

Analysis Comprehensive Exam

September 2016

Please be brief but justify your answers, citing relevant theorems.

1. Let (X, Σ, μ) be a measure space, and $(E_n)_{n \geq 0}$ be a sequence of μ -measurable sets.

(a) Show that the set

$$E = \{x \in X : x \in E_n \text{ for infinitely many } n\}$$

is μ -measurable.

(b) (Borel-Cantelli Lemma) Show that if $\sum_{n \geq 0} \mu(E_n) < \infty$, then $\mu(E) = 0$.

(c) Let $(f_n(x))$ be a sequence of μ -measurable functions such that

$$\sum_{n \geq 0} \mu \{x : |n^2 f_n(x)| > 2\} < \infty.$$

Prove that the series $\sum_{n \geq 0} f_n(x)$ converges μ -a. e.

2. Let $g_n(x, y)$ be a sequence of Lebesgue measurable functions on the unit square

$$\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Let $f_n^x(y) = g_n(x, y)$.

(a) Define what it means for the sequence $g_n(x, y)$ to converge to a measurable function $g(x, y)$ in Lebesgue measure.

(b) Show that if for a.e. $x \in [0, 1]$, the sequence of functions $f_n^x(y)$ converges to $f^x(y) = g(x, y)$ in (1-dimensional) Lebesgue measure, then $g_n(x, y)$ converges to $g(x, y)$ in measure.

(c) Show that the converse is false by constructing a sequence $g_n(x, y)$ converging to 0 in Lebesgue measure such that for no $x \in [0, 1]$, $f_n^x(y)$ converges to zero in Lebesgue measure.

Hint: First, construct a sequence $h_n(x)$ of functions on $[0, 1]$ which converges to zero in measure, but does not converge pointwise at any point of $[0, 1]$. Use the sequence to easily construct the required sequence $g_n(x, y)$.

3. Let $1 \leq p < q < \infty$. Let

$$B_n = \left\{ f \in L^p([0, 1]) : \int_0^1 |f(t)|^q dt < n \right\}.$$

(a) Show that B_n has an empty interior in L^p .

(b) Show that B_n is closed in L^p

Hint: Use Fatou's Lemma.

(c) Show that L^q is a subset of the first category in L^p .

- (d) Let $\frac{1}{r} + \frac{1}{p} = 1$. Assume that g is a measurable function on $[0, 1]$ which satisfies the following property:

For every $f \in L^p([0, 1])$, $f \notin L^q([0, 1])$, we have $fg \in L^1([0, 1])$.

Show that $g \in L^r([0, 1])$.

Hint: Use the sequence $g_n(x) = \min(|g(x)|, n)$, and the Uniform Boundedness Principle.

4. (a) What is the general formula for the Fourier transform of a function f on \mathbb{R} ? The Fourier inversion formula? The Plancherel formula?
- (b) Explain how to make the definition of Fourier transform precise in each of the following cases: f belongs to $L^1(\mathbb{R})$, to $L^2(\mathbb{R})$, to the Schwartz space $\mathcal{S}(\mathbb{R})$, or its dual space $\mathcal{S}'(\mathbb{R})$.
- (c) In particular, what does it mean to take the Fourier transform of the function

$$f(x) = 1 + 2x + 3x^2 + \cdots + (n+1)x^n?$$

Calculate it explicitly.

5. Use contour integration to evaluate

$$\int_0^\infty \frac{\sin x}{x(x^2 + 4)} dx.$$

6. Determine how many roots the equation $e^z = 3z + 1$ has in $|z| < 1$ and prove your answer briefly.