Algebra Exam (3 hours)

Tuesday, September 29, 2020

The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.

Good Luck!
Problem 1.

(a) Suppose that $p > q$ are prime numbers and that $G$ is a finite group of order $p^nq$ for some $n \geq 1$. Prove that $G$ contains a unique normal subgroup of index $q$.

(b) Suppose that $G$ is a finite $p$-group and $N \neq 1$ a normal subgroup of $G$. Prove that $N \cap Z(G) \neq 1$, where $Z(G)$ denotes the centre of $G$.

Problem 2. Suppose that $R$ is a commutative ring that is moreover an integral domain.

(a) If $R$ is finite (as set), show that $R$ is a field.

(b) If $R$ is artinian, show that $R$ is a field. (We say that $R$ is artinian if any descending chain $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ of ideals of $R$ stabilises, i.e. $I_k = I_{k+1}$ for all sufficiently large $k$.)

(c) Give an example of a (commutative) artinian ring that is not a field.

Problem 3.

(a) Determine all $F_2[x]$-modules $M$, up to isomorphism, such that $\dim_{F_2} M = 2$. Your list should not contain any duplicates.

(b) Write $(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \otimes (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z})$ as direct sum of cyclic groups.

Problem 4.

(a) Prove that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$.

(b) Let $\theta$ be a root of $p(x)$. Describe all the elements in the field $\mathbb{Q}(\theta)$ uniquely in terms of $\theta$.

(c) Compute $(1 + \theta)^{-1}$ in the field $\mathbb{Q}(\theta)$.

Problem 5.

(a) State the correspondence between groups and fields given by the Fundamental Theorem of Galois Theory.

(b) Suppose that $K/Q$ is the splitting field of a monic integral polynomial of degree 3, and that $\text{Gal}(K/Q) \cong S_3$. How many intermediate fields $\mathbb{Q} \subset E \subset K$ are there? For how many of these is $K/E$ Galois, and for how many is $E/Q$ Galois?

Problem 6.

(a) What is an affine algebraic set, its coordinate ring, and a morphism between affine algebraic sets?

(b) State the Hilbert Nullstellensatz, and describe the correspondence between geometry and algebra that it provides.