DEPARTMENT OF MATHEMATICS
(Full) Analysis Comprehensive Exam 2025
September 15, 2025 - 6:00 - 11:00 p.m.
BA6183

NO AIDS ALLOWED. Passing score is 80 percent.

Last name ....

First name ....

Email ....

1. Suppose $E \subset \mathbb{R}$ has positive Lebesgue measure, and A is a countable dense subset of $\mathbb{R}$. Show that $m(\mathbb{R} \setminus \bigcup_{a \in A}(a + E)) = 0$.

2. Let $f \in L^1(\mathbb{R})$. Show that there is a subsequence $n_k \to \infty$ such that $f(\cdot + 1/n_k) \to f$ a.e.

3. Suppose $f : [1, \infty) \to \mathbb{R}$ be a C^1 function satisfying $f(1) = 0$ and $f' \in L^2$. Show that $f(x)/x \in L^2([1, \infty))$.

4. State the open mapping theorem and the closed graph theorem. You do not need to prove it.

5. Define the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ and show that it is a continuous linear mapping from $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$. Is this mapping injective or surjective? Why?

6. Do the following functions define tempered distributions? Prove or disprove your claim.

   - $f(x) = x^2 \sin(x)$.
   - $f(x) = e^x$.

7. Use the residue at infinity to find the integrals

$$\int_{|z|=4} \frac{dz}{(z-3)(z^5-1)} \qquad \text{and} \qquad \int_{|z|=2} \frac{dz}{(z-3)(z^5-1)}.$$

8. Let $f(z)$ denote a doubly periodic meromorphic function with group of periods Γ, where Γ is generated by $e_1,\ e_2 \in \mathbb{C}$ (linearly independent over $\mathbb{R}$). Suppose that

   a) $f(z)$ has zeros (all simple) precisely at the points $(me_1 + ne_2)/2$, where $m, n \in \mathbb{Z}$ and $m + n$ is odd;

   b) $f(z)$ has poles (all simple) precisely at the points $(pe_1 + qe_2)/2$, where $p, q \in \mathbb{Z}$ and $p + q$ is even.

   Show that $f(z)$ is a constant multiple of the function

$$\frac{\wp'(z)}{\wp(z) - a_3}, \qquad \text{where } \ a_3 = \wp((e_1 + e_2)/2).$$

9. a) Show that the infinite product

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

represents an entire function with simple zeros at the negative integers.

b) Define $H(z)$ by

$$\frac{1}{H(z)} = ze^{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Prove that

$$\frac{d}{dz}\left(\frac{H'(z)}{H(z)}\right) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

10. Suppose that $f(z)$, $g(z)$ are meromorphic functions on $\mathbb{C}$ with no common poles, such that $f^n + g^n = 1$.

   a) Prove that, if $n > 2$, then f and g are constant.

   b) What can we say in the case $n = 2$?

   c) What can we say in the case $n = 3$, if we allow common poles?