The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.
Problem 1. Let $G$ be a group and let $Z(G)$ denote its centre.

1. Show that if $G/Z(G)$ is cyclic then $G$ is Abelian.

2. Prove that if the group $\text{Aut}(G)$ of automorphisms of $G$ is cyclic, then $G$ is Abelian.

Problem 2. Define a “Principal Ideal Domain (PID)” and a “Unique Factorization Domain (UFD)” and show that every PID is a UFD. If you need to use the lemma that an increasing chain of ideals in a PID must become constant at some point (i.e., that a PID is “Noetherian”), prove it.

Problem 3. Let $V = \mathbb{F}^n$ be an $n$-dimensional vector space over some field $\mathbb{F}$, let $T : V \rightarrow V$ be a linear transformation, let $R = \mathbb{F}[x]$ denote the ring of polynomials in a variable $x$ with coefficients in $\mathbb{F}$, and consider $V$ as an $R$-module by setting $xv = Tv$ for any $v \in V$. Let $\pi : R^n \rightarrow \mathbb{F}^n$ be the morphism of $R$-modules defined by mapping the standard basis elements $e_i$ of $R^n$ to their obvious counterparts in $\mathbb{F}^n$. Propose a set of $n$ generators $r_i$ of $\ker \pi$ and prove in detail that your proposed $r_i$ indeed generate $\ker \pi$.

Problem 4. Let $\overline{\mathbb{Q}}$ be an algebraic closure of the rational numbers. Suppose that $\alpha, \beta \in \overline{\mathbb{Q}}$ are such that $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are Galois extensions of $\mathbb{Q}$.

1. Prove that $\mathbb{Q}(\alpha, \beta)$ is a Galois extension of $\mathbb{Q}$.

2. Suppose that $\text{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ and $\text{Gal}(\mathbb{Q}(\beta)/\mathbb{Q})$ are Abelian. Prove that $\text{Gal}(\mathbb{Q}(\alpha, \beta)/\mathbb{Q})$ is abelian.

3. If $F \subset \overline{\mathbb{Q}}$ is a finite extension of $\mathbb{Q}$, we say that $F$ is Abelian over $\mathbb{Q}$ if $F$ is Galois over $\mathbb{Q}$ and $\text{Gal}(F/\mathbb{Q})$ is Abelian. Prove that if $K \subset \overline{\mathbb{Q}}$ is a finite Galois extension of $\mathbb{Q}$, there exists a maximal Abelian subfield $F$ of $K$ that is Abelian over $\mathbb{Q}$.

Note: In your solution to this question, please do not quote theorems about Galois groups of composites of field extensions. Feel free to use other results from Galois theory.

Problem 5. Let $G$ be a finite group and let $\chi = \chi_\rho$ be the character of an irreducible complex representation $\rho : G \rightarrow GL(V)$ of $G$. (In other words, $V$ is a simple $\mathbb{C}[G]$-module.) Let $H$ be a proper subgroup of $G$ and let $\chi_1, \ldots, \chi_r$ be the characters of the distinct isomorphism (that is, equivalence) classes of irreducible complex representations of $H$. Let $\chi_H$ be the restriction of $\chi$ to $H$.

1. Show that $\chi_H = \sum_{j=1}^r n_j \chi_j$ for nonnegative integers $n_1, \ldots, n_r$.

2. Show that $\sum_{j=1}^r n_j^2 \leq |G|/|H|$.

3. Show that if $H$ is a normal subgroup of $G$, then all of the nonzero $n_j$'s are equal.

Problem 6. Let $I$ be an ideal in a commutative ring $R$ with 1. The radical of $I$, denoted by $\text{Rad}(I)$, is equal to $\{c \in R \mid c^n \in I \text{ for some } n \geq 1\}$. The ideal $I$ is primary if whenever $a, b \in R$ satisfy $ab \in I$, then $a \in I$ or $b \in \text{Rad}(I)$. Prove that $I$ is prime if and only if $I = \text{Rad}(I)$ and $I$ is primary.

Good Luck!