Analysis Comprehensive Exam
September 2014

Please be brief but justify your answers, citing relevant theorems. Sometimes a sketch can help!

1. Let (f_n) be a sequence of Lebesgue measurable functions on $\mathbb{R}$ that converges to f in L^1, that is,

$$\lim \int |f_n - f|\, dx = 0\,.$$

(a) If, in addition, each $f_n \in L^2$ and there exists a constant M such that $||f_n||_2 \leq M$, that is,

$$\left(\int |f_n|^2\, dx \right)^{\frac{1}{2}} \leq M \qquad \text{for all } n\,,$$

prove that $f \in L^2$.

(b) Does it follow that f_n converges to f in L^2? Please substantiate your claim!

(c) Prove that $\lim ||f_n - f||_p = 0$ for all p with $1 < p < 2$.

2. Let $K(x, y)$ be a complex-valued function in $L^2(\mathbb{R}^2)$, and set

$$Tf(x) = \int_{\mathbb{R}} K(x, y) f(y)\, dy\,.$$

(a) Show that $f \mapsto Tf$ defines a bounded linear operator on $L^2(\mathbb{R})$.
(*Hint:* Schwarz' inequality.)

(b) Find a formula for its adjoint, T^*.
(You will need to exchange some integrals. Please explain why you can do that.)

(c) In the special case where $K(x, y) = \frac{1}{\sqrt{\pi}} e^{-(x-y)^2}$, prove that the quadratic form

$$Q(f) = \int_{\mathbb{R}} \bar{f}(x)\, Tf(x)\, dx$$

is positive definite, that is, $Q(f) > 0$ for all $f \neq 0$.
(*Hint:* Fourier transform.)

3. (a) Define, in simple terms: What does it mean for a set $N \subset \mathbb{R}$ to have *measure zero*?
What does it mean for a set $M \subset \mathbb{R}$ to be *meager* (that is, of first category)?

(b) State the *Baire Category Theorem.*

(c) Write $\mathbb{R}$ as the disjoint union of a meager set and a set of measure zero, that is, construct M and N such that

$$M \cup N = \mathbb{R}\,, \qquad M \cap N = \emptyset\,,$$

where M is meager and N has measure zero.

4. (a) Let X be a normed vector space. Define the *dual space*, X^*, and its norm.

   (b) Define *weak convergence* in X.

   (c) When $X = L^p(\mathbb{R})$ for some $p \in [1, \infty]$, what can you say about X^*?

   (d) Fix p with $1 < p < \infty$, and consider a closed subspace $V \subset L^p(\mathbb{R})$. For $f \in L^p(\mathbb{R})$, let

$$d(f, V) = \inf_{v \in V} ||f - v||_p .$$

   Prove that there exists a function $v_0 \in V$ such that

$$d(f, V) = ||f - v_0||_p .$$

   (*Hint:* Consider a *minimizing sequence* (v_n) in V, and extract a weakly convergent subsequence.)

5. Evaluate via residues

$$\int_0^\infty \frac{x^{a-1}}{1+x} dx$$

   where $0 < a < 1$.

6. Suppose that Ω is a domain in $\mathbb{C}$, f_k is a sequence of analytic functions on Ω, $f_k \to f$ uniformly on compact subsets of Ω, and f has a zero of order N at $z_0 \in \Omega$. Show that there exists $\rho > 0$ such that for k sufficiently large, f_k has exactly N zeros counting multiplicities on $|z - z_0| < \rho$.