DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Comprehensive Exam
3 hours

September 6, 2017

There are 7 questions in total

1. Consider $\mathbb{R}^n$ equipped with the outer Lebesgue measure ν.

a) Prove or disprove: If $A \subset \mathbb{R}^n$ is uncountable, then $\nu(A) > 0$.

b) Prove or disprove: If $A \subset \mathbb{R}^n$ satisfies $\nu(A \cap C) = \nu(C)$ for every compact $C \subset \mathbb{R}^n$, then $\bar{A} = \mathbb{R}^n$. (here the bar denotes the topological closure)

c) Prove or disprove: If $f : \mathbb{R}^n \to \mathbb{R}^n$ is 1-Lipschitz, then $\nu(f^{-1}(A)) \geq \nu(A)$ for all $A \subset \mathbb{R}^n$.

2. Consider $\mathbb{R}$ equipped with the Lebesgue measure.

a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in L^2(\mathbb{R})$ and $f \notin L^4(\mathbb{R})$.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function such that

$$\left(\int_{\mathbb{R}} fg^3\right)^4 \leq \left(\int_{\mathbb{R}} g^4\right)^3$$

for all $g \in C_c(\mathbb{R})$. Prove that $f \in L^4(\mathbb{R})$ and $||f||_{L^4(\mathbb{R})} \leq 1$.

c) Prove that if $f \in H^1(\mathbb{R})$, then f is Hölder continuous with exponent $1/2$, i.e.

$$|f(x) - f(y)| \leq C|x - y|^{1/2}$$

for some $C < \infty$. (here $H^1(\mathbb{R}) \subset L^2(\mathbb{R})$ denotes the Sobolev space of functions with weak derivative in $L^2(\mathbb{R})$)

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a) State the Hahn-Banach theorem, the open mapping theorem, the closed graph theorem and the uniform boundness principle.

b) Prove that there is a natural linear isometry from a Banach space B to its double dual space $(B^*)^* = B^{**}$. Is this mapping surjective? Why?

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a) Define the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$, and show that it is a continuous linear mapping from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. Is this mapping surjective? Why?

b) Calculate the Fourier coefficients of the following functions of $x \in [0, 1]$.

i) $f(x) = 1$

ii) $f(x) = cos(2\pi x)$

iii) $f(x) = xe^{2\pi ix}$

5. Evaluate

$$\int_0^{2\pi} \frac{\cos\theta}{3+\cos\theta} d\theta \ .$$

6. Show that there is a complex analytic function defined on the set $U = \{z \in \mathbb{C} : |z| > 4\}$ whose derivative is

$$\frac{z}{(z-1)(z-2)(z-3)}.$$

Is there a complex analytic function on U whose derivative is

$$\frac{z^2}{(z-1)(z-2)(z-3)} \ ?$$

Explain your answer.

7. Let S be the half-strip $S = \{z = x + iy : |x| < 1, y > 0\}$ and let f be an analytic function defined on S such that
a) $|f(z)| \leq 2, \ z \in S$;
b) $\lim_{y\to\infty} f(iy) = 1$.
Prove that for any $0 < a < 1$, $\lim_{y\to\infty} f(x + iy) = 1$ uniformly for $|x| \leq a$.
(Hint: consider the family of functions $f_t : S \to \mathbb{C},\ f_t(z) := f(z + it), t \geq 0$.)