Analysis Comprehensive Exam
September 2013

Please be brief but justify your answers, citing relevant theorems.

1. Let $(f_n)_{n\geq 1}$ be a sequence of integrable functions that converges pointwise a.e. to an integrable function f.

   (a) Prove that
$$\lim_{n\to\infty}\left\{\int |f_n| - \int |f-f_n|\right\} = \int |f(x)|\,.$$
   (*Hint:* Use the triangle inequality.)

   (b) Argue that the above identity improves upon Fatou's lemma.

   (c) Give an example where the inequality in Fatou's lemma is strict.

2. True or False? Why? (Try to find one-line arguments.)

   (a) If $A \subset [0,1]$ is compact and its interior is empty, then A has Lebesgue measure zero.

   (b) Let $B \subset \mathbb{R}^2$ be measurable, and let
$$B_y = \{x \in \mathbb{R} \mid (x,y) \in B\}\,, \qquad B^x = \{y \in \mathbb{R} \mid (x,y) \in B\}$$
   be its horizontal and vertical cross sections. (A sketch may help.)
   If B_y has measure zero for almost every y, then B^x has measure zero for almost every x.

   (c) If a Lebesgue-measurable set $C \subset [0,1]$ satisfies
$$m(C \cap I) \geq \frac{1}{2} m(I)$$
   for every interval $I \subset [0,1]$, then $m(C) = 1$.

3. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space.

   (a) What does it mean for a sequence $(u_n)_{n\geq 1}$ to be an orthonormal basis of $\mathcal{H}$?
   Please give three equivalent criteria.

   (b) Let $f_n \rightharpoonup f$ be a weakly convergent sequence in $\mathcal{H}$. Prove that
$$\lim_{n\to\infty} ||f_n - f|| = 0 \quad \Longleftrightarrow \quad \lim_{n\to\infty} ||f_n|| = ||f||\,.$$

   (c) Give an example of a sequence that converges weakly, but not strongly in $\mathcal{H}$.

4. (a) Define the Fourier transform $\mathcal{F} : f \mapsto \hat{f}$ for integrable functions f on $\mathbb{R}$.

   (b) How would you compute $\hat{f}$ for a function $f \in L^2 \setminus L^1$ (such as $f(x) = x^{-1} \sin x$)? Please justify why your method works. (*Hint:* Plancherel's theorem.)

   (c) Prove the *Hausdorff-Young inequality*

$$||\hat{f}||_q \leq C_p ||f||_p \,, \quad 1 \leq p \leq 2$$

   for $q = 1 - 1/p$ and a suitable constant C_p by interpolating between $p = 1$ and $p = 2$.

5. Suppose that f is an entire function and $|f(z)| \leq A + B|z|^k$, where A, B are positive constants and k is a positive integer. Show that f is a polynomial. What can you say about its degree ?

6. (a) Evaluate

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2 + a^2} dx$$

   via residues, where $t \in \mathbb{R}$ and $a > 0$.

   (b) Suppose f is a holomorphic function on a domain $\Omega \subset \mathbb{C}$ and that the closed disc of centre a and radius r is contained in Ω. Let γ be the positively oriented circle with centre a and radius r, and supose f has no zero on γ. For $p = 1, 2, 3, \ldots$, what is the value of

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} z^p dz?$$