DEPARTMENT OF MATHEMATICS
University of Toronto

Complex Analysis Exam
1.5 hours

September 6, 2017

There are 4 questions, all of equal value.
Show all your work.

1. Evaluate
$$\int_0^{2\pi} \frac{\cos\theta}{3+\cos\theta} d\theta \ .$$

2. Show that there is a complex analytic function defined on the set $U = \{z \in \mathbb{C} : |z| > 4\}$ whose derivative is
$$\frac{z}{(z-1)(z-2)(z-3)}.$$
Is there a complex analytic function on U whose derivative is
$$\frac{z^2}{(z-1)(z-2)(z-3)} \ ?$$
Explain your answer.

3. Let S be the half-strip $S = \{z = x + iy : |x| < 1, y > 0\}$ and let f be an analytic function defined on S such that
a) $|f(z)| \leq 2,\ z \in S$;
b) $\lim_{y\to\infty} f(iy) = 1$.
Prove that for any $0 < a < 1$, $\lim_{y\to\infty} f(x + iy) = 1$ uniformly for $|x| \leq a$.
(Hint: consider the family of functions $f_t : S \to \mathbb{C},\ f_t(z) := f(z + it), t \geq 0$.)

4. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Prove that if $|f(z^2)| \leq 2|f(z)|$ for all $z \in \mathbb{C}$, then f is constant.