Algebra Qualifying Exam

Mathematics Department

University of Toronto

September, 2013

Answer as many questions as possible; you are not expected to do them all.

Three hours; no aids permitted; questions are of equal value; explain your answers.

1. Let G be a group and $G \to G \times G$ the diagonal map that sends g to (g, g) for each $g \in G$. When is the image a normal subgroup of $G \times G$, endowed with the componentwise operation?

2. (i) Show that any finite group with at least 3 elements admits an automorphism that is not the identity map.

(ii) Is the corresponding statement true if one replaces "group" with "ring"?

3. Let G be the group of invertible (2×2)–matrices with entries in the finite field with p elements, p a positive prime number.

(i) What is the size of a Sylow p–subgroup of G?

(ii) How many such Sylow p-subgroups are there in G?

4. Show that the index of the centre of a finite group is never a prime number (1 is not prime!)

5. If $H < G$ is a proper subgroup of a finite group G, show that $\cup_{g \in G}\, gHg^{-1}$ is not equal to G.

6. (i) Show that the matrices

$$A = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$$

with a, b, c integers, form a commutative ring R under the usual addition and multiplication of matrices.

(ii) Determine all invertible elements as well as all zero divisors of R.

7. (i) Prove that the multiplicative group of a finite field must be cyclic. *(You may use the Fundamental Theorem of abelian groups).*

(ii) Suppose $p \in \mathbb{Z}$ is an odd prime. State and prove necessary and sufficient conditions on p for -1 to be a square in $\mathbb{F}_p$.

(iii) State and prove necessary and sufficient conditions that -1 is a square in $\mathbb{F}_{p^2}$.

8. Find an example of an irreducible cubic polynomial in $\mathbb{F}_3[x]$.

9. (i) What is the Galois group of the splitting field of $f(x) = x^3 - 2x - 3$ over $\mathbb{Q}$?

(ii) Find an example of a cubic polynomial $f(x) \in \mathbb{Z}[x]$ whose splitting field has Galois group S_3 over $\mathbb{Q}$ but A_3 over $\mathbb{Q}(i)$.

10. Suppose D, D' are distinct non-squares in $\mathbb{Q}$. Show that $\mathbb{Q}(\sqrt{D}, \sqrt{D'})/\mathbb{Q}$ is an extension of degree either 2 or 4.

(ii) Show that the extension is of degree 4 if and only if DD' is a non-square in $\mathbb{Q}$.

(iii) In this case, where the extension is of degree 4, describe all the intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{D}, \sqrt{D'})$.

11. Prove or disprove: the fields $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{7})$ are isomorphic as fields over $\mathbb{Q}$.

12. Suppose F is a field and $f(x) \in F[x]$ is irreducible of degree n. For any $g(x) \in F[x]$, consider $h(x) = f(g(x))$. Show that every irreducible factor of $h(x)$ has degree divisible by n.