omega_p^1(g'(s))$, using that ω_p is a Lipschitz function we can see that

$$\lim_{m\to\infty} \frac{\omega_p(t_m(v)) - \omega_p(t)}{|t - t_m|} = \frac{\partial}{\partial v}\omega_p(t). \tag{4.7}$$

As $t_m(v) \in \tilde{\mathcal{A}}_{\Gamma,M}$, for each $m \in \mathbb{N}$ there exists a stationary geodesic network $\gamma_m : \Gamma \to M$ with respect to $g'(t_m(v))$ such that

$$\omega_p(t_m(v)) = \omega_p^1(g'(t_m(v))) = \mathrm{L}_{g'(t_m(v))}(\gamma_m)$$

and properties (1) to (6) above hold. By the reasoning used to prove that the $\mathcal{B}_{\Gamma,M}$ are closed, we can construct a stationary geodesic net $\gamma : \Gamma \to (M, g'(t))$ which is embedded when restricted to each connected component Γ_i of Γ, is the limit of (a subsequence of) the γ_m's in the C^2 topology and realizes the width $\omega_p^1(g'(t))$. Hence from (4.7) we get

$$\frac{\partial}{\partial v}\omega_p(t) = \lim_{m\to\infty} \frac{\mathrm{L}_{g'(t_m)}(\gamma_m) - \mathrm{L}_{g'(t)}(\gamma)}{|t - t_m|}.$$

As $\gamma|_{\Gamma_i}$ is an embedded stationary geodesic net with respect to $g'(t)$ for each $1 \le i \le P$ and $g'(t)$ is bumpy, $\Pi_i : \mathcal{S}^q(\Gamma_i) \to \mathcal{M}^q$ is a diffeomorphism from a neighborhood U_i of $(g'(t), [\gamma_i])$ to a neighborhood $W_i = \Pi_i(U)$ of $g'(t)$. Denote Ξ_i its inverse. As there exists $m_0 \in \mathbb{N}$ such that $g'(t_m) \in W = \bigcap_{i=1}^P W_i$ and $[\gamma_m\big|_{\Gamma_i}] \in U_i$ for every $m \ge m_0$, we deduce that $[\gamma_m\big|_{\Gamma_i}] = \Xi_i(g'(t_m(v)))$ for each $1 \le i \le P$ if m is sufficiently large. Let us define $\Xi : W \to \hat{\Omega}(\Gamma, M)$ as $\Xi(g) = h$ where $h|_{\Gamma_i} = \Xi_i(g)$. Thus by Lemma 4.4.4

$$\begin{aligned}
\frac{\partial}{\partial v}\omega_p(t) &= \lim_{m\to\infty} \frac{\mathrm{L}_{g'(t_m)}(\Xi(g'(t_m))) - \mathrm{L}_{g'(t)}(\Xi(g'(t)))}{|t_m - t|} \\
&= \frac{\partial}{\partial v}\Big|_{s=t} \mathrm{L}(g'(s), \Xi(g'(s))) \\
&= \frac{1}{2}\int_{\gamma_v} \mathrm{tr}_{\gamma_v, g'(t)} \frac{\partial g'}{\partial v}(t)\, \mathrm{dL}_{g'(t)}\,.
\end{aligned}$$

Where $\gamma_v = \Xi(g'(t))$ is the one constructed before. Observe that γ_v depends on v and that the previous formula holds for each $v \in \mathbb{R}^N$, $|v| = 1$. Notice that each γ_v is a stationary geodesic network with respect to $g'(t)$, and as $g'(t)$ is bumpy there are countably many possible $\gamma_v's$, say $\{h_j\}_{j\in\mathbb{N}}$. This induces a map $F : \mathbb{R}^N \to \mathbb{N}$ defined as $F(0) = 1$ and if $w \ne 0$ then $F(w) = j$ where $\gamma_{\frac{w}{|w|}} = h_j$. By Lemma 4.4.3 we can obtain $m \in \mathbb{N}$ and a basis $w_1, ..., w_N$ of $\mathbb{R}^N$ with the property $\gamma(w_i) = m$ for every $1 \le i \le N$. Therefore if we set $v_i := \frac{w_i}{|w_i|}$, $v_1, ..., v_N$ is still a basis and by definition $\gamma_{v_i} = h_m$ for every i. By linearity of directional derivatives, denoting $\gamma = h_m$ we deduce that

$$\frac{\partial}{\partial v}\omega_p(t) = \frac{1}{2}\int_{\gamma} \mathrm{tr}_{\gamma, g'(t)} \frac{\partial g'}{\partial v}(t)\, \mathrm{dL}_{g'(t)}$$

for every unit $v \in \mathbb{R}^N$, which completes the proof. □

Proposition 4.4.5. *Let M be a closed manifold and let g be a C^q Riemannian metric on M, $q \ge 3$. Let $\gamma_1, ..., \gamma_k$ be a finite collection collection of connected, embedded stationary geodesic networks on (M, g) whose domains are good weighted multigraphs and let $\mathcal{U} \subseteq \mathcal{M}^q$ be an open neighborhood of g. Then there exists $g' \in \mathcal{U}$ such that $\gamma_1, ...\gamma_k$ are non-degenerate stationary geodesic nets with respect to g'.*

Proof. Following [31, Lemma 4], we will consider conformal perturbations of the metric of the form

$g_\varepsilon(x) = e^{-2\varepsilon\phi(x)}g(x)$. Let us denote $\tilde{\gamma} = \bigcup_{i=1}^k \gamma_i$, $\tilde{\Gamma} = \bigcup_{i=1}^k \Gamma_i$ (where $\gamma_i : \Gamma_i \to M$) and $\tilde{\mathscr{E}}$ the set of edges of $\tilde{\gamma}$. Notice that $\tilde{\gamma} : \tilde{\Gamma} \to M$ is a stationary geodesic network whose edges may overlap, even non-transversally. Given $E \in \mathscr{E}$, let $\mathrm{Reg}(\tilde{\gamma}_E)$ be the set of interior points of $\tilde{\gamma}_E$ which are not points of transverse intersection with any other edge $\tilde{\gamma}_E$. We define a finite poset

$$\mathcal{P} = \{\bigcap_{i=1}^{l} \mathrm{Reg}(\tilde{\gamma}_{E_i}) \neq \emptyset : E_1, ..., E_l \in \tilde{\mathscr{E}},\ E_i \neq E_j\ \forall i \neq j\}$$

which is the collection of finite non-empty intersections of sets in $\{\mathrm{Reg}(\tilde{\gamma}_E) : E \in \mathscr{E}\}$, with the order given by the inclusion. Denote by $\mathcal{P}'$ the set of minimal elements in $\mathcal{P}$. Observe that if α, α' are two different elements of $\mathcal{P}'$ then they are disjoint. Given $\alpha \in \mathcal{P}'$, write $\alpha = \bigcap_{i=1}^l \mathrm{Reg}(\tilde{\gamma}_{E_i})$ in the unique way such that $\alpha \cap \tilde{\gamma}_E = \emptyset$ for every $E \in \tilde{\mathscr{E}} \setminus \{E_1, ..., E_l\}$. Pick $t_\alpha \in \alpha$ for every $\alpha \in \mathcal{P}'$, and let $\eta > 0$ be such that the geodesic balls $B_\alpha = B(p_\alpha, \eta)$ verify

- $B_\alpha \cap B_{\alpha'} = \emptyset$ if $\alpha \neq \alpha'$.
- $B_\alpha \cap \gamma_E = \emptyset$ if $E \notin \{E_1, ..., E_l\}$.
- $B_\alpha \cap \gamma_{E_i} \subseteq \alpha$ for every $i = 1, ..., l$.
- There exists a diffeomorphism $\rho_\alpha : B_\alpha \to \mathbb{R}^n$ such that $\rho_\alpha(\gamma_{E_i} \cap B_\alpha) = \rho_\alpha(\alpha \cap B_\alpha) = \{(t, 0, 0, ..., 0) : t \in \mathbb{R}\}$ for each $i = 1, ..., l$.

Denote $B'_\alpha = B(p_\alpha, \frac{\eta}{2})$. Observe that for each $E \in \tilde{\mathscr{E}}$ there exists at least one $\alpha \in \mathcal{P}'$ such that $\alpha \subseteq \tilde{\gamma}_E$. Choose such an α for each $E \in \mathscr{E}$ and denote $B_E = B_\alpha$ and $B'_E = B'_\alpha$. We can now proceed to define the function ϕ which will induce the one-parameter family of metrics $g_\varepsilon(x) = e^{-2\varepsilon\phi(x)}$ mentioned before.

For each $\alpha \in \mathcal{P}'$, let $\psi_\alpha : M \to \mathbb{R}$ be a smooth function with $0 \leq \psi_\alpha \leq 1$, $\mathrm{spt}(\psi_\alpha) \subseteq B_\alpha$ and $\psi_\alpha \equiv 1$ in B'_α. Let $f_\alpha : B_\alpha \to \mathbb{R}$ be given in local coordinates under the chart (B_α, ρ_α) by $f_\alpha(x) = \sum_{i=2}^n x_i^2$. We define $\phi = \sum_{\alpha \in \mathcal{P}'} \psi_\alpha f_\alpha$. An easy computation shows that $D\phi$ vanishes along $\tilde{\gamma}$ and in local coordinates $\mathrm{Hess}_{\tilde{\gamma}(t)}\, \phi(X, Y) = \psi_\alpha(x) \sum_{i=2}^n x_i y_i$ if $\tilde{\gamma}(t) \in B_\alpha$, $X = (x_1, ..., x_n)$ and $Y = (y_1, ..., y_n)$; and $\mathrm{Hess}_{\tilde{\gamma}(t)}\, \phi \equiv 0$ if $t \notin \bigcup_{\alpha \in \mathcal{P}'} B_\alpha$. In particular, if $\tilde{\gamma}(t) \in B'_\alpha$ for some $\alpha \in \mathcal{P}'$ then $\mathrm{Hess}_{\tilde{\gamma}(t)}\, \phi(X, X) = 0$ if and only if $X \in \langle \dot{\tilde{\gamma}}_{E_j}(t) \rangle$ for some (or equivalently, for every) $j \in \{1, ..., l\}$ where $\alpha = \bigcap_{i=1}^l \mathrm{Reg}(\tilde{\gamma}_{E_i})$.

Therefore we know that ϕ and $D\phi$ vanish along each γ_i. Hence by [3, Theorem 1.159], the $\gamma_1, ..., \gamma_k$ are still stationary with respect to $g_\varepsilon(x) = e^{-2\varepsilon\phi(x)}g(x)$. Fix $\gamma = \gamma_i : \Gamma \to M$ with set of vertices $\mathscr{V}$ and set of edges $\mathscr{E}$. We assume that Γ is good* (i.e. every vertex has at least 3 different incoming edges), the case when γ is an embedded closed geodesic can be handled with the same method using the ellipticity of its Jacobi operator. As discussed in Section 4.3, the stability operator of γ with respect to g is the map $L : H^2(\gamma) \to L^2(\gamma)$ given by

$$L(J) = ((L_E(J))_{E \in \mathscr{E}}, (B_v(J))_{v \in \mathscr{V}})$$

where

$$L_E(J) = -\frac{n(E)}{l(E)}\left[\ddot{J}_E^{\perp} + R(\dot{\gamma}, J_E^{\perp}), \dot{\gamma}\right]$$
$$B_v(J) = \sum_{(E,i):\pi_E(i)=v} (-1)^{i+1}\frac{n(E)}{l(E)}\dot{J}_E^{\perp}(i).$$

Let us compute which is the change in the Jacobi operator along γ when we switch from the metric g to g_ε. We will denote L^ε the operator corresponding to g_ε. Using [3, Theorem 1.159] and the fact that $D\phi = 0$ along γ_i, we can see that $B_v^\varepsilon = B_v$ for all $v \in \mathscr{V}$, and that

$$L_E^\varepsilon(J) = -\frac{n(E)}{l(E)}(\ddot{J}_E^{\perp} + R(\dot{\gamma}, J_E^{\perp})\dot{\gamma} + \varepsilon \operatorname{Hess}\phi(J_E^{\perp}))$$

where the covariant derivatives and the curvature tensor R are taken with respect to the metric g, and at each point $p \in M$, $\operatorname{Hess}_p(\phi) : T_pM \to T_pM$ is the linear transformation such that the Hessian of ϕ at p is given by $(X, Y) \mapsto \langle \operatorname{Hess}_p \phi(X), Y\rangle_g$.

We know from Section 4.3 that each $L^\varepsilon : H^2(\gamma) \to L^2(\gamma)$ admits a non-decreasing sequence of eigenvalues $\lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq ... \leq \lambda_Q^\varepsilon \leq ...$ which are characterized by

$$\lambda_i^\varepsilon = \inf_W \max_{J\in W\setminus\{0\}} \frac{\langle L^\varepsilon(J), \iota(J)\rangle}{\langle \iota(J), \iota(J)\rangle}$$

where the infimum is taken over all i-dimensional subspaces W of $H^2(\gamma)$. Also, the map $\varepsilon \mapsto L^\varepsilon$ is continuous; therefore λ_i^ε varies continuously with ε for every $i \in \mathbb{N}$. We will use these facts to show that for sufficiently small values of $\varepsilon > 0$, 0 is not an eigenvalue of L^ε.

Let Q be the unique natural number such that $0 = \lambda_Q < \lambda_{Q+1}$ (here $\lambda_i := \lambda_i^0$). Denote S the sum of the eigenspaces corresponding to $\lambda_1, ..., \lambda_Q$. Let $J \in S$, $J \neq 0$. Then we have

$$\begin{aligned}\frac{\langle L^\varepsilon(J), \iota(J)\rangle}{\langle \iota(J), \iota(J)\rangle} &= \frac{\langle L(J), \iota(J)\rangle}{\langle \iota(J), \iota(J)\rangle} - \frac{\sum_{E\in\mathscr{E}}\int_E \frac{n(E)}{l(E)}\varepsilon\langle \operatorname{Hess}_{\gamma(t)}(\phi)(J_E^{\perp}(t)), J_E^{\perp}(t)\rangle_g dt}{\langle \iota(J), \iota(J)\rangle}\\ &\leq -\varepsilon\sum_{E\in\mathscr{E}}\frac{n(E)}{l(E)}\frac{\int_E\langle \operatorname{Hess}_{\gamma(t)}(\phi)(J_E^{\perp}(t)), J_E^{\perp}(t)\rangle_g dt}{\langle \iota(J), \iota(J)\rangle}\\ &\leq 0\end{aligned}$$

because $\operatorname{Hess}_{\gamma(t)}\phi \geq 0$ for every $t \in \Gamma$. Suppose there is equality for some $J \in S\setminus\{0\}$. Then the two inequalities should be equalities. From the first one we deduce that J is Jacobi along γ for the metric g, and thus it verifies $\ddot{J}_E^{\perp} + R(\dot{\gamma}, J_E^{\perp})\dot{\gamma} = 0$ for every $E \in \mathscr{E}$. From the second one, by considering the values of t for which $\gamma(t) \in B'_E$, we see that $J_E^{\perp}$ is a null vector of $\operatorname{Hess}_{\gamma(t)}\phi$ along $\gamma_E \cap B'_E$ and therefore $J_E^{\perp} = 0$ on $\gamma_E \cap B'_E$; and as it satisfies the Jacobi equation this implies $J_E^{\perp} = 0$ for every $E \in \mathscr{E}$. Thus J must be parallel and hence $J = 0$ as $H^2(\gamma)$ does not contain non-trivial parallel vector fields. But this is a contradiction because we chose $J \in S \setminus \{0\}$. Hence we just proved that

$$\frac{\langle L^\varepsilon(J), \iota(J)\rangle}{\langle \iota(J), \iota(J)\rangle} < 0$$

for every $J \in S \setminus \{0\}$. As S is finite dimensional and $\frac{\langle L^\varepsilon(J), \iota(J)\rangle}{\langle \iota(J), \iota(J)\rangle}$ is invariant under rescaling of J, the compactness of the unit ball in S implies that there exists $c(\varepsilon) > 0$ such that

$$\frac{\langle L^\varepsilon(J), \iota(J)\rangle}{\langle \iota(J), \iota(J)\rangle} \leq -c(\varepsilon)$$

for every $J \in S \setminus \{0\}$. By the min-max characterization of the eigenvalues for L^ε, we see that $\lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq ... \leq \lambda_Q^\varepsilon \leq c(\varepsilon) < 0$. If we also choose ε sufficiently small so that $\lambda_{Q+1}^\varepsilon > 0$, we get that for $0 < \varepsilon < \varepsilon(\gamma)$, γ is nondegenerate with respect to g_ε. Taking $0 < \varepsilon < \min\{\varepsilon(\gamma_i) : 1 \leq i \leq k\}$ such that $g_\varepsilon \in \mathcal{U}$ and defining $g' := g_\varepsilon$ we get the desired result.

□

Lemma 4.4.6. *Given $\eta > 0$ and $N \in \mathbb{N}$, there exists $\varepsilon > 0$ depending on η and N such that the following is true: for any Lipschitz function $f : I^N \to \mathbb{R}$ satisfying*

$$|f(x) - f(y)| \leq 2\varepsilon$$

for every $x, y \in I^N$, and for any subset $\mathcal{A}$ of I^N of full measure, there exist $N+1$ sequences of points $\{y_{1,m}\}_m, \cdots, \{y_{N+1,m}\}_m$ contained in $\mathcal{A}$ and converging to a common limit $y \in (-1,1)^N$ such that:

- *f is differentiable at each $y_{i,m}$,*
- *the gradients $\nabla f(y_{i,m})$ converge to $N+1$ vectors $v_1, \cdots, v_{N+1}$ with*

$$d_{\mathbb{R}^N}(0, Conv(v_1, \cdots, v_{N+1})) < \eta.$$

Proof. See [31, Lemma 3]. □

4.5 Proof of the Main Theorem

Fix an n-dimensional closed manifold M. We are going to consider several choices and constructions over M. Let g be a C^∞ Riemannian metric on M. Let $\varepsilon_1 > 0$ be a positive constant such that $\varepsilon_1 < \text{injrad}(M, g)$, where $\text{injrad}(M, g)$ is the injectivity radius of (M, g). Let K be an integer and $\hat{B}_1, ..., \hat{B}_K$ be disjoint domains in M, with piecewise smooth boundary, such that the union of their closures covers M. Let $B_1, ..., B_K$ be some open neighbourhoods of $\hat{B}_1, ..., \hat{B}_K$ respectively with the property that each of them is contained in a geodesic ball of radius of ε_1. Denote $\mathcal{M}^q$ the space of all C^q Riemannian metrics on M. For each $1 \leq k \leq K$, we define a smooth function $0 \leq \phi_k \leq 1$, $\text{spt}(\phi_k) \subseteq B_k$ such that

$$\phi_k = \begin{cases} 1 \text{ on } \hat{B}_k \\ 0 \text{ on } B_k^c \end{cases}.$$

Consider also the partition of unity $\psi_k = \frac{\phi_k}{\sum_{l=1}^K \phi_l}$. We denote

$$\mathcal{C}_{g,\tilde{K},\varepsilon_1} := \{(K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\})\}$$

the set of all possible choices as above with $K \geq \tilde{K}$. Notice that $\mathcal{C}_{g,\tilde{K},\varepsilon_1}$ is non-empty, as we can always find a sufficiently fine triangulation of (M, g). We claim that the following property holds:

Proposition 4.5.1. *For any metric $g \in \mathcal{M}^\infty$, for every $\varepsilon_1 > 0$, $\tilde{K} > 0$ and any choice of*

$$S = (K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}) \in \mathcal{C}_{g,\tilde{K},\varepsilon_1}$$

there is a metric $\tilde{g} \in \mathcal{M}^\infty$ arbitrarily close to g in the C^∞ topology such that the following holds: there are stationary geodesic networks $\gamma_1, ..., \gamma_J$ with respect to $\tilde{g}$ whose connected components are nondegenerate (according to Definition 4.2.13) and coefficients $\alpha_1, ..., \alpha_J \in [0,1]$ with $\sum_{j=1}^J \alpha_j = 1$ satisfying

$$\Big| \sum_{j=1}^{J} \alpha_j \fint_{\gamma_j} \psi_k dL_{\tilde{g}} - \fint_M \psi_k d\operatorname{Vol}_{\tilde{g}} \Big| < \frac{\varepsilon_1}{K} \tag{4.8}$$

for every $k = 1, ..., K$.

In the proof, we will need to measure the distance between two rescaled functions. In order to do that, we introduce the following definition.

Definition 4.5.2. *We say that two functions $f, g : (-\delta, \delta)^K \to \mathbb{R}$ are ε-close if*

$$\|\frac{1}{\delta} f_\delta - \frac{1}{\delta} g_\delta\|_\infty < \varepsilon$$

where $f_\delta, g_\delta : (-1,1)^K \to \mathbb{R}$ are given by $f_\delta(s) = f(\delta s)$ and $g_\delta(s) = g(\delta s)$.

Remark 4.5.3. *Observe that $\frac{1}{\delta} f_\delta$ is differentiable at $s \in (-1,1)^K$ if and only if f is differentiable at $\delta s \in (-\delta, \delta)^K$ and in that case $\nabla(\frac{1}{\delta} f_\delta)(s) = \nabla f(\delta s)$.*

Proof of Proposition 4.5.1. Let $g \in \mathcal{M}^\infty$, $\tilde{K} \in \mathbb{N}$ and $\varepsilon_1 > 0$. Fix $(K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}) \in \mathcal{C}_{g,\tilde{K},\varepsilon_1}$. Let $\mathcal{U}$ be a C^∞ neighborhood of g. Choose $\varepsilon_0' > 0$ sufficiently small and $q \geq K + 3$ sufficiently large so that if $g' \in \mathcal{M}^\infty$ satisfies $\|g - g'\|_{C^q} < \varepsilon_0'$, then $g' \in \mathcal{U}$. Let $\varepsilon' \leq \varepsilon_0'$ be a positive real number (which we will have to shrink later in the argument). Our goal is to show that there exists $\tilde{g} \in \mathcal{M}^\infty$ such that $\|\tilde{g} - g\|_{C^q} < \varepsilon_0'$ and (4.8) holds for some stationary geodesic nets $\gamma_1, ..., \gamma_J$ (whose connected components are nondegenerate) with respect to $\tilde{g}$ and some coefficients $\alpha_1, ..., \alpha_J$.

Consider the following K-parameter family of metrics. For a $t = (t_1, ..., t_K) \in (-1,1)^K$, we define

$$\hat{g}(t) = e^{2\sum_k t_k \psi_k} g.$$

At $t = 0$, for each k, we have

$$\begin{aligned} \frac{\partial}{\partial t_k}\Big|_{t=0} \operatorname{Vol}(M, \hat{g}(t)) &= \frac{\partial}{\partial t_k}\Big|_{t=0} \int_M (e^{2\sum_k t_k \psi_k(x)})^{\frac{n}{2}} \operatorname{dVol}_g \\ &= \int_M n\psi_k(x) \operatorname{dVol}_g . \end{aligned}$$

As t goes to zero, we have the following expansion

$$\operatorname{Vol}(M, \hat{g}(t))^{\frac{1}{n}} = \operatorname{Vol}(M, g)^{\frac{1}{n}} + \sum_{k=1}^{K} t_k \operatorname{Vol}(M, g)^{-\frac{n-1}{n}} \int_M \psi_k(x) \operatorname{dVol}_g + R(t) \tag{4.9}$$

where $|R(t)| \leq C_1 \|t\|^2$ if $t \in (-1,1)^K$, where C_1 is a constant which depends only on g (this can be checked by computing the second order partial derivatives of $t \mapsto \operatorname{Vol}(M, \hat{g}(t))^{\frac{1}{n}}$ and using the fact that $e^{-n}\operatorname{Vol}(M,g)\leq \operatorname{Vol}(M,\hat{g}(t))\leq e^{n}\operatorname{Vol}(M,g)$ as $\operatorname{Vol}(M,\hat{g}(t))=\int_M e^{n\sum_k t_k\psi_k(x)}\,\mathrm{dVol}_g$). Following [31] we can define the following function

$$f_0(t)=\frac{\operatorname{Vol}(M,\hat{g}(t))^{\frac{1}{n}}}{\operatorname{Vol}(M,g)^{\frac{1}{n}}}-\sum_{k=1}^{K}t_k\fint_M\psi_k(x)\,\mathrm{dVol}_g\,.$$

Because of (4.9), $|f_0(t)-1|=|\frac{R(t)}{\operatorname{Vol}(M,g)^{\frac{1}{n}}}|\leq C_2\|t\|^2$ for every $t\in(-1,1)^K$; where $C_2=\frac{C_1}{\operatorname{Vol}(M,g)^{\frac{1}{n}}}$ depends only on g (as C_1 and other constants C_i to be defined later).

By the previous, f_0 is $C_2\varepsilon'$-close to 1 in $(-\delta,\delta)^K$ if $\delta<\varepsilon'$ (see Definition 4.5.2). Let $\delta<\varepsilon'$ be such that $\hat{g}:(-\delta,\delta)^K\to\mathcal{M}^q$ is an embedding and $\|\hat{g}(t)-g\|_{C^q}<\frac{\varepsilon'}{2}$ for every $t\in(-\delta,\delta)^K$. We can slightly perturb $\hat{g}$ in the C^∞ topology to another embedding $g':(-\delta,\delta)^K\to\mathcal{M}^q$ applying Proposition 4.4.1. We can assume $\|g'(t)-\hat{g}(t)\|_{C^q}<\frac{\varepsilon'}{2}$ and $\|\frac{\partial g'}{\partial v}-\frac{\partial\hat{g}}{\partial v}\|_{C^q}<\varepsilon'$ for every $t\in(-\delta,\delta)^K$ and $v\in\mathbb{R}^K:|v|=1$. Consider the function

$$f_1(t)=\frac{\operatorname{Vol}(M,g'(t))^{\frac{1}{n}}}{\operatorname{Vol}(M,g)^{\frac{1}{n}}}-\sum_{k}t_k\fint_M\psi_k(x)\,\mathrm{dVol}_g\,.$$

By the properties of g', there exists $C_3>0$ such that f_1 is $C_3\varepsilon'$-close to the constant function equal to 1 on $(-\delta,\delta)^K$.

Now we will use the Weyl law for 1-cycles in n-manifolds, which asserts that for every metric g' on M

$$\lim_{p\to\infty}\omega_p^1(M^n,g')p^{-\frac{n-1}{n}}=\alpha(n,1)\operatorname{Vol}(M^n,g')^{\frac{1}{n}}. \tag{4.10}$$

The normalized p-widths $p^{-\frac{n-1}{n}}\omega_p^1(g'(t))$ are uniformly Lipschitz continuous on $(-\delta,\delta)^K$ by [27, Lemma 3.4]. Hence, by (4.10) the sequence of functions $t\mapsto p^{-\frac{n-1}{n}}\omega_p^1(M,g'(t))$ converges uniformly to the function $t\mapsto a(n)\operatorname{Vol}(M,g'(t))^{\frac{1}{n}}$. This implies that for the previously defined $\delta>0$, there exists $p_0\in\mathbb{N}$ such that $p\geq p_0$ implies

$$|p^{-\frac{n-1}{n}}\omega_p^1(M,g'(t))-\alpha(n,1)\operatorname{Vol}(M,g'(t))^{\frac{1}{n}}|<\delta\varepsilon'$$

and hence

$$|\frac{\omega_p^1(M,g'(t))}{\alpha(n,1)p^{\frac{n-1}{n}}\operatorname{Vol}(M,g)^{\frac{1}{n}}}-\frac{\operatorname{Vol}(M,g'(t))^{\frac{1}{n}}}{\operatorname{Vol}(M,g)^{\frac{1}{n}}}|<C_4\delta\varepsilon'$$

for every $t\in(-\delta,\delta)^K$. The previous means that $h(t)=\frac{\omega_p^1(M,g'(t))}{\alpha(n,1)p^{\frac{n-1}{n}}\operatorname{Vol}(M,g)^{\frac{1}{n}}}-\frac{\operatorname{Vol}(M,g'(t))^{\frac{1}{n}}}{\operatorname{Vol}(M,g)^{\frac{1}{n}}}$ is $C_4\varepsilon'$-close to 0 in $(-\delta,\delta)^K$ and therefore as f_1 is $C_3\varepsilon'$-close to 1, by triangle inequality we have that

$$f_2(t)=\frac{\omega_p^1(M,g'(t))}{\alpha(n,1)p^{\frac{n-1}{n}}\operatorname{Vol}(M,g)^{\frac{1}{n}}}-\sum_{k=1}^{K}t_k\fint_M\psi_k(x)\,\mathrm{dVol}_g$$

is $C_5\varepsilon'$-close to 1 if $p\geq p_0$, for some $C_5>0$.

On the other hand, by our choice of g' using Proposition 4.4.1, there exists a full measure subset $\mathcal{A}\subseteq(-\delta,\delta)^K$ such that for each $t\in\mathcal{A}$ and $p\in\mathbb{N}$ the map $t\mapsto\omega_p^1(g'(t))$ is differentiable at t and there exists a stationary geodesic net $\gamma_p(t)$ with respect to $g'(t)$ so that

1. $\omega_p^1(g'(t))=\mathrm{L}_{g'(t)}(\gamma_p(t))$

2. $\frac{\partial}{\partial v}(\omega_p^1 \circ g'(s))|_{s=t} = \frac{1}{2}\int_{\gamma_p(t)} \mathrm{tr}_{\gamma_p(t),g'(t)} \frac{\partial g'}{\partial v}(t)\,\mathrm{dL}_{g'(t)}$.

Define $f_3 : (-1,1)^K \to \mathbb{R}$ as $f_3(t) = \frac{1}{\delta} f_2(\delta t)$. We know that $\|f_3 - 1\|_{\infty,(-1,1)^K} < C_5\varepsilon'$. Now we want to use Lemma 4.4.6. In order to do that we will need to impose more restrictions on ε'. Let $\eta > 0$. Let $\varepsilon > 0$ be the one depending on η and $N = K$ according to Lemma 4.4.6. Choose ε' small enough so that $C_5\varepsilon' < \varepsilon$, $\varepsilon' < \eta$ and $\varepsilon' \leq \varepsilon_0'$. Observe that this allows us to define $\delta > 0$ and $p_0 \in \mathbb{N}$ with all the properties in the construction above. Then we have

$$|f_3(x) - f_3(y)| \leq 2\varepsilon$$

for every $x, y \in (-1,1)^K$. As f_3 is Lipschitz, we can apply Lemma 4.4.6 to f_3 and the full measure subset $\mathcal{A}' = \{\frac{t}{\delta} : t \in \mathcal{A}\}$. After passing to $(-\delta,\delta)^K$ by rescaling and using Remark 4.5.3, we get $K+1$ sequences of points $\{s_{1,m}\}_m, ..., \{s_{K+1,m}\}_{m\in\mathbb{N}}$ contained in $\mathcal{A}$ and converging to a common limit $s \in (-\delta,\delta)^K$ such that:

1. f_2 is differentiable at each $s_{j,m}$.

2. The gradients $\nabla f_2(s_{j,m})$ converge to $K+1$ vectors $v_1, ..., v_{K+1}$ with

$$d_{\mathbb{R}^N}(0, \mathrm{Conv}(v_1, ..., v_{K+1})) < \eta.$$

Let $\alpha_1, ..., \alpha_{K+1} \in [0,1]$ be such that $\sum_{j=1}^{K+1} \alpha_j = 1$ and $|\sum_{j=1}^{K+1} \alpha_j v_j| < \eta$. Then if m is sufficiently large,

$$|\sum_{j=1}^{K+1} \alpha_j \nabla f_2(s_{j,m})| < \eta$$

and hence

$$|\sum_{j=1}^{K+1} \alpha_j \frac{\partial f_2}{\partial t_k}(s_{j,m})| < \eta$$

for every $k = 1, ..., K$. But using the definition of f_2 and denoting $\gamma_{j,m} = \gamma_p(s_{j,m})$,

$$\begin{aligned}\frac{\partial f_2}{\partial t_k}(s_{j,m}) &= \frac{\frac{\partial}{\partial t_k}\omega_p^1(M, g'(s))|_{s=s_{j,m}}}{\alpha(n,1)\,\mathrm{Vol}(M,g)^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \fint_M \psi_k(x)\,\mathrm{dVol}_g \\ &= \frac{\int_{\gamma_{j,m}} \mathrm{tr}_{\gamma_{j,m},g'(s_{j,m})} \frac{\partial g'}{\partial t_k}(s_{j,m})\,\mathrm{dL}_{g'(s_{j,m})}}{2\alpha(n,1)\,\mathrm{Vol}(M,g)^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \fint_M \psi_k(x)\,\mathrm{dVol}_g\,.\end{aligned}$$

As the lengths $\mathrm{L}_{g'(s_{j,m})}(\gamma_{j,m}) = \omega_p(g'(s_{j,m}))$ of the $\gamma_{j,m}$'s are uniformly bounded, passing to a subsequence we can obtain stationary geodesic networks $\gamma_1, ..., \gamma_{K+1}$ with respect to $g'(s)$ verifying

$$\lim_{m\to\infty} \gamma_{j,m} = \gamma_j \tag{4.11}$$

in the varifold topology for every $j = 1, ..., K+1$. Hence from the previous,

$$|\sum_{j=1}^{K+1} \alpha_j \frac{\int_{\gamma_j} \mathrm{tr}_{\gamma_j,g'(s)} \frac{\partial g'}{\partial t_k}(s)\,\mathrm{dL}_{g'(s)}}{2\alpha(n,1)\,\mathrm{Vol}(M,g)^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \fint_M \psi_k(x)\,\mathrm{dVol}_g\,| \leq \eta$$

for every $k = 1, ..., K$. Using that $\|\hat{g}(t) - g\|_{C^q} < \frac{\varepsilon'}{2}$, $\|g'(t) - \hat{g}(t)\|_{C^q} < \frac{\varepsilon'}{2}$ and $\|\frac{\partial g'}{\partial v} - \frac{\partial \hat{g}}{\partial v}\|_{C^q} < \varepsilon'$ for every $t \in (-\delta, \delta)^K$ and $v \in \mathbb{R}^K : |v| = 1$; and the fact that $\varepsilon' < \eta$, we can see that there exists a constant $C_6 > 0$ such that

$$|\sum_{j=1}^{K+1} \alpha_j \frac{\int_{\gamma_j} \mathrm{tr}_{\gamma_j, \hat{g}(s)} \frac{\partial \hat{g}}{\partial t_k}(s) \, \mathrm{dL}_{\hat{g}(s)}}{2\alpha(n,1) \operatorname{Vol}(M, g'(s))^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \fint_M \psi_k(x) \, \mathrm{dVol}_{g'(s)} | \leq C_6 \eta.$$

By definition of $\hat{g}$, $\frac{\partial \hat{g}}{\partial t_k}(s) = 2\psi_k \hat{g}(s)$ thus

$$|\sum_{j=1}^{K+1} \alpha_j \frac{\int_{\gamma_j} \psi_k \, \mathrm{dL}_{\hat{g}(s)}}{\alpha(n,1) \operatorname{Vol}(M, g'(s))^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \fint_M \psi_k(x) \, \mathrm{dVol}_{g'(s)} | \leq C_6 \eta. \tag{4.12}$$

Combining (4.12) with the fact that $\|g'(s) - \hat{g}(s)\|_{C^q} < \frac{\varepsilon'}{2}$,

$$|\sum_{j=1}^{K+1} \alpha_j \frac{\int_{\gamma_j} \psi_k \, \mathrm{dL}_{g'(s)}}{\alpha(n,1) \operatorname{Vol}(M, g'(s))^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \fint_M \psi_k(x) \, \mathrm{dVol}_{g'(s)} | \leq C_7 \eta. \tag{4.13}$$

But we know that $\mathrm{L}_{g'(s)}(\gamma_j) = \omega_p^1(g'(s))$ for every $j = 1, ..., K+1$, so

$$\begin{aligned} &|\frac{\int_{\gamma_j} \psi_k \, \mathrm{dL}_{g'(s)}}{\alpha(n,1) \operatorname{Vol}(M, g'(s))^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \fint_{\gamma_j} \psi_k \, \mathrm{dL}_{g'(s)} | = \\ &|\fint_{\gamma_j} \psi_k \, \mathrm{dL}_{g'(s)} || \frac{\omega_p^1(g'(s))}{\alpha(n,1) \operatorname{Vol}(M, g'(s))^{\frac{1}{n}} p^{\frac{n-1}{n}}} - 1| \leq \\ &| \frac{\omega_p^1(g'(s))}{\alpha(n,1) \operatorname{Vol}(M, g'(s))^{\frac{1}{n}} p^{\frac{n-1}{n}}} - 1| \leq \eta \end{aligned}$$

if $p \geq p_1$ for some $p_1 \in \mathbb{N}$, because of the Weyl law and the fact $0 \leq \psi_k \leq 1$. Hence from (4.13),

$$|\sum_{j=1}^{K+1} \alpha_j \fint_{\gamma_j} \psi_k \, \mathrm{dL}_{g'(s)} - \fint_M \psi_k \, \mathrm{dVol}_{g'(s)} | \leq C_8 \eta$$

for some constant C_8 depending only on g. Let us take $\eta = \frac{\varepsilon_1}{2C_8 K}$ and $p \geq \max\{p_0, p_1\}$.

Notice that $\|g'(s) - g\|_{C^q} \leq \|g'(s) - \hat{g}(s)\|_{C^q} + \|\hat{g}(s) - g\|_{C^q} < \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} < \varepsilon_0'$. Let us represent each γ_i as a map $\gamma_i : \Gamma_i \to M$ where each connected component of the weighted multigraph Γ_i is good and the restrictions of γ_i to those connected components are embedded (here we are using Remark 4.2.12). The metric $g'(s)$ has all the properties required by Proposition 4.5.1 except that the components of the γ_i's may not be non-degenerate and may not be C^∞ (in principle they are only C^q). Using Proposition 4.4.5, we can change $g'(s)$ for another C^q metric $\overline{g}$ which still verifies $\|\overline{g} - g\|_{C^q} < \varepsilon_0'$, and has the property that $\gamma_1, ..., \gamma_{K+1}$ are non-degenerate stationary geodesic nets with respect to $\overline{g}$. If on top of that we apply the Implicit Function Theorem (see [27, Lemma 2.6]), we can find a C^∞ metric $\tilde{g}$ close enough to $\overline{g}$ in the C^q topology so that $\|\tilde{g} - g\|_{C^q} < \varepsilon_0$ which admits stationary geodesic networks $\tilde{\gamma}_1, ..., \tilde{\gamma}_{k+1}$ whose connected components are nondegenerate and verify

$$|\sum_{j=1}^{K+1} \alpha_j \fint_{\tilde{\gamma}_j} \psi_k \, \mathrm{dL}_{\tilde{g}} - \fint_M \psi_k \, \mathrm{dVol}_{\tilde{g}} | < \frac{\varepsilon_1}{K}$$

for every $k = 1, ..., K+1$. This completes the proof.

$\square$

Now we will show that Proposition 4.5.1 implies Theorem 4.1.5. Given $g \in \mathcal{M}^\infty$, $\varepsilon_1 > 0$, $\tilde{K} \in \mathbb{N}$ and $S \in \mathcal{C}_{g,\tilde{K},\varepsilon_1}$ we will denote $\mathcal{M}(g, \varepsilon_1, \tilde{K}, S)$ the set of all metrics $\tilde{g} \in \mathcal{M}^\infty$ such that $\|\tilde{g} - g\|_{C^q} < \varepsilon_1$ (computed with respect to g) and there exist stationary geodesic networks $\gamma_1, ..., \gamma_J$ with respect to $\tilde{g}$ whose connected components are nondegenerate (according to Definition 4.2.13) and coefficients $\alpha_1, ..., \alpha_J \in [0,1]$ with $\sum_{j=1}^J \alpha_j = 1$ such that (4.8) holds for every $k = 1, ..., K$. By the Implicit Function Theorem, $\mathcal{M}(g, \varepsilon_1, \tilde{K}, S)$ is open (see [27, Lemma 2.6]). Therefore given $\varepsilon_1 > 0$ and $\tilde{K} \in \mathbb{N}$ the set

$$\mathcal{M}(\varepsilon_1, \tilde{K}) = \bigcup_{g \in \mathcal{M}^\infty} \bigcup_{S \in \mathcal{C}_{g,\varepsilon_1,\tilde{K}}} \mathcal{M}(g, \varepsilon_1, K, S)$$

is open and by Proposition 4.5.1 it is also dense in $\mathcal{M}^\infty$. Define

$$\tilde{\mathcal{M}} = \bigcap_{m \in \mathbb{N}} \mathcal{M}(\frac{1}{m}, m)$$

which is a generic subset of $\mathcal{M}^\infty$ in the Baire sense. We are going to prove that if $\tilde{g} \in \tilde{\mathcal{M}}$ then there exists a sequence of equidistributed stationary geodesic networks with respect to $\tilde{g}$.

Fix $\tilde{g} \in \tilde{\mathcal{M}}$. By definition, given $m \in \mathbb{N}$ there exists $g \in \mathcal{M}^\infty$ such that $\tilde{g} \in \mathcal{M}(g, \frac{1}{m}, m, S)$ for some $S \in \mathcal{C}_{g,\frac{1}{m},m}$. Therefore, $\tilde{g}$ belongs to a $\frac{1}{m}$ neighborhood of g in the C^K topology; and there exist $J = J_m \in \mathbb{N}$, stationary geodesic networks $\gamma_{m,1}, ..., \gamma_{m,J_m}$ with respect to $\tilde{g}$ and coefficients $\alpha_{m,1}, ..., \alpha_{m,J_m} \in [0,1]$ with $\sum_{j=1}^{J_m} \alpha_{m,j} = 1$ satisfying

$$|\sum_{j=1}^{J_m} \alpha_{m,j} ⨍_{\gamma_{m,j}} \psi_k(x)\, \mathrm{dL}_{\tilde{g}} - ⨍_M \psi_k(x)\, \mathrm{dVol}_{\tilde{g}}\,| < \frac{1}{mK} \tag{4.14}$$

for every $k = 1, ..., K$. Let $f \in C^\infty(M, \mathbb{R})$. We want to obtain a formula analogous to the previous one but replacing ψ_k by f, which will imply the following proposition.

Proposition 4.5.4. *Let $\tilde{g} \in \tilde{\mathcal{M}}$. For each $m \in \mathbb{N}$, there exists $J = J_m$ depending on m, integers $\{c_{m,j}\}_{1 \le j \le J_m}$ and stationary geodesic networks $\{\gamma_{m,j}\}_{1 \le j \le J_m}$ such that*

$$|\frac{\sum_{j=1}^{J_m} c_{m,j} \int_{\gamma_{m,j}} f\, \mathrm{dL}_{\tilde{g}}}{\sum_{j=1}^{J_m} c_{m,j}\, \mathrm{L}_{\tilde{g}}(\gamma_{m,j})} - ⨍_M f\, \mathrm{dVol}_{\tilde{g}}\,| \le \frac{D(f)}{m}$$

for every $f \in C^\infty(M, \mathbb{R})$, where $D(f) > 0$ is a constant depending only on f and the metric $\tilde{g}$.

Proof. Given $m \in \mathbb{N}$, consider as above $g \in \mathcal{M}^\infty$ and $S \in \mathcal{C}_{g,\frac{1}{m},m}$ such that $\tilde{g} \in \mathcal{M}(g, \frac{1}{m}, m, S)$. Define $J_m \in \mathbb{N}$, stationary geodesic networks $\gamma_{m,1}, ..., \gamma_{m,J_m}$ with respect to $\tilde{g}$ and coefficients $\alpha_{m,1}, ..., \alpha_{m,J_m}$ such that (4.14) holds. Taking $S = (K, \{\hat{B}_k\}_k, \{B_k\}_k, \{\phi_k\}_k) \in \mathcal{C}_{g,\frac{1}{m},m}$ into account, let us choose points $q_1, ..., q_K$ with $q_k \in \hat{B}_k$ for each $k = 1, ..., K$. The idea will be to approximate the integral of $f(x)$ by the integral of the function $\sum_{k=1}^K f(q_k)\psi_k(x)$. First of all, by using (4.14) we can see that

$$|\sum_{j=1}^{J_m} \alpha_{m,j} \fint_{\gamma_{m,j}} [\sum_{k=1}^{K} f(q_k)\psi_k(x)] d\,\mathrm{L}_{\tilde{g}} - \fint_M [\sum_{k=1}^{K} f(q_k)\psi_k(x)] d\,\mathrm{Vol}_{\tilde{g}}\,| < \frac{D_1}{m} \tag{4.15}$$

where $D_1 = \|f\|_\infty = \max\{f(x) : x \in M\}$ depends only on f (and not on m, g or S). On the other hand, given $x \in M$

$$\begin{aligned}
|f(x) - \sum_{k=1}^{K} f(q_k)\psi_k(x)| &= |f(x)\sum_{k=1}^{K}\psi_k(x) - \sum_{k=1}^{K} f(q_k)\psi_k(x)| \\
&= |\sum_{k=1}^{K} f(x)\psi_k(x) - f(q_k)\psi_k(x)| \\
&\leq \sum_{k : x\in B_k} |f(x) - f(q_k)||\psi_k(x)| \\
&= \sum_{k : x\in B_k} |\nabla_{\tilde{g}} f(c_k)| d_{\tilde{g}}(x, q_k)\psi_k(x) \\
&\leq \frac{2\|\nabla_{\tilde{g}} f\|_\infty}{m} \sum_{k=1}^{K}\psi_k(x) \\
&= \frac{2\|\nabla_{\tilde{g}} f\|_\infty}{m}.
\end{aligned}$$

We used the Mean Value Theorem and the fact that $\mathrm{supp}(\psi_k) \subseteq B_k$ and $\mathrm{diam}_{\tilde{g}}(B_k) \leq 2\mathrm{diam}_g(B_k) \leq \frac{2}{m}$ for every i. Combining this and (4.15) we get

$$|\sum_{j=1}^{J_m} \alpha_{m,j} \fint_{\gamma_{m,j}} f\, \mathrm{dL}_{\tilde{g}} - \fint_M f\, \mathrm{dVol}_{\tilde{g}}\,| < \frac{D_2}{m} \tag{4.16}$$

where D_2 depends only on f and $\tilde{g}$. Let us choose integers $c_{m,j}, d_m \in \mathbb{N}$ such that

$$|\frac{\alpha_{m,j}}{\mathrm{L}_{\tilde{g}}(\gamma_{m,j})} - \frac{c_{m,j}}{d_m}| < \frac{1}{mJ_m\,\mathrm{L}_{\tilde{g}}(\gamma_{m.j})}.$$

Then it holds

$$\begin{aligned}
|\sum_{j=1}^{J_m} \alpha_{m,j} \fint_{\gamma_{m,j}} f\, \mathrm{dL}_{\tilde{g}} - \sum_{j=1}^{J_m} \frac{c_{m,j}}{d_m} \int_{\gamma_{m,j}} f\, \mathrm{dL}_{\tilde{g}}\,| &\leq \sum_{j=1}^{J_m} |\frac{\alpha_{m,j}}{\mathrm{L}_{\tilde{g}(\gamma_{m.j})}} - \frac{c_{m_j}}{d_m}||\int_{\gamma_{m,j}} f\, \mathrm{dL}_{\tilde{g}}\,| \\
&\leq \sum_{j=1}^{J_m} \frac{1}{mJ_m\,\mathrm{L}_{\tilde{g}}(\gamma_{m,j})}\|f\|_\infty\,\mathrm{L}_{\tilde{g}}(\gamma_{m,j}) \\
&= \frac{D_1}{m}
\end{aligned}$$

and hence by (4.16) and triangle inequality we get

$$|\sum_{j=1}^{K_m} \frac{c_{m,j}}{d_m} \int_{\gamma_{m,j}} f\, \mathrm{dL}_{\tilde{g}} - \fint_M f\, \mathrm{dVol}_{\tilde{g}}\,| < \frac{D_3}{m}$$

where $D_3 = D_2 + D_1$ depends only on f and $\tilde{g}$. On the other hand,

$$
\begin{aligned}
&|\sum_{j=1}^{J_m} \frac{c_{m,j}}{d_m} \int_{\gamma_{m,j}} f \, \mathrm{dL}_{\tilde{g}} - \frac{\sum_{j=1}^{J_m} c_{m,j} \int_{\gamma_{m,j}} f \, \mathrm{dL}_{\tilde{g}}}{\sum_{j=1}^{J_m} c_{m,j} \, \mathrm{L}_{\tilde{g}}(\gamma_{m,j})}| \leq \\
&|\frac{1}{d_m} - \frac{1}{\sum_{j=1}^{J_m} c_{m,j} \, \mathrm{L}_{\tilde{g}}(\gamma_{m,j})}||\sum_{j=1}^{J_m} c_{m,j} \int_{\gamma_{m,j}} f \, \mathrm{dL}_{\tilde{g}}| \leq \\
&|\frac{1}{d_m} - \frac{1}{\sum_{j=1}^{J_m} c_{m,j} \, \mathrm{L}_{\tilde{g}}(\gamma_{m,j})}|\sum_{j=1}^{J_m} c_{m,j} \|f\|_\infty \, \mathrm{L}_{\tilde{g}}(\gamma_{m,j}) = \\
&D_1|\sum_{j=1}^{J_m} \frac{c_{m,j}}{d_m} \, \mathrm{L}_{\tilde{g}}(\gamma_{m,j}) - 1| \leq \frac{D_1}{m}
\end{aligned}
$$

because $|\sum_{j=1}^{J_m} \frac{c_{m,j}}{d_m} \mathrm{L}_{\tilde{g}}(\gamma_{m,j}) - 1| < \frac{1}{m}$. Hence

$$
|\frac{\sum_{j=1}^{J_m} c_{m,j} \int_{\gamma_{m,j}} f \, \mathrm{dL}_{\tilde{g}}}{\sum_{j=1}^{J_m} c_{m,j} \, \mathrm{L}_{\tilde{g}}(\gamma_{m,j})} - \fint_M f \, \mathrm{dVol}_{\tilde{g}}| \leq \frac{D_4}{m}
$$

for a constant D_4 depending only on f and $\tilde{g}$, as desired. □

Given $\tilde{g} \in \tilde{\mathcal{M}}$, using Proposition 4.5.4 we can find a sequence of finite lists of connected embedded stationary geodesic nets $\{\beta_{m,1}, ..., \beta_{m,K_m}\}_{m\in\mathbb{N}}$ with respect to $\tilde{g}$ satisfying the following: given $f \in C^\infty(M,\mathbb{R})$, if we denote $X_{m,j} = \int_{\beta_{m,j}} f \, \mathrm{dL}_{\tilde{g}}$ and $\bar{X}_{m,j} = \mathrm{L}_{\tilde{g}}(\beta_{m,j})$, then

$$
|\frac{\sum_{j=1}^{K_m} X_{m,j}}{\sum_{j=1}^{K_m} \bar{X}_{m,j}} - \alpha| \leq \frac{D(f)}{m} \tag{4.17}
$$

where $\alpha = \fint_M f \, \mathrm{dVol}_{\tilde{g}}$ and $D(f)$ is a constant depending only on f. The lists $\{\beta_{m,j}\}_{1\leq j\leq K_m}$ are obtained from the lists $\{\gamma_{m,j}\}_{1\leq j\leq J_m}$ and the coefficients $\{c_{m,j}\}_{1\leq j\leq J_m}$ from Proposition 4.5.4 by decomposing each $\gamma_{m,j}$ as a union of embedded stationary geodesic networks whose domain is a good weighted multigraph (see Remark 4.2.12) and listing each of them $c_{m,j}$ times. From the $X_{m,j}$'s and the $\bar{X}_{m,j}$'s, we want to construct two sequences $\{Y_i\}_{i\in\mathbb{N}}, \{\bar{Y}_i\}_{i\in\mathbb{N}}$ such that

- For all i, there exist integers $m(i), j(i)$ (chosen independently of f) with $Y_i = X_{m(i),j(i)}$ and $\bar{Y}_i = \bar{X}_{m(i),j(i)}$,
- It holds
$$
\lim_{k\to\infty} \frac{\sum_{i=1}^k Y_i}{\sum_{i=1}^k \bar{Y}_i} = \alpha.
$$

This can be done as in [31, p. 437-439] and gives us a sequence $\{\gamma_i\}_{i\in\mathbb{N}}$ of connected embedded stationary geodesic networks with respect to $\tilde{g}$ (defined as $\gamma_i = \beta_{m(i),j(i)}$), which is constructed independently of the constant $D(f)$. It holds

$$
\lim_{k\to\infty} \frac{\sum_{i=1}^k \int_{\gamma_i} f \, \mathrm{dL}_{\tilde{g}}}{\sum_{i=1}^k \mathrm{L}_{\tilde{g}}(\gamma_i)} = \fint_M f \, \mathrm{dVol}_{\tilde{g}}
$$

for every $f \in C^\infty(M, \mathbb{R})$. This gives us the desired equidistribution result and completes the proof of Theorem 4.1.5.

4.6 Equidistribution of primitive closed geodesics in 2-manifolds

In this section we show that the proof of Theorem 4.1.5 combined with the work of Chodosh and Mantoulidis in [7] (where they show that the p-widths on a surface are realized by collections of primitive closed geodesics) imply Theorem 4.1.1. The strategy to show this result will be to follow the proof of Theorem 4.1.5 replacing "embedded stationary geodesic network" by "primitive closed geodesic". The main change needed in the proof is the following version of Proposition 4.4.1:

Proposition 4.6.1. *Let M be a closed 2-manifold. Let $g : I^N \to \mathcal{M}^q$ be a smooth embedding, $N \in \mathbb{N}$. If $q \geq N+3$, there exists an arbitrarily small perturbation in the C^∞ topology $g' : I^N \to \mathcal{M}^q$ such that there is a full measure subset $\mathcal{A} \subseteq I^N$ with the following property: for any $p \in \mathbb{N}$ and any $t \in \mathcal{A}$, the function $s \mapsto \omega_p^1(g'(s))$ is differentiable at t and there exist primitive closed geodesics $\gamma_p^1, ..., \gamma_p^P : S^1 \to M$ such that the following two conditions hold*

1. $\omega_p^1(g'(t)) = \sum_{i=1}^P L_{g'(t)}(\gamma_p^i(t))$.
2. $\frac{\partial}{\partial v}(\omega_p^1 \circ g')\big|_{s=t} = \frac{1}{2}\sum_{i=1}^P \int_{\gamma_p^i} \operatorname{tr}_{\gamma_p^i, g'(t)} \frac{\partial g'}{\partial v}(t)\, \mathrm{dL}_{g'(t)}$.

Proof. We are going to adapt the proof of Proposition 4.4.1 by introducing some necessary changes. A priori, the easiest way to do this seems to be substituting "stationary geodesic network" by "finite union of primitive closed geodesics" everywhere and use the Bumpy metrics theorem for almost embedded minimal submanifolds proved by Brian White in [48]. Nevertheless, there is not an easy condition (analog to conditions (1) to (7) in the proof of Proposition 4.4.1) that we can impose on a sequence of primitive closed geodesics to converge to another primitive closed geodesic without classifying them by their self-intersections and the angles formed there (we want to rule out the possibility of converging to a primitive closed geodesic traversed several times). Therefore, what we will do is to treat the primitive closed geodesics as a certain class of stationary geodesic networks, and then proceed as with Proposition 4.4.1.

To each primitive closed geodesic $\gamma : S^1 \to M$ we can associate a connected graph $\Gamma = S^1/\sim$ where $\sim$ is the equivalence relation $s \sim t$ if and only if $\gamma(s) = \gamma(t)$. This induces a map $f : \Gamma \to M$ defined as $f([t]) = \gamma(t)$. Observe that the as the self-intersections of γ are transverse, the vertices of Γ are mapped precisely to those self-intersections and the map $f : \Gamma \to M$ is injective. Moreover, Γ is a good multigraph and $f : \Gamma \to (M, g)$ is an embedded stationary geodesic network. We replace the set $\{\Gamma_i\}_{i\in\mathbb{N}}$ which in the proof of Proposition 4.4.1 is the set of all good connected multigraphs by the countable set of pairs $\mathcal{P} = \{(\Gamma, r)\}$ where Γ is a good multigraph which can be obtained as $\Gamma = S^1/\sim$ from a primitive closed geodesic $\gamma : S^1 \to (M, g)$ with respect to some metric g as before and r is the set of pairs $((E_1, i_1), (E_2, i_2))$ such that $\pi_{E_1}(i_1) = \pi_{E_2}(i_2)$ and $(-1)^{i_1+1}\frac{\dot{f}_{E_1}(i_1)}{|\dot{f}_{E_1}(i_1)|_g} = (-1)^{i_2}\frac{\dot{f}_{E_2}(i_2)}{|\dot{f}_{E_2}(i_2)|_g}$ (in other words, r contains the necessary information to reparametrize the geodesic net $f : \Gamma \to M$ as an immersed closed geodesic $\gamma : S^1 \to (M, g)$). Observe that if $(\Gamma, r) \in \mathcal{P}$ and $f : \Gamma \to (M, g)$ is an embedded stationary geodesic network verifying $(-1)^{i_1+1}\frac{\dot{f}_{E_1}(i_1)}{|\dot{f}_{E_1}(i_1)|_g} = (-1)^{i_2}\frac{\dot{f}_{E_2}(i_2)}{|\dot{f}_{E_2}(i_2)|_g}$ for every $((E_1, i_1), (E_2, i_2)) \in r$ then $f : \Gamma \to (M, g)$ can be reparametrized as an immersed closed geodesic $\gamma : S^1 \to (M, g)$ whose self intersections occur precisely at the points $\{f(v) : v \text{ vertex of } \Gamma\}$.

Taking the previous into account, instead of the $\mathcal{B}_{\Gamma,M}$ in the proof of Proposition 4.4.1 we will work with the following. Consider the set of pairs (Γ, r) where Γ is a graph, $\Gamma = \bigcup_{i=1}^{P} \Gamma_i$ as a union of connected components, $r = (r_i)_{1\leq i\leq P}$ and $(\Gamma_i, r_i) \in \mathcal{P}$ for every $1 \leq i \leq P$. Given such a pair (Γ, r) and a natural number $M \in \mathbb{N}$ we define $\mathcal{B}_{\Gamma,r,M}$ to be the set of all $t \in (-1,1)^N$ such that there exists a stationary geodesic network $f : \Gamma \to (M, g'(t))$ verifying

1. For each $1 \leq i \leq P$, $f_i = f|_{\Gamma_i}$ is an embedding and verifies the relations $(-1)^{i_1+1}\frac{\dot{f}_{i,E_1}(i_1)}{|\dot{f}_{i,E_1}(i_1)|_{g'(t)}} = (-1)^{i_2}\frac{\dot{f}_{i,E_2}(i_2)}{|\dot{f}_{i,E_2}(i_2)|_{g'(t)}}$ for every $((E_1, i_1), (E_2, i_2)) \in r_i$.
2. $\|f_i\|_3 \leq M$ for every $1 \leq i \leq P$.
3. $F_1(g'(t), f_i) \geq \frac{1}{M}$ for every $1 \leq i \leq P$.
4. $F_2^{(E_1,i_1),(E_2,i_2)}(g'(t), f_i) \leq 1 - \frac{1}{M}$ for every $1 \leq i \leq P$, and every pair $(E_1, i_1) \neq (E_2, i_2) \in \mathscr{E}_i \times \{0, 1\}$ such that $\pi_{E_1}(i_1) = \pi_{E_2}(i_2)$.
5. $d^E_{(g'(t),f_i)}(s) \geq \frac{1}{M}$ for every $1 \leq i \leq P$, $E \in \mathscr{E}_i$ and $s \in E$.
6. $d^{E,E'}_{(g'(t),f_i)}(s) \geq \frac{1}{M}$ for every $1 \leq i \leq P$, $E \neq E' \in \mathscr{E}_i$ and $s \in E$.
7. $\omega^1_p(g'(t)) = \mathrm{L}_{g'(t)}(f)$.

Therefore, same as in Proposition 4.4.1 we have $I^N = \bigcup_{\Gamma,r,M} \mathcal{B}_{\Gamma,r,M}$ because of the fact showed in [7] that the p-widths on surfaces are realized by unions of primitive closed geodesics; and each $\mathcal{B}_{\Gamma,r,M}$ is closed. The rest of the proof follows exactly as in Proposition 4.4.1 if we replace the pairs (Γ, M) by the triples (Γ, r, M). □

One more remark is necessary to adapt the proof of Proposition 4.5.1. The sequences $(\gamma_{j,m})_m$ in (4.11) have length uniformly bounded by some $L > 0$ and consist of finite unions of primitive closed geodesics. This implies that the number of closed geodesics whose union is $\gamma_{j,m}$ is also bounded (independently on m). Thus by applying Arzela-Ascoli to each of those components we can get a subsequence whose limit is not only a stationary geodesic net but also a union of closed curves with uniform convergence in C^0. The rest of the proof follows that of Theorem 4.1.5 word for word.

Bibliography

[1] F. Almgren, *The homotopy groups of the integral cycle groups*, Topology (1962), 257–299.

[2] ———, *The theory of varifolds*, Mimeographed notes, Princeton (1965).

[3] A. L. Besse, *Einstein manifolds*, Springer-Verlag, 1987.

[4] R. Bowen, *The equidistribution of closed geodesics*, American Journal of Mathematics **94** (1972), no. 2, 413–423.

[5] E. Calabi and J. Cao, *Simple closed geodesics on convex surfaces*, Journal of Differential Geometry **36** (1992), no. 3, 517–549.

[6] D. Chen, *On the C^∞ closing lemma for Hamiltonian flows on symplectic 4-manifolds*, preprint (2019), `https://arxiv.org/abs/1904.09900`.

[7] O. Chodosh and C. Mantoulidis, *The p-widths of a surface*, Publications Mathématiques de l'IHÉS **137** (2023), 245–342, `https://doi.org/10.1007/s10240-023-00141-7`.

[8] Otis Chodosh and Christos Mantoulidis, *Minimal surfaces and the Allen–Cahn equation on 3-manifolds: index, multiplicity, and curvature estimates*, Annals of Mathematics **191** (2020), no. 1, 213 – 328.

[9] V. Climenhaga, G. Knieper, and K. War, *Closed geodesics on surfaces without conjugate points*, preprint (2020), `https://arxiv.org/abs/2008.02249`.

[10] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., 1969.

[11] W. H. Fleming, *Flat chains over a finite coefficient group*, Transactions of the American Mathematical Society **121** (1966), no. 1, 160–186.

[12] P. Gaspar and M.A.M. Guaraco, *The Allen–Cahn equation on closed manifolds*, Calc. Var. **57** (2018), no. 101, `https://doi.org/10.1007/s00526-018-1379-x`.

[13] M. Gromov, *Dimension, non-linear spectra and width*, Geometric Aspects of Functional Analysis **1317** (1986/87), 132–184.

[14] ———, *Isoperimetry of waists and concentration of maps*, Geometric and Functional Analysis **13** (2003), 178–215.

[15] L. Guth, *Minimax problems related to cup powers and Steenrod squares*, Geometric and Functional Analysis **18** (2009), 1917–1987.

[16] L. Guth and Y. Liokumovich, *Parametric inequalites and Weyl law for the volume spectrum*, preprint (2022), `https://arxiv.org/abs/2202.11805`.

[17] J. Hass and F. Morgan, *Geodesic nets on the 2-sphere*, Proceedings of the American Mathematical Society **124** (1996), no. 12, 3843–3850.

[18] A. Hatcher, *Spectral sequences.*

[19] Morris W. Hirsch, *Differential topology*, Springer-Verlag New York Inc., 1976.

[20] Hasan Inci, Kappeler Thomas, and Topalov Peter, *On the regularity of the composition of diffeomorphisms*, Memoirs of the American Mathematical Society **1062** (2013).

[21] K. Irie, *Equidistributed periodic orbits of C^∞-generic three-dimensional Reeb flows*, Journal of Symplectic Geometry **19** (2018), no. 3, 531–566.

[22] K. Irie, F. C. Marques, and A. Neves, *Density of minimal hypersurfaces for generic metrics*, Annals of Mathematics **187** (2018), 963–972.

[23] Kei Irie, *Dense existence of periodic reeb orbits and ECH spectral invariants*, Journal of Modern Dynamics **9** (2015), no. 1, 357–363.

[24] D. Ketover, Y. Liokumovich, and A. Song, *On the existence of minimal Heegaard surfaces*, preprint (2019).

[25] Xinze Li and Bruno Staffa, *On the equidistribution of closed geodesics and geodesic nets*, Transactions of the American Mathematical Society **376** (2023), no. 12, 8825–8855, `https://www.ams.org/journals/tran/2023-376-12/S0002-9947-2023-09028-6/S0002-9947-2023-09028-6.pdf`.

[26] Y. Liokumovich, F. C. Marques, and A. Neves, *Weyl law for the volume spectrum*, Annals of mathematics **187** (2018), 933–961.

[27] Yevgeny Liokumovich and Bruno Staffa, *Generic density of geodesic nets*, Selecta Mathematica - New Series **30** (2024), no. 14, `https://link.springer.com/article/10.1007/s00029-023-00901-7`.

[28] Tsoy-Wo Ma, *Banach-Hilbert spaces, vector measures and group representations*, World Scientific Pub Co Inc, 2002.

[29] F. Marques and A. Neves, *Min-max theory and the Willmore conjecture*, Annals of Mathematics **179** (2014), 683–782.

[30] ———, *Existence of infinitely many minimal hypersurfaces in positive Ricci curvature*, Inventiones mathematicae **209** (2017).

[31] F. C. Marques, A. Neves, and A. Song, *Equidistribution of minimal hypersurfaces for generic metrics*, Inventiones mathematicae **216** (2019), 421–443.

[32] A. Nabutovsky and R. Rotman, *Volume, diameter and the minimal mass of a stationary 1-cycle*, Geometric and Functional Analysis **14** (2004), 748–790.

[33] Alexander Nabutovsky and Fabian Parsch, *Geodesic nets: Some examples and open problems*, Experimental Mathematics **32** (2023), no. 1, 1–25.

[34] Minoru Nakaoka, *Cohomology mod p of symmetric products of spheres*, Journal of the Institute of Polytechnics, Osaka City University. Series A: Mathematics **9** (1958), no. 1, 1 – 18.

[35] A. Pigati and D. Stern, *Minimal submanifolds from the abelian Higgs model*, Inventiones mathematicae **223** (2021), no. 3, 1027–1095.

[36] J. T. Pitts, *Regularity and singularity of one dimensional stationary integral varifolds on manifolds arising from variational methods in the large*, in Symposia Mathematica **XIV** (1973), 465–472.

[37] ———, *Existence and regularity of minimal surfaces on Riemannian manifolds*, Princeton University Press, 1981.

[38] M. Pollicott, *Asymptotic distribution of closed geodesics*, Israel Journal of Mathematics **52** (1985), 209–224.

[39] B. Schapira and S. Tapie, *Narrow equidistribution and counting of closed geodesics on noncompact manifolds*, Groups, Geometry, and Dynamics, European Mathematical Society **15** (2021), no. 3, 1085–1101.

[40] Stephen Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–868.

[41] A. Song, *Existence of infinitely many minimal hypersurfaces in closed manifolds*, Annals of Mathematics **197** (2023), 859–895.

[42] Antoine Song, *A dichotomy for minimal hypersurfaces in manifolds thick at infinity*, Annales Scientifiques de l École Normale Supérieure (2019).

[43] ———, *A dichotomy for minimal hypersurfaces in manifolds thick at infinity*, Annales Scientifiques de l École Normale Supérieure (2019).

[44] B. Staffa, *Parametric coarea inequality for* 1*-cycles*, preprint (2024).

[45] ———, *Weyl law for* 1*-cycles*, preprint (2024).

[46] Bruno Staffa, *Bumpy metrics theorem for geodesic nets*, Journal of Topology and Analysis (2023), `https://doi.org/10.1142/S1793525323500425`.

[47] B. White, *The space of minimal submanifolds for varying Riemannian metrics*, Indiana University Mathematics Department **40** (1991), no. 1, 161–200.

[48] ———, *On the bumpy metrics theorem for minimal submanifolds*, American Journal of Mathematics **139** (2017), no. 4, 1149–1155.

[49] Brian White, *On the bumpy metrics theorem for minimal submanifolds*, Amer. J. Math. **139** (2017), no. 4, 1149–1155.

[50] Xin Zhou, *On the Multiplicity One Conjecture in min-max theory*, Annals of Mathematics **192** (2020), no. 3, 767 – 820.

[51] Xin Zhou and Jonathan J Zhu, *Min–max theory for constant mean curvature hypersurfaces*, Inventiones mathematicae **218** (2019), no. 2, 441–490.