DEPARTMENT OF MATHEMATICS
University of Toronto
Complex Analysis Comprehensive Examination
September 28, 2020   120 minutes

1. Formulate and prove a version of Schwarz's reflection principle for harmonic functions.

2. Suppose that $f(z)$ is meromorphic in an open subset Ω of $\mathbb{C}$, and that $K \subset \Omega$ is a compact set with oriented boundary Γ. Assume that $f(z)$ does not take the value a on Γ and has no poles on Γ. Use the residue theorem to determine what is computed by the integral
$$\frac{1}{2\pi i}\int_\Gamma \frac{z^p f'(z)}{f(z)-a}dz\,,$$
where p is a positive integer.

3. (a) Show that the infinite product
$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$
represents an entire function with simple zeros at the negative integers.

   (b) Define $H(z)$ by
$$\frac{1}{H(z)} = ze^z\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}.$$
Prove that
$$\frac{d}{dz}\left(\frac{H'(z)}{H(z)}\right) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}.$$

4. Given $M > 0$, let $\mathcal{S}_M$ denote the set of functions $f(z)$ which are holomorphic on the open unit disk D, continuous on the closed unit disk, and satisfy
$$\int_0^{2\pi}|f(e^{i\theta})|\,d\theta \le M\,.$$
Show that $\mathcal{S}_M$ is a normal family.