USRA project: A constructive decomposition of BMO

January 8, 2024

The most basic non-trivial example of singular integral is the Hilbert transform, defined, for $f \in L^2(\mathbb{R})$ as

$$Hf(x) := p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy = \lim_{\varepsilon \to 0} \frac{1}{\pi}\int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\,dy.$$

It is called a singular integral, because, if it were not for the "principal value", the integrand (barely) fails to be integrable. It is indeed a (somewhat deep) old theorem that this expression makes sense (in $L^2(\mathbb{R})$ and a.e.) and that $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, with Lebesgue measure.

A natural question arose as to for which weights $0 \le w \in L^1_{loc}(\mathbb{R})$ is $H : L^2(\mathbb{R}, w(x)dx) \to L^2(\mathbb{R}, w(x)dx)$. This question has important applications e.g. in boundary value problems in PDE. This question has two known answers, both of which characterize such weights. The first one, due to Muckenhoupt and Wheeden, using a real variables proof, is that $H : L^2(\mathbb{R}, w(x)dx) \to L^2(\mathbb{R}, w(x)dx)$ if and only if the weight w belongs to the so-called Muckenhoupt A_2 class, which means that

$$[w]_{A_2} \equiv \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{w(x)}\,dx\right) < \infty,$$

where the supremum is taken over intervals $Q \subset \mathbb{R}$, and $|Q|$ denotes the Lebesgue measure (length) of Q.

The second characterization, proved by Cotlar and Sadosky using complex analysis, states that $H : L^2(\mathbb{R}, w(x)dx) \to L^2(\mathbb{R}, w(x)dx)$ if and only if $\log(w) = u + Hv$, where $u, v \in L^\infty(\mathbb{R})$ and $\| v \|_\infty < \frac{\pi}{2}$.

There is no known direct proof of the equivalence of the Cotlar-Sadosky condition and the Muckenhoupt A_2 condition. The best known bound obtained yields $\| v \|_\infty < \pi$ (the implication Cotlar-Sadosky implies A_2 is easy, the difficulty lies in the converse implication).

Note that $w \in A_2$ is essentially equivalent to $\log(w) \in BMO$, the space of functions with bounded mean oscillation. Indeed, $BMO = \{\lambda \log(w) : \lambda \in \mathbb{R}, w \in A_2\}$. So any advance in proving directly that Muckenhoupt A_2 implies Cotlar-Sadosky should translate in a constructive understanding of a decomposition of BMO functions (a deep, well-known problem, only partially understood). This is the project proposed (to advance in such direction, possibly using new ideas from quasi-conformal maps), and would be joint work with my coauthor Eero Saksman from the Univeristy of Helsinki.

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