DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

Thursday, September 7, 2017, 1–4 PM

The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.

Please give statements of results used in your solutions.

Good Luck!

Problem 1.

(a) Let G be a group of order 55 acting on a set X of cardinality 24. Show that there exists an $x \in X$ that is fixed by every element of G (i.e. $gx = x$ for all $g \in G$).

(b) Let G be a simple group of order 168. Show that there exists an injective homomorphism $G \hookrightarrow S_8$. Does there exist an injective homomorphism $G \hookrightarrow S_6$?

Problem 2. Suppose R is a commutative ring with 4 elements that contains $\mathbb{F}_2$ ($= \mathbb{Z}/2\mathbb{Z}$) as subring.

(a) Explain briefly why and how R is an $\mathbb{F}_2$-vector space in a natural way.

(b) By choosing a suitable vector space basis of R, or otherwise, show that there is a ring isomorphism $R \cong \mathbb{F}_2[x]/(f(x))$ for some polynomial $f(x) \in \mathbb{F}_2[x]$ of degree 2.

(c) Classify all possibilities for the ring R, up to isomorphism. How many are there?

Problem 3. Suppose R is a commutative ring and M is an R-module. An R-submodule N of M is called *pure* if $rN = N \cap rM$ for all $r \in R$.

(a) Show that any direct summand of M is pure.

(b) Assume that R is an integral domain. If M/N is torsion-free show that N is pure. Prove the converse when M is in addition assumed to be torsion-free.

(Recall that a direct summand is a submodule N such that $M = N \oplus N'$ for some submodule N', and that a module M over an integral domain R is called torsion-free if $rm = 0$ implies $r = 0$ or $m = 0$ for $r \in R$ and $m \in M$.)

Problem 4. Let L be a finite Galois extension of a field K of characteristic zero. Let $G = \mathrm{Gal}(L/K)$ be the Galois group of L over K.

For parts (a) and (b) ONLY, assume that $K = \mathbb{Q}$.

(a) Prove that if L is not contained in the real numbers, then $|G|$ is even.

(b) (Still assuming $K = \mathbb{Q}$), suppose that there exists $\alpha \in L$ such that the minimal polynomial $m_\alpha \in \mathbb{Q}[x]$ of α over $\mathbb{Q}$ has at least one real root and at least one nonreal root. Prove that G is nonabelian. (*Hint*: Results from Galois theory give information about an action of G on the roots of m_α.)

Problem 4, cont'd. For parts (c) and (d) (which are independent of parts (a) and (b)), suppose that $n \geq 2$, a_0, $a_1, \ldots, a_{n-1} \in L$ and

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

For $\sigma \in G$, let

$$\sigma(f)(x) = x^n + \sigma(a_{n-1})x^{n-1} + \cdots + \sigma(a_1)x + \sigma(a_0).$$

Let $g(x) = \prod_{\sigma \in G} \sigma(f)(x)$.

(c) Prove that $g \in K[x]$.

(d) Prove that if $L = K(a_0, a_1, \ldots, a_{n-1})$ and f is irreducible in $L[x]$, then g is irreducible in $K[x]$.

Problem 5. Let R be a commutative ring with 1. The ring R is Noetherian if R satisfies the ascending chain condition on ideals. That is, given a chain of ideals

$$I_1 \subseteq \cdots \subseteq I_{k-1} \subseteq I_k \subseteq I_{k+1} \subseteq \cdots$$

of R, there exists an n such that $I_k = I_n$ for all $k \geq n$. In each case below, prove or disprove that the ring R is Noetherian.

(a) $R = \mathbb{Z}[x, y]$.

(b) $\mathbb{Z}[x, y]/(x - y)$.

(c) $R = \{\, f(x) \in \mathbb{R}[x] \mid f(0) \in \mathbb{Q} \,\}$.

Problem 6. Let G be a finite group, V an n-dimensional complex vector space ($n \geq 1$) and $\rho : G \to GL(V)$ a representation of G.

(a) Prove that there exists an inner product $\langle \cdot, \cdot \rangle$ on V such that

$$\langle\, \rho(g)v_1,\, \rho(g)v_2 \,\rangle = \langle\, v_1,\, v_2 \,\rangle \qquad \forall\, g \in G,\, v_1, v_2 \in V.$$

(b) Prove that if ρ is irreducible and z belongs to the centre Z of G, then $\rho(z)$ is a scalar multiple of the identity operator on V. What can you say about properties of the scalar?

(c) Prove that if ρ is irreducible, then $n \leq \sqrt{|G|/|Z|}$. (*Hint*: Use character theory.)