Department of Mathematics

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Algebra Exam

Best of Luck!

1. Let $n \geq 5$ be an integer. You may assume that the alternating group A_n is simple.

(a) Prove that the only nontrivial proper normal subgroup of the symmetric group S_n is A_n.

(b) Prove that the only proper subgroup of index $< n$ in S_n is A_n.

(c) Describe a proper subgroup of S_n having index n. Describe a proper subgroup of S_4 having index 3.

2. Let G be a finite group which acts on a finite set S. Let V denote a vector space over $\mathbb{C}$ with basis $\{v_s\}_{s \in S}$ indexed by the elements in S. We obtain a representation $\rho : G \to \mathrm{GL}(V)$, where $\rho(g)(v_s) := v_{gs}$. Let $\chi : G \to \mathbb{C}$ be the character (i.e., trace) of ρ.

(a) Define the action of G on $S \times S$ given by $g(s,t) := (gs, gt)$. Prove that the character of the representation corresponding to this new action is equal to χ^2.

(b) Prove that the representation ρ contains exactly c copies of the trivial representation, where c is the number of orbits of G on S.

3. Let R be a commutative ring. We say that an R-module M is *simple* if $M \neq 0$ and the only R-submodules of M are 0 and M.

(a) Let F be a field. What are the (isomorphism classes of) simple F-modules?

(b) Let M_1 and M_2 be simple R-modules, and let $f : M_1 \to M_2$ be a homomorphism of R-modules. Prove that f is either zero or an isomorphism.

(c) Prove that M is a simple R-module if and only if M is isomorphic to R/I for some *maximal ideal* I of R.

(d) What are the isomorphism classes of simple $\mathbb{C}[x]$-modules? (Prove that no two modules in your list are isomorphic.)

4.

(a) Suppose that K is an extension of a field F. What does it mean for $\alpha \in K$ to be algebraic over F? Transcendental over F?

(b) If α is algebraic, what is its minimal polynomial over F? What properties does it have?

(c) If $f(x) \in F[x]$ is irreducible over F, prove that $E = F[x]/(f(x))$ is a field. Find a root α of f in E, and prove that $f(x)$ is the minimal polynomial of α over F.

5.

(a) State the correspondence between groups and fields given by the Fundamental Theorem of Galois theory.

(b) State as many properties of the correspondence as you can.

(c) Suppose that $K/\mathbb{Q}$ is the splitting field of a monic irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree 3, with $\mathrm{Gal}(K/\mathrm{Q}) \cong S_3$. How many intermediate fields

$$\mathbb{Q} \subsetneq E \subsetneq K$$

are there such that K/E is Galois, and how many with $E/\mathbb{Q}$ Galois? How many pairs of fields

$$\mathbb{Q} \subsetneq E_1, E_2 \subsetneq K$$

are there with $E_1 \cdot E_2 = K$, and how many with $E_1 \cap E_2 = \mathbb{Q}$?

6. State the Hilbert Nullstellensatz, and describe the correspondence between geometry and algebra that it provides.